ON SCHUR ALGEBRAS, DOTY COALGEBRAS AND QUASI-HEREDITARY ALGEBRAS

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Abstract

Motivated by Doty's Conjecture we study the coalgebras formed from the coefficient spaces of the truncated modules $\mathrm{Tr}^{\lambda}E$. We call these the Doty Coalgebras $D_{n,p}(r)$. We prove that $D_{n,p}(r) = A(n,r)$ for $n = 2$, and also that $D_{n,p}(r) = A(\pi, r)$ with π a suitable saturated set, for the cases;
i) $n = 3$, $0 \leq r \leq 3p - 1$, $6p - 8 \leq r \leq n^2(p-1)$ for all p;
ii) $p = 2$ for all n and all r;
iii) $0 \leq r \leq p - 1$ and $nt - (p-1) \leq r \leq nt$ for all n and all p;
iv) $n = 4$ and $p = 3$ for all r.

The Schur Algebra $S(n,r)$ is the dual of the coalgebra $A(n,r)$, and $S(n,r)$ we know to be quasi-hereditary. Moreover, we call a finite dimensional coalgebra quasi-hereditary if its dual algebra is quasi-hereditary and hence, in the above cases, the Doty Coalgebras $D_{n,p}(r)$ are also quasi-hereditary and thus have finite global dimension. We conjecture that there is no saturated set π such that $D_{3,p}(r) = A(\pi, r)$ for the cases not covered above, giving our reasons for this conjecture.

Stepping away from our main focus on Doty Coalgebras, we also describe an infinite family of quiver algebras which have finite global dimension but are not quasi-hereditary.

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Contents

Introduction

This thesis is concerned with the Polynomial Representation Theory of the General Linear Group as studied in Green [11]. Its motivation is a conjecture put forward by S. Doty, known as Doty's Conjecture, concerning the truncated symmetric powers of the general linear group. It states that a matrix, the Modular Kostka matrix, is non-singular. If true, then from its inverse and Steinberg's Tensor Product Theorem, one could obtain all irreducible polynomial characters of GL_n in characteristic p.

The focus of our research is on these truncated symmetric powers $\mathrm{Tr}^{\lambda}E$, and the coalgebras defined from their coefficient spaces, the so-called Doty Coalgebras, $D_{n,p}(r) = \sum \mathrm{cf}\,(\mathrm{Tr}^{\lambda}E)$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i$ $= r$, $\lambda_1 \geq \lambda_2 geq \ldots \geq \lambda_n$ and $\lambda_1 \leq n(p-1)$. The dual of a coalgebra is an algebra, and as shown in [11], we have that the Schur Algebra $S(n,r)$ is the dual of the coalgebra $A(n,r)$, which consists of homogeneous polynomials of degree r in n^2 variables. Our other motivation for this thesis is the notion of a Quasi-Hereditary Algebra (see for example [4]), and indeed it is the case that the Schur algebra is quasi-hereditary, and thus, as its dual, $A(n,r)$ is a quasi-hereditary coalgebra.

This thesis proves the following;

i) $D_{2,p}(r) = A(2,r)$ for all primes p;
ii) $D_{3,p}(r) = A(\pi,r)$ for all primes p, $0 \leq r \leq 3p-1$, $6p-8 \leq r \leq n^2(p-1)$ and π a suitable saturated set;
iii) $D_{n,2}(r) = A(\pi,r)$ for all n and π a suitable saturated set;
iv) $D_{n,p}(r) = A(\pi,r)$ for all p and all n, and for $0 \leq r \leq p-1$ and $n^2-(p-1) \leq r \leq n^2(p-1)$ with π a suitable saturated set;
v) $D_{4,3}(r) = A(\pi,r)$ for $0 \leq r \leq n^2(p-1)$ with π a suitable saturated set.

We therefore have that in all these cases, the corresponding Doty Coalgebra is quasi-hereditary, and thus has finite global dimension.

We now give a general overview of the structure of this thesis. Chapter 1 first states the Doty Conjecture and then goes on to define the notion of algebraic groups and coalgebras, and shows how the dual of an algebra is indeed a coalgebra. It then goes on to define the coalgebra $A(n,r)$ and the Schur Algebra $S(n,r)$ stating how they are each others dual. Quasi-Hereditary

Algebras are then defined, along with finite global dimension, and it is shown how all quasi-hereditary algebras have finite global dimension. We then explain why the Schur algebra is quasi-hereditary.

Chapter 2 focuses on defining the Doty Coalgebras, which initially involves introducing the Symmetric and Exterior Algebras, and then the aforementioned truncated modules $\mathrm{Tr}^{\lambda}E$ and their coefficient spaces. To be able to prove the cases above, we must show that the coefficient spaces of the tilting modules of $A(n,r)$ (or respectively $A(\pi,r)$) are contained in the coefficient spaces of the truncated modules. With this in mind we then have a section on Classification by Highest Weights, which defines the irreducible GL_n-modules $L(m,n)$, and then a section on tilting modules, giving the definition of these modules, classifying them for SL_2, and then stating why it is enough to show that their coefficient spaces arise in the coefficient spaces of the truncated modules.

Chapter 3 then proves the case $D_{2,p}(r) = A(2,r)$ for all primes p, whilst Chapter 4 proves $D_{3,p}(r) = A(\pi,r)$ for all primes p, $0 \leq r \leq 3p-1$, $6p-8 \leq r \leq n^2(p-1)$ and π a saturated set. The case for $n = 3$ is much more complex, and thus it is necessary to introduce p-cores and core classes, and then classify all core classes for $0 \leq r \leq 5p-5$. Tilting modules are then reintroduced and we prove a number of theorems on tilting truncated modules. We then introduce some facts on the filtration multiplicities $(T(\lambda) : \nabla(\mu))$, and then state the method we will use to prove the case $n = 3$, which incorporates an algorithm for calculating the character of the truncated modules, the Littlewood-Richardson Rule.

Chapter 5 considers the 'missing' range for the case $n = 3$, namely $3p \leq r \leq 6p-9$, and explains why we conjecture that $D_{3,p}(r) \neq A(\pi,r)$ for this range. It also takes a brief look at the case $n \geq 4$ which incorporates the proof of $D_{n,p}(r) = A(\pi,r)$ for all p and all n, and for $0 \leq r \leq p-1$, with π a saturated set. We then give reasons why we conjecture that $D_{n,p}(r) \neq A(\pi,r)$ for $n \geq 4$ and $r > p$ except for when $n = 4$ and $p = 3$. Finally, Chapter 5 proves the case $D_{n,2}(r) = A(\pi,r)$ for all n and π a suitable saturated set.

Chapter 6 then steps away from Doty Coalgebras. As stated above, we have that all quasi-hereditary algebras have finite global dimension. The reverse of this is not necessarily true, and in our research on quasi-hereditary alegbras we came across a paper by Dlab and Ringel [3], which gives an example of an 11-dimensional serial algebra which has finite global dimension but is not quasi-hereditary. Studying this example we were able to find an infinite family of such algebras, all of which are quiver algebras and thus can be displayed as a nice picture. This chapter defines this family of quiver algebras, proving why they have finite global dimension, but are not quasi-hereditary.

Chapter 1

Preliminaries

1.1 Doty's Conjecture, Algebraic Groups, Coalgebras and Algebras

AIM: This section states Doty's Conjecture, defines algebraic groups, coalgebras and their comodules. It shows how we get from an algebra to a coalgebra and how an algebraic group can be viewed as a coalgebra. Doty's Conjecture is stated first as this was the motivation behind the research, we will go on to define the modules used in the conjecture.

CONJECTURE 1.1.1 Doty's Conjecture [9, Conjecture 2.6]
The modular Kostka numbers are defined as follows: $K'_{\mu\lambda} = [\mathrm{Tr}^{\lambda}E : L(\mu)]$, for $\mathrm{Tr}^{\lambda}E$ the truncated module and $L(\mu)$ the irreducible polynomial GL_n-module of highest weight μ. Then Doty's Conjecture states that the modular Kostka matrix $K' = (K'_{\mu\lambda})$, with rows and columns indexed by the set of all partitions λ of length $\leq n$, and bounded by $n(p-1)$ (fixed in some order), is non-singular for all n and all primes p.

We now go on to define an algebraic group.

DEFINITION 1.1.2 Let $GL_n(k)$ be the group of invertible matrices over k an algebraically closed field, and $M_n(k)$ be the space of all $n \times n$ matrices over k.
A k-valued function f on $M_n(k)$ is a polynomial function if
$f \in k[c_{11}, c_{12}, ..., c_{nn}]$ with $c_{ij}(g) = (i, j)$th entry of g, for $g \in GL_n(k)$.
A subgroup $G \subseteq GL_n(k)$ is a linear algebraic group if there is a set A of polynomial functions on $M_n(k)$ such that

$$G = \{g \in GL_n(k) |\ f(g) = 0 \ \ \forall f \in A\}.$$

REMARK 1.1.3 [24, Theorem 2.3.5]
Every affine algebraic group over k is isomorphic to a closed subgroup of $GL_n(k)$, for some n.

DEFINITION 1.1.4 Let k be an algebraically closed field. Let $G = GL_n(k)$, and H a subgroup of G. Then

$$k[G] = k[c_{11}, c_{12}, ..., c_{nn}, (\det)^{-1}]$$

where $c_{ij}(g) = (i, j)$th entry of g.
Now fix $f_1, ..., f_r \in k[G]$. Suppose $H = \{g \in G| \ f_1(g) = \cdots = f_r(g) = 0\}$ then H is a closed set in the Zariski topology [26], and $k[H] = \{f \downarrow_H \ | \ f \in k[G]\}$.

We now give some definitions on coalgebras and comodules.

DEFINITION 1.1.5 A k-coalgebra is a triple (C, δ, ϵ) where C is a k-vector space, and $\delta : C \to C \otimes C$, and $\epsilon : C \to k$ are linear maps such that

$$(\delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \delta)\delta \qquad \text{coassociativity}$$

$$(\epsilon\otimes)\delta = (\mathrm{id} \otimes \epsilon)\delta = \mathrm{id} \qquad \text{counit}$$

where δ is the comultiplication or diagonal map, and ϵ is the counit or augmentation map.

A subspace $C' \subset C$ is a subcoalgebra if $\delta(C') \subseteq C' \otimes C'$. Note that C' is a coalgebra itself by restricting δ to $\delta' : C' \to C' \otimes C'$ and ϵ to $\epsilon' : C' \to k$.

DEFINITION 1.1.6 A right C-comodule is a pair (V, τ) where V is a k-vector space and $\tau : V \to V \otimes C$ is a linear map such that

$$(\tau \otimes \mathrm{id}_C)\tau = (\mathrm{id}_V \otimes \delta)\tau \qquad \text{and} \qquad (\mathrm{id}_V \otimes \epsilon)\tau = \mathrm{id}_V.$$

We call τ the structure map of V.

