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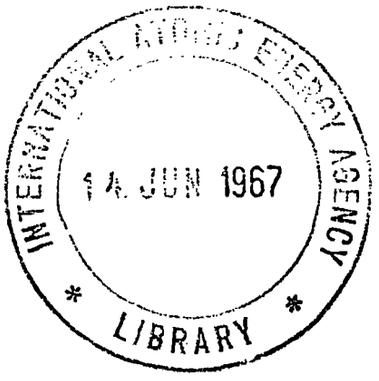
LUCAS HEIGHTS

MULTIPARTICLE COLLISIONS

PART 1. ANGULAR MOMENTUM EIGENSTATES

by

J.L. COOK



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ABSTRACT

The current situation with regard to relativistic representation of multiparticle angular momentum eigenstates is reviewed and it is concluded that no generally satisfactory formalism exists. Difficulties with the formalism are outlined and a general method of construction of partial wave amplitudes is put forward.

CONTENTS

1. INTRODUCTION

One of the most important problems that confront workers in the theory of multiparticle reactions for both low and high energy physics is to understand the role that the law of conservation of angular momentum plays in determining the analytic structure of scattering and production amplitudes. For example, some authors have advocated that the inclusion of these laws in the Fermi statistical theory of high energy multiparticle production processes might remove discrepancies between this theory and experiment. Of course, there is [illegible] general difficulties encountered when trying to relate scattering amplitudes with production amplitudes. However, the moderate overall successes of the statistical theory imply that purely kinematic constraints are a significant factor in determining the behaviour of production amplitudes, so it appears reasonable to expect that the inclusion of the conservation laws for rotational motion would provide further insight into the structure of matrix elements. It would be especially provident if the general structure of inelastic scattering theory could be ascertained by doing so, without having to delve into the extremely difficult question of interactions. This could be done by deriving a general multiparticle partial wave formation.

The work of Eden et al. (1961), and Landshoff et al. (1961) provides evidence that the powerful Mandelstam representation of scattering amplitudes is probably valid, even in energy regions where inelastic processes occur. Lardner (1961) has shown that the contribution to the scattering amplitude of the three-particle intermediate states separately satisfies the Mandelstam representation, provided one assumes analyticity in a momentum transfer variable for the five-point function. Landshoff and Treiman (1961) have maintained that this assumption may be untenable. None of these authors, however, considered the consequences of projecting out the eigenstates of angular momentum of three-particle systems.

Instead of simply looking at analyticity of amplitudes in terms of the linear energy-momentum variables, one should strive towards a valid expansion of production amplitudes into an infinite series of partial waves representing eigenstates of the total and subsets of angular momentum. To obtain the angular dependence of particle production, one can use two frames of reference, each oriented in a simple way with respect to initial and final states. Then the unitary properties of the scattering matrix together with the known behaviour of the multiparticle eigenstates with respect to rotations of axes could be employed to determine the structure of the partial wave amplitudes. We are, however, faced with one very great difficulty. No rigorously satisfactory derivation of the

relativistic eigenstates exists, and the objections to those put forward in the literature are reviewed in this paper. It is shown that if these eigenstates could be determined, the structure of the various partial wave amplitudes could be found.

2. KINEMATICAL CONSTRAINTS

Consider the constraints imposed by conservation of linear energy-momentum upon a multiparticle production process. The dynamical variables defined in this section refer to the overall barycentric system, and the masses are taken to be arbitrary within the limitations imposed by the process being a physical one. Let the direction of the i^{th} particle in the initial state define the z-direction of one polar co-ordinate system, and the direction of the j^{th} particle in the final state define the z direction of a second similar set of co-ordinates. Altogether, there are 4n components of four-momentum for a total of n particles. Let p_i and p_j be the three-momentum of the i^{th} particle in the initial state and the j^{th} particle in the final state. In the barycentric system:

