Generation and Detection of a Two-Photon Binomial Schrödinger Cat in a Cavity

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We introduce the “Binomial Schrödinger Cat”, formed by a quantum superposition of two orthogonal generalized binomial states of electromagnetic field with maximum number of photons \( N \). In particular, using resonant atom-cavity interactions, we propose a non-conditional scheme to generate a two-photon \( (N = 2) \) binomial Schrödinger cat in a single-mode high-\( Q \) cavity. We also give two single-shot schemes to detect the generated “cat” state, by exploiting suitable probe atoms that distinguish the two components and reveal the coherence of the superposition. We finally discuss the implementation of the proposed schemes.

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The Schrödinger cat notion has been introduced as a quantum superposition of macroscopically different states \([1, 2, 3]\). In quantum optics, a Schrödinger cat state is usually meant as a superposition of two coherent states \([4]\). Several schemes have been proposed to generate such a state, for example in a dispersive medium \([3]\), in a nanomechanical resonator \([6]\) and in a microwave cavity \([4]\). A free-propagating light pulse was also recently prepared in a Schrödinger cat state \([5]\). In the context of cavity quantum electrodynamics (CQED), superpositions of two coherent states have been generated by dispersive coupling between a circular Rydberg atom and the cavity field, the quantum decoherence of the superposition being there observed by probe atoms \([6]\). Nevertheless, superpositions of two coherent states \(|\alpha\rangle\) and \(|\alpha'\rangle\) can never be made exactly orthogonal, since \(|\langle\alpha|\alpha'\rangle|^2 = \exp\{-|\alpha' - \alpha|^2\}\). Therefore, different coherent states in a quantum superposition are not completely distinguishable. In the CQED experiment above, for example, it is necessary to adjust the detuning between the atomic transition and the cavity frequency to partially distinguish the two components of the superposition. So, the possibility of obtaining Schrödinger cat like states formed by orthogonal, completely distinguishable states with nonzero mean fields would be of interest.

Electromagnetic field states suited to this purpose are the generalized binomial states \([10]\). They are characterized by a finite maximum number of photons \( N \) and interpolate between the coherent state and the number state. Among the \( N \)-photon generalized binomial states it is always possible to find an orthogonal couple \([11]\). Moreover, in the CQED framework, this couple can be generated by resonant atom-cavity interactions \([12, 13]\). Therefore, binomial states may be useful to study the general problem of the classical-quantum border and the quantum measurement, provided that we can generate and reveal a their quantum superposition.

In this work we introduce the \( N \)-photon “Binomial Schrödinger Cat” (NBSC) formed by a quantum superposition of an orthogonal couple of \( N \)-photon generalized binomial states. We then propose, in particular, non-conditional schemes to generate and reveal a 2BSC in a high-\( Q \) cavity, that exploit resonant atom-cavity interactions. Finally, we briefly analyze the implementation of the proposed schemes.

The dynamics of the resonant interaction between a two-level atom and the single-mode cavity field is well described by the Jaynes-Cummings Hamiltonian \( H_{JC} = \hbar \omega \sigma_z/2 + \hbar \omega a^\dagger a + i \hbar g (\sigma_z a - a^\dagger \sigma_-) \), where \( \omega \) is the resonant cavity field mode, \( g \) the atom-field coupling constant, \( a \) and \( a^\dagger \) the field annihilation and creation operators, \( \sigma_z = \ket{\uparrow}\bra{\uparrow} - \ket{\downarrow}\bra{\downarrow} \), \( \sigma_- = (\sigma_z)^\dagger = \ket{\downarrow}\bra{\uparrow} \), and \( \sigma_+ = \ket{\uparrow}\bra{\downarrow} \) the pseudo-spin atomic operators, \( \ket{\uparrow} \) and \( \ket{\downarrow} \) being respectively the excited and ground state of the two-level atom. The Hamiltonian \( H_{JC} \) generates the time evolutions \([14]\)

\[
\begin{align*}
|\uparrow n\rangle & \to \cos(\sqrt{n + 1}t)|\uparrow n\rangle - \sin(\sqrt{n + 1}t)|\downarrow n + 1\rangle, \\
|\downarrow n\rangle & \to \cos(\sqrt{n}t)|\downarrow n\rangle + \sin(\sqrt{n}t)|\uparrow n - 1\rangle,
\end{align*}
\]

where \(|\uparrow n\rangle \equiv |\uparrow\rangle|n\rangle, |\downarrow n\rangle \equiv |\downarrow\rangle|n\rangle\) and \(a^\dagger a|n\rangle = n|n\rangle\).