DEFINITIONS 1.1.7 [6, Section 1.1 and 1.2]
(i) Let C and C' be two coalgebras. Then $\phi : C \to C'$ is a morphism of coalgebras if $\epsilon' \circ \phi = \epsilon$ and $(\phi \otimes \phi) \circ \delta = \delta' \circ \phi$
(ii) Let (V, τ) and (V', τ') be two C-comodules. Then $\phi : V \to V'$ is a morphism of comodules if $(\phi \otimes \mathrm{id}_C)\tau = \tau' \circ \phi$.
(iii) Let C be a coalgebra. Then a coideal of C is a subspace I such that $\delta(I) \subseteq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$.
(iv) Let J be a set and C_j a coalgebra for each $j \in J$. Then we have that the direct sum $C = \bigoplus_{j \in J} C_j$ is a coalgebra with $\delta : C \to C \otimes C$ such that $\delta(\sum_{j \in J} x_j) = \sum_{j \in J} \delta_j(x_j)$ and $\epsilon : C \to k$ such that $\epsilon(\sum_{j \in J} x_j) = \sum_{j \in J} \epsilon(x_j)$, for $x_j \in C_j$.

We now show how a coalgebra is the dual of an algebra.

DEFINITION 1.1.8 Let (C, δ, ϵ) be a k-coalgebra and R a k-algebra. Then $A = \mathrm{Hom}_k(C, R)$ is an associative k-algebra with multiplication given by the convolution product

$$(f * g)(c) = \sum_i f(c_i)g(c'_i)$$

for $f, g \in A$, where $\delta(c) = \sum_i c_i \otimes c'_i$ and with identity $1_A(c) = \epsilon(c)1_R$. In particular the linear dual $C^* = \mathrm{Hom}_k(C, k)$ is the dual algebra of C.

The following defines the invariant matrix, needed for the next proposition which shows how we get from a comodule to a module.

DEFINITION 1.1.9 Let (V, τ) be a right C-comodule and let $\{v_i\}_{i\in I}$ be a k-basis of V. The elements $c_{ij} \in C$, for $i, j \in I$, are defined by

$$\tau(v_j) = \sum_{i\in I} v_i \otimes c_{ij}$$

for $j \in I$. We call the matrix (c_{ij}) the invariant matrix afforded by the basis $\{v_i\}_{i\in I}$.

PROPOSITION 1.1.10 *[6, Proposition 2.2a]*
Let C be a coalgebra and (V, τ) a C-comodule. Let $f \in C^*$ and $v \in V$ and define $fv \in V$ by

$$fv = (id \otimes f)\tau(v) = (id \otimes f)(\sum_i v_i \otimes x_i) = \sum_i id(v) \otimes f(x_i) = \sum_i f(x_i)v_i.$$

In particular, if $\{v_i\}_{i\in I}$ is a basis for V then $fv_i = \sum_{i\in I} f(c_{ji})v_j$ for (c_{ij}) the invariant matrix. Then the product fv makes the right C-comodule V into a (unital) left C^*-module.

The next remark gives well known canonical isomorphisms between duals of comodules which will be needed later in the proof of Theorem 4.1.3.

REMARK 1.1.11 Let C be a coalgebra, and let V, W be C-comodules. Then
i) $(V \otimes W)^* \cong V^* \otimes W^*$,
ii) $\mathrm{Hom}_C(V, W) \cong \mathrm{Hom}_C(W^*, V^*)$.

PROPOSITION 1.1.12 *[6, Proposition 2.1b]*
(i) If $f : C \to D$ is a morphism of coalgebras then $f^* : D^* \to C^*$ where $f^*(\alpha) = \alpha \circ f$ is a morphism of algebras.
(ii) The assignment $C \mapsto C^*$ determines a contravariant functor from coalgebras to algebras which restricts to an equivalence of categories between finite dimensional coalgebras and algebras.

Finally we show how an algebraic group can be viewed as a coalgebra.

PROPOSITION 1.1.13 *[24, Exercise 2.1.2]*
Let G be an algebraic group over k and algebraically closed field.
Then we can define $R = k[G]$ with $\epsilon : k[G] \to k$ such that $\epsilon(f) = f(1)$ and also with $m^* = \delta : k[G] \to k[G] \times k[G]$, the comorphism of the multiplication map $m : G \times G \to G$.
Thus we have a triple (R, δ, ϵ) where δ and ϵ satisfy the following:
(i) $(\delta \otimes id_R) \circ \delta = (id_R \otimes \delta) \circ \delta$
(ii) $(\epsilon \otimes id_R) \circ \delta = (id_R \otimes \epsilon) \circ \delta = id_R$,
and hence (R, δ, ϵ) is a coalgebra.

1.2 The coalgebra $A(n,r)$ and the Schur Algebra $S(n,r)$

AIM: In this section we define the coalgebra $A(n,r) \subset k[G]$, and also the Schur Algebra $S(n,r)$, the dual algebra of $A(n,r)$. We also consider a different way of viewing the Schur Algebra, resulting in a theorem by Schur.

DEFINITION 1.2.1 Let $G = GL_n(k)$, with $k[G]$ as in Definition 1.1.4.
Let $\underline{n} = \{1, 2, \ldots, n\}$, then for each pair $i, j \in \underline{n}$, let $c_{ij} \in k[G]$ be the function which associates to each $g \in G$ its (i,j)-coefficient $g_{i,j}$.
Let $A(n) = k[c_{ij} \mid 1 \leq i, j \leq n]$, the algebra of polynomial functions in n^2 variables.

Then $A(n,r) \subseteq A(n)$, consists of the elements expressible as homogeneous polynomials of degree r in the c_{ij}.

DEFINITION 1.2.2 Let $I(n,r)$ be the set of all maps $i : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$. Then for $i, j \in I(n,r)$ write c_{ij} for $c_{i_1 j_1} c_{i_2 j_2} \ldots c_{i_r j_r}$.
Then $A(n,r) = k\text{-span } \{c_{i,j} |\ i = (i_1, \ldots, i_r), j = (j_1, \ldots, j_r) \in I(n,r)\}$

REMARK 1.2.3 [11, Section 2.1] The dimension of $A(n,r)$ is as follows;

$$\dim\ A(n,r) = \begin{pmatrix} n^2 + r - 1 \\ r \end{pmatrix}$$

$A(n,r)$ has a coalgebra structure as a subcoalgebra of $k[G]$, giving rise to an algebra structure on the dual $A(n,r)^*$.

After defining the coalgebra $A(n,r)$ we now move on to the Schur Algebra $S(n,r)$.

DEFINITION 1.2.4 The Schur Algebra $S(n,r) = A(n,r)^*$ where $S(n,r)$ has basis $\{\xi_{ij} \mid i,j \in I(n,r)\}$ dual to $\{c_{ij} \mid i,j \in I(n,r)\}$, the basis for $A(n,r)$.

However we can also look at $S(n,r)$ in a different way as now shown.

DEFINITION 1.2.5 Let E be a k-vector space with basis $\{e_1, \ldots, e_n\}$, so $\dim E = n$. Then $G = GL_n(k)$ acts on E as follows

$$ge_\nu = \sum_{\mu \in \underline{n}} g_{\mu\nu} e_\mu = \sum_{\mu \in \underline{n}} c_{\mu\nu}(g) e_\nu$$

for all $g \in G$ and $\nu \in \underline{n}$.

Now, for $i = (i_1, i_2, \ldots, i_r) \in I(n,r)$ where $1 \leq i_1, \ldots, i_r \leq n$, we write e_i for $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}$. Then $\{e_i \mid i \in I(n,r)\}$ is a basis of $E^{\otimes r}$. Now suppose $G = GL_n(k)$ then we have the action

$$G \times E^{\otimes r} \to E^{\otimes r}$$

$$ge_j = ge_{j_1} \otimes \ldots \otimes ge_{j_r}$$

where

$$\begin{aligned} ge_j &= \sum_{i \in I(n,r)} g_{i_1 j_1} \ldots g_{i_r j_r} e_i \\ &= \sum_{i \in I(n,r)} c_{i,j}(g) e_i \ \ \forall g \in G, j \in I(n,r) \end{aligned}$$

We also have the action by the symmetric group $E^{\otimes r} \times \mathrm{Sym}(r) \to E^{\otimes r}$ given by $e_i \pi = e_{i\pi}$. Note that the action is associative with $(e_i \sigma)\pi = (e_{i\sigma})\pi = e_{i\sigma\pi} = (e_i)(\sigma\pi)$.

THEOREM 1.2.6 *[11, Theorem 2.6c] By Schur's Theorem*

$$S(n,r) \cong End_{\mathrm{Sym}(r)}(E^{\otimes r}).$$

1.3 Quasi-hereditary Algebras and finite global dimension

AIM: Definiton 1.3.1 up to Proposition 1.3.14 introduces the notion of a quasi-hereditary algebra. There is a reasonable amount of detail needed for this, with the introduction of standard and costandard modules, high-weight categories, saturated sets and hereditary ideals. In sections 1.3.15-1.3.19 we define projective dimension and global dimension, and Proposition 1.3.21 states that any quasi-hereditary algebra has finite global dimension.
We begin with the definition of a weight space.

DEFINITION 1.3.1 (i) Let $X(n) = \mathbb{Z}^n$, so $\alpha = (\alpha_1, \ldots, \alpha_n) \in X(n)$ for $\alpha_i \in \mathbb{Z}$. Let $GL_1^n = GL_1(k)^n = GL_1(k) \times \ldots \times GL_1(k)$ (n times), and let V be a GL_1^n-module, with comodule structure map $\tau : V \to V \otimes k[GL_1^n]$.
Now, for $(x_1, \ldots, x_n) \in GL_1(k)^n$ we define $t_i(x_1, \ldots, x_n) = x_i$ where $1 \leq i \leq n$. Moreover for $\alpha \in X(n)$ we put $t^\alpha = t_1^{\alpha_1} \ldots t_n^{\alpha_n}$. We can then define the corresponding weight space

$$V^\alpha = \{v \in V |\ \tau(v) = v \otimes t^\alpha\},$$

and we call α a weight of V if $V^\alpha \neq 0$.

We then move on to the definition of a dominance order.

DEFINITION 1.3.2 Let $\alpha, \beta \in X(n)$, with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$. Let $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Then, we define a dominance order, written $\alpha \trianglelefteq \beta$ if
1) $|\alpha| = |\beta|$
2) $\alpha_1 + \ldots + \alpha_a \leq \beta_1 + \ldots + \beta_a$ for all $1 \leq a \leq n$
Moreover, for $\lambda \in X(n)$ we say that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is dominant if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

DEFINITION 1.3.3 We have
(i) $\Lambda(n, r) = \{\alpha \in \mathbb{N}_0^n, \alpha = (\alpha_1, \ldots, \alpha_n) :\ |\alpha| = r\}$
(ii) $X^+(n) = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\ :\ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n\}$.
(iii) $\Lambda^+(n, r) = \Lambda(n, r) \cap X^+(n)$

The following defines certain properties of modules.

DEFINITION 1.3.4 We define mod(A) to be the category of all finitely-generated A-modules.

