Magnetic rotating brane in \((n+1)\)-dimensional Einstein-Maxwell-dilaton gravity

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We construct two classes of magnetic rotating solutions in \((n+1)\)-dimensional Einstein-Maxwell-dilaton gravity with Liouville-type potential. These solutions are neither asymptotically flat nor (A)dS. The first class of solutions represent a \((n+1)\)-dimensional spacetime with a longitudinal magnetic field and \(k\) rotation parameters. We find that these solutions have no curvature singularities and no horizons, but have a conic geometry. We show that when one or more of the rotation parameters are non zero, the spinning brane has a net electric charge that is proportional to the magnitude of the rotation parameters. The second class of solutions represent a spacetime with an angular magnetic field and \(\kappa\) boost parameters. These solutions have no curvature singularities, no horizons, and no conical singularity. We find that the net electric charge of these traveling branes with one or more nonzero boost parameters is proportional to the magnitude of the velocity of the brane. We also use the counterterm method inspired by AdS/CFT correspondence and calculate the conserved quantities of the solutions.

I. INTRODUCTION

There has been great interest in recent years in dilaton gravity. It is important to investigate how the properties of black holes/branes are modified when a dilaton field is present. Some efforts have been done to construct exact solutions of Einstein-Maxwell-dilaton gravity. For example, exact dilaton black hole solutions in the absence of dilaton potential have been constructed by many authors \([1, 2, 3, 4]\). In the presence of Liouville-type potential, static charged black hole solutions have been discovered with positive \([5]\), zero or negative constant curvature horizons \([6]\). Recently, properties of these black hole solutions which are not asymptotically AdS or dS, have been studied \([7]\).

The exact solutions mentioned above are all static. Exact rotating solutions to the Einstein equation coupled to matter fields with curved horizons are difficult to find except in a limited

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number of cases. Indeed, rotating solutions of EMd gravity with curved horizons have been obtained only for some limited values of the coupling constant such as $\alpha = \sqrt{3}$ \[8\] and $\alpha = \sqrt{8/3}$ \[9\].

For general dilaton coupling, the properties of rotating charged dilaton black holes only with infinitesimally small charge \[10\] or small angular momentum in four \[11\] and five \[12\] dimensions have been investigated. For arbitrary values of angular momentum and charge only a numerical investigation has been done \[13\]. When the horizons are flat, charged rotating dilaton black string solutions, in four-dimensional EMd gravity have also been constructed \[14\]. Recently, we generalized these solutions, to $(n+1)$-dimensional EMd gravity for an arbitrary dilaton coupling and $k$ rotation parameters \[15\]. These solutions are not black holes and represent black branes with flat horizons.

On the other hand, there are many papers which are dealing directly with the issue of spacetimes generated by string source that are horizonless and have non trivial external solutions. Static uncharged cylindrically symmetric solutions of Einstein gravity in four dimensions with vanishing cosmological constant have been considered in \[16\]. Similar static solutions in the context of cosmic string theory have been found in \[17\]. All of these solutions \[16, 17\] are horizonless and have a conical geometry; they are everywhere flat except at the location of the line source. An extension to include the electromagnetic field has also been done \[18\]. Asymptotically anti de Sitter (AdS) spacetimes generated by static and spinning magnetic sources in three and four dimensional Einstein-Maxwell gravity with negative cosmological constant have been investigated in \[19, 20\].

The generalization of these asymptotically AdS magnetic rotating solutions of the Einstein-Maxwell equation to higher dimensions \[21\] and higher derivative gravity \[22\] have also been done. In the context of electromagnetic cosmic string, it has been shown that there are cosmic strings, known as superconducting cosmic strings, that behave as superconductors and have interesting interactions with astrophysical magnetic fields \[23\]. The properties of these superconducting cosmic strings have been investigated in \[24\]. Superconducting cosmic strings have also been studied in Brans-Dicke theory \[25\], and in dilaton gravity \[26\]. Although exact magnetic rotating solutions in three dimensions have been considered \[27\], two classes of magnetic rotating solutions in four-dimensional EMd gravity with Liouville-type potential have also been constructed by one of us \[28\]. These solutions are not black holes, and represent spacetimes with conic singularities. Our aim in this paper is to construct new exact horizonless solutions of $(n+1)$-dimensional EMd gravity for an arbitrary value of coupling constant and investigate their properties.

