RICE UNIVERSITY

Response and First-Passage Statistics of Nonlinear Structural Models under Evolutionary Stochastic Loads

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Master of Science

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May, 2009

Abstract

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In the first half of the thesis, a novel approach is developed for determining the response of a lightly damped nonlinear single-degree of freedom system to a random excitation with an evolutionary broad-band power spectrum. The new approach is based on the coupling of the concepts of stochastic averaging and equivalent linearization. The nonlinearities can be either of the hysteretic or of the 'zero-memory' kind. Moreover, approximate analytical relationships for evaluating the response variance are derived for a number of oscillators. The efficiency and accuracy of the approach is demonstrated by pertinent digital simulations.

In the second half of the thesis, an approximate analytical approach is presented for examining the first-passage problem in context with the response of a class of lightly damped nonlinear oscillators to broad-band random excitations. A Markovian approximation both of the response amplitude envelope and of the response energy envelope is considered. This modeling leads to a backward Kolmogorov equation which governs the evolution of the survival probability of the oscillator. The Kolmogorov equation is solved approximately by employing a Galerkin approach. A set of confluent hypergeometric functions is used as an orthogonal basis for the expansions which are involved in the application of the Galerkin approach. The reliability of the derived analytical solution is demonstrated by comparisons of digital data derived by Monte Carlo simulation.

Acknowledgements

Given the opportunity of this thesis, I wish to express my sincere gratitude to my advisor and thesis committee chair, *Professor Pol. D. Spanos*. His insightful suggestions together with his constructive comments made this research effort an enjoyable adventure. He provided me with all the "freedom" a graduate student could ask for. I am honestly grateful to him for "protecting and mentoring my dreams".

Moreover, I would like to express my thanks to the committee members, *Professor Satish Nagarajaiah* and *Assistant Professor Jamie Padgett*, for participating in the committee, reviewing the manuscript, and providing valuable suggestions. I deeply appreciate their encouragement and interest in my work.

The financial support from *Rice University* during the years of my graduate studies is acknowledged with pleasure.

Finally, I would like to express my gratitude to my family for their encouragement, and trust in my skills, but mostly for their unconditional love. Specifically, I am indebted to my father, *Athanasios Kougioumtzoglou*, for encouraging me to embrace a mathematical way of thinking; and to my mother, *Vasiliki Liggri*, for urging me to make this thinking more colorful.

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Chapter 1

Introduction

1.1 Motivation and objectives

Over the past decades considerable interest has been developed in random vibration analysis of dynamic systems used as models in structural engineering. In general, the use of the term random vibration analysis suggests the determination of the response statistics of a system subjected to a stochastic load. The interest shown by the engineering community in the advances in the random vibration field can be attributed to the fact that structural loads caused by earthquakes, sea waves, blast events, and winds may be realistically described on a stochastic basis.

The main characteristic of a stochastic excitation is that the exact time history of some future loading cannot be predicted, nor can the exact time history of the response. Consequently, the two random processes (the excitation and the response) must be characterized by quantities representing average values. In fact, these are expected values of random variables or, in a more intuitive manner, averages across ensembles containing all possible time histories of the process. In a random vibration analysis the primary goal is to compute average quantities of the response, such as the mean value and the power spectral density, from knowledge of similar characteristics of the excitation.

If the system under consideration is linear, the general methodology which has been developed for bearing on deterministically posed problems can be readily extended and applied to random vibration analysis. For example, any possible time history of the response process may be expressed as a functional of a time history of the excitation process. This is essentially the Duhamel convolution integral. Exploiting this relationship, only the mean value function of the excitation must be known in order to predict the mean value function of the response.

Linear dynamic models are appealing for many structural engineering applications. However, in several cases involving strong dynamic excitations, such as seismic loads, structural components are expected to exhibit severely nonlinear behavior. In general, nonlinear behavior of a structural system is associated either with material or geometrical aspects. The number of nonlinear random vibration problems which lend themselves to exact solutions is strikingly limited. In fact, when considering a non-stationary problem, exact solutions are almost non-existent. Thus, the predominant approach for determining, with any preselected reliability, the response statistics of nonlinear structural systems under random excitation is the Monte Carlo method. This approach involves purely numerical random experiments. Specifically, a large number of time history samples are numerically simulated and are considered representative of an infinite ensemble of possible time histories. Clearly, the computational cost increases almost linearly with the number of records, compared to the accuracy which increases with the square root of the numbers of records simulated. Hence, there are cases, especially for multi-degree of freedom systems, where the Monte Carlo approach can be computationally prohibitive.

In this context, in the first half of this thesis a novel approach is suggested for determining the response statistics for nonlinear oscillators under evolutionary excitation. The approach combines the concepts of equivalent linearization, and of Markovian modeling of the response by stochastic averaging to yield a simple expression for the response variance. The versatility of the approach is highlighted by applications to hysteretic and non-hysteretic nonlinear oscillators.

In the second half of the thesis, the first-passage problem is considered for nonlinear oscillators under random excitation. Employing the ideas of equivalent linearization and stochastic averaging a partial differential equation governing the evolution of the survival probability is obtained. The solution is determined by resorting to a Galerkin scheme.

1.2 Organization of the thesis

The thesis consists of five chapters followed by the list of cited references. Excluding chapter 1 and 2, which play an introductory role, and chapter 5, which contains the concluding remarks, the remaining chapters each presents an independent research topic. Therefore, they are self-contained and include an introductory section followed by the analytical derivations, verified by digital simulations.

Chapter 1 serves as an introduction to the thesis presenting the motivation and objectives of the current research effort. Moreover, the contents of the thesis are briefly outlined.

In chapter 2 the task of reviewing the existing literature on nonlinear random vibration problems is undertaken. First, a historical perspective of the birth of stochastic calculus is presented. From the early work of physicists on the Brownian motion to the modern engineering applications, various scientific branches had to contribute for the theory of stochastic mechanics to be set on a solid foundation. Furthermore, recent developments in the nonlinear random vibration field are briefly outlined and discussed. Analytical or numerical approaches employed over the years to yield solutions are presented. These include the methods of Markovian modeling of the process, equivalent linearization, moment closure, perturbation, series expansion, Monte Carlo simulation, and of numerical integration of SDEs.

Chapter 3 contains a novel approach for determining the non-stationary response of nonlinear oscillators under evolutionary broad-band excitations. The new approach comprises the elements of stochastic averaging and statistical linearization. First, a linearization procedure produces an equivalent time-dependent frequency and damping factor. Then, taking into account the equivalent elements, a simple first-order ordinary differential equation is derived for the response variance. This becomes feasible by employing a Markovian model for the response, and assuming a time-dependent Rayleigh distribution for the response amplitude. Moreover, approximate analytical relationships for evaluating the response variance are derived for a number of oscillators. The analytical results concern oscillators of the Duffing, the piecewise linear stiffness, the hysteretic bilinear and Preisach kind. The reliability of the approach is demonstrated through Monte Carlo simulations. To this aim, evolutionary excitations possessing a modulated Gaussian white noise spectrum, a Kanai-Tajimi spectrum and a non-separable spectrum have been considered.

In chapter 4 a method is presented for obtaining survival probability estimates of a class of lightly damped oscillators. First, the approach is developed considering Gaussian white noise excitation. At the end of the chapter the approach is generalized to include broad-band non-stationary random excitations. The approach is based on modeling the response amplitude envelope by a Markovian process. This modeling leads to a backward Kolmogorov equation which is satisfied by the survival probability. In fact, modeling the response energy envelope by a Markovian process is also considered to illustrate the superiority of the latter modeling in cases of stiffness nonlinearities. For the case of a linear oscillator, the solution of the backward Kolmogorov equation has been determined in the technical literature by using the technique of separation of variables. This procedure leads to a boundary value problem whose eigenfunctions are a set of orthogonal confluent hypergeometric functions. Thus, the solution is cast in the form of an infinite series expansion, where the unknown time-dependent coefficients must be determined. In fact, the orthogonality of the hypergeometric functions is crucial for applying a Galerkin scheme to obtain the unknown time functions as the solution of a system of simultaneous linear differential equations with time variant coefficients. The applicability and reliability of the proposed method is demonstrated by considering Duffing and Van Der Pol types of oscillators.

Concluding remarks along with suggestions for future work are provided in chapter 5; a list of cited references follows.

Chapter 2

Nonlinear random vibration literature review

2.1 Historical review

Since the invention of modern calculus by Gottfried Wilhelm Leibniz (1684) and Isaac Newton (1687), differential and integral equations have been used in the applied sciences, engineering, economics, and even social sciences to describe the current state of a system and predict its evolution in time. A simplified approach would suggest that the coefficients and the input to these equations are known quantities. In other words, when the level of uncertainty related to these systems is relatively small, the associated problems may be formulated in terms of averages. The aforementioned deterministic approach, however, cannot be expected to realistically describe systems where the level of uncertainty is severe. In fact, insufficient information, poor interpretation of underlying mechanisms, and inherent randomness of the system result in defining problems which possess random coefficients and input. In such cases, a probabilistic approach constitutes a rational basis for system analysis and design. These problems are referred to as stochastic problems, and the corresponding differential equations as stochastic differential equations.

Differential equations for stochastic processes often appear in the form

$$\dot{X}(t) = f\left(t, X(t)\right) + G\left(t, X(t)\right)\eta(t), \quad X(t_0) = c, \tag{2.1}$$

where the function $(\eta(t))$ represents a stochastic process of the white-noise type. This process cannot exist in the conventional sense, since its covariance function should be the Dirac delta function yielding an infinite variance. Therefore, the white noise process $(\eta(t))$ must be viewed as a mathematical idealization for describing random rapid fluctuations which are essentially uncorrelated for different time instants.

Langevin (1908) was the first one to study such equations describing the Brownian motion of a particle in a fluid following the original work by Einstein (1905). Einstein developed a partial differential equation whose solution yielded the approximation of the time-evolving probability density function related to the position of a particle under Brownian motion. Another approach to determine the time evolution of the position of the particle was followed by Langevin. Defining $(X(t))$ as a component of the velocity of a free particle, Langevin's equation is

$$\dot{X}(t) = -aX(t) + \sigma\eta(t), \quad \alpha > 0, \tag{2.2}$$

where the part $(-aX(t))$ represents the influence due to dynamic friction. The term $(\sigma\eta(t))$ represents the force exerted on the particle by the molecular collisions. Taking into account the vast number of molecular collisions per unit time, this term is in fact a rapidly varying one, justifiably idealized as white noise.

Although white noise is just a generalized stochastic process, the integral $\left(\int_0^t \eta(s)ds\right)$ can be defined as the Wiener process. The Wiener process is a Gaussian stochastic process with continuous and nowhere differentiable sample functions possessing zero mean, $\left(E[W(t)]=0\right)$, and covariance $\left(E[W(t_1)W(t_2)]=\min(t_1,t_2)\right)$. Setting

$$dW = \eta(t)dt\,, \tag{2.3}$$

eq.(2.1) can be recast in the differential form

$$dX = f\left(t, X(t)\right)dt + G\left(t, X(t)\right)dW, \quad X(t_0) = c\,. \tag{2.4}$$

The corresponding to eq.(2.4) integral equation can be written in the form

$$X(t) = c + \int_{t_0}^{t} f\left(s, X(s)\right)ds + \int_{t_0}^{t} G\left(s, X(s)\right)dW. \tag{2.5}$$

Recognizing the fact that the second integral in eq.(2.5) cannot be treated as an ordinary Riemann-Stieltjes one, Ito (1951) defined such integrals for a broad class of functionals $\left(G(t)\right)$ creating a solid theoretical foundation for the theory of stochastic differential equations. Motivated, however, by the applicability of the rules of the classical Riemann-Stieltjes integral, Stratonovich (1966) defined a new stochastic integral. Despite certain mathematical drawbacks (e.g. see Arnold, 1974), this definition is frequently used in engineering applications due to the similarity with the Riemann-Stieltjes integral. Further discussion on the appropriate use of the stochastic integral in engineering problems can be found in the authoritative article by Gray and Caughey (1965).

The examples of random excitations acting upon structures are plentiful. Wind, blast, ocean waves, and earthquake loads fall into this category. In developing the probabilistic theory of structural dynamics, engineers inherited an adequate amount of knowledge from the early work of physicists on the subject of Brownian motion. The work by Einstein and Langevin was followed by several other contributions (e.g. Uhlenbeck and Ornstein, 1930). Structural engineers, however, are mainly interested in problems where the excitation and the dissipation forces are considered independent. In the case of the Brownian motion the two types of forces are related since they are both provided by the fluid medium.

The scientific branches of physics and astronomy were greatly benefited by the incorporation of probabilistic approaches, during the first half of the twentieth century (e.g. Chandrashekhar, 1943). Engineering applications first occurred in the area of communication theory to address the problem of noisy signals (Rice, 1944; Rice, 1945; Middleton, 1960). The analysis and design of structural systems followed (Bolotin, 1961), leading to the birth of the new area of random vibrations (Crandall, 1958; Crandall, 1963a; Crandall and Mark, 1963;

Robson, 1963; Lin, 1967; Nigam, 1983; Newland, 1993; Lin and Cai, 1995; Elishakoff, 1999).

2.2 Solution techniques in nonlinear random vibration problems

Mechanical systems or civil engineering structures can be mathematically modeled through a set of differential, integro-differential or difference equations. These systems are often characterized by uncertainties in terms of structural properties, loading conditions etc. Moreover, bearing in mind that no real system is exactly linear, the complexity of the formulation increases. In fact, nonlinearities may arise in various forms. For instance, the structural components of a building exhibit considerable hysteretic behavior, strongly nonlinear in character, during an earthquake event. Ultimately, if a non-linear model of the system is adopted together with a stochastic process model of the excitation, then the dynamic model takes the form of an ordinary differential relationship between the input (e.g. excitation) and the output (e.g. response). Since both the input and the output are random in nature, one faces the problem of solving a nonlinear stochastic differential equation (e.g. Arnold, 1973; Soong, 1973; Grigoriu, 2002; Oksendal, 2003).

Taking into account the fact that there is no general mathematical framework and methodology for analytical solutions of nonlinear stochastic differential equations, several approximate approaches have been developed (e.g.

Spanos and Lutes, 1986; Lin et al., 1986; Roberts and Dunne, 1988; Proppe et al., 2003).

2.2.1 Markov methods

Since the early work of the scientists on the Brownian motion, it has been well understood that broad-band excitations can be adequately modeled in terms of Markov processes. This is an assumption which enables one to utilize the theory of continuous Markov processes. Indeed, when the excitation is a Markov random process, the state transition of the probability density function of the response is governed by a partial differential equation, the Fokker-Planck (F-P) equation. This probability density function which characterizes the response of the system can be also used to determine the response moments and mean-crossing rates, valuable measures in assessing the reliability of the system. Thus, when the excitation of the system is approximated as white noise, the theoretical framework of the Markov processes can be employed to study nonlinear vibration problems. In cases where the white noise assumption is unjustifiable, pre-filters operating on white noise processes can be introduced to produce excitations possessing the desired power spectra.

Exact solutions for linear and nonlinear systems for the stationary case (the time derivative term of the F-P equation is set to zero) can be found in Risken (1984). In general, the analytical solutions of the F-P equation are quite limited. Some solutions exist for a small class of stationary nonlinear problems in two dimensions (e.g. Dimentberg, 1982). Solutions for special systems in higher dimensions can be found in Lin and Cai (1995). As far as non-stationary problems are concerned, the analytical solutions are even scarcer and require special forms of the nonlinear functions (e.g. Caughey and Diens, 1961).

The stochastic averaging scheme constitutes a potent analytical framework for approximately determining the probability density function of the response of nonlinear systems. In this scheme, rapidly fluctuating functions are averaged to provide simplified equations for slowly fluctuating quantities. A review of the method may be found in Roberts and Spanos (1986). For lightly damped oscillators and broad-band excitations, this approach enables the two-dimensional Markov process governing the response to be replaced by a one-dimensional Markov process governing the response envelope amplitude process. Consequently, the reduction in dimension of the F-P equation simplifies the determination of solutions for non-stationary nonlinear problems. The aforementioned scheme is also valuable in calculating first-passage statistics (e.g. Roberts, 1986), beneficial to the reliability assessment of the system. The first-passage problem has been related to the F-P equation by Crandall (1970) and to the Backward-Kolmogorov (B-K) equation by Yang and Shinozuka (1970). Recently, Spanos et al. (2004) and Wang et al. (2009) have employed the stochastic averaging scheme to yield the response statistics of a Preisach hysteretic system. Moreover, Spanos et al. (2007) have combined stochastic averaging with a Galerkin scheme to obtain the probability density function of the response of a class of lightly damped nonlinear oscillators. Recent developments concerning the method of stochastic averaging can be found in Lin (1986), Zhu

(1988), Red-Horse and Spanos (1992), Lin and Cai (1995, 2000), Huang et al. (2000, 2002).

The path integral solution (PIS) technique has been a numerical approach to approximately solve the F-P equation. The basic characteristic of the procedure is that the evolution of the probability density function is computed in short time steps. The method has been used by Naess and Moe (1996) to determine the non-stationary response of a hysteretic bilinear oscillator. Related advancements include the work by Yu et al. (1997), Naess and Mo (2000). The method has also been used to derive reliability statistics (e.g. Cai and Lin, 1998; Iourtchenko et al., 2008).

2.2.2 Equivalent linearization

The standard method of stochastic equivalent linearization was independently introduced by Booton (1953), Kazakov (1954) and Caughey (1959, 1963). This method can be viewed as a natural extension of the original approach widely used to deal with deterministic problems (e.g. Bogoliubov and Mitropolsky, 1963). The concept of the method suggests replacing the nonlinear function by an equivalent linear one. The difference between the two functions is then minimized in an appropriate sense.

To perform equivalent linearization, the probability distribution of the response random process must be known. Obviously, this is not the case, and therefore, the assumption that the response process is Gaussian is usually adopted. The validity and efficiency of the method has been demonstrated through numerous applications. The technique was applied by Caughey (1960a) to derive the response statistics of an oscillator with bilinear hysteresis under stationary Gaussian white noise excitation. Making the assumption that the response is narrowband, an averaging technique was used to derive the equivalent stiffness and damping coefficients. Since then, the method has been generalized to cope with non-white transient excitations (e.g. Spanos, 1981a; Roberts, 1981b; Elishakoff, 2000; Roberts and Spanos, 2003; Proppe et al., 2003; Socha, 2005; Spanos et al., 2007).

After almost forty years since the technique was introduced in the field of stochastic mechanics, Socha and Pawleta (1994) and Elishakoff and Colanjanni (1997) independently claimed that the standard procedure harbors a subtle flaw. Consequently, they presented an alternative procedure where they changed the criterion for selecting linearization parameters, suggesting that their alternative was the correct one. A closure to the issue was brought by Crandall (2001) who concluded that there was no flaw in the standard method. In fact, the alternative procedure differs from the standard one in that it simply employs a different criterion for selecting the optimum linear approximation.

Ultimately, stochastic equivalent linearization has been proven to be a reliable and efficient method, which can be readily generalized to treat multi-degree of freedom systems. Generalization of the method to cope with cases where the response distribution deviates from being Gaussian is also a straightforward procedure (e.g. Crandall, 2004).

2.2.3 Moment closure methods

Using the initial equations of motion of the nonlinear system, equations for the moments of the response can be readily derived. However, the lower order moments appear to be coupled with the higher order ones. Considering additional equations for the higher order moments, even higher order moments are introduced. Therefore, the moment equations form an infinite hierarchy which cannot be solved exactly. As a result, a closure hypothesis must be assumed in order to obtain a soluble set of equations (e.g. Nigam, 1983; Soong and Grigoriu, 1992; Lin and Cai, 1995; Roberts and Spanos, 2003). The same problem also arises in linear systems subjected to parametric random excitations.

