

9-1-2002

# The Fourier Transform and Some Applications

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THE FOURIER TRANSFORM AND SOME APPLICATIONS

A Thesis

Presented to

The Faculty of the Department of Mathematics

Western Kentucky University

Bowling Green, Kentucky

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science

By

Christopher Matthew Zion

September 2002

## The Fourier Transform and Some Applications

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## Acknowledgements

I wish to thank Daniel Biles for all his time and dedication in directing this thesis. Dr. Biles worked hard the past year. Several of our meetings were review sessions because the thesis incorporated a variety of mathematics from several different classes. I want to thank Nezam Iraniparast and John Spraker for working with us and giving us excellent feedback. The whole mathematics department at Western Kentucky University deserves recognition because I would not have made it this far without their help.

I want to thank all my friends who have encouraged me. I thank my God, the Lord Jesus Christ, for answered prayers. I wish to thank my family also for their patience and support.



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# Chapter 1

## The Fourier Transform

### ■ 1.1 Introduction

In the early 1800's, Jean Baptiste Joseph Fourier first defined and used the function we call the Fourier transform. While the Fourier series is used for functions defined on a finite interval or periodic functions defined on all  $\mathfrak{R}$ , the Fourier transform is used for functions defined on all of  $\mathfrak{R}$ ; that is, the interval  $(-\infty, \infty)$ .  $\mathcal{F}[f]$  is called the Fourier transform and is defined by:

$$\mathcal{F}[f(t)](\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \quad (1.1.1)$$

for all real  $\omega$  such that the integral exists. One family of functions which have a Fourier transform are those defined on  $\mathfrak{R}$  with values in  $\mathbf{C}$  which satisfy two properties: (a) piecewise continuous and (b) absolutely integrable (see Theorem 1.1). We will refer to this family

of functions as  $G(\mathfrak{R})$ . We note that for this family we can interpret

$$\int_{-\infty}^{\infty} g(t) dt \text{ as } \lim_{R \rightarrow \infty} \int_{-R}^R g(t) dt, \text{ the Cauchy principal value.}$$

Actually, there are relatively few functions for which the Fourier transform can be found. But one of the reasons the Fourier transform is useful in physics is that many physical functions fortunately do have a Fourier transform.

Theorem 1.1 (Existence of the Fourier Transform) [10, p. 94]

For every function  $f \in G(\mathfrak{R})$ ,

1. F is defined for all  $\omega \in \mathfrak{R}$ .
2. F is a continuous function on  $\mathfrak{R}$ .
3.  $\lim_{\omega \rightarrow \pm\infty} F(\omega) = 0$ .

Proof of Theorem 1.1

1. Since  $|e^{-i\omega t}| = 1$  for all real  $t$  and  $\omega$ ,

$$\int_{-\infty}^{\infty} |f(t) e^{-i\omega t}| dt = \int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

Thus  $f(t) e^{-i\omega t}$  is absolutely integrable on  $\mathfrak{R}$  and is piecewise continuous. So F is defined for all real  $\omega$ .

2. To show F is continuous for  $\omega \in \mathfrak{R}$  we prove that

$$\lim_{h \rightarrow 0} [F(\omega + h) - F(\omega)] = 0. \text{ Applying the definition,}$$

$$\begin{aligned}
F(\omega + h) - F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i(\omega+h)t} dt - \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-i\omega t} [e^{-iht} - 1] dt.
\end{aligned}$$

For  $t, \omega \in \mathfrak{R}$ ,

$$\begin{aligned}
\lim_{h \rightarrow 0} f(t)e^{-i\omega t} [e^{-iht} - 1] &= f(t)e^{-i\omega t} \lim_{h \rightarrow 0} [e^{-iht} - 1] \\
&= f(t)e^{-i\omega t} \cdot 0 \\
&= 0.
\end{aligned}$$

Since  $|f(t)| \cdot |e^{-i\omega t}| \cdot |e^{-iht} - 1| \leq |f(t)| \cdot 1 \cdot 2 = 2|f(t)|$ , we may apply the Lebesgue Dominated Convergence Theorem from analysis,

$$\lim_{h \rightarrow 0} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} [e^{-iht} - 1] dt = 0.$$

So  $\lim_{h \rightarrow 0} [F(\omega + h) - F(\omega)] = 0$  and hence  $F$  is continuous at every point in  $\mathfrak{R}$ .

3. If the limit exists, we have by definition

$$\begin{aligned}
\lim_{\omega \rightarrow \pm\infty} F(\omega) &= \lim_{\omega \rightarrow \pm\infty} \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt \\
&= \lim_{\omega \rightarrow \pm\infty} \left[ \int_{-\infty}^{\infty} f(t) \cos \omega t dt + i \int_{-\infty}^{\infty} f(t) \sin \omega t dt \right].
\end{aligned}$$

To prove  $\lim_{\omega \rightarrow \pm\infty} F(\omega) = 0$  it is sufficient to show that  $\lim_{\omega \rightarrow \pm\infty} \int_{-\infty}^{\infty} f(t) \cos \omega t dt$

and  $\lim_{\omega \rightarrow \pm\infty} \int_{-\infty}^{\infty} f(t) \sin \omega t dt$  are both equal to zero. We will prove the

second result here. The proof of the first is analogous and is given in [10, p.96-97]. We know  $f$  is absolutely integrable. Choose  $\varepsilon > 0$ .

There exists an  $M > 0$  such that  $\int_{|t|>M} |f(t)| dt < \varepsilon$ . Thus

$$\left| \int_{|t|>M} f(t) \sin \omega t dt \right| \leq \int_{|t|>M} |f(t) \sin \omega t| dt \leq \int_{|t|>M} |f(t)| dt < \varepsilon.$$

Since  $f$  is piecewise continuous on  $[-M, M]$  there exists a partition  $-M = t_0 < t_1 < \dots < t_m = M$  such that the step function

$h(t) = f(t_k)$ ,  $t_{k-1} < t \leq t_k$ ,  $k = 1, 2, \dots, m$  satisfies  $\int_{-M}^M |f(t) - h(t)| dt < \varepsilon$ .

$$\text{Now, } \int_{-M}^M f(t) \sin \omega t dt = \int_{-M}^M [f(t) - h(t)] \sin \omega t dt + \int_{-M}^M h(t) \sin \omega t dt.$$

For all  $\omega \in \mathfrak{R}$ ,

$$\begin{aligned} \left| \int_{-M}^M [f(t) - h(t)] \sin \omega t dt \right| &\leq \int_{-M}^M |f(t) - h(t)| \cdot |\sin \omega t| dt \\ &\leq \int_{-M}^M |f(t) - h(t)| dt \\ &< \varepsilon. \end{aligned}$$

Now consider the second integral

$$\begin{aligned}
\left| \int_{-M}^M h(t) \sin \omega t dt \right| &= \left| \sum_{k=1}^m \int_{t_{k-1}}^{t_k} f(t_k) \sin \omega t dt \right| \\
&= \left| \sum_{k=1}^m f(t_k) \frac{\cos \omega t_{k-1} - \cos \omega t_k}{\omega} \right| \\
&\leq \sum_{k=1}^m |f(t_k)| \left| \frac{2}{\omega} \right| \\
&\leq \left| \frac{2m}{\omega} \right| \max_{-M \leq t \leq M} |f(t)|.
\end{aligned}$$

For sufficiently large  $\omega$ ,  $\left| \frac{2m}{\omega} \right| \max_{-M \leq t \leq M} |f(t)| < \varepsilon$ . Thus

$$\left| \int_{-\infty}^{\infty} f(t) \sin \omega t dt \right| \leq 2\varepsilon < 3\varepsilon \text{ and the theorem is proved.}$$

Let's work an example to illustrate the Fourier transform.

**Example 1.1** Find the Fourier transform of the function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

First apply the definition and break up the integral over three intervals for which the first and third are zero. So we get

$$\mathcal{F} [f(t)](\omega) = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_0^1 e^{-i\omega t} dt = \left[ \frac{e^{-i\omega t}}{-i\omega} \right]_0^1 = \frac{ie^{-i\omega}}{\omega} - \frac{i}{\omega}.$$

Now apply the identity  $e^{i\theta} = \cos \theta + i \sin \theta$  for our result,

$$F(\omega) = \frac{\sin \omega}{\omega} + i \left( \frac{\cos \omega}{\omega} - \frac{1}{\omega} \right).$$

**Example 1.2** Find the Fourier transform of the function  $f(t) = e^{-|t|}$ .

Apply the definition to the function  $f(t)$ :

$$\mathcal{F}[f(t)](\omega) = F(\omega) = \int_{-\infty}^{\infty} e^{-|t|} e^{-i\omega t} dt.$$

Combine the exponentials to get

$$F(\omega) = \int_{-\infty}^{\infty} e^{-|t|-i\omega t} dt.$$

By properties of even and odd functions,

$$F(\omega) = 2 \int_0^{\infty} e^{-t} \cos \omega t dt.$$

Integrate by parts twice to get

$$\int_0^{\infty} e^{-t} \cos \omega t dt = \left. \frac{-\cos \omega t + \omega \sin \omega t}{1 + \omega^2} e^{-t} \right|_0^{\infty}.$$

Using the fact that  $\lim_{t \rightarrow \infty} e^{-t} = 0$ , we have

$$\int_0^{\infty} e^{-t} \cos \omega t dt = \frac{1}{1 + \omega^2}.$$

So the Fourier transform of  $f(t) = e^{-|t|}$  is  $F(\omega) = \frac{2}{1 + \omega^2}$ .

**Example 1.3** Find the Fourier transform of the Dirac delta function,

$\delta(t)$ , defined by  $\delta(t) = 0, t \neq 0$ , and  $\int_{-\infty}^{\infty} \delta(t) dt = 1$ . [5, p.2]

One representation of the delta function is  $\delta(t) = \lim_{\varepsilon \rightarrow 0} \begin{cases} \frac{1}{\varepsilon} & -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2} \\ 0 & \text{elsewhere} \end{cases}$ . The Fourier transform of the Dirac delta function

can then be found as follows.

$$\begin{aligned}
 \mathcal{F}[\delta(t)] &= \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt \\
 &= \lim_{\varepsilon \rightarrow 0} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{1}{\varepsilon} e^{-i\omega t} dt \\
 &= \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} e^{-i\omega t} dt \right) \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{-i}{\varepsilon \omega} \left( e^{\frac{\varepsilon \omega i}{2}} - e^{-\frac{\varepsilon \omega i}{2}} \right) \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{-i}{\varepsilon \omega} \left( 2i \sin\left(\frac{\omega \varepsilon}{2}\right) \right) \\
 &= \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon \omega} \sin\left(\frac{\omega \varepsilon}{2}\right) \\
 &= 1.
 \end{aligned}$$

In Table 1.1 are a few functions and their Fourier transforms. For more details about these and other examples or exercises refer to Andrews, Bracewell, Pinkus, Pinsky, and Wienberger. However, for



consistency, the transforms shown in the table are found using definition (1.1.1).

*Table 1.1*

$f(t)$	$F(\omega)$
$\begin{cases} 1, & 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases}$	$\frac{\sin \omega}{\omega} + i \left( \frac{\cos \omega}{\omega} - \frac{1}{\omega} \right)$
$e^{- t }$	$\frac{2}{1 + \omega^2}$
$e^{-at^2}$	$\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$ , $a$ is a nonzero constant
$\frac{1}{t^2 + 1}$	$\pi e^{- \omega }$
$e^{-at} h(t)$	$\frac{1}{a - i\omega}$ , $a$ is a constant
$\frac{\sin t}{t}$ , $t \neq 0$	$\begin{cases} \pi &  \omega  < 1 \\ 0 &  \omega  > 1 \end{cases}$
$\delta(t - \xi)$	$e^{-i\omega\xi}$

## ■ 1.2 Properties and Formulas

Finding the Fourier transform from the definition can be difficult and tedious. Fortunately, we can find certain transforms by

use of the properties of the Fourier transform without going to the formal definition. Listed below in Table 1.2 are many of the important properties, which hold under the appropriate assumptions.

