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Young Tableaux with Applications to Representation Theory and Flag Manifolds

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ABSTRACT

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We outline the use of Young tableaux to describe geometric and algebraic objects using combinatorial methods. In particular, we discuss applications to representations of the symmetric group and the general linear group, flag varieties, and Schubert varieties. We also describe some recent work, including proofs of the Saturation Conjecture and a theorem on the eigenvalues of sums of Hermitian matrices.

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Contents

1. Introduction

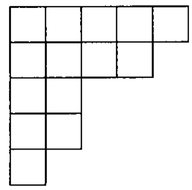
Young tableaux are relatively simple objects to define and describe, but they seem to have a deep relationship with problems in algebra and geometry. In this paper, we will describe the construction of Young tableaux and the Littlewood-Richardson numbers, and then show how we can apply these to describe objects in representation theory and algebraic geometry.

Generally, Sections 2, 3, and 4 contain the basic construction and properties of Young tableaux, and the following sections are applications of these ideas. In Section 2, we define Young tableaux and give three constructions for the product of two tableaux. This is then used to define the tableau ring. In Section 3, we describe a correspondence between matrices and pairs of tableaux. Section 4 uses the results of Sections 2 and 3 to define the Littlewood-Richardson number, and to give several equivalent combinatorial constructions. In Section 5, we construct the irreducible representations of the symmetric group S_n. We then calculate the character of these representations using a correspondence with the symmetric polynomials. In Section 6, we construct the irreducible representations of the general linear group $GL(E)$. In Section 7, we describe the flag varieties of nested subspaces. Then using the constructions from Section 6, we describe their relationship with the representations of $GL(E)$. In Section 8, we describe Schubert varieties using Young diagrams. In Section 9, we discuss more general Littlewood-Richardson rules and describe some recent results. In particular, we give proofs of the Saturation Conjecture and a theorem on the eigenvalues of sums of Hermitian matrices.

2. Young Tableaux

Definition 2.1. *Given an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, the* Young diagram *associated to partition λ is a collection of cells arranged in left-justified rows, where the i^{th} row has λ_i cells.*

We will require that partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ be weakly-decreasing, so in particular, a Young diagram consists of rows of weakly-decreasing numbers of boxes. For example, for the partition $\lambda = (5, 4, 2, 2, 1)$, we have the Young diagram



We will use the terms partition and Young diagram essentially interchangeably, depending on context.

Definition 2.2. (i) *A* filling *of a Young diagram is a Young diagram in which each cell contains an integer.*

(ii) *A* numbering *is a filling in which each entry is distinct.*

(iii) *A* Young tableau T *is a filling on a Young diagram* λ *such that:*

(a) *the filling is weakly increasing from left to right along each row,*

(b) *the filling is strictly increasing down each column.*

Here, we say that the tableau T *has* shape λ.

(iv) *A* standard tableau *is a tableau on shape* $\lambda \vdash n$ *in which the numbers* 1 *through* n *each occur once.*

For example, the Young diagram $\lambda = (5, 4, 2, 2, 1)$ can have tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 & 3 \\
\hline
2 & 3 & 3 & 3 \\
\cline{1-4}
4 & 4 \\
\cline{1-2}
5 & 6 \\
\cline{1-2}
6 \\
\cline{1-1}
\end{array}$$

and standard tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 10 & 11 & 14 \\
\hline
3 & 4 & 12 & 13 \\
\cline{1-4}
5 & 8 \\
\cline{1-2}
6 & 9 \\
\cline{1-2}
7 \\
\cline{1-1}
\end{array}$$

Notice that standard tableaux must have strictly increasing rows as well as columns.

Definition 2.3. *Given a diagram λ, the* conjugate *diagram $\tilde{\lambda}$ is the diagram obtained by flipping λ across its diagonal. Given a filling T on λ, the* transpose *filling T^τ on $\tilde{\lambda}$ is the filling obtained by flipping the filling T across its diagonal.*

Note that, given a tableau with distinct entries, the conjugate is also a tableau, so in particular, the conjugate of a standard tableau is always a standard tableau. However, the conjugate of an arbitrary tableau need not be a tableau, as in the following example.

Example 2.4.

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \qquad T^\tau = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

If we wish to describe the entries of a tableau T, one method is to count the number of entries with each value. For an ordered sequence $\mu = (\mu_1, \ldots, \mu_l)$, we say that T has *content* μ if T has μ_1 1's, μ_2 2's, and so on. Notice that $\sum \mu_i$ is the total number of cells in T, so if we require that μ be a decreasing sequence, it is also a partition of n, the number of cells of T. Since the content does not necessarily determine the placement of the entries of T, there may be multiple distinct tableaux with shape λ and content μ. The *Kostka number* $K_{\lambda\mu}$ is the number of tableaux with shape λ and content μ.

Definition 2.5. *Given a tableau T and integer m equal to the largest entry in T, we can define a monomial*

$$x^T = \prod_{i=1}^{m} x_i^{\#\text{ times } i \text{ appears in } T}.$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ and an integer $m \geq k$, the Schur polynomial *(in m variables) s_λ is defined to be the sum of these monomials over all tableau on λ drawn* *from the alphabet* $\{1, \ldots, m\}$

$$s_\lambda(x_1, \ldots, x_m) = \sum_T x^T.$$

Example 2.6. *The partition* $\lambda = (2, 1)$ *has the following tableaux on the alphabet* $\{1, 2, 3\}$.

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

Therefore, λ *has Schur polynomial on the variables* (x_1, x_2, x_3) *given by*

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

We also have a few distinguished examples that are simple to compute, but will be valuable for later calculations.

Example 2.7. *We denote the diagram consisting of a single row of* n *cells by* $\lambda = (n)$. *Since any set of* n *integers describes a unique tableau on* (n)*, the Schur polynomial consists of all polynomials in* m *variables of degree* n,

$$s_{(n)}(x_1, \ldots, x_m) = x_1^n + x_1^{n-1} x_2 + \cdots + x_m^n.$$

This is the n^{th} complete symmetric polynomial *in* m *variables, denoted* $h_n(x_1, \ldots, x_m)$.

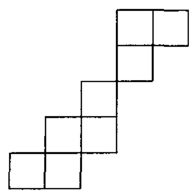
Example 2.8. *We denote the diagram consisting of a single column of* n *cells by* $\lambda = (1^n)$. *Any strictly increasing sequence of* n *integers will give a tableaux on* (1^n), *so this has Schur polynomial*

$$s_{(1^n)}(x_1, \ldots, x_m) = \prod_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

This is the n^{th} elementary symmetric polynomial *in* m *variables, denoted* $e_n(x_1, \ldots, x_m)$.

Definition 2.9. *Given partitions* $\lambda = (\lambda_1, \dots, \lambda_m)$ *and* $\mu = (\mu_1, \dots, \mu_n)$ *such that* $\lambda_i \geq \mu_i$ *for all* i*, we say that* μ *is* contained *in* λ*, written* $\mu \subset \lambda$*. A* skew diagram *is a diagram obtained from a Young diagram by removing the cells corresponding to another Young diagram contained in it. For* $\mu \subset \lambda$*, we denote the skew diagram formed by removing* μ *from* λ *with* λ/μ*. A* skew tableau *is a filling on a skew diagram such that entries are weakly increasing left to right along rows and strictly increasing down columns.*

Example 2.10. *For the diagrams* $\lambda = (5, 4, 3, 3, 2)$ *and* $\mu = (3, 3, 2, 1)$*, we have skew diagram*

$\lambda/\mu =$ 

It will be valuable to define a few partial orderings on partitions. We already have one such ordering $\subset$, given by inclusion. Another ordering is the *lexicographic ordering*, denoted $\leq$. For partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$, we have $\lambda \leq \mu$ if the first i such that $\lambda_i \neq \mu_i$ has $\lambda_i < \mu_i$. Notice that this is slightly different from inclusion ordering, as lexicographic ordering is a total ordering. A third ordering on partitions is *domination ordering* $\trianglelefteq$. We have $\lambda \trianglelefteq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

for all i.

If $\lambda \subset \mu$, we have that $\lambda_i \leq \mu_i$ for all i, so it follows that $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i, and so $\lambda \trianglelefteq \mu$ as well. Similarly, if $\lambda \trianglelefteq \mu$, then if we consider the first i such that $\lambda_1 + \cdots + \lambda_i \neq \mu_1 + \cdots + \mu_i$, we must have $\lambda_i < \mu_i$, and so $\lambda \leq \mu$. Therefore, we have the implications

$$\lambda \subset \mu \Rightarrow \lambda \trianglelefteq \mu \Rightarrow \lambda \leq \mu$$

though none of these implications can necessarily be reversed.

Now we wish to define some operations on tableaux. Our goal is to describe a ring on the set of tableaux. We will begin by defining the product of two tableaux and describing its properties.

First, we will need the *row-insertion* algorithm. Given a tableau T and an integer x, we define a new tableau $T \leftarrow x$ constructed in the following way. If x is at least as large as all entries in the first row, then we add x in a new cell at the end of the first row. Otherwise, there is some entry in the first row larger than x; find the first such entry x_1 and replace it with x (note that this preserves the tableau property along the row and column, since $x < x_1$). Then take x_1 and repeat this step on the second row. If we continue in this manner through each row of the tableau, we are left with a tableau that has one additional cell.

Example 2.11. *Let* $T = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3&3&4\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}$. *We will construct the tableau given by* $T \leftarrow 2$. *The following shows the sequence of "bumps", along with the elements that are bumped at each step.*

$$\boxed{2}\ \begin{array}{|c|c|c|}\hline 1&2&\mathbf{3}\\\hline 3&3&4\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} \to \boxed{3}\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&\mathbf{4}\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} \to \boxed{4}\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&3\\\hline 4&\mathbf{5}\\\cline{1-2} 6\\\cline{1-1}\end{array} \to \boxed{5}\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&3\\\hline 4&4\\\cline{1-2} \mathbf{6}\\\cline{1-1}\end{array} \to \boxed{6}\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&3\\\hline 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

$$\to \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&3\\\hline 4&4\\\cline{1-2} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$$

So $T \leftarrow 2 = \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&3\\\hline 4&4\\\cline{1-2} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$.

As noted above, this procedure will give us a new tableau. We can also describe how this is an invertible process: given the resulting tableau and the cell that has been added, we can reverse each step to recover the original tableau and the added entry.

Using this row-insertion procedure, we can define a *product tableau* $T \cdot U$. Given tableau T with shape λ and tableau U with shape μ, the product $T \cdot U$, is defined to be the tableau

$$T \cdot U = ((\cdots((T \leftarrow x_1) \leftarrow x_2)\cdots) \leftarrow x_{s-1}) \leftarrow x_s,$$

where the elements $x_1, \ldots, x_s$ are the entries of U, indexed from left to right, and from the bottom row to the top. (This indexing will motivate our definition of the "word" of a tableau.)

Example 2.12. *Let* $T = \begin{ytableau} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$ *and* $U = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$. *Then, indexing the entries of U from left to right, bottom to top, we have* $x_1 = 3$, $x_2 = 1$, *and* $x_3 = 2$. *Then we have the product*

$$T \cdot U = ((T \leftarrow 3) \leftarrow 1) \leftarrow 2 = \left(\left(\begin{ytableau} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau} \leftarrow 3 \right) \leftarrow 1 \right) \leftarrow 2$$

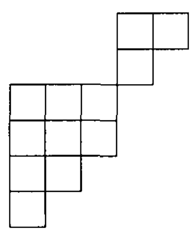
$$= \left(\begin{ytableau} 1 & 2 & 3 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau} \leftarrow 1 \right) \leftarrow 2 = \begin{ytableau} 1 & 1 & 3 & 3 \\ 2 & 3 & 4 \\ 3 & 5 \\ 4 \\ 6 \end{ytableau} \leftarrow 2$$

$$= \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$$

Since the result of a row-insertion is a tableau, the product defined in this way will be a tableau as well. Since we are inserting each of the entries of U into the tableau T, the number of cells in the product $T \cdot U$ is the sum of the numbers of cells in T and U. To show the other properties of the product, we will need to define two alternate methods of constructing the product tableau.

Our second method for calculating $T \cdot U$ will come from constructing a skew tableau from T and U. Given two non-trivial Young diagrams $\mu \subset \lambda$, we define two types of cells. An *inside corner* is a cell contained in the smaller diagram μ such that the cells to the right and below are not contained in μ. An *outside corner* is a cell contained in the larger diagram λ such that the cells to the right and below are not contained in λ.

Example 2.13. *Given shapes* $\lambda = (5, 4, 3, 3, 2, 1)$ *and* $\mu = (3, 3)$, *we have skew diagram*

$\lambda/\mu =$ 

This has one inside corner, the third cell in the second row of μ, *and* 5 *outside corners, on the last cell of the first, second, fourth, fifth, and sixth rows of* λ.

In order to transform a skew tableau into a Young tableau, we describe a procedure called *sliding.* Given a skew tableau S on λ/μ, choose an inside corner and consider the cells directly to the right and below it. Choose the smaller entry and move it into the inside corner, leaving a "hole". If both entries are equal, choose the cell below the hole. Now repeat this process on the hole, comparing the cells directly to the right and below the hole. This procedure continues until the hole is moved to an outside corner, and then it is removed from the skew tableau.

Example 2.14. *Consider the skew tableau* $\begin{ytableau} \none & \none & 1 & 2 \\ \none & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 5 \end{ytableau}$*. Choose the second cell of the first row as the inside corner we will slide out of the skew tableau. Then we have the following sequence of slides.*

$$\begin{ytableau} \none & \, & 1 & 2 \\ \none & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & \, & 2 \\ \none & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & 2 & 2 \\ \none & 2 & \, & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & 2 & 2 \\ \none & 2 & 3 & 3 \\ 2 & 3 & \, & 4 \\ 4 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & 2 & 2 \\ \none & 2 & 3 & 3 \\ 2 & 3 & 4 & \, \\ 4 & 4 & 5 \end{ytableau} \longrightarrow \begin{ytableau} \none & 1 & 2 & 2 \\ \none & 2 & 3 & 3 \\ 2 & 3 & 4 \\ 4 & 4 & 5 \end{ytableau}$$

After sliding an inside corner out of the skew diagram, we can continue this process and slide another inside corner out of the diagram until we are left with a Young tableau. The resulting tableau is called the *rectification* of the skew tableau S, denoted $\mathrm{Rect}(S)$.

Example 2.15. *Consider the skew tableau* $S = \begin{ytableau} \none & \none & \none & 1 & 2 \\ \none & \none & \none & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$ *. Then we obtain the rectification of S through the following sequence of slides. At each step, the inside corner that we are sliding out of the skew tableau is marked with a dot.*

$$\begin{ytableau} \none & \none & \none & 1 & 2 \\ \none & \none & \cdot & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau} \longrightarrow \begin{ytableau} \none & \none & \cdot & 1 & 2 \\ \none & \none & 3 & 3 \\ 1 & 2 & 4 \\ 3 & 3 \\ 4 & 5 \\ 6 \end{ytableau} \longrightarrow \begin{ytableau} \none & \none & 1 & 2 \\ \none & \cdot & 3 & 3 \\ 1 & 2 & 4 \\ 3 & 3 \\ 4 & 5 \\ 6 \end{ytableau} \longrightarrow \begin{ytableau} \none & \none & 1 & 2 \\ \cdot & 2 & 3 & 3 \\ 1 & 3 & 4 \\ 3 & 5 \\ 4 \\ 6 \end{ytableau} \longrightarrow \begin{ytableau} \none & \cdot & 1 & 2 \\ 1 & 2 & 3 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$$

$$\longrightarrow \begin{ytableau} \cdot & 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$$

So $\mathrm{Rect}(S) = \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$.

Note that the rule for sliding each cell is chosen to preserve the tableau property. At each step, the rows are still weakly increasing, and columns are still strictly increasing. Therefore, it follows that the result of rectification is a tableau. Also, notice that in this rectification procedure, we make a choice of a sequence of inside corners to slide. However, we will show later that the rectification of a skew tableau is independent of this sequence of slides. Therefore, the rectification of a given skew tableau is a unique Young tableau.

Now we will use this rectification procedure to define another method for calculating the product tableau. For tableaux T and U, define $T * U$ to be the skew tableau constructed by taking an $m \times n$ rectangle of (empty) cells, where m is the number of rows of U and n is the number of columns of T, and placing T below it and U to the right of it.

Example 2.16. *Let* $T = \begin{ytableau} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$ *and* $U = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$. *Then we have skew tableau*

$$T * U = \begin{ytableau} \none & \none & \none & 1 & 2 \\ \none & \none & \none & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau} .$$

Then we can define the product tableau to be

$$T \cdot U = \mathrm{Rect}(T * U).$$

Example 2.17. *Let* $T = \begin{ytableau} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$ *and* $U = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$. *Then we have*

$$T \cdot U = \mathrm{Rect}(T * U) = \mathrm{Rect}\left(\begin{ytableau} \none & \none & \none & 1 & 2 \\ \none & \none & \none & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau} \right)$$

Then by Example 2.15, $T \cdot U = \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$ *, which agrees with our calculation of* $T \cdot U$ *from Example 2.12.*

It remains to be shown that this definition of the product tableau agrees with the first definition.

Now we will define the third method for calculating the product tableau. For this, we will need to define a few new objects. For a tableau T, the *(row) word* of T, denoted $w(T)$, is the sequence of entries of T, read from left to right, from the bottom row to the top. Since rows and columns in a tableau are increasing, we can recover the original tableau exactly from its word by considering the increasing sequences in $w(T)$, even without knowing the shape. If we break the word at each letter that is strictly greater than the next, we will have the rows of the tableau, listed in reverse order.

Example 2.18. *For the tableau* $T = \begin{ytableau} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 4 & 5 \\ 6 \end{ytableau}$, $w(T) = 6\ 4\ 5\ 3\ 3\ 4\ 1\ 2\ 3$. *This word breaks up into increasing sequences as* $6 \mid 4\ 5 \mid 3\ 3\ 4 \mid 1\ 2\ 3$, *and so we can read off the rows of* T.

We can similarly define a word from the columns of a tableau. For a tableau T, the *column word* of T, denoted $w_{\mathrm{col}}(T)$ is the word consisting of the entries of T read from bottom to top along each column, beginning with the left-most column. Our discussion will focus mainly on row words, but we will utilize column words in the construction of a linear order on tableaux. In general, whenever we refer to the word of a tableau, we mean its row word.

The construction of the row word of a tableau is clearly motivated by the first definition of the product tableau. The order of entries in the word is precisely the order of indexing and insertion for the product tableau. Therefore, our discussion of the product of two words will be motivated by the relationship to the first definition of the product tableau.

First, consider how we row insert a single letter into a word of increasing letters. Let u and v be words and x' and x be letters such that $u\ x'\ v = u \cdot x' \cdot v$ is an increasing word. Then we will use the rule that

$$(u \cdot x' \cdot v) \cdot x \longrightarrow x' \cdot u \cdot x \cdot v \qquad \text{if } u \leq x < x' \leq v.$$

This is precisely the insertion rule for tableau that bumps the first element in the row that is strictly larger than the inserted element.

Given this rule, we can describe how the words of two tableaux relate to the product tableau. Here, "$\cdot$" will mean juxtaposition of words. First, break down u into increasing sequences. Then use the insertion rule on the entries of v in the order they appear in v.

Example 2.19. *Consider the words* $u = 6\ 4\ 5\ 3\ 3\ 4\ 1\ 2\ 3 = (6)(45)(334)(123)$ *and* $v = 3\ 1\ 2 = (3)(12)$ *(these are the words of the tableau from Example 2.12). Then the insertion rule on tableaux gives us the following sequence of words.*

$$\begin{aligned} u \cdot v &\longrightarrow (6)(45)(334)(123) \cdot 312 \longrightarrow (6)(45)(334)(1233) \cdot 12 \\ &\longrightarrow (6)(45)(334) \cdot 2 \cdot (1133) \cdot 2 \longrightarrow (6)(45) \cdot 3 \cdot (234)(1133) \cdot 2 \\ &\longrightarrow (6) \cdot 4 \cdot (35)(234)(1133) \cdot 2 \longrightarrow (6)(4)(35)(234)(1133) \cdot 2 \end{aligned}$$

$$\to (6)(4)(35)(234)\cdot 3\cdot(1123) \to (6)(4)(35)\cdot 4\cdot(233)(1123)$$

$$\to (6)(4)\cdot 5\cdot(34)(233)(1123) \to (6)(45)(34)(233)(1123)$$

We can also define the word of a skew tableau in the same way. This is just the entries of the skew tableau read from left to right, bottom to top. Of course, now we cannot reconstruct the skew tableau from its word, and a word will not correspond to a unique skew tableau. However, we can define an equivalence relation on tableau words and skew tableau words that will make every word equivalent to a unique tableau word. We will have to describe the operations involved in transforming the word of two tableaux to the word of their product. Note that all the operations in the above calculation can be broken down into two basic operations. Therefore, we will just need to consider the following simple product tableaux.

$$\boxed{y}\boxed{z}\cdot\boxed{x} = \begin{array}{|c|c|}\hline x & z \\ \hline y & \\ \cline{1-1}\end{array} \qquad \text{for } x < y \leq z$$

$$\boxed{x}\boxed{z}\cdot\boxed{y} = \begin{array}{|c|c|}\hline x & y \\ \hline z & \\ \cline{1-1}\end{array} \qquad \text{for } x \leq y < z$$

These follow from the rules for insertion, and will motivate the following transformations on words.

$$i : yzx \mapsto yxz \qquad \text{for } x < y \leq z$$

$$j : xzy \mapsto zxy \qquad \text{for } x \leq y < z$$

These maps, along with their inverses, are the *elementary Knuth transformations*, and two words are said to be *Knuth equivalent*, denoted $u \equiv v$, if they differ only by a sequence of Knuth transformations.

Example 2.20. *Consider the word* $u = 6\ 4\ 5\ 3\ 3\ 4\ 1\ 2\ 3\ 3\ 1\ 2$, *the word of the skew tableau* S *from Example 2.15. We will show that this is Knuth equivalent to the word of* $\operatorname{Rect}(S)$. *At each step, the* 3 *letters underlined are those transformed by the given Knuth transformation.*

$$\begin{aligned}
6\ 4\ 5\ 3\ 3\ 4\ 1\ 2\ \underline{3\ 3\ 1}\ 2 \xrightarrow{i} 6\ 4\ 5\ 3\ 3\ 4\ 1\ \underline{2\ 3\ 1}\ 3\ 2 &\xrightarrow{i} 6\ 4\ 5\ 3\ 3\ 4\ 1\ 2\ 1\ \underline{3\ 3\ 2} \\
&\xrightarrow{i} 6\ 4\ 5\ 3\ 3\ 4\ 1\ \underline{2\ 1\ 3}\ 2\ 3 \xrightarrow{i^{-1}} 6\ 4\ 5\ 3\ 3\ 4\ 1\ \underline{2\ 3\ 1}\ 2\ 3 \\
&\xrightarrow{i} 6\ 4\ 5\ 3\ 3\ 4\ \underline{1\ 2\ 1}\ 3\ 2\ 3 \xrightarrow{j} 6\ 4\ 5\ 3\ 3\ 4\ 2\ 1\ \underline{1\ 3\ 2}\ 3 \\
&\xrightarrow{j} 6\ 4\ 5\ 3\ 3\ 4\ \underline{2\ 1\ 3}\ 1\ 2\ 3 \xrightarrow{j^{-1}} 6\ 4\ 5\ 3\ 3\ \underline{4\ 2\ 3}\ 1\ 1\ 2\ 3 \\
&\xrightarrow{j^{-1}} 6\ 4\ 5\ \underline{3\ 3\ 2}\ 4\ 3\ 1\ 1\ 2\ 3 \xrightarrow{i} 6\ 4\ 5\ 3\ 2\ \underline{3\ 4\ 3}\ 1\ 1\ 2\ 3 \\
&\xrightarrow{j} 6\ 4\ 5\ \underline{3\ 2\ 4}\ 3\ 3\ 1\ 1\ 2\ 3 \xrightarrow{j^{-1}} 6\ 4\ 5\ 3\ 4\ 2\ 3\ 3\ 1\ 1\ 2\ 3
\end{aligned}$$

This is the word of $\operatorname{Rect}(S)$, *so we have* $w(S) \equiv w(\operatorname{Rect}(S))$.

