A Scale Variational Principle of Herglotz

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Abstract

The Herglotz problem is a generalization of the fundamental problem of the calculus of variations. In this paper, we consider a class of non-differentiable functions, where the dynamics is described by a scale derivative. Necessary conditions are derived to determine the optimal solution for the problem. Some other problems are considered, like transversality conditions, the multi-dimensional case, higher-order derivatives and for several independent variables.

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1 Introduction

The calculus of variations deals with optimization of a given functional, whose algebraic expression is the integral of a given function, that depends on time, space and the velocity of the trajectory:

$$x \mapsto \int_a^b L(t, x(t), \dot{x}(t))\, dt.$$

The variational principle of Herglotz can be seen as an extension of such classical theories, but instead of an integral, we have the functional as a solution of a differential equation (see [9, 10]):

$$\begin{cases} \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad \text{with } t \in [a, b], \\ z(a) = z_a. \end{cases}$$

Without the dependence of z, we can convert this problem into a calculus of variations problem. In fact, integrating the differential equation

$$\dot{z}(t) = L(t, x(t), \dot{x}(t))$$

from a to b, we obtain

$$z(b) = \int_a^b \left[L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] dt.$$

Recently, more advances were made namely proving Noether's type theorems for the variational principle of Herglotz (see e.g. [5, 6, 7, 8, 9, 12]). The aim of this paper is to consider the Herglotz problem, but the trajectories $x(\cdot)$ may be non-differentiable functions. We believe that this situation may model more efficiently certain physical problems, like fractals.

The organization of the paper is the following. In Section 2 we define what is a scale derivative, following the concept as presented in [2], and we present some of its main properties, like the algebraic rules, integration by parts formula, etc. In Section 3 we prove our new results. After presenting the Herglotz scale problem, we prove a necessary condition that every extremizer must fulfill. Some generalizations of the main result are also presented to complete the study.

2 Scale calculus

We review some definitions and the main results from [2] that we will need. For more on the subject, see references [1, 2, 3].

From now on, let α, β, h be reals in $]0,1[$ with $\alpha+\beta>1$ and $h \ll 1$, and consider $I := [a-h, b+h]$.

Definition 1. *Let $f : I \to \mathbb{R}$ be a function. The delta derivative of f at t is defined by*

$$\Delta_h[f](t) := \frac{f(t+h)-f(t)}{h}, \quad \text{for} \quad t \in [a-h, b],$$

and the nabla derivative of f at t is defined by

$$\nabla_h[f](t) := \frac{f(t)-f(t-h)}{h}, \quad \text{for} \quad t \in [a, b+h].$$

If f is differentiable, then

$$\lim_{h\to 0} \Delta_h[f](t) = \lim_{h\to 0} \nabla_h[f](t) = f'(t).$$

These two operators can be combined into a single one, where the real part is the mean value of such operators, and the complex part measures the difference between them.

Definition 2. *The h-scale derivative of f at t is given by*

$$\frac{\Box_h f}{\Box t}(t) = \frac{1}{2}\left[(\Delta_h[f](t) + \nabla_h[f](t)) + i\,(\Delta_h[f](t) - \nabla_h[f](t))\right], \quad \text{for} \quad t \in [a,b]. \tag{1}$$

For complex valued functions f, such definition is extended by

$$\frac{\Box_h f}{\Box t}(t) = \frac{\Box_h \mathrm{Re} f}{\Box t}(t) + i\frac{\Box_h \mathrm{Im} f}{\Box t}(t).$$

We now explain how to drop the dependence on the parameter h in the definition of the scale derivative. First, consider the set $C^0_{conv}([a,b]\times]0,1[,\mathbb{C})$ of the functions $g \in C^0([a,b]\times]0,1[,\mathbb{C})$ for which the limit

$$\lim_{h\to 0} g(t,h)$$

exists for all $t \in [a,b]$, and let E be a complementary space of $C^0_{conv}([a,b]\times]0,1[,\mathbb{C})$ in $C^0([a,b]\times]0,1[,\mathbb{C})$.

Define π the projection of $C^0_{conv}([a,b]\times]0,1[,\mathbb{C}) \oplus E$ onto $C^0_{conv}([a,b]\times]0,1[,\mathbb{C})$,

$$\begin{array}{rccc} \pi : & C^0_{conv}([a,b]\times]0,1[,\mathbb{C}) \oplus E & \to & C^0_{conv}([a,b]\times]0,1[,\mathbb{C}) \\ & g := g_{conv} + g_E & \mapsto & \pi(g) = g_{conv}. \end{array}$$

Using these definitions, we arrive at the main concept of [2].