DEFINITION 1.3.5 Let A be an algebra with M an A-module.
i) The socle of M, notated $\mathrm{soc}(M)$, is defined to be the unique largest semisimple submodule of M.
ii) We define the radical of M, denoted $\mathrm{rad}(M)$, to be the intersection of all proper maximal submodules of M. The head of M, $\mathrm{hd}(M) = M/\mathrm{rad}M$.

DEFINITION 1.3.6 Let A be a finite dimensional algebra with M a finitely-generated A-module.
Choose P_M a projective A-module such that $\mathrm{hd}(P_M) \cong \mathrm{hd}(M)$. Then P_M is the projective cover of M.
Choose I_M an injective A-module such that $\mathrm{soc}(I_M) \cong \mathrm{soc}(M)$. Then I_M is the injective envelope of M.

DEFINITION 1.3.7 Let A be a finite dimensional k-algebra with simples $\{L(\lambda) \mid \lambda \in \Lambda^+\}$, $P(\lambda)$ the projective cover of $L(\lambda)$ and $I(\lambda)$ the injective envelope of $L(\lambda)$.
Then if $X \in \mathrm{mod}(A)$ we define $[X : L(\lambda)]$ to be the multiplicity of $L(\lambda)$ as a composition factor of X.

The following looks at maximal and minimal submodules of an A-module.

DEFINITION 1.3.8 (i) Let $\pi \subseteq \Lambda^+$. Then $V \in \mathrm{mod}(A)$ belongs to π if all composition factors of V belong to $\{L(\lambda)| \ \lambda \in \pi\}$.

(ii) Among all submodules of arbitrary V in π there is a unique maximal submodule $U \subseteq V$ such that U belongs to π. We write $O_\pi(V) = U$.

(iii) Among all submodules of arbitrary V in π there is a unique minimal submodule $W \subseteq V$ such that V/W is in π. We write $O^\pi(V) = W$.

It is then necessary to define standard and costandard modules.

DEFINITION 1.3.9 Fix a partial ordering $\leq$ on Λ^+. For $\lambda \in \Lambda^+$ define $\pi(\lambda) = \{\mu \in \Lambda^+| \ \mu < \lambda\}$.
Let $M(\lambda)$ be the unique maximal submodule of $P(\lambda)$, and define $K(\lambda) = O^{\pi(\lambda)}(M(\lambda))$. Then we define the standard modules by $\Delta(\lambda) = P(\lambda)/K(\lambda)$, and the costandard modules by $\nabla(\lambda)/L(\lambda) = O_{\pi(\lambda)}(I(\lambda)/L(\lambda))$.

The next three definitions look at filtrations, high weight categories, and hereditary ideals, all needed to define Quasi-Hereditary Algebras.

DEFINITION 1.3.10 An A-module filtration $0 = X_0 \leq X_1 \leq \ldots \leq X_r = X$ is a Δ-filtration if $X_i/X_{i-1} = 0$ or $X_i/X_{i-1} \cong \Delta(\lambda)$ for some $\lambda \in \Lambda^+$. If such a filtration exists we write $X \in \mathcal{F}(\Delta)$.
One similarly defines ∇-filtration.

DEFINITION 1.3.11 We say mod(A) is a high weight category (with respect to ordering $\leq$) if for all $\lambda \in \Lambda^+$,
(i) $I(\lambda)/\nabla(\lambda) \in \mathcal{F}(\nabla)$,
(ii) whenever $(I(\lambda)/\nabla(\lambda) : \nabla(\mu)) \neq 0$ for $\mu \in \Lambda^+$ then $\mu > \lambda$.

We call the elements of Λ^+ the dominant weights.
This can be defined analogously via $P(\lambda)$ and $\Delta(\lambda)$.

DEFINITION 1.3.12 An ideal H of A is called a hereditary ideal if the following conditions are satisfied;
(i) H is a projective left A-module,
(ii) $\mathrm{Hom}_A(H, A/H) = 0$,
(iii) $HJ(A)H = 0$ where $J(A)$ is the radical of A.

Finally we can define a Quasi-Hereditary Algebra.

DEFINITION 1.3.13 A is a quasi-hereditary algebra if there exists a chain of ideals

$$A = H_0 > H_1 > \ldots > H_n = 0$$

with H_i/H_{i+1} hereditary in A/H_{i+1} for $1 \leq i \leq n$.

PROPOSITION 1.3.14 *[4, Proposition A3.7]*
Suppose mod(A) *is a high weight category with respect to the partial ordering* $\leq$. *Write out the elements of* Λ^+ *as* $\lambda_1, \ldots, \lambda_n$ *such that* $i < j$ *when* $\lambda_i < \lambda_j$, *and define* $\pi(i) = \{\lambda_1, \ldots, \lambda_i\}$ *where* $1 \leq i \leq n$.
Then $A > O^{\pi(1)}(A) > \ldots > O^{\pi(n)}(A) = 0$ *is a hereditary chain of ideals.*
Thus A *is quasi-hereditary.*

We now move on to projective and global dimension, to be able to show all Quasi-Hereditary Algebras have finite global dimension.

DEFINITION 1.3.15 The projective dimension of an $A-$module M is denoted by $\mathrm{Pd}_A(M)$, and is defined as the smallest positive integer n such that $\mathrm{Ext}_A^{n+1}(M, C) = 0$ for all $A-$modules C. If no such n exists, then we say $\mathrm{Pd}_A(M) = \infty$. Equivalently, the projective dimension of M is the smallest integer n such that M has a projective resolution

$$0 \to P^n \to P^{n-1} \to \cdots \to P^0 \to M \to 0,$$

where the P^i are all projective $A-$modules.

DEFINITION 1.3.16 The left global dimension of a finite dimensional algebra A is denoted l.gl.dim A and is defined as

$$\text{l.gl.dim } A = \sup\ \{\text{Pd}_A(M) \mid M \text{ is a finitely-generated left } A\text{-module}\}.$$

Analogously, for the right global dimension of A we have

$$\text{r.gl.dim } A = \sup\ \{\text{Pd}_A(M) \mid M \text{ is a finitely generated right } A\text{-module}\}.$$

DEFINITION 1.3.17 We therefore say that an algebra A has finite global dimension provided there exists some A-module M such that $\text{Pd}_A(M) = n$ and all modules have projective dimension less than or equal to n. If no such module M exists then the algebra has infinite global dimension.

DEFINITION 1.3.18 A ring R is called left Noetherian if every left ideal of R is finitely generated. Similarly, a ring R is right Noetherian if every right ideal of R is finitely generated. We say a ring is Noetherian if it is both left and right Noetherian.

PROPOSITION 1.3.19 *[18, Page 58][13, Proposition 4.7]*
(i) If a ring R is left and right Noetherian, then the left and right global dimensions are equal.
(ii) Suppose R is a finite dimensional algebra over a field k. Then R is both left and right Noetherian.

DEFINITION 1.3.20 We define $l(\lambda) = l$ to be the length of the longest chain $\lambda_0 < \lambda_1 < \ldots < \lambda_l = \lambda$ in Λ^+, and define

$$l(\Lambda^+) = \{\text{maximum of the lengths } l(\lambda) \mid \lambda \in \Lambda^+\}.$$

We then show how all quasi-hereditary algebras have finite global dimension.

PROPOSITION 1.3.21 *[4, Proposition A2.3]*
Let A be a quasi-hereditary algebra. Then $\text{Ext}^i_A(L(\lambda), L(\mu)) = 0$ for $\lambda, \mu \in \Lambda^+$ and $i > l(\lambda)+l(\mu)$, and hence A has finite global dimension bounded by $2l(\Lambda^+)$.

The following remark will be the basis of our final chapter.

REMARK 1.3.22 [3, Example Page 283] The above is not an if and only if statement; there exist algebras of finite global dimension which are not quasi-hereditary.

1.4 The Schur Algebra is Quasi-hereditary

AIM: This section is fairly self-explanatory. We explain why the Schur algebra is quasi-hereditary, defining its standard and costandard modules. We also highlight that this therefore means the Schur algebra has finite global dimension.

DEFINITION 1.4.1 [10, 3.3.1]
The simple $S(n,r)$-modules are $\{L(\lambda) \mid \lambda \in \Lambda^+(n,r)\}$.

DEFINITION 1.4.2 [10, 3.3.2] If $a \geq 1$ is an integer, we denote by $S^a(E)$ the a-th symmetric power of E. More generally, let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of r, then the tensor product of symmetric powers $S^\lambda(E) = S^{\lambda_1}(E) \otimes \ldots \otimes S^{\lambda_n}(E)$ is a $GL_n(k)$-module and hence an $S(n,r)$-module.
The canonical epimorphism $\Psi : E^{\otimes r} \to S^\lambda(E)$ is an $S(n,r)$-homomorphism.

We now go on to define the module $D_{\lambda,k}$, which requires a build up of definitions, we start with the diagram of λ and a bijective λ-tableau.

DEFINITION 1.4.3 [11, 4.2] We call the diagram of λ, the subset

$$[\lambda] = \{(s,t) \mid 1 \leq s, 1 \leq t \leq \lambda_s\}$$

of $\mathbb{Z} \times \mathbb{Z}$. A λ-tableau is a map from $[\lambda]$ to a set, and as $[\lambda]$ has r elements there exists at least one bijection $T : [\lambda] \to \underline{r}$. We choose one bijection and call it the bijective λ-tableau $T = T^\lambda$.

We now move on to define the column stabilizer of T.

DEFINITION 1.4.4 [11, 4.2] If the image under T of (s,t) is $x(s,t)$ we can depict T as follows;

$$\begin{array}{lllll} x(1,1) & x(1,2) & \ldots & \ldots & x(1,\lambda_1) \\ x(2,1) & x(2,2) & \ldots & x(2,\lambda_2) & \\ x(3,1) & x(3,2) & \ldots & & \\ \ldots & \ldots & & & \end{array}$$

So, every element of $\underline{r}$ appears once in the above depiction, and we say $x(s,t)$ is in row s and column t of T. The column stabilizer $C(T)$ of T is the subgroup of $\mathrm{Sym}(r)$ consisting of all $\pi \in \mathrm{Sym}(r)$ which preserve the columns of the above depiction.

We now go on to introduce the bideterminants $(T_i : T_j)$.

DEFINITION 1.4.5 [11, 4.3] If $i = (i_1, \ldots, i_r)$ is an element of $I(n,r)$ we denote the λ-tableau $iT : [\lambda] \to n$ by T_i.
Now let k be an infinite field, and let i, j be elements of $I(n,r)$. Then we define an element $(T_i : T_j) = (T_i : T_j)_k$ of $A(n,r)$ by the formula

$$(T_i : T_j) = \sum_{\pi \in C(T)} \operatorname{sgn}(\pi) c_{i,j\pi} = \sum_{\pi \in C(T)} \operatorname{sgn}(\pi) c_{i\pi,j}.$$

Where $\operatorname{sgn}(\pi)$ is the sign of π. Then, apart from the interchange of rows and columns, $(T_i : T_j)$ is a bideterminant [11, 4.3a].

After defining the following element of $I(n,r)$ we can then define the module $D_{\lambda,k}$.