The following proposition follows directly from the definition of the Knuth transformations.

Proposition 2.21. *For tableau* T *and letter* x,

$$w(T \leftarrow x) \equiv w(T) \cdot x.$$

This has the following immediate corollary, where the product tableau is defined by row-insertion.

Corollary 2.22. *For tableaux* T *and* U,

$$w(T \cdot U) \equiv w(T) \cdot w(U).$$

Now we wish to show the following proposition, which we will use to show the equivalence of the definitions of the product tableau.

Proposition 2.23. *Sliding preserves Knuth equivalence of words on a skew tableau.*

Proof. For a slide, we are moving an inside corner of a skew tableau to an outside corner, following the sliding procedure. At each step of a slide, we have a skew tableau, with a "hole" corresponding to the inside corner we are sliding out. Since horizontal slides give the same word, we only need to consider how the word of the skew tableau changes on a vertical slide. Consider the slide operation

$$\begin{array}{|c|c|c|}\hline u & & y \\ \hline v & x & z \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline u & x & y \\ \hline v & & z \\ \hline \end{array}$$

where u, y, v, x, and z are letters, and the diagrams are increasing across rows and strictly increasing down columns. Since we are sliding x up into the hole, we must have $u < v \leq x \leq y < z$. The words of these diagrams are related by the sequence of Knuth transformations

$$vxzuy \xrightarrow{j^{-1}} vxuzy \xrightarrow{i} vuxzy \xrightarrow{j} vuzxy \xrightarrow{j} vzuxy$$

so this slide preserves Knuth equivalence.

Now consider the general case of this slide, where u, y, v, and z are (increasing) words. Proceed by induction on the length of u and v. We have shown the case when u and v have length 1. Let $u = u_1\ u_2 \cdots u_p$ and $v = v_1\ v_2 \cdots v_p$, and let $u' = u_2 \cdots u_p$ and $v' = v_2 \cdots v_p$.

Then from the induction hypothesis, we have $v'xzu'y \equiv v'zu'xy$. Also, from Proposition 2.21, we have $v_1v'xzu_1 \equiv v_1u_1v'xz$, given by row-inserting u_1, and so

$$vxzuy = v_1v'xzu_1u'y \equiv v_1u_1v'xzu'y \equiv v_1u_1v'zu'xy \equiv v_1v'zu_1u'xy$$

where the last step is given by the equivalence $v_1v'zu_1u' \equiv v_1u_1v'zu'$, from row-inserting u_1. Therefore, $vxzuy \equiv vzuxy$, and so the proposition follows.

□

This proposition gives us the following theorem.

Theorem 2.24. *Every word is Knuth equivalent to the word of a unique Young tableau.*

Proof. By Proposition 2.23, if any two skew tableau are related by a sequence of slides, their words are Knuth equivalent. This means that, since we can rectify any skew tableau to obtain a Young tableau, a skew tableau is Knuth equivalent to the word of a tableau. Given any word $w = w_p \cdots w_1$, where w_i are the increasing sequences in w, we can construct a skew tableau S with word $w(S) = w$ simply by staggering the rows of the skew tableau,

$$S = \begin{array}{cccc} & & & \boxed{w_1} \\ & & \boxed{w_2} & \\ & \boxed{\cdot} & & \\ \boxed{w_p} & & & \end{array}$$

Since no two distinct tableaux are Knuth equivalent, the fact that the rectification of a skew tableau is unique (which we yet have to show) proves the theorem.

□

This leads us to our third definition of the product tableau. For tableaux T and U, let $T \cdot U$ be the unique tableau whose word is Knuth equivalent to $w(T) \cdot w(U)$.

We can now show that each of our notions of the product tableau are equivalent.

Theorem 2.25. *The three definitions of the product tableau agree.*

Proof. We have already shown that the third construction agrees with the first, in Corollary 2.22. Now consider the second construction, where $T \cdot U = \mathrm{Rect}(T * U)$. Since $w(T * U) = w(T) \cdot w(U)$, the equivalence of the second and third constructions of the product tableau follows from Proposition 2.23.

□

We must still show that if a word $w \equiv w(T)$ for some tableau T, the tableau T is uniquely determined (or equivalently, that $\mathrm{Rect}(S)$ is unique for any skew tableau S). For this proof, we must define a few new concepts. Given a word w on alphabet $[m]$, let $L(w, k)$ be the largest number obtained as the sum of the lengths of k disjoint increasing sequences in w.

Example 2.26. *The word* $w = 1\ 3\ 5\ 6\ 2\ 4\ 7\ 8\ 9$ *has* $L(w, 1) = 6$, *obtained from the sequence* $1\ 3\ 4\ 7\ 8\ 9$, *and* $L(w, 2) = 9$, *obtained from the sequences* $1\ 2\ 4\ 7\ 8\ 9$ *and* $3\ 5\ 6$.

Notice that our choices of sequences that achieve the maximum lengths are not unique, and also that the k sequences that achieve $L(w, k)$ are not necessarily obtained from any $k - 1$ sequences that obtain $L(w, k - 1)$.

For the word w of a tableau T, it is easy to read off the sequences that give us $L(w, k)$. If $w = w(T) = w_p \cdots w_1$, where the w_i are increasing sequences corresponding to the rows of T, w_1 gives the longest increasing sequence in w, since $L(w, 1)$ is at most the number of columns of T. By the same argument, $L(w, k)$ is determined by the first k rows of T. This demonstrates the following proposition.

Proposition 2.27. *Let* T *be a tableau of shape* $\lambda = (\lambda_1, \ldots, \lambda_p)$ *and* $w = w(T)$. *Then for each* $k \geq 1$,

$$L(w, k) = \sum_{i=1}^{k} \lambda_k.$$

The following proposition will be the main idea behind the remainder of the proof of Theorem 2.24.

Proposition 2.28. *For Knuth equivalent words* $w \equiv w'$,

$$L(w, k) = L(w', k)$$

for all $k \geq 1$.

Proof. We will show this by proving that the Knuth transformations do not change the total sums $L(w, k)$. Consider the words $w = u \cdot y\ x\ z \cdot v$ and $w' = u \cdot y\ z\ x \cdot v$ where u and v are arbitrary words and $x < y \leq z$. Then $w \equiv w'$ by Knuth transformation i. The only difference between w and w' are that x and z are transposed. This means that any increasing sequence in w' is also an increasing sequence in w, so $L(w, k) \geq L(w', k)$ for all k.

The reverse is also true, except when one of the sequences we have chosen in order to count $L(w, k)$ includes both x and z. Suppose we have such a sequence $u_1 \cdot x\ z \cdot v_1$ where u_1 and v_1 are sequences in u and v. If y is not already included in another sequence, then we can instead choose the sequence $u_1 \cdot y\ z \cdot v_1$, and so we have another sequence of the same length. If y is already included in another sequence, say $u_2 \cdot y \cdot v_2$, then we can replace these two sequences in w with the sequences $u_2 \cdot y\ z \cdot v_1$ and $u_1 \cdot x \cdot v_2$ in w', and so the sequences have the same total length.

Similarly, if we have words $w = u \cdot x\ z\ y \cdot v$ and $w' = u \cdot z\ x\ y \cdot v$, where $x \leq y < z$, then $w \equiv w'$ by the Knuth transformation j. Again, consider a sequence $u_z \cdot x\ z \cdot v_1$. If y is not already included in a sequence, then use the sequence $u_1 \cdot x\ y \cdot v_1$. If y is included in another sequence $u_2 \cdot v_1$, replace these with the sequences $u_1 \cdot x\ y \cdot v_2$ and $u_2 \cdot z \cdot v_1$ in w'. Therefore, any set of k increasing sequences in w has an equivalent set of k increasing sequences in w', so the proposition follows.

□

With the previous two propositions, we can describe the shape of the tableau whose word is Knuth equivalent to any given word. Given an arbitrary word, we can determine the values of $L(w, k)$ for each $k \geq 1$. Then for any word we can construct from this word by Knuth transformations, we must have $L(w, k) = L(w', k)$, by Proposition 2.28. In particular, if we can recover a tableau word from w, it has the same lengths of increasing sequences, so by Proposition 2.27, this gives us the shape of the tableau. Now all we need to do to completely determine the tableau of a given Knuth equivalence class of words is to find its content.

We can determine the content of the tableau by looking at where the largest entry in the word will go. If we delete the largest entry in the word (choosing the right-most entry if there are ties), we can then recover the tableau determined by the other letters. We will get a tableau with one less cell than the original word. Since we have removed the largest entry, this must fall in the missing cell. We can then repeat this on the second largest element, and so on until we have recovered the tableau.

To show that this works, we will need the following proposition.

Proposition 2.29. *If $w \equiv w'$ and w_1 and w'_1 are the words obtained by removing the p largest and q smallest letters of w and w', then $w_1 \equiv w'_1$.*

Proof. Consider the case where we remove the largest letter from both w and w', as removing the largest p letters follows from this. Let $w = u \cdot y\ x\ z \cdot v$ and $w' = u \cdot y\ z\ x \cdot v$, where u and v are arbitrary words and $x < y \leq z$, so $w \equiv w'$ by the Knuth transformation i.

If the largest letter is in either u or v, then clearly removing it will not change the equivalence. Otherwise, the largest letter is z, and so $w_1 = u \cdot y\ x \cdot v = w'_1$.

The argument is identical for words $w \equiv w'$ equivalent by the Knuth transformation j, and for the case where we remove the smallest letter in both w and w'.

□

Now we can complete the proof of Theorem 2.24. We need to show that for an arbitrary word w, there is a unique tableau T such that $w \equiv w(T)$.

Let w be an arbitrary word, and let T be a tableau such that $w \equiv w(T)$. By Propositions 2.27 and 2.28, the shape of this tableau is uniquely determined by the values of each $L(w,k)$. Let x be the largest letter in w, and let w_1 be the word obtained by removing x from w (choosing the right-most entry in the word if there are duplicates). Also, let T_1 be the tableau obtained from T by removing the entry x (again, choosing the right-most entry if there are duplicates).

Proceed by induction on the length of the word w. By Proposition 2.29, $w_1 \equiv w(T_1)$, and by the induction hypothesis, T_1 is the only such tableau. Since we have the shape of T and the content of T_1, the element we removed from T must go in the cell missing from T_1, so T is the only possible tableau such that $w \equiv w(T)$. Therefore, the word w determines the tableau T uniquely.

□

This completes the proof of Theorems 2.24 and 2.25, and so we may treat the three constructions of the product tableau as equivalent. Also, this shows that for any arbitrary skew tableau S, the rectification $\mathrm{Rect}(S)$ is uniquely determined. Therefore, the rectification procedure we defined above is independent of which sequence of inside corners we choose.

Now that we have a well-defined notion of the product of two tableaux, we can proceed to define the tableau ring. Let $M = M_m$ be the set of all Knuth equivalence classes of words on the alphabet $[m]$. Equivalently, we can write $M_m = F_m/R$, where F_m is the set of all words on alphabet $[m]$ and R are the relations given by the Knuth transformations. For words $w, v \in M_m$, define the product $w \cdot v$ by juxtaposition of words. This makes M_m a monoid, with identity given by the empty word .

By Theorems 2.24 and 2.25, the set of tableau with entries in $[m]$ and product given by the product tableau is isomorphic to M_m, so is also a monoid. Therefore, we can define the *tableau ring* $R_{[m]}$ to be the ring on the monoid of tableaux with entries in $[m]$. $R_{[m]}$ has basis given by the set of all tableaux with entries in $[m]$, so we can define a canonical homomorphism $R_{[m]} \to \mathbb{Z}[x_1, \ldots, x_m]$ given by $T \mapsto x^T$.

3. Robinson-Schensted-Knuth Correspondence

We now define an object which will help us describe some of the fundamental relations involving tableaux. Let w be an arbitrary word, and define $P(w)$ to be the tableau whose word is Knuth equivalent to w. As we have shown, this tableau is unique, so we can calculate $P(w)$ through whatever manner is most convenient. The canonical method for constructing $P(w)$ is simply by row-inserting each element. For $w = x_1\ x_2 \cdots x_p$, we have

$$P(w) = \big(\big(\cdots\big(\big(\boxed{x_1} \leftarrow x_2\big) \leftarrow x_3\big) \leftarrow \cdots\big) \leftarrow x_{p-1}\big) \leftarrow x_p.$$

As we have shown, the row-insertion process is reversible, so if we know the order in which cells are added to the tableau, we can recover the word that created it. This idea will motivate the following object. Given an arbitrary word w, we can construct the corresponding tableau $P(w)$ using the canonical procedure above. At the same time, we will construct a second tableau $Q(w)$ with the same shape as $P(w)$, called the *insertion tableau*, that records when each cell is added to the tableau. When we insert the i^{th} letter into the tableau, the corresponding cell in $Q(w)$ will be labeled with an i. The result will be a standard tableau with precisely the information we need to recover the word w used to define $P(w)$.

Example 3.1. *Consider the word $w = 1\ 3\ 3\ 4\ 2\ 3\ 1\ 1\ 2$. We will construct $P(w)$ and $Q(w)$ simultaneously. Let P_i and Q_i be the tableaux formed from row-inserting the first i letters of w. Then we have the following sequence of pairs (P_i, Q_i).*

$$\xrightarrow{\cdot 1} \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \xrightarrow{\cdot 3} \quad \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \quad \xrightarrow{\cdot 3} \quad \begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\xrightarrow{\cdot 4} \quad \begin{array}{|c|c|c|c|}\hline 1&3&3&4\\\hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} \quad \xrightarrow{\cdot 2} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5\\\cline{1-1}\end{array}$$

$$\xrightarrow{\cdot 3} \quad \begin{ytableau} 1 & 2 & 3 & 3 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 \end{ytableau} \xrightarrow{\cdot 1} \begin{ytableau} 1 & 1 & 3 & 3 \\ 2 & 4 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 \\ 7 \end{ytableau}$$

$$\xrightarrow{\cdot 1} \quad \begin{ytableau} 1 & 1 & 3 & 3 \\ 2 & 2 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{ytableau} \quad \xrightarrow{\cdot 2} \quad \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 & 9 \\ 7 & 8 \end{ytableau}$$

Then we have $P(w) = \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 4 \end{ytableau}$ *and* $Q(w) = \begin{ytableau} 1 & 2 & 3 & 4 \\ 5 & 6 & 9 \\ 7 & 8 \end{ytableau}$.

Given a pair $P(w)$ and $Q(w)$, the word w can be recovered by reversing the row-insertion process. Indeed, any pair (P, Q) of tableaux on the same shape λ with Q a standard tableau can be used to construct a word w such that $P = P(w)$ and $Q = Q(w)$. Therefore, we have a one-to-one correspondence between words of length n taken from the alphabet $[m]$ and ordered pairs of tableaux (P, Q) on the same shape with n cells, where P has entries from $[m]$ and Q is a standard tableau. This result is known as the *Robinson-Schensted correspondence* [10].

We may generalize this to allow that Q be an arbitrary tableau with the same shape as P. Then the word w is recovered in an analogous way. Given the pair (P_i, Q_i) at each step of the reverse row-insertion process, we recover (P_{i-1}, Q_{i-1}) by reverse row-inserting the cell with the largest element in Q_i (choosing the cell furthest to the right if there are ties). Let u_i be the element removed from Q_i on this step, and v_i be the element reverse row-inserted from P_i. Then we construct the following array

$$\omega = \begin{pmatrix} u_1 & u_2 & \cdots & u_p \\ v_1 & v_2 & \cdots & v_p \end{pmatrix}$$

which describes this process.

In the case where Q is a standard tableau, the top row of ω will be $(1\ 2 \cdots p)$, and the bottom row will be the word w such that $P(w) = P$ and $Q(w) = Q$. In this case, we call the array a *word*, since it just describes the word w.

If $\omega = \binom{u}{v}$ is a two-row array constructed using this reverse row-insertion procedure, then by definition, we have $u_1 \leq u_2 \leq \cdots \leq u_p$. Further, if $u_{i-1} = u_i$, then the cell removed from P_i is strictly to the right of the cell to be removed from P_{i-1}, and so $v_{k-1} \leq v_k$. If we have an arbitrary two-row array ω satisfying these two conditions, then we say that ω is in *lexicographic order.* We can also define an ordering on such arrays. We say that $\binom{u}{v} \leq \binom{u'}{v'}$ if $u \leq u'$, or $u = u'$ and $v \leq v'$, using the lexicographic order on words.

What we have shown is that, given an arbitrary two-row array ω in lexicographic order, we can construct a tableau pair (P, Q) with the same shape using the row-insertion procedure to form P, and recording the inserted cells to form Q. Since this process is reversible, the correspondence between such arrays and tableau pairs is one-to-one.

Example 3.2. *Consider the array* $\omega = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 1 & 2 & 3 & 2 & 2 \end{pmatrix}$. *We will construct the tableau pair corresponding to* ω.

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \longrightarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\longrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

Therefore, ω *corresponds to the tableau pair* $(P, Q) = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \right)$.

We now define another construction that will associate a matrix A to any tableau pair (P, Q). Let $\omega = \binom{u}{v}$ be a two-row array in lexicographic order, with u a word on the alphabet $[m]$ and v a word on the alphabet $[n]$. Define an $m \times n$ matrix A by

$$A = [a_{ij}] = [\#\text{ times } (u_i, v_i) = (i,j)]_{ij}$$

That is, the ij^{th} entry is the number of columns of ω equal to (i,j).

Example 3.3. *Consider the array* $\omega = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 1 & 2 & 3 & 2 & 2 \end{pmatrix}$. *This has associated matrix*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

A collection of pairs $\{(i,j)\}$ will uniquely determine the matrix A, since we simply count the number of times the pair (i,j) appears. Such a collection of pairs also uniquely defines a two-row lexicographic array, since we can let the pairs (i,j) be the columns of the array, and then order the columns by the lexicographic ordering. Therefore, we have a one-to-one correspondence between two-row lexicographic arrays and $m \times n$ matrices A with positive integer entries. We therefore also have a one-to-one correspondence between the matrices A and tableau pairs (P, Q) of the same shape, where P has entries in $[n]$ and Q has entries in $[m]$. This defines the *Robinson-Schensted-Knuth correspondence* [7].

4. Littlewood-Richardson Rule

Now we wish to describe the Littlewood-Richardson rule. For this, we will need the following proposition.

Proposition 4.1. *Let* $\omega = \begin{pmatrix} u_1 & u_2 & \cdots & u_p \\ v_1 & v_2 & \cdots & v_p \end{pmatrix}$ *be a two-row array in lexicographic order, and let* (P, Q) *be the tableau pair associated to* ω *by the Robinson-Schensted-Knuth correspondence. Let* T *be an arbitrary tableau, and consider the tableau defined* *by the row-insertions*

$$U = (\cdots((T \leftarrow v_1) \leftarrow v_2) \leftarrow \cdots) \leftarrow v_p.$$

At the same time, place the entries $u_1, \ldots, u_p$ *in a skew tableau corresponding to each row-insertion. Then these entries form a skew tableau* S *such that* $\operatorname{Rect}(S) = Q$.

Proof. Let U be an arbitrary tableau with the same shape as T and entries taken from an alphabet whose letters are strictly smaller than the letters u_i in S (we may allow these entries to be negative). By the Robinson-Schensted-Knuth correspondence, the tableau pair (T, U) corresponds to some lexicographic array, say $\begin{pmatrix} s_1 & \cdots & s_n \\ t_1 & \cdots & t_n \end{pmatrix}$.

Then the array $\begin{pmatrix} s_1 & \cdots & s_n & u_1 & \cdots & u_m \\ t_1 & \cdots & t_n & v_1 & \cdots & v_m \end{pmatrix}$ corresponds to the pair $(T \cdot P, V)$, where V is the tableau constructed by combining the tableau U and the skew tableau S. Invert this array to get $\begin{pmatrix} t_1 & \cdots & t_n & v_1 & \cdots & v_m \\ s_1 & \cdots & s_n & u_1 & \cdots & u_m \end{pmatrix}$, and then put the columns in lexicographic order. The resulting array corresponds to the pair $(V, T \cdot P)$. Let k be the word in the second row of this array. By construction, this is Knuth equivalent to $w(V)$. Similarly, if we invert ω to get $\begin{pmatrix} v_1 & v_2 & \cdots & v_p \\ u_1 & u_2 & \cdots & u_p \end{pmatrix}$ and then reorder the columns by lexicographic order, the resulting array corresponds to (Q, P), and the word h on the second row is Knuth equivalent to $w(Q)$. We can obtain h from k by removing the s_i's. Alternately, if we remove the n smallest letters from $w(V)$, we will get the word $w(S)$. Since the n smallest letters of $w(V)$ are the s_i's, this implies that $w(S) = w(Q)$, by Proposition 2.29. Therefore, $\operatorname{Rect}(S) = Q$, by Theorem 2.24.

□

The construction given by this proposition requires a bit of explanation. Consider the following example.

Example 4.2. *Consider the array* $\omega = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 1 & 2 & 3 & 2 & 2 \end{pmatrix}$ *from Example 3.2. We showed that this corresponds to the tableau pair* $(P, Q) = \left(\begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 \end{ytableau} \right)$. *Let* $T = \begin{ytableau} 2 & 3 \\ 4 \end{ytableau}$, *and we will calculate the skew tableau* S *given by the proposition.*

$$\begin{ytableau} 2 & 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} \\ {} \end{ytableau} \longrightarrow \begin{ytableau} 2 & 2 \\ 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} \\ {} \\ 1 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} \\ {} \\ 1 \\ 2 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 2 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} & 2 \\ {} \\ 1 \\ 2 \end{ytableau}$$

$$\longrightarrow \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} & 2 & 2 \\ {} \\ 1 \\ 2 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 2 & 2 \\ 2 & 3 \\ 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} & 2 & 2 \\ {} & 3 \\ 1 \\ 2 \end{ytableau}$$

$$\longrightarrow \begin{ytableau} 1 & 2 & 2 & 2 & 2 \\ 2 & 3 \\ 3 \\ 4 \end{ytableau} \quad \begin{ytableau} {} & {} & 2 & 2 & 3 \\ {} & 3 \\ 1 \\ 2 \end{ytableau}$$

So the skew tableau described by the proposition is $S = \begin{ytableau} \none & \none & 2 & 2 & 3 \\ \none & 3 \\ 1 \\ 2 \end{ytableau}$. *We can check that this rectifies to* Q.

$$\begin{ytableau} \none & \none & 2 & 2 & 3 \\ \cdot & 3 \\ 1 \\ 2 \end{ytableau} \longrightarrow \begin{ytableau} \none & \cdot & 2 & 2 & 3 \\ 1 & 3 \\ 2 \end{ytableau} \longrightarrow \begin{ytableau} \cdot & 2 & 2 & 3 \\ 1 & 3 \\ 2 \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 2 & 3 \\ 2 & 3 \end{ytableau}$$

So $\mathrm{Rect}(S) = Q$, *as claimed.*

Now we wish to consider the following problem: given partitions $\lambda \vdash n$, $\mu \vdash m$, and $\nu \vdash r$ and a tableau V of shape ν, how many pairs (T, U) of tableaux on λ and μ are there such that $V = T \cdot U$? This will be our motivating problem for the Littlewood-Richardson rule. As we will see, there are several equivalent ways of formulating this problem. We already have one such alternative question from one of our alternate definitions of the product tableau. This question is equivalent to determining the number of skew tableau on the shape $\lambda * \mu$ that rectify to V.