Definition 3. *The scale derivative of $f \in C^0(I,\mathbb{C})$, denoted by $\frac{\Box f}{\Box t}$, is defined by*

$$\frac{\Box f}{\Box t}(t) := \left\langle \frac{\Box_h f}{\Box t} \right\rangle (t), \quad t \in [a,b], \tag{2}$$

where

$$\left\langle \frac{\Box_h f}{\Box t} \right\rangle (t) := \lim_{h\to 0} \pi\left(\frac{\Box_h f}{\Box t}(t)\right).$$

Definition 4. *Given $f : I^n = [a-nh, b+nh] \to \mathbb{C}$, define the higher-order scale derivative of f by*

$$\frac{\Box^n f}{\Box t^n}(t) = \frac{\Box}{\Box t}\left(\frac{\Box^{n-1} f}{\Box t^{n-1}}\right)(t), \quad t \in [a,b],$$

where $\frac{\Box f^1}{\Box t^1} := \frac{\Box f}{\Box t}$ and $\frac{\Box f^0}{\Box t^0} := f$.

We will adopt the notation $\Box^n f(t)$ instead of $\frac{\Box^n f}{\Box t^n}(t)$ when there is no danger of confusion.
Scale partial derivatives are also considered here. They are defined as in the standard case.

Definition 5. *Let $f : \prod_{i=1}^{n}[a_i - h, b_i + h] \to \mathbb{R}$ be a function. Define, for each $i \in \{1, \dots, n\}$,*

$$\Delta_h^i[f](t_1, \dots, t_n) := \frac{f(t_1, \dots, t_{i-1}, t_i + h, t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n)}{h},$$

for $t_i \in [a_i - h, b_i]$ and for $t_j \in [a_j - h, b_j + h]$ if $j \neq i$, and

$$\nabla_h^i[f](t_1, \dots, t_n) := \frac{f(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, t_i - h, t_{i+1}, \dots, t_n)}{h},$$

for $t_i \in [a_i, b_i + h]$ and for $t_j \in [a_j - h, b_j + h]$, if $j \neq i$. The h-scale partial derivative of f with respect to the $i-th$ coordinate is given by

$$\frac{\Box_h f}{\Box t_i}(t_1, \dots, t_n) = \frac{1}{2}\left[\left(\Delta_h^i[f] + \nabla_h^i[f]\right) + i\left(\Delta_h^i[f] - \nabla_h^i[f]\right)\right],$$

for $t_i \in [a_i, b_i]$.

The definition of partial scale derivatives $\Box f / \Box t_i$ is clear.
In what follows, we will denote

$$C_\Box^n([a,b], \mathbb{K}) := \{f \in C^0(I^n, \mathbb{K}) \mid \frac{\Box^k f}{\Box t^k} \in C^0(I^{n-k}, \mathbb{C}), k = 1, 2, \dots, n\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

Definition 6. *Let $f \in C^0(I, \mathbb{C})$ and $\alpha \in]0,1[$. We say that f is Hölderian of Hölder exponent α if there exists a constant $C > 0$ such that, for all $s, t \in I$,*

$$|f(t) - f(s)| \leq C|t - s|^\alpha,$$

and we write $f \in H^\alpha(I, \mathbb{C})$, or simply $f \in H^\alpha$ when there is no danger of mislead.

We say that $f(t_1, \dots, t_n) \in H^\alpha$ if $f(t_1, \dots, t_{i-1}, \cdot, t_{i+1}, \dots, t_n) \in H^\alpha$, for all $i \in \{1, \dots, n\}$ and for all $t_j \in [a_j, b_j]$, $j \neq i$.

