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TWO-LEVEL METHOD FOR UNSTEADY NAVIER-STOKES
EQUATIONS BASED ON A NEW PROJECTION

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Abstract

A two-level algorithm for the two dimensional unsteady Navier-Stokes equations based on a new projection is proposed and investigated. The approximate solution is solved as a sum of a large eddy component and a small eddy component, which are in the sense of the new projection, constructed in this paper. These two terms advance in time explicitly. Actually, the new algorithm proposed here can be regarded as a sort of postprocessing algorithm for the standard Galerkin method (SGM). The large eddy part is solved by SGM in the usual $L^2$—based large eddy subspace while the small eddy part (the correction part) is obtained in its complement subspace in the sense of the new projection. The stability analysis indicates the improvement of the stability comparing with SGM of the same scale, and the $L^2$—error estimate shows that the scheme can improve the accuracy of SGM approximation for half order. We also propose a numerical implementation based on Lagrange multiplier for this two-level algorithm.

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1. **Introduction.** In a recent paper [9], a new approximate inertial manifold and related postprocessing procedure of the standard Galerkin method (SGM) approximation to the steady Navier-Stokes (NS) equations is proposed and investigated by constructing a new projection. For the usual two-level algorithms for the NS equations (including nonlinear Galerkin methods), the small eddy approximating (postprocessing) procedures are often accomplished in the small eddy subspace in the sense of the usual $L^2$ inner product, for example, algorithms in [2], [5], [8], [10], [11], [12] and etc. And for the algorithm in [9], one postprocesses the SGM approximation in certain new small eddy subspace which is constructed based upon the solution information obtained by the SGM approximation and generally consists of both the large and small eddy components in the sense of the usual $L^2$ inner product. Since the construction of the new small eddy subspace is based upon the solution information from the SGM approximation, the new algorithm is expected to have a better performance. Since the small eddy component, obtained by such new algorithm, consists of both the large and small eddy components in the sense of the usual $L^2$ inner product, that is, the small eddy component obtained by the algorithm in [9] will not only supply the SGM approximation with the usual small eddy component (the truncation part in the sense of usual $L^2$ inner product) but also do certain correction for the SGM approximation itself in the usual large eddy subspace, it is reasonable for us to hope that this kind of algorithm may do better job than usual ones.

In this paper, we will use the similar ideal in [9] to construct a new two-level algorithm for the unsteady NS equations based on a certain new small eddy subspace. Actually, we will show that the new small eddy subspace is a time dependent tangent space of a certain manifold associated to the NS equations. First, we get an approximate solution $u_{n+1}^m$ in the usual large eddy subspace $H_m$ at time $t_{n+1}$. Then we rewrite the NS equation at $t_{n+1}$ as:

\[(1.1)\]

$$ F^{n+1}(u(t_{n+1})) = 0, $$

where $u(t_{n+1})$ stands for the exact solution of the NS equations in certain Hilbert space $H$ at this moment. Different from the usual two-level method which usually supplies certain approximation of truncation part of $u_{m+1}^n$ in the usual $L^2$-based small eddy subspace, we intend to provide an approximation of the truncation part in certain approximate tangent space $\hat{V}^{n+1}$ of the manifold $f = F^{n+1}(u)$ at $u = u_{m+1}^n$. That is, we want to provide a suitable approximation $\hat{w}^{n+1}$ of the increment $u(t_{n+1}) - u_{m+1}^n$ in $\hat{V}^{n+1}$. Therefore, the key issue is to construct a projection and its associated approximate incremental subspace $\hat{V}^{n+1}$. Once we solve this issue, the construction of our two-level method is obvious. Indeed, our two-level algorithm, based on the above consideration, can be regarded as some postprocessing procedure to the SGM. On the coarse level $H_m$, we actually need to solve an explicit SGM equation to get a large eddy approximation $u_{m+1}^n$. The only difference from the SGM is that we supply the SGM approximation $u_{m}^n$ at the previous time step with the usual projection of $\hat{w}^n$ in $H_m$. On the fine level we have to solve an equation in $H_m + \hat{V}^{n+1}$ to get the final approximation $u_{m+1}^n + \hat{w}^{n+1}$ at time step $n+1$ by almost the same scheme as the explicit SGM. Our stability and error analysis shows that the new algorithm has better stability properties than the explicit SGM on the fine level $H_m + \hat{V}^{n+1}$ and can improve the accuracy.
of the SGM approximation on coarse level for half order. Another attractive thing is that the numerical implementation is very simple and there is almost no programming necessity if the explicit SGM code is at hand. For the sake of simplicity of the analysis, we consider only the spectral case.

The paper is organized as follows. In section 2, a detailed functional setting of NS equations is presented. In section 3, we construct the new projection and state our new two-level algorithm based on this projection. Section 4 and section 5 are the stability and error analysis of the proposed algorithm. Finally, we address some issues on numerical implementation in section 6.

2. The NS Equations. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\Gamma$. We consider the time-dependent NS equations describing the flow of a viscous incompressible fluid confined in $\Omega$:

$$
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
\nabla \cdot u = 0 \quad (x,t) \in \Omega \times \mathbb{R}^+,
$$

These equations are supplemented with the homogeneous Dirichlet boundary conditions

$$
u|_{\Gamma} = 0 \quad \forall (x,t) \in \Gamma \times \mathbb{R}^+,
$$
or the periodic boundary conditions

$$u \text{ is } \Omega - \text{periodic},$$

when $\Omega$ is a two dimensional torii. Here $u$ is the velocity field, $p$ is the pressure, $f$ represents the time-dependent density of body forces and $\nu > 0$ is the kinetic viscosity.

In the rest of this paper, we will consider only the Dirichlet boundary conditions case and all the results are true for the periodic boundary conditions case. Indeed, the periodic boundary conditions case is a little bit easier in analysis.

To write the problem in a functional form, we introduce the following linear vector space:

$$H = \{ v \in L^2(\Omega) : \nabla \cdot v = 0 \text{ in weak sense, } v \cdot n = 0 \},$$

where $L^2(\Omega) = L^2(\Omega)^2$ and $n$ denotes the unit outward normal vector of $\Gamma$. This space is a Hilbert space when equipped with the usual $L^2$–inner product and related norm:

$$(u,v) = \int_{\Omega} u \cdot v dx, \quad |v| = (u,u)^{1/2}.$$ 

We denote by $P$ the $L^2$–orthogonal projection from $L^2(\Omega)$ to $H$. It is convenient to introduce the Stokes operator $A = P(-\Delta)$, which is an unbounded linear positive operator on $H$ with compact inverse and whose domain is denoted by $D(A)$. Obviously, $A$ has the following countable positive sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$ and its associated eigenvectors $\phi_1, \phi_2, \cdots$, which form a complete orthogonal basis of $H$. In addition, it is classical that we can define the power operator $A^s$ for any $s \in \mathbb{R}$, whose domain is

$$D(A^s) = \{ v \in H : v = \sum_{i=1}^{\infty} v_i \phi_i, \sum_{i=0}^{\infty} \lambda_i^{2s} |v_i|^2 < +\infty, v_i \in \mathbb{R} \}.$$
which is a Hilbert space if it is equipped with the following nature inner product and related norm:

\[(u, v)_s = (A^s u, A^s v), \quad |A^s v| = (A^s v, A^s v)^{1/2}.\]

\(|A^s|\) is an equivalent norm of \(\| \cdot \|_{2s}\), the usual norm of Sobolev space \(H_0^{2s}(\Omega)^2\), at least for \(s \leq 1\). We often denote \(V = D(A^{1/2})\).

Projecting NS equations by \(P\) leads to the NS equations of functional form in \(H\):

\[
\frac{du}{dt} + \nu Au + B(u, u) = f, \quad u(0) = u_0,
\]

where \(B(u, v) = P[(u \cdot \nabla)v]\) and we assume \(f \in L^\infty(R^+, H)\). For any given positive integer \(M\), let us denote by \(P_M\) the \(L^2\)-orthogonal projection from \(H\) to the following finite dimensional subspace

\[H_M = \text{span}\{\phi_1, \phi_2, \cdots, \phi_M\}.\]

It is obvious that \(P_M\) is also an orthogonal projection in the sense of any \(D(A^s)-\)inner product. Let us denote by \(Q_M = I - P_M\) and the following properties are classical (see [3])

\[
|P_M A^\beta v| \leq \lambda_{M+1}^{\alpha-\beta}|A^\alpha v|, \quad |Q_M A^\alpha v| \leq \lambda_M^{\alpha-\beta}|A^\beta v| \quad \forall \alpha \leq \beta, v \in D(A^\beta).
\]

It is easy to verify that

\[
|P_M A^s v| + |Q_M A^s v| \leq \sqrt{2}|A^s v| \quad \forall v \in D(A^s).
\]

We also define the usual trilinear form

\[b(u, v, w) = (B(u, v), w) \quad \forall u, v, w \in V,
\]

and recall some properties of it (see [14]) which are extensively used in the rest of this paper

\[
b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in V,
\]

\[
|b(u, v, w)| \leq c_1 \begin{cases} |A^{s_1} u| |A^{s_2} v| |A^{s_3} w|, \\ |u| |A^{\frac{s_2}{2}} v| |w|, \\ |u| |A^{\frac{s_3}{2}} v| |w|, \\ |u| |A^{\frac{s_2}{2}} v| |w|, \end{cases}
\]

where \(c_1 > 0\) is a constant independent of \(u, v, w; s_1, s_2, s_3 \geq 0, s_1 + s_2 + s_3 \geq 1\) and \((s_1, s_2, s_3)\) can not be equal to \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\) when “=” is applied.

We conclude this section by recalling some other inequalities in Sobolev spaces whose combination with the above estimates (2.5) can produce more estimates which are useful in later analysis; they are Agmon’s inequality

\[
|v|_\infty \leq c_{1.1} |v|^{\frac{1}{2}} |A v|^\frac{5}{2} \quad \forall v \in D(A),
\]

the Sobolev interpolation inequality

\[
|A^s v| \leq c_{1.2} |v|^{\frac{1}{2}} |A^s v|^\frac{3}{2} \quad \forall v \in V,
\]
Brezis-Gallouet inequality [1]
\[ |v|_\infty \leq c_{1,3}|A^*v|(1 + \ln \frac{|A^*v|^2}{|\hat{A}^*v|^2})^\frac{1}{2} \quad \forall v \in D(A), \]
and the special case of Brezis-Gallouet inequality in finite dimensional subspace
\[ (2.8) \quad |v|_\infty \leq c_{1,3} L_M |A^*v| \quad \forall v \in H_M, \]
where \( L_M = (1 + \ln \frac{\lambda_M}{\lambda_1})^{\frac{1}{2}}, c_{1,1}, c_{1,2} \) and \( c_{1,3} \) are positive constants independent of \( v \) and \( M \). To avoid having too many constants, we regard \( c_{1,1}, c_{1,2} \) and \( c_{1,3} \) as unity from now on and this will not cause any significant difference. It is obvious that \( L_M \) changes very slowly as \( M \) changes thus it behaves like a constant comparing with \( \lambda_M \). Using (2.3), (2.5) and (2.8), it is easy to verify the following special estimates of the trilinear form:
\[ (2.9) \quad \frac{|b(u,v,w)|}{|b(w,v,u)|} \leq \sqrt{2c_1 L_M |A^*u||A^*v||w|} \quad \forall u,v \in V, w \in H_M. \]