Before we derive any answers to this question, we will need to define a few objects. Let U' be a tableau on shape μ and V' be a tableau on shape ν, and define

$$\mathcal{S}(\nu/\lambda, U') = \{S \text{ skew tableau on } \nu/\lambda : \mathrm{Rect}(S) = V\},$$

$$\mathcal{T}(\lambda, \mu, V') = \{[T, U] : T \text{ tableau on } \lambda, U \text{ tableau on } \mu, T \cdot U = V'\}.$$

The following proposition gives a relationship between these two sets.

Proposition 4.3. *Let U' be a tableau on μ and V' be a tableau on ν. Then there is a one-to-one correspondence*

$$\mathcal{S}(\nu/\lambda, U') \leftrightarrow \mathcal{T}(\lambda, \mu, V').$$

Proof. Let $[T, U] \in \mathcal{T}(\lambda, \mu, V')$. Let $\begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$ be a lexicographic array corresponding to the pair (U, U'). Let S be the skew tableau constructed by applying the process from Proposition 4.1 to the tableau T and this array. By definition, $T \cdot U = V'$ has shape ν, so by Proposition 4.1, $\mathrm{Rect}(S) = U'$, and so $S \in \mathcal{S}(\nu/\lambda, U')$.

Now let $S \in \mathcal{S}(\nu/\lambda, U')$. Let T' be a tableau on λ such that all letters in T' are smaller than all letters in S (again, we may allow negative integer entries). Let W be the tableau constructed by combining the tableau T' on λ with the skew tableau S on ν/λ. By the Robinson-Schensted-Knuth correspondence, the pair (V', W) corresponds uniquely to some lexicographic array, say

$$\omega = \begin{pmatrix} t_1 & \cdots & t_n & u_1 & \cdots & u_m \\ x_1 & \cdots & x_n & v_1 & \cdots & v_m \end{pmatrix}.$$

Consider the first n columns of this array, $\begin{pmatrix} t_1 & \cdots & t_n \\ x_1 & \cdots & x_n \end{pmatrix}$. Since the array is in lexicographic order, these come from the columns with the smallest n entries in the first row of ω. By construction, these correspond to the entries of T', so the array $\begin{pmatrix} t_1 & \cdots & t_n \\ x_1 & \cdots & x_n \end{pmatrix}$ corresponds to the pair (T, T') for some tableau T on shape λ. Then it follows that the array $\begin{pmatrix} u_1 & \cdots & u_m \\ v_1 & \cdots & v_m \end{pmatrix}$ corresponds to (U, U') for some tableau U on μ. Since (V', W) corresponds to array ω, it follows that $T \cdot U = V'$. Therefore, we have $[T, U] \in \mathcal{T}(\lambda, \mu, V')$, so we have shown the correlation.

□

Notice that in the statement of the proposition and the proof given, the correlation between $\mathcal{S}(\nu/\lambda, U')$ and $\mathcal{T}(\lambda, \mu, V')$ is independent of the choice of tableaux U' and V'. Therefore, we define the *Littlewood-Richardson number* $c^{\nu}_{\lambda\mu}$ to be the cardinality of these sets

$$c^{\nu}_{\lambda\mu} = |\mathcal{S}(\nu/\lambda, U')| = |\mathcal{T}(\lambda, \mu, V')|.$$

We then have the following immediate result.

Corollary 4.4. *The number*

$$c^{\nu}_{\lambda\mu} = |\mathcal{S}(\nu/\lambda, U')| = |\mathcal{T}(\lambda, \mu, V')|$$

depends only on the partitions λ, μ, and ν.

We can now describe some objects with the same cardinality.

Corollary 4.5. *The following sets have cardinality $c^{\nu}_{\lambda\mu}$.*

(i) $\mathcal{S}(\nu/\lambda, U')$ *for arbitrary tableau U' on μ*

(ii) $\mathcal{T}(\lambda, \mu, V')$ *for arbitrary tableau V' on ν*

(iii) $\mathcal{S}(\nu/\mu, T')$ *for arbitrary tableau T' on λ*

(iv) $\mathcal{T}(\mu, \lambda, V')$ *for arbitrary tableau V' on ν*

(v) $\mathcal{S}(\tilde{\nu}/\tilde{\lambda}, \tilde{U}')$ *for arbitrary tableau $\tilde{U}'$ on $\tilde{\mu}$*

(vi) $\mathcal{T}(\tilde{\lambda}, \tilde{\mu}, \tilde{V}')$ *for arbitrary tableau $\tilde{V}'$ on $\tilde{\nu}$*

(vii) $\mathcal{S}(\lambda * \mu, V')$ *for arbitrary tableau V' on ν*

Proof. (i) and (ii) are restatements of the definition using Corollary 4.4, and since $T \cdot U = \mathrm{Rect}(T * U)$, (vii) is equivalent to (ii). If we require U' to be a standard tableau on μ, then its transpose $\tilde{U}'$ is a standard tableau on $\tilde{\mu}$, and so there is a map from elements $\mathcal{S}(\nu/\lambda, U'$ to $\mathcal{S}(\tilde{\nu}/\tilde{\lambda}, \tilde{U}')$ defined by taking the transpose. This proves (v), and so (vi) follows by Proposition 4.3.

Since $\widetilde{\lambda * \mu} = \tilde{\mu} * \tilde{\lambda}$, we have that

$$c^{\nu}_{\lambda\mu} = |\mathcal{S}(\lambda * \mu, V')| = |\mathcal{S}(\tilde{\mu} * \tilde{\lambda}, \tilde{V}')| = |\mathcal{T}(\tilde{\mu}, \tilde{\lambda}, \tilde{V}')| = |\mathcal{T}(\mu, \lambda, V')|$$

so (iv) follows. Then (iii) follows from Proposition 4.3.

□

Now define $S_\lambda = S_\lambda[m] \in R_{[m]}$ to be the sum of all tableau on shape λ with alphabet $[m]$. By definition, the image of S_λ by the canonical homomorphism $R_{[m]} \to \mathbb{Z}[x_1, \ldots, x_m]$ is the Schur polynomial $s_\lambda(x_1, \ldots, x_m)$. As we will see, there is a very close relationship between these two objects which will let us essentially treat them as equivalent. For now, consider the following application of the Littlewood-Richardson numbers.

Corollary 4.6. *For partitions λ and μ,*

$$S_\lambda \cdot S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$$

where the sum is taken over all partitions ν.

Since $c^\nu_{\lambda\mu} \neq 0$ only if $\lambda, \mu \subset \nu$ and $|\lambda| + |\mu| = |\nu|$, this sum will always be finite. The corollary is essentially just the observation that the number of ways a tableau V of shape ν can be written as a product $T \cdot U$ where T has shape λ and U has shape μ is $c^\nu_{\lambda\mu}$.

Since we have the homomorphism $R_{[m]} \to \mathbb{Z}[x_1, \ldots, x_m]$, taking $S_\lambda \mapsto s_\lambda$, there is an identical Littlewood-Richardson rule for Schur polynomials, where

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

Now define $S_{\nu/\lambda} = S_{\nu/\lambda}[m]$ to be the sum

$$S_{\nu/\lambda} = \sum_S \mathrm{Rect}(S)$$

where the sum is taken over all skew tableau S on the shape ν/λ with alphabet $[m]$. Since $c^\nu_{\lambda\mu} = |\mathcal{S}(\nu/\lambda, U')|$, a similar argument gives us the following corollary.

Corollary 4.7. *For partitions ν and λ,*

$$S_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda\mu} S_\mu$$

where the sum is taken over all partitions μ.

We will now describe yet another interpretation for the Littlewood-Richardson number. First, we will need a couple of constructions.

Definition 4.8. *Let $w = x_1 \cdots x_n$ be a word on the alphabet $[m]$. If for all $p \leq n$, the last p letters of w contain i_1 1's, i_2 2's, and so on, up to i_m m's, with $i_1 \geq i_2 \geq \cdots \geq i_m$, then w is a* reverse lattice word. *If a skew tableau S has $w(S)$ a reverse lattice word, then it is a* Littlewood-Richardson skew tableau.

The following proposition tells us that reverse lattice words are preserved by Knuth equivalence.

Proposition 4.9. *Let w and w' be Knuth equivalent words. If w is a reverse lattice word, then w' is also a reverse lattice word.*

Proof. First, consider the case where $w = u \cdot y\ z\ x \cdot v$ and $w' = u \cdot y\ x\ z \cdot v$ for words u and v, and letters $x < y \leq z$. Since w is a reverse lattice word, we are concerned with whether or not this transformation changes the sequences of consecutive integers. If $x < y < z$, then there are no consecutive integers that are transposed, so the result is still a reverse lattice word. Now suppose that $i = x$ and $i + 1 = y = z$. Since w is a reverse lattice word, v must contain an i. Therefore, w' is still a reverse lattice word.

Now consider the case when $w = u \cdot z\ x\ y \cdot v$ and $w' = u \cdot x\ z\ y \cdot v$ with $x \leq y < z$. Again, if $x < y < z$, then there are no changes in the sequences of consecutive integers, so consider the case where $i = x = y$ and $i + 1 = z$. Since w is a reverse lattice word, it follows that v must also be a reverse lattice word. Since $x\ z\ y$ still satisfies the reverse lattice property, w' is a reverse lattice word.

□

For a partition μ, we define $U(\mu)$ to be the tableau with shape and content μ such that the cells in the i^{th} row all contain the entry i.

Proposition 4.10. *A skew tableau S with content μ is a Littlewood-Richardson skew tableau if and only if* $\operatorname{Rect}(S) = U(\mu)$.

Proof. First, consider the case of a tableau T with content μ that has $w(T)$ a reverse lattice word. The last letter of the word must be 1, so the last entry of the first row is 1. This implies that the first row contains only 1's. This argument continues to the other rows of the tableau, so $T = U(\mu)$.

Now suppose that S is an arbitrary Littlewood-Richardson skew tableau with content μ. Since rectification preserves Knuth equivalence, $w(S) \equiv w(\mathrm{Rect}(S))$. By Proposition 4.9, $w(\mathrm{Rect}(S))$ is a reverse lattice word. $\mathrm{Rect}(S)$ is a tableau, so by the above argument, $\mathrm{Rect}(S) = U(\mu)$.

□

Now we can give the following equivalent definition of the Littlewood-Richardson number.

Proposition 4.11. *The number of Littlewood-Richardson skew tableau on the shape ν/λ with content μ is $c^{\nu}_{\lambda\mu}$.*

Proof. This follows as a result of Proposition 4.10. Letting $U' = U(\mu)$, we have

$$c^{\nu}_{\lambda\mu} = |\mathcal{S}(\nu/\lambda, U(\mu))| = |\{\text{skew tableau } S \text{ on } \nu/\lambda : \mathrm{Rect}(S) = U(\mu)\}$$

□

5. Representations of S_n

Now we wish to describe some applications of Young tableaux to representation theory. First, we will construct representations of the symmetric group.

Let the symmetric group S_n be the group of automorphisms on $[n]$. Let T be a numbering on a Young diagram λ with n cells and containing the entries $\{1, \ldots, n\}$ with no repeats. S_n acts on such a numbering in the natural way: for $\sigma \in S_n$, let $\sigma \cdot T$ be the numbering on λ such that if a cell in T contains the letter a, then the corresponding cell in $\sigma \cdot T$ contains $\sigma(a)$.

For a given numbering T, we define the *row group* of T to be the subgroup $R(T) \leq S_n$ that preserves the rows of T (that is, entries of T are permuted only along rows). Then, by inspection, for $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have

$$R(T) \cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}.$$

Similarly, define the *column group* of T to be the subgroup $C(T) \leq S_n$ that preserves the columns of T.

Proposition 5.1. *For numbering T on shape $\lambda \vdash n$ and $\sigma \in S_n$,*

$$R(\sigma \cdot T) = \sigma \cdot R(T) \cdot \sigma^{-1}$$

$$C(\sigma \cdot T) = \sigma \cdot C(T) \cdot \sigma^{-1}$$

Proof. Consider $T = \begin{array}{|c|}\hline w_1 \\ \hline \vdots \\ \hline w_k \\ \hline \end{array}$, where w_i is the word consisting of the entries or row i. For $\gamma \in R(\sigma \cdot T)$, we have

$$\sigma^{-1} \cdot \gamma \cdot \sigma(T) = \sigma^{-1} \cdot \gamma \begin{array}{|c|}\hline \sigma w_1 \\ \hline \vdots \\ \hline \sigma w_k \\ \hline \end{array} = \sigma^{-1} \begin{array}{|c|}\hline \sigma w_1' \\ \hline \vdots \\ \hline \sigma w_k' \\ \hline \end{array} = \begin{array}{|c|}\hline w_1' \\ \hline \vdots \\ \hline w_k' \\ \hline \end{array}$$

where w_i' is simply a rearrangement of the letters of w_i. Therefore, $\sigma^{-1} \cdot R(\sigma \cdot T) \cdot \sigma \subset R(T)$. Since we can obtain any permutation of the rows of T in this way, the first equality follows. We can make the same argument to show the second equality, where we consider the words consisting of the columns of T.

□

Now define the following ordering on the set of numberings on n boxes (not necessarily of the same shape). For numberings T on λ and T' on λ', we say that $T < T'$

if either $\lambda < \lambda'$, or $\lambda = \lambda'$ and the largest entry on which the two numberings differ occurs earlier in $w_{\text{col}}(T')$ than in $w_{\text{col}}(T)$.

Notice that, if T is a standard tableau, then for any nontrivial element $p \in R(T)$ or $q \in C(T)$, neither $p \cdot T$ nor $q \cdot T$ is a tableau. With the ordering on numberings, we can say something a bit more specific.

Proposition 5.2. *Let T be a standard tableau, and let $p \in R(T)$ and $q \in C(T)$. Then*

$$p \cdot T \geq T \quad \textit{and} \quad q \cdot T \leq T.$$

Proof. Consider how p and q change the column word of T. The columns of T are strictly increasing, so any nontrivial permutation q, a larger letter will appear higher up in a column of $q \cdot T$ than in T. Since column words read from bottom to top, this means the larger element appears earlier in T than in $q \cdot T$, so $q \cdot T < T$.

Similarly, the rows of T are strictly increasing, so p sends a larger element to the left in its row. Since column words are read from the left column to the right, this means that $p \cdot T > T$.

□

The following proposition will be useful for many later calculations.

Proposition 5.3. *Let T be a numbering on shape λ and T' be a numbering on λ'. If λ does not strictly dominate λ', then either*

(i) *there are distinct entries a and b in the same column of T and the same row of T', or*

(ii) *$\lambda = \lambda'$ and there exist $p' \in R(T')$ and $q \in C(T)$ such that $p' \cdot T' = q \cdot T$.*

Proof. If (i) is not true, then for each pair a and b of entries contained in the same row of T', a and b appear in different columns of T.

Consider the first row of T'. By this assumption, all elements in the first row are in different columns of T. Therefore, we may choose $q_1 \in C(T)$ that takes these elements to the first row in T. Then the first row of $q_1 \cdot T$ contains all the letters in the first row of T'. We can repeat this procedure to find $q_2 \in C(q_1 \cdot T) = C(T)$ that takes all letters in the second row of T' to the second row of $q_1 \cdot T$, and so on for each row of T. Then we will have numbering $q_k \cdots q_1 \cdot T$ on λ such that the i^{th} row contains all the letters in the i^{th} row of T'.

This implies that $\lambda_1 + \cdots + \lambda_i \geq \lambda'_1 + \cdots + \lambda'_i$ for each i, so $\lambda' \trianglelefteq \lambda$. Since we assumed that λ does not strictly dominate λ', this further implies that $\lambda = \lambda'$. This shows that for $q = q_k \cdots q_1 \in C(T)$, $q \cdot T$ and T' have the same entries in each row. Then simply choose the element $p' \in R(T')$ that permutes the rows of T' to get $q \cdot T = p' \cdot T'$.

□

Corollary 5.4. *Let T and T' be standard tableau on the same shape with $T' > T$. Then there is a pair a and b of letters that appear in the same column of T and the same row of T'.*

Proof. Suppose we have $T' > T$ standard tableau on the same shape, and assume the corollary is false, so there are no such pairs of entries. Since T and T' have the same shape, they satisfy (ii) in Proposition 5.3, so there are $p' \in R(T')$ and $q \in C(T)$ such that $p' \cdot T' = q \cdot T$.

By Proposition 5.2, we have $p' \cdot T' \geq T'$ and $q \cdot T \leq T$. Since $T \neq T'$, p' and q cannot both be trivial, so we must have $T' < T$, contradicting our assumption.

□

Definition 5.5. *Given a numbering T on shape λ, a* tabloid *$\{T\}$ is the equivalence class of numberings on λ such that a numbering $T' \sim T$ if $T' = p \cdot T$ for some $p \in R(T)$.*

Note that the action of S_n on tableaux T extends to tabloids, with

$$\sigma \cdot \{T\} = \{\sigma \cdot T\}$$

for $\sigma \in S_n$. Clearly, the definition of a tabloid is motivated by the type of argument used in the proof of Proposition 5.1, that in many cases, we will not be concerned with the action of $R(T)$ on a numbering. Indeed, we can see that the orbit of $\{T\}$ under the action of S_n is isomorphic to $S_n/R(T)$.

Now let $A = \mathbb{C}[S_n]$ be the group ring of S_n, and define the following objects. For a numbering T on shape $\lambda \vdash n$ containing the entries $\{1, \ldots, n\}$ with no repeats, we have

$$a_T = \sum_{p \in R(T)} p, \qquad b_T = \sum_{q \in C(T)} \operatorname{sgn}(q) q, \qquad c_T = b_t \cdot a_T$$

as elements of A. These are the *Young symmetrizers* of the numbering T.

We wish to calculate the representations of S_n. Equivalently, we can calculate all left A-modules. Therefore, for λ a partition of n, define M^λ to be the vector space with basis given by the tabloids on shape λ. S_n acts on these tabloids, so S_n also acts on M^λ, making M^λ an A-module. Then we would like to determine the structure of M^λ and use this to determine the irreducible representations of S_n.

With this in mind, we define $v_T \in M^\lambda$ to be the element

$$v_T = b_T \cdot \{T\} = \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\}.$$

Then define the *Specht module* $S^\lambda \subset M^\lambda$ to be the vector subspace with basis

$$\{v_T : T \text{ numbering on } \lambda\}.$$

The following propositions will allow us to describe S^λ and M^λ.

Proposition 5.6. *Fix numbering T on shape $\lambda \vdash n$, and let $p \in R(T)$ and $q \in C(T)$. Then*

$$p \cdot a_T = a_T \cdot p = a_T, \qquad q \cdot b_T = b_T \cdot q = \operatorname{sgn}(q) b_T.$$

Proof. These can be verified by calculation.

$$p \cdot a_T = p \cdot \sum_{r \in R(T)} r = \sum_{r \in R(T)} p \cdot r = \sum_{r' \in R(T)} r' = a_T$$

The computation to show that $a_T \cdot p = a_T$ is precisely the same.

Similarly, for the second equality we have

$$\begin{aligned} q \cdot b_T = q \cdot \sum_{c \in C(T)} \operatorname{sgn}(c) c &= \sum_{c \in C(T)} \operatorname{sgn}(c) q \cdot c = \sum_{c \in C(T)} \frac{\operatorname{sgn}(q)\operatorname{sgn}(c) q \cdot c}{\operatorname{sgn}(q)} \\ &= \operatorname{sgn}(q) \sum_{c \in C(T)} \operatorname{sgn}(q \cdot c) q \cdot c = \operatorname{sgn}(q) \sum_{c' \in C(T)} \operatorname{sgn}(c') c' = \operatorname{sgn}(q) b_T \end{aligned}$$

Then $b_T \cdot q = b_T$ follows with a similar calculation.

□

Proposition 5.7. *For a numbering T,*

$$a_T \cdot a_T = |R(T)| a_T \qquad b_T \cdot b_T = |C(T)| b_T$$

Proof. Since $p \cdot a_T = a_T$, we have

$$a_T \cdot a_T = \left(\sum_{p \in R(T)} p \right) \cdot a_T = \sum_{p \in R(T)} p \cdot a_T = \sum_{p \in R(T)} a_T = |R(T)| a_T$$

The proof of the second equality is precisely the same.

□

Since a group has the same representations on its conjugacy classes, it will be useful to break up the classes of S_n. Since we can define any element of S_n as a product of disjoint cycles, the conjugacy classes of S_n consist of elements whose cycles have the same length. Therefore, a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ specifies a particular conjugacy class of S_n consisting of the permutations with cycles of length λ_i for each i. Let $C(\lambda)$ denote the conjugacy class determined by λ.

Proposition 5.8. *For partition λ,*

$$|C(\lambda)| = \frac{n!}{z(\lambda)},$$

where $z(\lambda) = \prod_r r^{m_r} m_r!$ and m_r is the number of times r appears in λ.

Proof. We show this by counting the number of ways of choosing the letters in each cycle, and then the number of ways of ordering those elements within each cycle.

First, place the letters $\{1, \ldots, n\}$ in cycles of lengths $\lambda_1, \ldots, \lambda_k$, without considering ordering within the cycle. The number of such choices are

$$\begin{aligned}\binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2}\binom{n-\lambda_1-\lambda_2}{\lambda_3}\cdots\binom{\lambda_{k-1}+\lambda_k}{\lambda_{k-1}}\binom{\lambda_k}{\lambda_k}&\\ = \left(\frac{n!}{(n-\lambda_1)!\lambda_1!}\right)\left(\frac{(n-\lambda_1)!}{(n-\lambda_1-\lambda_2)!\lambda_2!}\right)&\cdots\left(\frac{(\lambda_{k-1}+\lambda_k)!}{\lambda_{k-1}!\lambda_k!}\right)\\ = \left(\frac{n!}{\lambda_1!}\right)\left(\frac{1}{\lambda_2!}\right)\cdots\left(\frac{1}{\lambda_k!\lambda_{k-1}!}\right) = \frac{n!}{\lambda_1!\cdots\lambda_k!}&\end{aligned}$$

where each binomial corresponds to choosing the length of a row.

Now we need to count the number of ways to order each row. First, assume that when we write down the word representing a cycle, we always write the smallest letter first. Then the number of h-cycles on h distinct letters is

$$1 \cdot (h-1) \cdot (h-2) \cdots 2 \cdot 1 = (h-1)!$$

corresponding to choosing each letter in the cycle.

Then the number of ways to assign these n letters to cycles is

$$\left(\frac{n!}{\lambda_1!\cdots\lambda_k!}\right)(\lambda_1-1)!(\lambda_2-1)!\cdots(\lambda_k-1)! = \frac{n!}{\lambda_1\cdots\lambda_k}.$$

Two permutations that differ only in the ordering of their cycles are equal. Therefore, this over counts permutations with more than one cycle of length λ_i. Taking this into account, we have

$$\frac{n!}{\lambda_1 \cdots \lambda_k} \cdot \frac{1}{\begin{array}{l}\#\text{ permutations}\\ \text{of rows of}\\ \text{equal length}\end{array}} = \frac{n!}{\lambda_1 \cdots \lambda_k \prod_r m_r!} = \frac{n!}{\prod_r r^{m_r} m_r!} = \frac{n!}{z(\lambda)}$$

□

Proposition 5.9. *For numbering T on shape $\lambda \vdash n$ and $\sigma \in S_n$,*

$$\sigma \cdot v_T = v_{\sigma \cdot T}.$$

Proof. This follows from calculating the product. On the left, we have

$$\sigma \cdot v_T = \sigma \cdot (b_T \cdot \{T\}) = \sigma \cdot \left(\sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \right) = \sum_{q \in C(T)} \operatorname{sgn}(q)\{\sigma \cdot q \cdot T\}$$

Since $C(\sigma \cdot T) = \sigma \cdot C(T) \cdot \sigma^{-1}$, on the right hand side, we have

$$\begin{aligned} v_{\sigma \cdot T} &= \sum_{q \in C(\sigma \cdot T)} \operatorname{sgn}(q)\{q \cdot (\sigma \cdot T)\} = \sum_{q' \in C(T)} \operatorname{sgn}(\sigma \cdot q' \cdot \sigma)\{(\sigma \cdot q' \cdot \sigma^{-1}) \cdot (\sigma \cdot T)\} \\ &= \sum_{q' \in C(T)} \operatorname{sgn}(q')\{\sigma \cdot q' \cdot T\} = \sigma \cdot v_T \end{aligned}$$

□

Now Proposition 5.3 gives us the following

Proposition 5.10. *Let T be a numbering on λ and T' a numbering on λ', such that λ does not strictly dominate λ'. If there is a pair (a, b) of letters in the same column of T and the same row of T', then $b_T \cdot \{T'\} = 0$. Otherwise, $b_T = \pm v_T$.*

Proof. Suppose we have a pair (a, b) of letters in the same column of T and the same row of T'. Let $t \in S_n$ be the transposition between a and b. Then since $t \in C(T)$,

$$b_T \cdot t = \mathrm{sgn}(t) b_T = -b_T.$$

Then we have

$$b_T \cdot \{T'\} = b_T \cdot (t \cdot \{T'\}) = -b_T \cdot \{T'\}$$

and so $b_T \cdot \{T'\} = 0$.

