Global dynamics of cosmological scalar fields – Part II

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Abstract

This is the second part of integrability analysis of cosmological models with scalar fields. Here, we study systems with conformal coupling, and show that apart from four cases, where explicit first integrals are known, the generic system is not integrable. We also comment on some chaotic properties of the system, and the issues of integrability restricted to the real domain.

1 Introduction

Conformally coupled fields were subject to more rigorous integrability analysis, as opposed to minimally coupled ones, thanks to the natural form of their Hamiltonian. As will be shown
in the next section, the kinetic part is of natural form, albeit indefinite, and the potential is polynomial (in the case of real fields).

Ziglin proved that the system given by (7) is not meromorphically integrable when \( \Lambda = \lambda = 0 \) and \( k = 1 \) [18]. His methods were then used by Yoshida to homogeneous potentials which is the case for the system when \( k = 0 \) [14, 15, 16, 17]. Later, Yoshida’s results were sharpened by Morales-Ruiz and Ramis [11], and used by the present authors in [9] to obtain countable families (restrictions on \( \lambda \) and \( \Lambda \)) of possibly integrable cases. Also recently, more conditions for integrability have been given in [2], although only for non-zero spatial curvature \( k \) and a generic value of energy, that is, when the particular solution is a non-degenerate elliptic function (in particular not for zero energy).

Our work shows, that the conjecture of that paper is in fact correct – as shown in Section 4 – the system is only integrable in two cases (with the above assumptions). We go further than that and show that for a generic energy value, a spatially flat \( (k = 0) \) universe is only integrable in four cases. Also, the particular case of zero energy is analysed and new, simple conditions on the model parameters are found. Finally, we check that when \( E = k = 0 \), the problem remains open, as the necessary conditions for integrability are fulfilled.

When it comes to numerical studies of the problem, there are various results, most notably chaotic behaviour [6], but also fractal structure and chaotic scattering [13]. However, it remains unclear whether the widely exercised complex rotation of the variables changes the system’s integrability. Even for very simple systems it was shown [3] that there might exist smooth integrals, which are not real-analytic. This question is especially vital since our Universe clearly has real initial conditions and dynamics.

In what follows, we derive the Hamiltonian system for the conformally coupled scalar field, and proceed to analyse its integrability. For the basics of the theory used, and relevant, more detailed literature see the first part [12].

## 2 Conformally coupled scalar fields

The procedure of obtaining the Hamiltonian is the same as in the case of minimally coupled fields, only this time the action is

\[
\mathcal{I} = \frac{c^4}{16\pi G} \int \left[ \mathcal{R} - 2\Lambda - \frac{1}{2} \left( \nabla_{\alpha} \tilde{\psi} \nabla^\alpha \psi + \frac{m^2}{\hbar^2} |\psi|^2 + \frac{1}{6} \mathcal{R} |\psi|^2 \right) - \frac{\lambda}{4!} |\psi|^4 \right] \sqrt{-g} d^4x, \quad (1)
\]

where an additional coupling to gravity through the Ricci scalar \( \mathcal{R} \), and a quartic potential term with constant \( \lambda \) are present, as opposed to the minimal scenario. The cosmological constant \( \Lambda \) and the mass \( m \) remain as previously. We keep the same notation as before and express the involved quantities in comoving coordinates and conformal time to get

\[
\mathcal{L} = 6(a'' + Ka^2) - \frac{1}{2} a'' a |\psi|^2 + \frac{1}{2} |\psi'|^2 a^2 + \frac{m^2}{2\hbar^2} a^4 |\psi|^2 - \frac{\lambda}{4!} a^4 |\psi|^4 - 2\Lambda a^4 - \frac{1}{2} Ka^2 |\psi|^2, \quad (2)
\]

from which we remove a full derivative, and introduce new field variables \( \psi = \sqrt{\mathcal{I}_2} \phi \exp(i\theta)/a \) to obtain

\[
\mathcal{L} = 6 \left[ \phi'^2 + \phi^2 \theta'^2 - a'^2 + K(a^2 - \phi^2) - \frac{m^2}{\hbar^2} a^2 \phi^2 - \frac{\Lambda}{3} a^4 - \lambda \phi^4 \right]. \quad (3)
\]
The associated Hamiltonian is

\[ H = \frac{1}{24} \left( p^2 + \frac{1}{\phi^2} p_a^2 - p_a^2 \right) + 6 \left[ K (\phi^2 - a^2) + \frac{m^2}{\hbar^2} a^2 \phi^2 + \lambda \phi^4 + \frac{\Lambda}{3} a^4 \right]. \tag{4} \]

