Entangled, coherent, and squeezed states of a single SU(2) qutrit

Sinem Binicioğlu\textsuperscript{1}, M. Ali Can\textsuperscript{2}, Alexander A. Klyachko\textsuperscript{3}, and Alexander S. Shumovsky\textsuperscript{1}

\textsuperscript{1}Department of Physics, Bilkent University, Bilkent, Ankara, 06800 Turkey
\textsuperscript{2}Jet Propulsion Laboratory, California Institute of Technology, Pasadena CA 91109-8099, USA
\textsuperscript{3}Department of Mathematics, Bilkent University, Bilkent, Ankara, 06800 Turkey

Using the single SU(2) qutrit (spin-1-like system), we show that entanglement may take place for a single particle with respect to its internal degrees of freedom, in other words, beyond the conventional requirements of nonlocality and nonseparability. We show that the SU(2) or spin coherent states do not manifest entanglement, while the so-called squeezed spin states are entangled. We reveal the principle difference between the spin coherent and spin squeezed states in terms of quantum correlations. A number of physical realizations of the SU(2) qutrit is discussed.

PACS numbers: 03.67.Mn, 03.65.Fd

I. INTRODUCTION

The two main objectives of this paper are on the one hand to discuss relation between the notions of entanglement, coherence, and squeezing and on the other hand to investigate realization of entanglement as a physical phenomenon in the case of a single particle.

The single SU(2) qutrit that we consider below represents a thoroughly enlightening example. The point is that such a system can be associated with a single local object, like a particle, and at the same time manifests certain properties of two qubits.

By definition \footnote{SU(2)} qutrit is a physical system with three-dimensional states space $\mathcal{H}$ and dynamical group $SU(\mathcal{H}) = SU(3)$. Following Ref. \footnote{2}, we can treat $\mathcal{H}$ as spin one representation of smaller group SU(2) and call the resulting three-dimensional quantum system SU(2) qutrit. From the physical point of view, this object is similar to a spin-1 particle.

It seems to be natural to compare entangled, coherent, and squeezed states of the same system. In fact, the specific manifestation of quantum uncertainties (quantum fluctuations) is a characteristic trait of all three types of the states.

For example, Glauber coherent state of Bose fields \footnote{3} manifests the minimal amount of quantum fluctuations of the field quadratures. Its generalization on the case of spin-like systems \footnote{4} is also characterized by the minimal amount of quantum uncertainties (also see Refs. \footnote{5} and \footnote{6}).

In turn, the squeezed states of Bose field \footnote{7} assume that the uncertainty of one of the field quadratures is lower than the minimal uncertainty, which is often called the standard quantum limit or shot noise limit, due to the increase of uncertainty of another quadrature.

There is no disagreement among the experts on definition of spin coherent states. At the same time, there is a dissent concerning definition of spin squeezed states.

For example, a straightforward approach proposed by Wodkiewicz \footnote{8} fondly copies the main idea of squeezing of Bose fields. Namely, the definition is based on the Heisenberg uncertainty relation for components of spin operator

\begin{equation}
V_x(\psi)V_y(\psi) \geq \frac{1}{4}|\langle \psi | S_z | \psi \rangle|^2.
\end{equation}

Here $V_{\ell}$ denotes the variance of spin projection onto direction $\ell$

\begin{equation}
V_{\ell}(\psi) \equiv \langle \psi | S_{\ell}^2 | \psi \rangle - \langle \psi | S_\ell | \psi \rangle^2.
\end{equation}

According to Ref. \footnote{10}, state $\psi$ is squeezed if for some coordinate system one of the uncertainties either $V_x(\psi)$ or $V_y(\psi)$ is less than the standard quantum limit of $\frac{1}{2}\langle \psi | S_z | \psi \rangle$. An undoubted weak point of this definition is that the coherent spin state $|s\rangle$ with definite spin projection $s$ onto some axis $\ell$ has zero uncertainty $V_{\ell} = 0$ and should be considered as squeezed one \footnote{10}.