DEFINITION 1.4.6 [11, 4.3 Example 2 and 4.4] Let ℓ be the element of $I(n,r)$ whose λ-tableau is

$$T_\ell = \begin{bmatrix} 1 & 1 & \ldots & \ldots & 1 \\ 2 & 2 & \ldots & 2 & \\ 3 & 3 & \ldots & & \\ \ldots & & & & \end{bmatrix}$$

Then we define $D_{\lambda,k}$ as the k-span of the bideterminants $(T_\ell : T_i)$ for all $i \in I(n,r)$. Hence $D_{\lambda,k}$ is a subspace of $A(n,r)$.

REMARK 1.4.7 [11, 4.4] The module $D_{\lambda,k}$ is independent of the choice of T.

Having defined $D_{\lambda,k}$ we now go on to show how it can be viewed as an $S(n,r)$-submodule of $A(n,r)$, first showing how $A(n,r)$ is a bimodule for $S(n,r)$.

PROPOSITION 1.4.8 *[11, 4.4] Let $h, j \in I(n,r)$, then the space $A(n,r)$ is a bimodule for $S(n,r)$ via the following;*

$$\xi \circ c_{h,j} = \sum_{i \in I} \xi(c_{i,j}) c_{h,i},$$

$$c_{h,j} \circ \xi = \sum_{i \in I} \xi(c_{h,i}) c_{i,j}.$$

Now, if we replace h by $\ell\pi$, multiply by $s(\pi)$ and sum over all $\pi \in C(T)$, then, for all $\xi \in S(n,r)$, we get

$$\xi \circ (T_{\ell,j}) = \sum_{i \in I} \xi(c_{i,j})(T_\ell : T_i).$$

It thus follows $D_{\lambda,k}$ is a left $S(n,r)$-submodule of $A(n,r)$.

We now move on to the module $V_{\lambda,k}$.

DEFINITION 1.4.9 [11, 2.7 Example 1] For the module $E^{\otimes r}$ we can define a non-singular bilinear form

$$<,>: E^{\otimes r} \times E^{\otimes r} \to k$$

by $< e_i, e_j >= \delta_{ij}$ for all $i, j \in I(n, r)$. This bilinear form is contravariant and we call it the canonical form on $E^{\otimes r}$.

DEFINITION 1.4.10 Let $\lambda \in \Lambda^+(n, r)$, then we denote by N, the kernel of the $S(n, r)$-epimorphism $\phi : E^{\otimes r} \to D_{\lambda,k}$ where $e_j \mapsto (T_{\ell,j})$ for all $j \in I(n, r)$. We have an exact sequence in $M(n, r)$

$$0 \to N \to E^{\otimes r} \to D_{\lambda,k} \to 0.$$

Then we let $V_{\lambda,k}$ be the orthogonal compliment to N, relative to the canonical form $<,>$ on $E^{\otimes r}$. So

$$V_{\lambda,k} = \{x \in E^{\otimes r} \mid \; < x, N >= 0\}.$$

PROPOSITION 1.4.11 *[11, 5.1] The module $V_{\lambda,k} \cong (D_{\lambda,k})^{\circ}$.*

Proof. The canonical form $<,>$ we know to be contravariant, and N is an $S(n, r)$-submodule of $E^{\otimes r}$. Hence $V_{\lambda,k}$ is also a submodule of $E^{\otimes r}$. Moreover, since $<,>$ is non-singular, we can define a non-singular, contravariant form

$$(,) : V_{\lambda,k} \times D_{\lambda,k} \to k$$

by $(x, \phi(y)) =< x, y >$ for all $x \in V_{\lambda,k}$ and $y \in E^{\otimes r}$. Hence $V_{\lambda,k} \cong (D_{\lambda,k})^*$. □

THEOREM 1.4.12 *[10, Theorem 3.4]*
The Schur Algebra $S(n, r)$ is quasi-hereditary with respect to the dominance order and with $\Delta(\lambda) \cong V_{\lambda,k}$ and $\nabla(\lambda) \cong D_{\lambda,k}$.

COROLLARY 1.4.13 *The Schur Algebra has finite global dimension.*

Chapter 2

The Doty Coalgebras

2.1 The Symmetric and Exterior Algebras

AIM: The Doty Coalgebras are the focus of our research, and thus we spend a reasonable amount of time defining them. The symmetric and exterior algebras are themselves $GL_n(k)$-modules and they play a crucial role in defining these Doty coalgebras.
We first define the Symmetric Algebra.

DEFINITION 2.1.1 [8, Introduction] Let $G = SL(2)$, then there exists an irreducible G-module V_m of dimension $m+1$ for $m = 1, 2, \ldots$. Let $e_1, \ldots, e_m$ be a basis for V_{m-1}, then we define the symmetric algebra $S(E) = k[e_1, e_2, \ldots, e_m]$.

We have that $E = ke_1 + ke_2 + \ldots + ke_m \subseteq S(E)$.
There is also a G-action by algebra automorphisms, extending the G-module structure on E and hence we can write $S(E)$ as a G-module decomposition via:

$$S(E) = \bigoplus_{r=0}^{\infty} S^r(E)$$

where $S^r(E) = k\text{-span}\{e_{i_1} \cdot \ldots \cdot e_{i_r} \mid i_1, \ldots, i_r \in \underline{n}\}$.
In particular for $n = 2$ we have

$$S^r(E) = k\text{-sp}\{e_1^r, e_1^{r-1}e_2, e_1^{r-2}e_2^2, \ldots, e_1^2 e_2^{r-2}, e_1 e_2^{r-1}, e_2^r\}.$$

We now introduce the tensor algebra and then define the Exterior Algebra.

DEFINITION 2.1.2 Consider the tensor algebra

$$T(E) = k \ \oplus \ E \ \oplus \ E \otimes E \ \oplus \ E \otimes E \otimes E \ \oplus \ldots$$

with multiplication

$$(x_1 \otimes \ldots \otimes x_r) \cdot (y_1 \otimes \ldots \otimes y_s) = x_1 \otimes \ldots \otimes x_r \otimes y_1 \otimes \ldots \otimes y_s.$$

Let I be the ideal generated by all $x \otimes y - y \otimes x$ where $x, y \in E$. Then we also have that the symmetric algebra $S(E) = T(E)/I$.
Now, let J be the ideal generated by all the $x \otimes y + y \otimes x$ and the $x \otimes x$ for all $x, y \in E$. Then we define the exterior algebra to be

$$\Lambda(E) = T(E)/J.$$

The elements $\hat{x} = x + J$ satisfy $\hat{x}\hat{y} = -\hat{y}\hat{x}$ and $\hat{x}^2 = 0$ for $x, y \in E$.
We write $x_1 \wedge x_2 \wedge \ldots \wedge x_r$ for $x_1 \otimes x_2 \otimes \ldots \otimes x_r + J \in \Lambda(E)$.
If E has a basis $e_1, \ldots, e_n$ then $\Lambda(E)$ has basis $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}$ for $i_1 < i_1 < \ldots < i_r$. As in the case for $S(E)$, we also have $\Lambda(E) = \bigoplus_{r=0}^{\infty} \Lambda^r(E)$ where

$$\Lambda^r(E) = k\text{-span}\{e_{i_1} \wedge \ldots \wedge e_{i_r} | \; i_1 < \ldots < i_r\}.$$

2.2 The Doty Coalgebras

AIM: For C a coalgebra and V a C-comodule we can define the coefficient space $\mathrm{cf}\,(V)$, which is a subcoalgebra of C. In this section we define this coefficient space and give further information about its properties. We then return to the symmetric algebra $S(E)$ and with the use of an ideal $I = \oplus_r I_r$ we form a set of irreducible $GL_n(k)$-modules $\bar{S}^r E = S^r E/I_r$. We then form the truncated modules $\mathrm{Tr}^{\lambda} E = S^{\lambda_1} E \otimes \ldots \otimes S^{\lambda_n} E$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition of r into n parts. In 2.2.13 we then define the Doty coalgebras $D_{n,p}(r) = \sum_{\lambda} \mathrm{cf}\,(\mathrm{Tr}^{\lambda} E)$. We complete this section with an example of the case $D_{2,2}(r)$.

DEFINITION 2.2.1 [12, Section 1.2] Let X and Y be k-spaces with bases $\{\xi_i\}_{i \in I}$ and $\{\eta_j\}_{j \in J}$. Then for all $u \in X \otimes Y$ we define x_i and y_i by $u = \sum_{j \in J} x_j \otimes \eta_j = \sum_{i \in I} \xi_i \otimes y_i$ for some uniquely determined $x_i \in X,\; y_i \in Y$. If $U = \{u_\lambda\}_{\lambda \in \Lambda}$ is a subset of $X \otimes Y$ then in particular

$$u_\lambda = \sum_{\lambda \in \Lambda,\; j \in J} x_{\lambda j} \otimes \eta_j = \sum_{\lambda \in \Lambda,\; i \in I} \xi_i \otimes y_{\lambda i}.$$

We then define the left span of U to be

$$\mathcal{L}(U) = k\text{-sp}\{x_{\lambda j} \mid \lambda \in \Lambda, j \in J\}$$

a subspace of X, and define the right span of U to be

$$\mathcal{R}(U) = k\text{-sp}\{y_{\lambda i} \mid \lambda \in \Lambda, i \in I\}$$

a subspace of Y.

LEMMA 2.2.2 *[12, Section 1.2] Let (V, τ) a right C-comodule. Then for any subset U of V the $\mathcal{L}(\tau(U))$ is a subcomodule of V containing U.*

Having defined the left and right span, we can now define the coefficient space of a module.

DEFINITION 2.2.3 The coefficient space cf (V) is $\mathcal{R}(\tau(V))$. So for $\{v_i\}_{i\in I}$ a basis for V and $\tau : V \to V \otimes C$ such that $\tau(v_j) = \sum_{i\in I} v_i \otimes c_{ij}$ for $j \in I$ then

$$\text{cf}\,(V) = k\text{-sp}\{c_{ij} \mid i, j \in I\}.$$

The following two lemmas give further information on the coefficient space.

LEMMA 2.2.4 *[12, 1.2c]* cf (V) *is independent of the choice of basis of V.*

LEMMA 2.2.5 *[12, 1.2e]* cf (V) *is a subcoalgebra of C.*

We now give an example of a coefficient space.

EXAMPLE 2.2.6 Let $G = GL_2(k)$. Let $V = k\text{-sp}\{v_1, v_2\}$ the natural G-module, and

$$k[G] = k[c_{11}, c_{12}, c_{21}, c_{22}, \det^{-1}].$$

Now, V has structure map $\tau : V \to V \otimes k[G]$ with

$$\tau(v_1) = \sum_{i=1}^{2} v_i \otimes c_{i1} = v_1 \otimes c_{11} + v_2 \otimes c_{21}$$

$$\tau(v_2) = \sum_{i=1}^{2} v_i \otimes c_{i2} = v_1 \otimes c_{21} + v_2 \otimes c_{22}.$$

Then cf $(V) = \mathcal{R}(\tau(V)) = k\text{-sp}\{c_{11}, c_{12}, c_{21}, c_{22}\}$.
Note that if we take two different basis elements for V, say $\{w_1, w_2\}$ we still get cf $(V) = \mathcal{R}(\tau(V)) = k\text{-sp}\{c_{11}, c_{12}, c_{21}, c_{22}\}$.