We can see that \( \theta \) is a cyclical variable because we took the potential to depend on the modulus of \( \psi \) only, so we write a constant instead of the respective momentum \( p_\theta = \omega \).

Finally, we express everything in dimensionless quantities, rescaling the constants, but also the time and momenta (as they are in fact time derivatives), which results in rescaling the whole Hamiltonian. We do this as follows

\[ m^2 \rightarrow m^2 \hbar^2 |K|, \quad \Lambda \rightarrow \frac{3}{2} \Lambda |K|, \quad \lambda \rightarrow \frac{1}{2} \lambda |K|, \quad p_x^2 \rightarrow 144 \hbar^2 |K|^2|K|, \quad H \rightarrow \frac{1}{12 \sqrt{|K|}} H, \tag{5} \]

when \( K \neq 0 \), and using another of the dimensional constants otherwise. Thus, eliminating a multiplicative constant, the Hamiltonian reads

\[ H = \frac{1}{2} \left( p_\phi^2 - p_a^2 \right) + 12 \left[ k (\phi^2 - a^2) + \frac{\omega^2}{\phi^2} + m^2 \phi^2 \right] + \frac{1}{4} \left( \Lambda a^4 + \lambda \phi^4 \right), \tag{6} \]

with \( k \in \{ -1, 0, 1 \} \) (\( K = k |K| \)); \( \omega, \lambda, \Lambda, m^2 \in \mathbb{R} \), and \( H = 0 \) in any physically possible setup. However, the addition of radiation component, which scales like \( a^{-4} \) in the original action, leads to a constant in the Hamiltonian. Thus, it justifies studying energy levels other than zero as well.

We note that for \( m = 0 \) the system decouples, and is trivially integrable as shown in Appendix A. That is why we will assume \( m \neq 0 \) henceforth. We will also take \( \omega = 0 \), that is, consider a scalar field equivalent to a real field after a unitary rotation in the complex \( \psi \) plane.

We change the field variables into the standard \( q \) and \( p \) ones for further computation, taking

\[ a = q_1, \quad p_a = p_1, \quad \phi = q_2, \quad p_\phi = p_2. \tag{7} \]

The Hamiltonian is then

\[ H = \frac{1}{2} \left( -p_1^2 + p_2^2 \right) + V, \]

\[ V = \frac{1}{2} \left[ k(-q_1^2 + q_2^2) + m^2 q_1^2 q_2^2 \right] + \frac{1}{4} \left( \Lambda q_1^4 + \lambda q_2^4 \right). \tag{8} \]

### 3 Known integrable families

There are four known cases when the system has an additional first integral, independent of the Hamiltonian. They were found by applying the so called ARS algorithm basing on the Painlevé analysis [1]. The following table summarises those results.

<table>
<thead>
<tr>
<th>solvability case</th>
<th>( k )</th>
<th>( \Lambda )</th>
<th>( m^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0, ±1</td>
<td>( \Lambda = \lambda )</td>
<td>( m^2 = -3\lambda )</td>
</tr>
<tr>
<td>(2)</td>
<td>0, ±1</td>
<td>( \Lambda = \lambda )</td>
<td>( m^2 = -\Lambda )</td>
</tr>
<tr>
<td>(3)</td>
<td>0</td>
<td>( \Lambda = 16\lambda )</td>
<td>( m' = -6\lambda )</td>
</tr>
<tr>
<td>(4)</td>
<td>0</td>
<td>( \Lambda = 8\lambda )</td>
<td>( m^2 = -3\lambda )</td>
</tr>
</tbody>
</table>
And the respective integrals of the systems are