To avoid this confusion, Kitagawa and Ueda \footnote{10} have proposed another definition, which takes into account only uncertainties $V_{\ell}(\psi)$ in directions $\ell$ orthogonal to the mean spin vector $\bar{s} = \langle \psi | \mathbf{S} | \psi \rangle$. For a coherent state the variance $V_{\ell}(\psi)$ is equal to $s/2$ for all such directions. Spin state $\psi$ is said to be squeezed if $V_{\ell}(\psi) < s/2$ for some direction $\ell \perp \bar{s}$. The variance $V_{\ell}(\psi)$ as a function of direction $\ell$ form an ellipse, degenerating into a circle for coherent states. For the squeezed state its small semiaxis should be less to $s/2$.

All coherent spin states are unitary equivalent and can be obtained from the state $|-s\rangle$ by action of a displacement operator \footnote{4} \footnote{6}. The latter, in conformity with Bosonic coherent states, is defined as an exponential of a skew Hermitian operator linear in spin rising and lowering operators.

The structure of squeezed spin states is more complicated and for high spins is largely unknown. However, in spin 1 system every squeezed spin state can be obtained from the “vacuum” state $|-s\rangle$ by action of a squeeze operator given by exponential of a quadratic expression in spin rising and lowering operators \footnote{10}. In a sense, it copies Stoler’s squeeze operator that has been considered in the context of squeezed states of Bose fields \footnote{8} [11].

Quantum correlations caused by the bilinearity of the generator of spin squeezed states are similar to that discussed in the context of quantum entanglement \footnote{12}. Recall that entanglement is a manifestation of quantum
fluctuations in a state where they come to their extreme. In this respect the entangled states are opposite to coherent ones where the quantum fluctuations are minimal [14, 24]. This suggests that squeezed spin states may be associated with entanglement [12, 14, 16]. We show that for spin 1 system squeezed states coincide with entangled ones. Thus the physical manifestation of spin squeezing discussed in [14] can be also understood as a manifestation of a single particle entanglement.

The paper is arranged as follows. Since our consideration is based on the dynamic symmetry approach to quantum entanglement, in Section II we briefly review the main ideas of the approach. In Section III, we discuss the entanglement properties of a single SU(2) qutrit. In Section IV we prove that the spin coherent states coincide with the squeezed spin states. In Section V, we show that entangled states of the SU(2) qutrit coincide with the squeezed spin states. In Section VI we consider some physical realizations of the single SU(2) qutrit entanglement. Finally, in Section VII we briefly summarize the obtained results.

II. REVIEW OF THE DYNAMIC SYMMETRY APPROACH

The gist of the approach has been developed in Refs. [14, 16, 18, 20] is that we should perform measurement of a certain basic observables to determine whether or not a given state is entangled. By definition [14, 16], those basic observables form an orthonormal basis of the Lie algebra $\mathcal{L}$, generating the dynamic symmetry group

$$ G = \exp(i\mathcal{L}) $$

of the system.

For example, two qubit system is given by action of the dynamical group $G = SU(2) \times SU(2)$ in Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with spin projector operators of the components as the basic observables.

It is known that the set of entangled states of a given dynamical system $G : \mathcal{H}$ is closed under stochastic local operations assisted by classical communications (SLOCC) [21, 22, 23]. They can be identified with operators from the complexified dynamic symmetry group [18, 22]

$$ g^c \in G^c = \exp(C \mathcal{L} \otimes \mathbb{C}). $$

It should be stressed that SLOCC neither create nor destroy entanglement, but only changes its amount.

Thus, an entangled state $\psi_E \in \mathcal{H}$ can be generated by SLOCC from a certain completely entangled state $\psi_{CE} \in \mathcal{H}$

$$ \psi_E = g^c \psi_{CE} $$

(2)

which carries the maximal amount of entanglement. The latter can be characterized by entanglement equation

$$ \langle \psi_{CE} | X | \psi_{CE} \rangle = 0, \quad \forall X \in \mathcal{L} $$

(3)

which is enough to check for the basic observables $X_i$, see Ref. [13, 14, 24] and references therein.