If there are no such pairs, then we have case (ii) of Proposition 5.3, so there are $p' \in R(T')$ and $q \in C(T)$ such that $p' \cdot T' = q \cdot T$. Therefore, since $p' \in R(T')$,

$$b_T \cdot \{T'\} = b_T \cdot \{p' \cdot T'\} = b_T \cdot \{q \cdot T\} = b_T \cdot q \cdot \{T\} = \mathrm{sgn}(q) b_T \cdot \{T\} = \pm v_T$$

□

Then, with Corollary 5.4, we have the following.

Corollary 5.11. *For T and T' standard tableau with $T < T'$, $b_T \cdot \{T'\} = 0$.*

Now we want to show that the S^λ are irreducible representations of S_n.

Proposition 5.12. *Let T be a numbering on λ and λ' a numbering such that $\lambda < \lambda'$. Then*

$$b_T \cdot M^\lambda = b_T \cdot S^\lambda = \mathbb{C} \cdot v_T, \qquad b_T \cdot M^{\lambda'} = b_T \cdot S^{\lambda'} = 0.$$

Proof. Since $M^{\lambda'}$ has basis consisting of the tabloids $\{T'\}$, the second equality follows directly from Corollary 5.11.

Now let $\{U\} \in M^\lambda$ be a basis element of the vector space. Then by Proposition 5.10, we have

$$b_T \cdot \{U\} = \begin{cases} 0 & \text{if } \exists \text{ a pair of letters in same row of } S \text{ and column of } T, \\ \pm v_T & \text{if there is no such pair} \end{cases}$$

Therefore, $b_T \cdot M^\lambda = \mathbb{C} \cdot v_T$

Also, since

$$b_T \cdot v_T = b_T \cdot b_T \cdot \{T\} = |C(T)| v_T$$

It follows that $b_T \cdot S^\lambda = \mathbb{C} \cdot v_T$.

□

This will give us our main theorem.

Theorem 5.13. *The irreducible representations of S_n are the S^λ for partitions $\lambda \vdash n$.*

Proof. Since the number of partitions of n is equal to the number of conjugacy classes of S_n, we only need to show that the representations S^λ are not isomorphic to each other and are irreducible.

By Proposition 5.9, $\sigma \cdot v_T = v_{\sigma \cdot T}$ for all $\sigma \in S_n$, so it follows that $S^\lambda = A \cdot v_T$. Also, for partitions $\lambda \neq \lambda'$, the number of tabloids in S^λ cannot equal the number of tabloids in $S^{\lambda'}$, so $S^\lambda \ncong S^{\lambda'}$.

Finally, suppose that $S^\lambda = V \oplus W$ for some vector spaces V and W. By Proposition 5.12,

$$\mathbb{C} \cdot v_T = b_T \cdot S^\lambda = b_T \cdot V \oplus b_t \cdot W$$

This is a 1-dimensional subspace, so if V and W are irreducible subspaces, it follows that either $v_T \in b_T \cdot V$ or $v_T \in b_T \cdot W$. Without loss of generality, assume $v_T \in b_T \cdot V$. Since

$$v_T = b_T \cdot \{T\} \in b_T \cdot V,$$

it follows that there is some $\{T'\} \in V$ such that

$$\{T'\} = p' \cdot \{T'\} = q \cdot \{T\}$$

for some $p' \in R(T')$ and $q \in C(T)$.

Since we assume V is indecomposable, it must be invariant under S_n. In particular, $\{T\} \in V$, and $c \cdot \{T\} = \{c \cdot T\} \in V$ for all $c \in C(T) \subseteq S_n$. Therefore,

$$v_T = \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \in V.$$

But then $S^\lambda = A \cdot v_T = V$, so S^λ is indecomposable, and thus irreducible.

□

We have now shown that the irreducible representations of S_n are precisely the S^λ over all λ partitions of n. Now we want to determine the structure of these representations. First, we will decompose M^λ.

Proposition 5.14. *Suppose $\mathcal{O} : M^\lambda \to M^{\lambda'}$ is a homomorphism. If $S^\lambda \notin \ker(\mathcal{O})$, then $\lambda' \trianglelefteq \lambda$.*

Proof. Let T be a numbering on shape λ, and assume $v_T \notin \ker(\mathcal{O})$. Then

$$\begin{aligned} b_T \cdot \mathcal{O}(\{T\}) &= \sum_{q \in C(T)} \operatorname{sgn}(q) q \cdot \mathcal{O}(\{T\}) = \sum_{q \in C(T)} \operatorname{sgn}(q)\mathcal{O}(\{q \cdot T\}) \\ &= \mathcal{O}\left(\sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \right) = \mathcal{O}(v_T) \neq 0 \end{aligned}$$

Then $b_T \cdot \{T'\} \neq 0$ for some numbering T' on λ'.

If $\lambda \neq \lambda'$ and λ does not dominate λ', then we have case (i) of Proposition 5.3, so there are entries a and b in the same row of T' and the same column of T. But then $b_T \cdot \{T'\} = 0$, by Proposition 5.10. This contradicts the calculation above, so $\lambda' \trianglelefteq \lambda$.

□

This will allow us to show the following decomposition of M^λ.

Proposition 5.15. *Given partition λ,*

$$M^\lambda \cong S^\lambda \oplus \bigoplus_{\nu \rhd \lambda} (S^\nu)^{\oplus k_{\nu\lambda}}$$

for some integers $k_{\nu\lambda}$.

Proof. Let $\nu \rhd \lambda$, and let $k_{\nu\lambda}$ be the number of copies of S^ν contained in M^λ. Let T be a numbering on shape λ. Then by Proposition 5.12,

$$b_T \cdot M^\lambda = b_T \cdot S^\lambda = \mathbb{C} \cdot v_T$$

so there is exactly one copy of S^λ in M^λ. Thus $k_{\lambda\lambda} = 1$.

Now suppose S^ν occurs in the decomposition of M^λ. Since M^ν contains a copy of S^ν, let $\pi : M^\nu \to S^\nu$ be the map defined by projection, and $i : S^\nu \to M^\lambda$ an imbedding. Then let $\mathcal{O} = i \circ \pi : M^\nu \to M^\lambda$. This is a homomorphism, with

$$\mathcal{O}(S^\nu) = i(\pi(S^\nu)) = i(S^\nu) \neq 0.$$

Then by Proposition 5.14, $\lambda \trianglelefteq \nu$, and so

$$M^\lambda \cong S^\lambda \oplus \bigoplus_{\lambda \lhd \nu} (S^\nu)^{k_{\nu\lambda}}.$$

□

Now we will give a basis for each S^λ. First, we will need the following lemma. For partition λ, we define f^λ to be the number of standard tableaux on shape λ.

Lemma 5.16.

$$\sum_\lambda (f^\lambda)^2 = n!$$

where the sum is taken over all partitions λ of n.

Proof. By the Robinson-Schensted-Knuth correspondence, there is a one-to-one correspondence between two-row arrays of words and tableau pairs (P, Q) on some shape λ. If we require that P and Q be standard tableaux, then the two-row array corresponding to it will have top row $(1\ 2\ \cdots n)$, and bottom row consisting of some word on the letters $\{1, \ldots, n\}$. Therefore, the array describes a permutation of $[n]$, so there are $n!$ such arrays.

This shows that there are $n!$ such tableau pairs, and so

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

□

Proposition 5.17. *S^λ has a basis consisting of the elements v_T for all standard tableaux T on shape λ.*

Proof. By construction, S^λ is spanned by the elements v_T, and

$$v_T = \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\}.$$

Therefore, we need to choose a set of numberings T as representative elements of each v_T. Without loss of generality, assume that each numbering T defining a tabloid $\{T\}$ are given with increasing rows.

Let T be a standard tableau on λ, and consider the orbit of $\{T\}$ by $C(T)$. Since $q \cdot \{T\} < \{T\}$ for any nontrivial $q \in C(T)$, simply choose the maximal element in the orbit to represent v_T. This shows that each standard tableau determines a unique element v_T, and so $\dim(S^\lambda) \geq f^\lambda$.

By Lemma 5.16, we have

$$\sum_\lambda (\dim S^\lambda)^2 \geq \sum_\lambda (f^\lambda)^2 = n!$$

so we must have equality. Therefore, $\dim(S^\lambda) = f^\lambda$ for each λ, and the proposition follows.

□

Example 5.18. *Consider the Young diagram $\lambda = (n)$, the row diagram. $M^{(n)}$ is generated by a single tabloid $\{T\} = \left\{\boxed{1}\boxed{2}\boxed{\cdot}\boxed{n}\right\}$. For any $\sigma \in S_n$,*

$$\sigma \cdot \{T\} = \{\sigma \cdot T\} = \{T\}$$

since σ can only permute elements along the row. Similarly, $v_T = b_T \cdot \{T\} = \{T\}$ generates all of $S^{(n)}$. Therefore, $M^{(n)} = S^{(n)}$ is the trivial representation $\mathbb{I}_n$ of S_n.

Example 5.19. *Let T be a numbering on λ. M^λ is generated by the tabloids $\{U\}$, over all standard tableau U on λ. Since we can obtain all such tableau by some permutation of $\{T\}$, M^λ has basis given by $\sigma \cdot \{T\}$ over all $\sigma \in S_n/R(T)$. Therefore,*

$$M^\lambda \cong \mathrm{Ind}_{R(T)}^{S_n}(\mathbb{I}_n) = \mathbb{C}[S_n] \otimes_{\mathbb{C}[R(T)]} \mathbb{C}$$

the induced representation of $\mathbb{I}_n$ from $R(T)$ to S_n.

We have constructed a complete set of irreducible representations of S_n. We will now describe the ring of representations. Let R_n denote the free abelian group of isomorphism classes of representations of S_n. Let $R = \bigoplus_{n=0}^\infty R_n$. We make a R a graded ring by defining a product $R_m \times R_n \to R_{m+n}$ given by

$$[V] \cdot [W] = \left[\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V \otimes W\right]$$

for $[V] \in R_m$ and $[W] \in R_n$. Also, we define inner product $\langle \cdot, \cdot \rangle$ on each R_n by

$$\langle [V], [W] \rangle = \sum_\lambda m_\lambda n_\lambda$$

where $V \cong \bigoplus_\lambda (S^\lambda)^{\oplus m_\lambda}$ and $W \cong \bigoplus_\lambda (S^\lambda)^{\oplus n_\lambda}$ If we let V and W have characters χ_V and χ_W, then by orthogonality, we have

$$\langle [V], [W] \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma^{-1}) \chi_W(\sigma)$$

Then by Proposition 5.8, we have

$$\begin{aligned}(1) \quad \langle [V], [W] \rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} \chi_V(\sigma^{-1}) \chi_W(\sigma) = \frac{1}{n!} \sum_{\mu} \chi_V(C(\mu)) \chi_W(C(\mu)) \cdot |C(\mu)| \\ &= \sum_{\mu} \frac{1}{z(\mu)} \chi_V(C(\mu)) \chi_W(C(\mu))\end{aligned}$$

Now we want to describe the relationship between representations of S_n and the Schur polynomials s_λ. For this reason, we will make a slight digression to give some facts about symmetric polynomials.

Earlier, we gave a definition of the Schur polynomial $s_\lambda(x)$ on m variables $x = (x_1, \ldots, x_m)$. We will need to define the following polynomials in a similar manner. Given partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, let

$$h_\lambda(x) = h_{\lambda_1}(x) \cdot h_{\lambda_2}(x) \cdots h_{\lambda_k}(x) \quad \text{where} \quad h_n(x) = \sum_{i_1 + \cdots i_m = n} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}.$$

The polynomials $h_n(x)$ are the n^{th} *complete symmetric polynomials.* Let

$$e_\lambda(x) = e_{\lambda_1}(x) \cdot e_{\lambda_2}(x) \cdots e_{\lambda_k}(x) \quad \text{where} \quad e_n(x) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

The polynomials $e_n(x)$ are the n^{th} *elementary symmetric polynomials.* Let

$$m_\lambda(x) = \sum_{\sigma \in S_m} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(n)}^{\lambda_n},$$

the *monomial symmetric polynomial* of λ. Finally, define

$$p_\lambda(x) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_k}(x) \quad \text{where} \quad p_n(x) = x_1^n + x_2^n + \cdots + x_m^n.$$

The $p_n(x)$ are the *Newton power sums.* We will need these to describe the set of symmetric functions. Therefore, we will want the following proposition.

Proposition 5.20. *The following sets are bases of the homogeneous symmetric polynomials of degree n in m variables.*

(i) $\{m_\lambda(x) : \lambda \vdash n$ *with at most* m *rows*$\}$

(ii) $\{s_\lambda(x) : \lambda \vdash n$ *with at most* m *rows*$\}$

(iii) $\{e_\lambda(x) : \lambda \vdash n$ *with at most* m *columns*$\}$

(iv) $\{h_\lambda(x) : \lambda \vdash n$ *with at most* m *columns*$\}$

(v) $\{h_\lambda(x) : \lambda \vdash n$ *with at most* m *rows*$\}$

Proof. Let p be a homogeneous symmetric polynomial of degree n in m variables, and order the terms of p in lexicographic order. Choose the first monomial $c_\lambda x^\lambda = c_\lambda x_1^{\lambda_1} \cdots x_m^{\lambda_m}$. Since the terms are ordered in lexicographic order, λ must be a partition. By symmetry, all the terms occurring in m_λ must appear in p. Then $p - c_\lambda m_\lambda$ is also a symmetric polynomial, and it is strictly smaller than p in the lexicographic ordering. This shows that the m_λ span the homogeneous symmetric polynomials.

Now suppose we have $\sum_\lambda a_\lambda m_\lambda = 0$, and choose the maximal λ in lexicographic order such that $a_\lambda \neq 0$. But then the sum must contain the term $a_\lambda x^\lambda$, and no other elements m_μ contains this term. This is impossible, so the m_λ forms a basis of the homogeneous symmetric polynomials.

Note that all the sets in (i)-(v) have the same cardinality (the number of partitions of n with m parts), so now we just need to show that (ii)-(v) span the homogeneous symmetric polynomials. The argument for the s_λ is essentially the same as for the m_λ. The proof for the e_λ is similar since the leading monomial of e_λ is $x^{\tilde{\lambda}}$.

For (iv), we can use the identity

$$h_i(x) - e_1(x)h_{i-1}(x) + e_2(x)h_{i-2}(x) - \cdots + (-1)^i e_i(x) = 0$$

This shows that (iii) and (iv) have the same span. Finally, (ii) and (v) have the same span using the determinantal formula for Schur polynomials

$$s_\lambda(x) = \det(h_{\lambda_i - i + j}(x))_{i,j}.$$

□

Now we define the ring of symmetric functions. Note that the calculation of each symmetric polynomial described above depends on the number of variables m we use. However, many times this will not matter, since they have the property that they specialize. That is, for $k < m$,

$$p(x_1, \ldots, x_k, 0, \ldots, 0) = p(x_1, \ldots, x_k).$$

Functions with this property are known as *symmetric functions.* Let Λ_n be the $\mathbb{Z}$-module generated by the symmetric functions of degree n. Then define $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$ to be the graded ring of symmetric functions with the natural product. We define an inner product $\langle \cdot, \cdot \rangle$ on Λ_n such that the Schur polynomials s_λ form an orthonormal basis.

Now for partitions λ and μ of the same integer n, define integers χ_μ^λ and ξ_μ^λ to be the coefficients in the following polynomials.

$$(2) \qquad p_\mu = \sum_\lambda \chi_\mu^\lambda s_\lambda \qquad p_\mu = \sum_\lambda \xi_\mu^\lambda m_\lambda$$

We can show that, with the given inner product,

$$(3) \qquad \langle h_\lambda, m_\lambda\rangle = 1, \qquad \langle h_\lambda, m_\mu\rangle = 0, \quad \lambda \neq \mu$$

$$\langle p_\lambda, p_\lambda\rangle = z(\lambda) \qquad \langle p_\lambda, p_\mu\rangle = 0, \quad \lambda \neq \mu$$

Then we have the following equivalent form for χ^λ_μ.

$$(4) \qquad s_\lambda = \sum_\mu \frac{\langle s_\lambda, p_\mu\rangle}{\langle p_\mu, p_\mu\rangle} p_\mu = \sum_\mu \frac{1}{z(\mu)} \chi^\lambda_\mu p_\mu$$

Similarly, we can calculate an equivalent equation involving ξ^λ_μ.

$$(5) \quad h_\lambda = \sum_\mu \frac{\langle h_\lambda, p_\mu\rangle}{\langle p_\mu, p_\mu\rangle} p_\mu = \sum_\mu \frac{1}{z(\mu)} \left\langle h_\lambda, \sum_\nu \xi^\nu_\mu m_\nu \right\rangle p_\mu = \sum_\mu \frac{1}{z(\mu)} \sum_\nu \xi^\nu_\mu \langle h_\lambda, m_\nu\rangle p_\mu$$
$$= \sum_\mu \frac{1}{z(\mu)} \xi^\lambda_\mu p_\mu$$

Proposition 5.21. *Let χ be the character of M^λ. Then $\chi(C(\mu)) = \xi^\lambda_\mu$.*

Proof. The character χ on $\sigma \in S_n$ is given by the number of tabloids that are fixed by σ. Since we can express σ as a product of disjoint cycles, σ fixes a tabloid $\{T\}$ provided that each cycle in σ contains the same letters as each row in $\{T\}$.

Let $r(p, q)$ be the number of cycles of length q whose elements lie in the p^{th} row of a tabloid. Then the number of tabloids fixed by σ is

$$\sum_{\{r(p,q)\}} \prod_{q=1}^{n} \frac{m_q!}{r(1, q)! \cdots r(n, q)!}$$

where the sum is taken over all sets $\{r(p, q)\}$ such that

$$r(p, 1) + 2r(p, 2) + 3r(p, 3) + \cdots + nr(p, n) = \lambda_p,$$

$$r(1, q) + r(2, q) + r(3, q) + \cdots + r(n, q) = m_q,$$

corresponding to the conditions that the p^{th} row contains λ_p cells and there are m_q cycles of length q. Also, for $\sigma \in C(\mu)$, we have

$$p_\mu(x) = \prod_{q=1}^{n} (x_1^q + \cdots + x_n^q)^{m_q} = \prod_{q=1}^{n} \sum \frac{m_q!}{r(1,q)! \cdots r(n,q)!} x_1^{qr(1,q)} \cdots x_n^{qr(n,q)}$$

from the multinomial expansion. Therefore, the number of tabloids fixed by σ is the coefficient of x^λ in p_μ. By the definition in Equation (2), this is ξ_μ^λ.

□

Now define a homomorphism $\phi : \Lambda \to R$ given by $\phi(h_\lambda) = [M^\lambda]$.

Theorem 5.22. *ϕ is an isometric isomorphism, and $\phi(s_\lambda) = [S^\lambda]$ for each λ.*

Proof. From Example 5.18, $\phi(h_{(n)}) = \left[M^{(n)}\right] = \mathbb{I}_n$ in S_n. By Proposition 5.20, Λ is a polynomial ring on the variables h_n. Therefore, ϕ is a homomorphism if the product on the representation ring satisfies

$$M^\lambda = M^{(\lambda_1)} \cdot M^{(\lambda_2)} \cdots M^{(\lambda_k)}.$$

From Example 5.19, we have $M^\lambda = \mathrm{Ind}_{R(T)}^{S_n}(\mathbb{I}_n)$, so by definition of the product on the ring of representations, we have

$$M^\lambda = \mathrm{Ind}_{R(T)}^{S_n}(\mathbb{I}_n) = \mathrm{Ind}_{S_{\lambda_1} \times \cdots \lambda_k}^{S_{\lambda_1 + \cdots + \lambda_k}}(\mathbb{I}_{\lambda_1} \otimes \cdots \otimes \mathbb{I}_{\lambda_k}) = M^{(\lambda_1)} \cdots M^{(\lambda_k)}$$

By Proposition 5.15, we have

$$\left[M^\lambda\right] = \left[S^\lambda\right] + \sum_{\nu \rhd \lambda} k_{\nu\lambda} \left[S^\nu\right]$$

Then since the classes $[S^\lambda]$ form a basis of R, the classes $[M^\lambda]$ also form a basis of R, by ordering the partitions λ. Therefore, ϕ is an isomorphism.

Now define $\psi : R \to \Lambda$ such that the composition $\psi \circ \phi = \text{id}$. Since $\phi(h_\lambda) = [M^\lambda]$, we must have

$$\psi([M^\lambda]) = h^\lambda = \sum_\mu \frac{1}{z(\mu)} \xi^\lambda_\mu p_\mu = \sum_\mu \frac{1}{z(\mu)} \chi_{M^\lambda}(C(\mu)) p_\mu$$

using Equation 5 and Proposition 5.21. Since the $[M^\lambda]$ forms a basis of R, we can define ψ for any $[V] \in R$ by the same formula

$$\psi([V]) = \sum_\mu \frac{1}{z(\mu)} \chi_V(C(\mu)) p_\mu.$$

Therefore, ψ is a homomorphism, and since ϕ is its inverse, ψ is the inverse of ϕ.

Now note that for $[V], [W] \in R$,

$$\begin{aligned}
\langle \psi([V]), \psi([W]) \rangle &= \left\langle \sum_\mu \frac{1}{z(\mu)} \chi_V(C(\mu)) p_\mu, \sum_\mu \frac{1}{z(\mu)} \chi_W(C(\mu)) p_\mu \right\rangle \\
&= \sum_\mu \frac{1}{z(\mu)^2} \chi_V(C(\mu)) \chi_W(C(\mu)) \langle p_\mu, p_\mu \rangle \\
&= \sum_\mu \frac{1}{z(\mu)} \chi_V(C(\mu)) \chi_W(C(\mu)) = \langle [V], [W] \rangle
\end{aligned}$$

using Equation (1) and (3). Then ψ is an isometry, and so ϕ is also an isometry.

Using the identities $h_\lambda = \sum_\mu K_{\mu\lambda} s_\mu(x) = s_\lambda + \sum_{\nu \rhd \lambda} K_{\nu\lambda} s_\nu$, and $[M^\lambda] = [S^\lambda] + \sum_{\nu \rhd \lambda} k_{\nu\lambda}[S^\nu]$, it follows that

$$\phi(h_\lambda) = \phi(s_\lambda) + \sum_{\nu \rhd \lambda} K_{\nu\lambda} \phi(s_\nu) = [S^\lambda] + \sum_{\nu \rhd \lambda} k_{\nu\lambda}[S^\nu]$$

In particular,

$$\phi(s_\lambda) = [S^\lambda] + \sum_{\nu \rhd \lambda} m_{\nu\lambda}[S^\nu]$$

for some set of coefficients $m_{\nu\lambda}$. Then using the inner product, we get

$$1 = \langle s_\lambda, s_\lambda \rangle = \langle \phi(s_\lambda), \phi(s_\lambda) \rangle = \left\langle [S^\lambda] + \sum_{\nu\lambda} [S^\nu], [S^\lambda] + \sum_{\nu\lambda} [S^\nu] \right\rangle$$
$$= \langle [S^\lambda], [S^\lambda] \rangle + \sum m_{\nu\lambda}^2 \langle [S^\nu], [S^\nu] \rangle = 1 + \sum m_{\nu\lambda}^2$$

Therefore, the $m_{\nu\lambda} = 0$, and so $\phi(s_\lambda) = [S^\lambda]$.