The normalized \( N \)-photon generalized binomial state in terms of the number states is given by \([10]\)

\[
|N, p, \phi\rangle = \sum_{n=0}^{N} \left[ \binom{N}{n} p^n (1-p)^{N-n} \right]^{1/2} e^{in\phi} |n\rangle,
\]

where \( 0 \leq p \leq 1 \) is the probability of single photon occurrence and \( \phi \) is the mean phase \([15]\). Using the orthogonality property of binomial states with the same \( N \) \([11]\), \( \langle N, p, \phi | N, 1-p, \pi + \phi \rangle = 0 \), we define our NBSC as

\[
|\Psi^{(N)}_S\rangle \equiv \sqrt{\mathcal{N}} |N, p, \phi, \eta |N, 1-p, \pi + \phi\rangle,
\]

where \( \eta \) is in general a complex number and \( \mathcal{N} = 1/\sqrt{1+|\eta|^2} \). The state \( |\Psi^{(N)}_S\rangle \) represents a superposition of two orthogonal generalized binomial states, having the same \( N \) but different \( p \). Therefore, it differs from the superposition of binomial states with the same \( N \) and \( p \) previously introduced \([10]\). It should be noted that, for \( |\eta| = 1 \), the NBSC of Eq. \((3)\) is maximal with \( \mathcal{N} = 1/\sqrt{2} \).
A couple of
entangled two-level atoms

2
1
Ramsey Zone $R_p$

Cavity C

FIG. 1: Experimental setup for the generation of the 2BSC. $R_p$ is the “preparing” Ramsey zone.

while for $p = 0, 1$ it is reduced to a superposition of the number states $|0\rangle, |N\rangle$.

At this point, we describe a possible generation scheme of a 2BSC in a cavity, whose experimental setup is sketched in Fig. 1. The cavity $C$ is initially taken in the vacuum state $|0\rangle$, and a couple of two-level atoms, namely 1 and 2, is prepared in the entangled state $|\psi\rangle = \mathcal{N}(|1\rangle|2\rangle + \eta_0|2\rangle|1\rangle)$, with $\eta_0$ real. Entangled atomic states of this form, with a given separation time $T_0$ between the two atoms, have already been obtained using a cavity as atomic entanglement catalyst \[17, 18\]. We assume that the separation time between the atoms is such that only an atom at a time crosses the entire apparatus. Each atom first crosses a “preparing” Ramsey zone $R_p$. The Ramsey zone interaction makes each atom undergo the transformations

$$
|\uparrow\rangle \xrightarrow{R} |\uparrow\rangle u = \cos(\theta/2)|\uparrow\rangle - e^{i\varphi} \sin(\theta/2)|\downarrow\rangle,
$$

$$
|\downarrow\rangle \xrightarrow{R} |\downarrow\rangle u = e^{-i\varphi} \sin(\theta/2)|\uparrow\rangle + \cos(\theta/2)|\downarrow\rangle,
$$

with $u \equiv (-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta)$. The parameters $\theta$ (“Ramsey pulse”) and $\varphi$ can be arbitrarily fixed by adjusting the classical field amplitude and the atom-field interaction time. Each atom then resonantly interacts with $C$ for a time $T_j$ ($j = 1, 2$). These atom-cavity interaction times can be obtained by selecting either different velocities for each atom or the same velocity for the two atoms (“monokinetic atomic beam”) and applying a Stark shift inside the cavity for a time such as to have the desired resonant interaction time \[17, 18\]. The appropriate atomic velocity may be selected by laser induced atomic pumping \[20\]. We shall show that a 2BSC state can be efficiently generated by appropriately choosing the Ramsey zone settings and the atom-cavity interaction times. To this purpose, we follow a method previously suggested to generate a two-photon generalized binomial state in a cavity \[18\].

