Abstract

In this paper we analyze the one-loop renormalization of the $\theta$-expanded pure SU(N) Yang-Mills theory. The renormalization condition fixes the allowed values of Seiberg-Witten freedom parameter $a$ to one of the two solutions: $a = 1$ or $a = 3$. We also show that in the case $a = 3$ the deformation parameter $\hbar$ has to be renormalized and it is asymptotically free.

Key words: Standard Model, Non-commutative Geometry, Renormalization, Regularization and Renormalons

1 Introduction

For some years it was believed that field theories defined on non-commutative Minkowski space were not renormalizable. Namely, if non-commutativity is canonical,

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hbar \theta^{\mu\nu} = \text{const},$$  \hspace{1cm} (1.1)

then the algebra generated by the coordinates $\hat{x}^{\mu}$, i.e. non-commutative Minkowski space and the fields on it can be represented by the algebra of functions

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on the ordinary $\mathbb{R}^4$ with the Moyal-Weyl product instead of the usual multiplication

$$\hat{\phi}(x) \star \hat{\psi}(x) = e^{\frac{i}{2} h \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \hat{\phi}(x) \hat{\psi}(y)|_{y \to x}.} \tag{1.2}$$

Furthermore, the integral can be defined straightforwardly and has the trace property. Thus one can formulate field theories with the action and the variational principle. However, in the quantization of these $\theta$-unexpanded theories (e.g., $\phi^4$) one, as a rule, meets the obstruction to renormalizability: the UV/IR mixing [1–3].

In (1.1) the deformation parameter $h$ has dimension $\text{length}^2$ or $\text{energy}^{-2}$, and can also be written as $h = 1/\Lambda_{\text{NC}}^2$, where the $\Lambda_{\text{NC}}$ represents the scale of non-commutativity.

Gauge theories can be extended to a non-commutative (NC) setting in different ways. In our model, the classical action is obtained via a two-step procedure. First, the action of the non-commutative Yang-Mills (NCYM) theory is equipped with a star-product carrying information about the underlying non-commutative manifold, and, second, the star-product and non-commutative fields are expanded in the non-commutativity parameter $h\theta$ using the Seiberg-Witten (SW) map [4]. In this approach [5–7], non-commutativity is treated perturbatively. The major advantage is that models with any gauge group and any particle content can be constructed [5,8–11], so we can construct the generalization of the standard model (SM), too. The action is gauge invariant; furthermore, it has been proved that the action is anomaly free whenever its commutative counterpart is also anomaly free [12].

In this paper, which is a continuation of two recent papers, [6] and [7], we analyze the renormalizability property of Yang-Mills theory on non-commutative space, where we confine ourselves to the $\theta$-expanded NC SU(N) gauge theory. Commutative gauge symmetry is the underlying symmetry of the theory and is present in each order of the $\theta$-expansion. Noncommutative symmetry, on the other hand, exists only in the full theory, i.e. after the summation.

There are a number of versions of the non-commutative standard model (NC-SM) in the $\theta$-expanded approach, [8–11]. The argument of renormalizability was previously included in the construction of field theories on non-commutative Minkowski space producing not only the one-loop renormalizable model [6], but the model containing one-loop quantum corrections free of divergences [7], contrary to previous results [13–16]. This ‘good’ behaviour of the $\theta$-expanded non-commutative SM gauge theory is our primary motivation to re-examine one-loop renormalizability aspect of the pure NC SU(N) gauge sector. We shall perform the analysis at first order in $\theta$, and for the fundamental representations of the matter field. Phenomenological consequences of this investigation are
certainly important [11,17,18].

The plan of the paper is the following. In Section 2 we briefly review the ingredients of the SW freedom relevant to this work and we give the final Lagrangian. In Section 3 the one-loop renormalizability of the pure NC SU(N) gauge theory is worked out. Section 4 is devoted to the ultraviolet asymptotic behaviour of NC SU(N) gauge theory. The discussion of the results and the concluding points are given in Section 5.

2 NC SU(N) gauge sector effective action

According to [5], the NC parameter \( \hat{\Lambda} \), the NC vector potential \( \hat{V}_\mu \) and the corresponding NC field strength \( \hat{F}_{\mu\nu} \) take their values in the enveloping algebra of the Lie algebra of the gauge group. As in ordinary theory, in the non-commutative case, symmetry is localized by the NC vector potential \( \hat{V}_\mu \) and the NC field strength \( \hat{F}_{\mu\nu} \)

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - i(\hat{V}_\mu \star \hat{V}_\nu - \hat{V}_\nu \star \hat{V}_\mu). 
\]

