ON THE APPROXIMATION OF FUNCTIONS OF A REAL VARIABLE AND ON QUASI-ANALYTIC FUNCTIONS[1]

I

SURVEY OF THE ORIGINS AND DEVELOPMENT OF THE THEORY OF APPROXIMATION[2]

1. *Weierstrass's first theorem. Remarks suggested by the title of the Note containing this theorem*

The recent investigations on the approximation of functions take their point of departure in a note of Weierstrass presented in 1885 to the Academy of Sciences at Berlin and containing two theorems of which this is the first: *Every function of x continuous in the interval (ab) is developable in a uniformly convergent series of polynomials in that interval.*

This is certainly a perfectly precise and extremely simple statement, of which the proof, as we shall see, is also quite simple. Yet the theorem seemed quite remarkable to the contemporaries of Weierstrass, and created considerable stir; and there is no doubt that it seemed quite remarkable

[1] Lectures delivered at the Rice Institute on December 16, 17, and 19, 1924, by Professor Charles de la Vallée Poussin of the University of Louvain. Translated from the French by Professor Griffith C. Evans of the Rice Institute.

[2] We have already treated analogous ideas in a lecture delivered at a meeting of the Swiss Mathematical Society, held at Fribourg, Feb. 24, 1918 (L'Enseignement Mathématique, t. XX, 1918, p. 23), and more recently, in a lecture given to the Accademia Romana dei Nuovi Lincei, at Rome, in 1923 (Lezioni publiche tenute nella Settimana Accademica, 26 aprile — 2 maggio 1923).

to Weierstrass himself. We find the proof of this in the very title which Weierstrass gave to his note, a title which excites questions which reach beyond the scope of the theorem which was proved. It runs as follows: *On the analytic representability of so-called arbitrary functions of a real variable.*[1] Is this really the same thing as the special proposition quoted above? The terms employed in this caption are so vague that it would seem that the first thing to do would be to find their precise meaning. What is meant, generally, by an "analytic representation"? What are the so-called "arbitrary" functions in question? What is this species of antinomy between the words "analytic" and "arbitrary" which Weierstrass believes he has dissolved? Indeed, the title selected by Weierstrass refers much more to contemporary concepts than to the real content of the memoir. We shall persuade ourselves of this by discovering what was understood at the time by "arbitrary function."

Previously, at the time of Euler, for instance, what one called a function was an algebraic expression. The simplest sort of function was a power of x, like x^2, x^3, . . . A function was thus defined by a certain process of calculation, a certain law which enabled one to proceed from the value of x to the value of the function, and this law was *the same* for all values of x. Afterwards were considered functions defined by more complicated formulas, for example, by a series of powers like the series of Maclaurin or Taylor; but mathematicians were persuaded that such a formula defined a unique law for proceeding from the value of the variable to that of the function. It was sufficient to know the function in an interval no matter how small in order to

[1] *Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen.* Berl. Ber. (1885) p. 633, p. 789.

deduce that law, and therefore, also, the knowledge of the function in an arbitrary interval.[1]

To this original concept of *analytic function* (that is, function given by a formula which binds its definitions for different values of x) is opposed the concept of *arbitrary function*, a concept in which precisely this liaison is suppressed. The definition of arbitrary function is *point wise;* it is made *point by point*, without there being any longer a dependence between the definitions of the function in two different points. It was natural to suppose that this *dissociation* of the function would, as a consequence, render its representation by one single formula impossible. Thus it was in early days that the two concepts of analytic and arbitrary functions were contrasted.

But the validity of this distinction was put in doubt in a remarkable manner, in the early part of the nineteenth century, by Fourier's researches on trigonometric series. In fact, Fourier showed that one could represent by a single trigonometric series, built up with a single set of coefficients, functions which were up to that time considered as different; e.g., $\sin x$ between 0 and π and $\cos x$ between π and 2π. Thus was set the problem of the analytic representation of so-called arbitrary functions.

Fourier's series did not completely solve the problem. Even a continuous function is expressible in a trigonometric series only by means of certain conditions which are not always satisfied. The problem of the analytic representation of a continuous function, given arbitrarily, was solved for the first time in the memoir in question, of Weierstrass; and this is the explanation of the somewhat ambitious title which the author gave to it.