The organization of our paper is as follows: In Sec \[II\] we have a brief review of the field equations and general formalism of calculating the conserved quantities. In Sec \[III\] we present the $(n+1)$-
dimensional rotating dilaton brane which produce longitudinal and angular magnetic fields, and investigate their properties. The last section is devoted to some concluding remarks.

II. FIELD EQUATIONS AND CONSERVED QUANTITIES

The action of Einstein-Maxwell dilaton gravity with one scalar field Φ with Liouville-type potential in (n + 1) dimensions can be written as

\[
I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left( R - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) - e^{-4\alpha\Phi/(n-1)} F_{\mu\nu} F^{\mu\nu} \right) - \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \Theta(\gamma),
\]  

(1)

where \( R \) is the Ricci scalar curvature and \( \Phi \) is the dilaton field and \( V(\Phi) \) is a potential for \( \Phi \). \( \alpha \) is a constant determining the strength of coupling of the scalar and electromagnetic fields, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor field and \( A_\mu \) is the vector potential. The last term in Eq. (1) is the Gibbons-Hawking boundary term. The manifold \( \mathcal{M} \) has metric \( g_{\mu\nu} \) and covariant derivative \( \nabla_\mu \). \( \Theta \) is the trace of the extrinsic curvature \( \Theta^{\mu\nu} \) of any boundary(ies) \( \partial\mathcal{M} \) of the manifold \( \mathcal{M} \), with induced metric \( \gamma_{ij} \). In this paper, we consider the action (1) with a Liouville type potential,

\[
V(\Phi) = 2\Lambda e^{4\alpha\Phi/(n-1)},
\]  

(2)

where \( \Lambda \) is a constant which may be refereed to as the cosmological constant, since in the absence of the dilaton field (\( \Phi = 0 \)) the action (1) reduces to the action of Einstein-Maxwell gravity with cosmological constant. The equations of motion can be obtained by varying the action (1) with respect to the gravitational field \( g_{\mu\nu} \), the dilaton field \( \Phi \) and the gauge field \( A_\mu \) which yields the following field equations

\[
R_{\mu\nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) + 2 \alpha \frac{e^{-4\alpha\Phi}}{n-1} \left( F_{\mu\eta} F^{\eta\nu} - \frac{1}{2(n-1)} g_{\mu\nu} F_{\lambda\eta} F^{\lambda\eta} \right),
\]  

(3)

\[
\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{\alpha}{2} \frac{e^{-4\alpha\Phi}}{n-1} F_{\lambda\eta} F^{\lambda\eta},
\]  

(4)

\[
\nabla_\mu \left( e^{\frac{-4\alpha\Phi}{n-1}} F^{\mu\nu} \right) = 0.
\]  

(5)

The conserved mass and angular momentum of the solutions of the above field equations can be calculated through the use of the substraction method of Brown and York [29]. Such a procedure
causes the resulting physical quantities to depend on the choice of reference background. For asymptotically \((A)dS\) solutions, the way that one deals with these divergences is through the use of counterterm method inspired by \((A)dS/CFT\) correspondence \[30\]. However, in the presence of a non-trivial dilaton field, the spacetime may not behave as either \(dS (\Lambda > 0)\) or \(AdS (\Lambda < 0)\). In fact, it has been shown that with the exception of a pure cosmological constant potential, where \(\alpha = 0\), no \(AdS\) or \(dS\) static spherically symmetric solution exist for Liouville-type potential \[4\]. But, as in the case of asymptotically \(AdS\) spacetimes, according to the domain-wall/QFT (quantum field theory) correspondence \[31\], there may be a suitable counterterm for the stress energy tensor which removes the divergences. In this paper, we deal with spacetimes with zero curvature boundary \[R_{abcd}(\gamma) = 0\], and therefore the counterterm for the stress energy tensor should be proportional to \(\gamma^{ab}\). Thus, the finite stress-energy tensor in \((n + 1)\)-dimensional Einstein-dilaton gravity with Liouville-type potential may be written as

\[
T^{ab} = \frac{1}{8\pi} \left[ \Theta^{ab} - \Theta \gamma^{ab} + \frac{n-1}{l_{\text{eff}}} \gamma^{ab} \right],
\]

where \(l_{\text{eff}}\) is given by

\[
l_{\text{eff}}^2 = \frac{(n-1)(\alpha^2 - n)}{2\Lambda} e^{-\frac{4\alpha\Phi}{n-1}}.
\]