The assumption that the response is a Gaussian random process can be considered as the first level of sophistication in the closure schemes (e.g. Ibrahim and Roberts, 1972). It can be shown that the Gaussian assumption yields results identical to those derived by the equivalent linearization approach. A method introduced by Er (2000), called multi-Gaussian closure, can be interpreted as an extension to the standard Gaussian closure. According to this approach, an approximate probability density function is constructed as a linear superposition of Gaussian probability density functions. However, in cases where the response process deviates considerably from being Gaussian, non-Gaussian closure schemes should be employed. To this aim, higher order levels of closure have been considered (e.g. Crandall, 1980; Wu and Lin, 1984; Soize, 1988; Hu, 1991). In a different case, the Gaussian assumption could lead to highly inaccurate results, especially when failure probabilities are concerned (Papadimitriou and Lutes, 1996).

2.2.4 Perturbation techniques

The perturbation method can be successfully used only when the system nonlinearities are considerably small. Crandall (1963) extended the approach, already used to treat deterministic vibration theory problems, to incorporate cases of stochastic excitations. The basic idea of the scheme relies on expanding the nonlinear solution in terms of powers of a nonlinearity quantifying parameter. The first term represents the solution to the equivalent linear problem and the subsequent ones express the influence of nonlinearity. However, including higher order in the expansion terms makes the calculations cumbersome and intractable. The perturbation technique has been used to predict the response of nonlinear oscillators by numerous researchers, such as Manning (1975), Iwan and Spanos (1978), Henriques (2007).

2.2.5 Series expansion

The Wiener-Hermite expansion has been used by Jahedi and Ahmadi (1983) to represent the excitation and the response of a Duffing oscillator. An iterative procedure has been employed to determine the kernel functions arising in the expansion. Further developments include the work by Orabi and Ahmadi (1987) and Roy and Spanos (1989). Moreover, Lee (1995) has applied a non-

Gaussian closure scheme based on Edgeworth series expansion to calculate the equivalent linearization coefficients.

Furthermore, Spanos and Donley (1991) considered replacing the nonlinear functions of the system equations by equivalent quadratic ones. As a result, they approximated the system response in terms of a Volterra series expansion. The extension to the multi-degree of freedom system followed (Spanos and Donley, 1992). Tognarelli et al. (1997b) and Spanos et al. (2003) have also applied the method using cubic nonlinear equivalent functions.

2.2.6 Monte Carlo simulation

The use of Monte Carlo methods as a research tool stems from work on the atomic bomb during the second world war. This work involved a direct simulation of the probabilistic problems pertaining to random neutron diffusion in fissile material. Since then, the method has been applied to almost every scientific branch due to its simplicity and versatility. The Monte Carlo approach is associated with the fact that a stochastic differential equation can be interpreted as an infinite set of independent deterministic differential equations (e.g. Soong, 1973). Therefore, instead of solving the stochastic differential equation, a family of deterministic problems is considered with values for the random parameters compatible with their statistical characteristics. Statistical analysis on the family of the derived solutions is then conducted.

The Monte Carlo approach has been widely used to predict the response statistics of randomly excited nonlinear systems (e.g. Shinozuka, 1972).

Moreover, the standard method can be coupled with variance reduction techniques to yield a greater degree of efficiency. To this aim, Au and Beck (2001, 2003), Olsen and Naess (2007) and Macke and Bucher (2003) have used importance sampling techniques in order to estimate the failure probabilities of nonlinear systems. A discussion on the various Monte Carlo digital simulation methods can be found in the review article by Spanos and Zeldin (1998).

2.2.7 Numerical integration of SDEs

The applicability and versatility of direct integration methods have made numerical schemes, such as the stochastic central difference method (To, 1986), an attractive approach. Stochastic differential equations which are driven by Gaussian white noise cannot be expressed in terms of the Riemann-Stieltjes integral, due to the rapidly fluctuating term, corresponding to the white noise process. Therefore, the contribution of the Ito (1951) or the Stratonovich integral (1966) is needed. It must be noted, however, that the solution corresponding to the Stratonovich interpretation and the solution related to the Ito interpretation are not identical. They are related, though, according to the Wong-Zakai theorem (Wong and Zakai, 1965). It is often preferable and more intuitive to describe an engineering problem using the Stratonovich formulation, and then use the Ito definition as an efficient formulation for solving the problem (see also Gray and Caughey, 1965).

The extension of the Euler numerical scheme for deterministic differential equations to stochastic differential equations was made by Maruyama (1955) leading to the Euler-Maruyama method. A discussion on the available numerical schemes can be found in the review article by Kloeden and Platen (1992).

Chapter 3

Nonlinear stochastic response under evolutionary excitation

3.1 Preliminary remarks

A broad class of structural systems is subject to excitations such as seismic motions, winds, and ocean waves which inherently possess the attribute of evolution in time. Therefore, to accurately predict the system behavior under this kind of loading, realistic modeling has involved representation of these phenomena by non-stationary stochastic processes. Associated with the notion of a non-stationary stochastic process is the concept of a separable or of a non-separable evolutionary power spectrum. The former relates to the evolution in time of the intensity of a process with time invariant energy-frequency relationship. The latter, which in general reflects a more realistic approach, encompasses the concept of 'local' energy distributions over frequency (Priestley, 1965; Priestley, 1967).

Attempts towards determining, either exactly or approximately, the response statistics of a linear oscillator under evolutionary excitation can be found in several references (e.g. Caughey and Stumpf, 1961; Hammond, 1968; Roberts 1971; Corotis and Vanmarcke, 1975; Spanos and Lutes, 1980; To, 1982; Spanos and Solomos 1983; Iwan and Hou, 1989; Conte and Peng, 1996). Caughey and Stumpf (1961) first studied the transient response of a linear oscillator under unit step modulated white noise. The evolutionary power spectrum of the response process of an oscillator subject to a unit step modulated stationary excitation was studied in Corotis and Vanmarcke (1975). Explicit expressions for the second moment statistics of the response were presented in Iwan and Hou (1989), where the results refer to white noise excitation modulated by step, boxcar and gamma envelope functions. Moreover, approximate analytical solutions for the response amplitude statistics of a lightly damped oscillator under evolutionary excitation were derived in (Spanos and Lutes, 1980; Spanos and Solomos, 1983).

On the other hand, limited progress has been made in terms of determining the stochastic response of nonlinear systems. One of the interesting approaches of treating nonlinear oscillators under evolutionary excitation has been the coupling of the equivalent linearization method with the decomposition of the covariance matrix of the input random process (Roberts and Spanos, 2003). In this regard, a Karhunen-Loeve spectral decomposition was used in Smyth and Masri (2002). It can be argued, though, that often the complexity of such approaches limits their versatility.

To this aim, a general approach is attempted in this chapter based on the assumed pseudo-harmonic behavior of the response. Relying on this property, an averaging scheme, first proposed by Stratonovich in the 1960s, is applied to yield a first-order stochastic differential equation (Ito equation) for the response amplitude. In section 3.2.1 the mathematical details of this approach are briefly reviewed. A detailed presentation of the averaging procedure can be found in references such as Bogoliubov and Mitropolski (1963), Stratonovich (1963, 1967), Spanos (1976) and Roberts and Spanos (1986).

In section 3.2.2 the Fokker-Planck (F-P) or forward Kolmogorov equation associated with the Ito one is derived (see also Arnold, 1974; Soong, 1973; Oksendal, 2003) having as frequency and damping elements the equivalent ones obtained by a linearization scheme. Using the F-P equation with the assumption that the probability density function of the response amplitude is a time-dependent Rayleigh one, a first-order ordinary differential equation for the response variance is derived.

In sections 3.3.1-3.3.4 the aforementioned procedure is applied to a number of hysteretic or non-hysteretic nonlinear oscillators resulting in approximate analytical expressions for computing the time-dependent response variance. The accuracy of the proposed method is verified by Monte Carlo simulation data in sections 3.4.1-3.4.2.

3.2 Mathematical formulation

3.2.1 Determination of the equivalent linear system time-dependent elements

Consider a nonlinear single-degree-of-freedom system whose motion is governed by the differential equation

$$\ddot{x} + \beta \dot{x} + z(t, x, \dot{x}) = w(t), \tag{3.1}$$

where a dot over a variable denotes differentiation with respect to time (t); ($z(t,x,\dot{x})$) is the restoring force which could be either hysteretic or depend only on the instantaneous values of (x) and ($\dot{x}$); (β) is a linear damping coefficient; and ($w(t)$) represents a Gaussian, zero-mean non-stationary random process possessing an evolutionary broad-band power spectrum, $S(\omega,t)$.

Adopting an equivalent linearization approach followed in Goto and Iemura (1973) and described in Roberts and Spanos (2003), the linearized counterpart of eq.(3.1) is

$$\ddot{x}+\beta(\mathrm{A})\dot{x}+\omega^2(\mathrm{A})x=w(t), \tag{3.2}$$

where the equivalent damping element and natural frequency are assumed to be functions of the amplitude (A) of the response in order to partly account for the effect of the nonlinearity. Assuming the case of a lightly damped system, it can be argued that the amplitude (A) is a slowly varying function with respect to time and therefore can be treated as a constant over one cycle of oscillation. Thus, defining the error between eq.(3.1) and eq.(3.2) as

$$\varepsilon=z(t,x,\dot{x})+[\beta-\beta(\mathrm{A})]\dot{x}-\omega^2(\mathrm{A})x, \tag{3.3}$$

the expressions

$$\beta(\mathrm{A}) = \beta + \frac{\oint \dot{x} z dt}{\oint \dot{x}^2 dt}, \tag{3.4}$$

and

$$\omega^2(\mathrm{A}) = \frac{\oint x z dt}{\oint x^2 dt} \tag{3.5}$$

are derived by applying an error minimization procedure in the mean square sense, where $(\oint\ )$ can be interpreted as 'an average over one cycle' operator. The nonlinear oscillator (3.1) exhibits a pseudoharmonic behavior described by

$$x(t) = A(t)\cos\left[\omega(A)t + \phi(t)\right], \tag{3.6}$$

and

$$\dot{x}(t) = -\omega(A)A(t)\sin\left[\omega(A)t + \phi(t)\right]. \tag{3.7}$$

Substituting eq.(3.6) and eq.(3.7) into eq.(3.4) and eq.(3.5) and considering $A(t)$ and $\phi(t)$ constant over one cycle yields

$$\beta(A) = \beta + \frac{S(A)}{A\omega(A)}, \tag{3.8}$$

and

$$\omega^2(A) = \frac{C(A)}{A}, \tag{3.9}$$

where

$$C(A) = \frac{1}{\pi}\int_0^{2\pi} \cos[\psi] z(t, A\cos\psi, -\omega(A)A\sin\psi)d\psi\,, \tag{3.10}$$

and

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} \sin[\psi] z(t, A\cos\psi, -\omega(A)A\sin\psi)d\psi\,. \tag{3.11}$$

Let the symbol $p(A,t)$ represent the probability density function of the amplitude (A) of the response process (x). Then, the equivalent time-dependent damping factor and natural frequency can be evaluated by taking expectations on the right-hand sides of equations (3.8) and (3.9), respectively. That is,

$$\beta_{eq}(t) = \beta + E\left[\frac{S(A)}{A\omega(A)}\right] = \beta + \int_0^{\infty} \frac{S(A)}{A\omega(A)} p(A,t)dA\,, \tag{3.12}$$

and

$$\omega_{eq}^2(t) = E\left[\frac{C(A)}{A}\right] = \int_0^\infty \frac{C(A)}{A} p(A,t)dA. \tag{3.13}$$

3.2.2 Markovian modeling of the response envelope

Taking into account the manner the time-dependent natural frequency and damping factor have been determined, it is obvious that they possess the characteristic of being slowly varying functions with respect to time. Therefore, it could be argued that the equivalent linear system can be recast in the form

$$\ddot{x} + \beta_{eq}(t)\dot{x} + \omega_{eq}^2(t)x = w(t). \tag{3.14}$$

The amplitude ($A(t)$) and the phase ($\theta(t)$) of the response (x) are introduced by the transformations

$$x(t) = A(t)\cos\left[\omega(t)t + \theta(t)\right], \tag{3.15}$$

and

$$\dot{x}(t) = -\omega(t)A(t)\sin\left[\omega(t)t + \theta(t)\right], \tag{3.16}$$

which lead to the equations

$$A^2(t) = x^2(t) + \left(\frac{\dot{x}(t)}{\omega_{eq}(t)} \right)^2 , \tag{3.17}$$

and

$$\theta(t) = -\omega_{eq}(t)t - \tan^{-1}\left(\frac{\dot{x}(t)}{\omega_{eq}(t)x(t)} \right). \tag{3.18}$$

Differentiating eq.(3.17) and eq.(3.18) and taking into account eq.(3.14) yields

$$\dot{A}(t) = -\beta_{eq}(t)A(t)\sin^2[\omega_{eq}(t)t + \theta(t)] - \frac{w(t)}{\omega_{eq}(t)}\sin[\omega_{eq}(t)t + \theta(t)]. \tag{3.19}$$

Relying once more on the assumption of light damping, further simplification of eq.(3.19) is obtained by a combination of deterministic and stochastic averaging (e.g. Spanos and Lutes, 1980) which results in the following first order stochastic differential equation that approximately governs the evolution in time of the amplitude $(A(t))$:

$$\dot{A}(t) = -\frac{1}{2}\beta_{eq}(t)A(t) + \frac{\pi S(\omega_{eq}(t),t)}{2A(t)\omega_{eq}^2(t)} - \frac{\left(\pi S(\omega_{eq}(t),t)\right)^{1/2}}{\omega_{eq}(t)}\eta(t). \tag{3.20}$$

In eq.(3.20), $\eta(t)$ is a zero mean and delta correlated process of intensity one, i.e., $E(\eta(t))=0$ and $E(\eta(t)\eta(t+\tau))=\delta(\tau)$, with $(\delta(\tau))$ being the Dirac delta function. The importance of eq.(3.20) lies in the fact that it is decoupled from the phase $(\theta(t))$. Thus, it is feasible for the amplitude process $(A(t))$ to be modeled as a one-dimensional Markov process.

3.2.3 Fokker-Planck equation

The Fokker-Planck equation that corresponds to eq.(3.20) is (e.g. Nigam, 1983)

$$\begin{aligned}\frac{\partial p(A,t)}{\partial t} &= -\frac{\partial}{\partial A}\left\{\left(-\frac{1}{2}\beta_{eq}(t)A+\frac{\pi S(\omega_{eq}(t),t)}{2A\omega_{eq}^2(t)}\right)p(A,t)\right\}+\ldots \\ &\frac{\pi S(\omega_{eq}(t),t)}{2\omega_{eq}^2(t)}\frac{\partial^2 p(A,t)}{\partial A^2}\end{aligned} \tag{3.21}$$

Following a similar approach as in Spanos and Lutes (1980), a solution of eq.(3.21) is attempted in the form

$$p(A,t)=\frac{A}{c(t)}e^{-\frac{A^2}{2c(t)}}. \tag{3.22}$$

where $(c(t))$ accounts for the time-dependent variance of the response process (x). Substituting eq.(3.22) into eq.(3.21) and manipulating yields

$$\dot{c}(t) = -\beta_{eq}(t)c(t) + \frac{\pi S\left(\omega_{eq}\left(c(t)\right),t\right)}{\omega_{eq}^2\left(c(t)\right)}. \tag{3.23}$$

Taking into account eq.(3.12) and eq.(3.13) it can be readily seen that eq.(3.23) constitutes a first-order ordinary differential equation for the variance of the process (x). Therefore, by approximating the probability density function of the non-stationary amplitude response by a time-dependent Rayleigh one, a simple expression has been derived in order to determine the variance of the response process.

3.3 Analytical Results

3.3.1 Piecewise linear oscillator

The first application concerns a system with piecewise linear stiffness. Mathematically, the stiffness function can be described as

$$z(x) = \begin{cases} \omega_0^2 x, & |x| \le x_0 \\ \omega_1^2 x + x_0(\omega_0^2 - \omega_1^2)sign(x), & |x| > x_0 \end{cases}. \tag{3.24}$$

where the initial stiffness is given by (ω_0^2). When the absolute value of the displacement exceeds (x_0), the stiffness changes to (ω_1^2). Equivalently, making use of the Heaviside function yields

$$z(x) = \omega_0^2 x + \left(1 - \left[H(x+x_0) - H(x-x_0)\right]\right)\left(\omega_1^2 x + x_0(\omega_0^2 - \omega_1^2)sign(x) - \omega_0^2 x\right), \tag{3.25}$$

where

$$H(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}. \tag{3.26}$$

Introducing the non-dimensional displacement $(y=x/x^{*})$ and the non-dimensional time quantity $(\tau=\omega_0 t)$, eq.(3.25) becomes

$$z(y)=y+\left(1-\left[H(y+1)-H(y-1)\right]\right)\left(sy+(1-s)sign(y)-y\right), \tag{3.27}$$

where (s) is the ratio of secondary to primary elastic slope. Evaluating the integrals in eqs.(3.8) and (3.9) yields

$$\beta(A)=\beta, \tag{3.28}$$

and

$$\omega^2(A)=\begin{cases}\dfrac{2\sqrt{1-\dfrac{1}{A^2}}\left(1-s\right)+A\pi s-2A(-1+s)\csc^{-1}(A)}{\pi A}, & \mathrm{A}>1,\\ 1, & \mathrm{A}<1\end{cases} \tag{3.29}$$

where

$$\csc^{-1}(A) = \sin^{-1}\left(\frac{1}{A}\right). \tag{3.30}$$

Using eqs.(3.12-3.13) results in the expressions

$$\omega_{eq}^2(t) = \left[1 - e^{-\frac{1}{2c(t)}} + \frac{\int_1^\infty \left(2\sqrt{1-\frac{1}{A^2}}(1-s) + A\pi s - 2A(-1+s)\csc^{-1}(A) \right) e^{-\frac{A^2}{2c(t)}} dA}{\pi c(t)} \right] \tag{3.31}$$

and

$$\beta_{eq}(t) = \beta \tag{3.32}$$

for the time-dependent equivalent frequency and damping factor.