Theorem 1. *For all $f \in H^\alpha$ and $g \in H^\beta$, we have*

$$\frac{\Box(f.g)}{\Box t}(t) = \frac{\Box f}{\Box t}(t).g(t) + f(t).\frac{\Box g}{\Box t}(t), \quad t \in [a,b].$$

Theorem 2. *Let $f \in C_\Box^1([a,b], \mathbb{R})$ be such that*

$$\lim_{h \to 0} \int_a^b \left(\frac{\Box_h f}{\Box t}\right)_E (t)\, dt = 0, \tag{3}$$

where $\frac{\Box_h f}{\Box t} := \left(\frac{\Box_h f}{\Box t}\right)_{conv} + \left(\frac{\Box_h f}{\Box t}\right)_E$. Then,

$$\int_a^b \frac{\Box f}{\Box t}(t)\, dt = f(b) - f(a).$$

As a consequence, we have the following integration by parts formula. If

$$\lim_{h \to 0} \int_a^b \left(\frac{\Box_h (f \cdot g)}{\Box t}\right)_E (t)\, dt = 0,$$

where $f \in H^\alpha$ and $g \in H^\beta$,then

$$\int_a^b \frac{\Box f}{\Box t}(t) \cdot g(t) dt = [f(t)g(t)]_a^b - \int_a^b f(t) \cdot \frac{\Box g}{\Box t}(t) dt.$$

3 The scale variational principle of Herglotz

The (classical) variational principle of Herglotz is described in the following way. Consider the differential equation

$$\begin{cases} \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad \text{with } t \in [a, b] \\ z(a) = z_a \\ x(a) = x_a,\ x(b) = x_b, \end{cases}$$

where x, z and L are smooth functions. We wish to find x (and the correspondent solution z of the system) such that $z(b)$ attains an extremum. The necessary condition is a second-order differential equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}},$$

for all $t \in [a, b]$. This can be seen as an extension of the basic problem of calculus of variations. If L does not depend on z, then integrating the differential equation along the interval $[a, b]$, we get

$$\begin{cases} \displaystyle\int_a^b \left[L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a}\right] dt \quad \rightarrow \quad \text{extremize} \\ x(a) = x_a,\ x(b) = x_b. \end{cases}$$

As is well known, many physical phenomena are characterized by non-differentiable functions (e.g. generic trajectories of quantum mechanics [4], scale-relativity without the hypothesis of space-time differentiability [11]). The usual procedure is to replace the classical derivative by a scale derivative, and consider the space of continuous (and non-differentiable) functions. The scale calculus of variations approach was studied in [1, 2, 3] for a certain concept of scale derivative $\Box x(t)$:

$$\begin{cases} \displaystyle\int_a^b L(t, x(t), \Box x(t)) \quad \rightarrow \quad \text{extremize} \\ x(a) = x_a,\ x(b) = x_b. \end{cases}$$

Motivated by this problem, we define the fundamental scale variational principle of Herglotz. First we need to define what extremum is.

Definition 7. *We say that $z \in C^1([a, b], \mathbb{C})$ attains an extremum at $t = b$ if $z'(b) = 0$.*

The problem is then stated in the following way. Consider the system

$$\begin{cases} \dot{z}(t) = L(t, x(t), \Box x(t), z(t)), \quad \text{with } t \in [a, b] \\ z(a) = z_a \\ x(a) = x_a,\ x(b) = x_b. \end{cases} \tag{4}$$

For simplicity, define

$$[x, z](t) := (t, x(t), \Box x(t), z(t)).$$

We assume that

1. the trajectories x are in $H^\alpha \cap C^1_\Box([a, b], \mathbb{R})$, $\Box x \in H^\alpha$ and the functional z in $C^2([a, b], \mathbb{C})$,
2. for each x, there exists a unique solution z of the system (4)
3. z_a, x_a, x_b are fixed numbers,
4. the Lagrangian $L : [a, b] \times \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C}$ is of class C^2.

Observe that the solution $z(t)$ actually is a function on three variables, to know $z = z(t, x(t), \Box x(t))$. When there is no danger of mislead, we will simply write $z(t)$. We are interested in finding a trajectory x for which the corresponding solution z is such that $z(b)$ attains an extremum. In particular, what necessary conditions such solutions must fulfill. These equations are called Euler-Lagrange equation types. Again, problem (4) can be reduced to the scale variational problem in case L is independent of z:

$$\int_a^b L\left[(t, x(t), \Box x(t)) + \frac{z_a}{b-a}\right] dt \quad \to \quad \text{extremize.}$$

Theorem 3. *If the pair (x, z) is a solution of problem (4), and $\frac{\partial L}{\partial \Box x}[x, z] \in H^\alpha(I, \mathbb{C})$ ($\alpha \in]0, 1[$), then (x, z) is a solution of the equation*

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial \Box x}[x, z](t)\right) = \frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t)\frac{\partial L}{\partial \Box x}[x, z](t), \tag{5}$$

for all $t \in [a, b]$.