\begin{align}
(1) & \quad \left\{ \begin{array}{l}
H = \frac{1}{2}(p_2^2 - p_1^2) + \frac{k}{2}(q_2^2 - q_1^2) - \frac{m^2}{12}(q_1^4 - 6q_1^2q_2^2 + q_2^4), \\
I = p_1p_2 + \frac{1}{3}(m^2(q_2^2 - q_1^2) - 3k),
\end{array} \right. \\
(2) & \quad \left\{ \begin{array}{l}
H = \frac{1}{2}(p_2^2 - p_1^2) + \frac{k}{2}(q_2^2 - q_1^2) - \frac{m^2}{4}(q_2^2 - q_1^2)^2, \\
I = q_1p_2 + q_2p_1,
\end{array} \right. \\
(3) & \quad \left\{ \begin{array}{l}
H = \frac{1}{2}(p_2^2 - p_1^2) - \frac{m^2}{24}(16q_1^4 - 12q_1^2q_2^2 + q_2^4), \\
I = (q_1p_2 + q_2p_1)p_2 + \frac{m^2}{6}q_1q_2^2(q_2^2 - 2q_1^2),
\end{array} \right. \\
(4) & \quad \left\{ \begin{array}{l}
H = \frac{1}{2}(p_2^2 - p_1^2) - \frac{m^2}{12}(8q_1^4 - 6q_1^2q_2^2 + q_2^4), \\
I = p_1^2 + \frac{m^2q_2^2}{3}\left[4q_1q_2p_1p_2 + q_2^2p_1^2 - (q_2^2 - 6q_1^2)p_2^2 + \frac{1}{12}q_2^2(q_2^2 - 2q_1^2)^2 \right].
\end{array} \right.
\end{align}

In this work, we will show, that the above are the only integrable cases, when \( m \neq 0 \). An important point to note is that there is a complete symmetry with respect to interchanging \( \Lambda \) and \( \lambda \). It is a consequence of the fact, that there exists a canonical transformation of the form

\begin{align}
p_1 & \rightarrow ip_1, \quad q_1 \rightarrow -iq_1, \\
p_2 & \rightarrow p_2, \quad q_2 \rightarrow q_2,
\end{align}

that changes the Hamiltonian to

\[ H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\left[k(q_2^2 + q_1^2) - m^2q_1^2q_2^2 \right] + \frac{1}{4}(\Lambda q_1^4 + \lambda q_2^4), \]

which is the same after swapping the indices. We shall use this form of \( H \), where the kinetic part is in the natural form, to make the use of some already existing theorems more straightforward.

### 4 Integrability of the reduced problem

It is possible to give stringent conditions for integrability of the system, by considering a reduced Hamiltonian. Namely, we can separate potential \( V \) into homogeneous parts of degree 2 and 4:

\[ V = V_{h2} + V_{h4}, \]

\begin{align}
V_{h2} & = \frac{1}{2}k(q_1^2 + q_2^2), \\
V_{h4} & = \frac{1}{4}\left[-2m^2q_1^2q_2^2 + \Lambda q_1^4 + \lambda q_2^4 \right].
\end{align}

The following fact is crucial in our considerations: if a potential \( V \) is integrable then its higheest order as well as the lowest order parts are also integrable. For the proof, see [5] This means that in our case if \( V \) given by (12) is integrable then \( V_{h2} \) and \( V_{h4} \) must also be integrable. \( V_{h2} \) is the
potential of the two-dimensional harmonic oscillator, thus, it is trivially integrable. However, the homogeneous part \( V_{h4} \) gives strong integrability restrictions for the whole potential \( V \). We will call \( V_{h4} \) the reduced potential and denote it by \( \hat{V} \).

Thus we effectively set \( k = 0 \), and are now in position to exercise known theorems concerning homogeneous potentials of two variables. In particular the complete analysis for degree 4 has been completed in [10].

In order to identify our potential with some of the list given in that paper, we have to check how many Darboux points there exist, and what are the values of parameters \( \Lambda, \lambda \) and \( m \) that give potentials equivalent to a particular family.