We start by solving the gauge field transformation closure condition [5] order by order in the parameter \( h \). The solutions up to the first order for the vector field and the field strength read

\[
\hat{V}_\mu(x) = V_\mu(x) - \frac{1}{4} h \theta^{\sigma\rho} \{ V_\sigma(x), \partial_\rho V_\mu(x) + F_{\rho\mu}(x) \} + \ldots 
\]

\[
\hat{F}_{\mu\nu}(x) = F_{\mu\nu} + \frac{1}{4} h \theta^{\sigma\rho} (2 \{ F_{\mu\sigma}, F_{\nu\rho} \} - \{ V_\sigma, (\partial_\rho + D_\rho) F_{\mu\nu} \}) + \ldots 
\]

The non-Abelian field strength and the covariant derivative are defined in the usual way \( F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu] \) and \( D_\mu = \partial_\mu - i[V_\mu, \cdot] \). The relations (2.2-2.3) between non-commutative and commutative gauge symmetries are known as the Seiberg-Witten maps, [4]. For zero non-commutativity, \( \hat{V}_\mu \) and \( \hat{F}_{\mu\nu} \) reduce to the usual vector potential \( V_\mu \) and the field strength \( F_{\mu\nu} \). The general SW map, which fulfils a number of requirements (hermiticity, etc), leading to a physically acceptable theory, can be found in [19,20].

Clearly, the solution (2.2) is not unique. Non-uniqueness is given by the transformation

\[
\hat{V}_\mu \rightarrow \hat{V}_\mu + X_\mu, \quad \hat{F}_{\mu\nu} \rightarrow \hat{F}_{\mu\nu} + D_\mu X_\nu - D_\nu X_\mu, 
\]

which one understands as freedom to define the physical fields \( V_\mu \) and the field strengths \( F_{\mu\nu} \), in accord with [7].
The action for the non-commutative gauge theory is given as usual by
\[ S = -\frac{1}{2} \text{Tr} \int d^4x \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}. \] (2.5)

The expansion of (2.5) in the deformation parameter \( h \) using SW map reads
\[ S_1 = -\frac{1}{2} \text{Tr} \int d^4x F_{\mu\nu} F^{\mu\nu} + h\theta^{\mu\nu} \text{Tr} \int d^4x \left( \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} - F_{\mu\rho} F_{\nu\sigma} \right) F^\rho^\sigma, \] (2.6)

As analyzed in [7], when the SW map (2.3) and the non-uniqueness (2.4) are taken into account, one obtains the following action up to first order in \( h \):
\[ S = -\frac{1}{2} \text{Tr} \int d^4x F_{\mu\nu} F^{\mu\nu} + h\theta^{\mu\nu} \text{Tr} \int d^4x \left( \frac{a}{4} F_{\mu\nu} F_{\rho\sigma} - F_{\mu\rho} F_{\nu\sigma} \right) F^\rho^\sigma, \] (2.7)

where \( a \) is an arbitrary real parameter. It is important to notice that the \( h \)-linear terms in (2.6) and (2.7) depend on the representation of gauge fields: they are proportional to the trace of the product of three group generators. This would not be of importance if gauge fields were Lie algebra valued, but in the NC case, they live in the enveloping algebra. Thus the non-commutative correction to the gauge field action depends on the representations of fields in a given theory.

To find the classical action, we follow [6]. For non-Abelian vector fields, from (2.7) we obtain
\[ S_{\text{NCYM}} = \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right. \right. \]
\[ \left. \left. + \frac{1}{4} h\theta^{\mu\nu} d^{abc} \left( \frac{a}{4} F_{\mu\nu}^a F_{\rho\sigma}^b - F_{\mu\rho}^a F_{\nu\sigma}^b \right) F^{c\rho\sigma} \right), \] (2.8)

where \( d^{abc} \) are totally symmetric coefficients of the SU(N) group. This action, for \( a = 1 \), corresponds to the classical action \( S_{\text{mNCSM}} \), i.e. to Eq. (24) constructed in [9]. Here, for the general SU(N) gauge group, \( a, b, c = 1, \ldots, N^2 - 1 \) are the group indices. Finally, note that in [7] the action of the type (2.8) was absent owing to the special choice of the representation SU(3)\(_C\) group where the coefficient \( \kappa^{abc}_3 \) was zero.

### 3 One-loop renormalization

To perform the one-loop renormalization of the NC SU(N) gauge part action (2.7), we apply, as before [6,7], the background-field method [21,22]. As we
have already explained the details of the method in [16], here we only discuss the points needed for this computation. The main contribution to the functional integral is given by the Gaussian integral. However, technically, this is achieved by splitting the vector potential into the classical-background and the quantum-fluctuation parts, that is, \( \phi_V \rightarrow \phi_V + \Phi_V \), and by computing the terms quadratic in the quantum fields. In this way we determine the second functional derivative of the classical action, which is possible since our interactions (2.8) are of the polynomial type. The quantization is performed by the functional integration over the quantum vector field \( \Phi_V \) in the saddle-point approximation around the classical (background) configuration \( \phi_V \).