$$\sum_{i}^{N_i} \underset{\sim}{p}_i = 0, \qquad \sum_{j}^{N_f} \underset{\sim}{p}_j = 0, \qquad N_i + N_f = n \quad , \qquad \text{.....(1)}$$

where N_i and N_f are the numbers of particles in the initial and final states respectively. These conditions remove six degrees of freedom from the system. Let E_i, E_j, M_i, M_j be the energies and masses respectively of particles in the initial and final states. The constraint imposed by the mass conditions:

$$E_i^{\,2} - \underset{\sim}{p}_i^{\,2} = M_i^{\,2}, \qquad E_j^{\,2} - \underset{\sim}{p}_j^{\,2} = M_j^{\,2} \quad , \qquad \text{.....(2)}$$

removes a further n degrees of freedom. The equation expressing conservation of energy:

$$\sum_{i=1}^{N_i} E_i = \sum_{j=1}^{N_f} E_j = W \quad , \qquad \text{.....(3)}$$

removes one degree of freedom, while the z-axis and two azimuthal angles can be chosen to remove the dependence of a transition amplitude upon three more angular variables. This leaves a total of 3n-10 degrees of freedom.

3. ANGULAR MOMENTUM REPRESENTATIONS

In this section, some of the various proposed angular momentum representations are discussed. The first example encountered in inelastic scattering theory is

the special case of the process (2 particles $\longrightarrow$ 3 particles). Quite a number of techniques have been put forward as a means of analysing this reaction. The most fully developed to date is that propounded by Smith (1960), based upon a fundamental paper by Delves (1958). Smith defines a matrix M_{ij} which transforms the set of particle configurational co-ordinates r_i into centre-of-mass co-ordinates ξ_j. That is:

$$\xi_j = M_{ji}\, r_i \quad . \qquad \text{.....(4)}$$

The centre-of-mass momenta q_j are given by the inverse transformation of the particle momenta p_i as:

$$q_j = (M^{-1})_{ji}\, p_i \quad . \qquad \text{.....(5)}$$

Smith's co-ordinate scheme could be generalised to relativistic sets of co-ordinates in two ways. Firstly, the definitions of r_i, ξ_j, p_i and q_j are extended to a four-vector representation. Then:

(i) the matrix M_{ji} can be the matrix transformation in terms of proper masses, or

(ii) it can be the matrix transformation in terms of physical masses.

Consider these alternatives.

By looking at the manner in which variables separate from the three-particle kinetic energy operator in the Schrodinger equation, Smith was able to perceive a hierarchy of multiparticle barycentric angular momentum operators, in the non-relativistic approximation. However, when one tries to generalise his scheme, whether in terms of barycentric co-ordinates defined relative to physical or proper masses, the construction of Lorentz invariant amplitudes runs into unresolved difficulties. These are listed below.

Consider the mass structure of the M_{ji} operator for a system of three particles with masses m_1, m_2, and m_3, that is:

$$\underline{M} = \begin{pmatrix} \frac{m_1}{M} & \frac{m_2}{M} & \frac{m_3}{M} \\ \frac{m_1+m_2}{M} & \frac{m_1+m_2}{M} & \frac{-m_3}{M} \\ \frac{m_1}{m_1+m_2} & \frac{-m_2}{m_1+m_2} & 0 \end{pmatrix} \qquad M = m_1 + m_2 + m_3 \quad . \qquad \text{.....(6)}$$

To generalise Smith's treatment, we require functions that are eigenstates of the invariant operator:

$$T = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3}\right)$$

$$= \frac{1}{2}\left(\frac{q_1^2}{\mu_1} + \frac{q_2^2}{\mu_2} + \frac{q_3^2}{\mu_3}\right) , \qquad .....(7)$$

where $\mu_1 = M$, $\mu_2 = (m_1 + m_2)m_3/M$, $\mu_3 = m_1m_2/(m_1 + m_2)$. In this scheme, particles (1) and (2) have been coupled, and their centre-of-mass separately coupled to particle (3). The two mass couplings are:

Physical Barycentric System

Advantages: (i) The correct number of degrees of freedom (five) is obtained directly from the co-ordinate transformation and constraints (1), (2), and (3).