*Table 1.2*

***Properties of the Fourier Transform*** [1, p.225]

1.2.1	$\mathcal{F}(c_1 f + c_2 g) = c_1 \mathcal{F}(f) + c_2 \mathcal{F}(g)$
1.2.2	$\mathcal{F}(f^{(n)}) = (i\omega)^n F(\omega) \quad n = 0, 1, 2, \dots$
1.2.3	$\mathcal{F}(t f(t)) = i F'(\omega)$
1.2.4	$\mathcal{F}(f(t-a)) = e^{i a \omega} \mathcal{F}(f(t))$
1.2.5	$\mathcal{F}(e^{i a t} f(t)) = \mathcal{F}[f(t)](\omega - a)$
1.2.6	$\mathcal{F}(f(at)) = \frac{1}{ a } \mathcal{F}[f(t)](\frac{\omega}{a}), \quad a \neq 0$

Proof (1.2.2) We shall prove the case  $n = 1$ . [10, p.105]

Assume  $f \in G(\mathfrak{R})$ . By the definition of the Fourier transform we

have  $\mathcal{F}(f') = \int_{-\infty}^{\infty} f'(t) e^{-i\omega t} dt$ , if the limit exists. Now use integration by

parts.

$$\mathcal{F}(f') = f(t) e^{-i\omega t} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\omega f(t) e^{-i\omega t} dt$$

$$\begin{aligned}
&= 0 + i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \quad (\lim_{|t| \rightarrow \infty} f(t) = 0, \text{ since } f \in G(\mathfrak{R})) \\
&= i\omega \mathcal{F}[f(t)](\omega) \\
&= i\omega F(\omega).
\end{aligned}$$

Proof (1.2.5)

$$\mathcal{F}[e^{iat} f(t)] = \int_{-\infty}^{\infty} e^{iat} f(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(t) e^{-i(\omega-a)t} dt = \mathcal{F}[f(t)](\omega-a).$$

### ■ 1.3 Real and Complex Outcomes

By symmetry properties we can determine from the domain of the function what the result will be in the codomain of the transform.

The results are summarized in Table 1.3 [2, p14].

*Table 1.3*

	FUNCTION	TRANSFORM
1.3.1	real and odd	imaginary and odd
1.3.2	complex and even	complex and even
1.3.3	real and asymmetrical	complex and asymmetrical
1.3.4	imaginary and asymmetrical	complex and asymmetrical
1.3.5	real even and imaginary odd	real

1.3.6	real odd and imaginary even	imaginary
1.3.7	even	even
1.3.8	odd	odd

Proof (1.3.1)

Let  $f$  be real and odd. Then  $\mathcal{F}[f](-\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt$ . Let  $t = -x$ . So

$$\begin{aligned} \text{we have } \mathcal{F}[f](-\omega) &= - \int_{\infty}^{-\infty} e^{-i\omega x} f(-x) dx \\ &= - \int_{-\infty}^{\infty} e^{-i\omega x} f(x) dx \\ &= - \mathcal{F}[f](\omega). \end{aligned}$$

Thus  $\mathcal{F}$  is odd.  $\mathcal{F}[f]$  can be written as  $\int_{-\infty}^{\infty} [\cos \omega t - i \sin \omega t] f(t) dt =$

$$\int_{-\infty}^{\infty} \cos \omega t f(t) dt - i \int_{-\infty}^{\infty} \sin \omega t f(t) dt.$$

In order to show that  $\mathcal{F}$  is purely imaginary, we need to show that  $\int_{-\infty}^{\infty} \cos \omega t f(t) dt = 0$ . Breaking up this integral over two intervals,

$$\begin{aligned} \int_{-\infty}^0 \cos \omega t f(t) dt + \int_0^{\infty} \cos \omega t f(t) dt &= - \int_{\infty}^0 \cos \omega(-t) f(-t) dt + \int_0^{\infty} \cos \omega t f(t) dt \\ &= - \int_0^{\infty} \cos \omega t f(t) dt + \int_0^{\infty} \cos \omega t f(t) dt \end{aligned}$$

$$= 0.$$

## ■ 1.4 The Inverse Fourier Transform

There are a wide variety of definitions used in the Fourier transform and its inverse. Typically, the Fourier transform is defined

by  $\frac{1}{\gamma} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$  and the inverse Fourier transform by

$\frac{\gamma}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$  for some  $\gamma > 0$  (or vice versa). Being consistent with

our definition of the Fourier transform, our inverse Fourier transform is:

$$\mathcal{F}^{-1}[F(\omega)](t) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega. \quad (1.4.1)$$

The Fourier transform and its inverse, regardless of the pair of definitions used, should “undo” each other just as inverse functions do.

### Theorem 1.2

If  $f \in G(\mathbb{R})$  and  $f$  is differentiable, then

$$\mathcal{F}^{-1}[\mathcal{F}(f)] = f.$$

### Proof of Theorem 1.2

(based on a hint in exercise 10.3.9 in [6])

$$F(\omega) = \int_{-\infty}^{\infty} f(\bar{x}) e^{-i\omega\bar{x}} d\bar{x}$$

$$\frac{1}{2\pi} F(\omega) e^{i\omega x} = \frac{1}{2\pi} e^{i\omega x} \int_{-\infty}^{\infty} f(\bar{x}) e^{-i\omega\bar{x}} d\bar{x}$$

$$\frac{1}{2\pi} \int_{-L}^L F(\omega) e^{i\omega x} d\omega = \frac{1}{2\pi} \int_{-L}^L e^{i\omega x} \int_{-\infty}^{\infty} f(\bar{x}) e^{-i\omega\bar{x}} d\bar{x} d\omega$$

$$= \frac{1}{2\pi} \int_{-L}^L \int_{-\infty}^{\infty} e^{i\omega x} e^{-i\omega\bar{x}} f(\bar{x}) d\bar{x} d\omega$$

$$= \frac{1}{2\pi} \int_{-L}^L \int_{-\infty}^{\infty} e^{i\omega(x-\bar{x})} f(\bar{x}) d\bar{x} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-L}^L e^{i\omega(x-\bar{x})} f(\bar{x}) d\omega d\bar{x}$$

(by Fubini's Theorem, see for example, [12, p.269])

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{e^{i\omega(x-\bar{x})}}{i(x-\bar{x})} \right]_{-L}^L f(\bar{x}) d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iL(x-\bar{x})} - e^{-iL(x-\bar{x})}}{i(x-\bar{x})} f(\bar{x}) d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \frac{2}{x-\bar{x}} \sin L(x-\bar{x}) d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} [f(x) + f(\bar{x}) - f(x)] \frac{2}{x-\bar{x}} \sin L(x-\bar{x}) d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \frac{2}{x-\bar{x}} \sin L(x-\bar{x}) d\bar{x}$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} [f(\bar{x}) - f(x)] \frac{2}{x - \bar{x}} \sin L(x - \bar{x}) d\bar{x}$$

Consider the first integral. Let  $u = L(x - \bar{x})$ ,  $du = -L d\bar{x}$ . The constant  $2f(x)$  can be pulled outside of the integral. Note that

$\int_{-\infty}^{\infty} \frac{\sin u}{u} du$  has the well-known value of  $\pi$ . So the first integral is

$$\frac{2f(x)}{2\pi} \cdot \pi = f(x).$$

Now apply integration by parts to the second integral. Let  $u =$

$$\frac{f(\bar{x}) - f(x)}{x - \bar{x}} \quad \text{and} \quad dv = \sin L(x - \bar{x}) d\bar{x}. \quad \text{So} \quad du =$$

$$\frac{(x - \bar{x})f'(\bar{x}) + [f(\bar{x}) - f(x)]}{(x - \bar{x})^2} d\bar{x} \quad \text{and} \quad v = -\frac{\cos L(x - \bar{x})}{L}. \quad \text{Thus we have}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} [f(\bar{x}) - f(x)] \frac{2}{x - \bar{x}} \sin L(x - \bar{x}) d\bar{x} = \frac{1}{2\pi} \left[ -\left( \frac{f(\bar{x}) - f(x)}{x - \bar{x}} \right) \frac{\cos L(x - \bar{x})}{L} \right]_{-\infty}^{\infty}$$

$$+ \int_{-\infty}^{\infty} \frac{\cos L(x - \bar{x})}{L} \left[ \frac{(x - \bar{x})f'(\bar{x}) + [f(\bar{x}) - f(x)]}{(x - \bar{x})^2} \right] d\bar{x}. \quad \text{Note that}$$

$$\frac{1}{2\pi} \left[ -\left( \frac{f(\bar{x}) - f(x)}{x - \bar{x}} \right) \frac{\cos L(x - \bar{x})}{L} \right]_{-\infty}^{\infty} \rightarrow 0 \quad \text{since} \quad \frac{f(\bar{x})}{\bar{x}} \rightarrow 0 \quad \text{as} \quad x \rightarrow \pm\infty.$$

So putting all this together we have  $\frac{1}{2\pi} \int_{-L}^L F(\omega) e^{i\omega x} d\omega = f(x) +$

$$\int_{-\infty}^{\infty} \frac{\cos L(x - \bar{x})}{L} \left[ \frac{(x - \bar{x})f'(\bar{x}) + [f(\bar{x}) - f(x)]}{(x - \bar{x})^2} \right] d\bar{x}. \quad \text{Let} \quad L \rightarrow \infty. \quad \text{Thus}$$

$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega = f(x) + 0$ . Combining this result with the first line

of the proof we discover that  $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \left[ \int_{-\infty}^{\infty} f(\bar{x}) e^{-i\omega \bar{x}} d\bar{x} \right] d\omega$  which

implies the desired result that  $f = \mathcal{F}^{-1} [\mathcal{F}(f)]$ .

Similarly, it can be shown under the appropriate assumptions that  $\mathcal{F}[\mathcal{F}^{-1}(F)] = F$ .

The inverse Fourier transform also has a linearity property and a multiplicative property called convolution. These are listed below in Table 1.4 [1, p.227] and [5, p.7].

*Table 1.4*

1.4.1	$\mathcal{F}^{-1}(c_1 F + c_2 G) = c_1 \mathcal{F}^{-1}(F) + c_2 \mathcal{F}^{-1}(G)$
1.4.2	$\mathcal{F}^{-1}(F(\omega)G(\omega)) = \int_{-\infty}^{\infty} f(u)g(x-u) du = f * g,$ <p>where <math>f = \mathcal{F}^{-1}(F)</math> and <math>g = \mathcal{F}^{-1}(G)</math>.</p>

## ■ 1.5 Relationship Between Fourier and Laplace Transforms

The Fourier transform is related to the well-known Laplace transform, and we shall explore the relationship in this section.



Sometimes the Laplace transform will be the natural choice in solving a differential equation even in applications where the Fourier transform can be used. Interestingly, the Fourier transform and Laplace transform can both be used together in the same problem. We will look at joint transforms later in chapter 4.

Start with the Fourier integral relation [1, p.186], which is valid at points of continuity for piecewise smooth and absolutely integrable functions.