In the particular case \(\alpha = 0\), the effective \(l_{\text{eff}}^2\) of Eq. \(7\) reduces to \(l^2 = -n(n-1)/2\Lambda\) of the \(AdS\) spacetimes. The first two terms in Eq. \(6\) is the variation of the action \(1\) with respect to \(\gamma_{ab}\), and the last term is the counterterm which removes the divergences. One may note that the counterterm has the same form as in the case of asymptotically \(AdS\) solutions with zero curvature boundary, where \(l\) is replaced by \(l_{\text{eff}}\). To compute the conserved charges of the spacetime, one should choose a spacelike surface \(B\) in \(\partial M\) with metric \(\sigma_{ij}\), and write the boundary metric in ADM (Arnowitt-Deser-Misner) form:

\[
\gamma_{ab} dx^a dx^b = -N^2 dt^2 + \sigma_{ij} \left( d\varphi^i + V^i dt \right) \left( d\varphi^j + V^j dt \right),
\]

where the coordinates \(\varphi^i\) are the angular variables parameterizing the hypersurface of constant \(r\) around the origin, and \(N\) and \(V^i\) are the lapse and shift functions, respectively. When there is a Killing vector field \(\xi\) on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. \(6\) can be written as

\[
Q(\xi) = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \xi^b,
\]

where \(\sigma\) is the determinant of the metric \(\sigma_{ij}\), \(\xi\) and \(n^a\) are the Killing vector field and the unit normal vector on the boundary \(B\). For boundaries with timelike (\(\xi = \partial/\partial t\)), rotational (\(\varsigma_i = \partial/\partial \phi^i\))
and translational Killing vector fields \( (\zeta_i = \partial/\partial x^i) \), one obtains the quasilocal mass, components of total angular and linear momenta as

\[
M = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \xi^b, \tag{9}
\]

\[
J_i = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \varsigma_i^b, \tag{10}
\]

\[
P_i = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \zeta_i^b, \tag{11}
\]

provided the surface \( B \) contains the orbits of \( \varsigma \). These quantities are, respectively, the conserved mass, angular and linear momenta of the system enclosed by the boundary \( B \). Note that they will both be dependent on the location of the boundary \( B \) in the spacetime, although each is independent of the particular choice of foliation \( B \) within the surface \( \partial M \).

### III. (N+1)-DIMENSIONAL MAGNETIC ROTATING SOLUTIONS

Our aim here is to obtain the \((n+1)\)-dimensional horizonless solutions of Eqs. (3)-(5). First, we construct a spacetime generated by a magnetic source which produces a longitudinal magnetic field. Second, we obtain a spacetime generated by a magnetic source that produces angular magnetic fields along the \( \phi_i \) coordinates.

#### A. Longitudinal magnetic field solutions with one rotation parameter

The generalization of the four-dimensional metric given by Dias and Lemos \[20\] in \((n+1)\) dimensions with one rotation parameter can be written as

\[
ds^2 = -\frac{\rho^2}{l^2} R^2(\rho) (\Xi dt - a d\phi)^2 + f(\rho) \left(\frac{a}{l} dt - \Xi d\phi\right)^2 + \frac{\rho^2}{l^2} R^2(\rho) dX^2, \tag{12}
\]

where \( \Xi = \sqrt{1 + a^2/l^2} \) and \( a \) is the rotation parameter. \( f(\rho) \) and \( R(\rho) \) are functions which should be determined and \( l \) has the dimension of length which is related to the cosmological constant \( \Lambda \) for the case of Liouville-type potential with constant \( \Phi \). The angular coordinates are in the range \( 0 \leq \phi \leq 2\pi \) and \( dX^2 = \sum_{i=1}^{n-2} (dx^i)^2 \) is the Euclidean metric on the \((n-2)\)-dimensional flat submanifold.