□

This proof will also allow us to determine the character of each $[S^\lambda]$.

Corollary 5.23. *Let χ be the character of S^λ. Then $\chi(C(\mu)) = \chi_\mu^\lambda$.*

Proof. From Equation (4) and the definition of ψ in the proof of Theorem 5.22, we have

$$s_\lambda = \sum_\mu \frac{1}{z(\mu)} \chi_\mu^\lambda p_\mu$$
$$= \psi([S^\lambda]) = \sum_\mu \frac{1}{z(\mu)} \chi_{S^\lambda}(C(\mu)) p_\mu$$

Thus, $\chi_{S^\lambda}(C(\mu)) = \chi_\mu^\lambda$.

□

Given the isomorphism ϕ, we can translate identities on Schur polynomials to identities about representations. In particular, the formula $h_\lambda = \sum_\mu K_{\mu\lambda} s_\mu(x)$ gives us the following restatement of Proposition 5.15.

Corollary 5.24. *Given a partition λ,*

$$[M^\lambda] \cong [S^\lambda] \oplus \bigoplus_{\nu \rhd \lambda} [S^\nu]^{\oplus K_{\nu\lambda}}$$

where the integers $K_{\nu\lambda}$ are the Kostka numbers.

Also, we have a Littlewood-Richardson rule for representations.

Corollary 5.25. *For partitions λ and μ,*

$$[S^\lambda] \cdot [S^\mu] \cong \bigoplus_\nu [S^\nu]^{\oplus c^\nu_{\lambda\mu}}.$$

6. Representations of $GL(E)$

Now we want to describe representations of the general linear group involving Young tableaux. Here, we will let $R = \mathbb{C}$, and E be a complex vector space of dimension m, and we consider the general linear group $GL(E) = GL_m\mathbb{C}$.

First, we define the notion of an *exchange* on a numbering. Let λ be a Young diagram and T be a filling on λ. Choose 2 columns of T, v_i and v_j, and a set of k cells from each column $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$. The exchange determined by these choices is the filling S on λ obtained from T by switching the sets of chosen cells in each column, keeping the order of the entries within the column.

Example 6.1. *Consider the filling*

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & \mathbf{2} & 3 & 4 & 5 \\ \hline 6 & 7 & \mathbf{8} & 9 \\ \cline{1-4} 3 & \mathbf{4} & \mathbf{2} \\ \cline{1-3} 1 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array},$$

and choose the second and third columns of T to make the exchange. Choose the first and third cells of the second column, and the second and third cells of the third column. The exchange determined by these choices is then

$$S = \begin{array}{|c|c|c|c|c|} \hline 1 & \mathbf{8} & 3 & 4 & 5 \\ \hline 6 & 7 & \mathbf{2} & 9 \\ \cline{1-4} 3 & \mathbf{2} & \mathbf{4} \\ \cline{1-3} 1 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}.$$

Now let $E^{\times n} = E \times \cdots \times E$ denote the Cartesian product of n copies of E. For $\lambda \vdash n$, let $E^{\times\lambda}$ denote the product $E^{\times n}$ such that an element of $E^{\times\lambda}$ is a Young diagram with cells labelled by an element $x \in E$. In other words, $E^{\times\lambda} = E^{\times n}$, where we specify that an element is a filling of λ with entries from E.

For R-module F, let $\phi : E^{\lambda} \to F$ be a map satisfying the following properties:

(i) ϕ is R-multilinear.

(ii) ϕ is alternating in the columns of λ.

(iii) For $v \in E^{\times\lambda}$,

$$\phi(v) = \sum_{w} \phi(w),$$

where the sum is taken over all w obtained from v by an exchange on two chosen columns of v and a choice of k cells from the right-most chosen column.

Example 6.2. *Let ϕ be a map satisfying (i)-(iii) above, and consider a filling*

$$T = \begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 3 & 4 & 2 \\ 1 & 5 \\ 6 \end{ytableau} \; .$$

Choose the second and third columns of T, and choose the second and third cells of the third column. Then ϕ is a map that satisfies

$$\phi\left(\begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & \mathbf{8} & 9 \\ 3 & 4 & \mathbf{2} \\ 1 & 5 \\ 6 \end{ytableau}\right) = \phi\left(\begin{ytableau} 1 & \mathbf{8} & 3 & 4 & 5 \\ 6 & \mathbf{2} & \mathbf{2} & 9 \\ 3 & 4 & \mathbf{7} \\ 1 & 5 \\ 6 \end{ytableau}\right) + \phi\left(\begin{ytableau} 1 & \mathbf{8} & 3 & 4 & 5 \\ 6 & 7 & \mathbf{2} & 9 \\ 3 & \mathbf{2} & \mathbf{4} \\ 1 & 5 \\ 6 \end{ytableau}\right) + \phi\left(\begin{ytableau} 1 & \mathbf{8} & 3 & 4 & 5 \\ 6 & 7 & \mathbf{2} & 9 \\ 3 & 4 & \mathbf{5} \\ 1 & \mathbf{2} \\ 6 \end{ytableau}\right)$$

$$+\,\phi\left(\begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 & \mathbf{8} & \mathbf{7} & 9 \\ 3 & \mathbf{2} & \mathbf{4} \\ 1 & 5 \\ 6 \end{ytableau}\right) + \phi\left(\begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 & \mathbf{8} & \mathbf{7} & 9 \\ 3 & 4 & \mathbf{5} \\ 1 & \mathbf{2} \\ 6 \end{ytableau}\right) + \phi\left(\begin{ytableau} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & \mathbf{4} & 9 \\ 3 & \mathbf{8} & \mathbf{5} \\ 1 & \mathbf{2} \\ 6 \end{ytableau}\right)$$

Now define the *Schur module* E^{λ} to be the universal target module for the maps ϕ, and define the map $E^{\times\lambda} \to E^{\lambda}$ by $v \mapsto v^{\lambda}$, such that for any map $\phi : E^{\times\lambda} \to F$

satisfying (i)-(iii), we have a unique map $\tilde{\phi} : E^\lambda \to F$ such that $\phi(v) = \tilde{\phi}(v^\lambda)$ for any $v \in E^{\times\lambda}$.

Consider the following important examples.

Example 6.3. *Let $\lambda = (1^n)$. Here, (iii) is not applicable, since there is only one column in (1^n). Then (ii) makes the product alternating, so we have $E^{(1^n)} = \wedge^n E$, the n^{th} exterior product of E.*

Example 6.4. *Let $\lambda = (n)$. (ii) does not apply, since there is only one row of (n). (iii) gives us the relation*

$$v_1 \otimes \cdots \otimes v_n = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for any $\sigma \in S_n$, and so

$$E^{(n)} = E^{\otimes n}/v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} = Sym^n E$$

the n^{th} symmetric power of E.

These two examples immediately suggest how we should begin to construct E^λ for general λ. Let λ be a partition. (i) tells us that E^λ must consist of $E^{\otimes n}$ the n-fold tensor product. If we add in the property (ii), then we need to quotient by the elements in which two entries in the same column are equal. Then the module with properties (i) and (ii) is

$$\wedge^{\mu_1} E \otimes \cdots \otimes \wedge^{\mu_l} E$$

where $\mu = \tilde{\lambda}$, so μ_i gives the length of the i^{th} column. Then let $Q^\lambda(E)$ be the submodule generated by all elements of the form $\wedge v - \sum \wedge w$, where the w is obtained from v by exchanges. If we quotient by the submodule $Q^\lambda(E)$, this will gives us property (iii), so

$$E^\lambda = \wedge^{\mu_1} E \otimes \cdots \otimes \wedge^{\mu_l} E/Q^\lambda(E).$$

Our construction also has an obvious map to the set of fillings on λ. If we let E have basis $\{e_1, \dots, e_m\}$, then we can define a map from any filling T on λ with entries from $[m]$ by letting $i \mapsto e_i$ for each cell of T. Let e_T be the element of E^λ obtained in this way. Then we have the following proposition.

Proposition 6.5. *Let E have basis $\{e_1, \dots, e_m\}$, and let F be the vector space generated by the elements e_T for all fillings T on λ with entries from $[m]$. Then $E^\lambda \cong F/Q$, where Q is generated by the elements*

(i) e_T*, where T has two equal entries in a column,*

(ii) $e_T + e_{T'}$*, where T' is obtained from T by switching two entries in the same column,*

(iii) $e_T - \sum e_U$*, where the sum is taken over all fillings U obtained from T by the exchange procedure.*

Proof. The elements e_T generate E^λ by linearity, so the map $F \to E^\lambda$ is a well-defined surjection. In particular, the e_T form a basis of $E^{\otimes\lambda}$. Clearly, if we quotient by the elements given by properties (i) and (ii), we obtain the module $\wedge^{\mu_1} E \otimes \cdots \otimes \wedge^{\mu_l} E$. Then the elements given by (iii) generate $Q^\lambda(E)$, and so we have $E^\lambda \cong F/E$, by the argument above.

□

Now let $Z = \begin{bmatrix} Z_{1,1} & \cdots & Z_{1,m} \\ \vdots & \ddots & \vdots \\ Z_{n,1} & \cdots & Z_{n,m} \end{bmatrix}$ be an $n \times m$ matrix of variables $Z_{i,j}$. Then define $R[Z] = R[Z_{1,1}, Z_{1,2}, \dots, Z_{n,m}]$ to be the polynomial ring in the variables $Z_{i,j}$. For indices $i_1, \dots, i_p \in [m]$ with $p \leq n$, define the element

$$D_{i_1,i_2,\dots,i_p} = \det \begin{bmatrix} Z_{1,i_1} & \cdots & Z_{1,i_p} \\ \vdots & \ddots & \vdots \\ Z_{p,i_1} & \cdots & Z_{p,i_p} \end{bmatrix}.$$

Now let T be a filling on shape λ with at most n rows and entries taken from $[m]$. Then define

$$D_T = \prod_{j=1}^{l} D_{T(1,j),T(2,j),\ldots,D(\mu_j,j)}$$

where $\mu = \tilde{\lambda}$ gives the lengths of the columns of T and $T(i,j)$ is the entry in the ij^{th} cell of T.

Example 6.6. *Let* $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$. *Then*

$$D_T = D_{T(1,1),T(2,1)} D_{T(1,2)} = D_{1,3} D_2 = \begin{vmatrix} Z_{11} & Z_{13} \\ Z_{21} & Z_{23} \end{vmatrix} \cdot |Z_{12}| = Z_{11}Z_{12}Z_{23} - Z_{12}Z_{13}Z_{21}$$

Notice that for a given filling T, the expression $D_{T(1,j),T(2,j),\ldots,D(\mu_j,j)}$ is the determinant term defined by a column of the filling, and the element D_T can be thought of as the product of these column determinants.

Now we want to define a map between E^λ and $R[Z]$. For the following proposition, we will need to use Sylvester's theorem, given here for reference.

Theorem 6.7 (Sylvester). *Let M and N be $p \times p$ matrices, and let $k \leq p$. Then*

$$\det(M) \cdot \det(N) = \sum \det(M') \cdot \det(N'),$$

where the sum is taken over all matrices M' and N' obtained from M and N by exchanging a set of k fixed columns of M with any k columns of N.

Proposition 6.8. *The map $\gamma : E^\lambda \to R[Z]$ given by $e_T \mapsto D_T$ for all T is a homomorphism.*

Proof. By Proposition 6.5, the elements e_T generate E^λ, so we just need to show that the images $\gamma(e_T) = D_T$ satisfy the relations (i)-(iii) given in Proposition 6.5.

The D_T are defined by matrix determinants defined by the columns of T, so the alternating properties of columns (i) and (ii) follow from the alternating property of determinants.

For property (iii), suppose we consider the exchange on two columns of a filling T. Let $i_1, \ldots, i_p$ be the entries of the first column and $j_1, \ldots, j_q$ be the entries of the second column of the exchange. Since λ is a partition, we must have $p \geq q$, so we can define matrices

$$M = \begin{bmatrix} Z_{1,i_1} & \cdots & Z_{1,i_p} \\ \vdots & \ddots & \vdots \\ Z_{p,i_1} & \cdots & Z_{p,i_p} \end{bmatrix} \quad \text{and} \quad N = \left[\begin{array}{ccc|c} Z_{1,j_1} & \cdots & Z_{1,j_q} & 0 \\ \vdots & \ddots & \vdots & \\ Z_{p,j_1} & \cdots & Z_{p,j_q} & I_{p-q} \end{array}\right]$$

where N has a $q \times (p-q)$ block of zeros in the upper right corner and I_{p-q} is the identity on the last $p-q$ variables. Then, from Sylvester's theorem, we have

$$\begin{aligned} D_T &= \prod_{j=1}^{l} D_{T(1,j),T(2,j),\ldots,D(\mu_j,j)} = D_{T^1} \cdots D_{T^l} = D_{T^1} \cdots D_{T^l} \cdot \det(M) \cdot \det(N) \\ &= D_{T^1} \cdots D_{T^l} \sum_{M',N'} \det(M') \cdot \det(N') = \sum D_S \end{aligned}$$

where the sum is over all S obtained from T by exchanges on the two chosen columns.

□

Proposition 6.9. *The elements e_T, as T varies over all tableaux on λ with entries from $[m]$, form a basis of E^λ.*

Proof. From Proposition 6.5, the e_T span E^λ. Therefore, we need to show that when we restrict to just the e_T for T tableaux, they still span E^λ and are linearly independent. By properties (i) and (ii) of Proposition 6.5, we can restrict T to fillings with strictly increasing columns.

Now consider an ordering $\succ$ on fillings of the same shape λ, where $S \succ T$ if for the right-most column on which they differ, the lowest cell on which they differ is larger in S than in T.

Suppose we have have a filling T with stricly increasing columns, but not increasing rows. Let the j^{th} column be the rightmost column on which there is a non-increasing row element, and let the k^{th} row be the lowest row in this column that is not increasing. That is, the k^{th} entry of the j^{th} column is strictly greater than the k^{th} entry of the $(j+1)^{\text{th}}$ column. Then let S be the exchange of the first k entries of the j^{th} and $(j+1)^{\text{th}}$ columns. Then in this ordering, $S \succ T$. By Proposition 6.5, $e_S = e_T$ in E^λ, so we may replace T with S. We continue this process until we find an element that is maximal in the ordering $\succ$. By construction, this will be a tableaux. Therefore, the elements e_T over all T tableaux span E^λ.

Now consider the elements D_T as T varies over all tableaux. Order the variables $Z_{i,j}$ lexicographically by index, and then order the monomials lexicographically. Consider the term $D_{i_1,\ldots,i_p}$ with $i_1 < \cdots < i_p$ (that is, corresponding to a strictly increasing column of T). With the lexicographic ordering, the largest monomial in this term is $Z_{p,i_p} \cdots Z_{2,i_2} Z_{1,i_1}$. Then the largest monomial in D_T is

$$\prod_{i_1 < \cdots < i_p} Z_{1,i_1} Z_{2,i_2} \cdots Z_{p,i_p} = \prod_{\substack{1 \le i \le k \\ 1 \le j \le m}} Z_{i,j}^{m_T(i,j)}$$

where $m_T(i,j)$ is the number of times the entry j appears in the i^{th} row of T.

Now put an ordering $\ll$ on tableaux of the same shape where $T \ll T'$ if on the first row on which T and T' differ, the left-most entry on which they differ is strictly smaller in T than in T'. Then if $T \ll T'$, then the smallest pair (i,j) in the lexicographic order such that $m_T(i,j) \neq m_{T'}(i,j)$ has $m_T(i,j) > m_{T'}(i,j)$. Then $D_T > D_{T'}$, since the largest monomial in D_T is larger than $D_{T'}$ in the ordering described.

Now suppose that the D_T are linearly dependent, so $\sum a_T D_T = 0$ for some coefficients a_T. Choose the smallest such T by the ordering above, then by the argument given, the largest monomial in D_T is larger than any other term in the sum. Thus, the sum cannot be zero, and so the D_T are linearly independent. Then by Proposition 6.8, the linear independence of the e_T follows.

□

By construction, any homomorphism $E \to F$ of R-modules will determine a homomorphism $E^\lambda \to F^\lambda$. In particular, an endomorphism $E \to E$ gives an endomorphism $E^\lambda \to E^\lambda$. so we can define a left action of $\mathrm{End}_R E$ on E^λ. The following proposition describes precisely how this map acts on elements e_T.

Proposition 6.10. *Let $g = (g_{ij}) \in M_m R$ and let T be a filling on λ with entries $j_1, \ldots, j_n$. Then*

$$g \cdot e_T = \sum g_{i_1,j_1} \cdot g_{i_2,j_2} \cdots g_{i_n,j_n} e_{T'}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of elements from $[m]$, and where T' is a filling obtained from T by replacing $j_1, \ldots, j_n$ in T with $i_1, \ldots, i_n$.

Proof. Consider a single cell of T containing the entry p. Then e_T contains the entry e_p. An element $g = (g_{ij}) \in M_m R$ acts on e_p by

$$g \cdot e_p = (g_{ij}) e_p = g_{1p} e_1 + g_{2p} e_2 + \cdots g_{mp} e_m$$

Therefore, by multilinearity, the action of g on e_T is

$$g \cdot e_T = \sum g_{i_1,j_1} \cdot g_{i_2,j_2} \cdots g_{i_n,j_n} e_{T'}$$

over the sum described above.

□

Now we wish to show that the spaces E^λ, over all λ partitions of n with entries from $[m]$, are the irreducible polynomial representations of $GL(E)$. Assume a basis $\{e_1, \ldots, e_m\}$ of E, so $G = GL(E) = GL_m\mathbb{C}$. Let $H \subseteq G$ be the subgroup consisting of diagonal matrices. Let $x = \mathrm{diag}(x_1, \ldots, x_m)$ denote the diagonal matrix with entries x_i. For representation V of G, an element $v \in V$ is a *weight vector* of V if there is an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that

$$x \cdot v = x_1^{\alpha_1} \cdots x_m^{\alpha_m} v = x^\alpha v$$

for all $x \in H$.

Since H is given by diagonal matrices, it follows that any representation V can be written as a direct sum of its weight spaces

$$V = \bigoplus V_\alpha = \bigoplus \{v \in V : x \cdot v = x^\alpha v \text{ for all } x \in H\}.$$

Let $B \subseteq G$ be the Borel subgroup consisting of all upper triangular matrices. A *highest weight vector* of V is a weight vector $v \in V$ such that

$$B \cdot v = \mathbb{C}^* \cdot V.$$

Proposition 6.11. *Let λ be a partition of n, and let $T = U(\lambda)$. Then e_T is the unique highest weight vector on E^λ.*

Proof. Let $T = U(\lambda)$, and let $g = (g_{ij}) \in B$ be an upper triangular matrix. By Proposition 6.10, we have the sum

$$g \cdot e_T = \sum g_{i_1,j_1} \cdot g_{i_2,j_2} \cdots g_{i_n,j_n} e_{T'}$$

over the given elements $e_{T'}$. Consider the possible T' appearing in this sum. In the first row of e_T, we have e_1 in every cell. Since $g_{ij} = 0$ for $i > j$, then the only element in this sum from e_1 must be e_1. Similarly, for the second row of e_T, we have e_2 in every cell. Again, since $g_{ij} = 0$ for $i > j$, e_2 can only map to e_1 or e_2. If we have an element $e_{T'}$ with an e_1 in the second row, since the first row contains all e_1, the alternating property tells us that $e_{T'} = 0$. Therefore, the only possibility is that e_2 maps to e_2. This argument continues to the other rows of e_T to show that

$$g \cdot e_T = \sum g_{i_1,j_1} \cdot g_{i_2,j_2} \cdots g_{i_n,j_n} e_{T'} = g_{11} \cdot g_{22} \cdots g_{nn} e_T$$

so $B \cdot e_T = \mathbb{C}^* \cdot e_T$, and so e_T is a highest weight vector of E^λ.

Now suppose we have another tableau $T \neq U(\lambda)$. Let the p^{th} row of T be the first row that contains an entry greater than p (that is, the first row in which T differs from $U(\lambda)$), and let q be the smallest entry in this row strictly greater than p. Let $g = (g_{ij}) \in B$ be the upper triangular matrix with

$$g_{ij} = \begin{cases} g_{ij} = 1 & i = j \\ g_{ij} = 1 & i = p, \quad j = q \\ g_{ij} = 0 & \text{otherwise} \end{cases}$$

Then let T' be the tableau obtained from T by replacing the entries q in the p^{th} row of T with p. Then $e_{T'}$ appears with coefficient 1 in the sum for $g \cdot e_T$. Since T' is also a tableau, $e_{T'}$ is linearly independent from e_T, so e_T is not a highest weight vector.

□

This proposition essentially proves the following theorem. Let $D = \wedge^m E$ be the *determinant representation*, and let $D^{\otimes k}$ be the 1-dimensional representation determined by $g \mapsto \det(g)^k$

Theorem 6.12. (i) *Let λ be a partition with at most m rows. Then E^λ is an irreducible representation of $GL(E)$ with highest weight λ.*

(ii) *The representations E^λ are all the irreducible polynomial representations of $GL(E)$.*

(iii) *For any m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$, there is a unique irreducible representation of $GL(E)$ with highest weight α.*

Proof. Since E^λ has unique highest weight vector, it is an irreducible representation of $GL(E)$. Since two representations are isomorphic precisely when they have the same weight vectors, these are all irreducible polynomial representations of $GL(E)$.

Now let $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. Consider the representation $E^\lambda \bigotimes D^{\otimes k}$ with λ and k such that $\lambda_i = \alpha_i - k \geq 0$ for all i. This is an irreducible representation, and by construction, it has weight $\{\lambda_1 + k, \ldots, \lambda_m + k\} = \alpha$. This is the unique such representation, by the same argument as above.

□

This theorem gives us all the polynomial representations of $GL(E)$. We now consider the characters of these representations. Let V be an arbitrary representation of $GL(E)$, and decompose V as a direct sum of weight spaces

$$V = \bigoplus_\alpha V_\alpha.$$

For $x \in H$, the character of V can be written as

$$\chi_V(x) = \mathrm{tr}_V(\mathrm{diag}(x)) = \sum_\alpha \dim(V_\alpha) x^\alpha$$

since the V^α are the weight spaces of V.

Proposition 6.13. *For any partition λ,*

$$\chi_{E^\lambda}(x) = s_\lambda(x).$$

Proof. For each tableau T, we have exactly one weight vector e_T, so

$$\chi_{E^\lambda}(x) = \sum x_T = s_\lambda(x).$$

That is, the character of E^λ is the Schur polynomial s_λ.

□

This suggests that there is also a Littlewood-Richardson rule for representations of $GL(E)$.

Corollary 6.14. *For partitions λ, μ,*

$$E^\lambda \otimes E^\mu \cong \bigotimes_\nu (E^\nu)^{\oplus c^\nu_{\lambda\mu}}.$$

7. Flag Varieties

We now consider some applications of the theory developed to the geometry of flag varieties.

As before, let E be a complex vector space of dimension m. We define the *Grassmannian* Gr^dE to be the set of subspaces of E of codimension d. For $F \in Gr^dE$, the quotient map $\wedge^d E \to \wedge^d(E/F)$ has kernel given by a hyperplane in $\wedge^d E$. Then the *Plücker embedding* is the map $\phi : Gr^dE \to \mathbb{P}^*(\wedge^d E)$ that sends F to this hyperplane, $\phi(F) = \ker(\wedge^d E \to \wedge^d(E/F))$.