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos[\omega(t-x)] dt d\omega, \quad (1.5.1)$$

Applying Euler's formula [4, p.50] for cosine,  $\cos y = \frac{e^{iy} + e^{-iy}}{2}$ , where  $y$  is a real number, we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) [e^{i\omega(t-x)} + e^{-i\omega(t-x)}] dt d\omega \\ &= \frac{1}{2\pi} \left[ \int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt d\omega + \int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{-i\omega(t-x)} dt d\omega \right] \\ &= \frac{1}{2\pi} \left[ \int_0^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt d\omega + \int_{-\infty}^0 \int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt d\omega \right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt d\omega \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt d\omega. \quad (1.5.2)$$

From the integral formula (1.5.2), which is the exponential form of Fourier's integral theorem, we can derive the pair of transform formulas

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

and

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

which are equivalent to our definition of the Fourier transform (1.1.1) and inverse Fourier transform (1.4.1), respectively. Now if  $f$  is related to another function  $g$  such that  $f(t) = e^{-ct} g(t) h(t)$  where  $c$  is a positive constant and  $h(t)$  is the Heaviside step function, then it follows from absolute integrability of  $f$  that  $\int_0^{\infty} e^{-ct} |g(t)| dt < \infty$ . Notice that by

$$\text{substitution, } e^{-ct} g(t) h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \int_{-\infty}^{\infty} e^{-(c-i\omega)x} g(x) h(x) dx d\omega. \text{ Since } h(x)$$

$= 1$  whenever  $x > 0$ , we can write equivalently,  $g(t) h(t) =$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(c-i\omega)t} \int_0^{\infty} e^{-(c-i\omega)x} g(x) dx d\omega. \text{ By the change of variables } p = c - i\omega \text{ we}$$

formally obtain  $g(t)h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \int_0^{\infty} e^{-px} g(x) dx dp$ . Thus we have

derived the following pair of transform formulas [1, p.228]

$$F(p) = \int_0^{\infty} e^{-px} g(x) dx \quad (1.5.3)$$

and

$$g(t)h(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} F(p) dp. \quad (1.5.4)$$

Observe that (1.5.3) is well-known to be the definition of the Laplace transform. Equation (1.5.4) can be thought of as an inversion formula for the Laplace transform. Thus, both the Laplace and Fourier transforms can be motivated from the Fourier integral theorem, providing a connection between the two transforms.

Here is another relationship between the Laplace transform and the Fourier transform. The condition  $\int_{-\infty}^{\infty} |f(t)| dt < \infty$  is too restrictive for some purposes. According to Duffy, the following has proven useful in electrical engineering. Modify the Laplace transform as follows [5, p.7]:

$$F(\omega) = \int_0^{\infty} f(t) e^{-i\omega t} dt,$$

where  $\text{Im}(\omega) < 0$ . (Compare with (1.5.3).) Note that under this definition, a function that is not absolutely integrable, like the Heaviside function, might have a transform. We can take this one step further. If we can find  $c_1, c_2$  with  $c_2 > c_1$ , such that  $e^{-c_1 t} |f(t)| \rightarrow 0$  as  $t \rightarrow \infty$  and  $e^{-c_2 t} |f(t)| \rightarrow 0$  as  $t \rightarrow -\infty$ , then we may define the generalized Fourier transform (or two-sided Laplace transform) by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,$$

where  $c_2 > -\text{Im}(\omega) > c_1$ . (Compare with (1.1.1).)

There are several other transforms that have a direct connection with the Fourier and/or Laplace transform. A few of these are the Hankel, Mellin, Hilbert, Abel, and z transforms. Read chapter 12 of Bracewell [2] for an in-depth discussion about how these linear transformations relate to the Fourier transform.

## Chapter 2

### The Inverse Transform and Complex Analysis

#### ■ 2.1 Finding the Inverse Transform with Complex Analysis

To be able to perform the inversion of a function containing complex variables we need to be able to apply some theorems. First we'll introduce the residue theorem from complex variables.

Theorem 2.1 The Residue Theorem [4, p.183]

Let  $C$  be a positively oriented, simple closed contour on which  $f$  is analytic except for a finite number of isolated singular points  $z_1, z_2, \dots, z_n$  on the interior of  $C$ . If  $K_1, K_2, \dots, K_n$  are the residues of  $f$  at those points, then

$$\int_C f(z) dz = 2\pi i (K_1 + K_2 + \dots + K_n).$$

Theorem 2.2 allows us to evaluate a contour integral in either the upper or lower half plane.

Theorem 2.2 [8, p.275]

Let  $f$  be an analytic function on an open set containing the closed upper half plane  $\mathcal{H} = \{z \in \mathbf{C} \mid \text{Im}(z) \geq 0\}$  (or lower half plane  $\mathcal{L} = \{z \in \mathbf{C} \mid \text{Im}(z) \leq 0\}$ ) except for a finite number of isolated singularities, none of which lie on the real axis, and there exist real constants  $M$ ,  $p$ , and  $R_0$  with  $p > 1$  and  $|f(z)| < \frac{M}{|z|^p}$  whenever  $z \in \mathcal{H}$  (or  $z \in \mathcal{L}$ ) and  $|z| \geq R_0$ .

Then

$$(i) \quad \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum \{ \text{residues of } f \text{ in } \mathcal{H} \}$$

$$(ii) \quad \int_{-\infty}^{\infty} f(x) dx = -2\pi i \sum \{ \text{residues of } f \text{ in } \mathcal{L} \}.$$

Proof (see Example 4.3.5 [8, p.273])

(i) Choose radius  $R \geq R_0$  so that all poles are enclosed in the contour with the half-circle  $\Gamma_R = \gamma_R + \mu_R$ , where  $\gamma_R$  is the line contour on the real axis traversed from left to right and  $\mu_R$  is the curved contour that runs counterclockwise around the poles. Then by the residue theorem,

$$\int_{\Gamma_R} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res } f(z)_{z=z_k}. \quad \text{Thus } \int_{\gamma_R} f(z) dz + \int_{\mu_R} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res } f(z)_{z=z_k}.$$

Observe that  $\left| \int_{\mu_R} f(z) dz \right| \leq \frac{M}{R^p} \pi R = \frac{M\pi}{R^{p-1}} \rightarrow 0$  as  $R \rightarrow \infty$ . Also

$$\int_{\gamma_R} f(z) dz \rightarrow \int_{-\infty}^{\infty} f(z) dz \text{ as } R \rightarrow \infty. \text{ Therefore } \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = \int_{-\infty}^{\infty} f(z) dz =$$

$$2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z).$$

The proof of (ii) is similar and uses a contour in the lower half plane.

Next we need a theorem to evaluate these residues. Theorem 2.3 allows us to calculate residues without going to the definition of a residue, which uses a Laurent series expansion.

Theorem 2.3 [4, p.190]

An isolated singular point  $z_0$  of a function  $f$  is a pole of order  $m$  if and

only if  $f(z)$  can be written in the form  $f(z) = \frac{\phi(z)}{(z-z_0)^m}$  where  $\phi(z)$  is

analytic and nonzero at  $z_0$ . Moreover,

$$(i) \quad \operatorname{Res}_{z=z_0} f(z) = \phi(z_0) \text{ if } m=1$$

$$(ii) \quad \operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \text{ if } m \geq 2.$$

**Example 2.1** Find the inverse Fourier transform of  $F(\omega) = \frac{1}{\omega^2 + 4\omega + 8}$ .

The inversion formula is  $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + 4\omega + 8} e^{i\omega t} d\omega$ . Note that

$\frac{e^{izt}}{z^2 + 4z + 8}$  has simple poles at  $z = -2 \pm 2i$ . Before we use Theorem 2.2,

we need to show the conditions hold. First we note the following.

Claim 1  $\left| \frac{e^{izt}}{z^2 + 4z + 8} \right| = \frac{e^{-bt}}{|z^2 + 4z + 8|}$ , where  $z = a + bi$ .

Proof: Trivial.

Claim 2 There exists an  $N$  such that for all with  $|z| > N$ ,

$$\frac{1}{|z^2 + 4z + 8|} < \frac{2}{|z|^2}.$$

Proof:  $\lim_{z \rightarrow 0} \left| \frac{1}{1 + 4z + 8z^2} \right| = 1$

$$\Rightarrow \lim_{z \rightarrow 0} \left| \frac{\frac{1}{z^2}}{\frac{1}{z^2} + \frac{4}{z} + 8} \right| = 1$$

$$\Rightarrow \lim_{z \rightarrow \infty} \left| \frac{z^2}{z^2 + 4z + 8} \right| = 1.$$

Thus  $\exists N \ni \forall z$  with  $|z| > N$ ,

$$\left| \frac{z^2}{z^2 + 4z + 8} \right| < 1 + 1$$

$$|z^2| < 2 |z^2 + 4z + 8|$$

$$\frac{2}{|z|^2} > \frac{1}{|z^2 + 4z + 8|}.$$

So  $\left| \frac{e^{izt}}{z^2 + 4z + 8} \right| = \frac{e^{-bt}}{|z^2 + 4z + 8|}$  (by Claim 1).

$$< \frac{1}{|z^2 + 4z + 8|} \text{ (for appropriate } t, b \text{ (see below)).}$$



$$< \frac{2}{|z|^2} \text{ (by Claim 2).}$$

Thus the conditions of Theorem 2.2 hold.

For  $t > 0$ , we use the upper half plane in which  $b > 0$ , so  $e^{-bt} < 1$ .

For  $t < 0$ , use the lower half plane in which  $b < 0$ , so  $e^{-bt} < 1$ . Thus  $e^{-bt} < 1$  in both cases.

Applying Theorem 2.2(i), the inverse Fourier transform for  $t > 0$  in the upper half plane is

$$\begin{aligned} \mathcal{F}^{-1} \left[ \frac{1}{z^2 + 4z + 8} \right] &= \frac{1}{2\pi} \cdot 2\pi i \operatorname{Res}_{z=-2+2i} \frac{e^{izt}}{z^2 + 4z + 8} \\ &= i \cdot \operatorname{Res}_{z=2+2i} \frac{e^{izt}}{(z - (-2 + 2i))(z - (-2 - 2i))} \\ &= i \cdot \left. \frac{e^{izt}}{z - (-2 - 2i)} \right]_{z=-2+2i} \\ &= i \cdot \frac{e^{i(-2+2i)t}}{(-2 + 2i) - (-2 - 2i)} \\ &= \frac{i \cdot e^{-2t-2ti}}{4i} \\ &= \frac{e^{-2t}}{4} \cdot e^{-2ti} \\ &= \frac{e^{-2t}}{4} (\cos 2t - i \sin 2t). \end{aligned}$$

For  $t < 0$  use the pole  $z = -2 - 2i$  in the lower half plane. The inverse Fourier transform using Theorem 2.2(ii) is

$$\begin{aligned}
 \mathcal{F}^{-1} \left[ \frac{1}{z^2 + 4z + 8} \right] &= \frac{1}{2\pi} \cdot -2\pi i \operatorname{Res}_{z=-2-2i} \frac{e^{izt}}{z^2 + 4z + 8} \\
 &= -i \cdot \operatorname{Res}_{z=-2-2i} \frac{e^{izt}}{(z - (-2 - 2i))(z - (-2 + 2i))} \\
 &= -i \cdot \left. \frac{e^{izt}}{z - (-2 + 2i)} \right]_{z=-2-2i} \\
 &= -i \cdot \frac{e^{i(-2-2i)t}}{(-2 - 2i) - (-2 + 2i)} \\
 &= \frac{-i \cdot e^{2t-2ti}}{-4i} \\
 &= \frac{e^{2t}}{4} \cdot e^{-2ti} \\
 &= \frac{e^{2t}}{4} (\cos 2t - i \sin 2t).
 \end{aligned}$$

Thus the inverse Fourier transform is the combined solution,

$$\mathcal{F}^{-1} \left[ \frac{1}{\omega^2 + 4\omega + 8} \right] = \frac{e^{-2|t|}}{4} (\cos 2t - i \sin 2t).$$

If a pole should lie on the real axis, we can still find the inverse Fourier transform. One method is to move the contour slightly off the real axis with an epsilon band (see example 2.2). This integral is often

referred to as Bromwich's integral after the English mathematician Thomas John l'Anson Bromwich.