3. New Projection and Its Associated Two-Level Algorithm. Let us denote by \( k > 0 \) the time step length. For any non-negative integer \( n \), we introduce
\[ t_n = nk, \quad u^n = P_M u(t_n), \quad \tilde{u}^n = Q_M \tilde{u}(t_n), \]
and apply \( P_M \) to (2.1) which we write at \( t = t_{n+1} \):
\[ F^{n+1}(u^{n+1}) = u^{n+1} - u^n + k v A u^{n+1} + k P_M B(u^{n+1} + \tilde{u}^{n+1}, u^{n+1} + \tilde{u}^{n+1}) - k P_M f^{n+1} - h^{n+1} = 0, \]
where
\[ h^{n+1} = \int_{t_n}^{t_{n+1}} (P_M(u_t(s) - u_t(t_{n+1})))ds. \]

For certain given positive integer \( m \) (we, of course, assume that \( M \) is large enough such that \( M > m \)), suppose \( u_{m}^{n+1} \in H_m \) is a certain approximation to the solution \( u(t_{n+1}) \). Denoting \( V_M = P_M V \), we define a bilinear form on \( V_M \times V_M : \forall w,v \in V_M \)
\[ (3.2) \quad L^{n+1}_M(w,v) = (w,v) + ka(w,v) + kb(u_{m}^{n+1}, w, v) + kb(w, u_{m}^{n+1}, v). \]
It is easy to verify that the following associated variational problem: for any given \( g \in V^* \), find \( w \in V_M \) such that
\[ L^{n+1}_M(w,v) = \langle g,v \rangle \quad \forall v \in V_M, \]
is well posed provided the later introduced conditions (4.2) are satisfied. Now we can define a new projection \( R^{n+1}_m : V_M \rightarrow H_m \) as follows: for any \( w \in V_M \), find \( R^{n+1}_m w \in H_m \) such that
\[ L^{n+1}_M(w - R^{n+1}_m w, v) = 0 \quad \forall v \in H_m. \]
Consequently, by this new projection the space \( V_M \) has the following decomposition:
\[ V_M = H_m + \tilde{V}_M^{n+1}, \]
where
\[ \hat{V}_{M}^{n+1} = \{ \hat{w} = (P_M - B_m^{n+1})w : \forall w \in V \}. \]

The following "orthogonal"-like property

\[ \mathcal{L}_{M}^{n+1}(\hat{w}, v) = 0 \quad \forall \hat{w} \in \hat{V}_{M}^{n+1}, \quad v \in H_m, \]

is obvious. For convenience, we will use \( \mathcal{L}_{M}^{n+1} \) and \( \hat{V}_{M}^{n+1} \) to represent \( \mathcal{L}_{M}^{n+1} \) and \( \hat{V}_{M}^{n+1} \) in the remainder, respectively. In the next lemma we will show that the similar property (2.2) of the usual \( L^2 \) projection \( P_m \) is also valid for this new projection \( R_m^{n+1} \).

**Lemma 3.1.** Assume that \( u_{m}^{n+1} \in H_m \) is a certain approximation to the solution of NS equations at time step \( n+1 \) and there exists a positive constant \( M_1 \) such that

\[ |A^\perp u_{m}^{n+1}| \leq 2M_1. \]

Then the projection \( R_m^{n+1} \) has the following properties:

\[ |P_m \hat{w}| \leq |Q_m \hat{w}|, \quad |\hat{w}| \leq \sqrt{2}|Q_m \hat{w}| \quad \forall \hat{w} \in \hat{V}^{n+1}, \]

provided the later introduced conditions (4.2) is held.

**Proof.** Thanks to the property (3.3), we have

\[ \mathcal{L}_{m}^{n+1}(\hat{w}, P_m \hat{w}) = 0 \quad \forall \hat{w} \in \hat{V}^{n+1}. \]

That is

\[ |P_m \hat{w}|^2 + k\nu|P_m A^\perp \hat{w}|^2 = -kb(u_{m}^{n+1}, P_m \hat{w}) - kb(P_m \hat{w} + Q_m \hat{w}, u_{m}^{n+1}, P_m \hat{w}). \]

Thanks to (2.4), (2.5) and (2.8), we have

\[ kb(u_{m}^{n+1}, Q_m \hat{w}, P_m \hat{w}) = -kb(u_{m}^{n+1}, P_m \hat{w}, Q_m \hat{w}) \leq c_1 k|u_{m}^{n+1}|_{\infty} |P_m A^\perp \hat{w}| |Q_m \hat{w}| \]

\[ \leq 2c_1 M_1 L_m k|P_m A^\perp \hat{w}| |Q_m \hat{w}| \leq \frac{k\nu}{3}|P_m A^\perp \hat{w}|^2 + \frac{3c_1^2 M_1^2 L_m^2 k}{\nu}|Q_m \hat{w}|^2, \]

\[ kb(P_m \hat{w}, u_{m}^{n+1}, P_m \hat{w}) \leq c_1 k|A^\perp u_{m}^{n+1}| |P_m \hat{w}| |P_m A^\perp \hat{w}| \leq 2c_1 M_1 k|P_m \hat{w}| |P_m A^\perp \hat{w}| \]

\[ \leq \frac{k\nu}{3}|P_m A^\perp \hat{w}|^2 + \frac{3c_1^2 M_1^2 k}{\nu}|P_m \hat{w}|^2, \]

\[ k|b(Q_m \hat{w}, u_{m}^{n+1}, P_m \hat{w})| \leq c_1 k|Q_m \hat{w}| |A^\perp u_{m}^{n+1}| |P_m \hat{w}|_{\infty} \leq 2c_1 M_1 L_m k|Q_m \hat{w}| |P_m A^\perp \hat{w}| \]

\[ \leq \frac{k\nu}{3}|P_m A^\perp \hat{w}|^2 + \frac{3c_1^2 M_1^2 L_m^2 k}{\nu}|Q_m \hat{w}|^2. \]

Therefore,

\[ (1 - \frac{3c_1^2 M_1^2 k}{\nu})|P_m \hat{w}|^2 \leq \frac{6c_1^2 M_1^2 L_m^2 k}{\nu}|Q_m \hat{w}|^2. \]

Thanks to (4.2), it holds

\[ |P_m \hat{w}|^2 \leq |Q_m \hat{w}|^2. \]
Then we can conclude the result of this lemma as long as the conditions (3.4) and (4.2) are satisfied. \(^{1}\)

Now we give the new two-level (postprocessing) algorithm: for \(u_m^0 = P_m u^0\) and \(\tilde{w}^0 = Q_m u^0\), find \(u_m^{n+1} \in H_m\) and \(\tilde{w}^{n+1} \in V^{n+1}\) such that

\[
(3.5) \quad (u_m^{n+1}, v) + ka(u_m^{n+1}, v) + kb(u_m^{n}, u_m^{n}, v) = k(f^{n+1}, v) + (u_m^{n} + \tilde{w}^n, v) \quad \forall v \in H_m,
\]

\[
(3.6) \quad (u_m^{n+1} + \tilde{w}^{n+1}, v) + ka(u_m^{n+1} + \tilde{w}^{n+1}, v) + kb(u_m^n + \tilde{w}^n, u_m^{n+1} + \tilde{w}^n, v) = k(f^{n+1}, v) + (u_m^{n} + \tilde{w}^n, v) \quad \forall v \in V^{n+1}.
\]

For the convenience of later analysis, we give another equivalent form of scheme (3.5)--(3.6). By using (3.3), we can rewrite (3.5) as:

\[
(3.7) \quad (u_m^{n+1}, v) - (u_m^n, v) + ka(u_m^{n+1}, v) = -kb(u_m^n, u_m^n, v) - kb(u_m^n, \tilde{w}^n, v)
- kb(\tilde{w}^n, u_m^n, v) - ka(\tilde{w}^n, v) + k(f^{n+1}, v) \quad \forall v \in H_m.
\]

Using (3.5), (3.6) can be rewritten as:

\[
(3.8) \quad (\tilde{w}^{n+1}, v) - (\tilde{w}^n, Q_m v) + ka(\tilde{w}^{n+1}, v) = -kb(u_m^n, u_m^n, Q_m v) - kb(u_m^n, \tilde{w}^n, v)
- kb(\tilde{w}^n, u_m^n, v) - kb(\tilde{w}^n, \tilde{w}^n, v) + k(Q_m f^{n+1}, v) \quad \forall v \in V^{n+1}.
\]

In the rest of this paper, we will use the following symbols:

\[
\tilde{u}^{n+1} = P_m u^{n+1}, \quad \tilde{u}^{n+1} = Q_m u^{n+1}, \quad e^{n+1} = \tilde{u}^{n+1} - u_m^{n+1}, \quad e^{n+1} = \tilde{u}^{n+1} - \tilde{w}^{n+1}, \quad \tilde{e}^{n+1} = u_m^{n+1} - u_m^n, \quad \tilde{e}^{n+1} = \tilde{u}^{n+1} - \tilde{w}^{n+1}.
\]

Remark 1. As assumed that \(u_m^{n+1}\) is certain approximation to \(u(t_{n+1})\) in \(H_m\), then a possible approach to get a more accurate approximation in \(V_M\) is to solve the following equations of Newton iteration

\[
(D_u F^{n+1}(u_m^{n+1})) (w^{n+1} - u_m^{n+1}), v) = (-F^{n+1}(u_m^{n+1}), v) \quad \forall v \in V_M.
\]

It is easy to verify \((D_u F^{n+1}(u_m^{n+1}) \phi, v) = L^{n+1}(\phi, v)\). Since \(u_m^{n+1}\) is the approximation of \(u(t_{n+1})\) in \(H_m\), we have \((F^{n+1}(u_m^{n+1}), v) \approx 0 \) for all \(v \in H_m\). Thus, it is very natural for us to seek a suitable approximation of \(w^{n+1}\) or an approximate increment \(w^{n+1} \approx u_m^{n+1} - u_m^{n+1}\) such that

\[
L^{n+1}(w^{n+1}, v) = L^{n+1}(w^{n+1} - R_m^{n+1} w^{n+1}, v) = 0 \approx L^{n+1}(w^{n+1} - u_m^{n+1}, v) \quad \forall v \in H_m,
\]

where \(R_m^{n+1}\) is the new projection from \(V_M\) to \(H_m\). That is also the reason why we want to seek the approximate increment in \(V^{n+1}\). In the usual two-level method, the incremental subspace \((P_m - P_m) V\) is a flat manifold in the time-"spatial" space while the incremental subspace \(V^t\) in our two-level method is a nonlinear manifold in the same space whose \(t = t_{n+1}\) section is a flat manifold \(V^{n+1}\) in the "spatial" space consisting of both the usual \(L^2\) based large eddy and small eddy components. It is obvious that the usual two-level method always corrects the large eddy approximation in the same direction (subspace) while our two-level method will postprocess the SGM approximation in a different direction (subspace).
at a different time step according to the large eddy information one has already had. In the rest of this paper, we always call the subspaces \( H_m \) and \( \tilde{V}^{n+1} \) the large eddy and small eddy subspaces, respectively.