THEOREM 2.2.7 *Let C be a coalgebra over an algebraically closed field, and V a simple C-comodule. Let $D =$ cf (V), then $D^* \cong M_n(k)$ where $n =$ dim V. Thus D^* is semisimple.*

Proof. Refer to [12, Section 1.3], and choose a dual basis for T. Then there is an isomorphism between the multiplication of these dual elements and matrix multiplication. □

EXAMPLE 2.2.8 Let $n = 2$, C a coalgebra and V a C-comodule. Let $D = \mathrm{cf}(V) = k\text{-sp}\{c_{11}, c_{12}, c_{21}, c_{22}\}$ as in Example 2.2.6. $M_2(k)$ has basis $e_{11}, e_{12}, e_{21}, e_{22}$ where e_{ij} is a 2×2 matrix with 1 in the (i, j)th position, and 0 everywhere else, and D^* has a dual basis $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$.
Multiplication on $M_2(k)$ is as follows; $e_{ij}e_{rs} = \delta_{jr}e_{is}$ where δ is the Kronecker delta. Now consider multiplication in D^*;

$$\begin{aligned}(\alpha_{ij}\alpha_{rs})(c_{ab}) &= (\alpha_{ij} \otimes \alpha_{rs})\delta(c_{ab}) \\ &= (\alpha_{ij} \otimes \alpha_{rs})(\sum_t c_{at} \otimes c_{tb}) \\ &= \sum_t \delta_{ia}\delta_{jt}\delta_{rt}\delta_{sb} \\ &= \delta_{ia}\delta_{rj}\delta_{sb} \\ &= \delta_{rj}\alpha_{is}(c_{ab}).\end{aligned}$$

So, $\alpha_{ij}\alpha_{rs} = \delta_{rj}\alpha_{is}$.
Then we have an isomorphism $\phi : D^* \to M_2(k)$ such that $\phi(\alpha_{rs}) = e_{rs}$, and if $\alpha \in D^*$ then

$$\begin{aligned}\alpha &= \lambda_{11}\alpha_{11} + \lambda_{12}\alpha_{12} + \lambda_{21}\alpha_{21} + \lambda_{22}\alpha_{22} \\ &= \alpha(c_{11})\alpha_{11} + \alpha(c_{12})\alpha_{12} + \alpha(c_{21})\alpha_{21} + \alpha(c_{22})\alpha_{22}.\end{aligned}$$

Then

$$\phi(\alpha) = \begin{pmatrix} \alpha(c_{11}) & \alpha(c_{12}) \\ \alpha(c_{21}) & \alpha(c_{22}) \end{pmatrix}$$

The following looks at coefficient spaces of simple comodules.

REMARK 2.2.9 [12, 1.3.2] For A a finite dimensional algebra over an algebraically closed field k, we have $A/J(A) \cong M_{n_1}(k) \oplus \ldots \oplus M_{n_r}(k)$, and for C a coalgebra with pairwise non-isomorphic simples $V_1, \ldots, V_m$ we have that $C \supseteq \mathrm{cf}\,(V_1) \oplus \ldots \oplus \mathrm{cf}\,(V_m)$ where the $\mathrm{cf}\,(V_i)$ are all simple subcoalgebras.

We now give an example of this.

EXAMPLE 2.2.10 (i) There exists a surjective homomorphism $E \otimes E \to S^2E$ and thus

$$\mathrm{cf}\,(S^2E) \subseteq \mathrm{cf}\,(E \otimes E) = \mathrm{cf}\,(E)\mathrm{cf}\,(E) = A(2, 1)A(2, 1) \subseteq A(2, 2).$$

(ii) Let $p \geq 5$ and let

$$C = A(2, 3) = k\text{-sp}\{c_{ij}c_{kl}c_{mn}\} \mid 1 \leq i, j, k, l, m, n \leq 2\}.$$

Then by Remark 1.2.3, dim $A(2, 3) = 20$. $A(2, 3)$ has two simple comodules; $S^3E = k\text{-sp}\{e_1^3, e_1^2e_2, e_1e_2^2, e_2^3\}$ and $E \otimes \Lambda^2E = k\text{-sp}\{e_1, e_2\} \otimes k\text{-sp}\{e_1 \wedge e_2\}$. Then

$$A(2, 3) = \mathrm{cf}\,(S^3E) \oplus \mathrm{cf}\,(E \otimes \Lambda^2E).$$

We now give two basic definitions which are required to define the Doty Coalgebras.

DEFINITION 2.2.11 Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of r into n parts. Then we call λ a proper partition if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

DEFINITION 2.2.12 A ring A is called graded if $A = \bigoplus_{r=0}^{\infty} A_r$ where all A_r are subspaces of A and we have that $A_r A_s \subseteq A_{r+s}$.
An ideal $I = \bigoplus I_r$ where $I_r = A_r \cap I$ is called homogeneous.

We are now able to define the Doty Coalgebras.

DEFINITION 2.2.13 [9, Section 1] We have from Definition 2.1.1 that

$$S(E) = k[e_1, e_2, \ldots, e_n] = \bigoplus_{r=0}^{\infty} S^r E.$$

Then $S(E)$ is a graded ring. Let I be the ideal generated by $e_1^p, e_2^p, \ldots, e_n^p$ for char $k = p$ prime. This is a GL_n-submodule, and is also a homogeneous ideal with $I = \bigoplus_{r=o}^{\infty} I_r$ where

$$I_r = \mathrm{sp}\{e_1^{m_1} \cdot \ldots \cdot e_n^{m_n} |\ m_1 + \ldots + m_n = r \text{ and some } m_j \geq p\}.$$

Then we have

$$\bar{S}(E) = S(E)/I = \bigoplus_{r=0}^{\infty} S^r E/I_r.$$

So let $\bar{S}^r E = S^r E/I_r$ which are all irreducible. Note that for $r > n(p-1)$ we have $\bar{S}^r E = 0$. Now let $t = n(p-1)$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $t \geq \lambda_1 \geq \ldots \geq \lambda_n \geq 0$. Then we define the truncated modules

$$\mathrm{Tr}^{\,\lambda} E = \bar{S}^{\lambda_1} E \otimes \ldots \otimes \bar{S}^{\lambda_n} E.$$

We then define the Doty Coalgebras to be

$$D_{n,p}(r) = \sum_{\lambda} \mathrm{cf}\,(\mathrm{Tr}^{\,\lambda} E)$$

where λ runs over all partitions of r into n parts, with $t \geq \lambda_1 \geq \ldots \geq \lambda_n \geq 0$.

We follow this with a detailed example of $D_{2,2}$.

EXAMPLE 2.2.14 Let $n = 2$ and $p = 2$. Then $\bar{S}(E) = \bigoplus_{r=0}^{\infty} S^r E/I_r$ and

$$\begin{aligned}
&\bar{S}^0 E = k \\
&\bar{S}^1 E = k\text{-sp}\{e_1, e_2\}/\{0\} = E \\
&\bar{S}^2 E = k\text{-sp}\{e_1^2, e_1e_2, e_2^2\}/k\text{-sp}\{e_1^2, e_2^2\} \cong \Lambda^2 E \\
&\bar{S}^3 E = k\text{-sp}\{e_1^3, e_1^2e_2, e_1e_2^2, e_2^2\}/k\text{-sp}\{e_1^3, e_1^2e_2, e_1e_2^2, e_2^2\} = 0.
\end{aligned}$$

Indeed $\bar{S}^r E = 0$ for $r > 2$, and thus $\bar{S}(E) \cong k \oplus E \oplus \Lambda^2 E$.
As $\lambda = (\lambda_1, \lambda_2)$ runs over all partitions of r into n parts, then in this case λ runs over the partitions $(2,2), (2,1), (2,0), (1,1), (1,0), (0,0)$.
Giving

$$\begin{aligned}
\mathrm{Tr}^{(2,2)} E &= \bar{S}^2 E \otimes \bar{S}^2 E = \Lambda^2 E \otimes \Lambda^2 E, \\
\mathrm{Tr}^{(2,1)} E &= \bar{S}^2 E \otimes \bar{S}^1 E = \Lambda^2 E \otimes E, \\
\mathrm{Tr}^{(2,0)} E &= \bar{S}^2 E \otimes \bar{S}^0 E = \Lambda^2 E \otimes k, \\
\mathrm{Tr}^{(1,1)} E &= \bar{S}^1 E \otimes \bar{S}^1 E = E \otimes E, \\
\mathrm{Tr}^{(1,0)} E &= \bar{S}^1 E \otimes \bar{S}^0 E = E \otimes k, \\
\mathrm{Tr}^{(0,0)} E &= \bar{S}^0 E \otimes \bar{S}^0 E = k \otimes k.
\end{aligned}$$

Now, as an example, let us calculate $\mathrm{cf}\,(\Lambda^2 E \otimes \Lambda^2 E)$. Well, take $g \in GL_2(k)$ and write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\begin{aligned}
g(e_1 \wedge e_2 \otimes e_1 \wedge e_2) &= g(e_1 \wedge e_2) \otimes g(e_1 \wedge e_2) \\
&= (ae_1 + ce_2 \wedge be_1 + de_2) \otimes (ae_1 + ce_2 \wedge be_1 + de_2) \\
&= (ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1)) \otimes (ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1)) \\
&= (ad - bc)e_1 \wedge e_2 \otimes (ad - bc)e_1 \wedge e_2 \\
&= \det(g)e_1 \wedge e_2 \otimes \det(g)e_1 \wedge e_2 \\
&= \det(g)^2 e_1 \wedge e_2.
\end{aligned}$$

Thus $\mathrm{cf}\,(\mathrm{Tr}^{(2,2)} E) = k(\det)^2$.
Similarly we can say that $\mathrm{cf}\,(E) = A(2,1)$ and $\mathrm{cf}\,(E \otimes E) = A(2,2)$. Note also that $k\det \subseteq A(2,2)$ and thus $\mathrm{Tr}^{(2,0)} E$ embeds in $\mathrm{Tr}^{(1,1)} E$. Therefore we have that

$$D_{2,2} = k \oplus A(2,1) \oplus A(2,2) \oplus A(2,1)\det \oplus k\det^2.$$

2.3 Classification by Highest Weights

AIM: To be able to move on to proving our theorem, that $D_{2,p}(r) = A(2, r)$, it is first necessary to introduce further information. This section defines weight spaces and also gives the classification of modules of highest weights which form an irreducible set of $GL_n(k)$-modules: the modules $\{L(m_1, \ldots, m_n)\}$. We again finish on some examples, referring back to the final example in the last section and this time giving the result as tensor products of these irreducible modules of highest weights.
We first define $X(T)$, the group of algebraic group homomorphisms.