Proof. Let ϵ be an arbitrary real, and consider variation functions of x of type $x(t) + \epsilon\eta(t)$, with $\eta \in H^\beta(I, \mathbb{R}) \cap C^1_\Box([a, b], \mathbb{R})$ ($\beta \in]0, 1[$), $\eta(a) = \eta(b) = \Box\eta(a) = 0$, and

$$\lim_{h\to 0}\int_a^b \left(\frac{\Box_h}{\Box t}\left(\lambda(t)\frac{\partial L}{\partial \Box x}[x, z](t)\eta(t)\right)\right)_E dt = 0.$$

The corresponding rate of change of z, caused by the change of x in the direction of η, is given by

$$\theta(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon\eta(t), \Box x(t) + \epsilon\Box\eta(t))\big|_{\epsilon=0}.$$

Then

$$\begin{aligned}
\dot\theta(t) &= \frac{d}{dt}\frac{d}{d\epsilon} z(t, x(t) + \epsilon\eta(t), \Box x(t) + \epsilon\Box\eta(t))\big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} L(t, x(t) + \epsilon\eta(t), \Box x(t) + \epsilon\Box\eta(t), z(t, x(t) + \epsilon\eta(t), \Box x(t) + \epsilon\Box\eta(t))\big|_{\epsilon=0} \\
&= \frac{\partial L}{\partial x}[x, z](t)\eta(t) + \frac{\partial L}{\partial \Box x}[x, z](t)\Box\eta(t) + \frac{\partial L}{\partial z}[x, z](t)\theta(t).
\end{aligned}$$

We obtain a first order linear differential equation on θ, whose solution is

$$\lambda(b)\theta(b) - \theta(a) = \int_a^b \lambda(t)\left[\frac{\partial L}{\partial x}[x, z](t)\eta(t) + \frac{\partial L}{\partial \Box x}[x, z](t)\Box\eta(t)\right] dt,$$

where

$$\lambda(t) = \exp\left(-\int_a^t \frac{\partial L}{\partial z}[x, z](\tau)d\tau\right).$$

Using the fact that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \lambda(t)\left[\frac{\partial L}{\partial x}[x, z](t)\eta(t) + \frac{\partial L}{\partial \Box x}[x, z](t)\Box\eta(t)\right] dt = 0.$$

Integrating by parts the second term, we obtain

$$\int_a^b \left[\lambda(t)\frac{\partial L}{\partial x}[x, z](t) - \frac{\Box}{\Box t}\left(\lambda(t)\frac{\partial L}{\partial \Box x}[x, z](t)\right)\right]\eta(t)dt + \left[\eta(t)\lambda(t)\frac{\partial L}{\partial \Box x}[x, z](t)\right]_a^b = 0.$$

Since $\eta(a) = \eta(b) = 0$, and η is an arbitrary function elsewhere,

$$\lambda(t)\frac{\partial L}{\partial x}[x, z](t) - \frac{\Box}{\Box t}\left(\lambda(t)\frac{\partial L}{\partial \Box x}[x, z](t)\right) = 0,$$

for all $t \in [a, b]$. Since the function $t \mapsto \lambda(t)$ is differentiable, and the function $t \mapsto \frac{\partial L}{\partial \Box x}[x, z](t)$ is in H^α, it follows that

$$\lambda(t)\left(\frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t)\frac{\partial L}{\partial \Box x}[x, z](t) - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial \Box x}[x, z](t)\right)\right) = 0.$$

Finally, since $\lambda(t) > 0$, for all t, we get

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial \Box x}[x,z](t)\right) = \frac{\partial L}{\partial x}[x,z](t) + \frac{\partial L}{\partial z}[x,z](t)\frac{\partial L}{\partial \Box x}[x,z](t),$$

for all $t \in [a,b]$. $\Box$

Remark 1. *Assume that the set of state functions x is $C^1([a,b],\mathbb{R})$. Then equation (5) becomes*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}[x,z](t)\right) = \frac{\partial L}{\partial x}[x,z](t) + \frac{\partial L}{\partial z}[x,z](t)\frac{\partial L}{\partial \dot{x}}[x,z](t),$$

which is the generalized variational principle of Herglotz as in [10].