[1] This is the property which is possessed by functions of a complex variable which we speak of now as *analytic*. We shall return to it in regard to quasi-analytic functions.

We may, nevertheless, point out that Weierstrass's theorem is not paradoxical at bottom. We are dealing in fact, only with continuous functions. But a continuous function is not completely dissociated; the function is completely determined by the values which it takes for rational values of x, and, therefore, by a denumerable infinity of conditions, since the values for x irrational may be inferred from the others in view of the continuity. Moreover, a series of polynomials contains a denumerable infinity of parameters. The infinity of parameters and the infinity of conditions are aggregates of the same power. Hence it is not surprising that such a series can represent any continuous function.

But in his caption Weierstrass does not say continuous; he merely says: *On the analytic representability of so-called arbitrary functions.* If then we take the title as it stands, we may properly ask what truth there may be in the statement of this general possibility.

The very special analytic representation which is used by Weierstrass is that by means of a series of polynomials. If further we suppose, with Weierstrass, that the series is uniformly convergent, it is an elementary theorem that that representation applies only to continuous functions.

Let us now discard the condition of uniformity of convergence and ask what will then be the functions which are developable in series of polynomials. The answer to this question is found in the fundamental investigations made by Baire. In his thesis (1899), Baire published a classification of functions of the highest importance. It runs as follows:

Continuous functions form the class 0; discontinuous functions which are limits of continuous functions form the class 1; the functions which are limits of functions of class 1 and are not of class 1, or class 0, are of class 2; and so on.

The functions which can be expressed in series of polynomials are those of classes 0 and 1, and no others. Baire in his thesis gave a celebrated theorem for the purpose of characterizing the functions of class 1, which bears his name,[1] and which is one of the most profound and fertile of the theory of functions of a real variable. Thanks to this theorem it is easy to define the functions which are not of class 1, and which, therefore, it is not possible to represent by a series of polynomials. Perhaps they will admit some other sort of analytic representation than this. But what then, speaking generally, do we mean by *analytic representation?*

Weierstrass would perhaps have been much embarrassed in replying to this question. The question is in fact rather vague. Nevertheless there is a paper, by Lebesgue, entitled: *On the functions which are representable analytically,*[2] where we can find the basis of an answer to the question. If we accept Lebesgue's point of view, and it would be difficult to do otherwise, the analytically representable functions are those which may be defined in terms of continuous functions by means of a denumerable infinity of processes of passing to the limit, — in other words, they are the ones which come under Baire's classification.

Now there exist, at least theoretically, functions which do not come under this classification. Thus it follows that the title of Weierstrass's memoir announces a possibility open to debate. Weierstrass is the one among mathematicians most concerned with rigour, and the one who has given the best models of it. Is it not, therefore, quite startling to see him put, as he did, such a precise and simple theorem under a somewhat ambiguous title, — one which leads to

[1] Baire's theorem: A function of class 1 is punctually discontinuous on every perfect set.

[2] Journal de Mathématiques, t. 60 (1905), p. 139.

the most thorny problems of mathematical philosophy, and moreover, one which, taken literally, would seem to be incorrect?

2. *Original proof of Weierstrass's theorem*

Let us turn now to the proof first given by Weierstrass for his theorem, reproduced by E. Picard in the first volume of his *Traité d'Analyse*, and become classic.

Weierstrass endeavors first, given a function $f(x)$ continuous in an interval ab, to construct a polynomial which comes arbitrarily near to the function in that interval. He takes his point of departure in the integral, familiar in the calculus of probabilities,

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt = 1.$$

By changing t into $t\sqrt{n}$, it becomes

$$\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2}dt = 1.$$

If we suppose that n is a positive integer increasing indefinitely, the only values of t which give an effective contribution to the integral are those infinitely near to zero. It follows that if α and β are two numbers of different signs, the integral

$$(1) \qquad \sqrt{\frac{n}{\pi}}\int_{\alpha}^{\beta} e^{-nt^2}dt$$

will have the same limit as the preceding when n tends to infinity, and this limit will therefore be unity.