**Example 2.2** [5, p.77] Find the inverse Fourier transform of  $\frac{1}{\omega^2 - a^2 \omega i}$ ,

where  $a \neq 0$ . By (1.4.1), inversion formula is  $\mathcal{F}^{-1}\left[\frac{1}{\omega^2 - a^2 \omega i}\right] =$

$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 - a^2 \omega i} e^{i\omega t} d\omega$ . Set  $z^2 - a^2 z i = 0$  to find the poles,  $z = 0$  and  $z =$

$a^2 i$ . We have a singularity on the real axis and on the positive imaginary axis. Notice that there are no singularities in the lower half plane, so  $f(t) = 0$  for the case when  $t < 0$ . Now, for  $t > 0$ , modify the inversion integral using Bromwich's integral to effectively move the pole  $z = 0$  off the real axis.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 - a^2 \omega i} e^{i\omega t} d\omega \approx \frac{1}{2\pi} \int_{-\infty - \varepsilon i}^{\infty - \varepsilon i} \frac{e^{i\omega t}}{\omega(\omega - a^2 i)} d\omega,$$

where  $\int_{-\infty - \varepsilon i}^{\infty - \varepsilon i}$  indicates the integral along the contour  $x = t$ ,  $y = -\varepsilon$ ,

$-\infty < t < \infty$ . Note that the limit of the second integral as  $\varepsilon \rightarrow 0$  gives the original  $f(t)$ . The inversion integral can be converted into the closed contour integral

$$\frac{1}{2\pi} \oint_C \frac{1}{z^2 - a^2 z i} e^{izt} dz$$

where both of the singularities are now within the contour and are essentially in the upper half plane. Thus by the residue theorem,

$$\begin{aligned}
 f(t) &= \frac{1}{2\pi} \left[ 2\pi i \left( \text{Res}(z=0) + \text{Res}(z=a^2i) \right) \right] \\
 &= \frac{1}{2\pi} \left[ 2\pi i \left( \left[ \frac{e^{zt}}{(z-a^2i)} \right]_{z=0} + \left[ \frac{e^{zt}}{z} \right]_{z=a^2i} \right) \right] \\
 &= \frac{1}{2\pi} \left[ 2\pi i \left( -\frac{1}{a^2i} + \frac{e^{-a^2t}}{a^2i} \right) \right] \\
 &= \frac{1}{a^2} (e^{-a^2t} - 1).
 \end{aligned}$$

The final solution can be written as  $f(t) = \frac{1}{a^2} (e^{-a^2t} - 1)h(t)$ , where  $h(t)$

represents the Heaviside step function, defined by  $h(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$ .

Another method of finding the inverse Fourier transform, when a pole lies on the real axis, is to apply a theorem.

Theorem 2.4 [8, p.287]

Let  $\mathcal{L}$  be the open lower half plane  $\{z \in \mathbf{C} \mid \text{Im}(z) < 0\}$ , and let  $f$  be analytic on an open set containing its closure  $\{z \in \mathbf{C} \mid \text{Im}(z) \leq 0\}$  except for finitely many isolated singularities. Suppose the ones on the real axis are simple poles. Then if either (i)  $f$  satisfies the conditions of Theorem 2.2(ii) (except for the poles on the axis) or (ii)  $f(z) = e^{-i\omega z} g(z)$

with  $\omega > 0$  and  $g$  satisfying  $g(z) \rightarrow 0$  as  $z \rightarrow 0$  in the half plane in the sense that for each  $\varepsilon > 0$  there is an  $R(\varepsilon)$  such that  $|g(z)| < \varepsilon$  whenever  $|z| \leq R(\varepsilon)$  and  $z \in \mathcal{L}$ , then the integral exists and

$$\int_{-\infty}^{\infty} f(z) dz = -2\pi i \Sigma \{\text{residues of } f \text{ in } \mathcal{L}\} - \pi i \Sigma \{\text{residues of } f \text{ on the real axis}\}.$$

(Similarly,  $\int_{-\infty}^{\infty} f(z) dz = 2\pi i \Sigma \{\text{residues of } f \text{ in } \mathcal{H}\} + \pi i \Sigma \{\text{residues of } f \text{ on the real axis}\}$ , for the upper half plane.)

**Example 2.3** Find the inverse Fourier transform of  $\frac{1}{\omega(\omega^2 + 1)}$ .

The inversion formula is  $\mathcal{F}^{-1} \left[ \frac{1}{\omega(\omega^2 + 1)} \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega(\omega^2 + 1)} e^{i\omega t} d\omega$ .

The poles are  $z = 0$  and  $z = \pm i$ . If  $t > 0$ , use the upper half plane containing the poles  $z = 0$  and  $z = i$ . Thus by Theorem 2.4,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega(\omega^2 + 1)} d\omega &= \frac{1}{2\pi} \left[ 2\pi i \left[ \text{Res}_{z=i} \frac{e^{izt}}{z(z^2 + 1)} \right] + \pi i \left[ \text{Res}_{z=0} \frac{e^{izt}}{z(z^2 + 1)} \right] \right] \\ &= \frac{1}{2\pi} \left[ 2\pi i \left[ \text{Res}_{z=i} \frac{e^{izt}}{z(z-i)(z+i)} \right] + \pi i \left[ \text{Res}_{z=0} \frac{e^{izt}}{z(z-i)(z+i)} \right] \right] \\ &= i \left[ \frac{e^{izt}}{z(z+i)} \right]_{z=i} + \frac{i}{2} \left[ \frac{e^{izt}}{(z-i)(z+i)} \right]_{z=0} \\ &= i \left[ \frac{e^{-t}}{i(2i)} \right] + \frac{i}{2} \left[ \frac{1}{-i(i)} \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{ie^{-t}}{2} + \frac{i}{2} \\
&= \frac{i(1-e^{-t})}{2}.
\end{aligned}$$

If  $t < 0$ , use  $z = 0$  and  $z = -i$ . Thus by Theorem 2.4,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega(\omega^2+1)} d\omega &= \frac{1}{2\pi} \left[ -2\pi i \left[ \operatorname{Res}_{z=-i} \frac{e^{izt}}{z(z^2+1)} \right] - \pi i \left[ \operatorname{Res}_{z=0} \frac{e^{izt}}{z(z^2+1)} \right] \right] \\
&= \frac{1}{2\pi} \left[ -2\pi i \left[ \operatorname{Res}_{z=-i} \frac{e^{izt}}{z(z-i)(z+i)} \right] - \pi i \left[ \operatorname{Res}_{z=0} \frac{e^{izt}}{z(z-i)(z+i)} \right] \right] \\
&= -i \left[ \frac{e^{izt}}{z(z-i)} \right]_{z=-i} - \frac{i}{2} \left[ \frac{e^{izt}}{(z-i)(z+i)} \right]_{z=0} \\
&= -i \left[ \frac{e^t}{-i(-2i)} \right] - \frac{i}{2} \left[ \frac{1}{-i(i)} \right] \\
&= \frac{ie^t}{2} - \frac{i}{2} \\
&= \frac{i(e^t-1)}{2}.
\end{aligned}$$

$$\text{Thus } \mathcal{F}^{-1} \left[ \frac{1}{\omega(\omega^2+1)} \right] = \begin{cases} \frac{i(1-e^{-t})}{2} & t > 0 \\ \frac{i(e^t-1)}{2} & t < 0 \end{cases}.$$

We will finish this chapter with an example of a multivalued function that involves branch cuts.

**Example 2.4** Find the inverse Fourier transform of the function

$$F(\omega) = \begin{cases} 0 & \omega < 0 \\ \frac{1}{\omega^{\frac{1}{3}}(\omega^2 + 1)} & \omega > 0 \end{cases} .$$

By the inversion formula,  $\mathcal{F}^{-1}(F(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega =$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^{\frac{1}{3}}(\omega^2 + 1)} e^{i\omega t} d\omega . \text{ For } t > 0 \text{ use the upper half plane with the given}$$

contour (see Figure 2.4.1). We have  $\int_C f(z) dz = \int_{\rho}^R \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz +$

$$\int_{C_R} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz + \int_{-R}^{-\rho} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz + \int_{C_\rho} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz, \text{ where } R > 1 \text{ and}$$

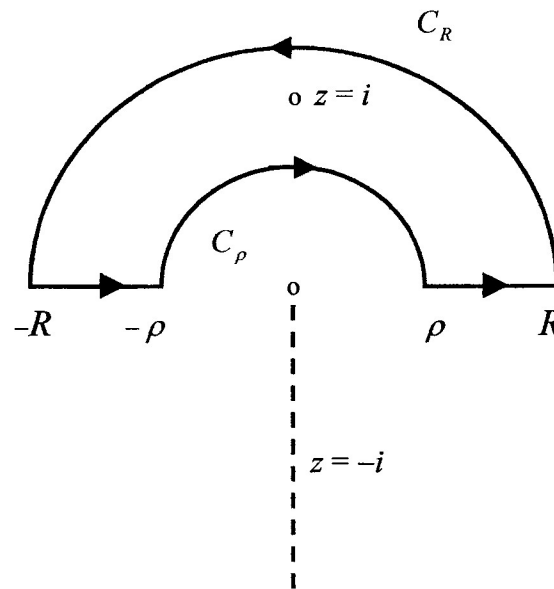


Figure 2.4.1

$\rho < 1$ . Place our branch cut so that  $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ . At  $\theta = 0$ , the function

$$f(z) = \frac{e^{\frac{1}{3}\ln r}}{r^2+1} e^{irt} = \frac{r^{\frac{1}{3}} e^{irt}}{r^2+1}. \quad \text{At } \theta = \pi, f(z) = \frac{e^{\frac{1}{3}(\ln r+i\pi)}}{(re^{i\pi})^2+1} =$$

$$\frac{e^{\frac{1}{3}\ln r} e^{\frac{1}{3}i\pi} e^{irt(e^{i\pi})}}{r^2 e^{2\pi i} + 1} = \frac{r^{\frac{1}{3}} e^{\frac{1}{3}i\pi} e^{-irt}}{r^2+1}. \quad \text{Now we shall show that the}$$

contribution of the circular arcs,  $C_R$  and  $C_\rho$ , to the contour integral is

$$\text{zero. Observe that } \left| \int_{C_R} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2+1)} dz \right| \leq \frac{e^{-(\ln z)t}}{R^{\frac{1}{3}}(R^2+1)} \cdot 2\pi R \leq \frac{1}{R^{\frac{1}{3}}(R^2+1)} \cdot 2\pi R$$

$$\rightarrow 0 \text{ as } R \rightarrow \infty. \quad \text{Also, } \left| \int_{C_\rho} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2+1)} dz \right| \leq \frac{e^{-(\ln z)t}}{\rho^{\frac{1}{3}}(\rho^2+1)} \cdot 2\pi \rho$$

$$\leq \frac{1}{\rho^{\frac{1}{3}}(\rho^2+1)} \cdot 2\pi \rho \rightarrow 0 \text{ as } \rho \rightarrow 0. \text{ Thus our contour integral becomes}$$

$$\int_0^\infty \frac{r^{\frac{1}{3}} e^{irt}}{r^2+1} dr + \int_{-\infty}^0 \frac{r^{\frac{1}{3}} e^{\frac{1}{3}i\pi} e^{-irt}}{r^2+1} dr. \quad (2.4.1)$$

By the residue theorem,  $\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) =$

$$2\pi i \operatorname{Res}_{z=i} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2+1)} = 2\pi i \operatorname{Res}_{z=i} \frac{e^{izt}}{z^{\frac{1}{3}}(z+i)(z-i)} = 2\pi i \cdot \frac{e^{izt}}{z^{\frac{1}{3}}(z+i)} \Bigg|_{z=i} = \frac{\pi e^{-t}}{i^{\frac{1}{3}}} =$$



$$\frac{\pi e^{-t}}{e^{\frac{1}{3}(\ln 1 + \frac{\pi i}{2})}} = \frac{\pi e^{-t}}{e^{\frac{\pi i}{6}}} = \frac{\pi e^{-t}}{\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}} = \frac{\pi e^{-t}}{\frac{\sqrt{3}}{2} + \frac{1}{2}i} = \frac{1}{2} \pi e^{-t}(\sqrt{3} - i). \quad \text{Use the}$$

change of variables  $r = -s$  to evaluate  $\int_{-\infty}^0 \frac{r^{\frac{1}{3}} e^{-\frac{1}{3}\pi i} e^{-irt}}{r^2 + 1} dr$ . Thus we have

$$e^{\frac{1}{3}\pi i} \int_{-\infty}^0 \frac{r^{\frac{1}{3}} e^{-irt}}{r^2 + 1} dr = -e^{\frac{1}{3}\pi i} \int_{\infty}^0 \frac{(-1)^{\frac{1}{3}} s^{\frac{1}{3}} e^{ist}}{s^2 + 1} ds = e^{\frac{1}{3}\pi i} \int_{\infty}^0 \frac{s^{\frac{1}{3}} e^{ist}}{s^2 + 1} ds =$$

$$-e^{\frac{1}{3}\pi i} \int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr. \quad \text{Back to (2.4.1), we now know that}$$

$$\int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr - e^{\frac{1}{3}\pi i} \int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr = \frac{1}{2} \pi e^{-t}(\sqrt{3} - i). \quad \text{Hence, } \int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr =$$

$$\frac{\pi e^{-t}(\sqrt{3} - i)}{2(1 - e^{\frac{1}{3}\pi i})} \text{ which simplifies to } -\pi e^{-t}i. \quad \text{Thus the inverse Fourier}$$

transform for  $t > 0$  is  $-\frac{1}{2} e^{-t}i$ .