(ii) The eigenstates are of the same construction as the non-relativistic ones.

(iii) The centres of momenta are the physically observable ones and their equations of motion describe the actual motion of the centres-of-mass of the system.

Disadvantages: (i) The invariant T of equation (7) is physically meaningless and represents neither the kinetic energy nor the total mass.

(ii) The statistical phase space factor required to obtain cross sections proves hopelessly complicated to evaluate.

(iii) The centre-of-mass energy (3) is not a direct eigenvalue of any conceivable component of the operator (7), and is extremely difficult to extract from the eigenvalue equation.

(iv) In quantum mechanics, the matrix M_{ji} would become a time-dependent operator because physical masses are proportional to total energies.

Proper Barycentric System

Advantages: (i) The phase space factors are readily evaluated.

(ii) The eigenvalue equation (7) has the meaning that T is half the total proper mass of the system.

(iii) The problem is a complete four-space analogue of the non-relative case.

(iv) The matrix M_{ji} is a C-number and not a differential operator.

Disadvantages: (i) The appropriate number of degrees of freedom is not obtained directly from the transformations and constraints. In

particular, the mass shell constraints (2) are most difficult to apply, and make the eigenvalue equation very involved.

(ii) The system described is not directly observable because the barycentric co-ordinates do not describe the motion of actual centres-of-mass.

(iii) The eigenstates of the four-component pseudo-Cartesian operators obtained contain Coupled Gegenbauer polynomials, whose argument is the four-space equivalent of an angle. These polynomials, because of the hyperbolic nature of the pseudo-angle argument, do not possess the orthogonality properties of the non-relativistic analogue over the physical angular region. Furthermore, the quantum number obtained as the relativistic analogue of orbital angular momentum is little understood and unrelated to experimental observations.

(iv) The transformation (4) yields relative time co-ordinates whose physical interpretation remains obscure.

Each approach has its own limitations which remove the possibility of application to multiparticle systems, so let us now consider some of the alternatives. Macfarlane (1961) has given a derivation of a partial wave representation in which he used the co-ordinates:

$$\underline{q}_1 = \underline{p}_1 + \underline{p}_2, \quad \underline{q}_2 = \underline{q}_1 + \underline{p}_3 , \qquad .....(8)$$

and defined a further set of space-like vectors which satisfy:

$$\underline{q}_j \cdot \underline{e}_j = 0; \quad \underline{e}_j^2 = -1 . \qquad .....(9)$$

By transforming the $\underline{e}_j$ to be energy-independent in the rest frames where the corresponding three-vector of $\underline{q}_j$ vanishes, he obtained a pair of angles describing the orientation of $\underline{e}_j$. These angles, together with the components of each $\underline{q}_j$, form his basic set of co-ordinates. He assumed that the ordinary spherical harmonic projection of the partial wave amplitudes with the angular orientations of $\underline{q}_j$ and $\underline{e}_j$ as arguments is valid. Some criticisms applicable to this representation are:

Advantages: (i) The phase space factor is very simple.

(ii) The eigenstates possess simple orthogonality properties.

Disadvantages: (i) There are n-3 time-like four-vectors and n-3 space-like four-vectors, hence there will be n-3 energies as arguments of the projected partial wave amplitude. This means that one cannot relate scattering to production in a simple way because

the scattering partial wave amplitude depends only on W. It would be advantageous to find the partial wave amplitude's dependence on energies other than W. just on the basis of a representation.

(ii) There exists no proof that the co-ordinates of $\underline{e}_j$ actually yield the normal spherical harmonics as eigenstates when the corresponding operators are separated from a kinetic energy or mass operator. Furthermore, the states have not been shown to represent eigenstates of physical orbital angular momentum operators.