First, an advantage of the background-field method is that it guarantees covariance, as in doing the path integral the local symmetry of the quantum field \( \Phi_V \) is fixed, while the gauge symmetry of the background field \( \phi_V \) is manifestly preserved.

Since we are dealing with gauge symmetry, our Lagrangian (2.8) is singular owing to its invariance under the gauge group. Therefore, a proper quantization of (2.8) requires the presence of the gauge fixing term \( S_{gf}[\phi] \) in the one-loop effective action

\[
\Gamma[\phi] = S_{cl}[\phi] + S_{gf}[\phi] + \Gamma^{(1)}[\phi], \quad S_{gf}[\phi] = -\frac{1}{2} \int d^4x (D_\mu \Phi_\nu^* \Phi^\mu)^2, \quad (3.9)
\]

producing the standard result of the commutative part of our action (2.7). In \( S_{gf} \) from (3.9) we have chosen the Feynman gauge ‘\( \alpha = 1 \)’.

The one-loop effective part \( \Gamma^{(1)}[\phi] \) is given by

\[
\Gamma^{(1)}[\phi] = \frac{i}{2} \log \det S^{(2)}[\phi] = \frac{i}{2} \text{Tr} \log S^{(2)}[\phi]. \quad (3.10)
\]

In (3.10), the \( S^{(2)}[\phi] \) is the second functional derivative of the classical action

\[
S^{(2)}[\phi] = \frac{\delta^2 S_{cl}}{\delta \phi_V_1 \delta \phi_V_2}. \quad (3.11)
\]

The structure of \( S^{(2)}[\phi] \) is

\[
S^{(2)} = \Box + N_1 + N_2 + T_2 + T_3 + T_4, \quad (3.12)
\]

where \( N_1, N_2 \) are commutative vertices, while \( T_2, T_3, T_4 \) are non-commutative ones. The indices denote the number of classical fields. The one-loop effective action computed by using the background-field method is
\[ \Gamma_{\theta, 2}^{(1)} = \frac{i}{2} \text{Tr} \log \left( I + \square^{-1}(N_1 + N_2 + T_2 + T_3 + T_4) \right) \]  
\[ = \frac{i}{2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \text{Tr} \left( \square^{-1}N_1 + \square^{-1}N_2 + \square^{-1}T_2 + \square^{-1}T_3 + \square^{-1}T_4 \right)^n. \]  

For dimensional reasons, the divergences in \( h\theta \)-linear order are all of the forms \( h\theta FV^4 \), \( h\theta F^2V^2 \) and \( h\theta F^3 \). Since the \( h\theta \)-3-, \( h\theta \)-4-, \( h\theta \)-5- and \( h\theta \)-6-vertices obtain divergent contributions, from the sum (3.13) we need to extract and compute only terms that contain up to three external field strengths.

As the conventions and the notation are the same as in [6], we only encounter and discuss the intermediate and the final results.

Using the previously introduced notation, the vertices read

\[ (N_1)^{ab\alpha\beta} = -2i(V^\mu)^{ab} g^{\alpha\beta} \partial_\mu, \]  
\[ (N_2)^{ab\alpha\beta} = -2 F^{abc} F^{\alpha\beta \gamma} - (V^\mu V^\nu)^{ab} g^{\alpha\beta}, \]  

which are the same as in the commutative case. Noncommutative vertices are

\[ (T_2)^{ab\alpha\beta} = \frac{h}{8} d^{abc} \left\{ \left[(a\theta^{\sigma\rho} F^{\sigma\rho}_{\gamma\nu} g^{\alpha\beta} - 2(\alpha - 1)\theta^{\alpha\mu} F^{3\beta\gamma} + 4\theta^{\alpha}\rho F^{3\beta\gamma}\rho g^{\mu\nu} 
+ 4\theta^{\mu}\rho F^{3\beta\gamma}\rho g^{\alpha\beta} \right] (\beta \leftrightarrow \nu) + [\alpha \leftrightarrow \beta] \right\} \partial_\mu \partial_\nu, \]  
\[ (T_3)^{ab\alpha\beta} = \frac{i h}{4} \{ d^{abc} \left[ -2a\theta^{\alpha\rho}(V^\nu)^{bc} F^{3\beta\gamma} - 2a\theta^{\beta\nu}(V^\mu)^{bc} F^{3\alpha\beta} \right. \]  
\[ \left. - a\theta^{\rho\sigma}(V^\mu)^{bc} F^{3\gamma\nu} + a\theta^{\sigma\rho}(V^\nu)^{bc} F^{3\gamma\nu} - 2\theta^{\alpha\rho}(V^\mu)^{bc} F^{3\beta\gamma} - 2\theta^{\beta\nu}(V^\mu)^{bc} F^{3\alpha\beta} \right. \]  
\[ + 2\theta^{\rho\sigma}(V^\mu)^{bc} F^{3\gamma\nu} + 2\theta^{\rho\sigma}(V^\nu)^{bc} F^{3\gamma\nu} \} \partial_\mu \partial_\nu, \]  
\[ (T_4)^{ab\alpha\beta} = \frac{h}{8} d^{abc} \left\{ \left[-4a\theta^{\alpha\rho}(V^\mu)^{bc} F^{3\beta\gamma} - a\theta^{\rho\sigma}(V^\mu)^{bc} F^{3\gamma\nu} + a\theta^{\sigma\rho}(V^\nu)^{bc} F^{3\gamma\nu} - 4\theta^{\alpha\rho}(V^\beta)^{ad} F^{3\gamma\mu} F^{3\alpha\beta} \right. \]  
\[ \left. + 4\theta^{\alpha\rho}(V^\nu)^{bc} F^{3\beta\gamma} - 4\theta^{\rho\sigma}(V^\mu)^{bc} F^{3\gamma\nu} \right\} \partial_\mu \partial_\nu. \]  