For  $t < 0$  use  $z = -i$  the lower half plane. See Figure 2.4.2. Again,

$$\text{we have } \int_C f(z) dz = \int_{\rho}^R \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz + \int_{C_R} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz + \int_{-R}^{-\rho} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz +$$

$$\int_{C_{\rho}} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2 + 1)} dz, \text{ but in the lower half plane. Place our branch cut so that}$$

$$-\frac{3\pi}{2} < \theta < \frac{\pi}{2}. \quad \text{Note that when we use } \theta = -\pi \text{ we get the same } f(z), \text{ except}$$

for the factor  $e^{\frac{1}{3}\pi i}$ , since  $e^{-\pi i} = e^{\pi i} = -1$ . By doing similar computations as before, we can show that the contour integral becomes

$$\int_0^{\infty} \frac{r^{-\frac{1}{3}} e^{irt}}{r^2+1} dr + \int_{-\infty}^0 \frac{r^{-\frac{1}{3}} e^{\frac{1}{3}\pi i} e^{-irt}}{r^2+1} dr. \quad (*)$$

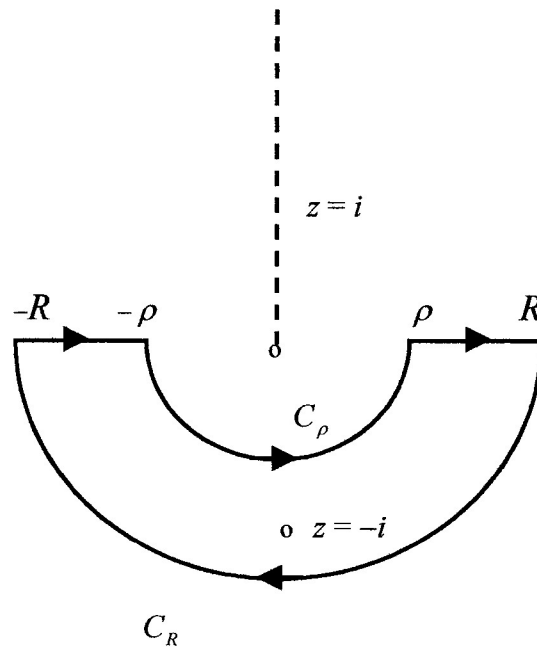


Figure 2.4.2

By the residue theorem,  $\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=-i} f(z) =$

$$2\pi i \operatorname{Res}_{z=-i} \frac{e^{izt}}{z^{\frac{1}{3}}(z^2+1)} = 2\pi i \operatorname{Res}_{z=-i} \frac{e^{izt}}{z^{\frac{1}{3}}(z+i)(z-i)} = 2\pi i \cdot \frac{e^{izt}}{z^{\frac{1}{3}}(z-i)} \Bigg|_{z=-i} = \frac{-\pi e^t}{(-i)^{\frac{1}{3}}} =$$

$$\frac{-\pi e^t}{e^{\frac{1}{3}(\ln 1 - \frac{\pi i}{2})}} = \frac{-\pi e^t}{e^{\frac{-\pi i}{6}}} = \frac{-\pi e^t}{\cos(\frac{-\pi}{6}) + i \sin(\frac{-\pi}{6})} = \frac{-\pi e^t}{\frac{\sqrt{3}}{2} - \frac{1}{2}i} = -\frac{1}{2}\pi e^t(\sqrt{3} + i). \quad \text{By}$$

applying the same change of variables as before we have

$$\int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr - e^{\frac{1}{3}\pi i} \int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr = -\frac{1}{2}\pi e^t(\sqrt{3} + i). \quad \text{Hence, } \int_0^{\infty} \frac{r^{\frac{1}{3}} e^{irt}}{r^2 + 1} dr =$$

$$\frac{-\pi e^t(\sqrt{3} + i)}{2\left(1 - e^{\frac{1}{3}\pi i}\right)} \text{ which simplifies to } -\pi e^t i. \quad \text{Thus the inverse Fourier}$$

transform for  $t < 0$  is  $-\frac{1}{2}e^t i$ . The inverse Fourier transform for the

function is  $\mathcal{F}^{-1}(F(\omega)) = -\frac{1}{2}e^{-|t|} i$ .

# Chapter 3

## Solving Differential Equations

### ■ 3.1 Using Fourier Transforms To Solve a DE

Here we will introduce the method of using Fourier transforms to solve a differential equation. Our strategy is as follows:

1. Take the Fourier transform of both sides of the DE.
2. Also take the Fourier transform of the boundary conditions.
3. Solve the transformed problem using appropriate methods.
4. Take the inverse transform, if possible, back to the original variables.

Now, we'll illustrate this method with an example involving the heat equation.

#### **Example 3.1**

Solve the PDE  $u_t = u_{xx}$  for  $-\infty < x < \infty$ ,  $t > 0$

subject to:  $u(x,0) = e^{-x^2}$  and  $u(x,t)$  bounded.

Take the Fourier transform of both sides, which transforms  $u(x,t)$  into  $U(\omega,t)$ .

$$\mathcal{F} [u_t] = \mathcal{F} [u_{xx}].$$

The variable  $t$  is fixed in the transform. Using Table 1.2 for the right hand side of the equation we get

$$U_t = -\omega^2 U.$$

Set our transformed PDE equal to zero,  $U_t + \omega^2 U = 0$ , and treat this new equation as a first order ODE of the form  $T' + aT = 0$  with solution  $T = ce^{-at}$ . So the general solution is  $U(\omega, t) = g(\omega)e^{-\omega^2 t}$ . Now take the Fourier transform of the first condition

$$\mathcal{F} [u(x, 0)] = \mathcal{F} [e^{-x^2}]$$

and look up the transform of  $e^{-x^2}$  from Table 1.1 to get

$$U(\omega, 0) = \sqrt{\pi} e^{-\frac{\omega^2}{4}}.$$

Direct substitution of  $t = 0$  into the general solution shows that  $U(\omega, 0)$  is also equal to  $g(\omega)$ . Thus

$$U(\omega, t) = \sqrt{\pi} e^{-\frac{\omega^2}{4}} e^{-\omega^2 t}.$$

The solution we want is the inverse Fourier transform of  $U(\omega, t)$ ,

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} \left( \sqrt{\pi} e^{-\frac{\omega^2}{4}} e^{-\omega^2 t} \right) \\ &= \mathcal{F}^{-1} \left( \sqrt{\pi} e^{-\frac{4t-1}{4}\omega^2} \right). \end{aligned}$$

Apply the formula  $\mathcal{F}^{-1}\left(\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}\right) = e^{-ax^2}$  and choose  $a = \frac{1}{4t+1}$  to get the final solution,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left(\sqrt{\frac{1}{4t+1}} \sqrt{\frac{\pi}{\frac{1}{4t+1}}} e^{-\frac{-4t-1}{4}\omega^2}\right) \\ &= \sqrt{\frac{1}{4t+1}} \mathcal{F}^{-1}\left(\sqrt{\frac{\pi}{\frac{1}{4t+1}}} e^{-\frac{-4t-1}{4}\omega^2}\right) \\ &= \sqrt{\frac{1}{4t+1}} e^{\frac{-x^2}{4t+1}}. \end{aligned}$$

**Example 3.2** Use the Fourier transform method to solve an ODE of the form  $y''(t) + b y'(t) + c y(t) = f(t)$  on  $-\infty < t < \infty$ , subject to the condition  $\lim_{|t| \rightarrow \infty} y(t) = 0$ .

First take the Fourier transform of both sides, assuming that  $f(t) \in \mathcal{G}(\mathbb{R})$ :

$$\mathcal{F} y''(t) + \mathcal{F} b y'(t) + \mathcal{F} c y(t) = \mathcal{F} f(t).$$

Now referring to properties in Table 1.2 we have

$$-\omega^2 Y(\omega) + b(i\omega)Y(\omega) + cY(\omega) = F(\omega),$$

which can be solved for  $Y(\omega)$ ,

$$Y(\omega) = \frac{F(\omega)}{-\omega^2 + bi\omega + c}.$$

To find the solution, apply the inverse Fourier transform,

$$y(t) = \mathcal{F}^{-1} \left( \frac{F(\omega)}{-\omega^2 + bi\omega + c} \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(\omega)}{-\omega^2 + bi\omega + c} e^{i\omega t} d\omega.$$

By convolution we can then find  $y(t)$  where the Green's function is

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{-\omega^2 + bi\omega + c} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-1}{\omega^2 - bi\omega - c} e^{i\omega t} d\omega.$$

The poles occur where  $z^2 - biz - c = 0$  so  $z = \frac{bi \pm \sqrt{-b^2 + 4c}}{2}$ . By

considering various values of  $b$  and  $c$ , there are several possible cases.

Let the discriminant  $-b^2 + 4c$  be denoted by  $D$ . Then we can summarize the outcomes in the chart below.

case:	$D > 0$	$D = 0$	$D < 0$
$\omega =$	$\frac{bi \pm d}{2},$	$\frac{b}{2}i$	$\frac{bi \pm di}{2},$
	$d = \sqrt{D}$		$d = \sqrt{-D}$
possibilities:	2 poles in $\mathcal{H}$	Double pole on the imaginary axis	2 poles in either $\mathcal{H}$ or $\mathcal{L}$
	2 poles in $\mathcal{L}$	Double pole at the origin	1 pole on imaginary axis and the other at the origin
	2 poles on the real axis		1 pole in each half plane

The author of the article at [16] did the case  $D < 0$  and  $b > 0$ .

We will calculate the solution for two other cases.