A useful physical quantity, reflecting important properties of entangled states, is the total uncertainty or total variance

$$ \forall (\psi) = \sum_i V_i(\psi) = \sum_i \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2, $$

where $\psi \in \mathcal{H}$. In irreducible quantum system $G : \mathcal{H}$ the Casimir operator

$$ C_\mathcal{H} = \sum_i X_i^2, $$

acts as a scalar which we denote by the same symbol $C_\mathcal{H}$. For example, for spin $s$ system the Casimir is equal to $s(s + 1)$. The entanglement equation (3) ensure maximality of the total variance

$$ \max \forall (\psi) = \forall (\psi_{CE}) = C_\mathcal{H}. $$

The general definition of entanglement based on equations (2) and (3) has a transparent physical meaning. In completely entangled state the system is at the center of its quantum fluctuations, and for such state the quantum fluctuations are maximal possible (and equal to the Casimir $C_\mathcal{H}$). In other words, entanglement can be interpreted as a manifestation of quantum fluctuations when they come to their extreme [14, 16, 20]. This definition leads to the following immediate corollaries.

- **The notion of entanglement does not require conventional assumption of nonlocality of the systems** [2, 18]. Remind that the condition of nonseparability that is often used as a definition of entanglement [22], assumes a priori the nonlocal nature of entanglement.

- **Entanglement is a relative phenomenon depending on the choice of basic observables, in other words, of the dynamic symmetry of the system** [20].

This allows to extend entanglement beyond composite systems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \cdots$. A single quantum particle defined in a non-factorable Hilbert space may manifest entanglement with respect to its intrinsic degrees of freedom. Let us stress a clear-cut distinction between the single-particle entanglement and so-called single-photon entanglement [24]. In the latter case, an external geometrical qubit created by the two equivalent outputs of a beam splitter is involved.

To illustrate the second corollary, consider a system, in which the Lie algebra of basic observables $\mathcal{L}$ contains a nontrivial subalgebra $\mathcal{L}' \subset \mathcal{L}$, so that the dynamic symmetry group $G$ has a subgroup $G' \subset G$.

$$ G' = \exp(i\mathcal{L}'). $$

Then, the system can be completely entangled with respect to observables from the basis of $\mathcal{L}'$ but unentangled with respect to basic observables from $\mathcal{L}$. An example is provided by *qutrits* with general symmetry SU(3) [1], which allows the reduced symmetry SU(2) $\subset$ SU(3). In the next section, we discuss application of the dynamic symmetry approach to a single SU(2) qutrit.
III. ENTANGLEMENT OF A SINGLE SU(2) QUTRIT

By definition, qutrit is a physical system with three-dimensional states space $\mathcal{H}$ and dynamical group $SU(3)$. With respect to this group all states are equivalent, and hence coherent. No entanglement is possible in such a system.

However if we treat $\mathcal{H}$ as spin one representation of smaller group $SU(2)$ there appears intrinsic difference between states. This opens a way to new physical phenomena like entanglement and squeezing. Below we always deal with spin group $G = SU(2)$ and denote by $\mathcal{H}_s$ its spin $s$ representation of dimension $2s + 1$. We are mostly interested in spin one system $\mathcal{H}_1$ called SU(2) qutrit. Fix the eigenbasis $|+\rangle$, $|0\rangle$, $|-\rangle$ of spin projection operator $S_z$. Then the other basic observables are given by the following matrices

\[
S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

There is a transparent formal correspondence between symmetric states of two qubits and single SU(2) qutrit, coming from the Clebsch-Gordon decomposition

\[
\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_1 \oplus \mathcal{H}_0.
\]

In the basis $|\uparrow\rangle$, $|\downarrow\rangle$ of $\mathcal{H}_2$, the spin one component $\mathcal{H}_1$ is given by the symmetric triplet

\[
\begin{align*}
|+1\rangle &= |\uparrow\rangle \\
|0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \\
|-1\rangle &= |\downarrow\rangle
\end{align*}
\]

while the scalar component $\mathcal{H}_0$ comes from the antisymmetric state

\[
|A\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle).
\]

It is clear that the symmetric state $|0\rangle$ in $\mathcal{H}_2$ is completely entangled as a state of two qubits. The same is true for the antisymmetric state $|A\rangle$.