Theorem 4. *Let the pair (x,z) be a solution of the problem (4), but now $x(b)$ is free. Then (x,z) is a solution of the equation*

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial \Box x}[x,z](t)\right) = \frac{\partial L}{\partial x}[x,z](t) + \frac{\partial L}{\partial z}[x,z](t)\frac{\partial L}{\partial \Box x}[x,z](t),$$

for all $t \in [a,b]$, and verifies the transversality condition

$$\frac{\partial L}{\partial \Box x}[x,z](b) = 0.$$

Proof. Following the proof of Theorem 3, the Euler-Lagrange equation is deduced. Then

$$\left[\eta(t)\lambda(t)\frac{\partial L}{\partial \Box x}[x,z](t)\right]_a^b = 0.$$

Since $\eta(a) = 0$ and $\eta(b)$ is arbitrary, we obtain the transversality condition. $\Box$

Multi-dimensional case

For simplicity, we considered so far one state function x only, but the multi-dimensional case $(x_1,\ldots,x_n)$ is easily studied.

Theorem 5. *Let $\alpha \in]0,1[$ and let the vector $(x_1,\ldots,x_n,z)$ be a solution of the problem: find $(x_1,\ldots,x_n)$ that extremizes $z(b)$, with*

$$\begin{cases} \dot{z}(t) = L(t, x_1(t),\ldots,x_n(t), \Box x_1(t),\ldots,\Box x_n(t), z(t)), & \text{with } t \in [a,b] \\ z(a) = z_a \\ x_i(a) = x_{ia},\ x_i(b) = x_{ib} \end{cases} \tag{6}$$

where, for all $i \in \{1,\ldots,n\}$,

1. *the trajectories x_i are in $H^\alpha \cap C^1_\Box([a,b],\mathbb{R})$, $\Box x_i \in H^\alpha$ and the functional z in $C^2([a,b],\mathbb{C})$,*
2. *z_a, x_{ia}, x_{ib} are fixed numbers,*
3. *$\frac{\partial L}{\partial \Box x_i}[x_1,\ldots,x_n,z] \in H^\alpha(I,\mathbb{C})$*
4. *the Lagrangian $L : [a,b] \times \mathbb{R}^n \times \mathbb{C}^{n+1} \to \mathbb{C}$ is of class C^2.*

Then, for all $i \in \{1,\ldots,n\}$, $(x_1,\ldots,x_n,z)$ is a solution of the equation

$$\frac{\Box}{\Box t}\left(\frac{\partial L}{\partial \Box x_i}[x_1,\ldots,x_n,z](t)\right) = \frac{\partial L}{\partial x_i}[x_1,\ldots,x_n,z](t) + \frac{\partial L}{\partial z}[x_1,\ldots,x_n,z](t)\frac{\partial L}{\partial \Box x_i}[x_1,\ldots,x_n,z](t),$$

for all $t \in [a,b]$.

Theorem 6. *Let the vector* $(x_1, \dots, x_n, z)$ *be a solution of the problem as stated in Theorem 5, but now* $x_i(b)$ *is free, for all* $i \in \{1, \dots, n\}$. *Then, for all* $i \in \{1, \dots, n\}$, $(x_1, \dots, x_n, z)$ *is a solution of the equation*

$$\frac{\square}{\square t}\left(\frac{\partial L}{\partial \square x_i}[x_1, \dots, x_n, z](t)\right) = \frac{\partial L}{\partial x_i}[x_1, \dots, x_n, z](t) + \frac{\partial L}{\partial z}[x_1, \dots, x_n, z](t)\frac{\partial L}{\partial \square x_i}[x_1, \dots, x_n, z](t),$$

for all $t \in [a, b]$, *and verifies the transversality condition*

$$\frac{\partial L}{\partial \square x_i}[x_1, \dots, x_n, z](b) = 0.$$

Higher-order derivatives case

Theorem 7. *Let* $\alpha \in]0, 1[$ *and let the pair* (x, z) *be a solution of the problem: find* x *that extremizes* $z(b)$, *with*

$$\begin{cases} \dot{z}(t) = L(t, x, \square x(t), \dots, \square^n x(t), z(t)), & \text{with } t \in [a, b] \\ z(a) = z_a \\ \square^i x(a) = x_{ia},\ \square^i x(b) = x_{ib}, & \text{for all } i \in \{0, \dots, n-1\}, \end{cases}$$

where

1. *the trajectories* x *are in* $H^\alpha \cap C^n_\square([a, b], \mathbb{R})$, $\square x \in H^\alpha$ *and the functional* z *in* $C^2([a, b], \mathbb{C})$,
2. z_a, x_{ia}, x_{ib} *are fixed numbers, for all* $i \in \{0, \dots, n-1\}$,
3. $\frac{\partial L}{\partial \square^i x}[x, z] \in H^\alpha(I^n, \mathbb{C})$, *for all* $i \in \{1, \dots, n\}$,
4. $[x, z](t) = (t, x, \square x(t), \dots, \square^n x(t), z(t))$ *and* $[x](t) = (t, x, \square x(t), \dots, \square^n x(t))$,
5. *the Lagrangian* $L : [a, b] \times \mathbb{R}^{n+1} \to \mathbb{R}$ *is of class* C^2.