Following the scheme of Fig. 1, the atom 1 crosses $R_p$ set with a “pulse” $\theta_1$ such that $\cos(\theta_1/2) \equiv \sqrt{p}$, $\sin(\theta_1/2) \equiv \sqrt{1-p}$, where $0 \leq p \leq 1$, and with an arbitrary value of $\varphi_1$. After a free evolution time $\tau_1$ between $R_p$ and $C$, the atom 1 interacts with $C$ for a given time $T_1 = (4m+1)\pi/2g$, $m$ non-negative integer. After the exit of the atom 1 from $C$, the atom 2 crosses the Ramsey zone $R_p$ and, after a free evolution time $\tau_2$ from $R_p$ to $C$, it interacts with $C$ for a time $T_2$. Note that, knowing the atomic velocities and the separation time $T_0$ between the two atoms, it is also possible to know when the atom 1 has come out of $R_p$, which can be then set for the atom 2 as desired. The field evolution is free during the time $T$ elapsed between the exit of the atom 1 from $C$ and the entrance of the atom 2 in $C$. After the passage of the atom 1, the $R_p$ parameters for the atom 2 must be reset at $\theta_2 = \theta_1 + \pi$, so to obtain $\cos(\theta_2/2) = -\sqrt{1-p}$ and $\sin(\theta_2/2) = \sqrt{p}$ in Eq. (4), and $\varphi_2 = \varphi_1 + \omega(t_1 + \tau_2 - \tau_1)$. From the time evolutions determined by Eqs. (1) and (4), together with Eq. (2), consecutively for the atoms 1 and 2, it is possible to see that, in order to obtain our 2BSC target state, the equalities $\sin(gT_2 + \pi/4) = 1$, $\sin(g\sqrt{T_2}) = 1$ must be simultaneously satisfied. It has been shown \[17\] that, choosing an appropriate interaction time $T_2$ of the atom 2 with $C$ ($T_2 = 41\pi/4g$), both equalities above are satisfied within the error due to the typical experimental interaction times uncertainties. With this choice for the interaction times $T_1, T_2$ and the $R_p$ parameters we obtain that, after the atom 2 has come out of the cavity, the initial total state $|\psi\rangle|0\rangle$ evolves into a factorized final total state $|\Psi_S^{(2)}\rangle|1\rangle|1\rangle$, where

$$
|\Psi_S^{(2)}\rangle = \mathcal{N}[|2, p, \phi\rangle + \eta_0 e^{i\gamma}|2, 1-p, \pi + \phi\rangle],
$$

with $\phi = -|\varphi_1 + \omega(t_1 + T_1)|$ and $\gamma = \omega(t_{R_2} - t_{R_1} - T_1)$, with $t_{R_1}, t_{R_2}$ being respectively the interaction times of the atoms 1 and 2 with $R_p$. By comparing Eqs. (5) and (3), we see that the state $|\Psi_S^{(2)}\rangle$, generated by our procedure, is the particular case $N = 2$ of the NBSC with $\eta = \eta_0 e^{i\gamma}$. In the final total state both atoms are in the ground state, so that our procedure to generate a 2BSC in a cavity does not require a final atomic measurement and it is a non-conditional, efficient one.

We shall now analyze the possibility to prove the generated $|\Psi_S^{(2)}\rangle$ state. We shall see that this is possible by exploiting two-level probe atoms that “read” the cavity field. Our considerations shall be made for the maximal 2BSC of Eq. (2). Therefore, we take $\eta_0 = \pm 1$ and the state $|\Psi_S^{(2)}\rangle$ is reduced to the form

$$
|\Psi_S^{(2)}\rangle = |2, p, \phi\rangle|2, 1-p, \pi + \phi\rangle/\sqrt{2}.
$$

For the Schrödinger cat state it is necessary to be able to find a measurement procedure that projects the total superposition state in one of the two components, permitting to distinguish each of them. In the following, we shall first describe a procedure that allows to distinguish the two components of the 2BSC of Eq. (0), i.e. the two binomial states $|2, p, \phi\rangle$, $|2, 1-p, \pi + \phi\rangle$. Successively, we propose a method that permits to reveal the coherence of the 2BSC.

**Distinction of the two components.** The experimental setup is illustrated in Fig. 2. It exploits two consecutive
probe atoms both in the ground state that interact with the apparatus one at a time. The cavity C is initially prepared in the 2BSC of Eq. (6) by the generation scheme previously described. The atom 1 resonantly interacts with C for an appropriate time $T_{P_1}$ ($T_{P_1} = 41\pi/4g$) and, after a delay time $t_1$ from C to $R_d$, it crosses the “decoding” Ramsey zone $R_d$ and finally it is measured by field ionization detectors. The $R_d$ parameters for the atom 1 are set so that $\cos(\theta_{d1}/2) = \sqrt{p}$, $\sin(\theta_{d1}/2) = \sqrt{1-p}$, $\varphi_{d1} = -\phi + \omega_1$, $p, \phi$ being coincident with those defining the two components of the 2BSC of Eq. (6). With these choices, using Eqs. (11) and (14) together with Eq. (2) for $N = 2$, after the atom 1 has crossed $R_d$ we get the evolutions (13):

$$|\downarrow_1\rangle|2, p, \phi\rangle \xrightarrow{T_{P_1}, R_d, e^{-i\varphi_{d1}} p, \phi\rangle |\downarrow_1\rangle,$$