Note that in some two-qubit systems the antisymmetric state $|A\rangle$ is not allowed. An example of some considerable interest is provided by the so-called biphoton (photon twins created at once and propagating in the same direction) \cite{biphoton}, where the antisymmetric state is forbidden by the Pauli principle. In this case, the two qubits correspond to polarization of the photons. Another example is given by the system of two two-level atoms interacting by dipole forces in the Dicke-Lamb limit \cite{dicke_lamb}.

Being interpreted as the spin-1 states with given projection of spin $s = \pm 1, 0$, the same symmetric triplet \cite{symmetric_triplet} can be associated with the single particle states. In this case, the above qubit interpretation should be referred to the intrinsic degrees of freedom of the particle.

It is clear that the SU(2) qutrit can be realized as a local object (particle). In this case, the notion of nonseparability of the single-particle state in the three-dimensional Hilbert space $\mathcal{H}_1$ is meaningless. The discussion of physical examples of the SU(2) qutrits we put off till Section VII.
to the basic observables. It can be easily seen that the following orthonormal states

\[ \frac{1}{\sqrt{2}}(|+1\rangle \pm |-1\rangle), \quad |0\rangle \]  

(6)
satisfy this equation and hence are completely entangled. They form a completely entangled basis of \( \mathcal{H}_1 \).

From the correspondence relations, it is seen that the states are also completely entangled as the two-qubit states in the symmetric sector of the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). Note once more that qubits here should be interpreted as the intrinsic degrees of freedom of the SU(2)-qutrit particle.

Thus, complete entanglement of a single SU(2) qutrit particle just reflects entanglement of intrinsic qubit degrees of freedom of this particle.

The fact that the spin-1 state \(|0\rangle\) with projection \(s = 0\) is completely entangled seems to be quite interesting. Possible interpretation of such entanglement will be discussed in Section VI. Here we consider as an example transformation of the state \(|0\rangle\) by SLOCC operator

\[ \exp(z S_z) = \frac{1}{2} \begin{pmatrix} \frac{e^{2 \alpha} - e^{-\alpha}}{2} & e^{\alpha} + e^{-\alpha} - 1 \\ e^{\alpha} - e^{-\alpha} & \frac{e^{2 \alpha} + e^{-2 \alpha}}{2} \end{pmatrix}, \]

where \(z\) is an arbitrary complex parameter. We get

\[ \exp(z S_z) |0\rangle = \frac{e^{\alpha} + e^{-\alpha}}{2} |0\rangle + \frac{e^{\alpha} - e^{-\alpha}}{2 \sqrt{2}} (|+1\rangle + |-1\rangle) \quad (7) \]

Taking into account that the amount of entanglement for an arbitrary SU(2) qutrit state \(\psi = \psi_{-1} |-1\rangle + \psi_0 |0\rangle + \psi_{+1} |+1\rangle\) is given by the concurrence

\[ C(\psi) = |\psi_0^2 - 2 \psi_{+1} \psi_{-1}| \quad (8) \]
on the vacuum state \(|\text{vac}\rangle\)

\[ |\alpha\rangle = D_\alpha |\text{vac}\rangle. \]

Here \(a\) and \(a^+\) denote the annihilation and creation operator for the Bose field under consideration and vacuum state obey the stability condition \(a |\text{vac}\rangle = 0\).

It is generally accepted that the spin version of Glauber coherent states can be defined in the similar fashion (for review, see Ref. [7]). Namely, let

\[ S_+ = S_x + i S_y, \quad S_- = S_x - i S_y \]

be the rising and lowering spin operators. Then, spin coherent state is defined by action of the displacement operator

\[ D_\alpha = \exp(\alpha S_+ - \alpha^* S_-), \quad \alpha \in \mathbb{C}, \quad (9) \]
on the lowest spin state \(|-s\rangle\), \(s \in \mathbb{C}\), \(s \in \mathbb{C}\)

\[ |\alpha\rangle = D_\alpha |-s\rangle. \quad (10) \]

The state \(|-s\rangle\) is considered as an analogue of the vacuum state \(|\text{vac}\rangle\) because \(S_- | -s\rangle = 0\). Note that in the case of SU(2) qutrit under consideration the state \(|-s\rangle = |-1\rangle\) is unentangled.