We say that a non-zero point \((q_1, q_2) = d\) is a Darboux point of the potential \( \hat{V}(q_1, q_2) \) when it satisfies the equation

\[
\hat{V}'(d) = \gamma d,
\]

where \( \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \). Such a point corresponds to a particular solution of the form

\[
q(\eta) = f(\eta)d, \quad p(\eta) = \dot{f}(\eta)d,
\]

with \( f(\eta) \) satisfying the differential equation (for degree 4 potential)

\[
\ddot{f}(\eta) = -\gamma f(\eta)^3.
\]

As explained in the first part of this paper, particular solutions allow for studying the variational equations along them, and yield necessary conditions for existence of additional first integrals. However, the major simplification discovered in [10] is that additionally there is only a finite number or parameter sets (or non-equivalent potentials) corresponding to integrable cases.

Following the cited paper’s exposition (and notation) we find that our potential has:

1. Four simple Darboux points, when \( \Lambda(m^2 + \Lambda)(m^2 + \lambda) \neq 0 \), and \( \Lambda \lambda \neq m^4 \). The only integrable cases are:
   
   (a) \( \lambda = \Lambda = -\frac{3}{4}m^2 \) (\( \hat{V} \) is equivalent to \( V_4 \)),
   
   (b) \( \lambda = -\frac{8}{3}m^2, \quad \Lambda = -\frac{1}{6}m^2 \) (\( \hat{V} \) is equivalent to \( V_5 \)),
   
   (c) \( \lambda = -\frac{8}{3}m^2, \quad \Lambda = -\frac{1}{3}m^2 \) (\( \hat{V} \) is equivalent to \( V_6 \)).

2. Three simple Darboux points, when \( \Lambda = 0 \), and \( \lambda(m^2 + \lambda) \neq 0 \). There are no integrable families here as \( I_{4,3} = \emptyset \).

3. Two simple Darboux points, when either \( \Lambda = \frac{m^4}{\lambda} \) and \( \lambda(m^2 + \lambda) \neq 0 \), or \( \Lambda = \lambda = 0 \). Again, no integrable families are present here because \( I_{4,2} = \emptyset \).

4. A triple Darboux point, when \( \Lambda = -m^2 \). Additionally there is a simple Darboux point when \( \lambda \neq 0 \). The potential is equivalent to \( V_3 \) and is only integrable when \( \lambda = -m^2 \).

There are two immediate implications that follow. Firstly, that the main system itself with \( k = 0 \) is only integrable in those four cases, and the respective first integrals are known, as given in the table. Secondly, as was shown in [2] those cases are the only ones which could be
integrable when $k \neq 0$. This happens because the integrability of the full potential implies the integrability of the homogeneous parts of the maximal and minimal degree (the latter is trivially solvable in our case).

As the table shows, when the potential is equivalent to $V_3$ (or, to be precise, its integrable subcase) or $V_4$, the second first integral is known; but $V_5$ and $V_6$ only have known integrals with zero curvature. And as was shown in [2], for $k = 1$, the values of $\Lambda$ and $\lambda$ are those of $V_5$ or $V_6$ forbid integrability. This is easily extended to the $k = -1$ case, since after the change of variables

$$q_i \rightarrow e^{i\pi/4} q_i, \quad p_i \rightarrow e^{-i\pi/4} p_i, \quad i = 1, 2,$$

we obtain a system with the sign of $k$ changed, but the ratios $m^2/\Lambda$ and $m^2/\lambda$ the same. Thus, concerning the conjecture of the quoted paper, our results for $k \neq 0$ enable us to state, that it is true, when rational integrability is considered.

However, the above considerations assume that the energy value is generic, so that the particular solution is a non-degenerate elliptic function. As stressed in the first part, this does not preclude the existence of an additional first integral on the physically crucial zero-energy level.