Let now $f(x)$ be a function of x continuous in the interval (a, b), and x a value between a and b. We form the integral

$$(2) \qquad F(x) = \sqrt{\frac{n}{\pi}}\int_{a}^{b} f(t)e^{-n(x-t)^2}dt.$$

It is the same as

$$(3) \qquad F(x) = \sqrt{\frac{n}{\pi}} \int_{a-x}^{b-x} f(t + x) e^{-nt^2} dt.$$

When n is allowed to become indefinitely great, we see, by comparison with (1), that the value of the integral (3) is a mean among the values taken by $f(x + t)$ as t ranges between the two limits of opposite sign $a - x$ and $b - x$. But since only the infinitely small values of t contribute anything, the mean value of $f(x + t)$ approaches $f(x)$ uniformly, the function being continuous. Hence $F(x)$ approaches $f(x)$ uniformly as n becomes infinite, through any interval interior to (a, b).

But the end in view is not yet attained, since $F(x)$ is not a polynomial; and this is in fact the objection to Weierstrass's method. In order to substitute a polynomial for $F(x)$ it is necessary to replace the exponential $e^{-n(t-x)^2}$, in the integral (2), by its uniformly convergent development in powers of $x - t$, and to keep in this development merely the number of terms necessary to get the desired approximation. By taking n sufficiently great, and taking in the development of the exponential a sufficient number of terms, we are able then to construct a polynomial which comes as near as we please to $f(x)$.

The problem of expressing $f(x)$ in a series of polynomials, and that of constructing a polynomial which approximates it to any desired degree are entirely equivalent. In fact, if P_n is a polynomial of degree n, and we have a sequence of polynomials $P_1, P_2, \ldots, P_n, \ldots$ approaching $f(x)$ uniformly as n becomes infinite, we also have an expression for $f(x)$ in a uniformly convergent series of polynomials, as follows:

$$f(x) = P_1 + (P_2 - P_1) + \ldots + (P_n - P_{n-1}) + \ldots$$

Hence Weierstrass's theorem is proved.

3. *Landau's integral*

The inconvenience of Weierstrass's method is that the function $F(x)$ defined by the integral (2) is not a polynomial in x. The properties of the integral which are used by Weierstrass relate to the presence of the exponential factor $\sqrt{n}\, e^{-n(x-t)^2}$, which has been called the *factor of discontinuity*. Hence it occurred to me to construct an integral similar to (2), but with another factor of discontinuity which should be a polynomial in x. Thus I was led to form an integral which since has received the name of *Landau's integral*, since Landau used it a few months before me for the same purpose.[1] However, this integral was considered even earlier by Hermite, and is found in his correspondence with Stieltjes, — a fact which Landau pointed out himself.

We replace the integral (1), which is the basis of Weierstrass's method, by the following,

$$\frac{k_n}{2}\int_{-1}^{+1}(1-t^2)^n dt = 1, \quad k_n = \frac{2\cdot 4 \ldots 2n}{1\cdot 3 \ldots 2n-1},$$

in which n is an integer. As before, when n increases indefinitely the only values of t important for the value of the integral are those infinitely near to zero.

Let now $f(x)$ be a function continuous in the interval (a, b). We shall assume this interval interior to $(0, 1)$. This hypothesis is indeed legitimate, since if it does not happen to be verified, we can satisfy it by a linear change of variable, which transforms a polynomial into another of the same degree.

With this understood, *Landau's integral* is the following:

$$P_{2n}(x) = \frac{k_n}{2}\int_0^1 f(t)\,[1-(x-t)^2]^n dt.$$

[1] *Über die approximation einer stetiger Funktion durch eine ganze rationale Funktion.* Rend. di Palermo, vol. 25 (1908), p. 337.

This can be substituted for the integral $F(x)$ of Weierstrass, for it yields exactly the same reasoning on the mean values. But it resolves precisely into a polynomial in x, $P_{2n}(x)$, of degree $2n$, which consequently approaches $f(x)$ as n becomes infinite. Thus the end is attained.

4. *Two problems studied by means of Landau's integral*

Landau's integral lends itself to the solution of interesting problems, of which I proposed to myself the two following: [1]

1. What is the order of approximation of $P_{2n}(x)$ to $f(x)$ relative to $1/n$ as n becomes infinite?
2. Are the successive derivatives of $f(x)$ represented approximately by those of the approximate polynomial $P_{2n}(x)$?

The first question, which Lebesgue [2] studied contemporaneously with me, is that of the *order of approximation*. It became the starting point of a theory which has since received great development, to which I will return in a few moments.