The Maxwell equation (5) can be integrated immediately to give

\[
F_{\phi\rho} = \frac{ql \Xi e^{\frac{4\phi}{\Xi R}}}{(\rho R)^{n-1}},
\]

\[
F_{t\rho} = -\frac{a}{\Xi l^2} F_{\phi\rho}. \tag{13}
\]
where \( q \), is an integration constant related to the electric charge of the brane. In order to solve the system of equations (3) and (4) for three unknown functions \( f(\rho) \), \( R(\rho) \) and \( \Phi(\rho) \), we make the ansatz

\[
R(\rho) = e^{2a\Phi/(n-1)}.
\]  

(14)

Using (14), the Maxwell fields (13) and the metric (12), one can easily show that equations (3) and (4) have solutions of the form

\[
f(\rho) = \frac{2\Lambda(\alpha^2 + 1)2b^{2\gamma}}{(n-1)(\alpha^2 - n)}\rho^{-2(\gamma-1)} + \frac{m}{\rho(n-1)(1-\gamma)-1} - \frac{2q^2(\alpha^2 + 1)2b^{-2(n-2)\gamma}}{(n-1)(\alpha^2 + n - 2)}\rho^{2(n-2)(\gamma-1)}
\]  

(15)

\[
\Phi(\rho) = \frac{(n-1)\alpha}{2(1 + \alpha^2)}\ln\left(\frac{b}{\rho}\right)
\]  

(16)

where \( b \) and \( m \) are arbitrary constants and \( \gamma = \alpha^2/(\alpha^2 + 1) \). One may note that in the particular case \( n = 3 \) these solutions reduce to the four-dimensional magnetic rotating dilaton black strings presented in [28]. In the absence of a non-trivial dilaton \( (\alpha = 0) \), the above solutions reduce to the \((n + 1)\)-dimensional horizonless rotating solutions presented in [21]. One can easily show that the gauge potential \( A_\mu \) corresponding to electromagnetic tensor (13) can be written as

\[
A_\mu = \frac{q\rho^{(3-n)\gamma}}{\Gamma \rho^l} \left( \frac{a}{\Gamma l^{\delta}} - \Xi l^{\delta^\phi} \right),
\]  

\[
\Gamma \equiv (n - 3)(1 - \gamma) + 1.
\]  

(17)

In order to study the general structure of these solutions, we first look for the curvature singularities in the presence of dilaton gravity. It is easy to show that the Kretschmann invariant \( R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \) and Ricci scalar diverge at \( \rho = 0 \), they are finite for \( \rho \neq 0 \) and goes to zero as \( \rho \rightarrow \infty \). Therefore one might think that there is a curvature singularity located at \( \rho = 0 \). However, as will see below, the spacetime will never reach \( \rho = 0 \). Second, we look for the existence of horizons, and therefore we searches for possible black brane solutions. The horizons, if any exist, are given by the zeros of the function \( f(\rho) = g^{\rho\rho} \). Let us denote the smallest positive root of \( f(\rho) = 0 \) by \( r_+ \).

The function \( f(\rho) \) is negative for \( \rho < r_+ \), and positive for \( \rho > r_+ \). Therefore, one may think that the hypersurface of constant time and \( \rho = r_+ \) is the horizon. However, this analysis is not correct. Indeed, one may note that \( g_{\rho\rho} \) and \( g_{\phi\phi} \) are related by \( f(\rho) = g_{\rho\rho}^{-1} = l^{-2}g_{\phi\phi} \), and therefore when \( g_{\rho\rho} \) becomes negative (which occurs for \( \rho < r_+ \)) so does \( g_{\phi\phi} \). This leads to an apparent change of signature of the metric from \((n - 1)+\) to \((n - 2)+\), and therefore indicates that we are using an incorrect extension. To get rid of this incorrect extension, we introduce the new radial coordinate
\( r^2 = \rho^2 - r_+^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_+^2} dr^2. \)