Since \(a S_+ - a^* S_- \in su(2)\), then the spin displacement operator amounts to an SU(2) rotation. In particular, every spin coherent state is just a state with minimal spin projection \(|-s\rangle\) onto some direction \(\ell\). Hence the spin displacement operator, in contrast to its bosonic counterpart, doesn’t create new physics. This assessment however depends on a specific physical realization of the SU(2) qutrit. For example, to produce the unitary transformation \(D_\alpha\) in biphoton system, one needs linear optical elements, rather than spatial rotation. In another realization of SU(2) qutrit as the symmetric part of two qubit system, the coherent states are just symmetric decomposable tensors \(|\alpha\rangle = \varphi \otimes \varphi\). One can find other physical incarnations of SU(2) qutrit in Section VI.

In any case the coherence and its opposite entanglement are SU(2) invariant properties, and they can’t be created or destroyed by the displacement operator. Note also that for any quantum system the total variance \(\mathcal{V}(\psi)\) attains its minimal value for coherent states, and maximum for completely entangled ones. For spin \(s\) system this gives

\[ s \leq \mathcal{V}(\psi) \leq s(s + 1). \quad (11) \]

V. SPIN SQUEEZING AND ENTANGLEMENT

As we’ve seen in the previous section a coherent spin \(s\) state \(|\alpha\rangle\) has definite spin projection onto direction

\[ \vec{s} = \langle \alpha | \vec{S} |\alpha\rangle \]

which can be taken as the quantization axis \(z\). Then \(|\alpha\rangle = |s\rangle\) is invariant with respect to rotations about \(z\)-axis, and hence the variance \(\mathcal{V}_z(\alpha)\) is independent of the
direction $\ell$ in $xy$ plane. On the other hand, the projection onto $z$-axis has definite value $s$ and therefore $V_z(\alpha) = 0$. Combining this with the value of the total variance of coherent state $V(\alpha) = s$, we end up with equation

$$V_\ell(\alpha) = s/2, \quad \forall \ell \perp \vec{s}. \quad (12)$$

Following Kitagawa and Ueda, we call spin state $\psi \in H_s$ to be squeezed iff $V_\ell(\psi) < s/2$ for some direction $\ell$ orthogonal to the mean spin vector $\vec{s} = \langle \psi | S | \psi \rangle$. The latter condition $\ell \perp \vec{s}$ is equivalent to $\langle \psi | S_\ell | \psi \rangle = 0$. This implies that in a coordinate system with $z$-axis along the mean spin vector $\vec{s}$ we always have

$$V_\ell(\psi) + V_y(\psi) \geq s, \quad (13)$$

and therefore at most one component $x$ or $y$ can be squeezed. Indeed, since $\langle \psi | S_x | \psi \rangle = \langle \psi | S_y | \psi \rangle = 0$, then

$$V_x(\psi) + V_y(\psi) = \langle \psi | S_x^2 | \psi \rangle + \langle \psi | S_y^2 | \psi \rangle$$

$$= s(s+1) - \langle \psi | S_z^2 | \psi \rangle \geq s(s+1) - s^2 = s.$$

As we’ll see later for spin 1 system both sides of (13) are equal. This makes a simile with light squeezing even more compelling.

Note also that the variance $V_\ell(\psi)$ is a quadratic form in coordinates of vector $\ell \perp \vec{s}$. The squeezing condition just tells that the minimal eigenvalue of its matrix should be less than $s/2$. Square roots of the eigenvalues are known as semi-axes of the uncertainty ellipse.

According to equation (12) the coherent states are not squeezed. In particular, there is no squeezing in spin $1/2$ system, where all states are coherent.

This observation can be extended to all states with definite spin projection $|j\rangle$, $j \neq 0$. Indeed, such a state is invariant with respect to rotation around $z$ axis. Hence the average spin vector $\vec{S}$ is parallel to $z$ axis and the variance $V_\ell(|j\rangle)$ is independent of direction $\ell \perp z$. Since $V_z(|j\rangle) = 0$ and by (11) the total variance is always $s$, we end up with inequality $V_\ell(|j\rangle) \geq s/2$ incompatible with squeezing.