3.3.2 Duffing oscillator

Consider the randomly excited Duffing oscillator

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x + \varepsilon\omega_0^2 x^3 = w(t), \quad \varepsilon > 0, \tag{3.33}$$

for which the function $(z(t,x,\dot{x}))$ is defined as

$$z(x) = \omega_0^2 x + \varepsilon \omega_0^2 x^3 . \tag{3.34}$$

Then, using eq.(3.8) and eq.(3.9), the amplitude-dependent equivalent natural frequency and damping term are found to be, respectively,

$$\beta(A) = \beta , \tag{3.35}$$

and

$$\omega^2(A) = \omega_0^2 (1 + \frac{3}{4} \varepsilon A^2) . \tag{3.36}$$

Substituting eq.(3.35) and eq.(3.36) into eq.(3.12) and eq.(3.13) respectively and taking into account eq.(3.22), the expressions

$$\beta_{eq}(t) = \beta , \tag{3.37}$$

and

$$\omega_{eq}^2(t) = \omega_0^2 (1 + \frac{3}{2} \varepsilon c(t)) \tag{3.38}$$

are obtained. Finally, the use of eqs.(3.37), (3.38) and (3.23) leads to a first-order differential equation of the variable $(c(t))$:

$$\dot{c}(t) = -2\beta c(t) + \frac{\pi S\left(\sqrt{\omega_0^2(1+\frac{3}{2}\varepsilon c(t))}, t\right)}{\omega_0^2(1+\frac{3}{2}\varepsilon c(t))}. \tag{3.39}$$

3.3.3 Bilinear oscillator

An oscillator that exhibits hysteretic behavior of the bilinear type will be considered. Thus, the equation of motion (3.1) becomes

$$\ddot{y} + \beta\dot{y} + ay + (1-a)z_0 = f(\tau), \tag{3.40}$$

where the non-dimensional displacement $(y = x/x^*)$ and the non-dimensional time quantity $(\tau = \omega_0 t)$ have been introduced; (x^*) is the critical value of the displacement at which yield first occurs; (ω_0) is the frequency of the oscillation corresponding to the primary elastic slope; (a) is the ratio of plastic to elastic stiffness; and (z_0) the hysteretic force corresponding to the elasto-plastic characteristic. The hysteretic force (z_0) can be represented in terms of a first-order differential equation (e.g. Suzuki and Minai, 1987a) as

$$\dot{z}_0 = \dot{y}\left[1 - H(\dot{y})H(z_0 - 1) - H(-\dot{y})H(-z_0 - 1)\right]. \tag{3.42}$$

Comparing eq.(3.1) and eq.(3.40) yields

$$z(t) = ay + (1-a)z_0 . \tag{3.43}$$

Using eqs.(3.8) and (3.9), the amplitude-dependent equivalent elements are

$$\beta(A) = \beta + \frac{(1-a)S_0(A)}{\sqrt{aA^2 + (1-a)AC_0(A)}} , \tag{3.44}$$

and

$$\omega^2(A) = a + (1-a)\frac{C_0(A)}{A} , \tag{3.45}$$

where

$$C_0(A) = \frac{1}{\pi}\int_0^{2\pi} \cos[\psi] z_0(A,t) d\psi , \tag{3.46}$$

and

$$S_0(A) = -\frac{1}{\pi}\int_0^{2\pi} \sin[\psi] z_0(A,t) d\psi . \tag{3.47}$$

A technique for evaluating the integrals in eqs.(3.46) and (3.47) can be found in (Caughey, 1960; Caughey, 1960a) which yields

$$C_0(A)=\begin{cases}\frac{A}{\pi}(\Lambda-0.5\sin(2\Lambda)), & \mathrm{A}>1\\ \mathrm{A}, & \mathrm{A}<1\end{cases}, \tag{3.48}$$

and

$$S_0(A)=\begin{cases}\frac{4}{\pi}(1-\frac{1}{A}), & \mathrm{A}>1\\ 0, & \mathrm{A}<1\end{cases}, \tag{3.49}$$

where

$$\cos 2\Lambda = 1-\frac{2}{A}. \tag{3.50}$$

Combining eqs.(3.44-3.50) and (3.12-3.13) yields the expressions

$$\omega_{eq}^2(t)=a+(1-a)\left[1-e^{-\frac{1}{2c(t)}}+\frac{1}{\pi c(t)}\int_1^\infty(\Lambda-0.5\sin 2\Lambda)Ae^{-\frac{A^2}{2c(t)}}dA\right], \tag{3.51}$$

and

$$\beta_{eq}(t)=\beta+\frac{4(1-a)}{\pi c(t)}\int_1^{\infty}\frac{\left(1-\frac{1}{A}\right)}{\sqrt{a+\frac{(1-a)}{\pi}(\Lambda-0.5\sin 2\Lambda)}}e^{-\frac{A^2}{2c(t)}}dA \tag{3.52}$$

for the time-dependent equivalent frequency and damping factor.

3.3.4 Preisach oscillator

Recently, an envelope-based approach has been applied by Spanos et al. (2004) to determine the response amplitude statistics of Preisach hysteretic systems under stationary Gaussian white noise excitation. The approach has been further extended in Wang et al. (2009) to yield response energy envelope statistics. Following the notation introduced in Spanos et al. (2004), the equation of motion (3.1) becomes

$$\ddot{x}+\beta\dot{x}+\varpi^2 x+f_H(t)=w(t), \tag{3.53}$$

where

$$\varpi=\sqrt{\omega_0^2+\omega_j^2}=\omega_j\sqrt{1+\phi}, \tag{3.54}$$

$$\omega_j=\sqrt{k_j}, \tag{3.55}$$

and

$$\phi = \omega_0^2 / \omega_j^2 . \tag{3.56}$$

As mentioned in Spanos et al. (2004), the Preisach restoring force can be divided in two terms; a linear part $(\omega_j^2 x)$ and a nonlinear one $(f_H(t))$ monitoring the memory of the system. Therefore, (ϕ) quantifies the stiffness of the linear counterpart of the Preisach element compared to the linear stiffness (ω_0^2) contribution. Introducing now the parameter

$$\psi = \frac{\varpi^2}{f_y^*}, \tag{3.57}$$

eq.(3.53) can be recast in the form

$$\ddot{x} + \beta \dot{x} + \varpi^2 \left(x + \psi d_H(t) \right) = w(t), \tag{3.58}$$

where $(d_H(t))$ the scaled hysteretic restoring force and

$$f_y^* = \frac{f_{y,\max} + f_{y,\min}}{2}, \tag{3.59}$$

where (f_y) is the yielding force. Defining the non-dimensional parameter (ν) as

$$\nu = \frac{f_{y,\max} - f_{y,\min}}{2f_y^*}, \tag{3.60}$$

and applying eqs.(3.8) and (3.9) for $\nu = 1$ yields

$$\beta(A) = \beta + \frac{\psi\bar{\omega}^2}{3\pi(1+\phi)^2\sqrt{\bar{\omega}^2 - \frac{\psi\bar{\omega}^2}{4(1+\phi)^2}A}}A \tag{3.61}$$

and

$$\omega^2(A) = \bar{\omega}^2 - \frac{\psi\bar{\omega}^2}{4(1+\phi)^2}A. \tag{3.62}$$

Equivalent expressions can be found for arbitrary values of (ν), though more complicated. Combining eqs.(3.61-3.62) and (3.12-3.13) leads to the expressions

$$\omega_{eq}^2(t) = \bar{\omega}^2\left(1 - \frac{\psi\sqrt{2\pi c(t)}}{8(1+\phi)^2}\right), \tag{3.63}$$

and

$$\beta_{eq}(t) = \beta + \frac{\psi\bar{\omega}^2}{3\pi(1+\phi)^2 c(t)}\int_0^\infty \frac{A^2}{\sqrt{\bar{\omega}^2 - \frac{\psi\bar{\omega}^2}{4(1+\phi)^2}A}} e^{-\frac{A^2}{2c(t)}} dA \tag{3.64}$$

for the time-dependent equivalent elements.

3.4 Numerical Applications

To assess the accuracy of the proposed method, digital simulations have been performed considering both separable and non-separable excitations. For each Monte-Carlo type simulation an ensemble size of 500 realizations has been used, whereas the value of 0.01 has been chosen for the ratio of critical damping (ζ).

3.4.1 Separable Processes

In the case of separable random processes the evolutionary power spectrum of the excitation is given by the equation

$$S(\omega,t) = |g(t)|^2 S_v(\omega), \tag{3.65}$$

where $(g(t))$ is a slowly varying time-dependent modulating function; $(S_v(\omega))$ is the power spectrum of a stationary process $(v(t))$. Under these circumstances, the excitation process can be recast in the form

$$w(t) = g(t)v(t). \tag{3.66}$$

In the ensuing steps, the simulation studies are performed choosing the modulating function to be

$$g(t) = k(e^{-at} - e^{-bt}), \tag{3.67}$$

in which $a = 0.25$; $b = 0.5$; and k is a normalization constant so that $g_{\max} = 1$.

3.4.1.1 Modulated Gaussian White Noise

The case where $S_v(\omega) = S_0, \quad 0 \le |\omega| \le \infty$ is first considered. Obviously, for the case of a modulated white noise excitation, there exist many tractable approaches for evaluating the response statistics. However, this simulation serves the purpose of comparing the proposed method to another equivalent linearization approach. The latter, equally simple to implement for modulated white noise, is generally expected to have greater accuracy, since it does not have the element of averaging. Extended presentation of the alternative method exists in Roberts and Spanos (2003), therefore, limited background information is included herein.

The simulation study is restricted to the case of a Duffing oscillator. Based on the assumption of Gaussian approximation of the response, eqs.(3.37) and (3.38) yield

$$\beta_{eq}(t) = \beta, \tag{3.68}$$

and

$$\omega_{eq}^2(t) = \omega_0^2(1+3\varepsilon c(t)) \tag{3.69}$$

for the non-stationary linearization approach. The variance of the response is then determined by solving the following set of coupled differential equations:

$$\begin{aligned}
&\frac{d}{dt}E\left(x^2\right)=2E\left(x\dot{x}\right)\\
&\frac{d}{dt}E\left(x\dot{x}\right)=-\left(\omega_0^2\left(1+3\varepsilon E\left(x^2\right)\right)\right)E\left(x^2\right)-\beta E\left(x\dot{x}\right)+E\left(\dot{x}^2\right)\\
&\frac{d}{dt}E\left(\dot{x}^2\right)=2\pi\left|g(t)\right|^2 S_0-2\left(\omega_0^2\left(1+3\varepsilon E\left(x^2\right)\right)\right)E\left(x\dot{x}\right)-2\beta E\left(\dot{x}^2\right)
\end{aligned} \quad . \tag{3.70}$$

The equivalent to eqs.(3.70) for the proposed method is eq.(3.23), which becomes

$$\dot{c}(t) = -2\beta c(t) + \frac{\pi\left|g(t)\right|^2 S_0}{\omega_0^2(1+\frac{3}{2}\varepsilon c(t))} . \tag{3.71}$$

The results obtained by eqs.(3.70) and (3.71), along with the digital data, are shown in Figs.(3.1) and (3.2). For the natural frequency (ω_0), the value 3.61 rad/s has been used, whereas the values $\varepsilon = 0.5$ and $\varepsilon = 1$ have been considered in Figs.(3.1) and (3.2), respectively.

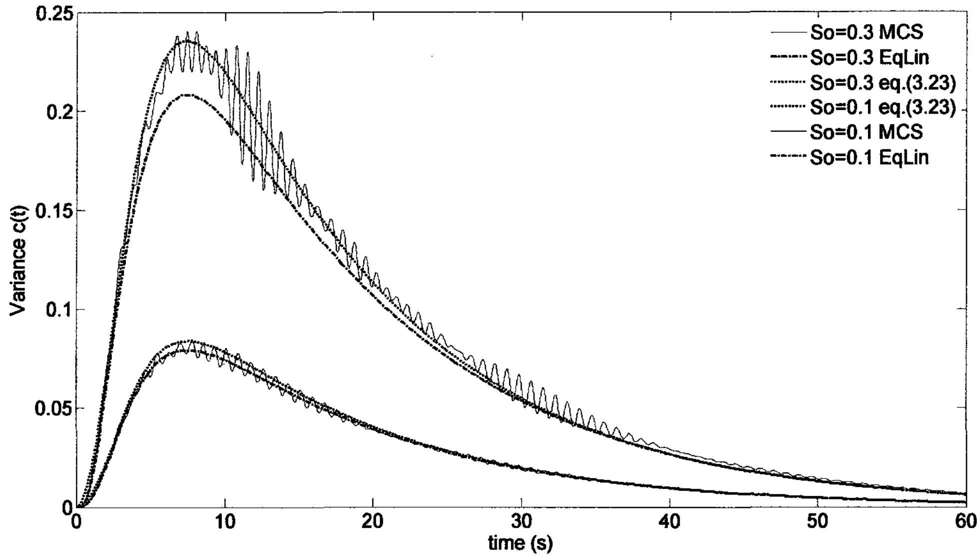

Fig.(3.1). Response Variance for a Duffing oscillator ($\varepsilon = 0.5$) under modulated Gaussian white noise. Comparison between MCS data (500 realizations), eq.(3.70) and eq.(3.71)

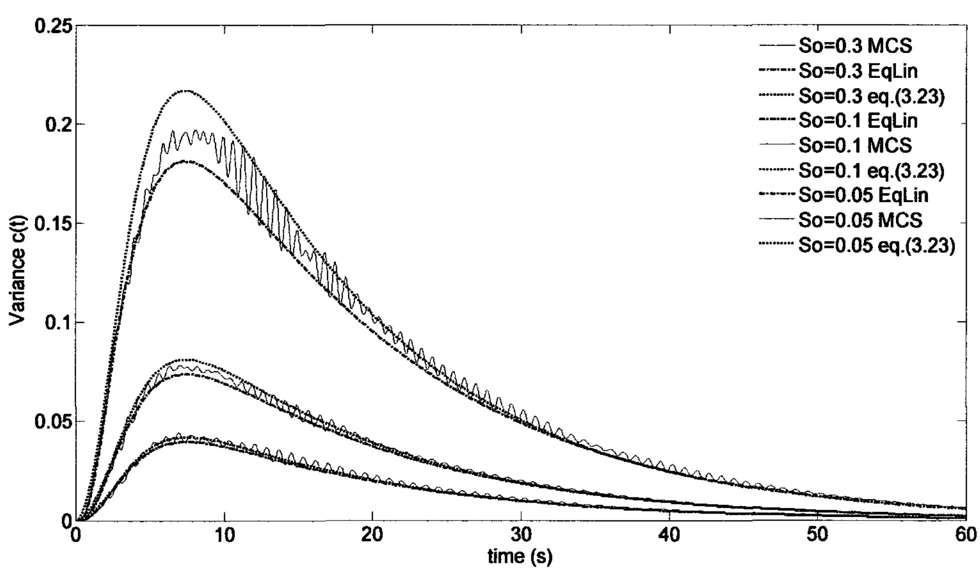

Fig.(3.2). Response Variance for a Duffing oscillator ($\varepsilon = 1$) under modulated Gaussian white noise. Comparison between MCS data (500 realizations), eq.(3.70) and eq.(3.71)

For small values of the power spectrum (S_0), it is seen that both methods are in excellent agreement with the Monte Carlo data. Furthermore, it is demonstrated that increasing the nonlinearity degree gradually results in divergence from the digital data as expected. However, the behavior of the new method indicates at least the same reliability level as the equivalent linearization one.

3.4.1.2 Modulated Kanai-Tajimi Spectrum

The modulated Kanai-Tajimi (Kanai, 1957; Tajimi, 1960) excitation has been frequently used in earthquake engineering applications. The following form for the power spectrum is considered

$$S_v(\omega) = S_1 \frac{(8\pi)^4 + 4(0.8)^2(8\pi)^2\omega^2}{\left((8\pi)^2 - \omega^2\right)^2 + 4(0.8)^2(8\pi)^2\omega^2}, \quad -\infty < \omega < \infty, \tag{3.72}$$

which corresponds to the squared modulus of the frequency response function of single-degree-of-freedom oscillator with prescribed stiffness and damping elements. Generating realizations of the process $v(t)$, being compatible with $(S_v(\omega))$, is possible by using an auto-regressive time series algorithm (e.g. Spanos and Zeldin, 1998), where a minimization procedure yields a Toeplitz system of linear equations known as the Yule-Walker equations. In Figs.(3.3-3.5), the time evolution of the response variance under modulated Kanai-Tajimi excitation is plotted. In Fig.(3.3), an oscillator possessing a piecewise linear stiffness is concerned. The value $(s = 2)$ is used. In Figs.(3.4) and (3.5), a Duffing, $(\varepsilon = 1)$ and a hysteretic bilinear one, $(a = 0.02, b = 0.1)$, are considered, respectively. The reliable behavior of the new method is demonstrated for different values of the input strength (S_1). Comparing the approach to Monte Carlo results, it can be argued that it successfully manages to follow the time evolution of the mean value of the variance, which is quite predictable taking into account the averaging procedure which is involved.

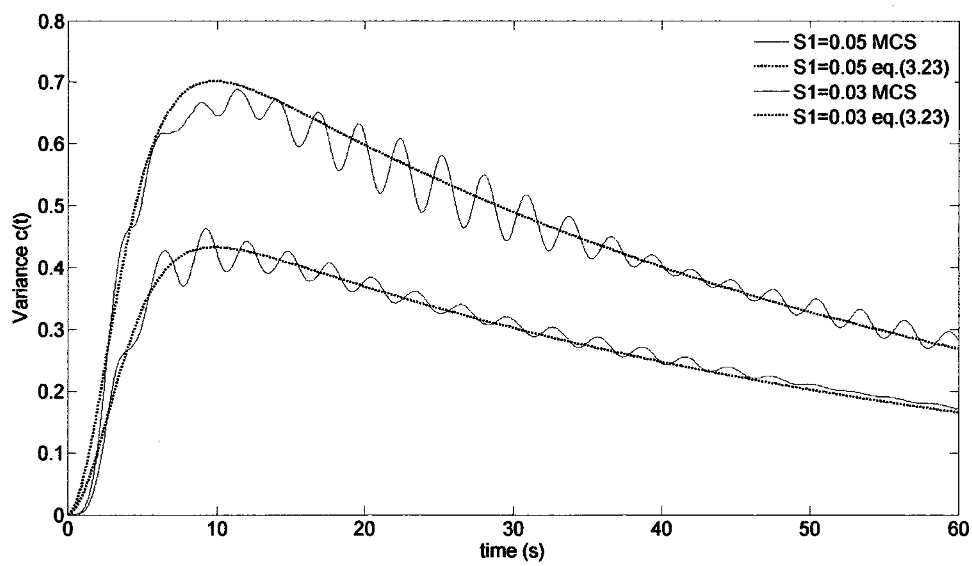

Fig.(3.3). Response Variance for an oscillator with Piecewise Linear Stiffness $(s = 2)$ under modulated Kanai-Tajimi spectrum. Comparison between MCS data (500 realizations) and eq.(3.23)

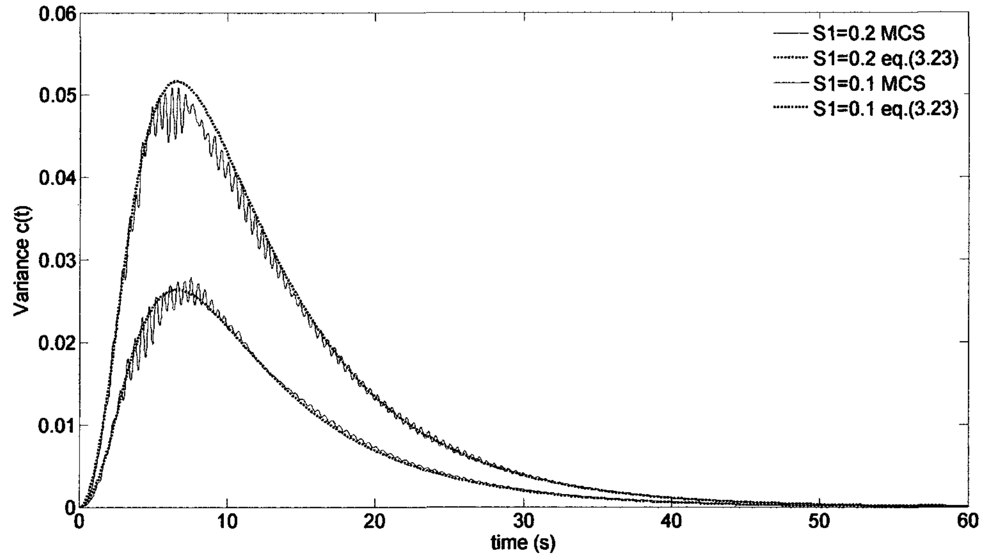

Fig.(3.4). Response Variance for a Duffing oscillator ($\varepsilon = 1$) under modulated Kanai-Tajimi spectrum. Comparison between MCS data (500 realizations) and eq.(3.23)

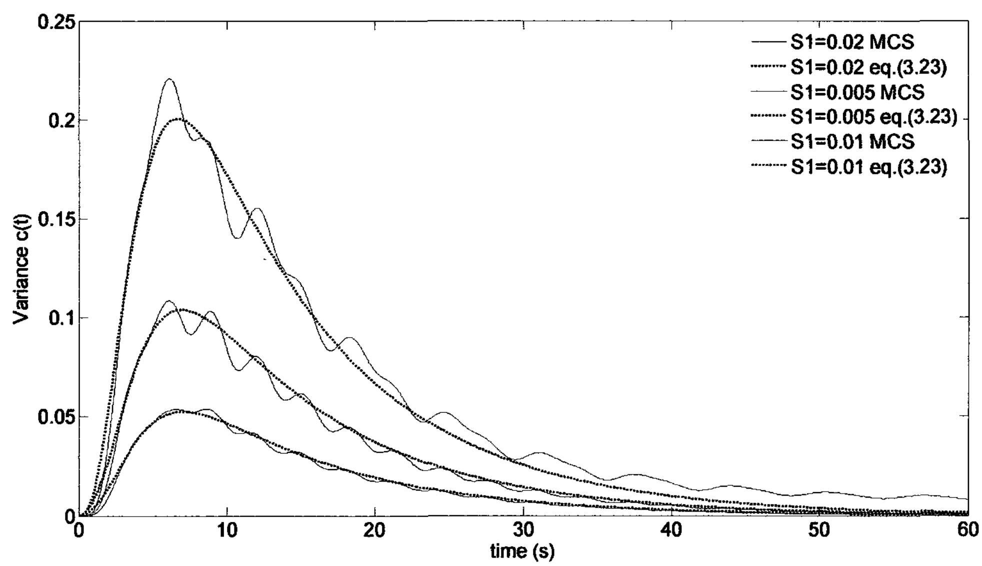

Fig.(3.5). Response Variance for a Bilinear oscillator ($a = 0.02, b = 0.1$) under modulated Kanai-Tajimi spectrum. Comparison between MCS data (500 realizations) and eq.(3.23)

3.4.2 Non-Separable Processes

The following non-separable power spectrum is considered

$$S(\omega,t) = S_2\left(\frac{\omega}{5\pi}\right)^2 e^{-0.15t}t^2 e^{-\left(\frac{\omega}{5\pi}\right)^2 t}, \quad t \geq 0, \quad -\infty < \omega < \infty . \tag{3.73}$$

This spectrum comprises some of the main characteristics of the seismic motion, such as decreasing of the dominant frequency with time. Realization records compatible with eq.(3.73) have been produced taking advantage of the concept of spectral representation of a stochastic process (e.g. Spanos and Zeldin, 1998; Shinozuka and Deodatis, 1991). In Figs.(3.6-3.8), the time evolution of the response variance under the non-separable process is plotted. Several values for the excitation level (S_2) are considered. Specifically, in Fig.(3.6), an oscillator possessing a piecewise linear stiffness is concerned. The value $(s = 2)$ has been used. In Figs.(3.7) and (2.8), a Duffing, $(\varepsilon = 1)$ and a hysteretic bilinear one, $(a = 0.02, b = 0.1)$, are examined, respectively. Again, the new method succeeds in capturing the average characteristics of the variance, while neglecting the oscillatory components.