With this new coordinate, the metric (12) is

\[
ds^2 = -\frac{r^2 + r_+^2}{l^2} R(r)^2 (\Xi dt - ad\phi)^2 + f(r) \left( \frac{a}{l} dt - \Xi d\phi \right)^2 + \frac{r^2 + r_+^2}{l^2} R(r)^2 dX^2,
\]

(18)

where the coordinates \( r \) and \( \phi \) assume the values \( 0 \leq r < \infty \) and \( 0 \leq \phi < 2\pi \), and \( f(r) \) and \( R(r) \) are now given as

\[
f(r) = \frac{2\Lambda(\alpha^2 + 1)b^2\gamma}{(n - 1)(\alpha^2 - n)} (r^2 + r_+^2)^{(1-\gamma)} + \frac{m}{(r^2 + r_+^2)|\Xi|^2(1-\gamma)^2},
\]

(19)

\[
R(r) = \frac{b^\gamma}{(r^2 + r_+^2)^{\gamma/2}}.
\]

(20)

The gauge potential in the new coordinate is

\[
A_\mu = \frac{q b^{(3-n)\gamma}}{2 \sqrt{(r^2 + r_+^2)^{\gamma/2}}} \left( \frac{a}{l} \delta_\mu - \Xi \delta_\mu \right).
\]

(21)

One can easily show that the Kretschmann scalar does not diverge in the range \( 0 \leq r < \infty \). However, the spacetime has a conic geometry and has a conical singularity at \( r = 0 \), since:

\[
\lim_{r \to 0} \frac{1}{r} \sqrt{g_{\varphi \varphi}} = \frac{ml(\alpha^2 + n - 2)}{2} r_+^{(n-1)(\gamma-1)} + \frac{2(1 + \alpha^2)}{(\alpha^2 - n)} \Lambda b^{2\gamma} r_+^{1-2\gamma} \neq 1.
\]

That is, as the radius \( r \) tends to zero, the limit of the ratio “circumference/radius” is not 2\( \pi \) and therefore the spacetime has a conical singularity at \( r = 0 \). The metric (18) and the other metric that we will present in this paper are neither asymptotically flat nor (anti)-de Sitter. Now we investigate the casual structure of the spacetime. As one can see from Eq. (15), there is no solution for \( \alpha = \sqrt{n} \) with a Liouville potential (\( \Lambda \neq 0 \)). The cases with \( \alpha > \sqrt{n} \) and \( \alpha < \sqrt{n} \) should be considered separately.

For \( \alpha > \sqrt{n} \), as \( r \) goes to infinity the dominant term in Eq. (19) is the second term, and therefore the function \( f(r) \) is positive in the whole spacetime, despite the sign of the cosmological constant \( \Lambda \), and is zero at \( r = 0 \). Thus, the solution given by Eqs. (18)-(20) exhibits a spacetime with conic singularity at \( r = 0 \).
For \( \alpha < \sqrt{n} \), the dominant term for large values of \( r \) is the first term, and therefore the function \( f(r) \) given in Eq. (19) is positive in the whole spacetime only for negative values of \( \Lambda \). In this case the solution represents a spacetime with conic singularity at \( r = 0 \). The solution is not acceptable for \( \alpha < \sqrt{n} \) with positive values of \( \Lambda \), since the function \( f(r) \) is negative for large values of \( r \).

Of course, one may ask for the completeness of the spacetime with \( r \geq 0 \) (or \( \rho \geq r_+ \)) [20, 32]. It is easy to see that the spacetime described by Eq. (18) is both null and timelike geodesically complete. In fact, we can show that every null or timelike geodesic starting from an arbitrary point can either extend to infinite values of the affine parameter along the geodesic or end on a singularity at \( r = 0 \). To do this, first, we perform the rotation boost \((\Xi - a\phi) \mapsto t; (at - \Xi l^2 d\phi) \mapsto l^2 d\phi\) in the \( t - \phi \) plane. Then the metric (18) becomes

\[
ds^2 = -\frac{r^2 + r_+^2}{l^2} R^2(r) dt^2 + \frac{r^2}{(r^2 + r_+^2)f(r)} dr^2 + l^2 f(r) d\phi^2 + \frac{r^2 + r_+^2}{l^2} R^2(r) dX^2.\]