**Remark 2.** Let us give a rough interpretation of the scheme (3.5)–(3.6). By using the bilinear form \( \mathcal{L}^{n+1}(.,. \cdot) \)

\[
b(u^{n+1}, u^{n+1}, v) = b(u^{n+1}, \delta^{n+1}, v) + b(\delta^{n+1}, u^{n}, v) + b(u^{n}, u^{n}, v),
\]

(3.1) can be rewritten as

\[
\mathcal{L}^{n+1}(\tilde{u}^{n+1}, v) + \mathcal{L}^{n+1}(\tilde{u}^{n+1}, v) - kb(u^{n+1}, u^{n+1}, v) - kb(u^{n+1}, u^{n+1}, v)
\]

\[
+ kb(u^{n}, u^{n}, v) + kb(u^{n+1}, \delta^{n+1}, v) + kb(\delta^{n+1}, u^{n}, v) + kb(u^{n+1}, \tilde{u}^{n+1}, v)
\]

\[
+ kb(\tilde{u}^{n+1}, u(t_{n+1}), v) = k(f^{n+1}, v) + (h^{n+1}, v) + (u^{n}, v) \quad \forall v \in \tilde{V}^{n+1},
\]

(3.9)

Taking \( v \in H_m \) in (3.9) and noticing (3.3) and the following substitution

\[
-kb(u^{n+1}, \tilde{u}^{n+1}, v) - kb(\tilde{u}^{n+1}, u^{n+1}, v) + kb(u^{n}, u^{n}, v) \\
= -kb(u^{n+1}, \tilde{u}^{n+1}, v) - kb(\tilde{u}^{n+1}, u^{n+1}, v) - kb(u^{n+1}, \tilde{u}^{n+1}, v) - kb(u^{n+1}, \tilde{u}^{n+1}, v) \\
+ kb(e^{n+1}, \tilde{u}^{n}, v) + kb(\tilde{u}^{n}, e^{n+1}, v) + kb(\tilde{u}^{n}, \tilde{u}^{n}, v) + kb(\tilde{u}^{n}, \tilde{u}^{n}, v),
\]

we have

\[
(\tilde{u}^{n+1}, v) + ka(\tilde{u}^{n+1}, v) + kb(e^{n+1}, \tilde{u}^{n}, v) + kb(\tilde{u}^{n}, e^{n+1}, v) + kb(\tilde{u}^{n}, \tilde{u}^{n}, v) \\
+ kb(\tilde{u}^{n}, \tilde{u}^{n}, v) + kG_1(\tilde{u}^{n+1}, v) + kG_2(\tilde{u}^{n+1}, v) \\
= k(f^{n+1}, v) + (h^{n+1}, v) + (\tilde{u}^{n+1}, v) \quad \forall v \in H_m,
\]

(3.10)

where

\[
G_1(\tilde{u}^{n+1}, v) = -b(u^{n+1}, \tilde{u}^{n+1}, v) - b(\tilde{u}^{n+1}, u^{n+1}, v) - b(\tilde{u}^{n+1}, \tilde{u}^{n+1}, v) \\
- b(\tilde{u}^{n+1}, \tilde{u}^{n+1}, v) + b(u^{n+1}, \delta^{n+1}, v) + b(\delta^{n+1}, u^{n+1}, v),
\]

\[
G_2(\tilde{u}^{n+1}, v) = b(u^{n+1}, \tilde{u}^{n+1}, v) + b(\tilde{u}^{n+1}, u(t_{n+1}), v).
\]

If we regard the terms that contain \( e^{n+1}, \delta^{n+1}, \tilde{u}^{n+1}, \tilde{u}^{n+1} \) and the terms \( kb(\tilde{u}^{n}, \tilde{u}^{n}, v) \) and \( (h^{n+1}, v) \) as high order small quantities and omit them, we can get the large eddy approximate equation (3.5) by substituting \( \tilde{u}^{n+1}, \tilde{u}^{n} \) and \( \tilde{u}^{n} \) with \( u^{n+1}, u^{n} \) and \( u^{n} \) respectively. On the other hand, if we take \( v \in V^{n+1} \)

in (3.9), we have

\[
(\tilde{u}^{n+1} + \tilde{u}^{n+1}, v) + ka(\tilde{u}^{n+1} + \tilde{u}^{n+1}, v) + kb(u^{n}, u^{n}, v) + kb(u^{n+1}, \delta^{n+1}, v) \\
+ kb(\delta^{n+1}, u^{n}, v) + kb(u^{n+1}, \tilde{u}^{n+1}, v) + kb(\tilde{u}^{n+1}, u(t_{n+1}), v) \\
= k(f^{n+1}, v) + (h^{n+1}, v) + (u^{n}, v) \quad \forall v \in \tilde{V}^{n+1}.
\]

(3.11)

Similar treatment for deriving (3.10) leads to the small eddy approximate equations (3.6).

4. **Stability Analysis.** We will establish the stability result of scheme (3.5)–(3.6) (or equivalent (3.7)–(3.8)) in this section. We will achieve this by several steps which are stated as several lemmas. For simplicity, we denote \( |f| = |f|_{L^\infty (R^d \times H)} \) and this will not cause any confusion according to the context.

First of all, we give several discrete counterparts of the Gronwall or uniform Gronwall inequalities.

**Lemma 4.1** (Discrete Gronwall Inequality[2]). Let \( d_n \) be a positive sequence satisfying

\[
\forall n \geq 0, \quad ad_{n+1} - \beta d_n \leq q,
\]

where \( \alpha, \beta, q \) are three positive constants with \( \alpha \neq \beta \). Then

\[
\forall n \geq 0, \quad d_n \leq \left( \frac{\beta}{\alpha} \right)^n (d_0 - \frac{q}{\alpha - \beta}) + \frac{q}{\alpha - \beta}.
\]
Lemma 4.2 (Discrete Uniform Gronwall Inequality[13]). Let $d^n$, $g^n$ and $q^n$ be three series satisfying
\[ \frac{d^{n+1} - d^n}{k} \leq g^n d^n + q^n, \quad \forall n \geq n_0 \]
and
\[ k \sum_{n=n_0}^{N+k_0} g^n \leq a_1, \quad k \sum_{n=n_0}^{N+k_0} q^n \leq a_2, \quad k \sum_{n=n_0}^{N+k_0} d^n \leq a_3, \quad \forall k_0 \geq n_0 \]
with $kN = r$. Then
\[ d^n \leq (a_2 + \frac{a_3}{r}) \exp(a_1), \quad \forall n \geq n_0 + N. \]

Lemma 4.3 (Discrete Usual Gronwall Inequality[13]). Let $d^n$, $g^n$ and $q^n$ be three series satisfying
\[ \frac{d^{n+1} - d^n}{k} \leq g^n d^n + q^n, \quad \forall n. \]
Then
\[ d^n \leq d^0 \exp(k \sum_{i=0}^{N} g^i) + k \sum_{i=0}^{N} q^i \exp(k \sum_{j=1}^{N} g^j), \quad \forall n \leq N + 1. \]

Now let us establish the stability theorem step by step.

Lemma 4.4. Assume that there exists a constant $M_1 > 0$ such that $|A\hat{\xi}u^m_n|^2 + |A\hat{\xi}\hat{w}^m|^2 \leq M_1^2$. Then, we have $|A\hat{\xi}u^m_{n+1}| \leq 2M_1$ as long as the later introduced conditions (4.2) are valid.

Proof. Once the conditions (4.2) are satisfied, the proof of this lemma is straightforward by the energy method and we leave to the readers. \]

This lemma guarantees the results of lemma 3.1 are valid in $V^{n+1}$ as long as $|A\hat{\xi}u^m_n|^2 + |A\hat{\xi}\hat{w}^m|^2 \leq M_1^2$.

Lemma 4.5. Suppose that there exists a constant $M_1 > 0$ such that
\[ |A\hat{\xi}u^m_n|^2 + |A\hat{\xi}\hat{w}^m|^2 \leq M_1^2 \quad \forall l \leq n. \]

Then there exists a constant
\[ M_0^2 = |u_0|^2 + \frac{20}{\nu^2 \lambda_1} |A\hat{\xi}f|^2, \quad (4.1) \]
such that
\[ |u^l_m|^2 + |\hat{w}^l|^2 \leq M_0^2 \quad \forall l \leq n + 1, \quad (4.2) \]
provided
\[ \frac{15c_1 M_1 L_M}{\nu \lambda_{m+1}} \leq \frac{1}{2}, \quad \frac{50c_1^2 M_1^2 L_M^2 k}{\nu} \leq \frac{1}{2}, \quad 10k

Proof. First of all, we do some large eddy estimates. Taking \( v = 2u^{n+1}_m \) in (3.7) and noticing

\[
b(u^n_m, \bar{w}^n_u, u^{n+1}_m) = -b(u^n_m, Q_m \bar{w}^{n+1} - \bar{w}^n_u, u^{n+1}_m) + b(u^n_m, Q_m \bar{w}^{n+1} - u^{n+1}_m).
\]

We get

\[
b(\hat{w}^n_u, u^n_m, u^{n+1}_m) = -b(Q_m \bar{w}^{n+1} - \hat{w}^n_u, u^n_m, u^{n+1}_m) + b(Q_m \bar{w}^{n+1} - u^{n+1}_m).
\]

we can get

\[
|u^{n+1}_m|^2 + |u^{n+1}_m - u^n_m|^2 + 2k\nu |A^\perp u^{n+1}_m|^2
\]

\[
= -2kb(u^n_m, u^{n+1}_m, u^{n+1}_m - u^n_m) + 2kb(u^n_m, Q_m \bar{w}^{n+1} - \bar{w}^n_u, u^{n+1}_m)
\]

\[
- 2kb(Q_m \bar{w}^{n+1} - \hat{w}^n_u, u^n_m, u^{n+1}_m) - 2kb(Q_m \bar{w}^{n+1} - u^{n+1}_m) - 2k\nu |Q_m \bar{w}^{n+1} - u^{n+1}_m| - 2k(f^{n+1}, u^{n+1}_m).
\]

Let us estimate the terms on the right-hand side of (4.3) one by one. Most of them are related to the estimates of the trilinear form \( b(\cdot, \cdot, \cdot) \). In the following estimates of the trilinear forms and the estimates of the trilinear forms in the rest of this paper, we will frequently use (2.2)–(2.9) and some combinations of them.

\[
2kb(u^n_m, u^{n+1}_m, u^{n+1}_m) = 2kb(u^n_m, u^{n+1}_m, u^{n+1}_m - u^n_m) \leq 2c_1 M_1 L_m k |A^\perp u^{n+1}_m| |u^{n+1}_m - u^n_m| \leq \frac{k\nu}{10} |A^\perp u^{n+1}_m|^2 + \frac{10c_1^2 M_1^2 L_m^2 k}{\nu} |u^{n+1}_m - u^n_m|^2,
\]

\[
2kb(Q_m \bar{w}^{n+1} - \bar{w}^n_u, u^n_m, u^{n+1}_m) \leq 2c_1 M_1 L_m k |Q_m \bar{w}^{n+1} - \bar{w}^n_u| |A^\perp u^{n+1}_m| \leq \frac{k\nu}{10} |A^\perp u^{n+1}_m|^2 + \frac{10c_1^2 M_1^2 L_m^2 k}{\nu} |Q_m \bar{w}^{n+1} - \bar{w}^n_u|^2,
\]

\[
2kb(Q_m \bar{w}^{n+1} - \hat{w}^n_u, u^n_m, u^{n+1}_m) \leq 2c_1 M_1 L_m k |Q_m \bar{w}^{n+1} - \hat{w}^n_u| |A^\perp u^{n+1}_m| \leq \frac{k\nu}{10} |A^\perp u^{n+1}_m|^2 + \frac{10c_1^2 M_1^2 L_m^2 k}{\nu} |Q_m \bar{w}^{n+1} - \hat{w}^n_u|^2,
\]

\[
2k\nu A^\perp u^{n+1}_m \leq 2k\nu A^\perp f^{n+1} \leq 2k |A^\perp f^{n+1}| |A^\perp u^{n+1}_m| \leq \frac{k\nu}{10} |A^\perp u^{n+1}_m|^2 + \frac{10k}{\nu} |A^\perp f^{n+1}|^2.
\]