## 5 Integrrability on the zero-energy hypersurface

We choose not to investigate the Darboux points, but the variational equations directly, as they are considerably simpler in this case. The Hamiltonian equations of (7) are

$$\dot{q}_1 = -p_1, \quad \dot{p}_1 = kq_1 - m^2 q_1 q_2^2 - \Lambda q_1^3,$$

$$\dot{q}_2 = p_2, \quad \dot{p}_2 = -(kq_2 + m^2 q_1 q_2 + \lambda q_2^3),$$

and they admit three invariant planes as was shown in [9]. They are

$$\Pi_k = \{(q_1, q_2, p_1, p_2) \in \mathbb{C}^4 \mid q_k = 0 \wedge p_k = 0\}, \quad k = 1, 2,$$

$$\Pi_3 = \{(q_1, q_2, p_1, p_2) \in \mathbb{C}^4 \mid q_2 = \alpha q_1 \wedge p_2 = -\alpha p_1\}, \quad \alpha^2 = -\frac{m^2 + \Lambda}{m^2 + \lambda}$$

Obviously two particular solutions are

$$\{q_1 = p_1 = 0, q_2 = q_2(\eta), p_2 = q_2'(\eta)\}, \quad 0 = \frac{1}{2} \left( p_2^2 + kq_2^2 + \frac{\Lambda}{2} q_2^4 \right),$$

$$\{q_2 = p_2 = 0, q_1 = q_1(\eta), p_1 = -q_1'(\eta)\}, \quad 0 = \frac{1}{2} \left( -p_1^2 - kq_1^2 + \frac{\Lambda}{2} q_1^4 \right),$$

and in order to find the third particular solution we make a canonical change of variables

$$(q_1, q_2, p_1, p_2)^T = B(Q_1, Q_2, P_1, P_2)^T$$

where symplectic matrix $B$ has the block structure

$$B = \left( \begin{array}{cc} \mathbb{A} & \mathbb{O} \\ \mathbb{O} & \mathbb{A}^T \end{array} \right), \quad \mathbb{A} = \left( \begin{array}{cc} -b & -a \\ a & b \end{array} \right), \quad \mathbb{O} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$$

$$\mathbb{O} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$$
and $a$ and $b$ are defined by

$$a = \sqrt{-\frac{m^2 + \Lambda}{2m^2 + \lambda + \Lambda}}, \quad b = \sqrt{\frac{m^2 + \lambda}{2m^2 + \lambda + \Lambda}}. \quad (22)$$

Let us introduce five quantities

$$\alpha_1 = 2m^2 + \lambda + \Lambda, \quad \alpha_2 = 3\lambda\Lambda + 2m^2(\lambda + \Lambda) + m^4, \quad \alpha_3 = \sqrt{(\lambda + m^2)(\lambda + m^2)},$$

$$\alpha_4 = \lambda^2 + \Lambda^2 - \Lambda m + m^4, \quad \alpha_5 = \lambda\Lambda - m^4. \quad (23)$$

Then, in the new variables, Hamiltonian $(7)$ has the form

$$H = \frac{1}{2} \left[ P_2^2 - P_1^2 + k(Q_2^2 - Q_1^2) \right] + \frac{1}{4\alpha_1} \left[ \alpha_5 Q_1^4 - 2\alpha_2 Q_1^2 Q_2^2 + 4(\lambda - \Lambda)\alpha_3 Q_1 Q_2^2 + 4\alpha_4 Q_2^4 \right]. \quad (24)$$

and the Hamiltonian equations read

$$\dot{Q}_1 = -P_1, \quad \dot{P}_1 = kQ_1 - \frac{1}{\alpha_1} \left[ \alpha_5 Q_1^3 - \alpha_2 Q_1 Q_2^2 + (\lambda - \Lambda)\alpha_3 Q_2^2 \right],$$

$$\dot{Q}_2 = P_2, \quad \dot{P}_2 = -kQ_2 + \frac{1}{\alpha_1} \left[ \alpha_2 Q_1^2 Q_2 - 3(\lambda - \Lambda)\alpha_3 Q_1 Q_2^2 - \alpha_4 Q_2^3 \right]. \quad (25)$$

Thus, the third particular solution can be seen to be

$$\{Q_2 = P_2 = 0, \ Q_1 = Q_1(\eta), \ P_1 = -Q_1'(\eta)\}, \quad 0 = \frac{1}{2} \left( -P_1^2 - kQ_1^2 + \frac{\alpha_5}{2\alpha_1} Q_1^4 \right). \quad (26)$$

Of course, this is only valid for $\alpha_1 \neq 0$ which is equivalent to $\alpha^2 \neq 1$. No restriction on the integrability can be obtained when $\alpha^2 = 1$, as we show at the end of this section.