(1) Suppose  $D > 0$  and  $b > 0$ . Then the poles  $z = \frac{d}{2} + \frac{b}{2}i$ ,  $-\frac{d}{2} + \frac{b}{2}i$  are in  $\mathcal{H}$ . Note: Butkov solved a similar ODE on page 278 using this case. The poles in that example were both in  $\mathcal{L}$ , which was one of the possibilities. By Theorem 2.2(i) where  $t > 0$ ,

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-1}{\omega^2 - ib\omega - c} e^{i\omega t} d\omega &= \frac{1}{2\pi} (2\pi i \sum \{\text{residues in } \mathcal{H}\}) \\
&= i \left( \text{Res}_{z=\frac{d}{2}+\frac{b}{2}i} \frac{-e^{izt}}{z^2 - ibz - c} + \text{Res}_{z=-\frac{d}{2}+\frac{b}{2}i} \frac{-e^{izt}}{z^2 - ibz - c} \right) \\
&= i \left( \left[ \frac{-e^{izt}}{z - \left(-\frac{d}{2} + \frac{b}{2}i\right)} \right]_{z=\frac{d}{2}+\frac{b}{2}i} + \left[ \frac{-e^{izt}}{z - \left(\frac{d}{2} + \frac{b}{2}i\right)} \right]_{z=-\frac{d}{2}+\frac{b}{2}i} \right) \\
&= i \left( \frac{-e^{i\left(\frac{d}{2}+\frac{b}{2}i\right)t}}{\left(\frac{d}{2} + \frac{b}{2}i\right) - \left(-\frac{d}{2} + \frac{b}{2}i\right)} + \frac{-e^{i\left(-\frac{d}{2}+\frac{b}{2}i\right)t}}{\left(-\frac{d}{2} + \frac{b}{2}i\right) - \left(\frac{d}{2} + \frac{b}{2}i\right)} \right) \\
&= i \left( \frac{-e^{\left(\frac{b}{2}+\frac{d}{2}i\right)t}}{d} + \frac{e^{\left(\frac{b}{2}-\frac{d}{2}i\right)t}}{d} \right) \\
&= \frac{e^{\left(\frac{b}{2}-\frac{d}{2}i\right)t} - e^{\left(\frac{b}{2}+\frac{d}{2}i\right)t}}{d} i
\end{aligned}$$



$$= \frac{2e^{-\frac{bt}{2}} \sin\left(\frac{dt}{2}\right)}{d}$$

For  $t < 0$ , we consider the lower half plane. There are no singularities there, so the integral is zero. Therefore the solution is

$$u(x,t) = \begin{cases} \frac{2e^{-\frac{bt}{2}} \sin\left(\frac{dt}{2}\right)}{d}, & t > 0 \\ 0, & t < 0 \end{cases} .$$

(2) Suppose that  $D > 0$  and  $b = 0$ . Then the poles  $\omega = \pm\sqrt{c}$  are on  $\Re$ .

Using Theorem 2.4 for  $t < 0$  we have,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-1}{\omega^2 - ib\omega - c} e^{i\omega t} d\omega &= \frac{1}{2\pi} (-\pi i \sum \{\text{residues on } \Re\}) \\ &= -\frac{i}{2} \left( \operatorname{Res}_{z=-\sqrt{c}} \frac{-e^{izt}}{z^2 - ibz - c} + \operatorname{Res}_{z=\sqrt{c}} \frac{-e^{izt}}{z^2 - ibz - c} \right) \\ &= -\frac{i}{2} \left( \operatorname{Res}_{z=-\sqrt{c}} \frac{-e^{izt}}{z^2 - c} + \operatorname{Res}_{z=\sqrt{c}} \frac{-e^{izt}}{z^2 - c} \right) \quad (\text{since } b = 0) \\ &= -\frac{i}{2} \left( \left( \frac{-e^{izt}}{z - \sqrt{c}} \right)_{z=-\sqrt{c}} + \left( \frac{-e^{izt}}{z + \sqrt{c}} \right)_{z=\sqrt{c}} \right) \\ &= \frac{-i\sqrt{c}}{4c} (e^{i\sqrt{c}t} - e^{-i\sqrt{c}t}) \\ &= \frac{-i\sqrt{c}}{4c} \cdot 2i \sin(\sqrt{c}t) \\ &= \frac{\sqrt{c} \sin(\sqrt{c}t)}{2c} . \end{aligned}$$

Similarly, for  $t > 0$  use the upper half plane and get  $-\frac{\sqrt{c} \sin(\sqrt{c}t)}{2c}$ . Thus

$$\text{the solution is } u(x,t) = -\frac{\sqrt{c} \sin(\sqrt{c}|t|)}{2c}.$$

**Example 3.3** [11]

Solve  $u_{rr} + \frac{2}{r}u_r = \frac{1}{c^2}u_{tt}$  subject to the boundary condition  $u_r = -\rho v_o \delta(t)$

at radius  $r = a$ .

Take the Fourier transform of both sides of the PDE

$$\mathcal{F}[u_{rr}] + \mathcal{F}\left[\frac{2}{r}u_r\right] = \frac{1}{c^2}\mathcal{F}[u_{tt}]$$

to get

$$U_{rr}(r,\omega) + \frac{2}{r}U_r(r,\omega) = -\frac{\omega^2}{c^2}U(r,\omega).$$

It is easily shown that the general solution is  $U(r,\omega) =$

$A(\omega)\frac{e^{ikr}}{r} + B(\omega)\frac{e^{-ikr}}{r}$ , where  $k = \frac{\omega}{c}$ . The inverse Fourier integral is

$$u(r,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A(\omega)\frac{e^{ikr}}{r} + B(\omega)\frac{e^{-ikr}}{r} \right] e^{i\omega t} d\omega. \text{ Physical considerations require}$$

$A(\omega) = 0$  (see [5] p.116). Now take the Fourier transform of the

boundary condition.

$$\mathcal{F}[u_r] = \mathcal{F}[-\rho v_o \delta(t)]$$

$$U_r(a,\omega) = -\rho v_o \mathcal{F}[\delta(t)] = -\rho v_o$$

We have  $U_r = \frac{B(\omega)e^{-ikr}(-ikr-1)}{r^2}$  and for the boundary condition, where  $r$

$$= a, U_r(a, \omega) = \frac{B(\omega)e^{-ika}(-ika-1)}{a^2} = -\rho v_0. \quad \text{Thus } B(\omega) = \frac{a^2 \rho v_0 e^{ika}}{ika+1} \text{ and we}$$

have  $U(r, \omega) = \frac{a^2 \rho v_0 e^{ika}}{ika+1} \frac{e^{-ikr}}{r}$  and hence

$$u(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a^2 \rho v_0 e^{\frac{i\omega a}{c}} e^{-\frac{i\omega r}{c}}}{(i\frac{\omega}{c}a+1)r} e^{i\omega t} d\omega = \frac{a^2 \rho v_0}{2\pi r} \int_{-\infty}^{\infty} \frac{e^{i\omega \left[ t - \frac{r-a}{c} \right]}}{1 + \frac{i\omega a}{c}} d\omega.$$

Evaluate this integral by the residue theorem. We have the simple pole

$z = \frac{c}{a}i$ . For  $t - \frac{r-a}{c} > 0$ , close the contour with a semicircle in the upper

half plane. By Theorem 2.2(i) we have  $\int_{-\infty}^{\infty} \frac{e^{i\omega \left[ t - \frac{r-a}{c} \right]}}{1 + \frac{i\omega a}{c}} d\omega = 2\pi i \operatorname{Res}_{z = \frac{c}{a}i} \frac{e^{iz \left[ t - \frac{r-a}{c} \right]}}{\frac{iza}{c} + 1}$

$$= \frac{2\pi i}{\left(\frac{ia}{c}\right)} \frac{e^{iz \left[ t - \frac{r-a}{c} \right]}}{z - \frac{c}{a}i} \Big|_{z = \frac{c}{a}i} = \frac{2\pi c}{a} e^{\frac{c}{a} \left[ t - \frac{r-a}{c} \right]}. \quad \text{Thus } u(r, t) = \frac{ac\rho v_0}{r} e^{\frac{c}{a} \left[ t - \frac{r-a}{c} \right]}.$$

For  $t - \frac{r-a}{c} < 0$  close the contour with a semicircle in the lower half

plane. Since the function is analytic at all points interior to and on this closed contour  $C$ , then by the Cauchy-Goursat theorem from complex

variables we have  $\oint_C \frac{e^{iz \left[ t - \frac{r-a}{c} \right]}}{\frac{iza}{c} + 1} dz = 0$ . Hence, our combined solution using

the Heaviside step function is  $u(r, t) = \frac{ac\rho v_0}{r} e^{\frac{c}{a} \left[ t - \frac{r-a}{c} \right]} h\left(t - \frac{r-a}{c}\right)$ .

**Example 3.4** [7]

Solve  $u_{xx} + u_{yy} - u = 0$  with  $-\infty < x < \infty$ ,  $0 < y < 1$  and subject to the boundary conditions  $u_y(x,0) = 0$  and  $u(x,1) = e^{-x} h(x)$ . Duffy [5] presents this exercise on page 120 and provides an outline to assist in finding the solution.

Take the Fourier transform of the PDE.

$$\mathcal{F}[u_{xx}] + \mathcal{F}[u_{yy}] - \mathcal{F}[u] = 0$$

$$-\omega^2 U + U_{yy} - U = 0$$

Now take the Fourier transform of the boundary conditions.

$$\mathcal{F}[u_y(x,0)] = \mathcal{F}[0] \quad \text{and} \quad \mathcal{F}[u(x,1)] = \mathcal{F}[e^{-x} h(x)]$$

$$U_y(\omega,0) = 0 \qquad U(\omega,1) = \frac{1}{1 + \omega i}$$

The characteristic equation for the differential equation is  $r^2 - \omega^2 - 1 = 0$

with solution  $r = \pm\sqrt{\omega^2 + 1}$ . Let  $m = \sqrt{\omega^2 + 1}$ . The general solution is

$$U(\omega, y) = A(\omega) \cosh my + B(\omega) \sinh my.$$

The partial with respect to  $y$  is  $U_y(\omega, y) = mA(\omega) \sinh my + mB(\omega) \cosh my$ .

So  $U_y(\omega,0) = mB(\omega) = 0$  which gives  $B(\omega) = 0$ . The general solution simplifies to  $U(\omega, y) = A(\omega) \cosh my$ . Now apply the second boundary

condition,  $U(\omega, 1) = A(\omega) \cosh m = \frac{1}{1 + \omega i}$  which gives  $A(\omega) = \frac{1}{(1 + \omega i) \cosh m}$ .

Hence, the solution is  $U(\omega, y) = \frac{1}{1 + \omega i} \frac{\cosh(y\sqrt{\omega^2 + 1})}{\cosh(\sqrt{\omega^2 + 1})}$ . The inverse

Fourier transform is  $\frac{1}{2\pi} \int_{-\infty}^{\infty} U(\omega, y) e^{i\omega x} d\omega$ . We now determine the poles to

use in applying the residue theorem. One of the poles occurs at  $z = i$ .

Since the hyperbolic cosine function is zero at the points  $\left(\frac{\pi}{2} + n\pi\right)i$

where  $n$  is an integer, we can calculate the other singularities using

algebra. The other simple poles occur at  $z = \pm \frac{i\sqrt{\pi^2(2n+1)^2 + 4}}{2}$ . Let

$\sigma_n = \sqrt{\pi^2(2n+1)^2 + 4}$ . (Following the proof of Theorem 2.2, that theorem

is valid for a countable number of singularities, if the limits exist.

Assuming that  $U(\omega, y) \in G(\mathfrak{R})$ , the limits do exist.)