A combination of the above 7 estimates with (4.3) admits

\[
|u^{n+1}_m|^2 + |u^{n+1}_m - u^n_m|^2 + 2k\nu |A^\perp u^{n+1}_m|^2 \leq \frac{1}{k\nu} |A^\perp u^{n+1}_m|^2 + \frac{10c_1^2 M_1^2 L_m^2 k}{\nu} |u^{n+1}_m - u^n_m|^2 + \frac{10c_1^2 M_1^2 L_m^2 k}{\nu} |Q_m w^{n+1} - w^n|^2 + \frac{10k}{\nu} |A^\perp f^{n+1}|^2.
\]
Next we have to do the small eddy estimates. Taking $v = 2\hat{w}^{n+1}$ in (3.8) leads to
\[
|\hat{w}^{n+1}|^2 + |\hat{u}^{n+1} - Q_m\hat{w}^{n+1}|^2 - |Q_m\hat{w}^{n+1}|^2 + 2kv|A^\frac{1}{2}\hat{w}^{n+1}|^2
= -2kb(u_m^n, u_m^{n+1}, Q_m\hat{w}^{n+1}) - 2kb(u_m^n, \hat{w}^{n+1}, \hat{w}^{n+1}) - 2kb(\hat{w}^{n}, u_m^n, \hat{w}^{n+1})
= -2kb(\hat{w}^{n+1}, \hat{w}^{n}, \hat{w}^{n+1}) + 2k(f^{n+1}, Q_m\hat{w}^{n+1}).
\]

Noticing
\[
|\hat{w}^{n+1} - Q_m\hat{w}^{n}|^2 + |P_m\hat{w}^{n}|^2 = |Q_m(\hat{w}^{n+1} - \hat{u}^{n})|^2 + |P_m\hat{w}^{n+1}|^2
= |Q_m\hat{w}^{n+1} - \hat{u}^{n}|^2 + |P_m\hat{w}^{n+1}|^2,
\]
we have
\[
|P_m\hat{w}^{n+1}|^2 + |\hat{w}^{n+1}|^2 + |Q_m\hat{w}^{n+1} - \hat{u}^{n}|^2 - |\hat{u}^{n}|^2 + 2kv|A^\frac{1}{2}\hat{w}^{n+1}|^2
= -2kb(u_m^n, u_m^{n+1}, Q_m\hat{w}^{n+1}) - 2kb(u_m^n, \hat{u}^{n+1}, \hat{w}^{n+1}) - 2kb(\hat{w}^{n}, u_m^n, \hat{w}^{n+1})
= -2kb(\hat{w}^{n+1}, \hat{w}^{n}, \hat{w}^{n+1}) + 2k(f^{n+1}, Q_m\hat{w}^{n+1}).
\]

We summarize the estimates on the right-hand side terms of (4.6) as:
\[
-2kb(u_m^n, u_m^{n+1}, Q_m\hat{w}^{n+1}) = 2kb(u_m^n, u_m^{n+1} - u_m^n, Q_m\hat{w}^{n+1}) - 2kb(u_m^n, u_m^{n+1}, Q_m\hat{w}^{n+1})
\leq 2c_1M_1L_m k|A^\frac{1}{2}\hat{w}^{n+1}|^2|u_m^{n+1} - u_m^n| + 2c_1M_1L_m \lambda_{m+1}^2 k|A^\frac{1}{2}\hat{u}^{n+1}|^2
\leq \frac{kv}{10} |A^\frac{1}{2}\hat{w}^{n+1}|^2 + \frac{10c_1^2M_1^2L_m^2 k}{\nu}|u_m^{n+1} - u_m^n|^2
+ \frac{\nu}{\lambda_{m+1}^2} k|A^\frac{1}{2}\hat{u}^{n+1}|^2,
\]

\[
-2kb(u_m^n, \hat{u}^{n+1}, \hat{w}^{n+1}) = 2kb(u_m^n, Q_m\hat{w}^{n+1} - \hat{u}^{n}, \hat{w}^{n+1}) - 2kb(u_m^n, \hat{u}^{n+1}, Q_m\hat{w}^{n+1}, \hat{w}^{n+1})
\leq 2c_1M_1L_m k|A^\frac{1}{2}\hat{w}^{n+1}|^2|Q_m\hat{w}^{n+1} - \hat{u}^{n}| + 2c_1M_1L_m \lambda_{m+1}^2 k|A^\frac{1}{2}\hat{u}^{n+1}|^2
\leq \frac{kv}{10} |A^\frac{1}{2}\hat{w}^{n+1}|^2 + \frac{10c_1^2M_1^2L_m^2 k}{\nu}|Q_m\hat{w}^{n+1} - \hat{u}^{n}|^2
+ \frac{\nu}{\lambda_{m+1}^2} k|A^\frac{1}{2}\hat{u}^{n+1}|^2,
\]

\[
-2kb(\hat{w}^{n}, u_m^n, \hat{w}^{n+1}) = -2kb(Q_m\hat{w}^{n+1} - \hat{u}^{n}, \hat{w}^{n+1}, u_m^n) + 2kb(Q_m\hat{w}^{n+1}, \hat{w}^{n+1}, u_m^n)
\leq 2c_1M_1L_m k|Q_m\hat{w}^{n+1} - \hat{u}^{n}|^2 + 2c_1M_1L_m \lambda_{m+1}^2 k|A^\frac{1}{2}\hat{w}^{n+1}||A^\frac{1}{2}\hat{u}^{n+1}|^2
\leq \frac{kv}{10} |A^\frac{1}{2}\hat{w}^{n+1}|^2 + \frac{10c_1^2M_1^2L_m^2 k}{\nu}|Q_m\hat{w}^{n+1} - \hat{u}^{n}|^2
+ \frac{\nu}{\lambda_{m+1}^2} k|A^\frac{1}{2}\hat{u}^{n+1}|^2,
\]

\[
-2kb(\hat{w}^{n}, \hat{u}^{n+1}, \hat{w}^{n+1}) = -2kb(\hat{u}^{n}, \hat{w}^{n+1}, Q_m\hat{w}^{n+1} - \hat{w}^{n}) + 2kb(\hat{u}^{n}, \hat{w}^{n+1}, Q_m\hat{w}^{n+1})
\leq 2c_1M_1L_m k|A^\frac{1}{2}\hat{w}^{n+1}||Q_m\hat{w}^{n+1} - \hat{w}^{n}| + 2c_1M_1L_m \lambda_{m+1}^2 k|A^\frac{1}{2}\hat{w}^{n+1}|^2
\leq \frac{kv}{10} |A^\frac{1}{2}\hat{w}^{n+1}|^2 + \frac{10c_1^2M_1^2L_m^2 k}{\nu}|Q_m\hat{w}^{n+1} - \hat{w}^{n}|^2
+ \frac{\nu}{\lambda_{m+1}^2} k|A^\frac{1}{2}\hat{u}^{n+1}|^2,
\]

\[
2k(Q_m f^{n+1}, \hat{w}^{n+1}) \leq 2k|Q_m A^\frac{1}{2} f^{n+1}||A^\frac{1}{2}\hat{w}^{n+1}| \leq \frac{kv}{10} |A^\frac{1}{2}\hat{w}^{n+1}|^2 + \frac{10k}{\nu} |Q_m A^\frac{1}{2} f^{n+1}|^2,
\]
A combination of the above 5 estimations with (4.6) leads to
\[
|\hat{w}^{n+1}|^2 + |Q_m\hat{w}^{n+1} - \hat{u}^{n}|^2 - |\hat{u}^{n}|^2 + 2kv|A^\frac{1}{2}\hat{w}^{n+1}|^2
\leq \frac{kv}{10} |A^\frac{1}{2}\hat{w}^{n+1}|^2 + \frac{6c_1M_1L_m}{\nu} k|A^\frac{1}{2}\hat{u}^{n+1}|^2
+ \frac{\nu}{\lambda_{m+1}^2} k|A^\frac{1}{2}\hat{u}^{n+1}|^2
+ |Q_m\hat{w}^{n+1} - \hat{w}^{n}|^2 + \frac{10c_1^2M_1^2L_m^2 k}{\nu} |Q_m\hat{w}^{n+1} - \hat{w}^{n}|^2
+ \frac{50k}{\nu} |Q_m A^\frac{1}{2} f^{n+1}|^2.
\]
Thanks to the conditions (4.2), the summation of (4.4) and (4.7) gives us

\begin{equation}
|u_m^{n+1}|^2 + |\tilde{u}^{n+1}|^2 + k\nu (|A^\frac{1}{2}u_m^{n+1}|^2 + |A^\frac{1}{2}\tilde{u}^{n+1}|^2) \\
\leq |u_m^n|^2 + |\tilde{u}^n|^2 + \frac{20k}{\nu} |A^{-\frac{1}{2}}f^{n+1}|^2,
\end{equation}

or

\begin{equation}
(1 + \lambda_k \nu)(|u_m^{n+1}|^2 + |\tilde{u}^{n+1}|^2) - (|u_m^n|^2 + |\tilde{u}^n|^2) \leq \frac{20k}{\nu} |A^{-\frac{1}{2}}f|^2.
\end{equation}

By using the discrete Gronwall inequality, we can derive from (4.9) that: $\forall 0 \leq l \leq n + 1$

\begin{equation}
(|u_{m,l}|^2 + |\tilde{u}_{m,l}|^2) \leq \left(\frac{1}{1 + k\nu \lambda_k}\right)^l |u_0|^2 + \frac{20}{\nu^2 \lambda_k} \left(1 - \frac{1}{1 + k\nu \lambda_k}\right)^l |A^{-\frac{1}{2}}f|^2.
\end{equation}

This ends the proof. \(\square\)

**Lemma 4.6.** Under the conditions of lemma 4.5, we can get a new positive constant $M_1'$ independent of $k$, $n$, $m$ and $M$ such that

\[|A^\frac{1}{2}u_{m,l}^n|^2 + |A^\frac{1}{2}\tilde{u}_{m,l}^n|^2 \leq M_1'^2 \quad \forall l \leq n + 1.\]

**Proof.** Taking $v = 2Au_m^{n+1}$ in (3.7) and using

\[b(u^n, u^n, Au^{n+1}) = b(u^{n+1}, u^n, Au^{n+1}) - b(u^{n+1} - u^n, u^n, Au^{n+1}),\]

\[b(u^n, \tilde{u}^n, Au^{n+1}) = -b(u^n, Q_m\tilde{u}^{n+1} - \tilde{u}^n, Au^{n+1}) + b(u^n, Q_m\tilde{u}^{n+1}, Au^{n+1}),\]

\[b(\tilde{u}^n, u^n, Au^{n+1}) = -b(Q_m\tilde{u}^{n+1} - \tilde{u}^n, u^n, Au^{n+1}) + b(Q_m\tilde{u}^{n+1}, u^n, Au^{n+1}),\]

admits

\begin{equation}
|A^\frac{1}{2}u_{m,l}^n|^2 + |A^\frac{1}{2}\tilde{u}_{m,l}^n|^2 - |A^\frac{1}{2}u_{m,l}^n|^2 + 2k\nu|Au_{m,l}^{n+1}|^2 \\
\leq 2k|b(u_m^{n+1} - u_m^n, u_m^n, Au^{n+1})| + 2k|b(u_m^{n+1} - u_m^n, Q_m\tilde{u}^{n+1}, Au^{n+1})| + 2k|b(Q_m\tilde{u}^{n+1} - \tilde{u}^n, u_m^n, Au^{n+1})| + 2k|b(Q_m\tilde{u}^{n+1}, u_m^n, Au^{n+1})| + 2k|b(Q_m\tilde{u}^{n+1}, u_m^n, Au^{n+1})| + 2k|b(Q_m\tilde{u}^{n+1}, u_m^n, Au^{n+1})|.
\end{equation}