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\[ + 2\theta^{\alpha\beta}(V_\mu)^{ad}(V_\nu)^{bc} F^{e\mu\nu} + 4\theta^{\alpha\rho}(V_\mu)^{ad}(V_\rho)^{bc} F^{e\beta\mu} \]
\[ + 2\theta^{\sigma\tau}(V_\mu)^{ad}(V_\sigma)^{bc} F^{e\alpha\beta} \] (\[a \leftrightarrow b, \alpha \leftrightarrow \beta\])
\[ + \int f^{abc}(2\alpha\theta^{\sigma\tau} F^{e\sigma\beta} + \alpha\theta^{\alpha\beta} F^{e\rho\sigma} \]
\[ + 4\theta^{rd\alpha} F^{e\beta\sigma} + 8\theta^{\alpha\rho} F^{e\beta\mu} F^{e}_{\mu\rho}\]. \tag{3.18}

The divergent parts are calculated in the momentum representation via dimensional regularization, by picking relevant terms out of the expansion (3.13). The resulting contributions are given by

\[
D_1^{\text{div}} = \frac{i}{2} \text{Tr} \left( (\Box^{-1} N_1)^2 (\Box^{-1} T_4) \right)^{\text{div}} \tag{3.19}
\]
\[= \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4 x \left[ \frac{a-3}{4} (\theta^{\mu\nu} F^{a}_{\alpha\nu} + \theta^{\alpha\mu} F^{a}_{\alpha\mu})(V_\mu V_\nu V_\rho V_\nu)^{bc} \right. \]
\[\left. + \frac{3a-4}{4} \theta^{\alpha\beta} F^{a}_{\alpha\beta}(V_\mu \nu \nu \nu V_\nu V_\nu)^{bc} \right],
\]
\[
D_2^{\text{div}} = -\frac{i}{2} \text{Tr} \left( (\Box^{-1} N_1)^3 (\Box^{-1} T_3) \right)^{\text{div}} \tag{3.20}
\]
\[= \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4 x \left[ \frac{7a-3a}{6} \theta^{\alpha\beta} F^{a}_{\alpha\beta}(V_\mu V_\nu V_\nu V_\nu + V_\mu V_\nu V_\nu V_\nu \right. \]
\[\left. + V_\mu V_\nu V_\nu V_\nu)^{bc} + \frac{3-2a}{6} (\theta^{\alpha\mu} F^{a}_{\alpha\nu} \right. \]
\[\left. + \theta^{\alpha\nu} F^{a}_{\alpha\mu})(V_\mu V_\nu V_\rho V_\rho + V_\mu V_\nu V_\nu V_\nu \right. \]
\[\left. + V_\mu V_\nu V_\nu V_\nu)^{bc} \right],
\]
\[
D_3^{\text{div}} = \frac{i}{2} \text{Tr} \left( (\Box^{-1} N_1)^4 (\Box^{-1} T_2) \right)^{\text{div}} \tag{3.21}
\]
\[= \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4 x \left[ \frac{7a-11}{12} \theta^{\alpha\beta} F^{a}_{\alpha\beta}(V_\mu V_\nu V_\nu V_\nu + V_\mu V_\nu V_\nu V_\nu \right. \]
\[\left. + V_\mu V_\nu V_\nu V_\nu)^{bc} + \frac{a-3}{12} (\theta^{\alpha\mu} F^{a}_{\alpha\nu} \right. \]
\[\left. + \theta^{\alpha\nu} F^{a}_{\alpha\mu})(2V_\rho V_\mu V_\nu V_\nu + 2V_\rho V_\mu V_\nu V_\nu \right. \]
\[\left. + V_\rho V_\nu V_\nu V_\rho + V_\rho V_\nu V_\nu V_\nu)^{bc} \right],
\]
\[
D_4^{\text{div}} = -\frac{i}{2} \text{Tr} \left( (\Box^{-1} N_2)(\Box^{-1} T_4) \right)^{\text{div}} \tag{3.22}
\]
\[= \frac{h}{(4\pi)^2 \epsilon} d^{abc} \int d^4 x \left[ \frac{4-3a}{4} \theta^{\alpha\beta} F^{a}_{\alpha\beta}(V_\mu V_\nu V_\nu V_\mu)^{bc} \right].
\]