Now apply the residue theorem. For  $x > 0$  use the upper half plane. Calculate the easier residue at  $z = i$  to get

$$\operatorname{Res}_{z=i} \frac{\cosh(y\sqrt{z^2 + 1}) e^{izx}}{i(z-i) \cosh(\sqrt{z^2 + 1})} = \frac{\cosh(y\sqrt{z^2 + 1}) e^{izx}}{i \cosh(\sqrt{z^2 + 1})} \Bigg|_{z=i} = -ie^{-x}.$$

Now calculate the sum of the other residues in the upper half plane.

$$\sum_{n=0}^{\infty} \operatorname{Res}_{z=\frac{i\sigma_n}{2}} U(z, y) e^{izx} = \sum_{n=0}^{\infty} \frac{\cosh(y\sqrt{z^2 + 1}) e^{izx}}{i(z-i) \cosh(\sqrt{z^2 + 1})} \Bigg|_{z=\frac{i\sigma_n}{2}}$$

Take the limit as  $z \rightarrow \frac{i\sigma_n}{2}$ . We observe that  $\lim_{z \rightarrow \frac{i\sigma_n}{2}} \frac{z - \frac{i\sigma_n}{2}}{\cosh(\sqrt{z^2 + 1})}$  can be

evaluated using L'Hospital's rule:  $\lim_{z \rightarrow \frac{i\sigma_n}{2}} \frac{\sqrt{z^2 + 1}}{\sinh(\sqrt{z^2 + 1}) z} = \frac{\sqrt{-\frac{\sigma_n^2}{4} + 1}}{\sinh \sqrt{-\frac{\sigma_n^2}{4} + 1} \frac{\sigma_n i}{2}}$

$= -\frac{\pi(2n+1)}{\sinh(\pi(\frac{2n+1}{2})i) \sigma_n} = \frac{\pi(2n+1)i}{(-1)^n \sigma_n}$ . The summation becomes

$$\sum_{n=0}^{\infty} \frac{\cosh(\frac{y\pi(2n+1)}{2}i)}{i(\frac{\sigma_n}{2}i - i)} \frac{\pi(2n+1)i}{(-1)^n \sigma_n} e^{i(\frac{\sigma_n}{2}i)x}. \text{ Applying the identity } \cosh(iz) = \cos z$$

and the fact that  $(-1)^n = \frac{1}{(-1)^n}$  we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi(2n+1)i}{\sigma_n(2 - \sigma_n)} e^{-\frac{\sigma_n}{2}x} \cos\left(\frac{(2n+1)\pi y}{2}\right).$$

So by Theorem 2.2(i) the inverse Fourier transform of  $U(\omega, y)$  for  $x > 0$  is

$$\begin{aligned} & \frac{1}{2\pi} \cdot 2\pi i \left( -ie^{-x} + \sum_{n=0}^{\infty} \frac{(-1)^n \pi(2n+1)i}{\sigma_n(2 - \sigma_n)} e^{-\frac{\sigma_n}{2}x} \cos\left(\frac{(2n+1)\pi y}{2}\right) \right) \\ &= e^{-x} + \sum_{n=0}^{\infty} \frac{(-1)^n \pi(2n+1)}{\sigma_n(\sigma_n - 2)} e^{-\frac{\sigma_n}{2}x} \cos\left(\frac{(2n+1)\pi y}{2}\right). \end{aligned}$$

For  $x < 0$ , use the infinite poles in the lower half plane. The calculation of the inverse Fourier transform is analogous except there is no "extra"

pole at  $z = i$ . The combined solution where  $\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$  is

$$u(x, y) = e^{-x} h(x) + 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{\sigma_n (\text{sgn}(x) \sigma_n - 2)} e^{-\frac{\sigma_n}{2}|x|} \cos\left(\frac{(2n+1)\pi y}{2}\right).$$

**Example 3.5** [3, p.283] Solve the ODE  $EI \frac{d^4 y}{dx^4} = q(x) - c y(x)$ ,  $-\infty < x < \infty$ ,

where  $q(x)$  is the external force per unit length on a beam resting on an elastic foundation,  $y(x)$  is the displacement, and  $c$ ,  $E$ , and  $I$  are constants.

Take the Fourier transform of both sides and solve.

$$\mathcal{F}\left[EI \frac{d^4 y}{dx^4}\right] = \mathcal{F}[q(x) - c y(x)]$$

$$EI \omega^4 Y(\omega) = Q(\omega) - c Y(\omega)$$

$$Y(\omega) = \frac{Q(\omega)}{EI \omega^4 + c}$$

Now find the inverse Fourier transform,

$$\mathcal{F}^{-1}\left(\frac{Q(\omega)}{EI \omega^4 + c}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{Q(\omega)}{EI \omega^4 + c} e^{i\omega x} d\omega.$$

Let  $q(x) = P \delta(x)$ , where  $P$  is a constant. Then  $Q(\omega) = \mathcal{F}[P \delta(x)] = P$ .

We can pull the constant outside of the integral,  $\frac{P}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{EI \omega^4 + c} d\omega$ .

Rename  $\frac{EI}{c}$  as  $\alpha^4$ . Then  $\mathcal{F}^{-1}\left(\frac{Q(\omega)}{EI \omega^4 + c}\right) = \frac{P}{2\pi c} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\frac{EI}{c} \omega^4 + 1} d\omega =$

$\frac{P}{2\pi c} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\alpha^4 \omega^4 + 1} d\omega$ . Make the change of variables  $\omega = \frac{s}{\alpha}$ . Then

$d\omega = \frac{1}{\alpha} ds$  and  $s = \alpha \omega$ . Our integral becomes  $\frac{P}{2\pi \alpha c} \int_{-\infty}^{\infty} \frac{e^{\frac{isx}{\alpha}}}{s^4 + 1} ds$ .

The poles occur where  $z^4 + 1 = 0$  or at the points  $z_1 = e^{\frac{\pi i}{4}}$ ,  $z_2 = e^{\frac{3\pi i}{4}}$ ,

$z_3 = e^{\frac{-3\pi i}{4}}$ ,  $z_4 = e^{\frac{-\pi i}{4}}$  with rectangular coordinates  $z_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ ,

$z_2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ ,  $z_3 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ ,  $z_4 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ . For  $x < 0$  use  $z_3 = e^{\frac{-3\pi i}{4}}$  and

$z_4 = e^{\frac{-\pi i}{4}}$  in the lower half plane. By Theorem 2.2 (ii), the integral

becomes

$$\begin{aligned}
& -2\pi i \left( \operatorname{Res}_{z=e^{\frac{3\pi i}{4}}} \frac{e^{\frac{izx}{\alpha}}}{z^4 + 1} + \operatorname{Res}_{z=e^{\frac{\pi i}{4}}} \frac{e^{\frac{izx}{\alpha}}}{z^4 + 1} \right) \\
&= -2\pi i \left( \left[ \frac{e^{\frac{izx}{\alpha}}}{z^4 + 1} (z - e^{\frac{-3\pi i}{4}}) \right]_{z=e^{\frac{3\pi i}{4}}} + \left[ \frac{e^{\frac{izx}{\alpha}}}{z^4 + 1} (z - e^{\frac{-\pi i}{4}}) \right]_{z=e^{\frac{\pi i}{4}}} \right) \\
&= -2\pi i \left( \left[ \frac{e^{\frac{izx}{\alpha}}}{(z - e^{\frac{\pi i}{4}})(z - e^{\frac{3\pi i}{4}})(z - e^{\frac{-\pi i}{4}})} \right]_{z=e^{\frac{3\pi i}{4}}} + \left[ \frac{e^{\frac{izx}{\alpha}}}{(z - e^{\frac{\pi i}{4}})(z - e^{\frac{3\pi i}{4}})(z - e^{\frac{-\pi i}{4}})} \right]_{z=e^{\frac{\pi i}{4}}} \right) \\
&= -2\pi i \left( \frac{e^{\frac{\sqrt{2}x}{2\alpha}}}{-2\sqrt{2}i + 2\sqrt{2}} \left( \cos \frac{\sqrt{2}x}{2\alpha} - i \sin \frac{\sqrt{2}x}{2\alpha} \right) + \frac{e^{\frac{\sqrt{2}x}{2\alpha}}}{-2\sqrt{2}i - 2\sqrt{2}} \left( \cos \frac{\sqrt{2}x}{2\alpha} + i \sin \frac{\sqrt{2}x}{2\alpha} \right) \right) \\
&= -2\pi i \left( \frac{(-2\sqrt{2}i - 2\sqrt{2})(\cos \frac{\sqrt{2}x}{2\alpha} - i \sin \frac{\sqrt{2}x}{2\alpha}) + (-2\sqrt{2}i + 2\sqrt{2})(\cos \frac{\sqrt{2}x}{2\alpha} + i \sin \frac{\sqrt{2}x}{2\alpha})}{-16} \right) e^{\frac{\sqrt{2}x}{2\alpha}} \\
&= \frac{\pi i}{8} \left( -4\sqrt{2}i \cos \frac{\sqrt{2}x}{2\alpha} + 4\sqrt{2}i \sin \frac{\sqrt{2}x}{2\alpha} \right) e^{\frac{\sqrt{2}x}{2\alpha}}
\end{aligned}$$



$= \frac{\sqrt{2}\pi}{2} \left( \cos \frac{\sqrt{2}x}{2\alpha} - \sin \frac{\sqrt{2}x}{2\alpha} \right) e^{\frac{\sqrt{2}x}{2\alpha}}$ . Multiplying this residue by  $\frac{P}{2\pi\alpha c}$  we get the

solution  $\frac{P\sqrt{2}}{4\alpha c} \left( \cos \frac{\sqrt{2}x}{2\alpha} - \sin \frac{\sqrt{2}x}{2\alpha} \right) e^{\frac{\sqrt{2}x}{2\alpha}}$ .

For  $x > 0$  use  $z_1 = e^{\frac{\pi i}{4}}$ ,  $z_2 = e^{\frac{3\pi i}{4}}$  the upper half plane. The calculation of the residues is similar and the integral is

$\frac{\sqrt{2}\pi}{2} \left( \cos \frac{\sqrt{2}x}{2\alpha} + \sin \frac{\sqrt{2}x}{2\alpha} \right) e^{-\frac{\sqrt{2}x}{2\alpha}}$ . The solution is  $\frac{P\sqrt{2}}{4\alpha c} \left( \cos \frac{\sqrt{2}x}{2\alpha} + \sin \frac{\sqrt{2}x}{2\alpha} \right) e^{-\frac{\sqrt{2}x}{2\alpha}}$ , for

this case. Using absolute values, we can combine the solutions from

both cases to give the final solution  $y(x) = \frac{P\sqrt{2}}{4\alpha c} \left( \cos \frac{\sqrt{2}|x|}{2\alpha} + \sin \frac{\sqrt{2}|x|}{2\alpha} \right) e^{-\frac{\sqrt{2}|x|}{2\alpha}}$ .

# Chapter 4

## Fourier and Laplace Joint Transforms

### ■ 4.1 The Joint Transform Method

We know that Fourier and Laplace transforms can be used separately to solve partial differential equations. Interestingly, they can be applied together for the same purpose. This undertaking requires a double inversion, which can be a difficult task. Techniques like the Cagnard-de Hoop method have been successfully used in finding the joint inverse.

In the Fourier-Laplace joint transform, we apply a Fourier transform to eliminate the spatial dimension and the Laplace transform to remove the time dependence. The resulting joint transform can then be solved using algebra or ordinary differential equation methods. Then we are faced with the double inversion. The order in which we do them depends on the character of the joint transform. The pattern we encounter in this chapter is: Fourier

transform, Laplace transform, inverse Laplace transform, inverse Fourier transform.

**Example 4.1** [3, p.606]

Solve  $u_{xx} - \frac{1}{c^2}u_{tt} = \delta(x - \xi)\delta(t - \tau)$ , where  $-\infty < x < \infty$  and  $t > 0$

subject to  $u(x,0) = 0$ ,  $u_t(x,0) = 0$ , and  $u$  approaches zero as  $x \rightarrow \pm\infty$ .

First, we will take the Fourier transform with respect to  $x$ .

$$\mathcal{F}[u_{xx}] - \frac{1}{c^2}\mathcal{F}[u_{tt}] = \delta(t - \tau)\mathcal{F}[\delta(x - \xi)]$$

$$-\omega^2 U(\omega, t) - \frac{1}{c^2}U_{tt}(\omega, t) = \delta(t - \tau)e^{-i\omega\xi}$$

The Fourier transform of the conditions are  $\mathcal{F}[u(x,0)] = \mathcal{F}[0]$  and

$\mathcal{F}[u_t(x,0)] = \mathcal{F}[0]$ , that is,  $U(\omega,0) = 0$  and  $U_t(\omega,0) = 0$ .