Then, (x, z) *is a solution of the equation*

$$\lambda(t)\frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^{n}(-1)^i \frac{\square^i}{\square t^i}\left(\lambda(t)\frac{\partial L}{\partial \square^i x}[x, z](t)\right) = 0,$$

for all $t \in [a, b]$.

Proof. Let $x(t)+\epsilon\eta(t)$ be a variation function of x, with $\epsilon \in \mathbb{R}$ and $\eta \in H^\beta \cap C^n_\square([a, b], \mathbb{R})$ ($\beta \in]0, 1[$). Also, assume that the variations fulfill the conditions:

1. for all $i = 0, \dots, n-1$, $\square^i \eta(a) = \square^i \eta(b) = 0$, and $\square^n \eta(a) = 0$,
2. for all $i = 1, 2, \dots, n$ and $k = 0, 1, \dots, i-1$,

$$\lim_{h \to 0} \int_a^b \left(\frac{\square_h}{\square t}\left(\lambda(t)\frac{\square^k}{\square t^k}\left(\frac{\partial L}{\partial \square^i x}[x, z](t)\right)\square^{i-k-1}\eta(t)\right)\right)_E dt = 0.$$

Define

$$\theta(t) = \frac{d}{d\epsilon}\, z(t, x(t) + \epsilon\eta(t), \square x(t) + \epsilon\square\eta(t), \dots, \square^n x(t) + \epsilon\square^n\eta(t))|_{\epsilon=0}\,.$$

Then

$$\dot{\theta}(t) = \frac{\partial L}{\partial x}[x, z](t)\eta(t) + \sum_{i=1}^{n}\frac{\partial L}{\partial \square^i x}[x, z](t)\square^i\eta(t) + \frac{\partial L}{\partial z}[x, z](t)\theta(t).$$

Solving this linear ODE, we arrive at

$$\int_a^b \lambda(t)\left[\frac{\partial L}{\partial x}[x, z](t)\eta(t) + \sum_{i=1}^{n}\frac{\partial L}{\partial \square^i x}[x, z](t)\square^i\eta(t)\right] dt = 0,$$

where

$$\lambda(t) = \exp\left(-\int_a^t \frac{\partial L}{\partial z}[x,z](\tau)d\tau\right).$$

Integrating by parts n times, we obtain the following:

$$\int_a^b \left[\lambda(t)\frac{\partial L}{\partial x}[x,z](t) + \sum_{i=1}^{n}(-1)^i \frac{\Box^i}{\Box t^i}\left(\lambda(t)\frac{\partial L}{\partial \Box^i x}[x,z](t)\right)\right]\eta(t)dt$$
$$+\left[\sum_{i=1}^{n}\sum_{k=0}^{i-1}(-1)^k \frac{\Box^k}{\Box t^k}\left(\lambda(t)\frac{\partial L}{\partial \Box^i x}[x,z](t)\right)\Box^{i-1-k}\eta(t)\right]_a^b = 0,$$

and rearranging the terms, we get

$$\int_a^b \left[\lambda(t)\frac{\partial L}{\partial x}[x,z](t) + \sum_{i=1}^{n}(-1)^i \frac{\Box^i}{\Box t^i}\left(\lambda(t)\frac{\partial L}{\partial \Box^i x}[x,z](t)\right)\right]\eta(t)dt$$
$$+\left[\sum_{i=1}^{n}\left[\sum_{k=i}^{n}(-1)^{k-i} \frac{\Box^{k-i}}{\Box t^{k-i}}\left(\lambda(t)\frac{\partial L}{\partial \Box^k x}[x,z](t)\right)\right]\Box^{i-1}\eta(t)\right]_a^b = 0.$$