\[+ \frac{2-a}{2} (\theta^{\alpha \mu} F_{\alpha \nu}^a + \theta_{\alpha \nu} F^{a \alpha \mu}) (V_\mu V^\rho V_\rho V_\nu)^{bc}\]
\[+ 2(a+1)i \theta_{\alpha \nu} F_{\beta \mu}^a (V^\mu F^{\alpha \beta} V_\nu)^{bc}\]
\[+ \frac{2-a}{2} i \theta^{\mu \nu} F_{\mu \nu}^a (V^\alpha F_{\alpha \beta} V_\beta)^{bc}\]
\[+ 2i \theta_{\alpha \beta} F_{\beta \mu}^a (V_\mu F^{\alpha \beta} V_\nu - V^\mu F_{\alpha \nu} V_\nu)^{bc}\]
\[+ i \theta_{\alpha \beta} F_{\mu \nu}^a (V^\mu F_{\alpha \beta} V_\nu - V_\alpha F^{\mu \nu} V_\beta)^{bc}\]
\[- 2N \theta_{\beta \mu} F_{\mu \nu}^a F_{\alpha \nu}^{bc} + N \theta^{\mu \nu} F_{\alpha \nu}^a F_{\beta \nu}^{bc} - aN \theta^{\mu \nu} F_{\mu \nu}^a F_{\alpha \nu}^{bc} - \frac{3N}{4} \theta^{\mu \nu} F_{\mu \nu}^a F_{\alpha \nu}^{bc} F^{\alpha \nu}\]
\[D_5^{\text{div}} = \frac{i}{2} \text{Tr} \left[ \left( \Box^{-1} N_2 \right)^2 \left( \Box^{-1} T_2 \right) \right]^{\text{div}} \]
\[= \frac{h}{(4\pi)^2 \epsilon} \int d^4x \left[ \frac{a-3}{2} (\theta^{\alpha \mu} F_{\alpha \nu}^a + \theta_{\alpha \nu} F^{a \alpha \mu}) (F_{\mu \rho} F^{\nu \rho})^{bc} \right. \]
\[+ \frac{3a-4}{4} \theta_{\alpha \beta} F_{\alpha \beta} (F_{\mu \nu} F^{\mu \nu})^{bc} \]
\[\left. + \frac{4a-7}{4} \theta_{\alpha \beta} F_{\alpha \beta} (V^\mu V_\mu V^\rho V_\rho)^{bc} \right], \quad (3.23)\]
\[D_6^{\text{div}} = \frac{i}{2} \text{Tr} \left[ \left( \Box^{-1} N_1 \right) \left( \Box^{-1} N_2 \right) \left( \Box^{-1} T_3 \right) \right]^{\text{div}} \]
\[+ \left( \Box^{-1} N_2 \right) \left( \Box^{-1} N_1 \right) \left( \Box^{-1} T_3 \right) \right]^{\text{div}} \]
\[= \frac{h}{(4\pi)^2 \epsilon} \int d^4x \left[ \frac{a-3}{2} \theta^{\alpha \mu} F_{\alpha \nu}^a (2V_\mu V_\rho V^\rho V_\nu + V_\nu V^\rho V_\mu V_\nu \right. \]
\[+ V_\rho V^\rho V_\nu V_\mu)^{bc} \]
\[+ \frac{5a-4}{4} \theta_{\alpha \beta} F_{\alpha \beta} (V^\mu V_\mu V^\rho V_\rho)^{bc} \]
\[+ \frac{3a-4}{4} \theta_{\alpha \beta} F_{\alpha \beta} (V^\mu V_\mu V^\rho V_\rho)^{bc} \]
\[- \frac{a}{2} \theta_{\alpha \beta} F_{\alpha \beta} (V^\mu V^\rho V_\mu V_\nu)^{bc} - \frac{a}{4} \theta_{\alpha \beta} F_{\alpha \beta} (F_{\mu \nu} F_{\mu \nu})^{bc} \]
\[- 2i(a+1) \theta_{\alpha \beta} F_{\mu \nu}^a (V_\nu F^{\beta \mu} V^\alpha)^{bc} \]
\[+ (a+1) \theta_{\alpha \beta} F_{\mu \nu}^a (F^{\alpha \nu} F^{\beta \mu})^{bc} \]
\[- 2i \theta^{\alpha \mu} F_{\alpha \nu}^a (V^\nu F_{\mu \rho} V_\rho)^{bc} + 2i \theta_{\alpha \mu} F_{\alpha \nu}^a (V_\mu F^{\nu \rho} V_\rho)^{bc} \]
\[ + \theta^\alpha \mu F^a_{\alpha \nu} (F^{\nu \rho} F_{\mu \rho})^{bc} + \theta^\alpha \nu F^a_{\alpha \mu} (F_{\mu \nu} F^{\nu \rho})^{bc} \]