An SU3 symmetry scheme has been proposed by Dragt (1965), but the projection of eigenstates does not leave a partial wave amplitude that is diagonal in the centre-of-mass total energy. Most promising is the helicity coupling scheme derived by Wick (1962), which has similar properties to Macfarlane's system, except that helicity eigenstates replace the normal orbital angular momentum eigenstates. The investigation of scattering and production as a single entity is rather inaccessible with this approach because the helicity scheme of partial wave phase shift analysis is not widely used and is a relatively undeveloped concept.

To summarise the state of available methods of partial wave analysis, no generally satisfactory scheme of coupling exists, and the interpretation of the relativistic angular momentum tensor remains obscure.

4. PHASE SPACE

To conserve angular momentum, and to ensure that the eigenstates of angular momentum are well defined and observable, they must be projected from an amplitude in such a way as to make the partial waves relativistically orthogonal and normalizable. To illustrate that this requirement is not readily obtained we can evaluate the volume element for the three-body final state in Figure 2. Lardner (1961) shows that it is given by:

$$d\Omega = d^4\underline{p}_3\, d^4\underline{p}_4\, d^4\underline{p}_5\, \delta(\underline{p}_3^2 - m_3^2)\, \delta(\underline{p}_4^2 - m_4^2)\, \delta(\underline{p}_5^2 - m_5^2)\, \delta(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4 - \underline{p}_5)$$

$$\times\ \theta(p_3^{\,o})\, \theta(p_4^{\,o})\, \theta(p_5^{\,o}) \quad , \qquad \text{.....(10)}$$

where $\underline{p}_i = (\underset{\sim}{q}_i, p_i^{\,o})$.

Elimination of the mass-shell δ-functions leads to:

$$d\Omega = \frac{1}{2^3} \frac{d^3q_3}{E_3} \frac{d^3q_4}{E_4} \frac{d^3q_5}{E_5}\, \delta(E_1 + E_2 - E_3 - E_4 - E_5)\, \delta(\underset{\sim}{q}_3 + \underset{\sim}{q}_4 + \underset{\sim}{q}_5)$$

$$\times\ \theta(E_3)\, \theta(E_4)\, \theta(E_5) \quad ,$$

and removal of the conservation of energy condition gives:

$$d\Omega = \frac{1}{2^3} \frac{d^3q_3}{E_3} \frac{d^3q_4}{E_4}\, q_5\, dz_{15}\, d\phi_{15}\, \delta(\underset{\sim}{q}_3 + \underset{\sim}{q}_4 + \underset{\sim}{q}_5)\, \theta(E_3)\, \theta(E_4)\, \theta(E_5) \quad ,$$

q_5, z_{15} and ϕ_{15} being the polar co-ordinates of $\underset{\sim}{q}_5$ relative to $\underset{\sim}{q}_1$ along the z-axis.

We can utilize the three components of the momentum conservation condition to remove $\underset{\sim}{q}_3$, $\underset{\sim}{q}_4$ or the set of co-ordinates (q_3, q_4, ϕ_{45}), this last giving:

$$d\Omega = \frac{1}{2^3} \frac{q_3^{\,2}\, q_4^{\,2}\, q_5^{\,2}}{E_3\, E_4\, E_5} \frac{dz_{35}\, dz_{45}\, d\phi_{35}\, dz_{15}\, d\phi_{15}}{q_3\sqrt{1 - z_{35}^{\,2}}\ .\ \Delta} \quad , \qquad \text{.....(11)}$$

where $\Delta = \begin{vmatrix} \frac{q_3}{E_3} & \frac{q_4}{E_4} & \frac{q_5}{E_5} \\ z_{35} & z_{45} & 1 \\ \sqrt{1-z_{35}^{\,2}} & -\sqrt{1-z_{45}^{\,2}} & 0 \end{vmatrix}$,

which is of the general form:

$$d\Omega = J_3\,(z_{35}, z_{45}, \phi_{35}, z_{15}, \phi_{15}, W)\, dz_{35}\, dz_{45}\, d\phi_{35}\, dz_{15}\, d\phi_{15} \quad . \qquad \text{.....(12)}$$

In general, a Jacobian J_f (3n - 10 variables) is obtained after eliminating the conservation laws from the volume element for an f-particle state. It turns out that ϕ_{15} is a redundant variable in the above example, and the matrix element is independent of it.