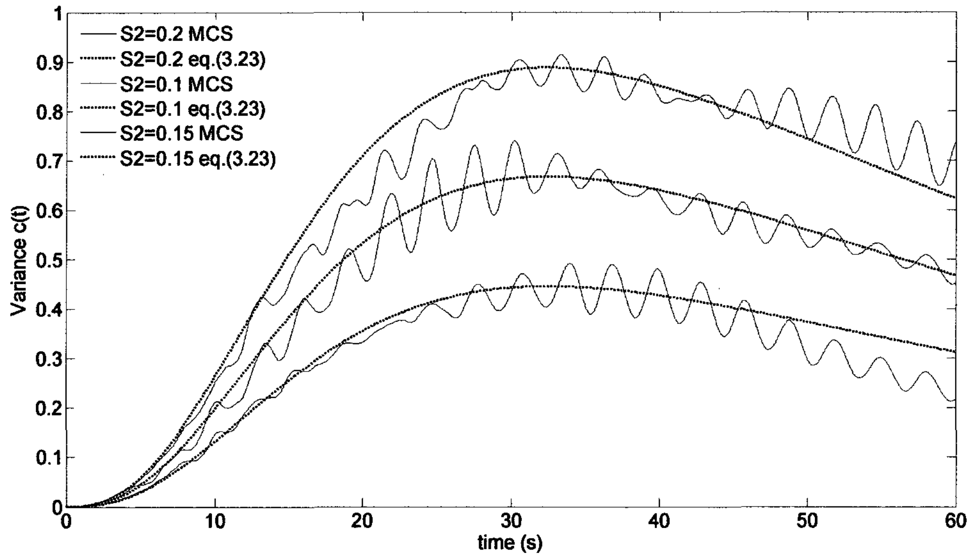

Fig.(3.6). Response Variance for an oscillator with Piecewise Linear Stiffness ($s = 2$) under Non-Separable excitation. Comparison between MCS data (500 realizations) and eq.(3.23)

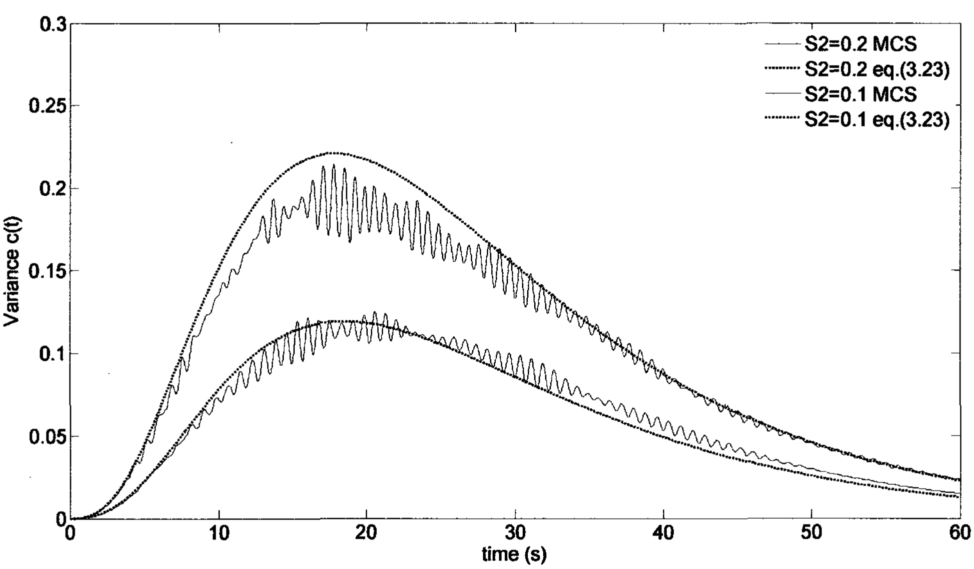

Fig.(3.7). Response Variance for a Duffing oscillator ($\varepsilon = 1$) under Non-Separable excitation. Comparison between MCS data (500 realizations) and eq.(3.23)

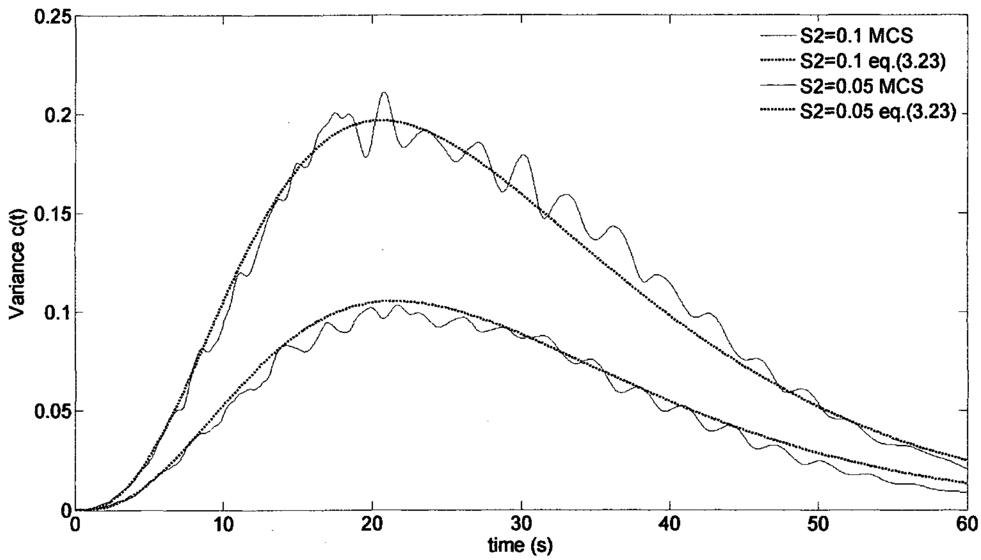

Fig.(3.8). Response Variance for a Bilinear oscillator $(a = 0.02, b = 0.1)$ under Non-Separable excitation. Comparison between MCS data (500 realizations) and eq.(3.23)

Chapter 4

First-Passage problem using a Galerkin approach

4.1 Preliminary remarks

In order to perform a reliability-based analysis of a mechanical or structural system, it is often desirable to estimate the probability that the response of the system reaches, and possibly crosses, a prescribed level for the first time. Clearly, the knowledge of such a probability would be beneficial to numerous practical applications, in terms of safety or risk assessment. This has led to considerable effort to address the aforementioned challenge, known as the first-passage problem.

Since this problem was first posed in the field of stochastic dynamics, several approximate solutions have been proposed with varying degree of success. An early approach by Coleman (1959) adopts the assumption that the level-crossings rate follows a Poisson distribution. In fact, this implies that the crossing events are independent. This is reasonable though only in the case where the level crossing is a rare event and quite unacceptable in the case of lightly damped systems as it is pointed out in Lin (1967). An advancement towards this direction (Vanmarcke, 1975) assumes a modified level-crossing rate which only asymptotically converges to the previous one as the level increases. Further, an approximate method for calculating the effect of clumping on the extreme response of lightly damped nonlinear systems may be found in Naess (1999).

Furthermore, since an analytical solution of the first-passage problem has not been possible, except for the case where the random phenomenon can be modeled as an one-dimensional Markov, or diffusion, process, efforts have been made to establish upper and lower bounds of the first-passage probability (e.g. Shinozuka, 1964). References to other improvements of this kind of approach can be found in Nigam (1983).

In general, several different approaches have been adopted over the past decades to encounter the problem. These range from the ones which include derivation of exact solutions (Kovaleva, 2009) or employ asymptotic analysis (Roy, 1997), to the more numerical ones (Sharp and Allen, 1998; Pichler and Pradlwarter, 2008). Moreover, since the first-passage problem lacks exact analytical solutions, one could argue that efficient implementations of the Monte Carlo method could yield reliable and applicable probability estimation procedures. Indeed, several attempts have focused on combining the basic idea of the Monte Carlo method with importance sampling procedures as in Au and Beck (2001, 2003) and Olsen and Naess (2007).

Recently, the path integral solution (PIS) technique, a numerical approach to approximately solve the Fokker-Planck (F-P) equation, has been used to derive reliability statistics (Cai and Lin, 1998; Iourtchenko et al., 2008). The basic characteristic of the approach is that the evolution of the probability density function is computed in short time steps.

Furthermore, Galerkin approaches have been proved to be a powerful tool, especially when utilizing the properties of orthogonal function bases. In fact, in Li and Ghanem (1998) first-passage statistics were computed using a polynomial chaos expansion in conjunction with a Galerkin projection scheme. In this regard, although Karhunen-Loeve (K-L) expansions often appear unattractive due to the computational cost of calculating K-L terms, wavelet bases can be used (Phoon et al., 2002) to enhance the conventional Galerkin approach to solve the Fredholm integral equation.

Undoubtedly, one of the most promising frameworks for bearing on the problem is associated with modeling the response as a one-dimensional Markov process. An extensive review on tackling the first–passage problem using diffusion methods can be found in Roberts (1986). Based on the assumption of pseudo-harmonic behavior of the response, an averaging procedure (Bogoliubov and Mitropolski, 1963; Stratonovich, 1963; Stratonovich, 1967) is employed to yield a first-order stochastic differential equation (Ito equation) governing the response amplitude. Related to the Ito equation is the backward Kolmogorov (B-K) partial differential equation.

In this chapter, the combination of the concepts of equivalent linearization (Roberts and Spanos, 2003; Proppe et al., 2003; Socha, 2005) and stochastic averaging (Lin, 1986; Roberts and Spanos, 1986; Zhu, 1988; Lin and Cai, 2000) yields a backward Kolmogorov (B-K) equation, having as frequency and damping elements the equivalent ones obtained by a linearization scheme. A general Galerkin mehodology is then applied to recover an approximate solution of the (B-K) equation.

In section 4.2 nonlinear oscillators subject to Gaussian white noise excitation are considered. In section 4.2.2 Markovian modeling of the energy envelope is also considered. The improved accuracy this alternative formulation offers is emphasized, especially in the case where stiffness nonlinearity is present. In section 4.3 the extension of the method to take into account evolutionary random loads is discussed. The accuracy of the proposed method is demonstrated through comparisons to Monte Carlo data for a number of nonlinear oscillators.

4.2 Nonlinear oscillators under Gaussian white noise excitation

4.2.1 Markovian modeling of the response amplitude envelope

In this section, the probability density function (PDF) of the first-passage time is determined by adopting the Galerkin approach used in Spanos et al. (2007) to obtain the non-stationary PDF of the response envelope. In this manner, a more general, improved framework is employed vis-à-vis a similar approach which has been used in Spanos (1982) to treat systems with damping nonlinearities.

4.2.1.1 Ito and backward Kolmogorov equations

Consider a nonlinear single degree of freedom system whose motion is governed by the differential equation

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega_0^2 x + \varepsilon f[x,\dot{x}] = w(t), \tag{4.1}$$

where ($f[x,\dot{x}]$) is an arbitrary nonlinear function which depends on the response displacement and velocity; (ζ_0) is the ratio of critical damping; and ($w(t)$) represents a Gaussian, zero-mean white noise random process possessing a power spectral density equal to (S_0).

Following an equivalent linearization approach as described in Roberts and Spanos (2003) and Goto and Iemura (1973), a linearized counterpart of eq.(4.1) is

$$\ddot{x}+2\omega_0\left[\zeta_0+\varepsilon\zeta_{eq}(\mathrm{A})\right]\dot{x}+\left[\omega_0^2+\varepsilon\omega_{eq}^2(\mathrm{A})\right]x=w(t), \tag{4.2}$$

where the equivalent damping element and natural frequency are assumed to be functions of the amplitude (A) of the response in order to partly account for the effect of the nonlinearity. Assuming the case of a lightly damped system, it can be argued that the amplitude (A) is a slowly varying function with respect to time and therefore can be treated as a constant over one cycle of oscillation. Thus, introducing the transformations

$$x(t)=A(t)\cos\left[\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}t+\phi(t)\right], \tag{4.3}$$

and

$$\dot{x}(t)=-\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}A(t)\sin\left[\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}t+\phi(t)\right], \tag{4.4}$$

and performing a mean square minimization procedure on the error between eq.(4.1) and eq.(4.2) the expressions

$$\omega_{eq}^2(A)=\frac{1}{\pi A}\int_0^{2\pi}\cos[\psi]f(A\cos\psi,-\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}A\sin\psi)d\psi \qquad (4.5)$$

and

$$2\omega_0\zeta_{eq}(A)=-\frac{1}{\pi A\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}}\int_0^{2\pi}\sin[\psi]f(A\cos\psi,-\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}A\sin\psi)d\psi \quad (4.6)$$

are derived for the equivalent damping and frequency elements. Moreover, substituting eqs.(4.3) and (4.4) into eq.(4.2), two coupled first-order stochastic differential equations are obtained governing the time evolution of the amplitude $(A(t))$ and the phase $(\theta(t))$. Nevertheless, relying on the assumption of light damping, modeling the amplitude process $(A(t))$ as a one-dimensional Markov process is feasible by applying a combination of deterministic and stochastic averaging (Bogoliubov and Mitropolski, 1963; Stratonovich, 1963; Stratonovich, 1967; Roberts and Spanos, 1986). This leads to decoupling from the phase the amplitude and to equation:

$$\dot{A}(t)=-\omega_0\left[\zeta_0+\varepsilon\zeta_{eq}(\mathrm{A})\right]A(t)+\frac{\pi S_0}{2A(t)\left(\omega_0^2+\varepsilon\omega_{eq}^2(A)\right)}-\frac{\left(\pi S_0\right)^{1/2}}{\sqrt{\omega_0^2+\varepsilon\omega_{eq}^2(A)}}\eta(t). \quad (4.7)$$

In eq.(4.7), $\eta(t)$ is a zero mean and delta correlated process of intensity one, i.e., $E(\eta(t))=0$ and $E(\eta(t)\eta(t+\tau))=\delta(\tau)$, with $(\delta(\tau))$ being the Dirac delta function.

Denoting by $(P(a,t))$ the probability that (A), starting from an initial value (a) never reaches the barrier level (B) during the time interval $[0,t]$, the following partial differential equation (B-K) associated with eq.(4.7) is satisfied

$$\frac{\partial P(a,t)}{\partial t}=-\left[\omega_0\left(\zeta_0+\varepsilon\zeta_{eq}(a)\right)a-\frac{\pi S_0}{2a\omega_n^2(a)}\right]\frac{\partial P(a,t)}{\partial a}+\left[\frac{\pi S_0}{2\omega_n^2(a)}\right]\frac{\partial^2 P(a,t)}{\partial a^2}, \qquad (4.8)$$

where

$$\omega_n^2(a)=\omega_0^2+\varepsilon\omega_{eq}^2(a). \qquad (4.9)$$

Taking into account the physical parameters of the problem, the following initial and boundary conditions are imposed:

$$P(a,0)=1, \qquad (4.10)$$

$$P(B,t)=0, \qquad (4.11)$$

and

$$P(0,t) = finite \tag{4.12}$$

4.2.1.2 Galerkin formulation

The derived backward Kolmogorov eq.(4.8) can be equivalently recast in the form

$$\frac{\partial P(a,t)}{\partial t} = L_{linear}[P(a,t)] + L_{nonlinear}\left[P(a,t)\right], \tag{4.13}$$

where

$$L_{linear}[.] = -\zeta_0\omega_0\left[a - \frac{\sigma_s^2}{a}\right]\frac{\partial[.]}{\partial a} + \zeta_0\omega_0\sigma_s^2\frac{\partial^2[.]}{\partial a^2}, \tag{4.14}$$

and

$$L_{nonlinear}[.] = -\left\{\zeta_0\omega_0\sigma_s^2\left[\frac{1}{a}\left(1 - \frac{\omega_0^2}{\omega_n^2(a)}\right)\right] + \varepsilon\omega_0\zeta_{eq}(a)\alpha\right\}\frac{\partial[.]}{\partial a} + \ldots$$
$$\zeta_0\omega_0\sigma_s^2\left\{\frac{\omega_0^2}{\omega_n^2(a)} - 1\right\}\frac{\partial^2[.]}{\partial a^2} \quad . \tag{4.15}$$

The rationale for this manipulation is that for $\varepsilon = 0$, eq.(3.8) takes the form

$$\frac{\partial P(a,t)}{\partial t} = L_{linear}[P(a,t)]. \tag{4.16}$$

In fact, eq.(4.16) along with the boundary conditions, eqs.(4.11-4.12), leads to a boundary value problem which can be recast as a Sturm-Liouville one (e.g. Spanos, 1982). As a result, the solution of eq.(4.16) is given in the form

$$P_{linear,B}(a,t) = \sum_{i=1}^{\infty} L_{i,B} \Phi_{i,B}[E, \lambda_{i,B}] e^{-2\omega_0 \zeta_0 \lambda_{i,B} t}, \tag{4.17}$$

where

$$E = \frac{1}{2}a^2. \tag{4.18}$$

In eq.(4.17), the variable $(\lambda_{i,B})$ denotes the i-th eigenvalue and the variable $(\Phi_{i,B}[E, \lambda_{i,B}])$ denotes the corresponding eigenfunction. Additional information regarding the form and the properties of these eigenfunctions can be found in Spanos (1980, 1982). In particular,

$$\Phi_{i,B}[E, \lambda_{i,B}] = M[-\lambda_{i,B}, 1, E], \tag{4.19}$$

where the symbol (M) denotes the confluent hypergeometric function. An important orthogonality condition can be derived based on its properties. That is,

$$\int_0^{\frac{1}{2}B^2} \Phi_{i,B}[E,\lambda_{i,B}]\Phi_{j,B}[E,\lambda_{j,B}]dE = 0, \quad i \neq j. \tag{4.20}$$