The second question, concerning the *differentiability* of the representation, was not new. Painlevé [3] had solved it in 1908 in the case of continuous derivatives. But, as I showed, Landau's integral furnishes a solution of the problem which is much better, and indeed almost perfect; the derivative of P_{2n} of any order p converges towards the derivative of the same order of $f(x)$ at the point x under the single condition of the existence of the latter derivative at

[1] *Sur l'approximation des fonctions de variables réelles et de leurs dérivées par des polynomes et des suites limitées de Fourier.* Bull. de l'Acad. Roy. de Belgique (Classe des Sciences), No. 3, Mars, 1908.

[2] *Sur la représentation approchée des fonctions.* Rend. di Palermo, vol. 26 (1908), p. 325.

[3] Comptes Rendus, t. 126 (1898), p. 459.

the point. That is quite the most remarkable property of Landau's integral, since as process of approximation of $f(x)$ it is only of mediocre value. In fact, the greatest value of Landau's integral for me was to suggest the construction of another integral which is its analogue for trigonometric approximation. This integral, analogous to Landau's, initiates investigations decidedly more interesting than Landau's integral itself. We may then naturally turn to the consideration of trigonometric approximation and the investigations which I have just mentioned.

5. *Trigonometric approximation and the analogue of Landau's integral*

The problem of trigonometric representation is as old as that of representation in terms of polynomials, and goes back to the original note of Weierstrass of 1885, that note containing a second theorem, as follows: *Every function, continuous and periodic, of period 2π, is developable in a uniformly convergent series of finite trigonometric expressions.*

A finite trigonometric expression of degree n is a polynomial of degree n in $\sin x$ and $\cos x$, or, what is the same thing, an expression of the form

$$\alpha_0 + (\alpha_1 \cos x + \beta_1 \sin x) + (\alpha_2 \cos 2x + \beta_2 \sin 2x) + \ldots + \ldots + (\alpha_n \cos nx + \beta_n \sin nx),$$

in which the coefficients α, β are constants.

It is well to notice that, whenever we are concerned with trigonometric approximation, the function $f(x)$ is always assumed to be *periodic*.

Weierstrass's second theorem has since received a large number of different demonstrations. But here I am only concerned with the one which I gave in 1908.[1] It is based

[1] Bull. Acad. Roy. de Belgique (Classe des Sciences), loc. cit.

on the consideration of an integral analogous to Landau's, possessing all of its advantages.

Let $f(x)$ be the continuous periodic function to be represented. The integral which defines a finite trigonometric expression $Q_n(x)$ of order n is the following:

$$Q_n(x) = l_n \int_0^{2\pi} f(t) \left(\cos \frac{x-t}{2}\right)^{2n} dt,$$

where the factor l_n is the reciprocal of the value of the well-known integral

$$\int_0^{2\pi} \cos^{2n} \frac{t}{2}\, dt = 4 \int_0^{\frac{\pi}{2}} \cos^{2n} t\, dt = \frac{1\cdot 3 \ldots (2n-1)}{2\cdot 4 \ldots 2n}\, 2\pi.$$

In this new integral $Q_n(x)$, the factor of discontinuity is $\cos^{2n} \dfrac{t-x}{2}$, which is a finite trigonometric expression of order n in x. We can treat this factor exactly in the same way as that which appears in Landau's integral, and we show in this way that $Q_n(x)$ converges to $f(x)$ when n becomes infinite, and possesses exactly the same properties as the polynomial $P_{2n}(x)$ of Landau with respect to the approximation and differentiability of the representation.

But what is most interesting in this new integral is that it leads, as I showed in my memoir of 1908, to the definition of a new method of summing divergent series.

6. *A new method of summing divergent series*

The trigonometric polynomial $Q_n(x)$, defined by the above integral, is connected in a very interesting way with the Fourier's series of $f(x)$. In fact, if we designate by a_k and b_k the Fourier constants for $f(x)$, to wit:

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx,$$

the expression $Q_n(x)$ is resolved into the finite trigonometric sum of order n

$$\frac{1}{2}a_0+\sum_{k=1}^{n}\frac{n(n-1)\dots(n-k+1)}{(n+1)(n+2)\dots(n+k)}(a_k\cos kx+b_k\sin kx).$$

Now the calculation of this sum and of its limit for $n=\infty$ constitutes a process of summation for the Fourier series. In fact, each term of this sum is obtained by multiplying the term of the same rank of the Fourier series by a numerical factor. These successive factors, for a given n, are continually decreasing, are < 1, and all approach 1 as n is let approach infinity. Since all these factors vanish from the $(n+1)^{th}$ on, this method yields the advantage of having a finite sum for each value of n. In accordance with what we have said, this process enables us to sum the Fourier series and its successive derivatives at a given point, under the single condition of the existence of these derivatives at the point.