One can produce squeezed spin states in the same fashion as the conventional Bose squeezed states. The latter are defined by action of the unitary squeeze operator

$$S_\xi = \exp \left[ \frac{1}{2} (\xi a^2 - \xi a^+)^2 \right], \quad \xi \in \mathbb{C}$$

on the vacuum state. Following Kitagawa and Ueda, we define spin squeezing operator in a similar way

$$S_\xi = \exp \left[ \frac{1}{2} (\xi S_z^2 - \xi S_z^2) \right]. \quad (14)$$

This is a unitary operator, but not from the group SU(2). For spin $1/2$ the operator is identical because in this case $S_\alpha^2 = S_\beta^2 = 0$. For spin equal to one, the squares of the rising and lowering operators are

$$S_+^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_-^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and the squeeze operator (14) takes the form

$$S_\xi = \begin{pmatrix} \cos |\xi| & 0 & -e^{i\varphi} \sin |\xi| \\ 0 & 1 & 0 \\ e^{-i\varphi} \sin |\xi| & 0 & \cos |\xi| \end{pmatrix},$$

where $\varphi = \arg \xi$. By action on the lowest state $|\!- \rangle$ of SU(2) qutrit we get

$$|\xi\rangle = S_\xi |\!- \rangle = -e^{i\varphi} \sin |\xi| |\!+ \rangle + \cos |\xi| |\!- \rangle. \quad (15)$$

It is well known that every spin 1 state is SU(2) equivalent to a state. Hence in this case the spin squeezing operator is powerful enough to produce the whole range of physically different states. This however is not the case for higher spins. Making use of equation (8) one can easily find the concurrence of the state (15)

$$C(\xi) = |\sin(2|\xi|)|.$$

Performing averaging of the basic observables we get

$$\langle \xi | S_{x,y} | \xi \rangle = 0, \quad \langle \xi | S_z | \xi \rangle = -\cos(2|\xi|),$$

so that the average spin direction coincides with $z$-axis. The uncertainties of the basic observables are given by equations

$$\begin{cases} V_x(\xi) = \frac{|1 - \sin(2|\xi|) \cos \varphi|}{2} \\ V_y(\xi) = \frac{|1 + \sin(2|\xi|) \cos \varphi|}{2} \\ V_z(\xi) = \sin^2(2|\xi|). \end{cases}$$

Note that changing the phase $\varphi$ amounts to a rotation in $xy$ plane. By choosing $\varphi = 0$ we get the extremal values of $V_x$ and $V_y$

$$V_x(\xi) = \frac{1 - \sin(2|\xi|)}{2}, \quad V_y(\xi) = \frac{1 + \sin(2|\xi|)}{2},$$

which are just squares of semi-axes of the uncertainty ellipse. It follows that the squeezing condition $V_x(\xi) < 1/2$ for spin $s = 1$ adds up to positivity of the concurrence $C(\xi) = |\sin(2|\xi|)| > 0$. Hence, for spin one system squeezed states coincide with entangled ones. Observe also equation $V_x(\xi) + V_y(\xi) = 1$ for spin one system, in contrast to general inequality (13).

Remarks. For higher spin relation between squeezing and entanglement is more complicate. One can show that for spin $s = 3/2$ all disentangled states have definite nonzero spin projection onto some direction. As we’ve seen above such states are not squeezed. Hence all squeezed state of spin $3/2$ are entangled. We expect that the same holds for any spin, i.e. squeezing is a stronger property then entanglement.

Starting from spin $s = 2$ there exist spherically symmetric completely entangled spin states $\psi$ for which the variance $V_\ell(\psi)$ is independent of direction $\ell$. For such a state $V_\ell(\psi) = s(s+1)/3 > s/2$, see Eq. (11). This gives an example of a completely entangled, but not squeezed state.
Observe that any state sufficiently close to a squeezed one is itself squeezed. Therefore the space of squeezed states has the same dimension as the space of all states. This is why for high spin one parametric spin squeeze operators \[14\] can’t generate all squeezed states.

One can find more on relation between spin squeezing and entanglement in [12].