Normal variational equations along those three solutions (in the position variables) are

$$\xi''(\eta) = \left[ -k - m^2 q(\eta)^2 \right] \xi(\eta),$$

$$\xi''(\eta) = \left[ k - m^2 q(\eta)^2 \right] \xi(\eta),$$

$$\xi''(\eta) = \left[ -k + \frac{\alpha_2}{\alpha_1} q(\eta)^2 \right] \xi(\eta), \quad (27)$$

where $q(\eta)$ is one of $\{q_1(\eta), q_2(\eta), Q_1(\eta)\}$, depending on the respective particular solution.

We will consider the $k = 0$ case first. Changing the independent variable to $z = q(\eta)^2$, all the NVE’s are reduced to the following

$$2z^2 \xi''(z) + 3z \xi'(z) + \rho \xi(z) = 0, \quad \rho = \frac{m^2}{\Lambda}, \quad \frac{m^2}{\lambda}, \quad -\frac{\alpha_2}{\alpha_5}, \quad (28)$$

whose solution is

$$\xi(z) = z^{-(1 \pm \sqrt{1 - 8\rho})/4}. \quad (29)$$

Note, that if any of $\Lambda$, $\lambda$ or $\alpha_5$ is zero, the corresponding particular solution is constant and cannot be used to restrict the problem’s integrability. Thus, we are left with the $E = k = 0$ case as potentially integrable.
When we assume \( k \neq 0 \), or equivalently \( k^2 = 1 \), and introduce the same independent variable \( z \) as before, the NVE’s read

\[
2z^2(\Lambda z - 2k)\xi''(z) + z(3\Lambda z - 4k)\xi'(z) + (m^2z + k)\xi(z) = 0, \\
2z^2(\lambda z + 2k)\xi''(z) + z(3\lambda z + 4k)\xi'(z) + (-m^2z + k)\xi(z) = 0, \\
2z^2\left(-\frac{\alpha_5}{\alpha_1}z + 2k\right)\xi''(z) + z\left(-3\frac{\alpha_5}{\alpha_1}z + 4k\right)\xi'(z) + \left(\frac{\alpha_2}{\alpha_1}z - k\right)\xi(z) = 0.
\]

(30)

First, let us observe that unlike in the previous case, when any of \( \Lambda, \lambda \) or \( \alpha_5 \) is zero, the system is not integrable. This happens, because the NVE then becomes the Bessel equation with its parameter \( n = 1 \) or \( n = i \), which is known not to possess Liouvillian solutions [8].

Assuming that none of those constants is zero, we rescale the variable \( z \) in the three equations with

\[
z \to \frac{2k}{\Lambda}z, \quad z \to -\frac{2k}{\lambda}z, \quad z \to \frac{2k\alpha_1}{\alpha_5}z,
\]

(31)

respectively, so that all three are transformed into a Riemann P equation of the form

\[
\xi''(z) + \left(\frac{1 - \delta - \delta'}{z} + \frac{1 - \gamma - \gamma'}{z - 1}\right)\xi'(z) + \left[\frac{\delta\delta'}{z^2} + \frac{\gamma\gamma'}{(z - 1)^2} + \frac{\beta\beta' - \delta\delta' - \gamma\gamma'}{z(z - 1)}\right]\xi(z) = 0,
\]