\[ + \frac{i}{2} \theta^\alpha \beta F^a_{\alpha \beta} (V^{\mu} F_{\mu \nu} V^{\nu})^{bc} - if^\alpha \beta F^a_{\mu \nu} (V^{\mu} F_{\alpha \beta} V^{\nu})^{bc} \],

\[ D_7^{\text{div}} = -\frac{i}{2} \text{Tr} \left( \sum (\Box^{-1} N_1)^2 (\Box^{-1} N_2) (\Box^{-1} T_2) \right)^{\text{div}}, \]

\[ = \frac{h}{(4\pi)^2 \epsilon} \int d^4 x \left[ \frac{18 - 11 a}{12} \theta^{\alpha \beta} F^a_{\alpha \beta} (2 V^{\mu} V^{\nu} V^{\nu} V^{\nu} + V^{\mu} V^{\nu} V^{\nu} V^{\mu})^{bc} \right. \]

\[ + \left. \frac{3 - a}{6} (\theta^{\alpha \mu} F^a_{\alpha \nu} + \theta^{\alpha \nu} F^{a \alpha \mu})(V^{\mu} V^{\nu} V^{\rho} V^{\rho})^{bc} \right. \]

\[ + V^{\mu} V^{\rho} V^{\nu} V^{\mu} - V^{\mu} V^{\rho} V^{\nu} V^{\rho})^{bc} \].

Their sum, that is the one-loop divergent part of the gauge boson interaction

\[ \sum_{i=1}^{7} D_i^{\text{div}} = \frac{N}{(4\pi)^2 \epsilon} h \theta^{\mu \nu} d^{abc} \int d^4 x \left( -\frac{25 a - 3}{48} F^a_{\mu \nu} F^b_{\rho \sigma} F^{c \rho \sigma} \right. \]

\[ + \frac{a + 21}{12} F^a_{\mu \nu} F^b_{\rho \sigma} F^{c \rho \sigma} \), \]

has been obtained after the dimensional regularization and summation of all the contributions. Therefore, the total divergent contribution to the effective action (2.8) is

\[ \Gamma^{\text{div}} = \frac{11}{6} \frac{N}{(4\pi)^2 \epsilon} \int d^4 x F^a_{\mu \nu} F^{\mu \nu a} \]

\[ + \frac{N}{(4\pi)^2 \epsilon} h \theta^{\mu \nu} d^{abc} \int d^4 x \left( -\frac{25 a - 3}{48} F^a_{\mu \nu} F^b_{\rho \sigma} + \frac{a + 21}{12} F^a_{\mu \nu} F^b_{\rho \sigma} \right) F^{c \rho \sigma}. \]

In the above expression the ghost contribution to the one-loop effective action is included.

We are interested in the renormalization of the theory. Our starting Lagrangian in \( D = 4 - \epsilon \) dimensional space-time has the following form:

\[ \mathcal{L} = -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} + \frac{1}{4} g \mu^{\epsilon/2} h \theta^{\mu \nu} d^{abc} \left( \frac{a}{4} F^a_{\mu \nu} F^{b}_{\rho \sigma} - F^a_{\mu \nu} F^{b}_{\mu \sigma} \right) F^{c \rho \sigma}, \]

where \( g \) is the gauge coupling constant and \( \mu \) is the subtraction point mass parameter or the so-called renormalization point. In order to cancel divergences, counter terms should be added to the starting action, which produces the bare Lagrangian from (3.27):
\[ \mathcal{L}_0 = -\frac{1}{4} F_{\mu \nu}^a F_{a \mu \nu} - \frac{11 N g^2}{6(4\pi)^2 \epsilon} F_{\mu \nu}^a F_{a \mu \nu} \]

\[ + \frac{1}{4} g \mu^{\epsilon/2} h \theta^{\mu \nu \rho \sigma} \left( \frac{a}{4} F_{\mu \nu}^b F_{a \rho \sigma} - F_{a \mu}^b F_{\nu \sigma} \right) F_{c \rho \sigma}^c \]

\[ - \frac{N g^3 \mu^{\epsilon/2}}{(4\pi)^2 \epsilon} h \theta^{\mu \nu \rho \sigma} d^{abc} \left( - \frac{25a - 3}{48} F_{\mu \nu}^a F_{a \rho \sigma} + \frac{a + 21}{12} F_{\mu \nu}^a F_{a \rho \sigma} \right) F_{c \rho \sigma}^c \]

\[ = -\frac{1}{4} F_{0 \mu \nu}^a F_{0 \mu \nu}^a + \frac{1}{4} g \mu^{\epsilon/2} h \theta^{\mu \nu \rho \sigma} d^{abc} \left[ \frac{a}{4} \left( 1 + \frac{25a - 3}{3a} \right) \frac{N g^2}{(4\pi)^2 \epsilon} F_{\mu \nu}^a F_{a \rho \sigma} \right] F_{c \rho \sigma}^c. \] (3.29)