Relying on the structure of eq.(4.13), an approximate solution can be constructed resorting to a Galerkin approach. To this aim, the solution for the nonlinear oscillator (4.1) is sought in the form

$$P_B(a,t) = P_{linear,B}(a,t) + \sum_{r=1}^{N} c_r(t)\Phi_{r,B}(a), \tag{4.21}$$

where the second term on the right hand side of eq.(4.21) accounts for the deviation of $(P_B(a,t))$ to $(P_{linear,B}(a,t))$ due to the nonlinearity. The time-dependent functions $(c_r(t))$ are to be determined, and the integer (N) denotes the truncation limit of the series expansion. Assuming that the system is initially at rest, the use of eqs.(4.10) and (4.21) yields

$$P_B(a,t=0) = 1 = P_{linear,B}(a,t=0) + \sum_{r=1}^{N} c_r(t=0)\Phi_{r,B}(a), \tag{4.22}$$

which implies that

$$c_r(t=0)=0, \quad r=0,1,...,N. \tag{4.23}$$

Applying a similar approach as in Spanos et al. (2007), substituting eq.(4.21) into eq.(4.13) yields for the residual error

$$\begin{aligned} R\left[a,c(t)\right] &= \sum_{r=1}^{N} \dot{c}_r(t)\Phi_{r,\mathrm{B}}(a) - \sum_{r=1}^{N} c_r(t)L_{linear,B}\left[\Phi_{r,\mathrm{B}}(a)\right] - ... \\ &\sum_{r=1}^{N} L_r e^{-2\zeta_0\omega_0\lambda_{r,B}t} L_{nonlinear,B}\left[\Phi_{r,\mathrm{B}}(a)\right] - \sum_{r=1}^{N} c_r(t)L_{nonlinear,B}\left[\Phi_{r,\mathrm{B}}(a)\right] \end{aligned} . \tag{4.24}$$

To determine the unknown $(c(t))$, an appropriately selected set of functions is employed. According to the Galerkin scheme, the projection of the residual error on this set yields a set of ordinary linear differential equations for the functions $(c(t))$. In this manner, selecting $\left(\Phi_{B,k}(a)e^{-\frac{1}{2}a^2}a\right)$ as weighting functions, the Galerkin principle takes the form

$$\int_0^B R\left[a,c(t)\right]\Phi_{B,k}(a)e^{-\frac{1}{2}a^2}ada = 0, \quad k=1,2,...,N. \tag{4.25}$$

Taking into account the orthogonality conditions (eq.(4.20)) and manipulating eq.(4.25) yields the following linear system

$$\underline{\dot{c}}(t) = \underline{\underline{\Phi}}\underline{c}(t) + \underline{d}(t) \tag{4.26}$$

where the components of the vector $(\underline{c}(t))$ are defined by eq.(4.23); the vector $(\underline{d}(t))$ has the form

$$\underline{d}(t)=\left\{\begin{array}{c}\dfrac{\sum_{r=1}^{N} L_r e^{-2\omega_0\zeta_0\lambda_{r,B}t}\int_0^B \Phi_{1,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}aL_{nonlinear,B}\left[\Phi_{r,\mathrm{B}}(a)\right]da}{\int_0^B \Phi_{1,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}ada}\\ \vdots \\ \dfrac{\sum_{r=1}^{N} L_r e^{-2\omega_0\zeta_0\lambda_{r,B}t}\int_0^B \Phi_{N,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}aL_{nonlinear,B}\left[\Phi_{r,\mathrm{B}}(a)\right]da}{\int_0^B \Phi_{N,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}ada}\end{array}\right\}, \tag{4.27}$$

and the matrix $(\underline{\underline{\Phi}})$ is given by

$$\underline{\underline{\Phi}}=\begin{bmatrix}\dfrac{-2\zeta_0\omega_0\lambda_{1,\mathrm{B}}+\mathrm{K}_{11}}{\int_0^B \Phi_{1,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}ada} & \cdots & \dfrac{\mathrm{K}_{1\mathrm{N}}}{\int_0^B \Phi_{1,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}ada}\\ \vdots & \ddots & \vdots \\ \dfrac{\mathrm{K}_{\mathrm{N}1}}{\int_0^B \Phi_{N,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}ada} & \cdots & \dfrac{-2\zeta_0\omega_0\lambda_{\mathrm{N,B}}+\mathrm{K}_{\mathrm{NN}}}{\int_0^B \Phi_{N,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}ada}\end{bmatrix}, \tag{4.28}$$

where

$$K_{ij}=\int_0^B \Phi_{i,\mathrm{B}}(\alpha)e^{-\frac{1}{2}a^2}aL_{nonlinear,B}\left[\Phi_{j,\mathrm{B}}(a)\right]da. \tag{4.29}$$

Note that in deriving eq.(4.28), the dimensionless variable

$$A = \frac{A}{\sigma_s}, \quad \sigma_s = \sqrt{\frac{\pi S_0}{2\zeta_0 \omega_0^3}} \tag{4.30}$$

has been introduced where (σ_x) represents the stationary standard deviation of the linear oscillator. Moreover, the following relationship

$$L_{linear,B}\left[\Phi_{i,\mathrm{B}}(a)\right] = -2\zeta_0\omega_0\lambda_{i,\mathrm{B}} \tag{4.31}$$

between the eigenfunctions and the eigenvalues has been taken into account. Having determined $(P_B(a,t))$ using eq.(4.21), the corresponding PDF for the first-passage time is obtained using the equation

$$p_B(a,t) = -\frac{dP_B(a,t)}{dt}. \tag{4.32}$$

4.2.1.3 Van Der Pol oscillator application

In this section, the preceding procedure is applied to a Van Der Pol oscillator whose equation of motion is given by the equation

$$\ddot{x} + 2\zeta_0\omega_0\left(-1 + \varepsilon x^2\right)\dot{x} + \omega_0^2 x = w(t). \tag{4.33}$$

Taking into consideration the transformation of eq.(4.26), straightforward application of eqs.(4.5) and (4.6) yields

$$\omega_{eq}(a) = 0\,, \tag{4.34}$$

and

$$\zeta_{eq}(a) = \zeta_0\left(-\frac{2}{\varepsilon}+\frac{a^2}{4}\right). \tag{4.35}$$

Furthermore, eq.(4.15) yields

$$L_{nonlinear}[\cdot] = -\left\{\varepsilon\omega_0\zeta_0\left(-\frac{2}{\varepsilon}+\frac{a^2}{4}\right)\alpha\right\}\frac{\partial[\cdot]}{\partial a}. \tag{4.36}$$

To assess the accuracy of the proposed procedure, a Van Der Pol oscillator possessing the following parameters is considered: $(\mathrm{B}=1, \zeta_0=0.01, \sigma_s=1, S_0=0.3, \varepsilon=3)$. In Fig.(4.1) the evolution in time of the series coefficients $(C_i(t),\;\; i=1,\ldots,5)$ is plotted. It can be seen that the influence of the terms in the series expansion becomes less dominant as the order of the terms increases. This is shown in Fig.(4.2) where the corresponding PDF of the Van Der Pol oscillator is evaluated. Direct comparison to Monte Carlo simulation (MCS) data shows a quite good agreement even for a small number of terms.

Specifically, despite the large value of the nonlinearity, 9 terms are enough to achieve a good agreement with MCS data. In fact, little improvement is obtained for a larger number of terms.

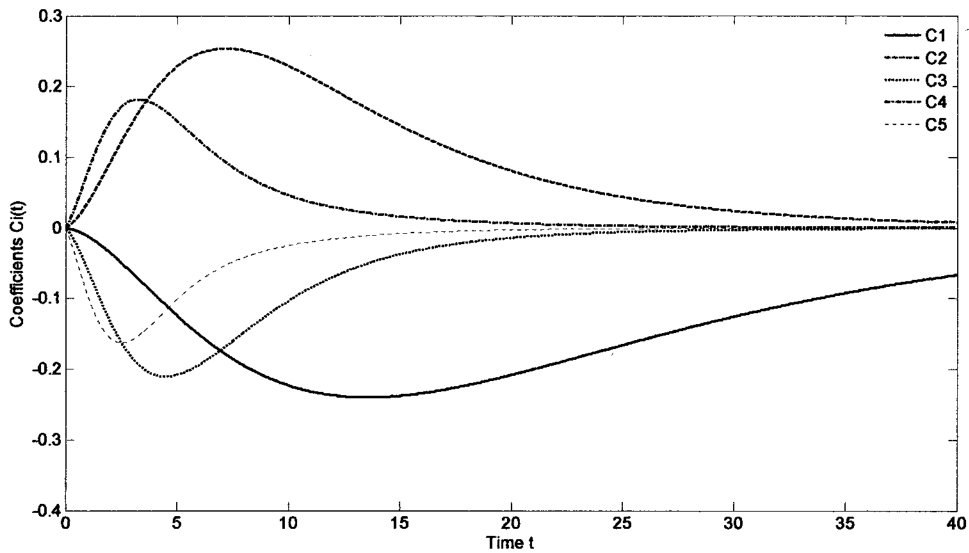

Fig.(4.1). Time-dependent series coefficients $\left(C_i(t),\quad i=1,\ldots,5\right)$ for a Van Der Pol oscillator $\left(\varepsilon=3\right)$

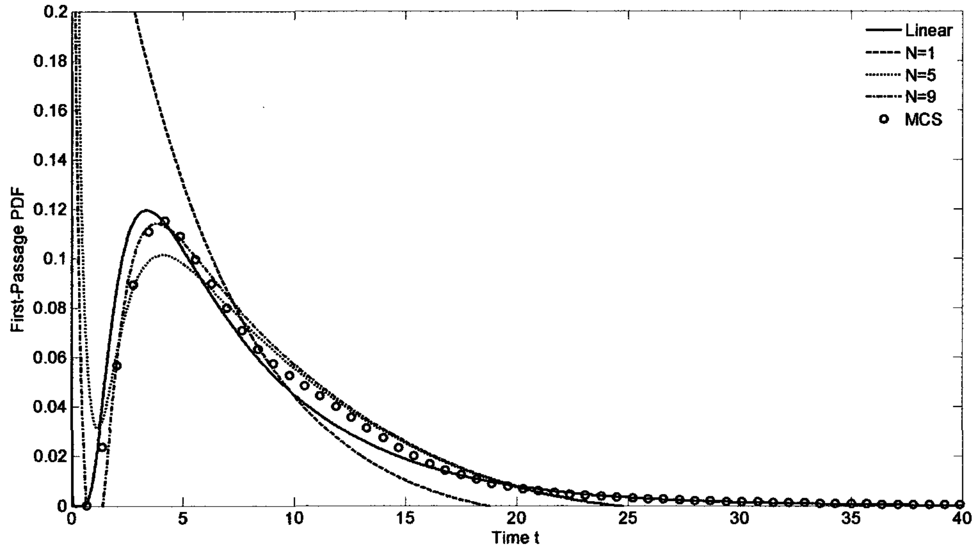

Fig.(4.2) First-Passage PDF for a Van Der Pol oscillator $(\varepsilon = 3)$. Comparison between MCS data (5000 realizations) and eq.(4.32)

Examining Fig.(4.2) it is noted that for small values of the time variable the theoretical data considerably deviate from the corresponding numerical simulations. It is obvious that the first-passage probability density of the oscillator is equal to zero for $t = 0$. However, this requirement necessitates the use of an infinite number of terms in the expansion in eq.(4.17) and therefore in eq.(4.21). The deviation observed in eq.(4.21) deteriorates in eq.(4.32) since differentiation takes place. However, there is no need to produce a smooth approximation at the vicinity of zero time, since the primary interest is directed to situations where the probability of first-passage time is higher than zero.

4.2.1.4 Duffing oscillator $(\varepsilon > 0)$ application

The case of a Duffing oscillator is considered whose motion is described by the equation

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega_0^2\left(1+\varepsilon x^2\right)x = w(t), \quad \varepsilon > 0. \tag{4.37}$$

Taking into consideration eqs.(4.5) and (4.6) yields

$$\omega_{eq}^2(a) = \frac{3}{4}\omega_0^2\alpha^2, \tag{4.38}$$

and

$$\zeta_{eq}(a) = 0. \tag{4.39}$$

Moreover, eq.(4.15) gives

$$L_{nonlinear}[.] = -\left\{\zeta_0\omega_0\left[\frac{1}{a}\left(1-\frac{\omega_0^2}{\omega_n^2(a)}\right)\right]\right\}\frac{\partial[.]}{\partial a} + \zeta_0\omega_0\left\{\frac{\omega_0^2}{\omega_n^2(a)}-1\right\}\frac{\partial^2[.]}{\partial a^2}. \tag{4.40}$$

The preceding formulation is applied to a Duffing oscillator possessing the parameter values: $(\mathrm{B}=1, \zeta_0 = 0.01, \sigma_s = 1, S_0 = 0.3, \varepsilon = 0.5)$. In Fig.(4.3) the time evolution of the series coefficients is shown, whereas in Fig.(4.4) the corresponding PDF for the first passage is shown. A second value for the nonlinearity parameter is also chosen $(\varepsilon = 1.0)$. The results are shown in Figs.(4.5) and (4.6). It can be readily seen that more accurate estimation is retrieved for the lower value of the nonlinearity, as expected. However, slight improvement is observed for values of $(N > 15)$. It should be mentioned that in case of high nonlinearity degree the approach is unavoidably affected by the approximations involved in the stochastic averaging procedure in both the derivation of the one dimensional Ito equation and the linearization of the system.

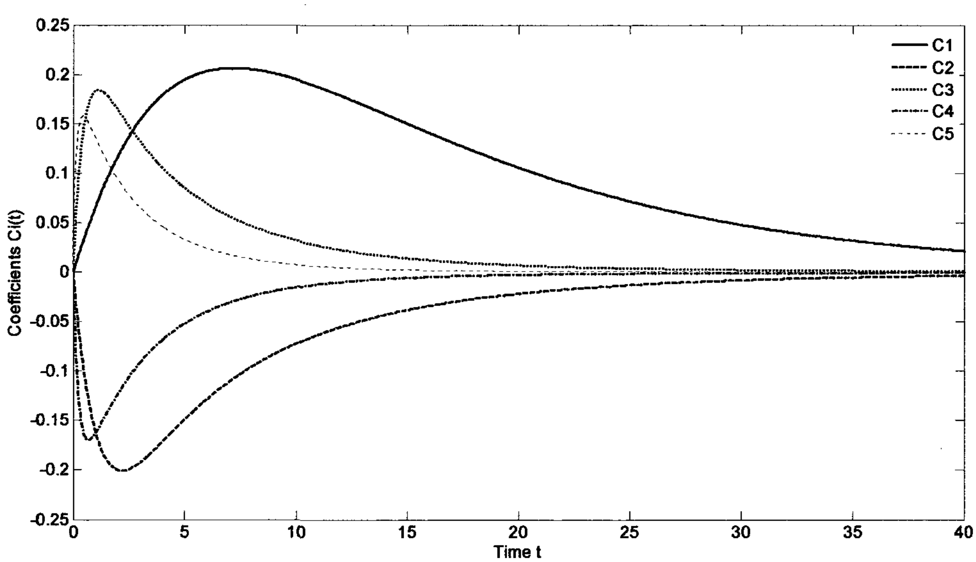

Fig.(4.3). Time-dependent series coefficients $(C_i(t),\ i = 1,\ldots,5)$ for a Duffing oscillator $(\varepsilon = 0.5)$

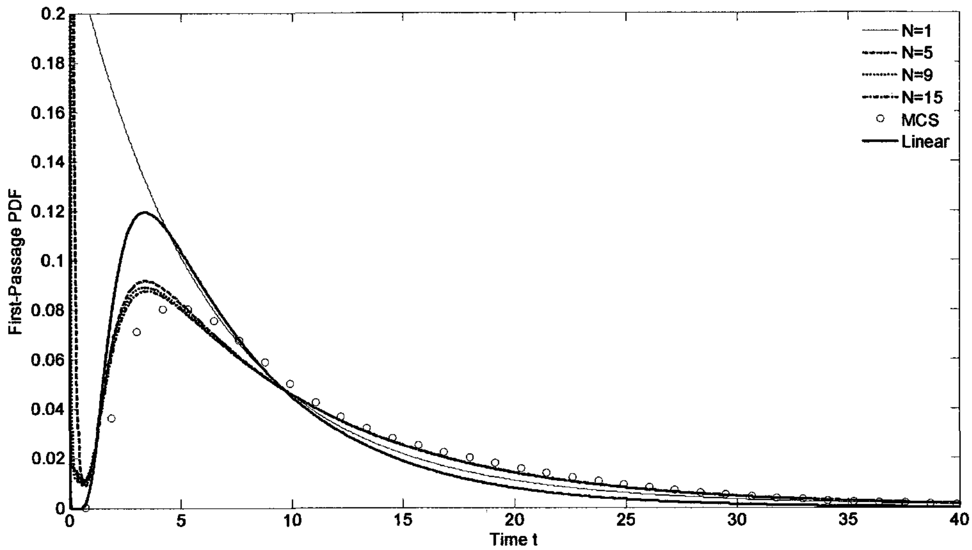

Fig.(4.4). First-Passage PDF for a Duffing oscillator $(\varepsilon = 0.5)$. Comparison between MCS data (5000 realizations) and eq.(4.32)

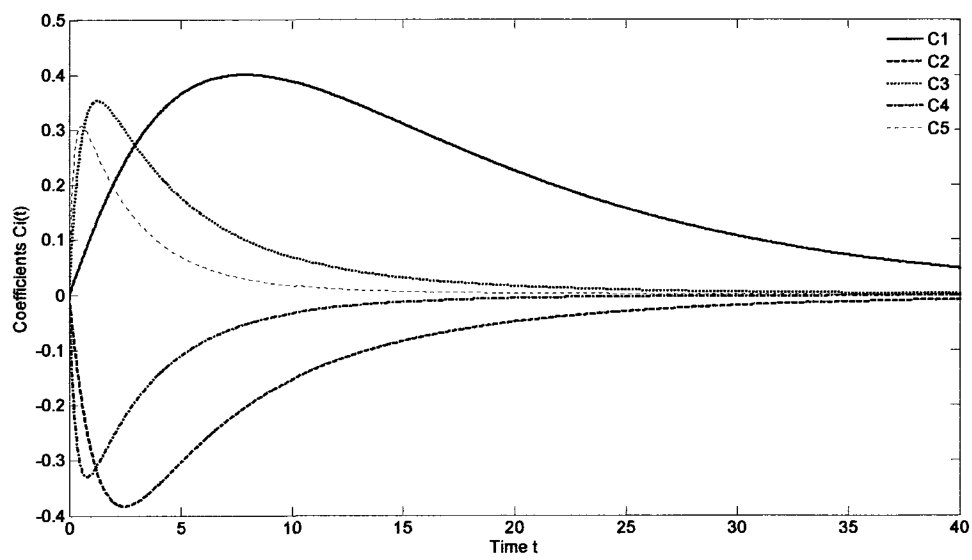

Fig.(4.5). Time-dependent series coefficients $(C_i(t),\ \ i = 1,\ldots,5)$ for a Duffing oscillator $(\varepsilon = 1.0)$

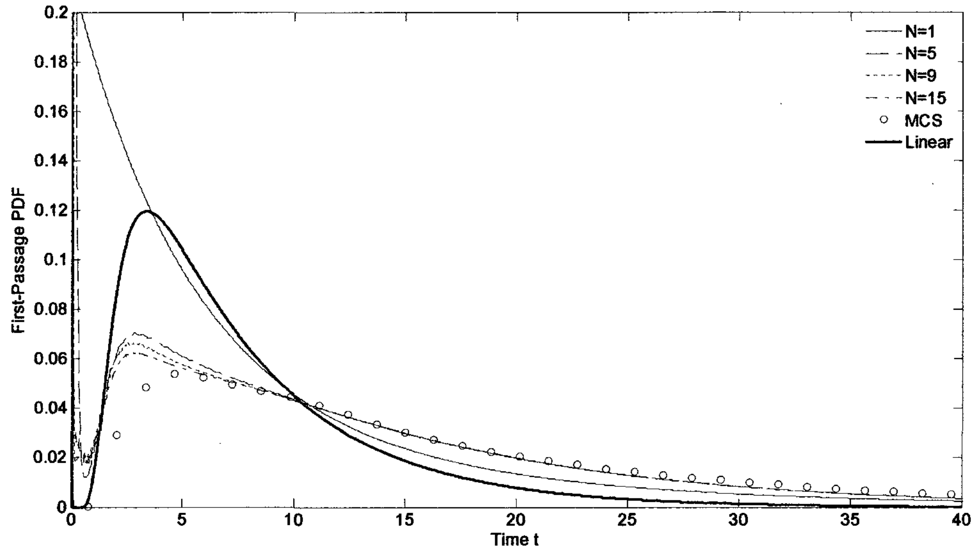

Fig.(4.6). First-Passage PDF for a Duffing oscillator $(\varepsilon = 1.0)$. Comparison between MCS data (5000 realizations) and eq.(3.32)

4.2.1.5 Softening Duffing oscillator $(\varepsilon < 0)$ application

The case of a softening Duffing oscillator is next concerned whose equation of motion is described by

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + \omega_0^2\left(1 + \varepsilon x^2\right)x = w(t), \quad \varepsilon < 0. \tag{4.41}$$

This kind of nonlinearity is associated with instability issues and therefore it has been treated as a special case in the literature (e.g. Roberts, 1986). In order to apply the aforementioned methodology to a softening Duffing oscillator, singularities which appear in eq.(4.40) should be taken into consideration.