For each right-hand side term of (4.11), we have

\[2kb(u_m^{n+1} - u_m^n, u_m^n, Au^{n+1}) \leq 2c_1k|u_m^{n+1}|_\infty |A^\frac{1}{2}u_m^n||Au_m^{n+1}| \\
\leq 2c_1k|u_m^{n+1}|^2 |A^\frac{1}{2}u_m^n| + |Au_m^{n+1}| \leq \frac{k\nu}{2} |Au_m^{n+1}|^2 + \frac{27c_1^2 M_2^2 k}{2\nu^3} |A^\frac{1}{2}u_m^n|^4,
\]

\[2kb(u_m^{n+1} - u_m^n, u_m^n, Au^{n+1}) \leq 2c_1 M_1 Lm |A^\frac{1}{2}(u_m^{n+1} - u_m^n)| |Au_m^{n+1}| \\
\leq \frac{k\nu}{10} |Au_m^{n+1}|^2 + \frac{10c_1^2 M_2^2 L_2 k}{\nu} |A^\frac{1}{2}(u_m^{n+1} - u_m^n)|^2,
\]

\[2kb(u_m^n, Q_m\tilde{u}^{n+1} - \tilde{u}^n, Au^{n+1}) \leq 2c_1 M_1 Lm |A^\frac{1}{2}(Q_m\tilde{u}^{n+1} - \tilde{u}^n)| |Au_m^{n+1}| \\
\leq \frac{k\nu}{10} |Au_m^{n+1}|^2 + \frac{10c_1^2 M_2^2 L_2 k}{\nu} |A^\frac{1}{2}(Q_m\tilde{u}^{n+1} - \tilde{u}^n)|^2,
\]

\[2kb(u_m^n, Q_m\tilde{u}^{n+1}, Au^{n+1}) \leq 2c_1 M_1 Lm |A^\frac{1}{2}(Q_m\tilde{u}^{n+1})| |Au_m^{n+1}| \\
\leq \frac{k\nu}{10} |Au_m^{n+1}|^2 + \frac{10c_1^2 M_2^2 L_2 k}{\nu} |A^\frac{1}{2}(Q_m\tilde{u}^{n+1})|^2,
\]
A combination of the above 8 estimates with (4.11) yields

\[
2k(b_m, Q_m \hat{w}^{n+1}, A_m^{n+1}) \leq 2c_1 M L \lambda_{m+1} \frac{1}{k} |Q_m A \hat{w}^{n+1}| |A_m^{n+1}| + c_1 M L \lambda_{m+1} \frac{1}{k} (|A \hat{w}^{n+1}|^2 + |A_m^{n+1}|^2),
\]

\[
2k(Q_m \hat{w}^{n+1} - \hat{w}^n, u_m, A_m^{n+1}) \leq 2c_1 M_1 [M_k |A \hat{w}^{n+1} - \hat{w}^n|] |A_m^{n+1}| \leq \frac{2c_1 M_1 \|M_k |A \hat{w}^{n+1} - \hat{w}^n|\| |A_m^{n+1}|}{k},
\]

\[
2k(Q_m \hat{w}^{n+1}, u_m, A_m^{n+1}) \leq 2c_1 M_1 |Q_m \hat{w}^n| + 2c_1 M_1 \lambda_{m+1} \frac{1}{k} |A \hat{w}^{n+1}| |A_m^{n+1}| \leq 2c_1 M_1 \lambda_{m+1} \frac{1}{k} |A \hat{w}^{n+1}| |A_m^{n+1}|,
\]

\[
2k(A \hat{w}^{n+1}, A_m^{n+1}) = -2k \lambda_{m+1} \frac{1}{k} |A \hat{w}^{n+1} - \hat{w}^n| |A_m^{n+1}| \leq \frac{2c_1 M_1 \|Q_m \hat{w}^{n+1} - \hat{w}^n\| |A_m^{n+1}|}{k},
\]

\[
2k(f^{n+1}, A_m^{n+1}) \leq 2k |f^{n+1}| |A_m^{n+1}| \leq \frac{2k \|f^{n+1}\| |A_m^{n+1}|}{k}. \tag{4.12}
\]

A combination of the above 8 estimates with (4.11) yields

\[
|A \hat{u}^{n+1}_m|^2 - |A \hat{u}^{n+1}_m|^2 + (1 - \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu}) |A \hat{u}^{n+1}_m|^2 + 2k \nu |A_m^{n+1}|^2 \leq \frac{27c_1^3 M_1^3 L_{m,k}^3}{20c_1^2 M_1^2 L_{m,k}^2} |A \hat{u}^{n+1}_m|^2 + 2c_1 M_1 L \lambda_{m+1} \frac{1}{k} (|A_m^{n+1}|^2 + |A \hat{w}^{n+1}|^2) + \frac{20c_1^2 M_1^2 L_{m,k}^2}{\nu} + 10c_1 \lambda_{m+1} |A \hat{w}^{n+1} - \hat{w}^n| + \frac{10k}{\nu} |f^{n+1}|^2.
\]

Taking \( v = 2A \hat{w}^{n+1} \) in (3.8) and using the similar formula as (4.5) yields

\[
|P_m A \hat{w}^{n+1} + |A \hat{w}^{n+1}|^2 + |A \hat{w}^{n+1} - \hat{w}^n|^2 + 2k \nu |A \hat{w}^{n+1}|^2 \leq -2k b(u_m, u_m, Q_m A \hat{w}^{n+1}) - 2k b(u_m, \hat{u}^n, A \hat{w}^{n+1}) - 2k b(\hat{u}^n, u_m, A \hat{w}^{n+1}) + 2k b(Q_m \hat{w}^{n+1}, A \hat{w}^{n+1}).
\]

The estimates of each term on the right-hand side of (4.12) are as follows:

\[
-2k b(u_m, u_m, Q_m A \hat{w}^{n+1}) = 2k b(u_m, u_m, Q_m A \hat{w}^{n+1}) - 2k b(u_m, u_m, Q_m A \hat{w}^{n+1}) \leq 2c_1 M_1 L \lambda_{m+1} |Q_m A \hat{w}^{n+1}| + c_1 M L \lambda_{m+1} |Q_m A \hat{w}^{n+1}| + c_1 M L \lambda_{m+1} |Q_m A \hat{w}^{n+1}| + c_1 M L \lambda_{m+1} |Q_m A \hat{w}^{n+1}| \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{u}^{n+1}_m|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4 \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{u}^{n+1}_m|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4.
\]

\[
-2k b(u_m, \hat{u}^n, A \hat{w}^{n+1}) = 2k b(u_m, Q_m \hat{w}^{n+1} - \hat{w}^n, A \hat{w}^{n+1}) - 2k b(u_m, Q_m \hat{w}^{n+1}, A \hat{w}^{n+1}) \leq 2c_1 M_1 L \lambda_{m+1} |Q_m \hat{w}^{n+1} - \hat{w}^n| |A \hat{w}^{n+1}| + 2c_1 M L |Q_m \hat{w}^{n+1} - \hat{w}^n| |A \hat{w}^{n+1}| \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{w}^{n+1}|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4 \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{w}^{n+1}|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4,
\]

\[
-2k b(\hat{u}^n, u_m, A \hat{w}^{n+1}) = 2k b(u_m, Q_m \hat{w}^{n+1} - \hat{w}^n, A \hat{w}^{n+1}) - 2k b(u_m, Q_m \hat{w}^{n+1}, A \hat{w}^{n+1}) \leq 2c_1 M_1 L \lambda_{m+1} |Q_m \hat{w}^{n+1} - \hat{w}^n| |A \hat{w}^{n+1}| + 2c_1 M L |Q_m \hat{w}^{n+1} - \hat{w}^n| |A \hat{w}^{n+1}| \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{w}^{n+1}|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4 \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{w}^{n+1}|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4,
\]

\[
-2k b(u_m, \hat{u}^n, A \hat{w}^{n+1}) = 2k b(u_m, Q_m \hat{w}^{n+1} - \hat{w}^n, A \hat{w}^{n+1}) - 2k b(u_m, Q_m \hat{w}^{n+1}, A \hat{w}^{n+1}) \leq 2c_1 M_1 L \lambda_{m+1} |Q_m \hat{w}^{n+1} - \hat{w}^n| |A \hat{w}^{n+1}| + 2c_1 M L |Q_m \hat{w}^{n+1} - \hat{w}^n| |A \hat{w}^{n+1}| \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{w}^{n+1}|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4 \leq \frac{10c_1^2 M_1^2 L_{m,k}^2}{\nu} |A \hat{w}^{n+1}|^2 + \frac{3k \nu}{4} |A \hat{w}^{n+1}|^2 + \frac{4c_1^3 M_1^3 L_{m,k}^3}{\nu^3} |A \hat{u}^{n+1}_m|^4,
\]
For a certain positive constant $k$, we have:

$$-2k b(\hat{w}^n, u_m^n, A\hat{w}^{n+1}) = 2kb(Q_m\hat{w}^{n+1} - \hat{w}^n, u_m^n, A\hat{w}^{n+1}) - 2kb(Q_mw^n, u_m^n, A\hat{w}^{n+1})$$

$$\leq 2c_1M_1\nu \frac{M_2^2 k}{\nu} |A^\frac{2}{\nu}(Q_m\hat{w}^{n+1} - \hat{w}^n)|^2 + 2c_1M_1 \nu \frac{k\nu |A\hat{w}^{n+1}|^2}{\nu \lambda_{m+1}^2}$$

$$-2k b(\hat{w}^n, \bar{w}, \hat{w}^n, A\hat{w}^{n+1}) = 2kb(Q_m\hat{w}^{n+1} - \hat{w}^n, \bar{w}, A\hat{w}^{n+1}) - 2kb(Q_m\hat{w}^{n+1}, \bar{w}, A\hat{w}^{n+1})$$

$$\leq 2c_1M_1\nu \frac{M_2^2 k}{\nu} |A^\frac{2}{\nu}(Q_m\hat{w}^{n+1} - \hat{w}^n)|^2 + 2c_1M_1 \nu \frac{k\nu |A\hat{w}^{n+1}|^2}{\nu \lambda_{m+1}^2}$$

Combining the above 5 estimates with (4.13), we obtain:

$$|A^\frac{2}{\nu}\hat{w}^{n+1}|^2 - |A^\frac{2}{\nu}\hat{w}^n|^2 + |A^\frac{2}{\nu}(Q_m\hat{w}^{n+1} - \hat{w}^n)|^2 + \frac{3}{2} \frac{6c_1M_1L}{\nu \lambda_{m+1}^2} k\nu |A\hat{w}^{n+1}|^2$$

$$\leq 10c_1^2 M_1^2 L_2^2 k |A^\frac{2}{\nu}(w_m^{n+1} - w_m^n)|^2 + \frac{30c_1^2 M_1^2 L_2^2 k}{\nu} |A^\frac{2}{\nu}(Q_m\hat{w}^{n+1} - \hat{w}^n)|^2$$

Now the summation of (4.12) and (4.14) and applying (4.2) leads to:

$$|A^\frac{2}{\nu}u_m^{n+1}|^2 + |A^\frac{2}{\nu}\hat{w}^{n+1}|^2 - |A^\frac{2}{\nu}u_m^n|^2 - |A^\frac{2}{\nu}\hat{w}^n|^2$$

$$\leq \frac{35c_1^4 M_2^2 k}{2\nu^3} |A^\frac{2}{\nu}u_m^{n+1}|^2 + \frac{20k}{\nu} |f|^{n+1}|^2$$

as long as the stability conditions (4.2) are satisfied.