It is easy to see that in order to keep the ratio of the coefficients of two terms from (3.29) the same as in the classical action (2.8), one has to impose the condition

\[ \left( -\frac{25a - 3}{48} : \frac{a + 21}{12} \right) = \frac{a}{4} : (-1). \] (3.30)

Interestingly enough, this equation has two solutions, \( a = 1 \) and \( a = 3 \). In our previous paper [6] the action (2.8) was discussed and renormalizability was proved for \( a = 1 \). In this case, the divergences are cancelled through redefinition of the gauge potential and the coupling constant. The case \( a = 3 \) is different since the deformation parameter \( h \) has to be renormalized. The bare gauge field, the coupling and the deformation parameter are defined as follows:

\[ V_0^\mu = V^\mu \sqrt{1 + \frac{22Ng^2}{3(4\pi)^2 \epsilon}}, \] (3.31)

\[ g_0 = \frac{g \mu^{\epsilon/2}}{\sqrt{1 + \frac{22Ng^2}{3(4\pi)^2 \epsilon}}}, \] (3.32)

\[ h_0 = \frac{h}{1 - \frac{2Ng^2}{3(4\pi)^2 \epsilon}}, \] (3.33)

with an arbitrary choice for the renormalization point \( \mu \). Any change in \( \mu \) is compensated by the corresponding change in the charge \( g \), the deformation parameter \( h \) and for the scale of the fields. The above result means that it is not possible to renormalize our action, for \( a = 3 \), only through the renormalization of the vector potential and the coupling constant.
In this section we investigate the high-energy behaviour of our theory (2.8) by employing the renormalization group equation (RGE) and compute the relevant $\beta$ functions. The RGE provides a framework within which we discuss the ultraviolet (UV) asymptotic behaviour of renormalizable gauge field theory (GFT), i.e. the behaviour of the relevant amplitudes in a physically uninteresting region, i.e. in a region for large $g$ and/or far from the origin.

Since the gauge coupling constant $g$ in our theory (2.8) depends on the renormalization point $\mu$ satisfying the same beta function

$$\beta_g = \mu \frac{\partial}{\partial \mu} g(\mu) = -\frac{11N g^3(\mu)}{3(4\pi)^2}, \quad (4.34)$$

as for the commutative Yang-Mills theory without fermions and with gauge independence in lowest order ($g^3$), our theory is UV stable. This means that (2.8) belongs to the class of asymptotically approaching free-field theories, or in short ‘asymptotically free theory’. The solution to (4.34) is the very well-known result [23,24]

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} = \frac{6\pi}{11N \ln \frac{\mu}{\Lambda}}. \quad (4.35)$$

In (4.35), $\Lambda$ is an integration constant not predicted by the theory, thus it is a free parameter to be determined from the experiment. The QCD (physical) interpretation of $\Lambda$ is that it represents the marking of the boundary between a world of quasi-free quarks and gluons and the world of protons, pions, and so on. For typical QCD energies $\mu$ with $m_b \ll \mu \ll m_t$, where fermions are included ($N_f = 5$), the study of hadronic production in $e^+ e^-$ annihilation at the Z resonance has given a direct measured value $\alpha_s(m_Z) = 0.12$ corresponding to $\Lambda = \Lambda_{\text{QCD}} \simeq 250$ MeV.

The $\beta$ function for the deformation parameter $h$ can be easily computed from (3.33) and (4.34):

$$\beta_h = \mu \frac{\partial}{\partial \mu} h(\mu) = -\frac{11N g^2(\mu)}{24\pi^2} h(\mu). \quad (4.36)$$

Since both $\beta$ functions (4.34) and (4.36) are negative, both the coupling $g$ and the parameter $h$ decrease with increasing energy.
Solving equation (4.36) we obtain

\[ h(\mu) = \frac{h_0}{\ln \frac{\mu}{\Lambda}} , \]  

(4.37)

which is the *running deformation parameter* \( h \). Here \( h_0 \) is an additional integration constant whose physical interpretation is going to be discussed later. From the above expression we see that with the increase of energy the deformation parameter decreases, which might seem counterintuitive in the view of Heisenberg uncertainty relations. However, there are many arguments for modification of uncertainty relations at high energy [25]. For example, if the commutation relation is

\[ [x, p] = i\hbar (1 + \beta p^2) , \]  

(4.38)

where \( \beta \) is a constant and has dimension energy\(^{-2} \), then one can easily see that in the region of the large momentum \( \Delta x \) grows linearly [26]

\[ \Delta x = \frac{\hbar}{2} \left( \frac{1}{\Delta p} + \beta \Delta p \right) . \]  

(4.39)

From this example it follows that large energies do not necessarily correspond to small distances, i.e. the behaviour of the running deformation parameter (3.33) does not imply that non-commutativity vanishes at small distances. This is related to the UV/IR correspondence.