Examining eqs.(4.9), (4.38) and (4.40) it is readily seen that the condition which must be satisfied is the following:

$$a < \sqrt{-\frac{4}{3\varepsilon}}, \quad \varepsilon < 0. \tag{4.42}$$

Interpreting eq.(4.42), the prescribed barrier level should not be greater than the amplitude level at which the amplitude-dependent equivalent natural frequency of the oscillator reaches the zero value. In other words, for a chosen barrier level value the oscillator should possess positive stiffness. Under these circumstances, the Galerkin scheme is applied to a softening Duffing oscillator under Gaussian white noise excitation possessing the following parameter values: $\left(\mathrm{B}=1, \zeta_0 = 0.01, \sigma_s = 1, S_0 = 0.3, \varepsilon = -0.5\right)$. In Fig.(4.7) the time evolution of the series coefficients is plotted, whereas in Fig.(4.8) the corresponding PDF for the first-passage time is plotted. A second value for the nonlinearity parameter is also chosen $(\varepsilon = -1.0)$. The high degree of nonlinearity can also be deduced by comparing the PDF which corresponds to the linear oscillator to the one corresponding to the nonlinear one. The results are shown in Figs.(4.9) and (4.10).

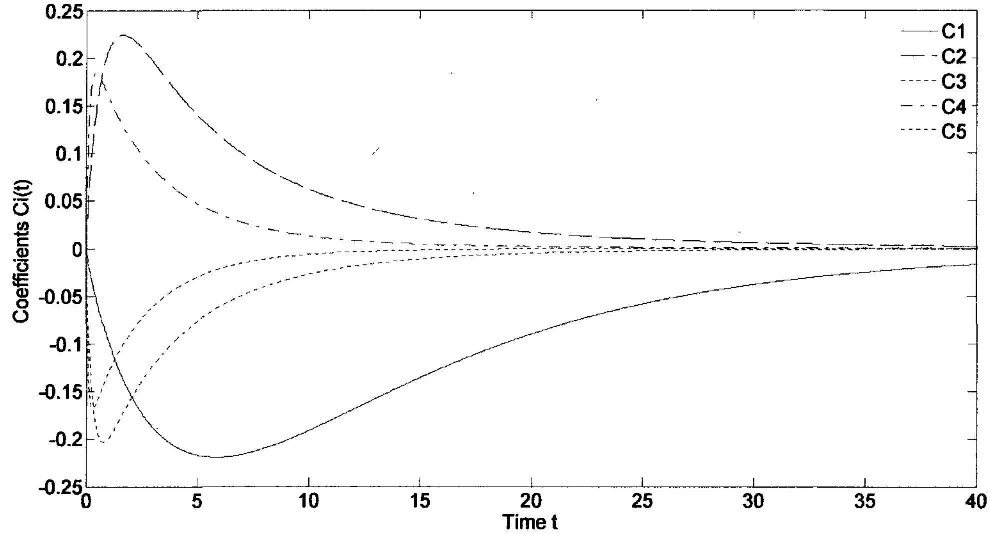

Fig.(4.7). Time-dependent series coefficients $(C_i(t),\quad i=1,...,5)$ for a softening Duffing oscillator $(\varepsilon=-0.5)$

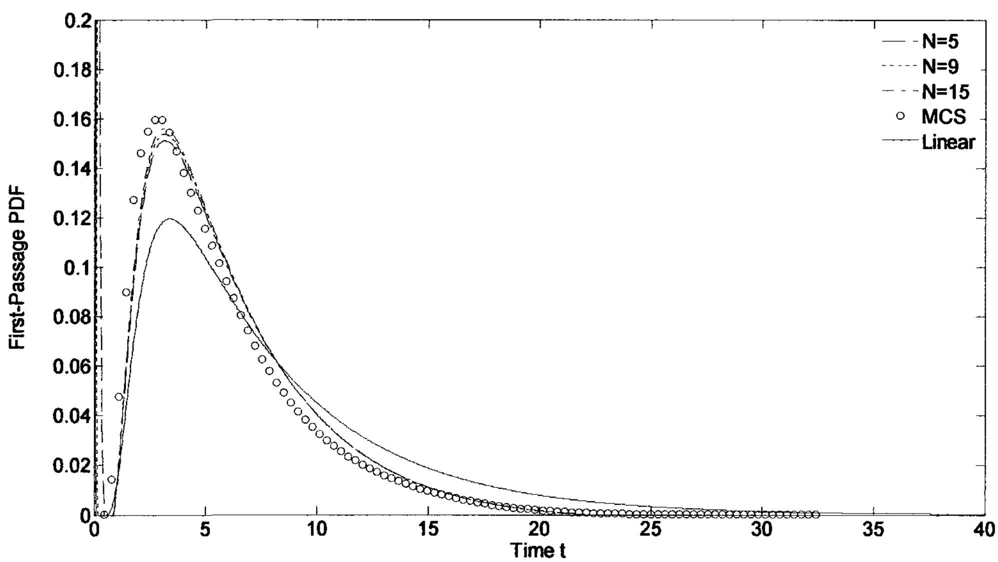

Fig.(4.8). First-Passage PDF for a softening Duffing oscillator $(\varepsilon=-0.5)$. Comparison between MCS data (5000 realizations) and eq.(4.32)

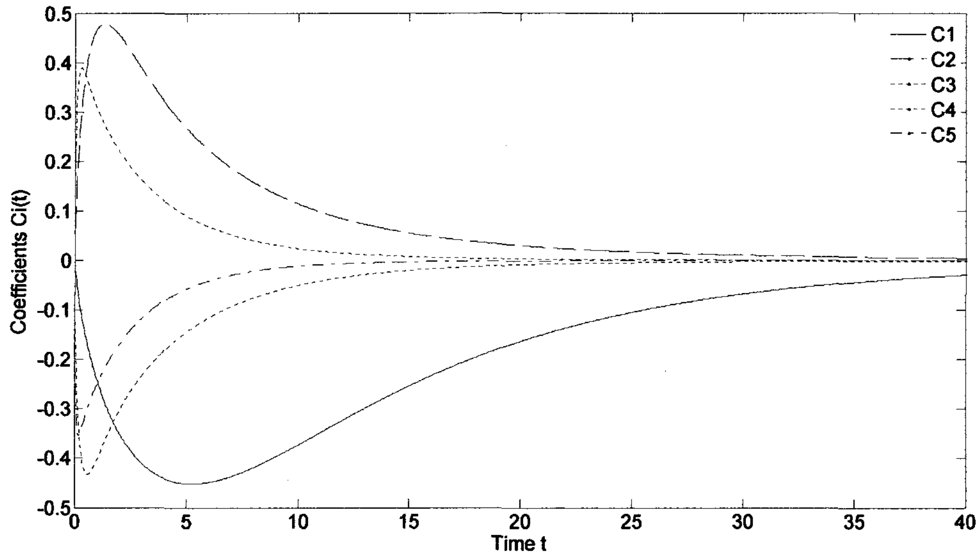

Fig.(4.9). Time-dependent series coefficients $\left(C_i(t),\quad i=1,\ldots,5\right)$ for a softening Duffing oscillator $(\varepsilon=-1.0)$

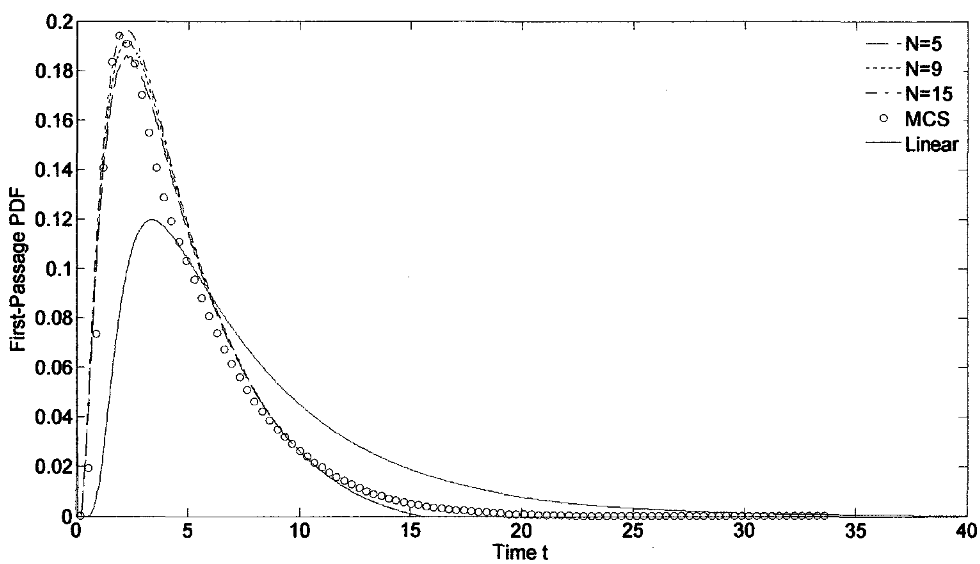

Fig.(4.10). First-Passage PDF for a softening Duffing oscillator $(\varepsilon=-1.0)$. Comparison between MCS data (5000 realizations) and eq.(4.32)

4.2.2 Markovian modeling of the response energy envelope

In order to circumvent the approximations that are inherent in the stochastic averaging procedure when applied to the case of the Duffing oscillator, an alternative generalized averaging scheme is performed according to Red-Horse and Spanos (1992).

4.2.2.1 Ito and backward Kolmogorov equations

Consider the class of non-linear oscillators described by the equation

$$\ddot{x} + 2\zeta_0\omega_0\dot{x} + g(x) = w(t), \tag{4.43}$$

where $(g(x))$ represents the non-linear stiffness of the system; ($w(t)$) represents a Gaussian, zero-mean white noise random process possessing a power spectral density equal to (S_0). Then, defining the potential energy of the oscillator as

$$u(x) = \int_0^x g(\lambda)d\lambda, \tag{4.44}$$

and considering the transformations

$$\dot{x} = -\sqrt{2V}\sin\phi, \tag{4.45}$$

and

$$u(x) = V\cos^2\phi\,, \tag{4.46}$$

eqs.(4.45) and (4.46) can be combined to yield

$$V = \frac{\dot{x}^2}{2} + u(x)\,, \tag{4.47}$$

and

$$\phi = -\tan^{-1}\left[\frac{\dot{x}}{\sqrt{2u(x)}}\right]. \tag{4.48}$$

where (V) represents the total energy envelope of the system. In the case of a linear oscillator the stochastic averaging procedure leads to decoupling the energy envelope from the variable (ϕ) which results in the following Ito equation for the variable (V)

$$\dot{V}(t) = 2\zeta_0\omega_0\left[\sigma_s^2\omega_0^2 - V\right] + \left(2\sigma_s\omega_0\sqrt{\zeta_0\omega_0 V}\right)\eta(t) \tag{4.49}$$

In eq.(4.49), $\eta(t)$ is a zero mean and delta correlated process of intensity one, i.e., $E\left(\eta(t)\right) = 0$ and $E\left(\eta(t)\eta(t+\tau)\right) = \delta(\tau)$, with $(\delta(\tau))$ being the Dirac delta function. The associated backward Kolmogorov equation has the form

$$\frac{\partial P(v,t)}{\partial t} = 2\zeta_0\omega_0\left[\sigma_s^2\omega_0^2 - v\right]\frac{\partial P(v,t)}{\partial v} + 2\sigma_s^2\zeta_0\omega_0^3 v\frac{\partial^2 P(v,t)}{\partial v^2}. \tag{4.50}$$

Following the procedure described in Spanos (1982) and defining the variable

$$E = \frac{v}{\omega_0^2}, \tag{4.51}$$

eq.(4.50) together with the initial and boundary conditions (eqs.(4.10-4.12)) leads to the following boundary value problem:

$$E\frac{d^2\Phi(E)}{dE^2} + (1-E)\frac{d\Phi(E)}{dE} + \lambda\Phi(E) = 0, \tag{4.52}$$

$$\Phi(0) = \mathit{finite}, \tag{4.53}$$

$$\Phi\left(\frac{B}{\omega_0^2}\right) = 0, \tag{4.54}$$

where it has been assumed that $\left(\sigma_s^2 = 1\right)$. This problem can be recast as a Sturm-Liouville one (e.g. Spanos, 1982). As a result, the solution of eq.(4.50) is given in the form

$$P_{linear,B}(v,t) = \sum_{i=1}^{\infty} L_{i,B} \Phi_{i,B}[E, \lambda_{i,B}] e^{-2\omega_0 \zeta_0 \lambda_{i,B} t}, \tag{4.55}$$

where in this case

$$L_{i,B} = \frac{\int_0^B \Phi_{i,B}(v, \lambda_{i,B}) e^{-\frac{v}{\omega_0^2}} \left(\frac{1}{\omega_0^2} \right) dv}{\int_0^B \Phi_{i,B}^2(v, \lambda_{i,B}) e^{-\frac{v}{\omega_0^2}} \left(\frac{1}{\omega_0^2} \right) dv}. \tag{4.56}$$

4.2.2.2 Galerkin formulation for a Duffing oscillator

In the case of a Duffing oscillator, that is

$$g(x) = \omega_0^2 \left(x + \varepsilon x^3 \right), \tag{4.57}$$

the corresponding backward Kolmogorov equation takes the form

$$\frac{\partial P(v,t)}{\partial t} = L_{linear}[P(v,t)] + L_{nonlinear}\left[P(v,t) \right], \tag{4.58}$$

where

$$L_{linear}[.] = 2\zeta_0 \omega_0 \left[\omega_0^2 - v \right] \frac{\partial [.]}{\partial v} + 2\zeta_0 \omega_0^3 v \frac{\partial^2 [.]}{\partial v^2}, \tag{4.59}$$

and

$$L_{nonlinear}[.] = \left\{2\zeta_0\omega_0\left[-v\Psi(v) - \omega_0^2 + v\right] + \pi S_0\right\}\frac{\partial[.]}{\partial v} + \ldots \\ 2\zeta_0\omega_0^3 v\left(\Psi(v) - 1\right)\frac{\partial^2[.]}{\partial v^2}, \tag{4.60}$$

and

$$\Psi(v) = \frac{4r}{3m^2}\left|(1+r) - 2\frac{E\left(\sqrt{\frac{(r-1)}{2r}}\right)}{K\left(\sqrt{\frac{(r-1)}{2r}}\right)}\right|, \tag{4.61}$$

with

$$r = m^2 + 1 \tag{4.62}$$

and

$$m^2 = \frac{4v\varepsilon}{\omega_0^2}. \tag{4.63}$$

The operator $\left(\mathrm{E}(.)\right)$ denotes the complete elliptic integral of the second kind and the operator $\left(\mathrm{K}(.)\right)$ denotes the complete elliptic integral of the first kind. Further details about the definition and the properties of the function $(\Psi(v))$ can be found in Red-Horse and Spanos (1992).

In this case the orthogonality condition derived based on the properties of the confluent hypergeometric function takes the following form

$$\int_0^{\frac{B}{\omega_0^2}} \Phi_{i,B}[E,\lambda_{i,B}]\Phi_{j,B}[E,\lambda_{j,B}]dE = 0, \quad i \neq j. \tag{4.64}$$

Relying once again on the Galerkin scheme, the solution for the nonlinear oscillator will be of the form

$$P_B(v,t) = P_{linear,B}(v,t) + \sum_{r=1}^{N} c_r(t)\Phi_{r,B}(v) \tag{4.65}$$

Following a similar procedure as in the case of the Markovian modeling of the response envelope and selecting $\left(\Phi_{B,k}(v)e^{-\frac{v}{\omega_0^2}}\frac{1}{\omega_0^2}\right)$ as weighting functions, the Galerkin approach takes the form

$$\int_0^B R\left[v,c(t)\right]\Phi_{B,k}(v)e^{-\frac{v}{\omega_0^2}}\frac{1}{\omega_0^2}dv = 0, \quad k = 1,2,...,N. \tag{4.66}$$

Taking into account the orthogonality conditions (eq.(4.64)) and manipulating eq.(4.66) yields a linear system equivalent to eq.(4.26).

This alternative formulation is applied to a Duffing oscillator possessing the following parameter values: $\left(\mathrm{B}=1, \zeta_0=0.01, \sigma_s=1, \omega_0^2=3.5, \varepsilon=3.0\right)$. In Fig.(4.11) the time evolution of the series coefficients is plotted, whereas in Fig.(4.12) the corresponding PDF for the first passage is plotted. It can be readily seen that a more accurate estimation is retrieved in comparison to the classical approach, despite the high value of the nonlinearity. This higher accuracy justifies the choice of the energy envelope formulation in cases where the nonlinearity appears in terms of stiffness.