Using the geodesic equation, one obtains

\[
i = \frac{l^2}{b^2(1 + r_+^2)} E, \quad x^i = \frac{l^2}{b^2(1 + r_+^2)} P^i, \quad \phi = \frac{1}{l^2 f(r)} L, \quad (22)
\]

\[
r^2 r^2 = (r^2 + r_+^2) f(r) \left[ \frac{l^2 (E^2 - P^2)}{b^2(1 + r_+^2)} - \eta \right] - \frac{r^2 + r_+^2}{l^2} L^2, \quad (23)
\]

where the dot denotes the derivative with respect to an affine parameter and \( \eta \) is zero for null geodesics and +1 for timelike geodesics. \( E, L, \) and \( P^i \) are the conserved quantities associated with the coordinates \( t, \phi, \) and \( x^i \) respectively, and \( P^2 = \sum_{i=1}^{n-2} (P^i)^2 \). Notice that \( f(r) \) is always positive for \( r > 0 \) and zero for \( r = 0 \).

First we consider the null geodesics \((\eta = 0)\). (i) If \( E^2 > P^2 \) the spiraling particles \((L > 0)\) coming from infinity have a turning point at \( r_{tp} > 0 \), while the nonspiraling particles \((L = 0)\) have a turning point at \( r_{tp} = 0 \). (ii) If \( E = P \) and \( L = 0 \), whatever is the value of \( r, \dot{r} \) and \( \dot{\phi} \) vanish and therefore the null particles move in a straight line in the \((n - 2)\)-dimensional submanifold spanned by \( x^1 \) to \( x^{n-2} \). (iii) For \( E = P \) and \( L = 0 \), and also for \( E^2 < P^2 \) and any value of \( L \), there is no possible null geodesic.

Second, we analyze the timelike geodesics \((\eta = +1)\). Timelike geodesics are possible only if \( l^2 (E^2 - P^2) > b^2 r_+^{2(1-\gamma)} \). In this case the turning points for the nonspiraling particles \((L = 0)\) are \( r_{tp}^1 = 0 \) and \( r_{tp}^2 \) given as

\[
r_{tp}^2 = \sqrt{b^{2\gamma} l^2 (E^2 - P^2)}^{1/(1-\gamma)} - r_+, \quad (24)
\]

while the spiraling \((L \neq 0)\) timelike particles are bound between \( r_{tp}^a \) and \( r_{tp}^b \) given by

\[
0 < r_{tp}^a < r_{tp}^b < r_{tp}^2. \quad (25)
\]
Therefore, we have confirmed that the spacetime described by Eq. (18) is both null and timelike geodesically complete.

**B. Longitudinal magnetic field solutions with all rotation parameters**

Our aim here is to construct the \((n+1)\)-dimensional longitudinal magnetic field solutions with complete set of rotation parameters. The rotation group in \(n+1\) dimensions is \(SO(n)\) and therefore the number of independent rotation parameters is \([n/2]\), where \([x]\) is the integer part of \(x\). We now generalize the above metric given in Eq. (18) with \(k \leq [n/2]\) rotation parameters. This generalized solution can be written as

\[
ds^2 = -r^2 dr^2 + \frac{r^2 + r_+^2}{l^2(\Xi^2 - 1)} R^2(r) \sum_{i<j} (a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2 + r_+^2}{l^2} R^2(r) dX^2,
\]

where \(\Xi = \sqrt{1 + \sum_i a_i^2/l^2}\), \(dX^2\) is the Euclidean metric on the \((n-k-1)\)-dimensional submanifold and \(f(r)\) and \(R(r)\) are the same as given in Eq. (19) and (20). The gauge potential is

\[
A_\mu = \frac{g b^{(3-n)\gamma}}{\Gamma(r^2 + r_+^2)^{1/2}} \left( \sqrt{\Xi^2 - 1} \delta_\mu^t - \frac{\Xi}{\sqrt{\Xi^2 - 1}} a_i \delta_\mu^i \right) \quad \text{(no sum on i)}. \tag{27}
\]

Again this spacetime has no horizon and curvature singularity. However, it has a conical singularity at \(r = 0\). One should note that these solutions reduce to those discussed in [21], in the absence of dilaton field \((\alpha = 0)\) and those presented in [28] for \(n = 3\).