Proposition 7.1. *The Plücker embedding is a bijection $Gr^dE \to W$, where $W \subseteq \mathbb{P}^*(\wedge^d E)$ is a variety defined by the equations*

$$(v_1 \wedge \cdots \wedge v_d) \cdot (w_1 \wedge \cdots \wedge w_d)$$
$$- \sum_{i_1 < \cdots < i_k} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_k \wedge \cdots \wedge v_d) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge w_{k+1} \wedge \cdots \wedge w_d) = 0$$

where the $v_i, w_j \in E$

Proof. The sum in the proposition is over all indices $i_1 < \cdots < i_k$, and we are taking the sum where we exchange the first k components of the element $w_1 \wedge \cdots \wedge w_d$

with each v_{i_p} in $v_1 \wedge \cdots \wedge v_d$, maintaining the order in the exchange. We will interpret this as a statement on Plücker coordinates.

Let $X_{i_1,\ldots,i_d} = e_{i_1} \wedge \cdots \wedge e_{i_d}$ where the e_i form a basis of E. Also, consider points of $\mathbb{P}^*(\wedge^d E)$ in the coordinates $x_{i_1,\ldots,i_d}$ where $i_1 < \cdots < i_d$. Let $V \in Gr^d E$, and express V as the kernel of a matrix $A : \mathbb{C}^m \to \mathbb{C}^d$ of rank d. Then we have a map $\wedge^d A : \wedge^d(\mathbb{C}^m) \to \mathbb{C}$ that takes a basis element $e_{i_1} \wedge \cdots \wedge e_{i_d}$ to the minor of A determined by the columns $i_1, \ldots, i_d$. The Plücker coordinates of an element in $\mathbb{P}^*(\wedge^d E)$ are this determinant.

With this in mind, we can see that the expressions in the proposition are equivalent to the equations in the Plücker coordinates

$$X_{i_1,\ldots,i_d} \cdot X_{j_1,\ldots,j_d} - \sum X_{i'_1,\ldots,i'_d} \cdot X_{j'_1,\ldots,j'_d} = 0$$

where the sum is over all $i'_1, \ldots, i'_d, j'_1, \ldots, j'_d$ obtained by exchanging a fixed set of k of the indices $j_1, \ldots, j_d$ with any k of the indices $i_1, \ldots, i_d$ keeping the order of the exchanged indices. By the alternating property, we may restrict to the sum over exchanges on the first k indices of $j_1, \ldots, j_d$.

Notice that these equations are precisely those given by Sylvester's theorem. Since the Plücker coordinates are given by determinant equations, by Sylvester's theorem they must satisfy these equations. Therefore, the map is well-defined. Also, by the construction of the Plücker coordinates, it is surjective.

To show that the map is injective, consider the spaces

$$V = \langle e_{p+1}, \ldots, e_m \rangle$$

$$W = \langle e_1, \ldots, e_r, e_{p+r+1}, \ldots, e_m \rangle$$

These have different Plücker coordinates for all $r \geq 1$, so their image in $\mathbb{P}^*(\wedge^d E)$ is given by a nontrivial quadratic equation. Thus, they have distinct images, so the map is injective.

□

Now consider the product

$$Gr^p E \times Gr^q E \subseteq \mathbb{P}^*(\wedge^p E) \times \mathbb{P}^*(\wedge^q E)$$

consisting of pairs (V, W) of subspaces. The *incidence variety* $Fl^{p,q}(E) \subseteq Gr^p E \times Gr^q E$ is the set of pairs (V, W) of subspaces of codimension p and q such that $V \subseteq W$. The following proposition shows that this object is indeed a variety, and gives the equations that define it.

Proposition 7.2. *$Fl^{p,q}(E) \subseteq Gr^p E \times Gr^q E$ is defined by the equations*

$$(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q)$$
$$- \sum_{i_1 < \cdots < i_k} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_k \wedge \cdots \wedge v_p) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge w_{k+1} \wedge \cdots \wedge w_q) = 0$$

for $v_1, \ldots, v_p, w_1, \ldots, w_q \in E$.

Proof. These equations looks very similar to the quadratic equations given in Proposition 7.1. Once again, we will show this by considering the Plücker coordinates. The equivalent statement on Plücker coordinates is

$$X_{i_1,\ldots,i_p} \cdot X_{j_1,\ldots,j_q} - \sum X_{i'_1,\ldots,i'_p} \cdot X_{j'_1,\ldots,j'_q} = 0 \tag{6}$$

where the sum is over all $i'_1, \ldots, i'_p, j'_1, \ldots, j'_q$ obtained by interchanging the first k indices of $j_1, \ldots, j_q$ with some set of k indices from $i_1, \ldots, i_p$, maintaining order among the exchanged indices.

First, note that $GL(E)$ acts in precisely the same way on both $Fl^{p,q}(E)$ and the quadratic equations. Both are preserved by the action of $GL(E)$, so we may choose any appropriate basis for E.

Suppose we have $(V, W) \in Fl^{p,q}(E)$, and let $V = \langle e_{p+1}, \ldots, e_m \rangle$ and $W = \langle e_{q+1}, \ldots, e_m \rangle$. From this choice, V and W each have only one nonzero Plücker coordinate, $x_{i_1,\ldots,i_p}$ and $x_{i_1,\ldots,i_q}$, and so equation (6) is trivially true.

Now suppose we have a pair (V, W) such that $W \not\subseteq W$. By the argument above, for arbitrary $r \geq 1$, we can let

$$V = \langle e_1, \ldots, e_r, e_{p+r+1}, \ldots, e_m \rangle \qquad W = \langle e_{q+1}, \ldots, e_m \rangle$$

Then consider the index sets $I = (r+1, \ldots, r+p)$ and $J = (1, \ldots, q)$, and let $k = 1$. Then the we can see that the term $X_I \cdot X_J$ appears with coefficient 1 in the left-hand side of equation (6), so the equation does not hold.

□

Now for a decreasing sequence $m \geq d_1 > d_2 > \cdots > d_s \geq 0$, define the *flag variety* $Fl^{d_1,\ldots,d_s}(E) \subseteq Gr^{d_1}E \times \cdots \times Gr^{d_s}E$ to be the set of nested subspaces

$$Fl^{d_1,\ldots,d_s}(E) = \{E_1 \subset E_2 \subset \cdots \subset E_s \subseteq E : \operatorname{codim}(E_i) = d_i, 1 \geq i \geq s\}.$$

By the Plücker embedding, it is therefore a subset of $\mathbb{P}^*(\wedge^{d_1}E) \times \cdots \times \mathbb{P}^*(\wedge^{d_s}E)$ as well.

Proposition 7.3. *$Fl^{d_1,\ldots,d_s}(E) \subseteq \mathbb{P}^*(\wedge^{d_1}E) \times \cdots \times \mathbb{P}^*(\wedge^{d_s}E)$ is defined by the quadratic equations given by (6) for $p \geq q$ in $\{d_1, \ldots, d_s\}$.*

Proof. This follows directly from Propositions 7.1 and 7.2.

□

Now we will show a connection between the flag varieties $Fl^{d_1,\ldots,d_s}(E)$ and the Schur modules E^λ. For this, we will need to define a few maps.

(i) Given a surjection $V \twoheadrightarrow W$ of vector spaces, there is a natural embedding $\mathbb{P}^*(W) \subseteq \mathbb{P}^*(V)$ that takes a hyperplane in W to its inverse image in V by the surjection.

(ii) There is an embedding $\mathbb{P}^*(V) \subseteq \mathbb{P}^*(\mathrm{Sym}^a V)$ taking the kernel of a surjection $V \twoheadrightarrow L$ to the kernel of the induced map $\mathrm{Sym}^a V \twoheadrightarrow \mathrm{Sym}^a L$. This is the a-fold *Veronese embedding.*

(iii) There is an embedding $\mathbb{P}^*(V_1) \times \cdots \mathbb{P}^*(V_s) \subseteq \mathbb{P}^*(V_1 \otimes \cdots \otimes V_s)$ taking the kernel of each surjection $V_i \twoheadrightarrow L_i$ to the kernel of the induced map $V_1 \otimes \cdots \otimes V_s \twoheadrightarrow L_1 \otimes \cdots L_s$. This is the *Segre embedding.*

The a_i-fold Veronese embedding applied to each $\mathbb{P}^*(\wedge^{d_i} E)$ gives us the embedding

$$\mathbb{P}^*(\wedge^{d_1} E) \times \cdots \times \mathbb{P}^*(\wedge^{d_s} E) \subseteq \mathbb{P}^*(\mathrm{Sym}^{a_1}(\wedge^{d_1} E)) \times \cdots \times \mathbb{P}^*(\mathrm{Sym}^{a_s}(\wedge^{d_s} E)).$$

Applying the Segre embedding to this gives us the embedding

$$\mathbb{P}^*(\mathrm{Sym}^{a_1}(\wedge^{d_1} E)) \times \cdots \times \mathbb{P}^*(\mathrm{Sym}^{a_s}(\wedge^{d_s} E)) \subseteq \mathbb{P}^*\left(\bigotimes_{i=1}^{s} \mathrm{Sym}^{a_i}(\wedge^{d_i} E)\right)$$

Also, the map $\bigotimes_{i=1}^{s} \mathrm{Sym}^{a_i}(\wedge^{d_i} E) \twoheadrightarrow E^\lambda$ gives us the embedding

$$\mathbb{P}^*(E^\lambda) \subseteq \mathbb{P}^*\left(\bigotimes_{i=1}^{s} \mathrm{Sym}^{a_i}(\wedge^{d_i} E)\right).$$

These maps give us the following diagram.

$$\begin{array}{ccccc}
Fl^{d_1,\ldots,d_s}(E) \subset & \prod_{i=1}^{s} Gr^{d_i} E \subset & & \prod_{i=1}^{s} \mathbb{P}^*(\wedge^{d_i} E) \\
 & & & \cap \\
\cap & & & \prod_{i=1}^{s} \mathbb{P}^*(\mathrm{Sym}^{a_i}(\wedge^{d_i} E)) \\
 & & & \cap \\
\mathbb{P}^*(E^\lambda) & \subset & & \mathbb{P}^*\left(\bigotimes_{i=1}^{s} \mathrm{Sym}^{a_i}(\wedge^{d_i} E)\right)
\end{array}$$

Since the same quadratic equations defining the $Fl^{d_1,\ldots,d_s}(E) \subseteq \prod_{i=1}^{s} \mathbb{P}^*(\wedge^{d_i}E)$ also define $\mathbb{P}^*(E^\lambda) \subseteq \mathbb{P}^*\left(\bigotimes_{i=1}^{s} \mathrm{Sym}^{a_i}(\wedge^{d_i}E)\right)$, the diagram commutes.

8. Schubert Varieties

We now describe some applications to the construction and analysis of Schubert varieties.

First, the flag $Fl^{m,m-1,\ldots,1}(E)$ is the *complete flag variety*, in which each flag contains a subspace of every dimension less than m. Let λ be a partition with at most r rows and n columns, and let $F_\bullet : 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = E$ be a complete flag. Then define

$$\Omega_\lambda = \Omega_\lambda(F_\bullet) = \{V \in Gr^n E : \dim(V \cap F_{n+i-\lambda_i} \geq i, 1 \leq i \leq r\}.$$

This is the *Schubert variety* defined by λ. We will be considering classes $\sigma = [\Omega_\lambda]$ in the cohomology group $H^{2|\lambda|}(Gr^n E)$. We will see that these classes have properties closely related to the Schur polynomials.

Define the *Schubert cell* Ω_λ° to be the elements $V \in Gr^n E$ such that

$$\dim(V \cap F_k) = i \quad \text{for} \quad n+i-\lambda_i \leq k \leq n+i-\lambda_{i+1}, \quad 0 \leq i \leq r$$

We will describe this further using a matrix construction. Let E have basis $e_1, \ldots, e_m$, and specify that $F_k = \langle e_1, \ldots, e_k \rangle$ for each subspace F_k in the flag. Then let $V \in \Omega_\lambda^\circ$ be spanned by the rows of an $r \times m$ matrix constructed in the following way. On row i, there is a 1 in the $(n+i-\lambda_i)^{\text{th}}$ position. All entries to the right of this entry in the same row are zero, and all other entries in the same column are zero. Fill in the entire matrix in this manner, and the entries that are not yet determined are arbitrary. For example, consider $\lambda = (6,4,3,2)$ with $r = 4$ and $n = 6$. This determines the 4×10

matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & 0 & * & 1 & 0 & 0 \end{bmatrix}$$

By this construction, the i^{th} row has $n - \lambda_i$ stars, so we have a map from Ω_λ° to the affine space of dimension $rn - |\lambda|$.

Now let $\tilde{F}_\bullet$ denote the flag $F_\bullet$ where now we let $\tilde{F}_k = \langle e_{m-k+1}, \ldots, e_m \rangle$ (that is, $\tilde{F}_k$ is given by the last k basis elements). Let $\tilde{\Omega}_\lambda$ and $\tilde{\Omega}_\lambda^\circ$ be the corresponding Schubert variety and Schubert cell. We parametrize these by a similar matrix construction. Construct an $r \times n$ matrix by placing a 1 in the $(r - i + 1, \lambda_i - i)^{\text{th}}$ position. Then place zeros to the left of this entry and in each entry in the same column. This is precisely the same matrix as constructed above, but rotated 180°. Now consider the following subspaces

$$A_i = F_{n+i-\lambda_i}, \quad B_i = \tilde{F}_{n+i-\mu_i} \quad C_i = A_i \cap B_{r+1-i} \quad 1 \leq i \leq r.$$

Note that the C_i are spanned by the e_j such that $i + \mu_{r+1-i} \leq j \leq n + i - \lambda_i$.

Proposition 8.1. *If Ω_λ and $\tilde{\Omega}_\mu$ are not disjoint, then $\lambda_i + \mu_{r+1-i} \leq n$ for all i.*

Proof. Let $V \in \Omega_\lambda \cap \tilde{\Omega}_\lambda^\circ$. Then $\dim(V \cap A_i) \geq i$ and $\dim(V \cap B_{r+1-i}) \geq r + 1 - i$ for all i. Since $\dim V = r$,

$$\dim(V \cap A_i) + \dim(V \cap B_{r+1-i}) - \dim V \geq i + (r + 1 - i) - r = 1$$

so the intersection $(V \cap A_i) \cap (V \cap B_{r+1-i}) = V \cap C_i$ has dimension at least 1, and so the proposition follows by the remark above.

□

Note that the requirement $\lambda_i + \mu_{r+1-i} \leq n$ for each $1 \leq i \leq r$ means that for Young diagrams λ and μ, the lengths of the first row of λ and the last row of μ sum to less than n, the lengths of the second row of λ and the second to last row of μ sum to less than n, and so on. This is equivalent to saying that we can rotate μ by 180° and place λ and μ in an $r \times n$ rectangle such that they do not overlap. In particular, if $|\lambda| + |\mu| = rn$, then Ω_λ and $\tilde{\Omega}_\mu$ intersect precisely when λ and μ fill this rectangle exactly. This proves the following duality theorem.

Proposition 8.2.

$$\sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \textit{if } \lambda_1 + \mu_{r+1-i} = n \quad \textit{for all } 1 \leq i \leq r \\ 0 & \textit{if } \lambda_1 + \mu_{r+1-i} > n \quad \textit{for any } i \end{cases}$$

Therefore, for partition λ, we define a *dual partition* μ by $\mu_i = n - \lambda_{r+1-i}$, and we say that σ_μ is the *dual class* to σ_λ.

This idea will also give us more general results. Notice that our matrix parametrizations give us the coordinates on each Schubert variety. Two Schubert varieties Ω_λ and $\tilde{\Omega}_\mu$ intersect precisely when their matrix parametrizations share common non-zero entries. With this interpretation, it is clear that two varieties intersect transversally precisely when their matrix parametrizations share only entries with 1. Similarly, the spaces contained in their intersection are those that can be parametrized by entries that are in both matrix parametrization. That is, they are parametrized by the arbitrary entries between the 1's on each row. Equivalently, if the partitions λ and μ can be placed on an $r \times n$ rectangle as described above, and there are k cells not contained in either μ or λ, no two of which are in the same column, then we have $\sigma_\mu \cdot \sigma_\lambda \cdot \sigma_k = 1$ where σ_k corresponds to the *special Schubert variety* $\Omega_{(k)}$. This proves *Pieri's formula.*

Proposition 8.3. (Pieri)

$$\sigma_\lambda \cdot \sigma_k = \sum \sigma_{\lambda'}$$

for λ' obtained from λ by adding k cells to λ, no two in the same column.

9. Further Relationships

We will now consider some related applications of Young tableaux. In particular, we consider a class of problems related to the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$. As we have shown, this number arises from problems involving the Schur polynomials, Young tableaux, and representation theory. Define the set

$$T_n = \{(\lambda, \mu, \nu) : \lambda, \mu, \nu \text{ partitions with at most } n \text{ parts}, c^\nu_{\lambda\mu} \neq 0\}.$$

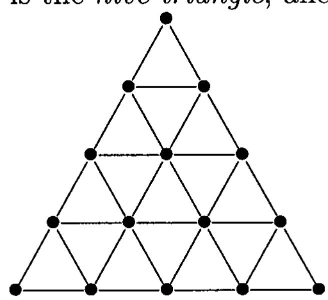
By Corollary 4.4, $(\lambda, \mu, \nu) \in T_n$ if there is a skew tableau S on ν/λ that rectifies to a tableau on μ. Equivalently, $(\lambda, \mu, \nu) \in T_n$ if there are tableaux (T, U) on λ and μ such that $T \cdot U$ is a tableau on ν. Corollary 4.5 and Proposition 4.11 give us a number of similar statements about the set T_n and Young tableaux. We have also shown applications to representation theory. From Corollary 5.25, $(\lambda, \mu, \nu) \in T_n$ if and only if the element $[S^\nu]$ appears in the irreducible decomposition of the product $[S^\lambda] \cdot [S^\mu]$ in the representation ring of S_n.

Now note that $T_n \subset \mathbb{Z}^{3n}$. We have the following saturation conjecture, proven by Knutson and Tao [8].

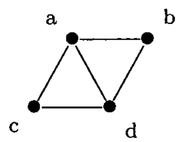
Theorem 9.1. (Saturation Conjecture) *For $(\lambda, \mu, \nu) \in \mathbb{Z}^{3n}$ and $N > 0$, $(\lambda, \mu, \nu) \in T_n$ if and only if $(N\lambda, N\mu, N\nu) \in T_n$.*

That is, T_n is saturated in $\mathbb{Z}^{3n}$. Here, we follow the simplified proof given by Buch [1]. For this, we will need to make a new construction, the hive model, and then show that this has a Littlewood-Richardson rule.

First, we construct a *hive* beginning with a triangular array of vertices, with $n+1$ vertices on each side. This is the *hive triangle*, and it contains n^2 small triangles.



A *rhombus* is the union of two adjacent small triangles. Let H be the set of vertices, and let $\mathbb{R}^H$ be the labellings of H by $\mathbb{R}$. On each rhombus, the *rhombus inequality* on $\mathbb{R}^H$ is the requirement that the sum of the labels of the two shared vertices is greater than or equal to the sum of the labels of the other two vertices. That is, for the rhombus in the following diagram, $a+d \geq b+c$.



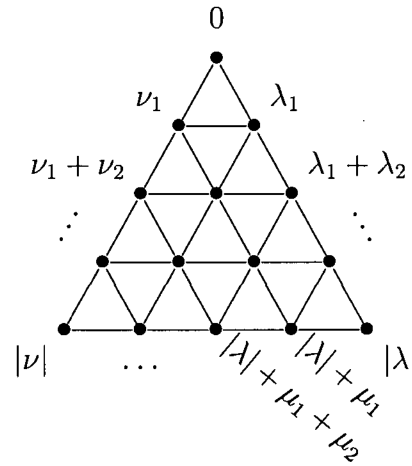
A *hive* is a labelling of a hive triangle that satisfies the rhombus inequalities on every rhombus. A hive is said to be *integral* if it has all integer labels. Let $C \subset \mathbb{R}^H$ be the convex cone of all possible hives on the hive triangle.

Let B be the set of vertices on the border, and $\rho : \mathbb{R}^H \to \mathbb{R}^B$ be the restriction to the border vertices. If $b \in \mathbb{R}^B$ is a border labelling, then $\rho^{-1}(b) \cap C$ consists of all hives on H with the border labelling b. Since C is a convex cone and the border labels give finite restrictions on the possible hive labellings, $\rho^{-1}(b) \cap C$ is a compact polytope. Therefore, we call this the *hive polytope* over b.

With this construction, we have the following theorem, due to Knutson and Tao [8].

Theorem 9.2. *Let λ, μ, and ν be partitions such that $|\nu| = |\lambda|+|\mu|$. Then the number of integral hives with the borders labeled with $\{0, \nu_1, \nu_1+\nu_2, \ldots, |\nu|\}$ on one side,*

$\{0, \lambda_1, \lambda_1 + \lambda_2, \dots, |\lambda|\}$ *on another side, and* $\{|\lambda|, |\lambda| + \mu_1, |\lambda| + \mu_1 + \mu_2, \dots, |\lambda| + |\mu|\}$ *on the third is* $c^{\nu}_{\lambda\mu}$.



Proof. Here, we follow the proof given by Fulton [1]. Let h be an integral hive with sides of length $n+1$. Let a^i_k be the labels on the hive, where k is the number of the row running from north to southwest, and i is the number of the row running from north to southeast, so our hive is indexed as in the following diagram.

$$
\begin{array}{ccccccc}
 & & & a^1_0 & & & \\
 & & a^2_0 & & a^1_1 & & \\
 & a^3_0 & & a^2_1 & & a^1_2 & \\
a^4_0 & & a^3_1 & & a^2_2 & & a^1_3 \\
a^{n+1}_0 & & a^n_1 & \cdots & a^2_{n-1} & & a^1_n
\end{array}
$$

Then, with labels indexed in this way, our partitions λ, μ, and ν that label the borders are given by the following differences.

$$\lambda_k = a^1_k - a^1_{k-1}$$

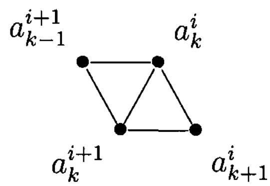
$$\mu_k = a^{k+1}_{n-k} - a^k_{n-k+1}$$

$$\nu_k = a^{k+1}_0 - a^k_0$$

Now define a set of sequences $\lambda^{(i)} = \left(\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_{n+1-i}^{(i)}\right)$ by the differences in the rows running from north to southeast

$$\lambda_k^{(i)} = a_k^i - a_{k-1}^i.$$

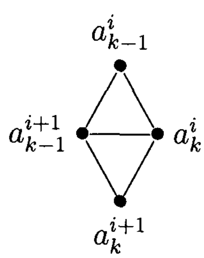
If we consider the rhombus inequalities on the hive, this will give us some relations on the $\lambda_k^{(i)}$. There are three different small rhombi on the hive. For the small rhombus



we have the inequality $a_k^i + a_k^{i+1} \geq a_{k-1}^{i+1} + a_{k+1}^i$, and so

$$\lambda_{k+1}^{(i)} = a_{k+1}^i - a_k^i \leq a_k^{i+1} - a_{k-1}^{i+1} = \lambda_k^{(i+1)}.$$

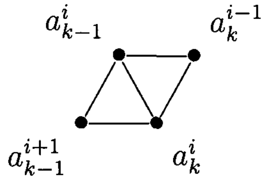
Similarly, on the small rhombus



we have the inequality $a_{k-1}^{i+1} + a_k^i \geq a_k^{i+1} + a_{k-1}^i$, which gives

$$\lambda_k^{(i+1)} = a_k^{i+1} - a_{k-1}^{i+1} \leq a_k^i - a_{k-1}^i = \lambda_k^{(i)}.$$

Finally, the third possible rhombus has the form



which gives us the inequality $a_{k-1}^i + a_k^i \geq a_k^{i-1} + a_{k-1}^{i+1}$. Therefore, $a_{k-1}^{i+1} - a_{k-1}^i \leq a_k^i - a_k^{i-1}$.