For a certain positive constant $r$, choose a positive integer $N$ such that $kN = r$. Thanks to (4.8), we have:

$$k \sum_{l=k_0}^{N+k_0} (|A^\frac{2}{\nu}u_m^l|^2 + |A^\frac{2}{\nu}\hat{w}^l|^2) \leq \frac{2M_0^2}{\nu} + \frac{20r}{\nu^2} |f|^2 \quad \forall N + k_0 \leq n + 1.$$

Let us denote:

$$a_2 = \frac{20r}{\nu^2} |f|^2, \quad a_3 = \frac{2M_0^2}{\nu} + a_2, \quad a_1 = \frac{35c_1^4 M_2^2}{2\nu^3} a_3.$$

By using the discrete uniform Gronwall inequality, we can get from (4.15) that:

$$|A^\frac{2}{\nu}u_m^l|^2 + |A^\frac{2}{\nu}\hat{w}^l|^2 \leq (a_2 + a_3) \exp(a_1) \quad \forall N \leq l \leq n + 1.$$

For $l < N$, by using the discrete usual Gronwall inequality we can obtain that:

$$|A^\frac{2}{\nu}u_m^l|^2 + |A^\frac{2}{\nu}\hat{w}^l|^2 \leq \left(|A^\frac{2}{\nu}u_0|^2 + \frac{20r}{\nu^2} |f|^2\right) \exp(a_1) \quad \forall 0 \leq l < N.$$
In the following lemma, we give some estimates of (5.3) for any $t$, which is independent of $k, n, m$ and $M$, such that the scheme (3.5)-(3.6) is $V-$stable provided $k$ and $m$ satisfy the stability conditions (4.2). That is

$$|A^+_\tau w^n_m|^2 + |A^+_\tau \delta^n|^2 \leq M_1^2, \quad \forall n \geq 0.$$

**Proof.** With the knowledge of lemma 4.5 and lemma 4.6, we can complete the proof of this theorem by induction.

From the definition of $M_1$ and $w_0^n = P_m u_0, \ddot{w}^0 = Q_m u_0 = (P_\ell - P_m)u_0$, we know that the result is true for $l = 0$. Suppose that the result is valid for $l = n > 0$, then by lemma 4.6 we can conclude that the result is also true for $l = n + 1$. And this ensures the validity of the result for every $n \geq 0$. □

It is obvious that the scheme (3.5)-(3.6) has weaker stability conditions than usual one level explicit SGM. In fact the stability conditions (4.2) of our scheme are similar to the stability conditions for explicit SGM in $H_m$, which is easier to be satisfied than the conditions for SGM in $H_\ell$. That means we could take larger time step length than usual explicit SGM in $H_\ell$.

**5. Error Analysis.** In this section, we will give some error analysis of our scheme. Throughout this section, we always assume that

$$|u^n_m|^2 + |\ddot{w}^n|^2, |u^n_m + \ddot{w}^n|, |u(t_n)|^2 \leq M_1^2,$$

$$|A^+_\tau u^n_m|^2 + |A^+_\tau \ddot{w}^n|^2, |A^+_\tau (u^n_m + \ddot{w}^n)|^2, |A^+_\tau u(t_n)|^2 \leq M_1^2,$$

$$|A^+_\tau \delta^{n+1}| = |A^+_\tau (u^{n+1} - u^n)| \leq \kappa_2 k, \quad \kappa_2 = |A^+_\tau u|_{L_\infty(R^m;H)} < +\infty,$$

$$|A^+_\tau \ddot{w}^{n+1}| < +\infty,$$

for any $n \geq 0$ and $t \geq 0$. From [5], we know that

$$|Q_m u(t)| \leq \frac{\kappa_0 L_m}{\lambda_{m+1}}, \quad \kappa_0 = \frac{\nu c M_1}{|Q_m f|}, \quad M_1 = M_1^1 + M_1^2,$$

$$|Q_m A^+_\tau u(t)| \leq \frac{\kappa_1 L_m}{\lambda_{m+1}}, \quad \kappa_1 = \frac{\nu c M_1}{|Q_m f|} + M_1^1 + \frac{M_1^2}{M_1^1}.$$

In the following lemma, we give some estimates of $|A^+_\tau \delta^{n+1}|$ and $|A^+_\tau \ddot{w}^{n+1}|$.

**Lemma 5.1.** If the stability conditions (4.2) and (5.1)-(5.2) are valid, furthermore we assume

$$\frac{8c^2 M_0 M_1 L_m^2 \lambda_m \nu k}{\lambda_{m+1}} \leq \frac{1}{2},$$

then

$$|A^+_\tau \delta^{n+1}|, \quad |A^+_\tau \ddot{w}^{n+1}| \leq \kappa_3 k,$$
where $\kappa_3$ is a positive constant independent of $k$, $n$, $m$, $M$ and will be defined in the proof.

Proof. Since

$$|A^N\bar{b}^{\alpha_n+1}| = |A^N(P_m\delta^{\alpha_n+1} - P_m\hat{\delta}^{\alpha_n+1})| \leq |A^N\hat{b}^{\alpha_n+1}| + |A^N P_m\hat{b}^{\alpha_n+1}|,$$

we only have to estimate $|A^N P_m\hat{b}^{\alpha_n+1}|$. Thanks to (3.3), we have

$$(\hat{\delta}^{\alpha_n+1}, v + k\alpha(\hat{\delta}^{\alpha_n+1}, v) + k\beta(u_n^{\alpha_n+1} - u_m^{\alpha_n}, v) \quad \forall v \in H_m.$$ 

Take $v = A^N P_m\hat{b}^{\alpha_n+1}$ in this equation, then

$$|A^N P_m\hat{b}^{\alpha_n+1}|^2 + k\nu |A^N P_m\hat{b}^{\alpha_n+1}|^2 = -k\beta(u_n^{\alpha_n+1}, P_m\hat{b}^{\alpha_n+1}, A^N P_m\hat{b}^{\alpha_n+1}) - k\beta(u_n^{\alpha_n+1}, Q_m\hat{b}^{\alpha_n+1}, A^N P_m\hat{b}^{\alpha_n+1}) - k\beta(\hat{b}^{\alpha_n+1}, u_m^{\alpha_n+1}, P_m\hat{b}^{\alpha_n+1}) - k\beta(Q_m\hat{b}^{\alpha_n+1}, u_m^{\alpha_n+1}, P_m\hat{b}^{\alpha_n+1}).$$

Noticing $Q_m\hat{b}^{\alpha_n+1} = Q_m\delta^{\alpha_n+1}$, for each term on the right-hand side of this equation, we have

$$kb(u_m^{\alpha_n+1}, P_m\hat{b}^{\alpha_n+1}, A^N P_m\hat{b}^{\alpha_n+1}) \leq c_1 M_0^2 M_1^2 \lambda_m^2 k |A^N\hat{b}^{\alpha_n+1}||A^N P_m\hat{b}^{\alpha_n+1}|$$

$$\leq \frac{k\nu}{6} |A^N P_m\hat{b}^{\alpha_n+1}|^2 + \frac{6c_1^2 M_0 M_1 \lambda_m^2 k^3}{\nu},$$

$$kb(u_n^{\alpha_n+1}, Q_m\hat{b}^{\alpha_n+1}, A^N P_m\hat{b}^{\alpha_n+1}) \leq c_1 M_0^2 M_1^2 \lambda_m^2 k^2 |A^N P_m\hat{b}^{\alpha_n+1}| |A^N Q_m\hat{b}^{\alpha_n+1}|$$

$$\leq \frac{k\nu}{6} |A^N P_m\hat{b}^{\alpha_n+1}|^2 + \frac{6c_1^2 M_0 M_1 \lambda_m^2 k^3}{\nu},$$

$$kb(\hat{b}^{\alpha_n+1}, u_m^{\alpha_n+1} - u_n^{\alpha_n}, A^N P_m\hat{b}^{\alpha_n+1}) \leq \left\{ \begin{array}{ll}
\frac{c_1 k |A^N(u_m^{\alpha_n} - u_m^{\alpha_n})||A^N\hat{b}^{\alpha_n}| |A^N P_m\hat{b}^{\alpha_n}|}{\lambda_m^{\alpha_n+1}} & \\
\frac{c_1 k |A^N(u_n^{\alpha_n} - u_m^{\alpha_n})||A^N\hat{b}^{\alpha_n}| |A^N P_m\hat{b}^{\alpha_n}|}{\lambda_m^{\alpha_n+1}} & \\
\end{array} \right.$$

$$\leq \frac{k\nu}{6} |A^N P_m\hat{b}^{\alpha_n+1}|^2 + \frac{6c_1^2 \kappa_0 \kappa_1 L_m^2 k^3}{\nu \lambda_m^{\alpha_n+1}},$$

$$kb(P_m\hat{b}^{\alpha_n+1}, u_m^{\alpha_n}, A^N P_m\hat{b}^{\alpha_n+1}) \leq c_1 M_1 k |A^N\hat{b}^{\alpha_n+1}| |A^N P_m\hat{b}^{\alpha_n+1}|$$

$$\leq \frac{k\nu}{6} |A^N P_m\hat{b}^{\alpha_n+1}|^2 + \frac{6c_1^2 M_1^2 k^3}{\nu} |A^N P_m\hat{b}^{\alpha_n+1}|^2.$$
From (3.7), it is easy to verify that

\[ |u^{n+1}_m - u^n_m| \leq \kappa_4 \lambda nk, \quad \kappa_4 = 2vM_1 + 3c_1 M_0 M_1 + |f|. \]

If we apply the condition (5.4) and define

\[ \kappa_3 = \kappa_2 + \sqrt{\kappa_2^2 + \frac{\kappa_0 \kappa_1}{M_0 M_1}} \kappa_2^2, \]

we can derive the result of this lemma.