Now we want to calculate conserved quantities of these solutions. Denoting the volume of the hypersurface boundary at constant \(t\) and \(r\) by \(V_{n-1} = (2\pi)^k \Sigma_{n-k-1}\), the mass and angular momentum per unit volume \(V_{n-1}\) of the black branes \((\alpha < \sqrt{n})\) can be calculated through the use of Eqs. (28) and (29). We find

\[
M = \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{(n-\alpha^2)\Xi^2 - (n-1)}{1 + \alpha^2} \right) m, \tag{28}
\]

\[
J_i = \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{n-\alpha^2}{1 + \alpha^2} \right) \Xi m a_i. \tag{29}
\]

For \(a_i = 0\) \((\Xi = 1)\), the angular momentum per unit volume vanishes, and therefore \(a_i\)'s are the rotational parameters of the spacetime. Of course, one may note that in the particular case \(n = 3\), these conserved charges reduce to the conserved charges of the magnetic rotating black string obtained in Ref. [28], and in the absence of dilaton field \((\alpha = \gamma = 0)\) they reduce to that obtained in [21].
C. Angular magnetic field solutions with one boost parameter

The generalization of four-dimensional spacetime with angular magnetic field to the case of an \((n + 1)\)-dimensional charged rotating dilaton black brane with one boost parameter can be written as

\[
ds^2 = -\frac{r^2 + r_+^2}{l^2} R^2(r) \left( \Xi dt - \frac{v}{l} dz \right)^2 + f(r) \left( \frac{v}{l} dt - \Xi dz \right)^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} + (r^2 + r_+^2) R^2(r) d\Omega^2,
\]

where \(\Xi = \sqrt{1 + v^2/l^2}\), \(d\Omega^2 = \sum_{i=1}^{n-2} (d\phi_i)^2\), and \(f(r)\) and \(R(r)\) are the same as given in Eq. \((19)\) and \((20)\). The coordinates \(r\) and \(\phi_i\)'s assume the values \(0 \leq r < \infty\) and \(0 \leq \phi_i \leq 2\pi\), and the gauge potential is given by

\[
A_\mu = \frac{qb^{(3-n)\gamma}}{r^2 (r_+^2)^{1/2}} \left( \frac{v}{l} \delta^{t\mu} - \Xi \delta^{z\mu} \right).
\]

Using the same arguments given for the case of longitudinal magnetic field solutions discussed in \((III A)\), one can show that this spacetime has no curvature singularity, no horizons and no conical singularity.

D. Angular magnetic field solutions with all boost parameters

Now we introduce the solutions of the Einstein-Maxwell dilaton gravity with no rotation parameter and \(\kappa\) boost parameters. The maximum number of boost parameters can be \(n - 2\). In this case the solution can be written as

\[
ds^2 = -\frac{r^2 + r_+^2}{l^2} R^2(r) \left( \Xi dt - l^{-1} \sum_{i=1}^{\kappa} v_i dx^i \right)^2 + f(r) \left( \sqrt{\Xi^2 - 1} dt - \frac{\Xi}{l\sqrt{\Xi^2 - 1}} \sum_{i=1}^{\kappa} v_i dx^i \right)^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)} + (r^2 + r_+^2) R^2(r) d\Omega^2,
\]

where \(\Xi = \sqrt{1 + \sum_i v_i^2/l^2}\), \(d\Omega^2 = \sum_{i=1}^{n-\kappa-1} (d\phi_i)^2\), and \(f(r)\) and \(R(r)\) are the same as given in Eq. \((19)\) and \((20)\). The gauge potential is given by

\[
A_\mu = \frac{qb^{(3-n)\gamma}}{r^2 (r_+^2)^{1/2}} \left( \sqrt{\Xi^2 - 1} \delta^{t\mu} - \frac{\Xi}{l\sqrt{\Xi^2 - 1}} v_i \delta^{z\mu} \right) \quad \text{(no sum on i).}
\]