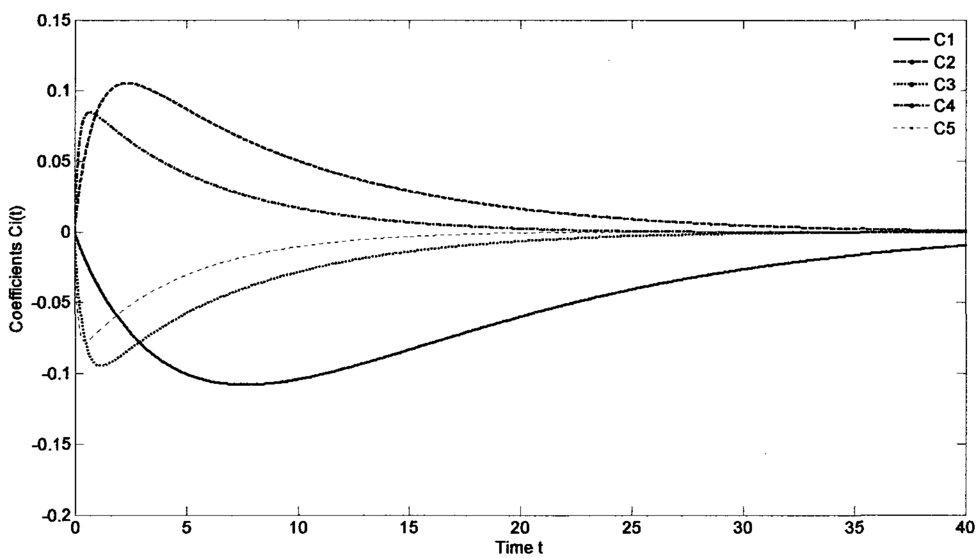

Fig.(4.11). Time-dependent series coefficients $\left(C_i(t), \quad i=1,\ldots,5\right)$ for a Duffing oscillator $(\varepsilon=3.0)$ (Energy Envelope Modeling)

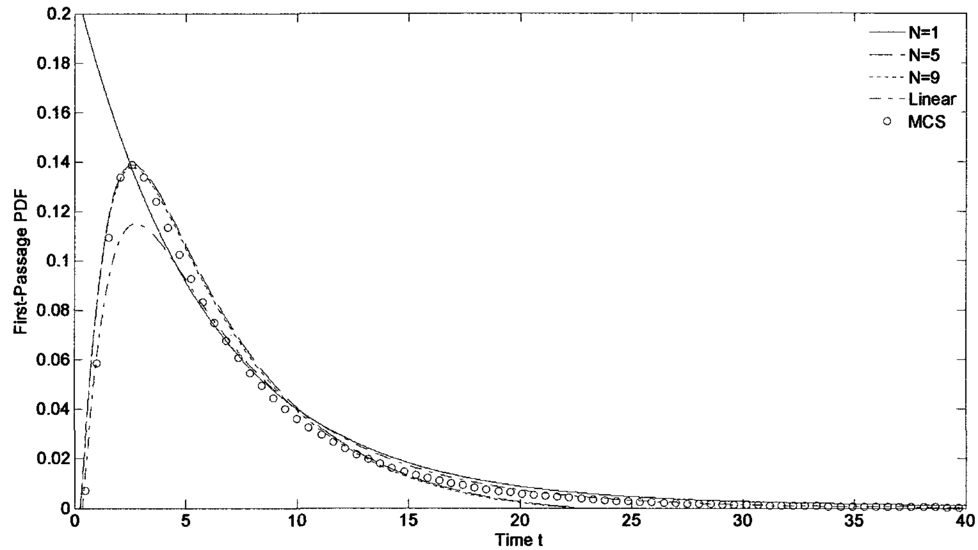

Fig.(4.12). First-Passage PDF for a Duffing oscillator $(\varepsilon = 3.0)$ (Energy Envelope Modeling). Comparison between MCS data (5000 realizations) and eq.(4.32)

4.3 Nonlinear oscillators under evolutionary excitation

4.3.1 Markovian modeling of the response amplitude envelope

Consider a nonlinear single degree of freedom system whose motion is governed by the differential equation

$$\ddot{x} + \beta\dot{x} + z(t,x,\dot{x}) = w(t), \tag{4.67}$$

where a dot over a variable denotes differentiation with respect to time (t); ($z(t,x,\dot{x})$) is the restoring force which could be either hysteretic or depend only on the instantaneous values of (x) and ($\dot{x}$); (β) is a linear damping coefficient;

and ($w(t)$) represents a Gaussian, zero-mean non-stationary random process possessing an evolutionary broad-band power spectrum, $S(\omega,t)$.

Then, following the approach proposed in chapter 3 the equivalent linearized counterpart of eq.(4.67) has the form

$$\ddot{x}+\beta_{eq}(t)\dot{x}+\omega_{eq}^2(t)x=w(t), \tag{4.68}$$

where $\left(\beta_{eq}(t)\right)$ and $\left(\omega_{eq}^2(t)\right)$ are given by eqs.(3.12) and (3.13) respectively. As a result, the corresponding to eq.(4.68) Ito equation is

$$\dot{A}(t)=-\frac{1}{2}\beta_{eq}(t)A(t)+\frac{\pi S(\omega_{eq}(t),t)}{2A(t)\omega_{eq}^2(t)}+\frac{\left(\pi S(\omega_{eq}(t),t)\right)^{1/2}}{\omega_{eq}(t)}\eta(t). \tag{4.69}$$

In eq.(4.69), $\eta(t)$ is a zero mean and delta correlated process of intensity one, i.e., $E\left(\eta(t)\right)=0$ and $E\left(\eta(t)\eta(t+\tau)\right)=\delta(\tau)$, with ($\delta(\tau)$) being the Dirac delta function. The associated backward Kolmogorov partial differential equation for the reliability function takes the form

$$\frac{\partial P}{\partial t}=-\frac{1}{2}\beta_{eq}(t)\left[a-R(t)\frac{1}{a}\right]\frac{\partial P}{\partial a}+\frac{1}{2}\beta_{eq}(t)R(t)\frac{\partial^2 P}{\partial a^2}, \tag{4.70}$$

where

$$R(t)=\frac{\pi S(\omega_{eq}(t),t)}{\beta_{eq}(t)\omega_{eq}^{2}(t)}. \tag{4.71}$$

4.3.2 Galerkin formulation

It is now possible to apply a Galerkin type scheme to solve eq.(4.70). Defining the variable (E) so that $E=\frac{1}{2}a^2$, eq.(4.70) takes the form

$$\begin{aligned}&\frac{1}{\beta_{eq}(t)R(t)}\frac{\partial P}{\partial t}=\left[1-\frac{E}{R(t)}\right]\frac{\partial P}{\partial E}+E\frac{\partial^2 P}{\partial E^2}\\ &\Leftrightarrow \\ &\frac{1}{\beta_{eq}(t)R(t)}\frac{\partial P}{\partial t}=[1-E]\frac{\partial P}{\partial E}+\left[1-\frac{1}{R(t)}\right]E\frac{\partial P}{\partial E}+E\frac{\partial^2 P}{\partial E^2}\end{aligned}. \tag{4.72}$$

Furthermore, observing that the confluent hypergeometric function satisfies the equation

$$[1-E]\frac{dM}{dE}+E\frac{d^2M}{dE^2}=-\lambda M, \tag{4.73}$$

and that the eigenfunctions M_i satisfy the orthogonality condition

$$\int_0^{\frac{B^2}{2}} M_i M_j e^{-E}dE=0, i\neq j \tag{4.74}$$

a solution is sought in the form

$$P=\sum_{i=1}^{\infty}M_i(E)T_i(t) \tag{4.75}$$

Substituting (4.75) into (4.72) and taking into account (4.73) and (4.74) yields

$$\dot{T}_j=-\beta_{eq}(t)R(t)\lambda_j T_j+\beta_{eq}(t)\left[R(t)-1\right]\sum_{i=1}^{\infty}T_i v_{ji}\ , \tag{4.76}$$

where

$$v_{ji}=\frac{\int_0^{\frac{B^2}{2}}E\frac{dM_i}{dE}M_j e^{-E}dE}{\int_0^{\frac{B^2}{2}}M_j^{\,2}e^{-E}dE}. \tag{4.77}$$

Taking into account that $\left(P(E,0)=1\right)$, the initial conditions in order to solve eq.(4.77) are

$$T_j(0)=\left(\int_0^{\frac{B^2}{2}}M_j e^{-E}dE\right)/\left(\int_0^{\frac{B^2}{2}}M_j^{\,2}e^{-E}dE\right). \tag{4.78}$$

4.3.3 Duffing oscillator application

Applying the aforementioned scheme to a Duffing oscillator $\left(\mathrm{B}=1,\zeta_0=0.01,\sigma_s=1,S_0=0.3,\varepsilon=0.5\right)$ the first-passage PDF is derived

(Fig.(12)). The agreement to Monte Carlo simulations is quite satisfactory. However, the need for fairly broad-band excitation spectra should be mentioned. The excitation spectrum is a time-modulated Gaussian white noise one of the form

$$S(\omega,t) = |g(t)|^2 S(\omega), \tag{4.79}$$

where

$$S(\omega) = S_0, \tag{4.80}$$

and

$$g(t) = k(e^{-at} - e^{-bt}), \tag{4.81}$$

in which $a = 0.05$; $b = 0.5$; k is a normalization constant so that $g_{\max} = 1$. In Fig.(4.13) the time-dependent variance $(V(t))$ is shown, whereas in Fig.(4.14) the time evolution of the series coefficients is plotted.

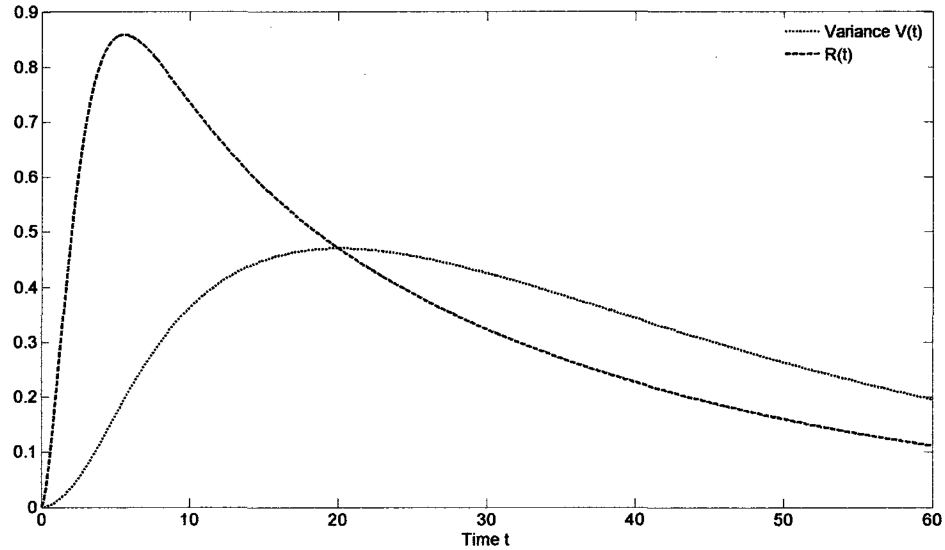

Fig.(4.13). Time-dependent functions $(V(t))$ and $(R(t))$ for a Duffing oscillator $(\varepsilon = 0.5)$ under modulated Gaussian white noise.

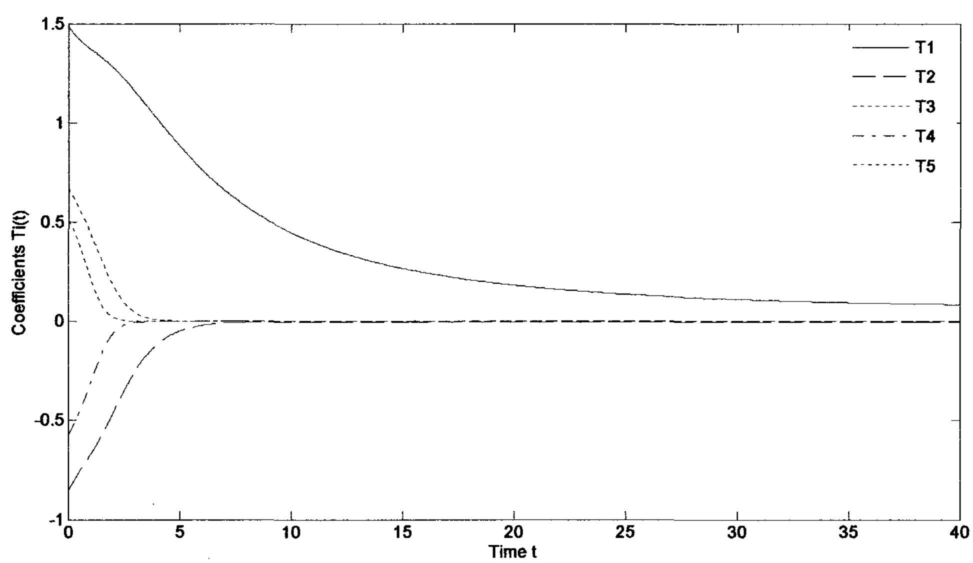

Fig.(4.14). Time-dependent series coefficients $(T_i(t),\ \ i = 1,\ldots,5)$ for a Duffing oscillator $(\varepsilon = 0.5)$ under modulated Gaussian white noise.

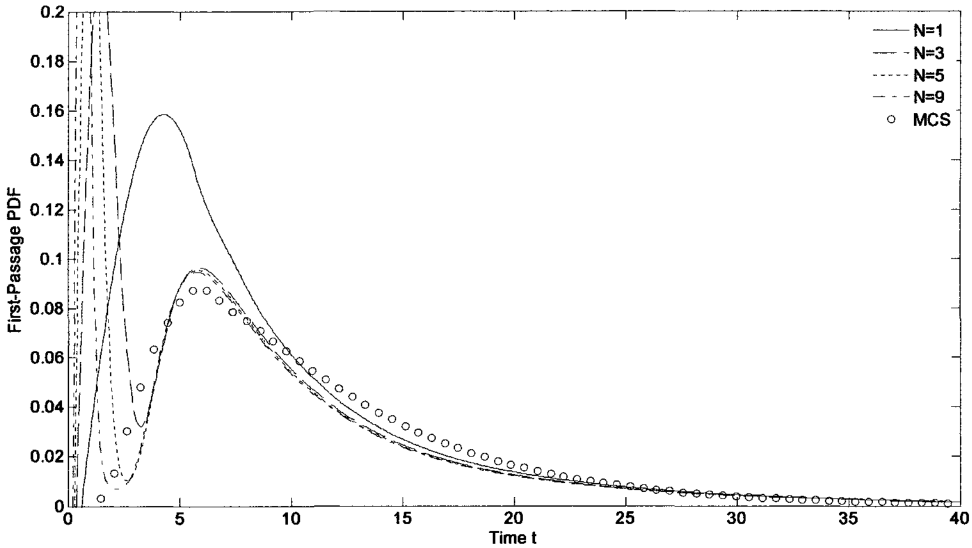

Fig.(4.15). First-Passage PDF for a Duffing oscillator $(\varepsilon = 0.5)$ under modulated Gaussian white noise. Comparison between MCS data (5000 realizations) and eq.(4.32).

Chapter 5

Concluding remarks

In this chapter, the main conclusions associated with the analytical formulations and the numerical results are presented and discussed. Furthermore, suggestions for further development of the proposed methods are also outlined.

In chapter 3 the non-stationary response of nonlinear oscillators under evolutionary excitations has been studied. A new approach has been proposed which comprises the elements of stochastic averaging and statistical linearization. Specifically, taking into account the equivalent time-dependent frequency and damping factor, a simple first-order ordinary differential equation has been derived for the response variance. For this purpose, a time-dependent Rayleigh distribution for the response amplitude has been assumed. Analytical expressions have been derived for a number of hysteretic and non-hysteretic nonlinear oscillators.

Extensive digital studies demonstrate the capacity of the approach to successfully capture the time evolution of the mean value of the variance, which is quite predictable taking into account the averaging procedure that is involved. In other words, the new approach succeeds in capturing the average characteristics of the variance, while neglecting the oscillatory components. As a general remark, increasing the nonlinearity degree and the excitation level gradually results in divergence from the digital data as expected. However, the behavior of the new method indicates at least the same reliability level as the standard equivalent linearization does (see section 3.4.1.1). It appears that the proposed approach performs well for a broad class of nonlinear, elastic and inelastic, oscillators. It affords the option of treating problems which involve non-separable and non-white excitation spectra without resorting to ad hoc pre-filtering or other spectral manipulation of the system excitation as is the case for many of existing linearization schemes (e.g. Roberts and Spanos, 2003). Furthermore, based on the demonstrated reasonable reliability of the proposed approach for determining the nonlinear response variance, it can be argued that the evolving Rayleigh distribution given by eq.(3.22) can be used as a logical approximation of the system response non-stationary probability density function. Obviously, it can be argued that the simplicity and versatility of the proposed method compensates for the possible limitations due to the assumption of a lightly damped system.

In chapter 4 an approximate analytical approach has been presented for examining the first-passage problem in context with the response of a class of lightly damped nonlinear oscillators to broad-band random excitations. A Markovian approximation both of the response amplitude envelope and of the response energy envelope has been considered. This modeling leads to a backward Kolmogorov equation which governs the evolution of the survival probability of the oscillator. The Kolmogorov equation is solved approximately by employing a Galerkin approach. A set of confluent hypergeometric functions is used as an orthogonal basis for the expansions that are involved in the application of the Galerkin approach.

As a general remark, it can be argued that the influence of the terms in the series expansion becomes less dominant as the order of the terms increases. Specifically, direct comparison to Monte Carlo simulation data shows a quite good agreement even for a small number of terms. For instance, as far as the Van Der Pol oscillator is concerned, even for large values of the nonlinearity, 9 terms are enough to achieve a good agreement with MCS data. In fact, little improvement is achieved for a larger number of terms.

Another common feature which has been observed is that for small values of the time variable the theoretical data deviate considerably from the corresponding numerical simulations. It is obvious that the first-passage probability density of the oscillator is equal to zero for $t = 0$. This requirement necessitates the use of an infinite number of terms in the expansion series, which is obviously not feasible. However, there is no critical need to produce a smooth approximation at the vicinity of zero time, since the primary interest is focused on situations where the probability of first-passage time is higher than zero.

Note that for the case of stiffness nonlinearities and having considered the case of a Duffing oscillator, the method involving Markovian modeling of the response amplitude envelope yields accurate results only in the case of small nonlinearities. In fact, it should be mentioned that in case of high nonlinearity degree the method is unavoidably affected by the approximations involved in the stochastic averaging procedure in both the derivation of the one dimensional Ito equation and the linearization of the system. The remedy of this problem is the Markovian modeling of the response energy envelope. It has been demonstrated that a more accurate estimation is derived in comparison to the classical approach, despite the high value of the nonlinearity. Thus, this higher accuracy justifies the choice of the energy envelope formulation in cases where the nonlinearity appears in terms of stiffness.

Finally, to apply the proposed approach to a softening Duffing oscillator, singularities should be taken into consideration. In fact, the prescribed barrier level should not be greater than the amplitude level at which the amplitude-dependent equivalent natural frequency of the oscillator reaches the zero value. In other words, for a chosen barrier level value the oscillator should possess positive stiffness.

As far as future research suggestions are concerned, an extension of the proposed response statistics estimation method could be possible by utilizing a wavelet representation of the non-stationary excitation and response processes (e.g. Basu and Gupta, 1998). The coupling of the concepts of equivalent linearization and wavelet transform may also be a feasible idea (e.g. Basu and Gupta, 1999). Extension of the approach to cope with response spectra estimation (e.g. Spanos and Failla, 2004) appears to be another logical objective.

References

Arnold L., 1974. *Stochastic Differential Equations: Theory and Applications.* New York: Wiley.

Au S. K., Beck J. L., 2001. First excursion probabilities for linear systems by very efficient importance sampling, *Probabilistic Engineering Mechanics, 16: 193-207.*

Au S. K., Beck J. L., 2003. Importance sampling in higher dimensions, *Structural Safety, 25: 139-163.*

Au S. K., 2009. Importance sampling for elasto-plastic systems using adapted process with deterministic control, *Int. J. Non-Linear Mech., 44: 190-199.*

Basu B., Gupta V. K., 1998. Seismic response of SDOF systems by wavelet modeling of non-stationary processes. *Journal of Engineering Mechanics, vol. 124: 1142-1150.*

Basu B., Gupta V. K., 1999. On equivalent linearization using wavelet transform. *Journal of Vibration and Acoustics, vol. 121: 429-432.*

Bogoliubov N., Mitropolski A., 1963. *Asymptotic Methods in the Theory of Non-Linear Oscillations*, Gordon & Breach, New York.

Bolotin V. V., 1965. *Statistical method in structural mechanics*. English translation, 1969. San Francisco: Holden-Day.

Booton R. C., 1953. The analysis of nonlinear central systems with random inputs. *Proc. Symposium on nonlinear circuit analysis, Polytechnic institute of Brooklyn.*

Cai G. Q., Lin Y. K., 1998. Reliability of nonlinear structural frame under seismic excitation. *Journal of Engineering Mechanics, vol. 124: 852-856.*

Caughey T. K., 1959. Response of a nonlinear string to random loading. *J. App. Mech., 26: 341-344.*

Caughey T. K., 1960. Sinusoidal Excitation of a System with Bilinear Hysteresis, *J. App. Mech., 27: 640-643.*

Caughey T. K., Diens J. K., 1961. Analysis of a nonlinear first-order system with a white noise input. Journal of Applied Physics, vol. 32: 2476-2479.

Caughey T. K., Stumpf H. J., 1961. Transient Response of a dynamic system under random excitation, *J. App. Mech., 27: 563-566.*

Caughey T. K., 1963. Equivalent linearization techniques. *J. Acoust. Soc. Am., 35: 1706-17.*

Chandrashekhar S., 1943. Stochastic problems in physics and astronomy. *Rev. Mod. Phys. 15(1): 1-89.* Reprinted in *Selected Papers on Noise and Stochastic Processes*, N. Wax, ed. 1954. New York, Dover: 1-91.