These are the results to which I attained in 1908. Since then, several skilful geometers have concerned themselves with this process of summation and have shown that it has a power, importance and interest which I did not expect.

The process was applied by Plancherel to the summation of the series of Laplace and Legendre.[1] Kogbetliantz has applied it quite recently to ultraspherical functions.[2] Gronwall has made investigations even more searching. Some years ago he showed that this method of summation has at least all the generality of those by the Cesaro means.[3]

[1] *Sur l'application aux séries de Laplace du procédé de M. de la Vallée Poussin.* Comptes Rendus, t. 152 (1911), p. 1226.—Rend. di Palermo, vol. 33 (1912), p. 41.

[2] *Sur la sommation des fonctions ultrasphériques par la méthode* (Σ_0) *de M. de la Vallée Poussin.* Rend. di Palermo, vol. 46 (1922), p. 146.

[3] *Über einige Summationsmethoden und ihre Anwendung auf die Fouriersche Reihe.* Journal für die reine und angewandte Mathematik, Bd. 147 (1917), p. 16.

But I have learned, through a kind communication from Dr. Gronwall, that he has continued his researches in this direction, and has obtained most important results, which he has not yet published. With his permission, I indicate one of them which throws considerable light on the question.

A series $u_0 + u_1 + u_2 + \ldots$ is summable in my way and has for its sum the value s if the expression

$$V_n = \sum_{k=0}^{n} \frac{n!\,n!}{(n-k)!\,(n+k)!}\,u_n$$

tends towards s as n becomes infinite. Gronwall found the generating identity for the V_n. It is the following:

$$\sum_0^\infty u_n x^n = \sqrt{1-y}\,\sum_0^\infty \frac{(2n)!}{2^{2n} n!\,n!}\,V_n y^n, \text{ where } y = \frac{4x}{(1+x)^2}.$$

This striking identity directed Gronwall along the road to interesting generalizations of my method of summation. I cannot elaborate them here; but the identity exhibits directly an important fact, viz., that if V_n has a limit, $\Sigma u_n x^n$ necessarily approaches the same limit, as x approaches unity.[1]

If we apply this remark to the summation of Fourier's series, it follows that Poisson's method of summation must offer all the advantages of my own. I noticed the fact in 1908 and proved it in my Memoir, retracing all the proofs in detail. It is seen, thanks to Gronwall's elegant formula, that it was only the particular application of a general property.

7. *The problem of the order of approximation*

The most important of the problems which have been attacked in the study of approximation is that of the *order*

[1] We see in fact that we have

$$\frac{1}{\sqrt{1-y}} = \sum_0^\infty \frac{(2n)!}{2^{2n} n!\,n!}\,y^n.$$

of approximation. Let us define first what we mean by *approximation.* For example, let a continuous function $f(x)$ be represented by means of a polynomial of degree n, and let $P_n(x)$ be such a polynomial. The difference $f - P_n$ is the *error* of the approximation, and is a function of x; its maximum value in the interval of representation is the *approximation* ρ_n. This positive number approaches 0 as $1/n$ approaches zero, if the polynomial P_n is well chosen. It is therefore an infinitesimal of a certain order with respect to $1/n$. The problem of the order of approximation is the following: *To determine the relation which exists between the order of approximation* ρ_n, *which* $f(x)$ *may admit for a finite expression of order* n, *and the differential properties of the function.*

I offered myself the beginnings of an answer to this very problem in 1908, while studying the approximation given by Landau's integral. I showed also that the function $|x|$ admits an approximation to the order of $1/n$ by a polynomial of degree n, and I raised the question of deciding whether or not that was the order of the best possible approximation.[1] This definite question had much more importance for the development of the subject than had the few isolated results which I had obtained, because that question caused the writing of the two most important memoirs on the subject, one by D. Jackson and the other by S. Bernstein.