Some properties of the J_f are:

(i) They are invariant under general Lorentz transformations. In particular, under ordinary rotations one can see that J_3 above is independent of $(z_{15}, \phi_{15}, \phi_{35})$ and all relative orientations of rotated axes can be specified by these three angles.

(ii) No appeal to special Lorentz frames of reference can remove the dependence of J_f on particle velocities, and under such transformations:

$$J_f . d\Omega = J_f' \, d\Omega' , \qquad \text{.....(13)}$$

where the dash denotes reference to the

Let A_{fi} be the amplitude for all transitions between a state of i particles, to a state of f particles. In order to make the eigenfunctions of total angular momentum orthogonal with respect to integrations over the variables in the above example, one should expand the amplitude $\sqrt{J_i.J_f.A_{fi}}$ directly into orthogonal partial waves. The Jacobians J_i, J_f are what is known as "phase space factors".

5. PROJECTIONS OF EIGENSTATES

The approach to the practical problem of the formal structure of a production amplitude is as follows: It is assumed that a set of angles and pseudo-angles may be defined in a manner (after Delves 1958) which, together with the centre-of-mass energy, W, represents all degrees of freedom. Angles which refer to relative orientations of particles and cannot be changed without altering the energy of one or more particles are described as non-Eulerian. The partial wave eigenstate of orbital angular momentum, however derived, is denoted by $R^{\ell}_{mm}(w)$, where w is a set of two angles defining relative orientations. When all such w are fixed, and rotations of axes in three-space are carried out, the w do not vary, while the effect on the set of co-ordinates is that three degrees of freedom which describe the orientation of the entire system in space are altered. These degrees of freedom are represented by Euler angles (α, β, γ) after Edmunds (1957), and we employ his rotation group operators $D^{\ell}_{m',m}(\alpha, \beta, \gamma)$ when determining the effect of rotations. α rotates the axes in the xy plane, β rotates the z-axis to align the old with the new z-axis, and γ further rotates the axes in the xy plane to complete the alignment.

It is assumed that for non-Eulerian angles, a generalized spherical harmonic $Y^M_L(\Omega)$ can be defined in terms of the individual $R^{\ell}_{m',m}$ in the same manner as occurs for non-relativistic theory:
That is,

$$Y^M_L(\Omega) = \prod_{v=1}^{f} \sum_{m''_v} (\ell_1 m_1 \ell_2 m_2 \ldots |LM)\, S^{\ell_v}_{m_v m''_v}(0, \theta_v, \phi_v) \qquad \ldots\ldots(14)$$

where $\theta_f = 0$, $\phi_f = 0$, $S^{\ell}_{m',m}(\Omega) = (i)^{\ell} \left(\frac{2\ell+1}{4\pi}\right)^{\frac{1}{2}} R^{\ell}_{m',m}(\Omega)$.

This is achieved by coupling the various substates $S^{\ell}_{m',m}(\Omega)$ with the aid of vector coupling coefficients $(\ell_1 m_1 \ell_2 m_2 \ldots |LM)$ to give a state of well-defined orbital angular momentum L, and z component M. In the treatment given by Macfarlane, one would have:

$$S^{\ell}_{om}(0, \theta, \phi) = (i)^{\ell} Y_{\ell}^{m}(\theta \phi) \quad , \qquad \ldots\ldots(15)$$

where $Y_{\ell}^{m}(\beta.\gamma)$ is the usual spherical harmonic and $\theta.\phi$ are invariantly defined pseudo-angles.