Since $\Box^i\eta(a) = \Box^i\eta(b) = 0$, for all $i \in \{0,\ldots,n-1\}$ and η is arbitrary elsewhere, we get

$$\lambda(t)\frac{\partial L}{\partial x}[x,z](t) + \sum_{i=1}^{n}(-1)^i \frac{\Box^i}{\Box t^i}\left(\lambda(t)\frac{\partial L}{\partial \Box^i x}[x,z](t)\right) = 0,$$

for all $t \in [a,b]$. □

Theorem 8. *Let the pair (x,z) be a solution of the problem as stated in Theorem 7, but now $\Box^i x(b)$ is free, for all $i \in \{0,\ldots,n-1\}$. Then, (x,z) is a solution of the equation*

$$\lambda(t)\frac{\partial L}{\partial x}[x,z](t) + \sum_{i=1}^{n}(-1)^i \frac{\Box^i}{\Box t^i}\left(\lambda(t)\frac{\partial L}{\partial \Box^i x}[x,z](t)\right) = 0,$$

for all $t \in [a,b]$, and verifies the transversality condition

$$\sum_{k=i}^{n}(-1)^{k-i} \frac{\Box^{k-i}}{\Box t^{k-i}}\left(\lambda(t)\frac{\partial L}{\partial \Box^k x}[x,z](t)\right) = 0 \quad \text{at} \quad t=b,$$

for all $i \in \{1,\ldots,n\}$.

Several independent variables case

We generalize Theorem 3 for several independent variables. First we fix some notations. The variable time is $t \in [a,b]$, $x = (x_1,\ldots,x_n) \in \Omega := \prod_{i=1}^n [a_i,b_i]$ are the space coordinates and the state function is $u := u(t,x)$.

Theorem 9. *Let $\alpha \in]0,1[$ and let the pair (u,z) be a solution of the problem: find u that extremizes $z(b)$, with*

$$\begin{cases} \dot{z}(t) = \displaystyle\int_\Omega L\left(t,x,u,\frac{\Box u}{\Box t},\frac{\Box u}{\Box x_1},\ldots,\frac{\Box u}{\Box x_n},z(t)\right)d^n x, & \text{with } t \in [a,b] \\ z(a) = z_a \\ u(t,x) \quad \text{takes fixed values}, & \forall t \in [a,b]\,\forall x \in \partial\Omega \\ u(t,x) \quad \text{takes fixed values}, & \forall t \in \{a,b\}\,\forall x \in \Omega, \end{cases} \tag{7}$$

where, for all $i \in \{1,\ldots,n\}$,

1. *the trajectories* u *are in* $H^\alpha(I \times \Omega, \mathbb{R}) \cap C^1_\square([a,b] \times \Omega, \mathbb{R})$, $\frac{\square u}{\square t}, \frac{\square u}{\square x_i} \in H^\alpha([a,b] \times \Omega, \mathbb{C})$ *and the functional* z *in* $C^2([a,b], \mathbb{C})$,

2. z_a *is a fixed number,*

3. $d^n x = dx_1 \dots dx_n$,

4. $\frac{\partial L}{\partial \square t}[u,z], \frac{\partial L}{\partial \square x_i}[u,z] \in H^\alpha(I \times \Omega, \mathbb{C})$, *where* $\frac{\partial L}{\partial \square t}[u,z]$ *denotes the partial derivative of* L *with respect to the variable* $\frac{\square u}{\square t}$, *and* $\frac{\partial L}{\partial \square x_i}[u,z]$ *denotes the partial derivative of* L *with respect to the variable* $\frac{\square u}{\square x_i}$, *and* $[u,z](t) = (t, x, u, \frac{\square u}{\square t}, \frac{\square u}{\square x_1}, \dots, \frac{\square u}{\square x_n}, z(t))$,

5. $L : [a,b] \times \Omega \times \mathbb{R} \times \mathbb{C}^{n+2} \to \mathbb{C}$ *is of class* C^2.

Then, (u,z) *is a solution of the equation*

$$\frac{\partial L}{\partial u}[u,z](t) + \frac{\partial L}{\partial \square t}[u,z](t) \int_\Omega \frac{\partial L}{\partial \square z}[u,z](t)\, d^n x - \frac{\square}{\square t}\left(\frac{\partial L}{\partial \square t}[u,z](t)\right) - \sum_{i=1}^n \frac{\square}{\square x_i}\left(\frac{\partial L}{\partial \square x_i}[u,z](t)\right) = 0,$$

for all $t \in [a,b]$ *and for all* $x \in \Omega$.