Now apply the Laplace transform to  $t$ ,

$$-\omega^2 \mathcal{L}[U](\omega, s) - \frac{1}{c^2}(s^2 \mathcal{L}[U](\omega, s) - sU(\omega,0) - U_t(\omega,0)) = e^{-\tau s} e^{-i\omega\xi}.$$

Using the initial conditions we have

$$-\omega^2 \mathcal{L}[U](\omega, s) - \frac{s^2}{c^2}\mathcal{L}[U](\omega, s) = e^{-\tau s} e^{-i\omega\xi}.$$

Thus the joint transform of this PDE is

$$\mathcal{L}[U](\omega, s) = \frac{-c^2 e^{-\tau s} e^{-i\omega\xi}}{c^2 \omega^2 + s^2}.$$

Now take the inverse Laplace transform  $U(\omega, t) = \mathcal{L}^{-1} \left( \frac{-c^2 e^{-\tau s} e^{-i\omega \xi}}{c^2 \omega^2 + s^2} \right)$

$= -c^2 e^{-i\omega \xi} \mathcal{L}^{-1} \left( \frac{e^{-\tau s}}{c^2 \omega^2 + s^2} \right)$ . By combining rules 23 and 10 from the

Laplace transform tables in [9], the inverse Laplace transform is

$$U(\omega, t) = \frac{-c^2 e^{-i\omega \xi}}{c \omega} \sin(c \omega (t - \tau)) h(t - \tau).$$

Take the inverse Fourier transform

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} \left( \frac{-c e^{-i\omega \xi}}{\omega} \sin(c \omega (t - \tau)) h(t - \tau) \right) \\ &= -c h(t - \tau) \mathcal{F}^{-1} \left( \frac{e^{-i\omega \xi}}{\omega} \sin(c \omega (t - \tau)) \right) \\ &= -c h(t - \tau) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega \xi}}{\omega} \sin(c \omega (t - \tau)) e^{i\omega x} d\omega. \end{aligned}$$

We know from [3, p.265] that  $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \sin \frac{ak}{k} e^{-ikx} dk = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ .

Thus  $u(x, t) = \frac{-c h(t - \tau)}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin(c \omega (t - \tau))}{\omega} e^{-i\omega(\xi - x)} d\omega =$

$\begin{cases} \frac{-c h(t - \tau)}{2}, & |\xi - x| < c(t - \tau) \\ 0, & |\xi - x| > c(t - \tau) \end{cases}$ . This solution can be rewritten as follows.

For the case when  $t < \tau$ , the function is 0 (since  $h(t - \tau) = 0$ ). When

$t > \tau$ , we have  $\frac{-c}{2}$  for  $|\xi - x| < c(t - \tau)$ , which can be expressed as

$\frac{-c}{2}(h(\xi - x + c(t - \tau)) - h(\xi - x - c(t - \tau)))$ . To summarize, the solution is

$$u(x,t) = \begin{cases} \frac{-c}{2}(h(\xi - x + c(t - \tau)) - h(\xi - x - c(t - \tau))), & t > \tau \\ 0, & t < \tau \end{cases}.$$

**Example 4.2** [14, p.197]

Solve  $EI u_{xxxx} + ku + m u_{tt} = P_0 \delta(x) \delta(t)$  with  $-\infty < x < \infty$ ,  $t > 0$  subject to the initial conditions  $u(x,0) = 0$  and  $u_t(x,0) = 0$  and  $u$  approaches zero as  $x \rightarrow \pm\infty$ .

The steps required to apply the Fourier-Laplace joint transform of this PDE are outlined in [5, p.276]. First take the Fourier transform of both sides,  $\mathcal{F}[EI u_{xxxx}] + \mathcal{F}[ku] + \mathcal{F}[m u_{tt}] = \mathcal{F}[P_0 \delta(x) \delta(t)]$ . Pulling out the constants we have  $EI \mathcal{F}[u_{xxxx}] + k \mathcal{F}[u] + m \mathcal{F}[u_{tt}] = P_0 \delta(t) \mathcal{F}[\delta(x)]$ , which gives us

$$EI \omega^4 U(\omega, t) + kU(\omega, t) + m U_{tt}(\omega, t) = P_0 \delta(t). \quad (4.2.1)$$

The Fourier transform of the initial conditions become  $U(\omega, 0) = 0$  and  $U_t(\omega, 0) = 0$ .

Now, apply the Laplace transform to (4.2.1),  $EI\omega^4 \mathcal{L}[U](\omega, s) + k \mathcal{L}[U](\omega, s) + m(s^2 \mathcal{L}[U](\omega, s) - sU(\omega, 0) - U_t(\omega, 0)) = P_0 \mathcal{L}[\delta(t)]$ . By applying the transformed initial conditions and factoring we get  $(EI\omega^4 + k + ms^2) \mathcal{L}[U](\omega, s) = P_0$ . Thus the joint transform of our PDE

$$\text{is } \mathcal{L}[U](\omega, s) = \frac{P_0}{EI\omega^4 + k + ms^2}. \quad (4.2.2)$$

Now we will apply the inverse Laplace transform to (4.2.2),

$$U(\omega, t) = \mathcal{L}^{-1}\left(\frac{P_0}{EI\omega^4 + k + ms^2}\right) = P_0 \mathcal{L}^{-1}\left(\frac{1}{EI\omega^4 + k + ms^2}\right). \text{ Let } a^2 = \frac{EI}{m}.$$

Thus  $U(\omega, t) = \frac{P_0}{m} \mathcal{L}^{-1}\left(\frac{1}{a^2\omega^4 + \frac{k}{m} + s^2}\right)$ . We shall use the well-known

formula  $\mathcal{L}^{-1}\left\{\frac{b}{s^2 + b^2}\right\} = \sin bt$ . Let  $b = \sqrt{a^2\omega^4 + \frac{k}{m}}$ . We then have

$$\frac{1}{\sqrt{a^2\omega^4 + \frac{k}{m}}} \frac{P_0}{m} \mathcal{L}^{-1}\left(\frac{\sqrt{a^2\omega^4 + \frac{k}{m}}}{s^2 + a^2\omega^4 + \frac{k}{m}}\right), \text{ which can be written as}$$

$$\frac{1}{\sqrt{a^2\omega^4 + \frac{k}{m}}} \frac{P_0}{m} \sin \sqrt{a^2\omega^4 + \frac{k}{m}} t. \quad (4.2.3)$$

The last step is to take the inverse Fourier transform of (4.2.3).

$$\begin{aligned} u(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{P_0}{m} \frac{\sin\left(t\sqrt{a^2\omega^4 + \frac{k}{m}}\right)}{\sqrt{a^2\omega^4 + \frac{k}{m}}} e^{i\omega x} d\omega \\ &= \frac{P_0}{2\pi m} \left[ \int_{-\infty}^{\infty} \frac{\sin\left(t\sqrt{a^2\omega^4 + \frac{k}{m}}\right)}{\sqrt{a^2\omega^4 + \frac{k}{m}}} \cos(\omega x) d\omega + i \int_{-\infty}^{\infty} \frac{\sin\left(t\sqrt{a^2\omega^4 + \frac{k}{m}}\right)}{\sqrt{a^2\omega^4 + \frac{k}{m}}} \sin(\omega x) d\omega \right] \end{aligned}$$

$$= \frac{P_0}{\pi m} \int_0^{\infty} \frac{\sin\left(t\sqrt{a^2\omega^4 + \frac{k}{m}}\right)}{\sqrt{a^2\omega^4 + \frac{k}{m}}} \cos(\omega x) d\omega.$$

Notice that since the first integrand was even and the second integrand was odd, we could reduce to the last integral. The original paper goes through seven additional steps to rewrite the solution with physical considerations. However, their solution is still in integral form. For our purposes, we express the solution as

$$u(x,t) = \frac{P_0}{\pi m} \int_0^{\infty} \frac{\sin\left(t\sqrt{a^2\omega^4 + \frac{k}{m}}\right)}{\sqrt{a^2\omega^4 + \frac{k}{m}}} \cos(\omega x) d\omega.$$

For another example of the Fourier-Laplace joint transform, the reader can consult the journal article [13]. In this paper, the authors apply the Fourier transform followed by the Laplace transform to Maxwell's equations, which govern radiation and propagation of electromagnetic fields. Using this method, integral representations for the electric field and the magnetic induction can be found.

## ■ 4.2 The Double Fourier Transform

There are several other joint transforms that we could explore. But since the emphasis of this paper is the Fourier transform we ask the question can the Fourier transform be used jointly with itself in

solving a PDE? The answer is yes. In certain situations, after taking the Fourier transform once and examining the result, it might be appropriate to use the Fourier transform a second time. This method is sometimes called a multiple Fourier transform [15, p.329] or the Fourier-Fourier transform. Since this method can be used in the same way that a joint transform is used but uses only one kind of transform, we will call it the double Fourier transform.

**Example 4.3**

Solve  $u_t - u_{xx} - u_{yy} = 0$  with  $t > 0$ ,  $-\infty < x < \infty$ ,  $-\infty < y < \infty$  and subject to  $u(x, y, 0) = f(x, y)$ . Assume  $f \in G(\mathbb{R})$ .

Take the Fourier transform of both sides with respect to  $x$ ,

$\mathcal{F}[u_t] - \mathcal{F}[u_{xx}] - \mathcal{F}[u_{yy}] = \mathcal{F}[0]$ , which is  $U_t + \omega^2 U - U_{yy} = 0$ . The

Fourier transform of the initial condition is  $\mathcal{F}[u(x, y, 0)] = \mathcal{F}[f(x, y)]$ ,

or  $U(\omega, y, 0) = F(\omega, y)$ .

Now apply the Fourier transform again, this time with respect to the variable  $y$ ,  $\mathcal{F}[U_t] + \mathcal{F}[\omega^2 U] - \mathcal{F}[U_{yy}] = \mathcal{F}[0]$ . So we have

$\hat{U}_t + \omega^2 \hat{U} + s^2 \hat{U} = 0$ , which can be expressed as  $\hat{U}_t + (\omega^2 + s^2) \hat{U} = 0$ .

The Fourier transform of the initial condition is now  $\mathcal{F}[U(\omega, y, 0)] =$



$\mathcal{F} [F(\omega, y)]$ , or  $\hat{U}(\omega, s, 0) = \hat{F}(\omega, s)$ . This ODE is an initial-value problem with solution  $\hat{U}(\omega, s, t) = \hat{F}(\omega, s) e^{-(\omega^2 + s^2)t}$ .

Take the inverse Fourier transform with respect to  $s$  and apply convolution (see 1.4.2),

$$\begin{aligned} \mathcal{F}^{-1}[\hat{U}(\omega, s, t)] &= \mathcal{F}^{-1}[\hat{F}(\omega, s) e^{-(\omega^2 + s^2)t}] \\ U(\omega, y, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\omega, \xi) e^{-\omega^2 t} e^{-\frac{(y-\xi)^2}{4t}} d\xi. \end{aligned}$$

Lastly, take the inverse Fourier transform with respect to  $\omega$ ,

$$\begin{aligned} \mathcal{F}^{-1}[U(\omega, y, t)] &= \mathcal{F}^{-1}\left[\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\omega, \xi) e^{-\omega^2 t} e^{-\frac{(y-\xi)^2}{4t}} d\xi\right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\omega, \xi) e^{-\omega^2 t} e^{-\frac{(y-\xi)^2}{4t}} d\xi\right] e^{i\omega x} d\xi d\omega. \quad (4.3.1) \end{aligned}$$

Note that  $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega, \xi) e^{-\omega^2 t} e^{i\omega x} d\omega = \mathcal{F}^{-1}(F(\omega, \xi) e^{-\omega^2 t})$  by definition of

the inverse Fourier transform. Thus by applying convolution again

we have  $\int_{-\infty}^{\infty} f(\eta, \xi) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-\eta)^2}{4t}} d\eta$ . Therefore, (4.3.1) can be rewritten

as  $\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\eta, \xi) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-\eta)^2}{4t}} d\eta \right] e^{-\frac{(y-\xi)^2}{4t}} d\xi$  to give us the final

solution,

$$u(x, y, t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta, \xi) e^{-\frac{(x-\eta)^2 - (y-\xi)^2}{4t}} d\eta d\xi.$$

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