(32)

with the following pairs of exponents \((\delta, \delta'), (\gamma, \gamma'), (\beta, \beta')\) at its singular points

\[
\left(\frac{1}{2}, -\frac{1}{2}\right), \quad \left(\frac{1}{2}, 0\right), \quad \left(\frac{1}{4} + \frac{1}{4}\sqrt{1 - \frac{8m^2}{\Lambda}}, \frac{1}{4} - \frac{1}{4}\sqrt{1 - \frac{8m^2}{\Lambda}}\right), \\
\left(\frac{i}{2} - \frac{i}{2}\right), \quad \left(\frac{1}{2}, 0\right), \quad \left(\frac{1}{4} + \frac{1}{4}\sqrt{1 + \frac{8m^2}{\lambda}}, \frac{1}{4} - \frac{1}{4}\sqrt{1 + \frac{8m^2}{\lambda}}\right), \\
\left(\frac{1}{2}, -\frac{1}{2}\right), \quad \left(\frac{1}{2}, 0\right), \quad \left(\frac{1}{4} + \frac{1}{4}\sqrt{1 + \frac{8\alpha_2}{\alpha_5}}, \frac{1}{4} - \frac{1}{4}\sqrt{1 + \frac{8\alpha_2}{\alpha_5}}\right).
\]

(33)

Using Kimura’s results on solvability of the Riemann P equation [7] we check when the difference of the exponents gives us integrable cases, and find that the parameters must belong to the following families

\[
m^2 = \pm \frac{l_1(1 + l_1)}{2}, \quad l_1 \in \mathbb{Z}, \\
m^2 = \pm \frac{l_2(1 + l_2)}{2}, \quad l_2 \in \mathbb{Z}, \\
\frac{\alpha_2}{\alpha_5} = 4l_3(1 + l_3), \quad l_3 \in \mathbb{Z}.
\]

(34)

Finally, we turn to see what happens when \( \alpha^2 = 1 \). It is sufficient to consider only \( \alpha = 1 \), as \( \alpha = -1 \) results from the change of sign of \( q_2 \) and \( p_2 \). We also notice, that on the particular solution the Hamiltonian necessarily vanishes, so it only applies on the zero-level of energy.

Since we take \( q_1 = q_2 \) and \( p_1 = p_2 \), the particular solution satisfies the equation

\[
\ddot{q} = q(\mu q^2 - k),
\]

(35)
where $\mu = \Lambda + m^2 = -\lambda - m^2$. Although the Hamiltonian is zero, the above equation has a first integral of

$$\bar{E} = \frac{1}{2} q^2 + \frac{1}{2} k q^2 - \frac{1}{4} \mu q^4.$$ (36)

The variational equations in the positional variables are

$$\xi''_1(\eta) = \left[ (3\mu - 2m^2)q(\eta)^2 - k \right] \xi_1(\eta) + 2m^2 q(\eta)^2 \xi_2(\eta),$$

$$\xi''_2(\eta) = -2m^2 q(\eta)^2 \xi_1(\eta) + \left[ (3\mu + 2m^2)q(\eta)^2 - k \right] \xi_2(\eta).$$ (37)

Next, we apply the change of variables

$$\zeta_1(z) = \xi_1(\eta) + \xi_2(\eta),$$

$$\zeta_2(z) = \xi_2(\eta),$$ (38)

with $z = q(\eta)^2$, so that we arrive at

$$\zeta''_1(z) + w_1(z)\zeta'_1(z) + w_2(z)\zeta_1(z) = 0,$$

$$\zeta''_2(z) + w_1(z)\zeta'_2(z) + w_2(z)\zeta_2(z) = w_3(z)\zeta_1(z),$$ (39)

where

$$w_1(z) = \frac{3\mu z^2 + 4E - 4kz}{2z(\mu z^2 + 4E - 2kz)}, \quad w_2(z) = \frac{-3\mu z + k}{2z(\mu z^2 + 4E - 2kz)}, \quad w_3(z) = \frac{-2m^2 z}{2z(\mu z^2 + 4E - 2kz)}.$$ (40)

It is now straightforward to check, using any symbolical package for solving differential equations, that the solution of these two equations is Liouvillean, that is, consists of algebraic functions and their integrals. At the same time, the expressions obtained are a few lines long, and since we do not need their explicit form, we choose no to quote them here.