VI. PHYSICAL REALIZATION OF SINGLE-PARTICLE ENTANGLEMENT

We already mentioned in Section II the example of single SU(2) qutrit entanglement associated with the biphotons [28] and two-level atoms in Lamb-Dicke limit [30]. Note that a biphoton, consisting of photon twins created at once and propagating along the same direction, is a local object that can be split into two photons by a beam splitter. Before such a decay, the biphoton polarization can be considered as an intrinsic degree of freedom for a single spin-1 object in direct analogy with general discussion in Section III. Note that the Pauli principle forbids the antisymmetric state of two photons [34].

In the case of two two-level atoms with dipole-dipole interaction in Lamb-Dicke limit of short interatomic distances, when the antisymmetric state is again forbidden [30], the system is only approximately local (at the distances exceeding the wavelength).

Note now that the notion of the SU(2) qutrit involves a huge number of physical objects of different nature. For example, the angular momentum \(j = 1\) and isospin \(I = 1\) manifest the same properties.

Remind that isospin is a term introduced to describe groups of particles which have nearly the same mass, such as the three pions (\(\pi\)-mesons) that compose the isos triplet \(I = 1\) [32]. The isospin projections are +1 for the positive, 0 for the neutral, and −1 for negative pions.

In view of the results listed in Section III, the state of the neutral meson \(\pi^0\) is completely entangled with respect to the isospin projection.

To interpret this entanglement in terms of the intrinsic degrees of freedom, we note that, according to the quark model (e.g., see [37]), the fundamental representation of the isospin-1 symmetry is given by two quark doublets, consisting of “up” (\(u\)) and “down” (\(d\)) quarks and antiquarks \(\bar{u}, \bar{d}\). Each doublet can be naturally interpreted as a qubit. The isos triplet states of pions have the following quark structure:

\[
\begin{align*}
|+1\rangle & \sim \pi^+ = ud \\
|0\rangle & \sim \pi^0 = (u\bar{u} + d\bar{d})/\sqrt{2} \\
|-1\rangle & \sim \pi^- = \bar{u}d
\end{align*}
\]

Thus, \(\pi\)-mesons represent the symmetric states of two quark qubits similar to the spin-1 states expressed in terms of spin-1/2 states. Therefore, the completely entangled isospin-1 state \(\pi^0\) can also be interpreted as the complete entanglement of two quark qubits, corresponding to the intrinsic degrees of freedom of this single particle. Let us stress that free quarks cannot exist. It also follows from the results of Section III that the charged pions \(\pi^\pm\) correspond to the SU(2) coherent and hence unentangled states.

Since the completely entangled states manifest the maximal amount of quantum uncertainties (in the case of pions, these are the quantum fluctuations of quarks), all one can expect is that the neutral meson \(\pi^0\) should be less stable than the charged pions \(\pi^\pm\), which is just the case [35, 36].

Another interesting example of a single particle with \(s = 1\) is provided by the deuteron, which is a nucleus of a deuterium atom, consisting of weakly bounded proton and neutron [37]. Note that, unlike \(\pi^0\) meson, this is a stable particle. Each nucleon in the deuteron can be considered as a qubit with respect to its spin \(\frac{1}{2}\). An experimental proof of the existence of intrinsic entanglement in deuteron and the use of it for quantum teleportation of spin states of massive particles has been reported recently Ref. [38].

Note that the entangled state of spin-1 particles with projection \(s = 0\) can be easily separated from the coherent states \(s = \pm 1\) by means of Stern-Gerlach apparatus.

It seems to be tempting to consider a photon as a single (2) qutrit. Note that although photon spin \(s = 1\), the absence of the rest mass allows only two spin states (helicities) usually associated with the photon polarization [34] (the photon polarization qubit).

At the same time, photons emitted by atomic, molecular, and nuclear transitions between the states characterized by a given value of the total angular momentum and parity carry these physical quantities due to the conservation laws [34, 39]. The representation of those multipoles photons is given by quantization of spherical waves emitted by a point-like source (atom, for example) [40]. The total angular momentum of photons consists of the spin and orbital parts:

\[
\vec{J} = \vec{S} + \vec{L}.
\]

Photons with total angular momentum \(j\) and parity \(P = (-1)^j\) are called the electric-type \(j\)-pole photons. Those photons have only two allowed values of the orbital angular momentum, namely \(\ell = j - 1\) and \(\ell = j + 1\). Thus, the orbital angular momentum of electric-type photons can also be considered as a qubit.