Coleman, J. J., 1959. Reliability of aircraft structures in resisting chance failure. *Operations Res. 7(5): 639-645*

Conte J. P., Peng B. F., 1996. An explicit closed-form solution for linear systems subjected to nonstationary random excitation, *Probabilistic Engineering Mechanics, 11: 37-50.*

Corotis R. B., Vanmarcke E. H., 1975. Time-Dependent spectral content of system response, *J. Eng. Mech. Div., Am. Soc. Civ. Eng., 101 (5): 213-224.*

Crandall S. H., ed., 1958. *Random vibration*, vol. I.Cambridge, MA: MIT Press.

Crandall S. H., ed., 1963a. *Random vibration*, vol. II.Cambridge, MA: MIT Press.

Crandall S. H., Mark W. D., 1963. *Random vibration in mechanical systems*. New York: Academic Press.

Crandall S. H., 1970. First-crossing probabilities of the linear oscillator. *Journal of Sound and Vibration, vol. 12: 285-299.*

Crandall S. H., 1980. Non-Gaussian closure for random vibration of nonlinear oscillators. *Int. J. Non-Linear Mech., 15: 303-313.*

Crandall S. H., 2001. Is stochastic equivalent linearization a subtly flawed procedure?. *Probabilistic Engineering Mechanics, 16: 169-176.*

Crandall S. H., 2004. On using non-Gaussian distributions to perform statistical linearization. *Int. J. Non-Linear Mech., 39: 1395-1406.*

Dimentberg M. F., 1982. An exact solution to a certain nonlinear random vibration problem. . *Int. J. Non-Linear Mech., 17: 231-236.*

Einstein A., 1905. On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat. *Annalen der Physik, Vol. 17: 549-569.*

Elishakoff I. E., Colajanni P., 1997. Stochastic linearization critically re-examined. *Chaos, Solitons, Fractals, 8: 1957-1972.*

Elishakoff I. E., 1999. *Probabilistic theory of structures.* New York: Dover Publications.

Elishakoff I. E., 2000. Stochastic linearization technique: A new interpretation and a selective review. *The Shock and Vibration Digest, vol. 32, 3: 179-188.*

Er G.K., 2000. The probabilistic solutions to nonlinear random vibrations of multi-degree of freedom systems. *J. App. Mech., ASME, vol. 67, 2: 355-359.*

Goto H., Iemura H., 1973. Linearization Techniques for Earthquake Response of Simple Hysteretic Structures, *Proceedings of the Japaneese Society of Civil Engineering, Vol.212: 109-119.*

Gray A. H., Caughey T. K., 1965. A controversy in problems involving random parametric excitation. *J. Math. and Phys., 44: 288-296.*

Grigoriu M., 2002. *Stochastic calculus: applications in science and engineering.* Birkhauser, Boston.

Hammond S. K., 1968. On the Response of Single and Multidegree of Freedom Systems to Non-Stationary Random Excitations, *Journal of Sound and Vibration, Vol. 7: 393-416..*

Henriques A. A., 2007. Efficient analysis of structural uncertainty using perturbation techniques. *Engineering Structures, vol. 30, 4: 990-1001.*

Hu S-L. J., 1991. Probabilistic independence and joint cumulants. *J. Eng. Mech., ASCE, vol. 117, 3: 640-652.*

Huang Z. L., Zhu W.Q., Suzuki Y., 2000. Stochastic averaging of strongly nonlinear oscillators under combined harmonic and white noise excitation. *Journal of Sound and Vibration, vol. 238: 233-256.*

Huang Z. L., Zhu W. Q., Ni Y. Q., Ko J. M., 2002. Stochastic averaging of strongly nonlinear oscillators under bounded noise excitation. *Journal of Sound and Vibration, vol. 254: 245-267.*

Ibrahim R. A., Roberts J. W., 1976. Broad band random excitation of a two-degree of freedom system with auto-parametric coupling. *Journal of Sound and Vibration, vol. 44, 3: 335-348.*

Ito K., 1951. On stochastic differential equations. New York, Amer. Math. Soc., (Memoirs, Amer. Math. Soc. No. 4).

Iourtchenko D., Mo E., Naess A., 2008. Reliabilbity of strongly nonlinear single degree of freedom dynamic systems by the path integration method. *Journal of Applied Mechanics, vol. 75*

Iwan W. D., Spanos P. D., 1978. Response envelope statistics for nonlinear oscillators with random Excitation, *ASME J. Appl. Mech., 100(1): 170-174.*

Iwan W. D., Hou Z. K., 1989. Explicit Solutions for the response of simple systems subjected to nonstationary random excitation, *Structural Safety (6): 77-86.*

Jahedi A., Ahmadi G., 1983. Application of Wiener-Hermite expansion to non-stationary random vibration of a Duffing oscillator. *Journal of Applied Mechanics, ASME, vol. 50: 436-442.*

Kanai K., 1957. Semi-empirical formula for the seismic characteristics of the ground. *University of Tokyo Bulletin of Earthquake Research Institute.*

Khasminskii, R. Z., 1968. On the averaging principle for stochastic differential equations. *Kiberrnetika, 4: 260-279.*

Kazakov I. E., 1954. Approximate method for the statistical analysis of nonlinear systems. *Trudy VVIA 394:1-52.*

Kloeden P. E., Platen E., 1992. Higher-order implicit strong numerical schemes for stochastic differential equations. J. Stat. Phys., vol. 66: 283-314.

Kolmogorov A. N., 1956. *Foundations of the theory of probability.* Chelsea Publishing Company.

Kovaleva A., 2009. An exact solution of the first-exit time problem for a class of structural systems, *doi:10.1016/j.probengmech.2009.01.002.*

Langevin P., 1908. *Sur la theorie du mouvement brownien.* C. R. Acad. Sc. Paris, 146: 530-533.

Lee J., 1995. Improving the equivalent linearization technique for stochastic Duffing oscillators. *Journal of Sound and Vibration, vol. 186: 846-855.*

Leibniz G. W., 1684. *Nova methodus pro maximis et minimis* (New method for maximums and minimums); translated in Struik, D. J., 1969. A Source Book in Mathematics, 1200–1800. Harvard University Press: 271–81.

Li R., Ghanem R., 1998. Adaptive polynomial chaos expansions applied to statistics of extremes in nonlinear random vibration, *Probabilistic Engineering Mechanics, 13: 125-136.*

Lin Y. K., 1967. *Probabilistic theory of structural dynamics*. New York: McGraw-Hill.

Lin Y. K., 1986. Some observations on the stochastic averaging method, *Probabilistic Engineering Mechanics, 1: 23-27.*

Lin Y. K., Kozin F., Wen Y. K., Casciati F., Schueller G. I., Der Kiureghian A., Ditlevsen O., Vanmarcke E. H., 1986. Methods of stochastic structural dynamics, *Structural Safety, 3: 167-194.*

Lin Y. K., Cai G. Q., 1995. *Probabilistic structural dynamics: advanced theory and applications.* McGraw-Hill, New York.

Lin Y. K., Cai G. Q., 2000. Some thoughts on averaging techniques in stochastic dynamics, *Probabilistic Engineering Mechanics, 15: 7-14.*

Loeve M., 1977. *Probability theory.* 4th Ed., Springer, Heidelberg.

Manning J. E., 1975. Response spectra for nonlinear oscillators. *Journal for Engineering for Industry, ASME, vol. 97: 1223-1226.*

Macke M., Bucher C., 2003. Importance sampling for randomly excited dynamical systems. *J. Sound and Vib., vol. 286, 2: 269-290.*

Maruyama G., 1955. Continuous Markov processes and stochastic equations. *Rend. Circ. Mat. Palermo, Vol. 4: 48-90.*

Middleton D., 1960. *An introduction to statistical communication theory*. New York: McGraw-Hill.

Naess A., Moe V., 1996. Stationary and non-stationary random vibration of oscillators with bilinear hysteresis. *Int. J. Non-Linear Mech., 31: 553-562.*

Naess A., 1999. Extreme response of nonlinear structures with low damping subjected to stochastic loading, *Journal of Offshore Mechanics and Arctic Engineering, 121: 255-260.*

Naess A., Moe V., 2000.Efficient path integration methodsfor nonlinear dynamic systems. *Probabilistic Engineering Mechanics, vol. 15, 221-231.*

Newland D. E., 1993. *Random vibrations, spectral and wavelet analysis.* Longman, Edinburgh.

Newton I., 1687. *Philosophiae naturalis principia mathematica*; (Translation of 1833). Vol. 1 (3. ed. (1726), with variant readings / assembled and ed. by Alexandre Koyré ed.). Harvard University Press. ISBN 0674664752.

Nigam, N. C., 1983. *Introduction to Random Vibrations*. MIT Press Series in Structural Mechanics.

Oksendal B., 2003. *Stochastic differential equations: an introduction with applications.* Springer, 6th edition.

Olsen A. I., Naess A., 2007. An importance sampling procedure for estimating failure probabilities of non-linear dynamic systems subjected to random noise, *Int. J. Non-Linear Mech., 42: 848-863.*

Orabi I. I., Ahmadi G., 1987. A functional series expansion method for the response analysis of a Duffing oscillator subjected to white noise excitations. *Int. J. Non-Linear Mech., 22: 451-465.*

Papadimitriou C., Lutes L. D., 1996. Stochastic cumulant analysis of MDOF systems with polynomial-type nonlinearities. *Probabilistic Engineering Mechanics, vol. 11, 1: 1-13.*

Papoulis A., 1984. *Probability, random variables and stochastic processes.* 2nd Ed., McGraw-Hill, New York.

Phoon K. K., Huang S. P., Quek S. T., 2002. Implementation of Karhunen-Loeve expansion for simulation using a wavelet-Galerkin scheme, *Probabilistic Engineering Mechanics, 17: 293-303.*

Pichler L., Pradlwarter H. J., 2008. Evolution of probability densities in the phase space for reliability analysis of nonlinear structures, *doi:10.1016/j.strusafe.2008.09.002.*

Priestley M. B., 1965. Evolutionary spectra and non-stationary processes. *Journal of the Royal Statist. Soc., Series B, 27: 204-237.*

Priestley M. B., 1967. Power Spectral Analysis of Non-Stationary Random Processes, *Journal of Sound and Vibration, Vol. 6: 86-97.*

Proppe C., Pradlwarter H. J., Schueller G. I., 2003. Equivalent linearization and Monte Carlo simulation in stochastic dynamics, *Probabilistic Engineering Mechanics, 18: 1-15.*

Red-Horse J. R., Spanos P. D., 1992. A generalization to stochastic averaging in random vibration, *Int. J. Non-Linear Mech., 27: 85-101.*

Rice S. O., 1944, 1945. *Mathematical analysis of random noise*. Bell Syst. Tech. J. 23: 282-332; 24: 46-156. Reprinted in *Selected Papers on Noise and Stochastic Processes*, N. Wax, ed. 1954. New York, Dover: 133-249.

Risken H., 1984. *Fokker-Planck equation method of solution and applications.* Springer-Verlag.

Roberts J. B., 1971. The covariance response of linear systems to nonstationary random excitation, *Journal of Sound and Vibration, 14 (3): 385-400.*

Roberts J. B., 1976. First-Passage probability for non-Linear oscillators, *J. Eng. Mech. Div., Am. Soc. Civ. Eng., Vol. 102, No. EM5: 851-866.*

Roberts J.B., 1981a. Response of nonlinear mechanical systems to random excitation: Part I, Markov methods. *Shock and Vibration Digest, vol. 13: 17-28.*

Roberts J.B., 1981b. Response of nonlinear mechanical systems to random excitation: Part II, Equivalent linearization and other methods. *Shock and Vibration Digest, vol. 13: 15-29.*

Roberts J. B., 1986. Response of an oscillator with nonlinear damping and a softening spring to non-white random excitation, *Probabilistic Engineering Mechanics, 1: 40-48.*

Roberts J. B., 1986. First-Passage probabilities for randomly excited systems: Diffusion Methods, *Probabilistic Engineering Mechanics, 1: 66-81.*

Roberts J. B., Spanos P. D., 1986. Stochastic Averaging: An Approximate Method of Solving Random Vibration Problems, *Int. J. Non-Linear Mech., 21: 111-134.*

Roberts J. B., Dunne J. F., 1988. Nonlinear random vibration in mechanical systems. *Shock Vib. Digest, 20: 16-25.*

Roberts J. B., Spanos P. D., 2003. *Random Vibration and Statistical Linearization.* New York: Dover Publications.

Robson J. D., 1963. *An introduction to random vibration.* Edinburgh: University Press.

Roy R. V., Spanos P. D., 1989. Wiener-Hermite functional representation of nonlinear stochastic systems. *Structural Safety, vol. 6: 187-202.*

Roy R. V., 1997. Asymptotic analysis of first-passage problems, *Int. J. Non-Linear Mech., 32: 173-186.*

Sharp W. D., Allen E. J., 1998. Numerical solution of first passage problem using an approximate Chapman-Kolmogorov relation, *Probabilistic Engineering Mechanics, 13: 233-241.*

Shinozuka M., 1964. Probability of structural failure under random loading. *J. Eng. Mech. Dev., Proc. ASCE 90(EM5): 147-170.*

Shinozuka M., 1972. Monte Carlo solution of structural dynamics. *Computational Statistics, vol. 2: 855-874.*

Shinozuka M., Deodatis G., 1991. Simulation of stochastic processes by spectral representation, *Appl. Mech. Rev., ASME, vol. 44, no4: 191-203.*

Smyth A.W., Masri S. F., 2002. Nonstationary response of nonlinear systems using equivalent linearization with a compact analytical form of the excitation process, *Probabilistic Engineering Mechanics, 17: 97-108.*

Socha L., Pawleta M., 1994. Corrected equivalent linearization of stochastic dynamic systems. Machine Dynam Problems, 7: 149-161.

Socha L., 2005. Linearization in analysis of nonlinear stochastic systems: Recent results-Part I: Theory, *Applied Mechanics Reviews, 58: 178-205.*

Soize C., 1988. Steady state solution of Fokker-Planck equation in higher dimension. *Probabilistic Engineering Mechanics, vol. 3, 4: 196-206.*

Solomos G. P., Spanos P. D., 1983. Structural reliability under evolutionary seismic excitation, *Soil Dynamics and Earthquake Engineering, Vol. 2, No2.*

Soong T. T., 1973. *Random differential equations in science and engineering.* New York: Academic Press.

Soong T. T., Grigoriu M., 1992. *Random vibration of mechanical and structural systems.* Prentice Hall, Englewood cliffs, New Jersey.

Spanos P. D., 1976. *Linearization Techniques for Non-Linear Dynamical Systems,* Report No. EERL, 70-04, Earthquake Engineering Research Laboratory, California Institute of Technology, Pasadena, CA

Spanos P. D., Iwan W. D., 1978b. On the existence and uniqueness of solutions generated by equivalent linearization. *Int. J. Non-Linear Mech., 13: 71-78.*

Spanos P. D., Lutes L. D., 1980. Probability of Response to Evolutionary Process, *J. Eng. Mech. Div., Am. Soc. Civ. Eng., Vol. 106, No. EM2: 213-224.*

Spanos P. D., 1980. On the computation of the confluent hypergeometric function at densely spaced points, *J. App Mech., 47: 683-685.*

Spanos P. D., 1981a. Stochastic linearization in structural dynamics. *Applied Mechanics Review, ASME, vol. 34: 1-8.*

Spanos P. D., 1982. Numerics for common first-passage problem, *J. Eng. Mech. Div., Am. Soc. Civ. Eng., Vol. 108, No. EM5: 864-882.*

Spanos P. D., 1982. Survival probability of non-linear oscillators subjected to broad-band random disturbances, *Int. J. Non-Linear Mech., 17: 303-317.*

Spanos P. D., Solomos G. P., 1983. Markov Approximation to Transient Vibration, *J. Eng. Mech., 109: 1134-1150.*

Spanos P. D., Solomos G. P., 1984. Barrier crossing due to transient excitation, *Journal of Engineering Mechanics, vol. 110, No. 1: 20-36.*

Spanos P. D., Lutes L. D., 1986. A primer of random vibration techniques in structural engineering. *Shock Vib. Digest, 18: 3-10.*

Spanos P. D., Donley M. G., 1991. Equivalent statistical quadratization for nonlinear systems. *Journal of Engineering Mechanics, 117: 1289-1310.*

Spanos P. D., Donley M. G., 1992. Nonlinear multi-degree of freedom system random vibration by equivalent statistical quadratization. *Int. J. Non-Linear Mech., 27: 735-748.*

Spanos P. D., Zeldin B. A., 1998. Monte Carlo treatment of random fields: A broad perspective, *Appl. Mech. Rev., ASME, vol. 51, no3: 219-237.*

Spanos P. D., Failla G., Di Paola M., 2003. Spectral approach to equivalent statistical quadratization and cubicization methods for nonlinear oscillators. *Journal of Engineering Mechanics, 129: 31-42.*

Spanos P. D., Failla G., 2004. Evolutionary spectra estimation using wavelets. *Journal of Engineering Mechanics, vol. 130: 952-960.*

Spanos P. D., Cacciola P., Muscolino G., 2004. Stochastic averaging of Preisach hysteretic systems, *J. Eng. Mech. Div., Am. Soc. Civ. Eng., 130 (11): 1257-1267.*

Spanos P. D., Sofi A., Di Paola M., 2007. Nonstationary Response Envelope Probability Densities of Nonlinear Oscillators, *J. App Mech., 74: 315-324.*

Stratonovich R. L., 1963. *Topics in the theory of Random Noise*, vol. I. New York: Gordon Breach

Stratonovich R. L., 1967. *Topics in the theory of Random Noise*, vol. II. New York: Gordon Breach

Stratonovich R. L., 1966. A new representation for stochastic integrals and equations. *SIAM J. Control, 4: 362-371.*

Suzuki Y., Minai R., 1987a. Application of Stochastic Differential Equations to Seismic Reliability Analysis of Hysteretic Structures, *Lecture Notes in Engineering, Y.K. Lin and R. Minai, Eds. Berlin: Springer*

Tajimi H., 1960. A statistical method for determining the maximum response of a building structure during an earthquake. *Proceedings of the 2nd World Conference on Earthquake Engineering, Tokyo and Kyoto, Japan.*

To C. W. S., 1982. Nonstationary random responses of a multi-degree-of-freedom system by the theory of evolutionary random spectra, *Journal of Sound and Vibration, 83 (2): 273-291.*

To C. W. S., 1986. The stochastic central difference method in structural dynamics. *Computers and Structures, vol. 23, 6: 813-818.*

Tognarelli M.A., Zhao M.A., Rao K. B., Kareem A., 1997b. Equivalent statistical quadratization and cubicization for nonlinear system. *Journal of Engineering Mechanics, vol. 123: 512-523.*

Uhlenbeck G. E., Ornstein L. S., 1930. On the theory of Brownian motion. *Physics Review, 36: 362-371.*

Vanmarcke E. H., 1975. On the distribution of the first-passage time for normal stationary random processes, *J. App Mech. 42: 215-220.*

Wang Y., Ying Z. G., Zhu W. Q., 2009. Stochastic averaging of energy envelope of Preisach hysteretic systems, *Journal of Sound and Vibration, 321: 976-993.*

Wong E., Zakai M., 1965. On the relation between ordinary and stochastic differential equations. *International Journal of Engineering Science, vol. 3, 2: 213-229.*

Wu W. F., Lin Y. K., 1984. Cumulant-neglect closure for nonlinear oscillators under random parametric and external excitations. *Int. J. Non-Linear Mech., 19: 349-362.*

Yang J. N., Shinozuka M., 1970. First-passage time problem. *The J. Acoust. Soc. Amer., vol. 47: 393-394.*

Yu J. S., Cai G. Q., Lin Y. K., 1997. A new path integration procedure based on Gauss-Legendre scheme. *Int. J. Non-Linear Mech., 32: 759-768.*

Zhu W. Q., 1988. Stochastic averaging methods in random vibration. *Applied Mechanics Review, vol. 41: 189-199.*