REMARK 2.3.1 [24, Section 2.5] Let

$$T = \{\text{diag}(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k^\times\}$$

where $k^\times$ is the multiplicative group of the field k. Define $X(T) = \text{Hom}(T, k^\times)$ the group of algebraic group homomorphisms. Now, given $\lambda = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we define a homomorphism $\tilde{\lambda} : T \to k^\times$ by

$$\tilde{\lambda}(\text{diag}(a_1, \ldots, a_n)) = a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}.$$

Write k_λ for the corresponding simple 1-dimensional module, so T acts on k by $tv = \tilde{\lambda}(t)v$. Then every irreducible module for T has the form k_λ for some $\lambda \in X(T)$. And as every T-module is semisimple then each rational T-module $V \cong \bigoplus_{i \in I} k_{\lambda_i}$ where $\lambda_i \in X(T)$.

The following returns to weight spaces which we follow with an example.

DEFINITION 2.3.2 In Definition 1.3.1 we defined the weight space V^α for $\alpha \in X(n)$. Analogously, for $\lambda \in X(T)$, we can define the λ-weight space

$$V^\lambda = \{v \in V \mid tv = \lambda(t)v \ \forall \ t \in T\}.$$

So, in the case $n = 2$ we have
Let $T = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \mid r, s \in k^\times \right\}$ and define

$$V^{(a,b)} = \{v \in V \mid tv = r^a s^b v \ \forall \ t \in T\}.$$

The fact that V is a polynomial module implies that $V = \bigoplus_{a,b \geq 0} V^{(a,b)}$.

DEFINITION 2.3.3 With V as in the definition above, then we define the character of V to be $\text{ch}\, V = \sum (\dim \ V^{(a,b)}) x_1^a x_2^b$.

EXAMPLE 2.3.4 (i) For $E = ke_1 + ke_2$ we have
$te_1 = re_1 = r^1 s^0 e_1$ and so $E^{(1,0)} = ke_1$ with dim $E^{(0,1)} = 1$, so

$$E = E^{(1,0)} \oplus E^{(0,1)}$$

and

$$\begin{aligned} \operatorname{ch} E &= \sum (\dim V^{(a,b)}) x_1^a x_2^b \\ &= x_1 + x_2. \end{aligned}$$

(ii) For $S^2 E = ke_1^2 + ke_1 e_2 + ke_2^2$ we have
$te_1^2 = (te_1)^2 = r^2 e_1^2$ and so $E^{(2,0)} = ke_1^2$ with dim $E^{(2,0)} = 1$
$te_1 e_2 = re_1 se_2$ and so $E^{(1,1)} = ke_1 e_2$ with dim $E^{(1,1)} = 1$
$te_2^2 = (te_2)^2 = s^2 e_2^2$ and so $E^{(0,2)} = ke_2^2$ with dim $E^{(0,2)} = 1$
so

$$\operatorname{ch} S^2 E = x_1^2 + x_1 x_2 + x_2^2.$$

(iii) $\operatorname{ch} \Lambda^2 E = x_1 x_2$.

DEFINITION 2.3.5 We say a module has highest weight $(m_1, \ldots, m_n)$, if, $(m_1, \ldots, m_n)$ is itself a weight, and, with respect to the dominance order, $(m_1, \ldots, m_n) \trianglerighteq (r_1, \ldots, r_n)$ for all other weights r.
For example with $E = E^{(1,0)} \oplus E^{(0,1)}$ we have that $(1,0) \trianglerighteq (0,1)$, and E has highest weight at $(1,0)$.

The above information now allows us to look at the following theorem.

THEOREM 2.3.6 *[11, Theorem 3.5a] Classification by Highest Weights*
For each $(m_1, \ldots, m_n)$ with $m_1 \geq m_2 \geq \ldots \geq m_n > 0$, there exists an irreducible module of highest weight $(m_1, m_2, \ldots, m_n)$. We call this module $L(m_1, m_2, \ldots, m_n)$. The $\{L(m_1, m_2, \ldots, m_n) \mid m_1 \geq \ldots \geq m_n\}$ form a full set of irreducible rational $GL_n(k)$-modules.

REMARK 2.3.7 For G-modules V_1 and V_2 we have;
(i) $\operatorname{ch}(V_1 \otimes V_2) = (\operatorname{ch} V_1)(\operatorname{ch} V_2)$
(ii) V_1, V_2 have the same composition factors (including multiplicities) if and only if $\operatorname{ch} V_1 = \operatorname{ch} V_2$

We finally return to our example from the last section.

EXAMPLE 2.3.8 (i) Consider $\Lambda^2(E) = k(e_1 \wedge e_2)$, then

$$\begin{aligned} t(e_1 \wedge e_2) &= te_1 \wedge te_2 \\ &= ae_1 \wedge be_2 \\ &= ab(e_1 \wedge e_2) \end{aligned}$$

Hence $\Lambda^2(E) = L(1,1)$.
(ii) $L(r,0)$ is the submodule of $S^r E$ generated by e_1^r.
(iii) $L(r+s,s) = L(r,0) \otimes \Lambda^2 E \otimes \ldots \otimes \Lambda^2 E$ (s times).
(iv) Consider Example 2.2.14 where $n = 2$ and $p = 2$, then

$$\bar{S}(E) = k \oplus E \oplus \Lambda^2 E = L(0,0) \oplus L(1,0) \oplus L(1,1),$$

and

$$\begin{aligned}
\mathrm{Tr}^{(2,2)}E &= \bar{S}^2 E \otimes \bar{S}^2 E = \Lambda^2 E \otimes \Lambda^2 E = L(1,1) \otimes L(1,1) \cong L(2,2),\\
\mathrm{Tr}^{(2,1)}E &= \Lambda^2 E \otimes E = L(1,1) \otimes L(1,0) \cong L(2,1),\\
\mathrm{Tr}^{(2,0)}E &= \Lambda^2 E \otimes k = L(1,1) \otimes L(0,0) \cong L(1,1),\\
\mathrm{Tr}^{(1,1)}E &= E \otimes E = L(2,0) + 2L(1,1),\\
\mathrm{Tr}^{(1,0)}E &= E \otimes k = L(1,0) \otimes L(0,0) \cong L(1,0),\\
\mathrm{Tr}^{(0,0)}E &= k \otimes k = L(0,0) \otimes L(0,0) \cong L(0,0).
\end{aligned}$$

2.4 Tilting Modules

AIM: Tilting modules play a crucial role in proving our theorem $D_{2,p}(r) = A(2,r)$, and so we introduce these now. In this section we also give a theorem that states if the coefficient spaces of the tilting modules of $A(2,r)$ are contained within $D_{2,p}(r)$, then we will have proven that $D_{2,p}(r) = A(2,r)$ and thus that $D_{2,p}(r)$ is quasi-hereditary and has finite global dimension, the result required. It therefore remains to find the coefficient spaces of these tilting modules!

DEFINITION 2.4.1 Define $\nabla(r,s) = S^{r-s}E \otimes k\det^{\otimes s}$ for $r \geq s$.
Then for $G = GL_2(k)$ a G-module M has a good filtration if there is a filtration $M = M_0 > M_1 > M_2 > \ldots > M_r = 0$ where each $M_i/M_{i+1} \cong \nabla(a,b)$, for some $(a,b) \in X^+(2)$.

DEFINITION 2.4.2 We call a finite dimensional rational module M, a tilting module, if M and M^* both have a good filtration.

REMARK 2.4.3 [5, Theorem 1.1, Proposition 1.2]
(i) For $\lambda = (r_1, \ldots, r_n)$ with $r_1 \geq \ldots \geq r_n$ there exists a tilting module $M(r_1, \ldots, r_n)$ with unique highest weight $(r_1, \ldots, r_n)$, (occurring once).
(ii) Any tilting module is isomorphic to a direct sum of these $M(r_1, \ldots, r_n)$.
(iii) The tensor product of two tilting modules is again a tilting module.

So far we have considered GL, but now, having defined a tilting module we classify them for SL_2.

THEOREM 2.4.4 *[5, Section 2 Ex. 2] Classifying the Tilting Modules for* SL_2
(i) For $r \leq p-1$*, then* $T(r) = L(r) = S^r E$*.*
(ii) For $r = p$*, then* $T(r) = T(p) = E \otimes S^{p-1}E$*, and there exists a short exact sequence*

$$0 \to \Lambda^2 E \otimes S^{p-2}E \to E \otimes S^{p-1}E \to S^p E \to 0$$

and we notate this by

$$T(p) = \frac{S^p E}{S^{p-2}E}.$$

(iii) For $p-1 \leq r \leq 2p-2$*, then*

$$T(p-1+a) = \frac{S^{p-1+a}E}{S^{p-1-a}E}$$

where $1 \leq a \leq p-1$*.*
(iv) For $p-1 \leq r \leq 2p-2+cp$*, let* $s = b+pc$ *where* $p-1 \leq b \leq 2p-2$ *and* $c \geq 0$*, then* $T(s) \cong T(b) \otimes T(c)^F$*.*

REMARK 2.4.5 For $GL_2(k)$, with $a \geq b$, then

$$T(a,b) \cong T(a-b,0) \otimes (\Lambda^2 E)^{\otimes b},$$

and so for $SL_2(k)$, $T(a,b) \downarrow_{SL_2(k)} = T(a-b)$.
In particular $T(r,0) \downarrow_{SL_2(k)} \cong T(r)$. For example $T(4,3) \cong T(1,0) \otimes (\Lambda^2 E)^{\otimes 3}$ and $T(4,3) \downarrow_{SL_2(k)} = T(1)$.

The following theorem is then crucial to proving our main theorem 3.1.1 for the case $n = 2$.

THEOREM 2.4.6 *Suppose* $\{T(\lambda) |\ \lambda \in \Lambda\}$ *is a full set of tilting modules for the coalgebra* $A(n,r)$*. Suppose also that* $\mathrm{cf}\,(T(\lambda)) \subseteq D_{n,p}(r)$ *for all* $\lambda \in \Lambda^+(n,r)$*.*
Then $D_{n,p}(r) = A(n,r)$*.*

Proof. By [7, Corollary 7.3] we have that $A(n,r) = \sum_{\lambda \in \Lambda^+(n,r)} \mathrm{cf}\,(T(\lambda))$, and hence if $\mathrm{cf}\,(T(\lambda)) \subseteq D_{n,p}(r)$ for all $\lambda \in \Lambda^+(n,r)$, then $A(n,r) \subseteq D_{n,p}(r)$. We also have by definition that $D_{n,p}(r) \subseteq A(n,r)$ and hence $A(n,r) = D_{n,p}(r)$. □

Chapter 3

The case $n = 2$ for all primes p

3.1 The Theorem for n=2

AIM: This short section gives the main theorem we will prove for the case $n = 2$, and explains briefly how we will go about proving this theorem.

THEOREM 3.1.1 *The Doty Coalgebras $D_{2,p}(r)$ are quasi-hereditary for all primes p and $0 \leq r \leq 2t$, where $t = 2(p-1)$, as*

$$D_{2,p}(r) = \begin{cases} A(2,r) & 0 \leq r \leq t \\ A(2,t-j)det^j & r = t+j,\ 0 \leq j \leq t \\ 0 & r > 2t \end{cases}$$

REMARK 3.1.2 By Theorem 2.4.6 we know that

$$D_{2,p}(r) = A(2,r) \text{ if cf } (T(\lambda)) \subseteq D_{2,p}(r)$$

for all tilting modules $T(\lambda)$ of $A(2,r)$. Thus we prove Theorem 3.1.1, by finding the coefficient spaces of these tilting modules within the Doty Coalgebra $D_{2,p}(r)$.