$$|\downarrow_1\rangle, 1-p, \pi + \phi\rangle \xrightarrow{T_{P_1}, R_d} |1-p, \pi + \phi\rangle |\downarrow_1\rangle,$$ (7)

where $|p, \phi\rangle$ indicates the one-photon generalized binomial state given by Eq. (2) and $\phi' = \phi - \omega T'$, $T'$ being the total field free evolution time. Notice that the field free evolution induces just a shift $-\omega T'$ of the mean phase $\phi$ in the generalized binomial state of Eq. (2), that does not change the binomial nature of the state. In Eq. (7) we see that the atom absorbs a photon, reducing the two-photon generalized binomial state in the one-photon generalized binomial state. The initial state $|\downarrow_1\rangle|\Psi_S^{(2)}\rangle$ then evolves in

$$|\phi\rangle = |p, \phi\rangle |\downarrow_1\rangle \pm e^{i(\gamma - \varphi_{d1})|1-p, \pi + \phi\rangle |\downarrow_1\rangle}/\sqrt{2},$$ (8)

where the atom is now entangled with the cavity field. By performing a measurement of the atom 1 final state, we see from Eq. (8) that the outcome $|\downarrow_1\rangle$ makes the cavity collapse in $|p, \phi\rangle$, while the outcome $|\uparrow_1\rangle$ makes the cavity collapse in $|1-p, \pi + \phi\rangle$. It is possible to verify the collapsed cavity field state by successively sending a probe atom 2, that resonantly interacts with C for a suitable time $T_{P_2}$ ($T_{P_2} = (4m + 1)\pi/2g$). After a delay time $t_2$ elapsed to go from C to $R_d$, the atom 2 crosses the Ramsey zone $R_d$, set with $\theta_{d2} = \theta_{d1}$, as above but with $\varphi_{d2} = -\phi + \omega(T' + t_2)$, $T'$ being here the time interval between the exit of the atom 1 from C and the entrance of the atom 2 in C. After the atom 2 has crossed the Ramsey zone $R_d$, using Eqs. (11) and (14) together with Eq. (2) for $N = 1$, we get the evolutions

$$|\downarrow_2\rangle|p, \phi\rangle \xrightarrow{T_{P_2}, R_d} e^{-i\varphi_{d2}} |0\rangle |\downarrow_2\rangle,$$

$$|\downarrow_2\rangle, 1-p, \pi + \phi\rangle \xrightarrow{T_{P_2}, R_d} |0\rangle |\downarrow_2\rangle.$$ (9)

From Eqs. (7), (8) and (9), we see that (i) the measurement of both atoms 1 and 2 in the excited levels $|\downarrow_1\rangle$ and $|\downarrow_2\rangle$ reveals the component $|2, p, \phi\rangle$ of the 2BSC of Eq. (6), while (ii) the measurement of both atoms in the ground levels $|\downarrow_1\rangle$ and $|\downarrow_2\rangle$ reveals the orthogonal component $|2, 1-p, \pi + \phi\rangle$. The probe atoms act here as “quantum mice”, changing their state if they find the “cat living” (state $|2, p, \phi\rangle$) or maintaining the same state if they find the “cat dead” (state $|2, 1-p, \pi + \phi\rangle$). We underline that, with our protocol, the evolutions of Eqs. (7) and (9) are true if and only if the initial cavity field state are respectively the two-photon and one-photon generalized binomial states appearing in those equations. Thus, repeating this single-shot procedure, we can distinguish the two components of the 2BSC $|\Psi_S^{(2)}\rangle$ of Eq. (6).

Detection of the coherence. The experimental setup is sketched in Fig. 3. Note the addition of a second Ramsey zone $R_c$, that has the function to decode the information about the coherence of the 2BSC. Once again, we use two consecutive probe atoms, namely 1 and 2, both initially in the ground state that interact with the apparatus one at a time. The cavity C is prepared in the 2BSC of Eq. (6). The atom 1 follows the same evolution described by Eqs. (7), (8) and (9) up to $R_d$ and, after a time $t_1'$ elapsed to go from $R_d$ to $R_c$, it enters $R_c$. After the atom 1 has come out of $R_c$, the atom 2 enters the apparatus and follows the same evolution described by Eq. (9) up to $R_d$. After crossing $R_d$, the atom 2 takes a time $t_2'$ to get $R_c$. Studying the time evolutions by Eqs. (11), (12), (14) and (16) and taking into account all the free evolutions, after the atom 2 leaves $R_c$ it is possible to see that the following total evolutions