It is essential that eigenstates such as (14) should be projected from transition amplitudes in such a way that total angular momentum is conserved between initial and final states. Therefore, the set of reference axes oriented in a special way with respect to particle trajectories in the final state may be related to a similar set oriented with respect to the initial state, by the rotation group operators, such that:

$$S^{\ell}_{mm''}(\theta_{in}, \phi_{in}) = D^{\ell}_{m'm}(\alpha, \beta, \gamma)\, S^{\ell}_{m'm''}(\theta_f, \phi_f). \qquad \ldots\ldots(16)$$

(θ_{in}, ϕ_{in}) and (θ_f, ϕ_f) are co-ordinates specifying the orientation of vectors characterising initial and final particles respectively. However, all degrees of freedom referring to relative particle orientations and momenta remain unaltered by the rotation (16).

To complicate matters, real space angular arguments are insufficient in number to fix the system in three-momentum space when there are more than three particles present, and they must be supplemented by pseudo-angles, as was done by Delves and Smith. As pointed out, a precise structure of these co-ordinates has not been determined.

Following the above general properties, we write the component of the amplitude which is invariant under three-space rotations of axes as the resultant eigenvector which arises from the coupling of all those spherical harmonics (14) which have arguments invariant under rotations.

$$\Psi_n(L, M, \Omega, \ldots \Omega_k) = \frac{1}{\sqrt{J_n}} \prod_{j=1}^{k} \sum_{m''_j = -|\ell_j + L_{j+1}|}^{|\ell_j + L_{j+1}|} (\ell_j m_j L_{j-1} | L_j M_j) \times S^{\ell_j}_{m_j m''_j}(\Omega_j), \qquad \ldots\ldots(17)$$

where (ℓ_j, m_j) refer to the orbital angular momentum and its z component respectively of a set of uncoupled eigenstates, while (L_j, M_j) refer to the resultant states, ordered in a specific hierarchy of coupling. The index k refers to the invariant degrees of freedom, of which there are 3n-6, where n is the number of particles in the state. There may be one or more angles involved in the Ω_j, but only one angle in each is a non-redundant degree of freedom. For scattering, one has k = 0, and so:

$$\Psi_2 (L.\ M,\ \Omega) = \frac{1}{\sqrt{2}} \quad .$$

The production amplitude ${}_fA_i$ is written as:

$$A_{f_i} = \sum_L \Psi_f (LM;\underset{\sim}{\Omega})\ {}_fA_i(LM;W)\ D^{L}_{MM'}.\ (W_{f_i})\ \Psi_i\ (LM';\underset{\sim}{\Omega}') \quad , \qquad .....(18)$$

where we have set (θ_f, ϕ_f) and (θ_{in}, ϕ_{in}) all to zero in (16) and where $\underset{\sim}{\Omega}$ contains 3f-6 components and $\underset{\sim}{\Omega}'$ contains 3i-6 components. ${}_fA_i(LM;W)$ is the production partial wave amplitude. For particles with spin, a matrix representation must be used. The argument W_{f_i} is (α, β, γ) as in (16).

One further vital property of the $\bar{\Psi}_n$ must be enforced before further progress can be made. They must be orthogonal functions when integrated over the phase space interval for the state, that is,

$$\int d\underset{\sim}{\Omega}.\ \Psi^*_n\ (LM;\underset{\sim}{\Omega}).\Psi_n\ (L'M';\underset{\sim}{\Omega}) = \prod_{j=1}^{k} \delta(\ell_j,\ell'_j)\ \delta(m_j,m'_j)$$

$$= \delta(L,L')\ \delta(M,M') \quad . \qquad .....(19)$$

This property will ensure that angular momentum is conserved between vertices in an interaction involving intermediate states.

6. CONCLUDING REMARKS

General properties of a partial wave production formalism have been outlined and the difficulties encountered in finding representations discussed. In Part II (issued as AAEC/TM358) the unitarity principle is investigated using methods described in this paper, and a compound state formalism is derived, similar to the one obtained from R matrix theory.

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