The proof splits into three parts;
(i) $0 \leq r \leq p-1$
(ii) $p \leq r \leq t$
(iii) $t+1 \leq r \leq 2t$.

Before we begin these parts of the proof we shall first do some general work on the Doty Coalgebra $D_{2,p}(r)$.

3.2 Generalisation of the $D_{2,p}(r)$ for all p and all r

AIM: This section describes the $\bar{S}^r E$ for $0 \le r \le t$ and then describes the truncated modules $\mathrm{Tr}^{\lambda} E$. We then give a full description of the $D_{2,p}(r) = \sum \mathrm{cf}\,(\mathrm{Tr}^{\lambda} E)$. Examples are given throughout the section.

THEOREM 3.2.1 *We have*

$$\bar{S}^r E \cong \begin{cases} S^r E, & \text{if } 0 \le r \le p-1 \\ S^{t-r} E \otimes \Lambda^2 E^{\otimes r-p+1}, & \text{if } p \le r \le t \end{cases}$$

Proof. For $0 \le r \le p-1$ we have $\bar{S}^r E = k\text{-sp}\{e_1^r, e_1^{r-1} e_2, \ldots, e_1 e_2^{r-1}, e_2^r\}/0 = S^r E$.

For $p \le r \le t$ we have;

$$\begin{aligned}
\bar{S}^r E &= k\text{-sp}\{e_1^r, e_1^{r-1}e_2, \ldots, e_1 e_2^{r-1}, e_2^r\}/ \\
&\quad\; k\text{-sp}\{e_1^r, e_1^{r-1}e_2, \ldots, e_1^p e_2^{r-p}, e_1^{r-p} e_2^p, \ldots, e_1 e_2^{r-1}, e_2^r\} \\
&= k\text{-sp}\{e_1^{p-1} e_2^p + I_p, e_1^{p-2} e_2^2 + I_p, \ldots, e_1^2 e_2^{p-2} + I_p, e_1 e_2^{p-1} + I_p\} \\
&= L(p-1, r-(p-1)) \ \text{ by Theorem 2.3.6} \\
&\cong L((p-1)-(r-(p-1)), 0) \otimes L(1,1)^{\otimes r-(p-1)} \\
&\cong S^{(p-1)-(r-(p-1))} E \otimes \Lambda^2 E^{\otimes r-(p-1)} \\
&\cong S^{2p-2-r} E \otimes \Lambda^2 E^{\otimes r-(p-1)} \\
&\cong S^{t-r} E \otimes \Lambda^2 E^{\otimes r-p+1}
\end{aligned}$$

□

EXAMPLE 3.2.2 Take $n = 2$ and $p = 5$, then $t = n(p-1) = 8$.
We have that $\bar{S}^r E = S^r E$ for $0 \le r \le 4 = p-1$,

$$\begin{aligned}
\bar{S}^5 E &= S^5 E / k\text{-sp}\{e_1^5, e_2^5\} \\
&= k\text{-sp}\{e_1^4 e_2 + I_5, e_1^3 e_2^2 + I_5, e_1^2 e_2^3 + I_5, e_1 e_2^4 + I_5\} \\
&= L(4,1) \\
&\cong L(3,0) \otimes L(1,1) \\
&\cong S^3 E \otimes \Lambda^2 E \\
&\cong S^{8-5} E \otimes \Lambda^2 E^{\otimes 5-5+1}.
\end{aligned}$$

$$\begin{aligned}
\bar{S}^6 E &= S^6 E / k\text{-sp}\{e_1^6, e_1^5 e_2, e_1 e_2^5, e_2^6\} \\
&= k\text{-sp}\{e_1^4 e_2^2 + I_6, e_1^3 e_2^3 + I_6, e_1^2 e_2^4 + I_6\} \\
&= L(4,2) \\
&\cong L(2,0) \otimes L(1,1)^{\otimes 2} \\
&\cong S^2 E \otimes \Lambda^2 E^{\otimes 2} \\
&\cong S^{8-6} E \otimes \Lambda^2 E^{\otimes 6-5+1}.
\end{aligned}$$

$$
\begin{aligned}
\bar{S}^7E &= S^7E/k\text{-sp}\{e_1^7, e_1^6e_2, e_1^5e_2^2, e_1^2e_2^5, e_1e_2^6, e_2^7\}\\
&= k\text{-sp}\{e_1^4e_2^3 + I_7, e_1^3e_2^4 + I_7\}\\
&= L(4,3)\\
&\cong L(1,0) \otimes L(1,1)^{\otimes 3}\\
&\cong S^1E \otimes \Lambda^2E^{\otimes 3}\\
&\cong S^{8-7}E \otimes \Lambda^2E^{\otimes 7-5+1}.
\end{aligned}
$$

$$
\begin{aligned}
\bar{S}^8E &= S^8E/k\text{-sp}\{e_1^8, e_1^7e_2, e_1^6e_2^2, e_1^5e_2^3, e_1^3e_2^5, e_1^2e_2^6, e_1e_2^7, e_2^8\}\\
&= k\text{-sp}\{e_1^4e_2^4 + I_8\}\\
&= L(4,4)\\
&= L(1,1)^{\otimes 4}\\
&= \Lambda^2E^{\otimes 4}\\
&= S^{8-8}E \otimes \Lambda^2E^{\otimes 8-5+1}.
\end{aligned}
$$

Having looked at the $\bar{S}^rE$ we now look at the number of ways we can partition r into $n = 2$ parts.

THEOREM 3.2.3 *We have* $D_{2,p}(r) = \sum \mathrm{cf}\,(\mathrm{Tr}^{\,(\lambda_1,\lambda_2)}E)$ *where* $\lambda_1 + \lambda_2 = r$ *and* $t \geq \lambda_1 \geq \lambda_2$.
Then the number of such partitions (λ_1, λ_2) *is as follows:*

$$
\begin{array}{ll}
\frac{r+1}{2} & \text{for } \ 0 \leq r \leq t, r\,odd\\
\frac{r+2}{2} & \text{for } \ 0 \leq r \leq t, r \ even\\
\frac{r+1}{2} - (r-t) & \text{for } \ t+1 \leq r \leq 2t, r \ odd\\
\frac{r+2}{2} - (r-t) & \text{for } \ t+1 \leq r \leq 2t, r \ even
\end{array}
$$

Proof. For r even, the number of ways of partitioning r into two parts is $\frac{r+2}{2}$, and for r odd, it is $\frac{r+1}{2}$.
For example;
$r = 1$ $\quad \frac{1+1}{2} = 1$, so 1 can be partitioned into $(1,0)$

$r = 2$ $\quad \frac{2+2}{2} = 2$, so 2 can be partitioned into $(2,0)$ and $(1,1)$

$r = 3$ $\quad \frac{3+1}{2} = 2$, so 3 can be partitioned into $(3,0)$ and $(2,1)$

$r = 4$ $\quad \frac{4+2}{2} = 3$, so 4 can be partitioned into $(4,0)$, $(3,1)$ and $(2,2)$

$r = 5$ $\quad \frac{5+1}{2} = 3$, so 5 can be partitioned into $(5,0)$, $(4,1)$ and $(3,2)$

So, we would have for $r/2 \leq \lambda_1 \leq r$ for r even, and $(r+1)/2 \leq \lambda_1 \leq r$ for r odd. However, in our case, $\lambda_1 \leq t$, thus restricting the number of partitions available. For example in the case above, when $p = 3$ then $t = 4$ and 5 partitions as $(4,1)$ and $(3,2)$ only.

To make this adjustment to the number of partitions, we find $r-t$ whenever $r \geq t+1$, (when $r \leq t$ we do not need to make an adjustment). Then for r even the number of partitions is $\frac{r+2}{2} - (r-t)$ and for r odd then number is $\frac{r+1}{2} - (r-t)$. $\square$

EXAMPLE 3.2.4 $n=2, p=3, t=4$

$r=1$ $\quad \frac{1+1}{2} = 1$, so 1 can be partitioned into $(1,0)$

$r=2$ $\quad \frac{2+2}{2} = 2$, so 2 can be partitioned into $(2,0)$ and $(1,1)$

$r=3$ $\quad \frac{3+1}{2} = 2$, so 3 can be partitioned into $(3,0)$ and $(2,1)$

$r=4$ $\quad \frac{4+2}{2} = 3$, so 4 can be partitioned into $(4,0)$, $(3,1)$ and $(2,2)$

$r=5$ $\quad \frac{5+1}{2} - (5-4) = 2$, so 5 can be partitioned into $(4,1)$ and $(3,2)$

$r=6$ $\quad \frac{6+2}{2} - (6-4) = 2$, so 6 can be partitioned into $(4,2)$ and $3,3)$

$r=7$ $\quad \frac{7+1}{1} - (7-4) = 1$, so 7 can be partitioned into $(4,3)$

$r=8$ $\quad \frac{8+2}{2} - (8-4) = 1$, so 8 can be partitioned into $(4,4)$

Now that we have an understanding of what the $\bar{S}^r E$ look like we can formulate the truncated modules $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ whose coefficient spaces sum together to give the Doty Coalgebras.

THEOREM 3.2.5 *The truncated modules* $\mathrm{Tr}^{\lambda}E$ *are equal to*
i) $S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes k\mathrm{det}^{(\lambda_1-p+1)+(\lambda_2-p+1)}$, *for* $p \leq \lambda_1, \lambda_2 \leq t$
ii) $S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes k\mathrm{det}^{(\lambda_1-p+1)}$, *for* $p \leq \lambda_1 \leq t, \quad 0 \leq \lambda_2 \leq p-1$
iii) $S^{\lambda_1}E \otimes S^{\lambda_2}E$, *for* $0 \leq \lambda_1, \lambda_2 \leq p-1$

Proof. (iii) For $0 \leq \lambda_i \leq p-1$, we have $\bar{S}^{\lambda_i}E = S^{\lambda_i}E$ for $i=1,2$.
Hence $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E = \bar{S}^{\lambda_1}E \otimes \bar{S}^{\lambda_2}E = S^{\lambda_1}E \otimes S^{\lambda_2}E$.