The problem can be set in two inverse formulations: *the direct problem*, the only one which I had attacked, has for its object to find the possible order of approximation in terms of the assumed properties of the function; *the inverse*

[1] *Sur la convergence des formules d'interpolation entre ordonnées équidistantes.* Bull. Ac. Roy. de Belgique (Classe des Sciences), 1104 (1908). This memoir ends with a note *Sur l'approximation par un polynome d'une fonction dont la derivée est à variation bornée,* the note which was the occasion of the question cited.

problem, more difficult, consists in repassing from a given supposedly possible order of approximation to the differential properties resulting for the function which is represented.

It is the memoir by D. Jackson [1] which answers most completely the direct question, and that of S. Bernstein [2] which answers most completely the inverse problem.

The results contained in these two memoirs constitute the essential matter of the volume of the Borel Collection which I had the privilege of publishing under the title: *Leçons sur l'approximation des fonctions de variables réelles.*[3] I combined the results obtained by the two authors above named, and filled them out in many points; I changed or simplified the proofs; but I contributed little in the way of new materials to the construction.

There will be found in that volume quite a number of results connected with the problem proposed, some concerning functions which possess merely a finite number of successive derivatives, others relative to functions which are indefinitely differentiable, or even analytic. It will suffice for me here to reproduce merely one of these results, — the one which makes apparent in most striking fashion the mutual dependence which exists between the order of approximation and the differential properties of the function. It is the subject of a theorem which I gave for the

[1] *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationnale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Inaugural Dissertation, Göttingen (1911). This memoir was crowned by the Academy of Sciences of Göttingen to which it had been presented in answer to a problem set by the Academy identical with mine.

[2] *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné.* Memoires publiés par la Classe des Sciences de l'Academie Royale de Belgique. Collec. in-4, 2e série, t. IV, 1912. Crowned memoir presented for a prize question set by the Class in 1911 at my instigation.

[3] Paris, Gauthier-Villars (1919).

first time in a meeting of the Swiss Mathematical Society at Fribourg in 1918,[1] and deals with the trigonometric representation of a periodic function $f(x)$.

If $f(x)$ admits a derivative of order r satisfying a Lipschitz condition of order α $(0 < \alpha < 1)$,[2] then whatever n, $f(x)$ admits a trigonometric representation of order n, with an approximation

$$\rho_n < \frac{M}{n^{r+\alpha}} \qquad (M \text{ const.}).$$

Conversely, if it is possible to satisfy this last condition for every n, $f(x)$ possesses a derivative of order r which satisfies a Lipschitz condition of order α.

This theorem deals with trigonometric approximation, but there is an analogous theorem for representation in terms of polynomials in an interval (a, b). In fact the two methods of representation lead each to the other, a fact which Bernstein brought out most clearly by his use of *trigonometric polynomials.*

We show in fact, following Bernstein, that the representation of $f(x)$ in terms of polynomials may be derived from its trigonometric representation. We can always assume that the interval of representation is $(-1, +1)$, since any other may be reduced to it by a linear transformation. We make then the transformation $x = \cos\phi$. We have then for the function $f(\cos\phi)$, which is periodic and an even function, the trigonometric development

$$f(\cos\phi) = a_0 + a_1 \cos\phi + \ldots + a_n \cos n\phi + \ldots$$

But $\cos n\phi$ is a polynomial of degree n in $\cos\phi$; and it is this polynomial $P_n(\cos\phi)$, which Bernstein calls a *trigonometric polynomial.* Hence if we return to the variable x,

[1] L'Enseignement Mathématique, t. 20 (1918), p. 23.

[2] That is to say, one can assign a constant A such that for all x and δ we have

$$|f^{(n)}(x+\delta) - f^{(n)}(x)| < A\delta^{\alpha}.$$

we have the redresentation of $f(x)$ in a series of trigonometric polynomials

$$f(x) = a_0 + a_1P_1(x) + \ldots + a_nP_n(x) + \ldots$$

The transformation moreover applies directly to a finite sum as well as to a series, a fact which justifies our statement.