The first two inequalities give us that $\lambda_{k+1}^{(i)} \leq \lambda_k^{(i+1)} \leq \lambda_k^{(i)}$, and so the sequences $\lambda^{(i)}$ are each partitions. Further, $\lambda_k^{(i-1)} \leq \lambda_k^{(i)}$ for each k, and so each partition contains the next $\lambda^{(i)} \supset \lambda^{(i+1)}$, so we have the sequence

$$\lambda = \lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)}.$$

We now construct *contratableaux* T on the partition λ. Let λ define a diagram with n rows, and with λ_1 cells in the last row, λ_2 cells in the second to last row and so on, with each row aligned to the right. In other words, this is the Young diagram on shape λ rotated by 180°. Since we have the containment $\lambda^{(i)} \supset \lambda^{(i+1)}$, and each of these are partitions contained in λ, we can define skew shapes $\lambda^{(i)}/\lambda^{(i+1)}$, where again we consider this as the usual skew shape on these partitions rotated by 180°. Now fill the diagram λ by placing the entry i in each cell of $\lambda^{(i)}/\lambda^{(i+1)}$. By the above argument, $\lambda_{k+1}^{(i)} \leq \lambda_k^{(i+1)}$ so $\lambda^{(i)}/\lambda^{(i+1)}$ does not contain two cells in the same column. Therefore, if we fill λ in this way, the columns will be strictly increasing, and the rows are weakly increasing by construction. Therefore, the contratableau T labelled in this way is also a skew tableau.

Now construct a word $\tilde{w}(T)$ on the contratableau in the same manner as for skew tableau. $\tilde{w}(T)$ is given by reading off the entries of T from right to left, starting with the bottom row.

Example 9.3. *Consider the following hive.*

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 12 & & 9 & & & & \\
 & & & 22 & & 20 & & 16 & & & \\
 & & 29 & & 28 & & 26 & & 22 & & \\
 & 35 & & 34 & & 33 & & 31 & & 27 & \\
40 & & 40 & & 39 & & 37 & & 33 & & 28
\end{array}$$

Here, the border labels correspond to the partitions $\lambda = (9, 7, 6, 5, 1)$, $\mu = (5, 4, 2, 1)$, *and* $\nu = (12, 10, 7, 6, 5)$.

From the construction above, we have the partitions

$$\lambda^{(1)} = (9, 7, 6, 5, 1)$$

$$\lambda^{(2)} = (8, 6, 5, 2)$$

$$\lambda^{(3)} = (6, 5, 4)$$

$$\lambda^{(4)} = (5, 5)$$

$$\lambda^{(5)} = (5)$$

$$\lambda^{(6)} = \emptyset$$

Then the contratableau T *constructed from the hive is*

$$T = \begin{array}{ccccccccc} & & & & & & & & \boxed{1} \\ & & & & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{2} \\ & & & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} & \boxed{3} \\ & & \boxed{1} & \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \\ \boxed{1} & \boxed{2} & \boxed{2} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{5} \end{array}$$

The word for T *is*

$$\tilde{w}(T) = 1223555551244444123333111221.$$

Let $U(\mu)$ be the usual tableau on the shape μ, where every cell of the i^{th} row of μ is labelled with i, and let $w(U(\mu))$ be its word.

Now we want to show that $\tilde{w}(T) \cdot w(U(\mu))$ is a reverse lattice word. Note that $U(\mu)$ is already a reverse lattice word, so we just need to show that the portion of $\tilde{w}(T) \cdot w(U(\mu))$ corresponding to the rows of T satisfies the reverse lattice word property.

Consider the k^{th} row from the bottom of T, and consider the point in this row where a cell is labelled i, and all cells to the left are labelled strictly less than i. Then this cell is on the outer edge of the skew diagram $\lambda^{(i)}/\lambda^{(i+1)}$. In particular, the number of times i appears in $\tilde{w}(T) \cdot w(U(\mu))$ to the right of this point is the number of times i appears in the last k rows of T, so

$$\begin{aligned}
&\left(\lambda_k^{(i)} - \lambda_k^{(i+1)}\right) + \left(\lambda_{k+1}^{(i)} - \lambda_{k+1}^{(i+1)}\right) + \cdots + \left(\lambda_{n+1-i}^{(i)} - 0\right) + \mu_i \\
&= \left(\lambda_k^{(i)} + \lambda_{k+1}^{(i)} + \cdots + \lambda_{n+1-i}^{(i)}\right) - \left(\lambda_k^{(i+1)} + \lambda_{k+1}^{(i+1)} + \cdots + \lambda_{n-i}^{(i+1)} + 0\right) + \mu_i \\
&= \left(a_k^i - a_{k-1}^i + a_{k+1}^i - a_k^i + \cdots + a_{n+1-i}^i - a_{n-i}^i\right) \\
&\qquad - \left(a_k^{i+1} - a_{k-1}^{i+1} + a_{k+1}^{i+1} - a_k^{i+1} + \cdots + a_{n-i}^{i+1} - a_{n-i-1}^{i+1}\right) + \left(a_{n-i}^{i+1} - a_{n+1-i}^i\right) \\
&= -a_{k-1}^i + a_{n+1-i}^i + a_{k-1}^{i+1} - a_{n-i}^{i+1} + a_{n-i}^{i+1} - a_{n+1-i}^i \\
&= a_{k-1}^{i+1} - a_{k-1}^i
\end{aligned}$$

Similarly, the number of times that $i-1$ occurs to the right of this point is

$$\begin{aligned}
&\left(\lambda_{k+1}^{(i-1)} - \lambda_{k+1}^{(i)}\right) + \left(\lambda_{k+2}^{(i-1)} - \lambda_{k+2}^{(i)}\right) + \cdots + \left(\lambda_{n+2-i}^{(i-1)} - 0\right) + \mu_{i-1} \\
&= \left(-a_k^{i-1} + a_{n+2-i}^{i-1}\right) - \left(-a_k^i + a_{n+1-i}^i\right) + \left(a_{n-i+1}^i - a_{n-i+2}^{i-1}\right) \\
&= a_k^i - a_k^{i-1}
\end{aligned}$$

Therefore, by the third rhombus inequality, we have $a_{k-1}^{i+1} - a_{k-1}^i \leq a_k^i - a_k^{i-1}$. Since this holds for any consecutive pair $i-1$ and i and any row of T, this shows that $\tilde{w}(T) \cdot w(U(\mu))$ is a reverse lattice word.

Conversely, if T is a contratableau on the shape λ such that $\tilde{w}(T) \cdot w(U(\mu))$ is a reverse lattice word, then we can derive the partitions $\lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)}$ by looking at the cells containing each entry i. Then we can construct the hive starting with the top label (which must be a 0), and then filling in the north to southeast rows using the paritions $\lambda^{(i)}$. The arguments above are reversible, so the hive constructed in this way must satisfy the rhombus inequalities.

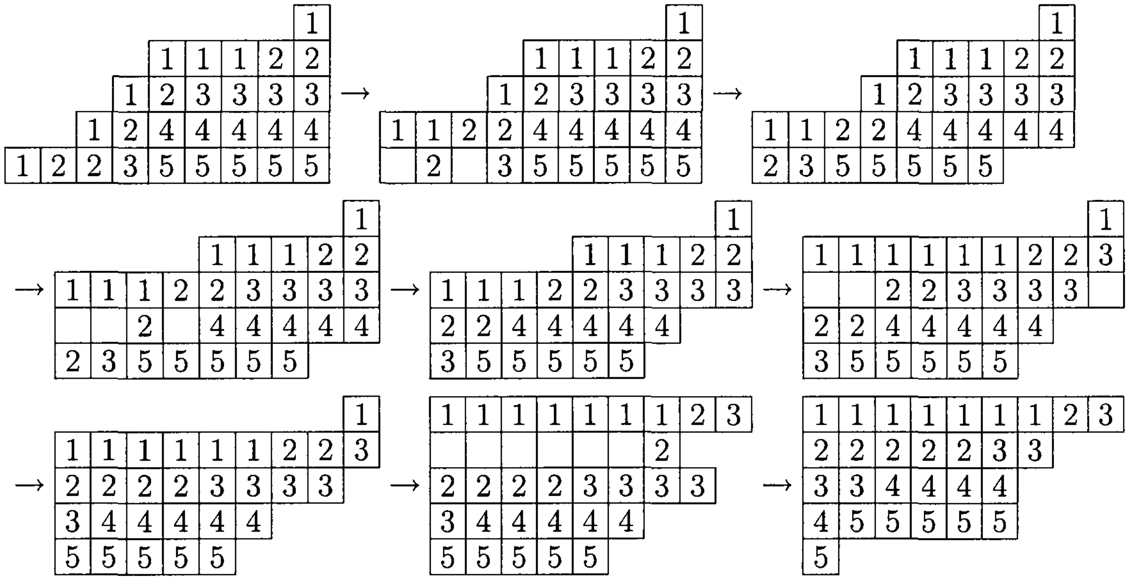
Note that the content of the word $\tilde{w}(T) \cdot w(U(\mu))$ is $\lambda + \mu$, and the content of T is $\nu - \mu$. Then by Theorem 2.24 and Proposition 4.10, $\tilde{w}(T) \cdot w(U(\mu)) \equiv w(U(\nu))$ if and only if $\tilde{w}(T) \cdot w(U(\mu))$ is a reverse lattice word.

Let $R = \mathrm{Rect}(T)$. Since T is a skew tableau, no element appears twice in the same column. Also, T is right-justified in each row. Therefore, we can rectify T in a specific way. First, rectify the last 2 rows of T. Since the rows are right-justified, this means that we perform the sliding procedure on the inside corners in the second to last row. Since every time we slide a cell to the left, another cell must replace it, the result is that the second to last row will have λ_1 cells, and so the last row now has λ_2 cells. Then the last row is rectified by sliding all cells to the left. Now rectify the last 3 rows. Perform the sliding procedure on the inside corners in the third to last row. Again, the result is that this row will have λ_1 cells. Then finish rectifying the last two rows, so now the last three rows have λ_1, λ_2, and λ_3 cells and are left-justified. If we continue this process, we can see that at each step of the sliding procedure, the lengths of each row is always going to be one of the parts of the partition λ, and so the result is a tableau on the shape λ. Further, each step respects the sliding procedure, so the result will be $R = \mathrm{Rect}(T)$. Therefore, R has shape λ.

Example 9.4. *We illustrate the rectification procedure described here with an example. Consider the contratableau from Example 9.3*

$$T = \begin{array}{ccccccccc} & & & & & & & & 1 \\ & & & & 1 & 1 & 1 & 2 & 2 \\ & & & 1 & 2 & 3 & 3 & 3 & 3 \\ & & 1 & 2 & 4 & 4 & 4 & 4 & 4 \\ 1 & 2 & 2 & 3 & 5 & 5 & 5 & 5 & 5 \end{array}$$

This gives us the following sequence of slides.



Then $\mathrm{Rect}(T)$ *has shape* λ.

Since $R = \mathrm{Rect}(T)$, we have $w(R) \equiv \tilde{w}(T)$ by Proposition 2.23. Therefore,

$$w(R) \cdot w(U(\mu)) \equiv w(U(\nu)).$$

By Theorem 2.25, this is equivalent to the requirement that $R \cdot U(\mu) = U(\nu)$.

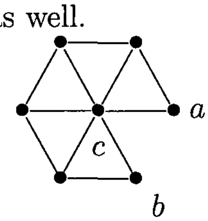
Now suppose that $R \cdot S = U(\nu)$ for some tableau S. The fact that $w(R) \cdot w(U(\mu)) \equiv w(U(\nu))$ implies that $S = U(\nu)$, by Theorem 2.25. Then we have a one-to-one correspondence between tableaux R and pairs (R, S) with shapes λ and μ such that $R \cdot S = U(\nu)$. By definition, this is the set $\mathcal{T}(\lambda, \mu, U(\nu))$, so by Corollary 4.4, the number of such tableaux is $c^{\nu}_{\lambda\mu}$.

□

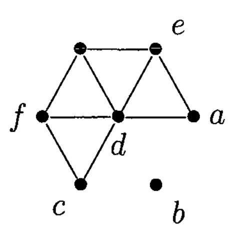
We now consider a hive as a graph on the hive triangle, where we let the label on a vertex be its height. Then let the height at a point on any edge or face of the graph be determined by a linear combination of the heights of adjacent vertices. The rhombus inequality then translates to the requirement that a rhombus cannot bend up across the edge between the shared vertices. If the entire graph satisfies the rhombus inequalities, then this describes a convex polygonal surface. We say a rhombus is *flat* if we have equality on the rhombus inequality.

Now define a *flatspace* to be a maximal connected union of triangles such that any rhombus contained in the union is flat. If a small triangle is not contained in any flat rhombus, then it is its own maximal flatspace. With this construction, the flatspaces of a graph must be disjoint, and their union is the entire hive.

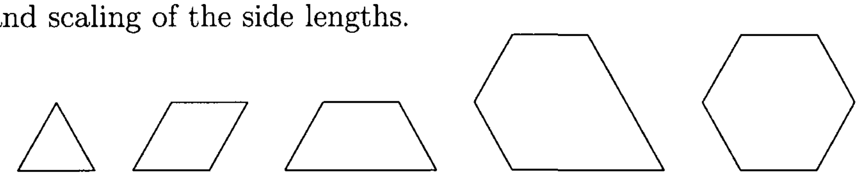
Note that a flatspace is a union of regular triangles, so the angle at each vertex of the flatspace must be a multiple of 60°. We consider the possible shapes of such a flatspace. Since a flatspace is flat, without loss of generality, we may assume that every vertex in the flatspace is at the same height (since this just involves reorienting the hive so the height vector is perpendicular to the flatspace). Consider the possible angles at a vertex of the flatspace. If this angle is 300°, then at this vertex, we have 5 triangles in the flatspace. Then, as in the following diagram, the vertices a and b must have the same height in order for these triangles to be in the flatspace, so the triangle abc is in the flatspace as well.



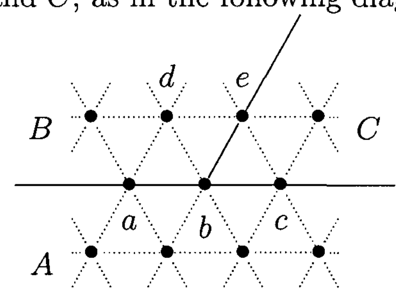
Therefore, a flatspace cannot have any 300° angles. Similarly, if a flatspace has a 240° angle, then there are 4 triangles in the flatspace at this vertex, as in the following diagram.



The vertices a, c, and d must be at the same height. Then in order for the triangles abd and bcd to not be in the flatspace, b must be at a different height. the rhombus inequality on $abcd$ implies that b has height greater than d. however, the rhombus inequalities on $abde$ and $bcfd$ both imply that b has height less than d. Thus, since each vertex is on the hive, b must be at the same height as the other vertices, and so the two missing triangles are also in the flatspace. Therefore, a flatspace cannot have a 240° outer angle either. Then the only possible outer angles on a flatspace are 60°, 120°, and 180°, and so there are only five possible shapes for a flatspace, up to rotation and scaling of the side lengths.

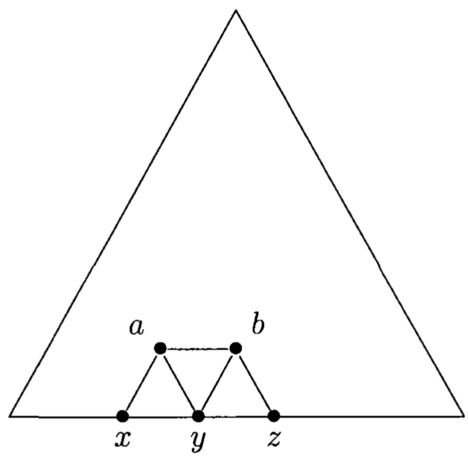


Now consider the case where we have a flatspace A sharing one of its edges with two distinct flatspaces B and C, as in the following diagram.



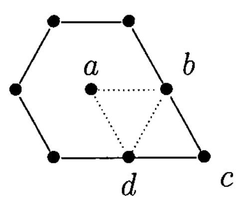
In this case, A has a straight edge bordering both B and C, so B and C must share an edge as shown. If we orient the hive such that the height vector is perpendicular to the flatspace C, the heights on each vertex of C are equal. Then the rhombus inequality on $bced$ implies that $d < b = c = d$. The rhombus inequality on $abed$ then implies that $a \leq d$. However, in order for a, b, and c to be on the edge of flatspace A with $b = c$, a must be at the same height. This implies that d has the same height as a, b, c, and e, and so the triangle bed is in the flatspace C. This violates our assumption that B and C are distinct flatspaces, so it follows that the edge of a flatspace is either on the border of the hive triangle, or is shared completely with another flatspace.

Now for adjacent vertices $x, y, z \in b$ the border of the hive, the rhombus inequalities specify that, as in the diagram, $a + y \geq x + b$ and $b + y \geq a + z$. Therefore, $y - x \geq b - a \geq z - y$.



A border b is said to be *regular* if whenever we have adjacent vertices $x, y, z \in b$, we have $y - x > z - y$. With the borders given by partitions as in Proposition 9.2, this is exactly the requirement that each partition is strictly decreasing. Notice that if the border of a hive is regular, then there are no flatspaces that share a a side of length greater than 1 with the border.

A non-empty set $S \subseteq H \setminus B$ is *increasable* if we can add a small amount ϵ to the labels of the hive vertices such that H is still a hive. Then it follows that the interior vertices of a hexagon flatspace form an increasable subset. However, if a flatspace has a $60°$ angle on its boundary, then this is not the case, since raising each interior vertex will violate the rhombus inequality on the rhombus at this vertex, as illustrated by the following diagram.

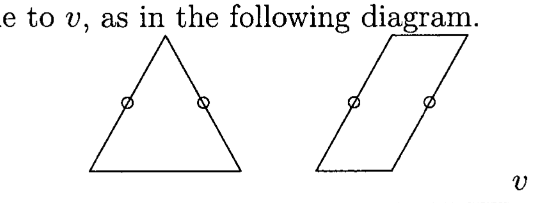


Here, the points a, b, c, and d are vertices on a flatspace, so they are all at the same height, under some orientation. Then raising the height of the interior point a will give the inequality $a + c > b + d$, which violates the rhombus inequality. Since all of the possible flatspace shapes except the hexagon have a $60°$ angle, the hexagon is the only flatspace with this property.

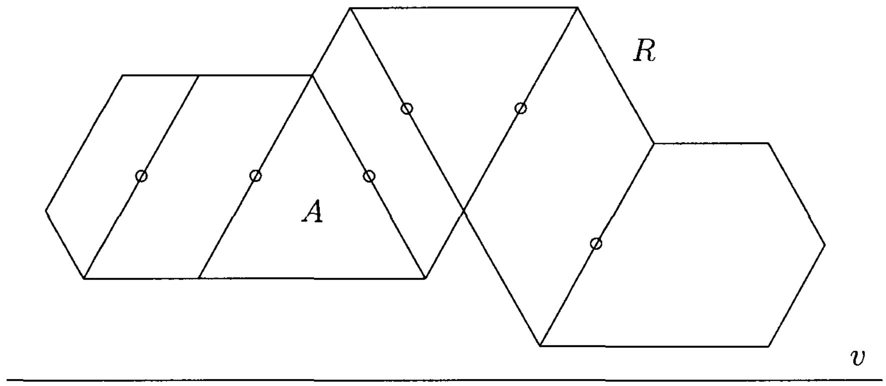
Proposition 9.5. *Let h be a hive with regular borders and no increasable subsets. Then any flatspace of h is either a small triangle or a small rhombus.*

Proof. Let h be a hive with regular borders, and suppose that h has a flatspace that is not either a small triangle or a small rhombus. If there is a hexagonal flatspace, then we have an increasable subset, by the remark above. Since these are the only possible flatspace shapes with edges all of length 1, h must have a flatspace with a side of length at least 2. Let m be the maximal length of a flatspace side among all the flatspaces in h, and let A be a flatspace with a side of length m. We will use A to construct a region R in the hive whose interior vertices form an increasable subset.

Mark a side of A that has length m, and choose a line v that runs at a $60°$ angle to this marked side. If A is a triangle or a parallelogram, we also mark the other side that forms a $60°$ angle to v, as in the following diagram.



We now construct the region R by adding the flatspaces that share a marked side with A. By the remark above, the side of a flatspace must be either shared with another flatspace or lie on the boundary of the hive. Since h has regular boundary and the marked side has length at least 2, the marked side cannot be on the boundary of h. Therefore, there is another flatspace that shares this side. Add this flatspace to the region, and if it is a triangle or a parallelogram, mark the side that is opposite the marked side in the direction of v. Continue adding new regions in this way until there are no more marked sides on the boundary of R.



Since we are only adding flatspaces in the direction of v, this process must end. Also, since a marked side must have length $m \geq 2$, no marked edge is on the border of the hive, since h has regular border.

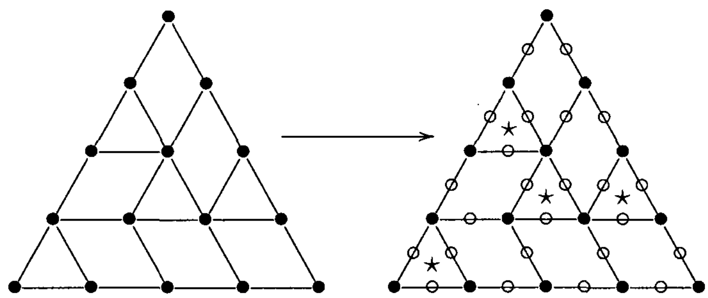
Now note that, by construction, each flatspace we add to R has a marked side adjacent to every vertex with a 60° angle on the boundary. Since every marked side borders another flatspace contained in R, the region R will not have any 60° angles on its boundary. Therefore, we will not have the difficulty described in the remark above.

Now we need to check that the interior points of R form an increasable subset. In particular, we need to check that increasing the heights of the interior points will not violate any rhombus inequalities for rhombi on the boundary of R. First, note that if a rhombus is shared between two flatspaces (that is, there is one triangle in each flatspace), then the rhombus inequality is strict. Therefore, we can add some nonzero amount to any of the vertices without violating it. Thus, we only need to consider rhombi that are entirely contained in a flatspace.

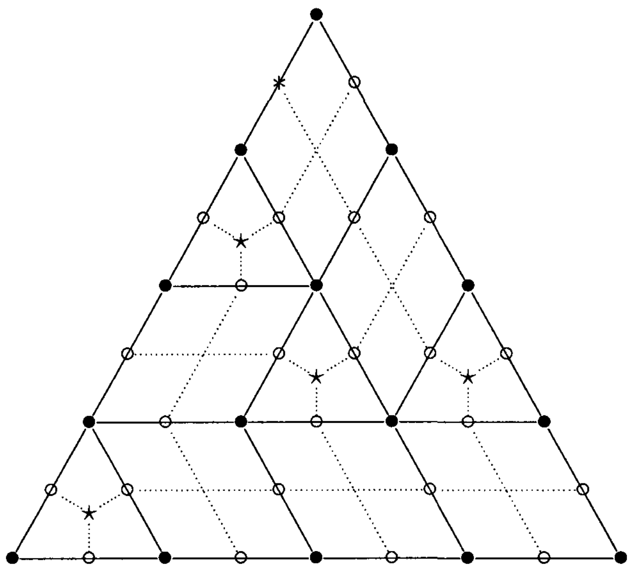
Now, the only problem with such a rhombus is if it has more of its obtuse vertices on the boundary of R than acute vertices. If both obtuse vertices are on the boundary of R, then by convexity of the flatspace, both lie on marked sides, and so at least one must be in the interior of R. If the rhombus has one obtuse vertex and no acute vertices on the border of R, by convexity of the flatspace, there must be an acute vertex on the border of the flatspace. Since this is not on the border of R, it must be on a marked side. Since the obtuse vertex is on the border of R, it must be at the end of this marked side. Since the other acute vertex is not on the border of R, it must lie on another marked side. But then there are two marked sides on this rhombus intersecting at a 120° angle. No such angles occur in our construction, so this is impossible. Thus, the interior vertices of R form an increasable subset on h.