The case of \(j = 1\) and parity \(P = -1\) corresponds to the electric dipole (E1) photons, which are probably the most widespread type of photons in the universe. The quantum state of E1 photons contains a certain linear combination of states with \(\ell = 0\) and \(\ell = 2\), so that the orbital angular momentum of those photons does not have a well defined value [34]. This also means that spin (polarization) and orbital momentum are strongly correlated and that the total angular momentum cannot be divided into spin and orbital contributions.

With respect to the total angular momentum \(j = 1\), a single E1 photon should be considered as the SU(2)
qutrit, whose intrinsic qubit degrees of freedom correspond to polarization and orbital angular momentum quits.

During the last decade, the orbital angular momentum of photons has attracted a great deal of experimental interest (e.g., see Ref. [41] and references therein). In particular, entanglement of photons with respect to their orbital angular momentum has been observed [12]. The photon beams far from the source were used in these experiments. At the same time, specific features of the dipole photons and correlation between spin and orbital parts of the angular momentum should be maximally visible at short distances (less than the wavelength) where spherical waves of photons cannot be successfully approximated by plane waves.

VII. CONCLUSION

Let us briefly discuss the obtained results.

We have examined quantum entanglement of a single SU(2) qutrit. The instructive significance of the system is determined by the fact that it allows twofold consideration, as a single spin-1 particle and as two qubits defined in the symmetric sector of the Hilbert space. The two qubits are associated here with the intrinsic degrees of freedom of the single spin-1 particle. The latter can manifest entanglement despite the fact that it does not fit conventional conditions of nonlocality and nonseparability.

We have shown that the spin-1 state with projection $s = 0$ manifests complete entanglement and can be used as the generic entangled state with respect to SLOCC.

We have proved that the SU(2) coherent state [12] is always disentangled and shows the minimal value of the total uncertainty.

We have shown that the spin squeezed state [15] defined in direct analogy to the case of a single-mode Bose field, manifests entanglement. The corresponding total uncertainty always exceeds the minimal value.

Although our results, connecting spin coherence with entangled states and spin squeezing with entanglement, were obtained for the special case of the SU(2) qutrit, they are valid in general case of quantum systems with arbitrary dynamic symmetry group $G$, for which the definition of complete entanglement [2] leads to the maximum of the total uncertainty, while the unentangled states manifest the minimal total uncertainty.

We have shown a number of physical systems that realize the SU(2) qutrit states and therefore can be prepared in the single-particle entangled states. Note here that entanglement is usually discussed in the context of quantum communication and information processing that requires at least bipartite system. The consideration of entanglement beyond the conditions of nonlocality and nonseparability seems to be of high importance for understanding of the physical nature of quantum entanglement that most probably is associated with manifestation of quantum uncertainties of basic observables at their extreme. It can also lead to new interpretation of a number of physical phenomena. The association of the low stability of $\pi^0$ meson with the maximal order of quantum fluctuations provides an example.

The possibility to observe the single-particle entangled states represents a problem of high importance and deserves special discussion. Let us note that the decay of a single SU(2) qutrit into two qubits may be used for this. For example, it is possible to expect that decay of deuteron prepared in the spin state with projection $s = 0$ should give rise to the symmetric completely entangled state of proton and neutron with respect to their spins.

According to our results, any single SU(2) qutrit has a natural entangled state $|s = 0\rangle$. Therefore, it is possible to find some other interesting examples of the single-particle entanglement, from the ordinary systems like $^{87}\text{Rb}$ and $^{23}\text{Na}$ spin-1 atoms, widely used in investigation of Bose-Einstein condensation, spin-1 states of Helium nuclei, and so on, to the exotic systems like vector meson and three spin-1 gauge bosons used in the Standard Model [43].

Acknowledgements

Part of this work was carried out (by M.A.C.) at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration (NASA). M.A.C. also acknowledges support from the Oak Ridge Associated Universities (ORAU) and NASA.