Thus, the third particular solution gives us no restrictions whatsoever, regardless of the value of $k$. This is only important when $k^2 = 1$, as the second case of the table tells us that the system is integrable, and the relation $\Lambda + \lambda + 2m^2 = 0$ holds, so that we can ignore the third condition in (34), which is in fact not satisfied.

6 Conclusions

Bringing the results together we can state the following properties of the system.

When we consider a generic energy hypersurface, and a spatially flat universe ($k = 0$), the equations are only integrable in the four cases listed in the table. Otherwise there exist no additional, meromorphic first integral.

On a generic energy hypersurface, in a spatially curved space-time ($k^2 = 1$), the only integrable cases are the ones with $\Lambda$ and $\lambda$ specified in the first two rows of the table. Other than that, there exist no additional, rational first integrals. This result can be strengthened to meromorphic integrals, although not for all values of the parameters, as described in [2].

On the zero energy hypersurface, if it is integrable, then either $k = 0$, or $k^2 = 1$ and additionally conditions [34] hold. Otherwise, the system is not meromorphically integrable.
In particular this means, that for $k^2 = 1$ if at least one of $\Lambda$ and $\lambda$ is zero, the system is non-integrable.

Of course, depending on the properties of the first integrals, we might get quite different results, and the requirement of meromorphicity or rationality is still very restricting. As described in the introduction, this leaves open the question of existence of real-analytic first integrals. Also we recall that physically the scale factor $a$ cannot even assume negative values, and some authors argue that when cosmological (instead of conformal) time is used, the evolution is not, in essence, chaotic [3]. Thus, we would like to stress that Liouvillian integrability is a mathematical property of the system, and often the methods used to study it require the complexification of variables. This means that when restricted to the narrower, physical domain, the dynamics might be much simpler. And in particular we might be interested in a particular trajectory whose behaviour is far from generic. It is no surprise then, that the dynamics of our system when restricted to $a > 0$ might appear regular. It should still be noted that the notion of chaos, although frequently associated with integrability, has not yet been successfully conflated with it. And that regular evolution is not necessarily integrable.

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Appendix A. Massless field

For $m = 0$ we can separately solve for each variable, so that we have

\[
E_1 = -\frac{1}{2} a^2 - \frac{1}{2} k a^2 + \frac{1}{4} 4 \Lambda a^4, \\
E_2 = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \omega^2 + \frac{1}{2} k \phi^2 + \frac{1}{4} \lambda \phi^4,
\]

with $E_1 + E_2 = E$ being the total energy. The first of these is immediately solved, when we substitute $v_1 = a^2$ to get

\[
\dot{v}_1^2 = 2 \Lambda v_1^3 - 4 k v_1^2 - 8 E_1 v_1, \tag{42}
\]

whose solution is

\[
v_1(\eta) = \frac{2}{\Lambda} \varphi(\eta - \eta_1; g_2, g_3) + \frac{2 k}{3 \Lambda}, \tag{43}
\]

with $\eta_1$ the integration constant and

\[
g_2 = \frac{4}{3} k^2 + 4 \Lambda E_1, \quad g_3 = \frac{8}{27} k^3 + \frac{4}{3} k \Lambda E_1. \tag{44}
\]

Of course, when $\Lambda = 0$ the Weierstrass function $\varphi$ reduces to a trigonometric function.
Similarly, for the other variable, we substitute \( v_2 = \phi^2 \) and obtain
\[
\dot{v}_2 = -2\lambda v^3 - 4kv^2 + 8E_2v - 4\omega^2,
\]
whose solution is
\[
v_2(\eta) = -\frac{2}{\lambda} \varphi(\eta - \eta_2; g_4, g_5) - \frac{2k}{3\lambda},
\]
where
\[
g_4 = \frac{4}{3} k^2 + 4\lambda E_2, \quad g_5 = \frac{8}{27} k^3 + \frac{4}{3} k\lambda E_2 + \lambda^2 \omega^2,
\]
and \( \eta_2 \) is the integration constant. As before, for \( \lambda = 0 \) the solution degenerates to trigonometric functions.

References