As we have already had occasion to notice, the theory of approximation has been the cause of the discovery of important theorems which retain a very considerable interest even outside that theory. One of the most remarkable in this way is a theorem about trigonometric expressions, discovered by Bernstein; it plays the essential rôle in the solution of the *inverse problem* mentioned above. It runs as follows:

If the absolute value of a trigonometric expression of order n does not exceed M, the absolute value of its derivative does not exceed nM, and, consequently, the absolute value of its p^{th} derivative does not exceed n^pM.

The algebraic character of this theorem was brought out by Marcel Riesz in 1914.[1] I also pointed out this algebraic character in my book, not being aware of Riesz's theorem at that time.

I have already given the most characteristic theorem on the order of approximation. It is interesting only for functions which possess merely a finite number of successive derivatives. In my book on approximation, I attacked the problem of obtaining results of similar precision for indefinitely differentiable functions, and in particular, for analytic functions. I was not able to solve it completely. I shall not describe these results here, because I am inclined to think that *for these cases* it is just as

[1] *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome.* Jahresberichte der deutschen Mathematiker-Vereinigung, Bd. 23 (1914), p. 354.

interesting to study the convergence of their Fourier series. This study, moreover, is made by similar methods, and will be the object of our second lecture.

8. *The best approximation*

Given a function $f(x)$ continuous in an interval (a, b), there exists a polynomial which furnishes the best possible approximation in this interval among all those of degree n and this polynomial is unique: it is called *the polynomial of minimum approximation* of degree n. This polynomial P_n is completely characterized by the fact that the difference $f - P_n$ takes on its maximum value $n + 2$ times, with alternation of sign. These results were given long ago by Tchebycheff,[1] and the proofs were made rigorous by Kirschberger and Borel.[2] Analogous theorems hold for the best trigonometric approximation of order n. The trigonometric expression of order n, S_n, which gives the best approximation is characterized by the fact that $f - S_n$ takes on its maximum value $2n + 2$ times, with alternation of sign, in its interval of periodicity. Certainly, Bernstein has succeeded in determining the polynomial of minimum approximation in some quite remarkable cases. But, in general, the exact determination of this polynomial is an inaccessible problem. There isn't any really practicable method for the approximate calculation of this polynomial; and of course there is the same state of affairs in the case of trigonometric approximation.

Nevertheless, it is quite important, when we consider an approximate expression of a certain order, to know if it gives a good approximation; that is to say, whether or

[1] *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions.* Mémoires de l'Acad. Imp. des Sciences de St. Petersbourg, Sciences Math. et Phys., Série 6, t. VII, 1859. Collected Works, vol. I.

[2] *Leçons sur les fonctions de variables réelles*, Paris, Gauthier-Villars (1905).

not the approximation obtained differs too much from the best possible one. To decide this, it is necessary to know a lower limit to the best approximation. There are, as a matter of fact, quite a number of criteria which I developed in my *Leçons sur l'approximation.* I shall mention the three most characteristic. In order of date, the first belongs to Lebesgue, the second to myself, and the third to Bernstein.

Lebesgue's criterion [1] concerns trigonometric approximation. It is stated as follows: *If the sum of order* n *of the Fourier series for the periodic function* $f(x)$ *does not exceed* $\phi(n)$, *the best approximation of order* n *is not less than* $A\phi(n)/\log n$, *where* A *is a numerical constant which can be determined once for all.*

A second criterion, which I published some months later,[2] concerns polynomial approximation. It may be stated as follows: *If the polynomial* Q_n *of degree* n *is such that the difference* $f - Q_n$ *takes values of alternating sign in* $n + 2$ *consecutive points in the interval* (a, b), *the smallest in absolute value of these* $n + 2$ *values is a lower limit for the best approximation.*

Finally, Bernstein's criterion [3] deals again with trigonometric representation. It is connected with the well-known minimal property of the Fourier sums. This is the statement of it: *Let* a_k, b_k *be the Fourier constants for the continuous periodic function* $f(x)$; *the best approximation for this function by a trigonometric sum of order* n *is not less than the square root of*

$$\frac{1}{2}\sum_{n+1}^{\infty}(a_k^2 + b_k^2).$$

We shall see in the next lecture that this criterion is particularly interesting when we apply it to analytic functions.

[1] Ann. de la Fac. des Sciences de Toulouse, série 3, t. 1 (1910).

[2] Bull. Acad. Roy. de Belgique Classe des Sciences (1910).

[3] S. Bernstein (1912), loc. cit.