□

Now let h be a hive that has only small triangular and small rhombus flatspaces. We construct a graph G from h in the following way. Divide h into its flatspaces. Put a star in the middle of each small triangle flatspace, and place a circle on every flatspace side.



Connect each star to the three vertices on each side of its triangle, and for each flat rhombus, connect the circles that are on opposing sides.



The result of this procedure is a graph on the hive. The following propositions will be used to prove the saturation conjecture.

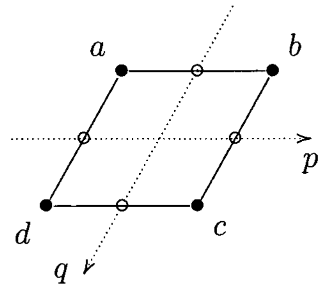
Proposition 9.6. *If h is a corner of the hive polytope $\rho^{-1}(\rho(h)) \cap C$, then G is an acyclic graph.*

Proof. We will show that if G has a cycle, then h is not a corner of its hive polytope. Let hive h have graph G, and suppose that G has a nontrivial cycle. Choose an orientation on this cycle and define $w(v)$ on each vertex $v \in H$ to be the number of times the cycle wraps around v in the counter-clockwise direction (that is, the winding number from the given loop). Since h has a cycle, there must be some nonzero $w(v)$ on the hive.

For $r \in \mathbb{R}$, define a new labelling $h_r \in \mathbb{R}^H$ by $h_r(v) = v + rw(v)$ (that is, add r times the winding number to each vertex).

Now let $\epsilon > 0$ such that on each rhombus where the rhombus inequality is strict on h, the rhombus inequality is also satisfied on h_r for any $|r| < \epsilon$. This is possible since there are only a finite number of such rhombi, so there must be a minimal nonzero ϵ with this property.

Since there are only small triangle and small rhombus flatspaces in our hive, we only have strict inequality in the rhombus inequality when the rhombus contains a small triangle flatspace. Therefore, in order to show that h_r is a hive for any $|r| < \epsilon$, we only need to show that it satisfies the rhombus inequality on the small rhombi in h where we have equality. Consider such a rhombus, where we have $a + c = b + d$.



Suppose our chosen loop crosses through this rhombus, and goes through the horizontal sides q times and through the vertical sides p times, as shown. Then, if we suppose that at vertex x we have $w(d) = t$ on the hive h, then by construction, the other vertices have the following winding numbers.

$$w(a) = t + p, \qquad w(b) = t + p + q, \qquad w(c) = t + q$$

Therefore, on the labelling h_r, we will have the vertex labels

$$h_r(a) = a + r(t + p), \qquad h_r(b) = b + r(t + p + q),$$

$$h_r(c) = c + r(t + q), \qquad h_r(d) = d + rt.$$

Then the left and right hand sides of the rhombus inequality will be

$$h_r(a) + h_r(c) = (a + rt + rp) + (c + rt + rq) = a + c + 2rt + rp + rq,$$

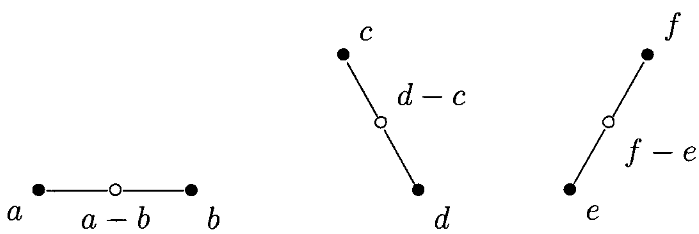
$$h_r(b) + h_r(d) = (b + rt + rp + rq) + (d + rt) = b + d + 2rt + rp + rq.$$

These are equal, so this is still a flat rhombus in h_r. This shows that h_r is a hive for any $r \in (-\epsilon, \epsilon)$, and so h is not a corner of the hive polytope $\rho^{-1}(\rho(h)) \cap C$.

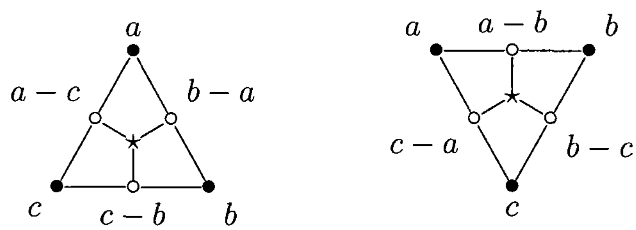
□

Proposition 9.7. *Let h be a corner of its hive polytope $\rho^{-1}(\rho(h)) \cap C$, and let h have flatspaces consisting of only small triangles and small rhombi. Then the labels H are determined by an integer linear combination of border labels.*

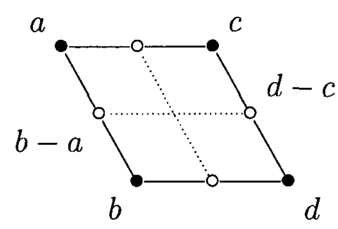
Proof. Let G be the graph of h. By Proposition 9.6, G is acyclic. Define a labelling on the circle vertices according to the adjacent vertices and the angle of its edge. For each edge containing a circle vertex, we label the circle vertex by the difference of the hive vertices on its edge, as in the following diagram.



With this labelling, the labels of the circle vertices around a small triangle flatspace sum to zero.



Also, if two circle vertices are connected by an edge, this means that they are on the edge of a small rhombus flatspace. In the following diagram, we therefore have $a + d = c + b$, and so $d - c = b - a$. Therefore, any two circle vertices connected by an edge have the same label.



Now we want to show that all the circle vertices labelled in this way are $\mathbb{Z}$-linear combinations of the border labels. Suppose this is not true, and let S be the set of circle vertices on G whose labels are not $\mathbb{Z}$-linear combinations of border labels and the star vertices that are connected to them.

Since G is acyclic, there must be a vertex $u \in S$ that is connected to at most one other vertex in S. If u is a circle vertex, then it is not on the border of the hive, or else it would be the difference of two border labels. Since G has endpoints only on the border of h, u must not be an endpoint of the graph. Therefore, it is connected to some other vertex $v \in G \setminus S$. Since S contains all the star vertices connected to u, v must be a circle vertex. Then u and v are circle vertices connected by an edge, so by the argument above, they must have the same label. This is a contradiction, since v has label given by an integer linear combination of border labels.

Now suppose that u is a star vertex. By assumption, it is connected to at most one other vertex of $v \in S$, so this must be a circle vertex. Therefore, u is connected to v and two other circle vertices x and y not contained in S. By the argument above, the sum of the labels on these circle vertices is zero, so $v = -x - y$. But then the label of v is an integer linear combination of border labels, contradicting our assumption that $v \in S$. Therefore, all circle vertices have labels given by $\mathbb{Z}$-linear combinations of border labels. Since the labels on the circle vertices are the differences of the labels on the adjacent hive vertices, this implies that the hive vertices also have labels given by integer linear combinations of border labels.

□

Now let ω be a linear functional on $\mathbb{R}^{H\setminus B}$ such that ω has a unique maximum on the set $\rho^{-1}(b) \cap C$ for any $b \in \rho(C)$. In this case, ω is said to be *generic.* This definition is motivated by a related construction in linear programming, which also gives us the following lemma.

Lemma 9.8. *The set of generic functionals are dense in* $\left(\mathbb{R}^{H\setminus B}\right)^*$.

Proof. Let $A = H \setminus B$ be the set of non-border vertices of the hive triangle, and let $\omega : A \to \mathbb{R}$ be some labelling of these vertices. Since the hive triangle is a regular triangular lattice, the rhombus inequality implies that the labels on the hive vertices form a concave function; that is, that for any $x, y \in Q = \mathrm{Conv}(A)$, we have

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y), \qquad 0 \leq t \leq 1.$$

Therefore, if the border labelling b and the labelling ω defines a hive, then ω is a concave function on $H \setminus B$. This means that ω makes the hive triangle a *coherent triangulation* of $Q = \mathrm{Conv}(A)$ (up to some small perturbation of ω).

In general, for any triangulation T of (Q, A) and function $\psi : A \to \mathbb{R}$, there is a unique map $g_{\psi,T} : Q \to \mathbb{R}$ given by $g_{\psi,T}(v) = \psi(v)$ for any vertex v of T and extended linearly across each simplex to non-vertex points. We then define the following object.

Definition 9.9. *For a triangulation T of (Q, A), we define $C(T)$ to be the cone in $\mathbb{R}^A$ of functions $\psi : A \to \mathbb{R}$ such that:*

(i) $g_{\psi,T} : Q \to \mathbb{R}$ *is concave.*

(ii) *For $v \in A$ that is not a vertex of T, we have $g_{\psi,T}(v) \geq \psi(v)$.*

The set of $C(T)$ for all coherent triangulations T of (Q, A) is a set of cones intersecting along their faces, forming a polygonal fan. Since the set of these $C(T)$ along with each intersecting face covers all of $\mathbb{R}^A$, this is a complete fan, known as the *secondary fan* of A. In the present case, the secondary fan of A contains all labellings of the interior vertices A.

Now consider $A \subset \mathbb{R}^{k-1}$ and let Vol be a fixed volume form on $\mathbb{R}^{k-1}$. Then for a triangulation T on (Q, A), the *characteristic function* $\phi_T : A \to \mathbb{R}$ is defined to be

$$\phi_T(v) = \sum_{\sigma : v \in \mathrm{Vert}(\sigma)} \mathrm{Vol}(\sigma).$$

The *secondary polytope* $\Sigma(A)$ is defined to be the convex hull in $\mathbb{R}^A$ of the characteristic functions ϕ_T over all triangulations T of (Q, A). The *normal cone* $N_{\phi_T}\Sigma(A)$ is the cone consisting of all linear forms $\psi : \mathbb{R}^A \to \mathbb{R}$ such that

$$\psi(\phi_T) = \max_{\phi \in \Sigma(A)} \psi(\phi),$$

that is, all linear forms ψ on $\Sigma(A)$ that achieve their maximum at ϕ_T. By a theorem of Gelfand, Kapranov, and Zelevinsky [5, §7.1], for any triangulation T, we have $C(T) = N_{\phi_T}\Sigma(A)$. This means that the normal fan of the secondary polytope $\Sigma(A)$ is precisely the secondary fan of A.

Since the secondary polytope $\Sigma(A)$ is a compact polytope, the set of linear functionals that do not take a unique maximum in $\Sigma(A)$ correspond to vectors orthogonal to the faces of $\Sigma(A)$ (where we consider the action of a linear functional to be orthogonal projection by a vector). Therefore, the set of such functionals has measure zero in $\left(\mathbb{R}^A\right)^*$, and so the set of linear functionals with a unique maximum on $\Sigma(A)$ is dense in $\left(\mathbb{R}^A\right)^*$.

□

We may now give a proof of the saturation conjecture. One direction is simple: if $(\lambda, \mu, \nu) \in T_n$, then by Theorem 9.2, there is an integral hive with border labelling given by (λ, μ, ν). We can obtain an integral hive with border labelling $(N\lambda, N\mu, N\nu)$

by multiplying all the labels of this hive by N. Therefore, $c_{N\lambda,N\mu}^{N\nu} \neq 0$, and so $(N\lambda, N\mu, N\nu) \in T_n$.

Now suppose $(N\lambda, N\mu, N\nu) \in T_n$. Then by Theorem 9.2, there is an integral hive with border labelling $(N\lambda, N\mu, N\nu)$. We can then obtain a hive with border labelling (λ, μ, ν) simply by dividing the the entries of this hive by N, though this will not necessarily be integral. Therefore, for b given by the partitions (λ, μ, ν), the hive polytope $\rho^{-1}(b) \cap C \neq \emptyset$. It remains to be shown that there is an integral hive in this hive polytope.

Let ω be a generic linear functional on $\mathbb{R}^{H \setminus B}$ that is given by a positive integer linear combination of the labels at non-border vertices. By Lemma 9.8, there must be such a functional. Since ω is generic, for each $b \in \rho(C)$, ω takes its maximum at a unique hive in $\rho^{-1}(b) \cap C$. Therefore, we can define a function $l : \rho(C) \to C$ such that $l(b)$ is the unique hive in $\rho^{-1}(b) \cap C$ where ω is maximal.

Let $b_1, b_2 \in \rho(C)$ such that $l(b_1)$ and $l(b_2)$ are contained in the same subcone of the secondary fan of A. Then $l(b_1 + b_2)$ is the maximal hive with respect to ω with border labelling $b_1 + b_2$. Let $h = l(b_1) + l(b_2)$, so we have $l(b_2) = h - l(b_1)$ a hive with border labelling b_2. Since ω is a linear functional, we have

$$\omega(l(b_2)) = \omega(h - l(b_1)) = \omega(h) - \omega(l(b_1)).$$

If h is not the maximal hive with border $b_1 + b_2$, then there is another hive k with the same border, and such that $\omega(k) > \omega(h)$. But then $k - l(b_1)$ is a hive with border b_2 and with

$$\omega(k - l(b_1)) = \omega(k) - \omega(l(b_1)) > \omega(h) - \omega(l(b_1)) = \omega(l(b_2)).$$

This contradicts our assumption that $l(b_2)$ is the maximal such hive, so we must have

$$h = l(b_1) + l(b_2) = l(b_1 + b_2).$$

Therefore, l is piecewise linear.

Since ω is given by a linear combination of the non-border vertex labels with positive integer coefficients, $l(b)$ cannot have any increasable subsets.

Now we want to show that the labels of $l(b)$ are $\mathbb{Z}$-linear combinations of the border labels of b. In particular, $l(b)$ is an integral hive if and only if b contains only integral labels. Let $b \in \rho(C)$ be a regular border. By Proposition 9.5, $l(b)$ has only small triangular and small rhombus flatspaces. Since l maximizes a generic linear functional on $\rho^{-1}(b) \cap C$, $l(b)$ must be a corner of its hive polytope $\rho^{-1}(b) \cap C$. Then by Proposition 9.7, all labels of $l(b)$ are integer linear combinations of the border labels b.

By the argument above, if $b \in \rho(C)$ is a non-regular border, then the labels on the border are not strictly increasing. However, we can perturb the border labels slightly to make it regular. Therefore, the regular borders are dense in $\rho(C)$. Since l is linear and continuous on the set of regular borders within each subcone of $\rho(C)$, it follows that l extends linearly to non-regular borders as well. Thus, l is defined by integer linear combinations of the border labels. Since we have shown that $\rho^{-1}(b) \cap C \neq \emptyset$, the hive polytope must contain an integral hive. By Theorem 9.2, we have $c^{\nu}_{\lambda\mu} \neq 0$, and so $(\lambda, \mu, \nu) \in T_n$.

□

Now consider the following problem. Let A, B, and C be $n \times n$ Hermitian matrices, and let $\alpha_1 \geq \cdots \geq \alpha_n$ be the eigenvalues of A, $\beta_1 \geq \cdots \geq \beta_n$ be the eigenvalues of B, and $\gamma_1 \geq \cdots \geq \gamma_n$ be the eigenvalues of C. Then, for which α, β, and γ can we find Hermitian matrices A, B, and C such that $A + B = C$? The answer is, surprisingly, related to the Littlewood-Richardson numbers.

Theorem 9.10. *If $(\alpha, \beta, \gamma) \in T_n$, then there is a triple of Hermitian matrices (A, B, C) with eigenvalues (α, β, γ) such that $A + B = C$.*

Proof. Here, we follow the proof given by Quieró, Santana, and de Sá [9]. For a partition α with n parts, we have V_α the representation of $GL(E)$ with highest weight α. As shown above, V_α is an irreducible polynomial representation, and the set of all such V_α contains all irreducible polynomial representations of $GL(E)$. From the Littlewood-Richardson rule, we have the product formula

$$V_\alpha \otimes V_\beta \cong \bigotimes_\nu c^\gamma_{\alpha\beta} V_\gamma.$$

Now consider the subgroup $U_n \leq GL(E)$ of $n \times n$ unitary matrices. Let $V_\alpha = V_\alpha|_{U_n}$ be the restriction of the representation V_α to U_n. This is also a representation of U_n, and by the Weyl unitary trick, these V_α are also nonisomorphic irreducible representations, so we have the same decomposition formula above.

Now consider $\mathfrak{u}_n$, the Lie algebra of U_n. U_n acts on $\mathfrak{u}_n$ by the adjoint representation

$$\mathrm{Ad}(\mathfrak{u}(X)) = uXu^{-1} \quad \text{for } u \in U_n, \quad X \in \mathfrak{u}_n.$$

We can also identify its dual, the coadjoint representation $\mathfrak{u}_n^*$, with $\mathfrak{u}_n$ via a fixed U_n-invariant inner product. Therefore, we can equivalently consider the orbits of U_n via the coadjoint representation as the orbits of Hermitian matrices via conjugation. Then we can parametrize these orbits via the spectra α of each conjugation orbit $\mathcal{O}_\alpha$.

Consider $A, B \in \mathfrak{u}_n$ with spectra α and β, and consider the set

$$\mathcal{O}_\alpha + \mathcal{O}_\beta = \left\{ uAu^{-1} + vBv^{-1} : u, v \in U_n \right\}.$$

Now identify $\mathfrak{u}_n$ with $\mathfrak{u}_n \oplus \mathfrak{u}_n$ via the diagonal embedding. Since $\mathfrak{u}_n^* \oplus \mathfrak{u}_n^* \cong (\mathfrak{u}_n \oplus \mathfrak{u}_n)^*$, this identification induces a map

$$q : \mathfrak{u}_n^* \oplus \mathfrak{u}_n^* \to \mathfrak{u}_n^*.$$

Let $\phi, \psi \in \mathfrak{u}_n^*$ corresponding to A and B. For any $C \in \mathfrak{u}_n$, we have

$$\langle q(A \oplus B), C\rangle = \langle A \oplus B, C \oplus C\rangle = \langle A, C\rangle + \langle B, C\rangle = \langle A + B, C\rangle$$

so it follows that $q(\phi, \psi) = A + B$. Then for a spectrum γ, we have $\mathcal{O}_\gamma \in \mathcal{O}_\alpha + \mathcal{O}_\beta$ if and only if

$$\mathcal{O}_\gamma \in q\left(\mathcal{O}_\alpha \times \mathcal{O}_\beta\right) = q\left(\mathcal{O}_{(\alpha,\beta)}\right)$$

where $\mathcal{O}_{(\alpha,\beta)}$ is the orbit in $\mathfrak{u}_n^* \oplus \mathfrak{u}_n^*$ with spectrum (α, β).

Now let $V_{\alpha\beta} = V_\alpha \boxtimes V_\beta$ be the exterior tensor product. This is the unique irreducible representation of $U_n \times U_n$ with highest weight (α, β). Restricting $V_{\alpha\beta}$ to U_n by the diagonal map gives the usual tensor product $V_\alpha \otimes V_\beta$.

By a theorem of Heckman [6, §7], if α and β are strictly decreasing partitions with n parts and V_γ occurs in $V_{\alpha\beta}|_{U_n}$, then $\mathcal{O}_\gamma$ occurs in $q\left(\mathcal{O}_{(\alpha,\beta)}\right)$. Therefore, if $c_{\alpha\beta}^\gamma \neq 0$, then $\mathcal{O}_\gamma$ occurs in $\mathcal{O}_\alpha + \mathcal{O}_\beta$.

Let $\mathrm{Sp}(\alpha, \beta)$ denote the sequences γ that occur as the spectra of matrices $C = A+B$, where A and B are $n \times n$ Hermitian matrices with spectra α and β. Then it is enough to show that $c_{\alpha\beta}^\gamma \neq 0$ implies that $\gamma \in \mathrm{Sp}(\alpha, \beta)$. The argument above and Heckman's theorem prove this for the case where α and β are strictly increasing partitions with n parts, so we only need to show that it extends to any such pairs of partitions.

For partition γ with n parts, define a function J such that

$$J\gamma = \gamma + (n-1, n-2, \ldots, 1, 0).$$

Now consider a hive triangle with border labels given by

$$((n-1, n-2, \ldots, 1, 0), (n-1, n-2, \ldots, 1, 0), (2n-2, 2n-4, \ldots, 2, 0)).$$

Index the vertices of the hive triangle with (k, j), where the $(k, j)^{\text{th}}$ vertex is on the k^{th} row from the top and is the j^{th} vertex from the right on this row. Then we can obtain a hive with the given border by labelling the $(k, j)^{\text{th}}$ vertex with

$$n(k+j) - \sum_{i=1}^{k} i - \sum_{i=1}^{j} i.$$

In particular, there is a hive with the given border.

Now suppose that $c^{\gamma}_{\alpha\beta} \neq 0$. By Theorem 9.2, there is a hive with border labelling (α, β, γ). Then we can obtain a hive with border labelling $(J\alpha, J\beta, JJ\gamma)$ by adding this hive to the one above. Since such a hive exists, $c^{J\gamma}_{J\alpha,J\beta} \neq 0$, by Theorem 9.2.

Let $\|\cdot\|$ denote the infinity norm on $\mathbb{R}^n$, so we have

$$\|J\alpha - \alpha\| = \|(n-1, n-2, \ldots, 1, 0)\| = n - 1$$

and

$$\|JJ\gamma - \gamma\| = \|(2n-2, 2n-4, \ldots, 2, 0)\| = 2n - 2.$$

For positive integer N, we define the following sequences of partitions

$$\alpha_N = \frac{1}{N} JN\alpha, \qquad \beta_N = \frac{1}{N} JN\beta, \qquad \gamma_N = \frac{1}{N} JJN\gamma.$$

By the same argument as above, $c^{JN\gamma}_{JN\alpha,JN\beta} \neq 0$. Since the sequences $JN\alpha$ and $JN\beta$ are strictly decreasing, we have $JJN\gamma \in \mathrm{Sp}(JN\alpha, JN\beta)$. Therefore, $\gamma_N \in \mathrm{Sp}(\alpha_N, \beta_N)$. Now note that

$$\|\gamma_N - \gamma\| = \frac{\|JJN\gamma - N\gamma\|}{N} = \frac{2(n-1)}{N} \longrightarrow 0$$
$$\|\alpha_N - \alpha\| = \|\beta_N - \beta\| = \frac{n-1}{N} \longrightarrow 0$$

as $N \to 0$. Therefore the sequences α_N, β_N, and γ_N converge to α, β, and γ. Since these sequences correspond to matrices in U_n, the fact that U_n is compact implies that the matrices corresponding to α, β, and γ are also contained in U_n. Therefore, $\gamma \in \mathrm{Sp}(\alpha, \beta)$.

$\square$

REFERENCES

[1] A.S. Buch and W. Fulton, The saturation conjecture (after A. Knutson and T. Tao), *L'Enseignement Mathématique* 46 (2000), 43–60.

[2] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1999.

[3] W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, *Bull. of the American Math. Soc.* 37:3 (2000), 209–249.

[4] W. Fulton and J. Harris, *Representation theory: A first course*, Graduate Texts in Mathematics, 129. Springer-Verlag, New York, 2004.

[5] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications. Birkhäuser Boston, Boston, 1994.

[6] G.J. Heckman, Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, *Invent. Math.* 67:2 (1982), 333–356.

[7] D.E. Knuth, Permutations, matrices, and generalized Young tableaux, *Pacific Journal of Mathematics.* 34:3 (1970), 709–727.

[8] A. Knutson and T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture, *Journal of the American Math. Soc.* 12:4 (1999), 1055–1099.

[9] J.F. Quieró, A.P. Santana and E.M. de Sá, Group representations and matrix spectral problems, *Linear and Multilinear Algebra* 46:1–2 (1999), 1–23.

[10] C. Schensted, Longest increasing and decreasing subsequences, *Canadian Journal of Mathematics* 13 (1961), 179–191.

[11] B. Sturmfels and R.R. Thomas, Variation of cost functions in integer programming, *Mathematical Programming* 77:3 (1997), 357–387.