Subtracting (3.5) from (3.10) admits

\begin{align*}
(e^{n+1}, v) + ka(e^{n+1}, v) - (e^n, v) &= -kb(e^{n+1}, \hat{u}^n, v) - kb(\delta^n, e^{n+1}, v) \\
-kb(e^n, \hat{u}^n, v) - kb(u^n_m, e^n, v) - kb(\hat{u}^n, \hat{u}^n, v) - kG_1(\delta^n, v) \\
-kG_2(\hat{u}^{n+1}, v) + (h^{n+1}, v) + (e^n, v) &\quad \forall v \in H_m.
\end{align*}

(5.5)

Thanks to (3.3), we have

\[ (e^n, v) = -ka(e^n, v) - kb(u^n_m, e^n, v) - kb(e^n, u^n_m, v) \quad \forall v \in H_m. \]

By using this relation, the above equations can be rewritten as

\begin{align*}
(e^{n+1}, v) + ka(e^{n+1}, v) - (e^n, v) &= -kb(e^{n+1}, \hat{u}^n, v) - kb(\delta^n, e^{n+1}, v) \\
-kb(e^n, \hat{u}^n, v) - kb(u^n_m, e^n, v) - ka(e^n, v) - kb(u^n_m, e^n, v) \\
-kb(e^n, u^n_m, v) - kb(\hat{u}^n, \hat{u}^n, v) - kG_1(\delta^n, v) \\
-kG_2(\hat{u}^{n+1}, v) + (h^{n+1}, v) &\quad \forall v \in \tilde{V}^{n+1}.
\end{align*}

(5.6)

Subsequently, by subtracting (3.6) from (3.11) we can obtain

\begin{align*}
(e^{n+1}, v) + ka(e^{n+1}, v) + (e^{n+1}, v) + ka(e^{n+1}, v) + kb(u^{n+1}, \delta^{n+1}, v) \\
+kb(\delta^{n+1}, u^n, v) + kb(e^n, e^n, u^n, v) + kb(u^n_m + \hat{u}^n, \hat{u}^n, v) + e^n, v + kG_2(\hat{u}^{n+1}, v) \\
= (e^n + e^n, P_m v) + (e^n + e^n, Q_m v) + (h^{n+1}, v) &\quad \forall v \in \tilde{V}^{n+1}.
\end{align*}

By using (5.5) and observing that

\[ (e^n + e^n, Q_m v) = (e^n, Q_m v), \quad (e^{n+1}, v) + ka(e^{n+1}, v) = (e^{n+1}, P_m v) + ka(e^{n+1}, P_m v), \]

the substitution of \((e^n + e^n, P_m v)\) in the above equations finally admits via careful calculations that

\begin{align*}
(e^{n+1}, v) + ka(e^{n+1}, v) - (Q_m e^n, v) &= kb(\hat{u}^n, e^{n+1}, P_m v) - kb(e^n, \hat{u}^n, Q_m v) \\
-kb(u^n_m, e^n, Q_m v) - kb(\hat{u}^n, e^n, v) + kb(e^{n+1} - e^n, \hat{u}^n, P_m v) \\
-kb(e^n, u^n_m, v) - kb(u^n_m + \hat{u}^n, e^n, v) + kG_3(\delta^{n+1}, v) - kG_2(\hat{u}^{n+1}, v) \\
+kb(\hat{u}^n, \hat{u}^n, P_m v) + (h^{n+1}, Q_m v) &\quad \forall v \in \tilde{V}^{n+1},
\end{align*}

(5.7)

where

\[ G_3(\delta^{n+1}, v) = G_1(\delta^{n+1}, P_m v) - b(u^{n+1}, \delta^{n+1}, v) - b(\delta^{n+1}, u^n, v) \\
= -b(u^n_m, \delta^{n+1}, P_m v) - b(\delta^{n+1}, u^n_m, P_m v) - b(\delta^{n+1}, \hat{u}^n, P_m v) \\
- b(\hat{u}^n, \delta^{n+1}, P_m v) - b(u^{n+1}, \delta^{n+1}, Q_m v) - b(\delta^{n+1}, u^n, Q_m v). \]

**Theorem 5.2.** Suppose that the stability conditions (4.2) in theorem 4.7, (5.4) and assumptions (5.1) and (5.2) are fulfilled. Then we have: \( \forall n \geq 0 \)

\[ |u(t_n) - (u^n_m + \hat{u}^n)| \leq (e^{-\frac{\alpha^n M_{n+1}}{2}} - 1)^{\frac{1}{2}} (\kappa_5 k + \kappa_6 L_m (L_m \lambda^{-\frac{1}{2}}_{n+1} + L_M \lambda^{-\frac{1}{2}}_{M+1})) + \kappa_0 L_M \lambda^{-\frac{1}{2}}_{M+1}, \]

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where $\kappa_5$ and $\kappa_6$ are positive constants independent of $k$, $m$, $M$ and $n$ which will be defined explicitly at the end of the proof of this theorem.

Proof. Taking $v = 2^{n+1}$ in (5.6) leads to

$$
|e^{n+1}|^2 + |e^{n+1} - e^n|^2 + 2k\nu|A^\frac{1}{2} e^{n+1}|^2 - |e^n|^2 = -2k k(e^{n+1}, \bar{u}^n, e^{n+1})
$$

$$
-2k k(e^n, \bar{u}^n, e^{n+1}) - 2k k(u_m^n, e^n, e^{n+1}) - 2k a(e^{n, e^{n+1}})
$$

$$
-2k k(u_m^n, e^n, e^{n+1}) - 2k k(e^n, u_m^n, e^{n+1}) - 2k k(\bar{u}^n, \bar{u}^n, e^{n+1})
$$

$$
-2k G_1(\delta^{n+1}, e^{n+1}) - 2k G_2(\bar{u}^n, e^{n+1}) + 2(h^{n+1}, e^{n+1}).
$$

For each term on the right-hand side of (5.8), we have

$$
2k k(e^{n+1}, \bar{u}^n, e^{n+1}) = -2k k(e^{n+1}, e^{n+1}, \bar{u}^n) \leq 2\sqrt{2} c_1 M_1 L M \lambda_{m+1}^{-\frac{1}{2}} |A^\frac{1}{2} e^{n+1}|^2,
$$

$$
2k k(e^n, \bar{u}^n, e^{n+1}) = 2k k(e^{n+1}, \bar{u}^n, e^{n+1}) - 2k k(e^{n+1} - e^n, \bar{u}^n, e^{n+1})
$$

$$
\leq 2c_1 M_1 k |A^\frac{1}{2} e^{n+1}| |e^{n+1}| + 2c_1 M_1 L k |A^\frac{1}{2} e^{n+1}| |e^{n+1} - e^n|
$$

$$
\leq \frac{2k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{10 c_1^2 M_1^2 L^2 k}{\nu} |e^{n+1} - e^n|^2,
$$

$$
2k k(u_m^n, e^n, e^{n+1}) = 2k k(u_m^n, e^{n+1}, e^{n+1} - e^n) \leq 2c_1 M_1 L M |A^\frac{1}{2} e^{n+1}| |e^{n+1} - e^n|
$$

$$
\leq \frac{2k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{10 c_1^2 M_1^2 L^2 k}{\nu} |e^{n+1} - e^n|^2,
$$

$$
2k a(e^{n+1}, e^n) = -2k a(Q m e^{n+1} - e^n, e^{n+1}) \leq 2c_1 \nu |\lambda_{m+1}^2 k| Q m e^{n+1} - e^n | |A^\frac{1}{2} e^{n+1}|
$$

$$
\leq \frac{2k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + 10 c_1^2 \nu |\lambda_{m+1}^2 k| Q m e^{n+1} - e^n|^2,
$$

$$
2k k(u_m^n, e^{n+1}, e^{n+1}) = 2k \left( -b(u_m^n, e^{n+1}, Q m e^{n+1}) + b(u_m^n, e^{n+1}, Q m e^{n+1} - e^n) \right),
$$

$$
\leq 2c_1 M_1 L M \lambda_{m+1}^{-\frac{1}{2}} |A^\frac{1}{2} e^{n+1}| |Q m A^\frac{1}{2} e^{n+1}| + 2c_1 M_1 L M |A^\frac{1}{2} e^{n+1}| |Q m e^{n+1} - e^n|
$$

$$
\leq \frac{c_1 M_1 L M}{\lambda_{m+1}^2} |\lambda_m^\frac{1}{2} A^\frac{1}{2} e^{n+1}|^2 + |A^\frac{1}{2} e^{n+1}|^2
$$

$$
+ \frac{k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{10 c_1^2 M_1^2 L^2 k}{\nu} |Q m e^{n+1} - e^n|^2,
$$

$$
2k k(\bar{u}^n, \bar{u}^n, e^{n+1}) \leq 2c_1 k |\bar{u}^n| |A^\frac{1}{2} \bar{u}^n| |A^\frac{1}{2} e^{n+1}|
$$

$$
\leq \frac{k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{20 c_1^2 M_1^2 L^2 k}{\nu |\lambda_{m+1}^2|},
$$

$$
2(h^{n+1}, e^{n+1}) \leq 2 |A^\frac{1}{2} h^{n+1}| |A^\frac{1}{2} e^{n+1}|
$$

$$
\leq \frac{k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{10 c_1^2 M_1^2 L^2 k}{\nu |\lambda_{m+1}^2|},
$$

The estimate of $G_1$ is nothing but some estimates of the trilinear forms containing $\delta^{n+1}$, $\bar{\delta}^{n+1}$ or $\hat{\delta}^{n+1}$. For each term in it, we have to obtain an estimate in terms of $|A^\frac{1}{2} \delta^{n+1}|$, $|A^\frac{1}{2} \bar{\delta}^{n+1}|$ or $|A^\frac{1}{2} \hat{\delta}^{n+1}|$. Since there is nothing difficult, we directly state the final estimate:

$$
2k G_1(\delta^{n+1}, e^{n+1}) \leq \frac{6k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{20 c_1^2 M_0 M_1 (2 \kappa_1^2 + \kappa_2^2)}{\nu} k^3.
$$

For $G_2$, we have

$$
2k G_2(\bar{u}^{n+1}, e^{n+1}) \leq \frac{2k \nu}{10} |A^\frac{1}{2} e^{n+1}|^2 + \frac{40 c_1^2 M_1^2 L^2 k}{\nu |\lambda_{M+1}^2|},
$$

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here we used the property (2.9). Then the combination of the above estimates with (5.8) yields

\[
|e^{n+1}|^2 + |e^{n+1} - e^n| + \frac{2}{5} |\kappa \nu |A^\frac{3}{2}e^{n+1}|^2 - |e^n|^2 \leq \frac{10c_1^2 M_2^2 k}{\nu} |e^{n+1}|^2
\]

\[
+ \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2}e^{n+1}|^2 + \frac{2c_1 M_1 L_m}{\nu} |e^{n+1}|^2 - |e^n|^2
\]

\[
+ \frac{20c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}} |e^{n+1}|^2 - |e^n|^2 + (10c_1^2 \nu \lambda_m k + \frac{30c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}}) |Q_m e^{n+1} - e^n|^2
\]

\[
+ \frac{10k_0}{\nu} |A^\frac{3}{2} u_m|^2 + \frac{20c_1^2 M_0 M_1 (c_1^2 + 2n_0^2)}{\nu \lambda_{m+1}} + \frac{20c_1^2 M_0 M_1 L_m^2 k}{\nu \lambda_{m+1}} + \frac{40c_1^2 M_0^2 L_m^2 L_m^2 k}{\nu \lambda_{m+1}}
\]

\[
(5.9)
\]

Taking \( v = 2e^{n+1} \) in (5.7) and noticing the relation (4.5), we have

\[
|P_m e^{n+1}|^2 + |e^{n+1}|^2 + |Q_m e^{n+1} - e^n|^2 + 2\kappa \nu |A^\frac{3}{2} e^{n+1}|^2 - |e^n|^2
\]

\[
= 2kb(\bar{u}^n, e^{n+1}, P_m e^{n+1}) - 2kb(e^n, u^n, Q_m e^{n+1}) - 2kb(u_m^n, e^n, Q_m e^{n+1})
\]

\[
- 2kb(\bar{u}^n, e^n, e^{n+1}) + 2kb(e^{n+1} - e^n, \bar{u}^n, P_m e^{n+1}) - 2kb(e^{n+1} - e^n, u^n, Q_m e^{n+1})
\]

\[
- 2kb(u_m^n, e^n, e^{n+1}) - 2kb(\bar{u}^n, \bar{u}^n, P_m e^{n+1}) + 2h(e^{n+1}, Q_m e^{n+1})
\]

\[
+ 2kG_0(\delta e^{n+1}, e^{n+1}) - 2kG_0(\bar{u}^n, Q_m e^{n+1})
\]