Owing to the necessity of the renormalization of the non-commutativity deformation parameter \( h \), the scale of non-commutativity \( \Lambda_{NC} \) becomes a function of energy \( \mu \) too,

\[ \Lambda_{NC}(\mu) = \Lambda_\theta \sqrt{\ln \frac{\mu}{\Lambda}} . \]  

(4.40)

Equivalently to (4.37), the scale \( \Lambda_{NC} \) becomes the *running scale of non-commutativity*. Here \( \Lambda_\theta \) is an additional integration constant, namely the dimension of energy, not predicted by the theory.

Even though that the physical interpretation of \( h_0 \) and/or \( \Lambda_\theta \) is not quite clear, it seems that, owing to the energy dependence, they have to be proportional to the scale of non-commutativity \( \Lambda_{NC} \). If one could think of \( h_0 \) and/or \( \Lambda_\theta \) as a boundary between a world of commutative fields (particles) and non-commutative quantum fields, then, according to (4.37) and (4.40), it would be obvious to assume that in a first approximation \( h_0 = 1/\Lambda_\theta^2 = 1/\Lambda_{NC}^2 \).

Considering typical QCD energies, the factor \( \sqrt{\ln(m_Z/\Lambda_{QCD})} \simeq 2.4 \), which
means that at such energies the scale of non-commutativity $\Lambda_{NC}$ is effectively shifted by a factor of $\simeq 2.4$ up.

5 Discussion and conclusion

We have constructed a version of the SU(N) model on non-commutative Minkowski space at first order in the deformation parameter $h$, which has the one-loop multiplicative renormalizable gauge sector.

We have shown in [6] that if the gauge fields are in the adjoint representation of SU(N), the action (2.6) is renormalizable. Trying to extend this result to the gauge group of the standard model $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$, we have seen [7] that the action of the type (2.6) cannot be renormalized. However, with a suitable choice of the representations of the gauge group, the theory is renormalizable and finite for the SW freedom parameter $a = 3$ [7]. This naturally poses a question: is the obtained result, $a = 3$, just an outcome of a specific interaction among gauge bosons in the NCSM or is there something generic about it?

In order to answer the above question, in this paper we reconsider the renormalizability of the ‘building blocks’, i.e. of the non-commutative SU(N) theories described by the action (2.8), for arbitrary values of the freedom parameter $a$. If we want to have renormalized theory, we have found that $a$ has to be either 1 or 3. In this paper we have analyzed the case $a = 3$. The relevant beta functions, $\beta_g$ and $\beta_h$, have been computed and they are both negative. The non-commutative deformation parameter $h$ becomes the running deformation parameter and vanishes for large energies. However, owing to the inverse square behaviour, the non-commutative scale runs according to (4.40). The function (4.40) is very smooth and mild, showing a small change of the scale of non-commutativity as the energy increases. We consider this property very welcome, because it shows a large degree of stability of our theory within a wide range of energy.

The one-loop renormalizability of the non-commutative SU(N) gauge sector is certainly a very encouraging result, both theoretically and experimentally. So far, this property has not concerned fermions: the results on the renormalizability of NC theories including the Dirac fermions are not yet positive, [14,16], i.e. till now fermions in non-commutative theory have still spoilt the stable behaviour of (4.34) and (4.36) owing to non-renormalizability.

Non-Abelian commutative theories are completely renormalizable. However, fermions spoil the stable behaviour of a gauge boson beta function, but they leave room to spare, and the theory becomes asymptotically free as long as
The possibility that something similar could happen in the case of the NCSM [7–11], at first order in the NC parameter $\hbar \theta^{\mu \nu}$, is a matter for other, fermion involving, studies.

The necessity of the $\hbar$ renormalization jeopardizes the previous hope that the NC SU(N) gauge theory might be renormalizable to all orders in $\theta^{\mu \nu}$. This means that most probably the theory would need to be renormalized independently order by order in the non-commutative parameter $\theta^{\mu \nu}$.

Anyhow, the present result could be an indication that the inclusion of fermions into a renormalizable non-commutative theory might be possible by a more careful choice of freedom parameters, of the representations of a gauge group and of the renormalization of the non-commutative parameter $\theta^{\mu \nu}$ as well.

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