Again this spacetime has no curvature singularity, no horizons, and no conical singularity. The conserved quantities of the spacetime \((32)\) are the mass and linear momentum. The mass and
linear momentum per unit volume $V_{n-1}$ of the black branes ($\alpha < \sqrt{n}$) can be calculated through the use of Eqs. (9) and (11). We find

$$M = \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{(n - \alpha^2)\Xi^2 - (n - 1)}{1 + \alpha^2} \right) m,$$

(34)

$$P_i = \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{n - \alpha^2}{1 + \alpha^2} \right) \Xi m v_i.$$

(35)

Finally, we calculate the electric charge of the solutions (26) and (32) obtained in this section. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces is

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{V^i}{N},$$

and the electric field is $E^\mu = g^{\mu\nu} \frac{\Xi}{\Xi_N} F_{\rho\nu} u^\nu$. Then the electric charge per unit volume $V_{n-1}$ can be found by calculating the flux of the electric field at infinity, yielding

$$Q = \frac{\sqrt{\Xi^2 - 1} q}{4\pi l^{n-2}}.$$

(36)

Note that the electric charge of the system per unit volume is proportional to the rotation parameter or boost parameter, and is zero for the case of a static solution. Again, in the particular case $n = 3$, these conserved charges reduce to the conserved charges of the magnetic rotating black string obtained in Ref. [28], and in the absence of dilaton field ($\alpha = \gamma = 0$) they reduce to the conserved charged of $(n + 1)$-dimensional horizonless rotating black brane solutions presented in [21].

**IV. CONCLUDING REMARKS**

Unfortunately, exact rotating solutions to the Einstein equation coupled to matter fields with curved horizons are difficult to find except in a limited number of cases. Indeed, rotating solutions of EMd gravity with curved horizons have been obtained only for some limited values of the coupling constant [8, 9]. For general dilaton coupling, the properties of rotating charged dilaton black holes only with infinitesimally small charge [10] or small angular momentum in four [11] and five [12] dimensions have been investigated. When the horizons are flat, exact charged rotating dilaton black string solutions, in four-dimensional [14] and $(n + 1)$-dimensional [15] EMd gravity have also been constructed. These solutions are not black holes and represent black string/brane with flat horizons. Two classes of magnetic rotating solutions in four-dimensional EMd gravity
with Liouville-type potential have also been constructed in \cite{28}. These solutions are not black holes, and represent spacetimes with conic singularities.

In this paper, we obtained two classes of $(n+1)$-dimensional exact magnetic rotating solutions of Einstein-Maxwell dilaton gravity in the presence of Liouville-type potential. These solutions are neither asymptotically flat nor (A)dS. In the presence of Liouville-type potential, we obtained exact solutions provided $\alpha \neq \sqrt{n}$. For $\alpha = 0$, these solutions reduce to the horizonless rotating black brane solutions of \cite{21}, while for $n = 3$, these solutions reduce to the four-dimensional magnetic rotating dilaton black string presented in \cite{28}. The first class of solutions represent a $(n+1)$-dimensional spacetime with a longitudinal magnetic field. We found that these solutions have no curvature singularities and no horizons, but have conic singularity at $r = 0$. We also confirmed that these solutions are both null and timelike geodesically complete. In fact, we showed that every null or timelike geodesic starting from an arbitrary point can either extend to infinite values of the affine parameter along the geodesic or end on a singularity at $r = 0$. In these spacetimes, when all the rotation parameter are zero (static case), the electric field vanishes, and therefore the brane has no net electric charge.

For the spinning brane, when one or more rotation parameters are nonzero, the brane has a net electric charge density which is proportional to the magnitude of rotation parameter given by $\sum_i a_i^2$. The second class of solutions represent a spacetime with angular magnetic field. These solutions have no curvature singularity, no horizon, and no conic singularity. Again, we found that the brane in these spacetimes have no net electric charge when all the boost parameters are zero. We also showed that, for the case of traveling brane with one or more nonzero boost parameters, the net electric charge of the brane is proportional to the magnitude of the velocity of the brane ($\sum_i v_i^2$).

We also used the counterterm method inspired by the AdS/CFT correspondence and calculated the conserved quantities of the two classes of solutions.

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