We summarize the estimate for each term on the right-hand side of (5.10) as:

\[
2kb(\bar{u}^n, e^{n+1}, P_m e^{n+1}) \leq \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2} e^{n+1}|^2 + |P_m A^\frac{3}{2} e^{n+1}|^2,
\]

\[
2kb(e^n, u^n, Q_m e^{n+1}) \leq \frac{2k}{10} |Q_m A^\frac{3}{2} e^{n+1}|^2 \leq \frac{10c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}} |e^{n+1} - e^n|^2
\]

\[
+ \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2} e^{n+1}|^2 + |Q_m A^\frac{3}{2} e^{n+1}|^2,
\]

\[
2kb(u_m^n, e^n, Q_m e^{n+1}) \leq \frac{2k}{10} |Q_m A^\frac{3}{2} e^{n+1}|^2 \leq \frac{10c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}} |e^{n+1} - e^n|^2
\]

\[
+ \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2} e^{n+1}|^2 + |Q_m A^\frac{3}{2} e^{n+1}|^2,
\]

\[
2kb(e^{n+1} - e^n, u^n, P_m e^{n+1}) \leq \frac{2k}{10} |P_m A^\frac{3}{2} e^{n+1}|^2 \leq \frac{10c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}} |e^{n+1} - e^n|^2
\]

\[
+ \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2} e^{n+1}|^2 + |Q_m A^\frac{3}{2} e^{n+1}|^2,
\]

\[
2kb(e^n, u^n, e^{n+1}) = -2kb(Q_m e^{n+1} - e^n, u^n, e^{n+1}) + 2kb(Q_m e^{n+1}, u^n, e^{n+1})
\]

\[
\leq \frac{2k}{10} |Q_m A^\frac{3}{2} e^{n+1}|^2 \leq \frac{10c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}} |e^{n+1} - e^n|^2
\]

\[
+ \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2} e^{n+1}|^2 + |Q_m A^\frac{3}{2} e^{n+1}|^2,
\]

\[
2kb(u_m^n + \bar{u}^n, e^n, e^{n+1}) \leq \frac{2k}{10} |A^\frac{3}{2} e^{n+1}|^2 \leq \frac{10c_1^2 M_2^2 L_m^2 k}{\nu \lambda_{m+1}} |e^{n+1} - e^n|^2
\]

\[
+ \frac{2\sqrt{2c_1 M_1 L_m}}{\nu \lambda_{m+1}} |A^\frac{3}{2} e^{n+1}|^2 + |Q_m A^\frac{3}{2} e^{n+1}|^2.
\]
Asymptotic property

Noticing the stability conditions (4.2), the summation of (5.9) and (5.11) yields

\[
2kG_2(\tilde{u}^{n+1}, Q_m e^{n+1}) \leq \frac{20\epsilon^2 M_0 M_1 (\kappa_2^2 + 2\kappa_3^2)}{\nu} k^3.
\]

The combination of the above estimates with (5.10) leads to

\[
|\epsilon^{n+1}|^2 + |Q_m e^{n+1} - \epsilon n|^2 + \frac{4}{5} k_{\nu} |A^\frac{1}{2} e^{n+1}|^2 - |\epsilon n|^2 \\
\leq \frac{50\epsilon^2 M_0^2 L_m^2 k}{\nu} |e^{n+1} - \epsilon n|^2 + \frac{(1 + \sqrt{2}) c_1 M_1}{\nu} k_{\nu} |A^\frac{1}{2} e^{n+1}|^2 \\
+ \frac{4\sqrt{2} c_1 M_1 L_m k_{\nu} (|A^\frac{1}{2} e^{n+1}|^2 + |A^\frac{1}{2} e^{n+1}|^2)}{\nu \lambda_m} + \frac{10k^3}{\nu} - |Q_m A^{\frac{1}{2}} u_{tt}|^2 \\
+ \frac{20\epsilon^2 M_0 M_1 (\kappa_2^2 + 2\kappa_3^2)}{\nu} k^3 \\
+ \frac{40\epsilon^2 M_0 M_1 (\kappa_2^2 + 2\kappa_3^2)}{\nu \lambda_m} L_m^2 k \\
+ \frac{20\epsilon^2 M_0 M_1 (\kappa_2^2 + 2\kappa_3^2)}{\nu \lambda_m} L_m^2 k.
\]

(5.11)

Noticing the stability conditions (4.2), the summation of (5.9) and (5.11) yields

\[
(|\epsilon^{n+1}|^2 + |\epsilon^{n+1}|^2) - (|\epsilon n|^2 + |\epsilon n|^2) \leq \frac{10\epsilon^2 M_0^2 k}{\nu} (|\epsilon^{n+1}|^2 + |\epsilon^{n+1}|^2) \\
+ \frac{\kappa_2^2 k^2}{\nu} + \frac{\kappa_2^2 L_m k}{\nu} (L_m^2 \lambda_{m+1}^{-3} + L_m^2 \lambda_{m+1}^{-2}),
\]

where

\[
\kappa_2^2 = 20 |A^{\frac{1}{2}} u_{tt}|^2 + 40\epsilon^2 M_0 M_1 (\kappa_2^2 + 2\kappa_3^2), \quad \kappa_3^2 = 40\epsilon^2 M_0 M_1 (\kappa_1^2 + 2M_1^2).
\]

Now by introducing

\[
\kappa_5 = \frac{\kappa_2^2}{10\epsilon^2 M_0^2}, \quad \kappa_6 = \frac{\kappa_2^2}{10\epsilon^2 M_1^2},
\]

and using the discrete Gronwall inequality, we can get the results. \(\Box\)

For the two dimensional NS equations, the eigenvalues of the Stokes operator \(A\) has the following asymptotic property

\[
\lambda_m \sim m.
\]

Obviously, to balance the spatial discretization error terms in the result of theorem 5.2, we should choose \(M\) and \(m\) such that \(m = cM^{\frac{1}{2}}\).
6. On Numerical Implementation. We discuss in this section the issues concerning the numerical implementation of scheme (3.5)–(3.6). The large eddy simulation is simple. In fact, it is nothing but a SGM with Euler forward time discretization except for the update of the approximation at previous time step in $H_m$ with $P_m\hat{v}^n$. The numerical implementation on this level is quite easy and we can directly use the standard code for SGM to do this job. The main difficulty comes from the small eddy simulation in the new time dependent small eddy subspace $\hat{V}^{n+1}$, which is determined by the $u_m^{n+1}$ related projection $R_m^{n+1}$. It is almost impossible for us to use the formula (3.6) directly in computation because it is really a time consuming procedure at every time step to compute $\hat{V}^{n+1}$. To avoid computing the small eddy subspace $\hat{V}^{n+1}$, we have to construct certain equivalent forms of the algorithm (3.5)–(3.6), whose numerical implementation should be as easy as general SGM. Fortunately, we can achieve this by introducing some Lagrange multiplier. This kind of technique can be found in many literatures, for example see [4],[6],[7] and etc.

By introducing a Lagrange multiplier $\omega^{n+1} \in H_m$, the algorithm (3.5)–(3.6) is equivalent to: find $(u_m^{n+1}, \hat{w}^{n+1}, \omega^{n+1}) \in H_m \times V_M \times H_m$ such that for any $(v_1, v_2, v_3) \in H_m \times V_M \times H_m$

\[
(u_m^{n+1}, v_1) + k\omega_m u_m^{n+1} + B(u_m^{n+1} - u_m^n) = k (f^{n+1}, v_1) + (u_m^n + \hat{w}^n, v_1),
\]

\[
(u_m^{n+1} + \hat{w}^{n+1}, v_2) + k\omega_m u_m^{n+1} + B(u_m^{n+1} + \hat{w}^n, v_2) = k (f^{n+1}, v_2) + (u_m^n + \hat{w}^n, v_2),
\]

\[
\mathcal{L}^{n+1}(\hat{w}^{n+1}, v_3) = 0.
\]

Let us denote

\[
b_1^n = kP_m[u_m^n + \hat{w}^n + f^{n+1} - B(u_m^n, u_m^n)],
\]

\[
b_2^n = kP_m[u_m^n + \hat{w}^n + f^{n+1} - B(u_m^n + \hat{w}^n, u_m^n + \hat{w}^n)].
\]

For an easy illustration, we regard $u_m^{n+1}$, $\hat{w}^{n+1}$, $\omega^{n+1}$, $b_1^n$ and $b_2^n$ as column vectors of their Fourier coefficients and rewrite the above system in the matrix form:

\[
D_1 u_m^{n+1} = b_1^n,
\]

\[
D_2 \hat{w}^{n+1} + D_2 E_m^T u_m^{n+1} + B^T E_m^T \omega^{n+1} = b_2^n,
\]

\[
E_m B \hat{w}^{n+1} = 0,
\]

where $D_i = \text{diag}\{1 + kv \lambda_1, 1 + kv \lambda_2, \cdots, 1 + kv \lambda_m\}$, $i = 1, 2, m_1 = m, m_2 = M; E_m = (I_{m \times m}, 0_{m \times (M-m)}), B$ is a matrix of order $M \times M$ which is corresponding to the operator $\cdot + kv + kP_m B(u_m^{n+1}, \cdot) + kP_m B(\cdot, u_m^{n+1})$. To see the above algebraic equations more clearly, we denote by $b_{1,1}^n$ and $b_{2,2}^n$ the vectors consisting of the first $m$ components of $b_1^n$ and the remainder $M - m$ components of $b_2^n$ respectively, and write the matrix $D_2$ and $B$ in block form and define matrices $C$ and $G$ as follows: $D_2 = \text{diag}(D_{11}, D_{22})$

\[
B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
D_{11}^{-1} & B_{11}^{-1} B_{12}^T \\
D_{22}^{-1} & B_{22}^{-1} B_{12}^T
\end{pmatrix},
\]

\[
G = B_{11} D_{11}^{-1} B_{11}^T + B_{12} D_{22}^{-1} B_{22}^T.
\]

where $B_{11} \in R^{m \times m}$, $B_{22} \in R^{(M-m) \times (M-m)}$, $B_{12}, B_{21}^T \in R^{m \times (M-m)}$, $D_{11} = D_1$ and $D_{22} = \text{diag}\{1 + kv \lambda_{m+1}, 1 + kv \lambda_{m+2}, \cdots, 1 + kv \lambda_M\}$. Then we have

\[
u_m^{n+1} = D_1^{-1} b_1^n,
\]

\[
\omega^{n+1} = G^{-1}(B_{11} D_{11}^{-1} b_{2,1}^n + B_{12} D_{22}^{-1} b_{2,2}^n - B_{11} D_1^{-1} b_1^n),
\]

\[
\hat{w}^{n+1} = D_2^{-1} b_2 - C \omega^{n+1} - E_m^T u_m^{n+1}.
\]
The only equation which needs solving is (6.2) and the solving of (6.1) and (6.3) is straightforward. Since matrix $G$ is symmetric non-negative definite and generally invertible and well conditioned provided $k$ is small enough, this makes the conjugate gradient solution of (6.2) inexpensive. Besides solving (6.2) and computing $b_{n1}$ and $b_{n2}$, a great part of computational efforts is used to form the matrix $B$. Fortunately, we only need to compute the sub-block $B_{11}$ and $B_{12}$ whose computing expenses is much less than the whole matrix $B$, especially when $M \gg m$.

Again, we want to point out that the programming expense of the scheme (3.5)–(3.6) (or equivalent (6.1)–(6.3)) is very cheap. (6.1) and (6.3) are actually the explicit SGM implementations in $H_m$ and $H_M$ and they can be carried out by the standard code for explicit SGM without any modification. Equations (6.2) can be solved by direct method (the Gaussian elimination) or any iterative method (for example, the conjugate gradient method) since the matrix $G$ is generally well conditioned.

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**REFERENCES**