$$|\downarrow_1\downarrow_2\rangle|\Psi_S^{(2)}\rangle_+ \rightarrow |0\rangle |\downarrow_1\downarrow_2\rangle + e^{i\alpha} |\downarrow_1\downarrow_2\rangle/\sqrt{2},$$

$$|\downarrow_1\downarrow_2\rangle|\Psi_S^{(2)}\rangle_- \rightarrow |0\rangle |\downarrow_1\downarrow_2\rangle + e^{i\beta} |\downarrow_1\downarrow_2\rangle/\sqrt{2},$$ (10)

are obtained, provided that the $R_c$ parameters are set from the beginning at $\theta_c = \pi/2$ and $\varphi_c = (\gamma - 2\phi + \omega T)/2$. 

FIG. 2: Experimental setup for distinguishing the two binomial states of the 2BSC. $R_d$ is the “decoding” Ramsey zone.

FIG. 3: Experimental setup for revealing the coherence of the 2BSC. $R_c$ is the “coherence decoding” Ramsey zone.
with \( \tau \equiv T' + t_1 + t_2 + t_2' \). All these free evolution times are determinable by the atomic velocities, the separation time \( T_0 \) between the two atoms and the distance between the interaction zones. In Eq. (11), \( \alpha \) and \( \beta \) indicate relative phases due to particular times involved in the procedure. From Eq. (10), we see that a “two-atom event” detection, after \( R_c \), permits to know the coherence (“sign”) of the superposition of Eq. (6). In fact, the measurement of parallel atomic states assures us the 2BSC inside \( C \) was \( |\Psi_S^{(2)}\rangle_+ \), while the measurement of antiparallel atomic states assures us the 2BSC inside \( C \) was \( |\Psi_S^{(2)}\rangle_- \). In principle, the probability of this “two-atom event” is equal to one, depending on the detectors efficiency, and we thus have a single-shot scheme revealing the coherence of a 2BSC, as desired.

**About the implementation.** We now briefly analyze the experimental feasibility of the proposed schemes. Our schemes require precise atom-cavity interaction times. However, the experimental uncertainty of the selected velocity \( \Delta v \) induces an error \( \Delta T \) on the interaction time such that \( \Delta T / T \approx \Delta v / v \). In current laboratory experiments it is possible to select a given atomic velocity such that \( \Delta v / v \leq 10^{-2} \). This error does not sensibly affect our schemes. We have also ignored the atomic or photon decay during the atom-cavity interactions. This assumption is valid if \( \tau_{at}, \tau_{cav} > T \), where \( \tau_{at}, \tau_{cav} \) are the atomic and photon mean lifetimes respectively and \( T \) is the interaction time. For circular Rydberg atomic levels and microwave superconducting cavities with quality factors \( Q \sim 10^8 - 10^{10} \), the required inequality on the mean lifetimes can indeed be satisfied, being \( \tau_{at} \sim 10^{-5} - 10^{-2}s \), \( \tau_{cav} \sim 10^{-4} - 10^{-1}s \) and \( T \sim 10^{-5} - 10^{-4}s \). Moreover, the typical mean lifetimes of circular Rydberg atomic levels \( \tau_{at} \) are such that the atoms do not decay during the entire sequence of the schemes. The separation time \( T_0 \) between the two atoms can be adjusted so that they cross the experimental apparatus one at a time, as required by our schemes. Finally, recent laboratory developments open promising perspectives for a better and easy control of a well-defined atom numbers sequence and for a high efficiency atomic detection in microwave CQED experiments.

In conclusion, we have introduced the “Binomial Schrödinger Cat” (BSC), formed by a quantum superposition of two orthogonal \( N \)-photon generalized binomial states of electromagnetic field. We have then given non-conditional schemes to generate and detect a two-photon BSC (2BSC) in a single-mode high-\( Q \) cavity by opportune resonant interactions of two-level atoms with the cavity field. The important feature of such a “cat state” is that, thanks to the orthogonality of the binomial states, we are able both to distinguish the two components of the 2BSC and reveal the coherence of the superposition by suitable probe atoms. We have also briefly analyze the implementation of the proposed schemes, showing how the typical experimental errors do not sensibly affect them. The results of this paper can provide both new knowledge about the foundations of quantum theory (measurement process, quantum-classical border) and applications in quantum information processing, in analogy with the superpositions of coherent states. The generation of BSC with \( N > 2 \) is possible and it will be treated somewhere else. The current CQED experimental improvements make our schemes very near to be realized in laboratory.

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