Proof. Let $u(t,x) + \epsilon\eta(t,x)$ be a variation function of u, with $\epsilon \in \mathbb{R}$ and $\eta \in H^\beta(I \times \Omega, \mathbb{R}) \cap C^1_\square([a,b] \times \Omega, \mathbb{R})$ ($\beta \in]0,1[$). Also, assume that the variations fulfill the conditions:

1. $\eta(t,x) = 0, \quad \forall t \in [a,b]\, \forall x \in \partial\Omega$,
2. $\eta(t,x) = 0, \quad \forall t \in \{a,b\}\, \forall x \in \Omega$,
3. $\frac{\square \eta}{\square t}(a,x) = \frac{\square \eta}{\square x_i}(a,x) = 0, \quad \forall x \in \Omega$,
4. for all $i = 1, 2, \dots, n$,

$$\lim_{h \to 0} \int_a^b \left(\frac{\square_h}{\square t}\left(\lambda(t)\frac{\partial L}{\partial \square t}[u,z](t)\eta(t)\right)\right)_E dt = 0.$$

and

$$\lim_{h \to 0} \int_a^b \left(\frac{\square_h}{\square x_i}\left(\lambda(t)\frac{\partial L}{\partial \square x_i}[u,z](t)\eta(t)\right)\right)_E dt = 0,$$

where

$$\lambda(t) = \exp\left(-\int_a^t \int_\Omega \frac{\partial L}{\partial z}[u,z](\tau)\, d^n x\, d\tau\right).$$

Let

$$\theta(t) = \frac{d}{d\epsilon} z\left(t, x, u + \epsilon\eta, \frac{\square u}{\square t} + \epsilon\frac{\square \eta}{\square t}, \frac{\square u}{\square x_1} + \epsilon\frac{\square \eta}{\square x_1}, \dots, \frac{\square u}{\square x_n} + \epsilon\frac{\square \eta}{\square x_n}\right)\Bigg|_{\epsilon=0}.$$

Proceeding with some calculations, we arrive at the ODE

$$\dot\theta(t) - \int_\Omega \frac{\partial L}{\partial z}[u,z](t)\, d^n x\; \theta(t) = \int_\Omega \frac{\partial L}{\partial u}[u,z](t)\eta + \frac{\partial L}{\partial \square t}[u,z](t)\frac{\square \eta}{\square t} + \sum_{i=1}^n \frac{\partial L}{\partial \square x_i}[u,z](t)\frac{\square \eta}{\square x_i}\, d^n x.$$

Solving the ODE, and taking into consideration that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \int_\Omega \lambda(t)\left[\frac{\partial L}{\partial u}[u,z](t)\eta + \frac{\partial L}{\partial \square t}[u,z](t)\frac{\square \eta}{\square t} + \sum_{i=1}^n \frac{\partial L}{\partial \square x_i}[u,z](t)\frac{\square \eta}{\square x_i}\right] d^n x\, dt = 0.$$

Integrating by parts, and considering the boundary conditions over η, we get

$$\int_a^b \int_\Omega \left[\lambda(t)\frac{\partial L}{\partial u}[u,z](t) - \frac{\square}{\square t}\left(\lambda(t)\frac{\partial L}{\partial \square t}[u,z](t)\right) - \sum_{i=1}^n \frac{\square}{\square x_i}\left(\lambda(t)\frac{\partial L}{\partial x_i}[u,z](t)\right)\right] \eta d^n x dt = 0.$$

By the arbitrariness of η, it follows that for all $t \in [a,b]$ and for all $x \in \Omega$,

$$\lambda(t)\frac{\partial L}{\partial u}[u,z](t) - \frac{\Box}{\Box t}\left(\lambda(t)\frac{\partial L}{\partial \Box t}[u,z](t)\right) - \sum_{i=1}^{n}\frac{\Box}{\Box x_i}\left(\lambda(t)\frac{\partial L}{\partial x_i}[u,z](t)\right) = 0.$$

Since $\lambda(t) > 0$, this condition implies that

$$\frac{\partial L}{\partial u}[u,z](t) + \frac{\partial L}{\partial \Box t}[u,z](t)\int_\Omega \frac{\partial L}{\partial \Box z}[u,z](t)\, d^n x - \frac{\Box}{\Box t}\left(\frac{\partial L}{\partial \Box t}[u,z](t)\right) - \sum_{i=1}^{n}\frac{\Box}{\Box x_i}\left(\frac{\partial L}{\partial \Box x_i}[u,z](t)\right) = 0,$$

for all $t \in [a,b]$ and for all $x \in \Omega$, and the theorem is proved. □

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