(ii) For $0 \leq \lambda_2 \leq p-1$ and $p \leq \lambda_1 \leq t$, we have $\bar{S}^{\lambda_1}E = S^{t-\lambda_1}E \otimes \Lambda^2 E^{\otimes \lambda_1-p+1}$ and $\bar{S}^{\lambda_2}E = S^{\lambda_2}E$. Hence

$$\begin{aligned}
\mathrm{Tr}^{(\lambda_1,\lambda_2)}E &= \bar{S}^{\lambda_1}E \otimes \bar{S}^{\lambda_2}E \\
&= S^{(t-\lambda_1)}E \otimes \Lambda^2 E^{\otimes \lambda_1-p+1} \otimes S^{\lambda_2}E \\
&= S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1-p+1} \\
&= S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \mathrm{det}^{\lambda_1-p+1}.
\end{aligned}$$

(i) For $p \le \lambda_i \le t$ we have $\bar{S}^{\lambda_i}E = S^{(t-\lambda_i)}E \otimes \Lambda^2 E^{\otimes \lambda_i - p+1}$ for $i = 1, 2$. Hence

$$\begin{aligned}
\mathrm{Tr}^{(\lambda_1,\lambda_2)}E &= \bar{S}^{\lambda_1}E \otimes \bar{S}^{\lambda_2}E \\
&= S^{(t-\lambda_1)}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1} \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes \lambda_2 - p+1} \\
&= S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1} \otimes \Lambda^2 E^{\otimes \lambda_2 - p+1} \\
&= S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes (\lambda_1 - p+1)+(\lambda_2-p+1)} \\
&= S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \det{}^{\otimes (\lambda_1 - p+1)+(\lambda_2-p+1)}
\end{aligned}$$

□

REMARK 3.2.6 For future use we will call;
$\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ with $p \le \lambda_i \le t$ for $i = 1, 2$; Category 1.
$\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ with $p \le \lambda_1 \le t, 0 \le \lambda_2 \le p-1$; Category 2.
$\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ with $0 \le \lambda_i \le p-1$ for $i = 1, 2$; Category 3.

Having generalised the $\bar{S}^r E$ and the $\mathrm{Tr}^{\lambda}E$ along with the partitions of r into n parts, we can now formulate the Doty Coalgebras.

THEOREM 3.2.7 *For r even let $\iota = \frac{r}{2}$ and for r odd let $\iota = \frac{r+1}{2}$. Then we have that $D_{2,p}(r)$ is equal to one of the following;*

i) $\sum_{\iota \le \lambda_1 \le r} \mathrm{cf}\,(S^{\lambda_1}E \otimes S^{\lambda_2}E)$ *if* $0 \le r \le p-1$
ii) $\sum_{\iota \le \lambda_1 \le p-1} \mathrm{cf}\,(S^{\lambda_1}E \otimes S^{\lambda_2}E) + \sum_{p \le \lambda_1 \le r} \mathrm{cf}\,(S^{t-\lambda_1}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1})$ *if* $p \le r \le t$
iii) $\sum_{\iota \le \lambda_1 \le t} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1})$ *if* $r = t+1$
iv) $\sum_{r-p+1 \le \lambda_1 \le t} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1}) + \sum_{\iota \le \lambda_1 \le r-p} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes (\lambda_1 - p+1)+(\lambda_2-p+1)})$ *if* $t+2 \le r \le t+(p-1)$
v) $\sum_{\iota \le \lambda_1 \le t} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes (\lambda_1 - p+1)+(\lambda_2-p+1)})$ *if* $t+p \le r \le 2t$

Proof. (i) $0 \le r \le p-1$
In this case, where $r = \lambda_1 + \lambda_2$, $\lambda_1 \le p-1 < t$, so we have no restrictions on the number of partitions and there will be $\frac{r+2}{2}$ for r even, and $\frac{r+1}{2}$ for r odd. Referring now to the $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$, then as $\lambda_1 \le p-1$ and $\lambda_2 \le \lambda_1$, then both $\lambda_1, \lambda_2 \le p-1$ and so all $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ will be from Category 3. Hence $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E = S^{\lambda_1}E \otimes S^{\lambda_2}E$.
Recall also that $\frac{r}{2} \le \lambda_1$ for r even and $\frac{r+1}{2} \le \lambda_1$ for r odd. So we have

$$D_{2,p}(r) = \sum_{\iota \le \lambda_1 \le r} \mathrm{cf}\,(S^{\lambda_1}E \otimes S^{\lambda_2}E) \text{ for } 0 \le r \le p-1.$$

(ii) $p \le r \le t$
Again, as $r \le t$, then $\lambda_1 \le t$ and so we have no restrictions on the number of partitions. Thus there will be $\frac{r+2}{2}$ for r even and $\frac{r+1}{2}$ for r odd.We now want

to know the range of λ_1 and λ_2 to know which categories the $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ will come from.
We have $p \leq r \leq t$, so similarly, $\lambda_1 \leq t$. Take r greatest, so $r = t$. As $t = 2(p-1)$ then t is always even. Thus the smallest λ_1 can be is $\frac{t}{2}$, when $r = t$. In this case $\lambda_2 = r - \lambda_1 = t - \frac{t}{2} = \frac{t}{2} = \frac{2(p-1)}{2} = p - 1$. So, $\lambda_2 < p$. Obviously, when $\lambda_1 = t$, then $\lambda_2 = t - \lambda_1 = 0$. So, $0 \leq \lambda_2 \leq p-1$.
When r is smallest, $r = p$ and p odd, so with $\frac{r+1}{2} \leq \lambda_1 \leq r$ then $\frac{p+1}{2} \leq \lambda_1 \leq p$. Hence, overall $\frac{p+1}{2} \leq \lambda_1 \leq t$. Thus, in this case, we have $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ from both Categories 1 and 2. So

$$\mathrm{Tr}^{(\lambda_1,\lambda_2)}E = S^{\lambda_1}E \otimes S^{\lambda_2}E \text{ for } \frac{p+1}{2} \leq \lambda_1 \leq p-1$$

and

$$\mathrm{Tr}^{(\lambda_1,\lambda_2)}E = S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes(\lambda_1 - p+1)} \text{ for } p \leq \lambda_1 \leq t.$$

Then, for $p \leq r \leq t$, we have

$$D_{2,p}(r) = \sum_{\iota \leq \lambda_1 \leq p-1} \mathrm{cf}\,(S^{\lambda_1}E \otimes S^{\lambda_2}E) + \sum_{p \leq \lambda_1 \leq r} \mathrm{cf}\,(S^{t-\lambda_1}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1}).$$

(iii) $r = t+1$
For $r = t+1$ there is a restriction on the number of partitions. As t is always even, then $t+1$ is always odd, so the number of partitions is $\frac{r+1}{2} - (r-t) = p-1$. With $r = t+1$ we have $\frac{r+1}{2} = \frac{t+1+1}{2} \leq \lambda_1 \leq t$. Now $\frac{t+1+1}{2} = \frac{t+2}{2} = \frac{t}{2} + 1 = \frac{2(p-1)}{2} + 1 = p-1+1 = p$, and thus $p \leq \lambda_1 \leq t$. When $\lambda_1 = p$, then $\lambda_2 = t+1-p = 2p-2+1-p = p-1$. Hence $1 \leq \lambda_2 \leq p-1$. So, in this case, the $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ come from Category 2, and thus for $r = t+1$,

$$D_{2,p}(r) = \sum_{\iota \leq \lambda_1 \leq t} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1}).$$

(iv) $t+2 \leq r \leq t+(p-1)$
As in (iii), $r < t+1$ and so there is a restriction on the number of partitions, namely $\frac{r+2}{2} - (r-t)$ for r even and $\frac{r+1}{2} - (r-t)$ for r odd. Now $r \geq t+2$ and so $\frac{t+2}{2} \leq \lambda_1 \leq t \Rightarrow p \leq \lambda_1 \leq t$. If $r = t+2$ then $\lambda_2 = r - \lambda_1 = t+2-p = 2p-2+2-p = p$, so $2 \leq \lambda_2 \leq p$ and thus we have $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ from both Categories 1 and 2.

Then

$$D_{2,p}(r) = \sum_{r-p+1 \le \lambda_1 \le t} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1})$$
$$+ \sum_{\iota \le \lambda_1 \le r-p} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes(\lambda_1-p+1)+(\lambda_2-p+1)})$$

for $t+2 \le r \le t+(p-1)$.

Note: In Category 2, the greatest λ_2 can be is $p-1$. So we have $\lambda_2 = r - \lambda_1 \le p-1 \Rightarrow r - \lambda_1 \le p-1 \Rightarrow -\lambda_1 \le p-1-r \Rightarrow \lambda_1 \ge r-p+1$.

(v) $t+p \le r \le 2t$
Again there is a restriction on the number of partitions, as in (iv). Also $\lambda_1 \ge \frac{r}{2}$ for r even and $\lambda_1 \ge \frac{r+1}{2}$ for r odd. In both cases $\lambda_1 \ge p$. To find the smallest λ_2 can be, take r least at $r = t+p$. We have $\lambda_1 \le t$, so $\lambda_2 = r - \lambda_1 = t+p-t = p$, and hence $\lambda_2 \ge p$, and we thus we have $\mathrm{Tr}^{(\lambda_1,\lambda_2)}E$ from Category 1 only.

So for $t+p \le r \le 2t$ we have that

$$D_{2,p}(r) = \sum_{\iota \le \lambda_1 \le t} \mathrm{cf}\,(S^{(t-\lambda_1)}E \otimes S^{(t-\lambda_2)}E \otimes \Lambda^2 E^{\otimes(\lambda_1-p+1)+(\lambda_2-p+1)}).$$

□

3.3 Proof of Theorem 3.1.1 part (i)

THEOREM 3.3.1 *The Doty Coalgebras $D_{2,p}(r) = A(2,r)$ for $0 \le r \le p-1$*

In Definition 2.3.3 we defined the character of a module. As we prove the above theorem, we will be viewing characters both in terms of SL_2-modules and in terms of GL_2-modules. We therefore prove the following two lemmas so we understand the relationship between GL and SL.

LEMMA 3.3.2 *Let X and Y be polynomial modules of degree r. Then*

$$\mathrm{Hom}_{GL}(X,Y) = \mathrm{Hom}_{SL}(X,Y).$$

Proof. We have $\mathrm{Hom}_{GL}(X,Y) \subseteq \mathrm{Hom}_{SL}(X,Y)$. Now let

$$Z = \{\mathrm{diag}(t,\ldots,t) \mid t \in k^*\}$$

with k algebraically closed. Then $GL = ZSL$. Take $z_t = \mathrm{diag}(t,t,\ldots,t)$, then we now make the following claim;

CLAIM 3.3.3 If V is a finite dimensional polynomial of degree r then

$$z_t v = t^r v$$

for $v \in V$.

Proof. Let $V = E^{\otimes r}$, then V is spanned by vectors $e = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}$. So

$$z_t e = (te_{i_1}) \otimes \ldots \otimes (te_{i_r}) = t^r e.$$

Now let $\pi : E^{\otimes r} \to S^\alpha E$, such that α has at most n parts and $|\alpha| = r$, then for $v \in S^\alpha E$ we have $v = \pi(u)$ where $u \in E^{\otimes r}$. Hence

$$z_t v = z_t \pi(u) = \pi(z_t u) = \pi(t^r u) = t^r \pi(u) = t^r v.$$

Now, every module embeds in a direct sum of these $S^\alpha E$ and hence with $\phi : V \hookrightarrow M = \oplus(S^\alpha E)$ then for $v \in V$ we have

$$\phi(z_t v) = z_t \phi(v) = t^r \phi(v) = \phi(t^r v)$$

and thus $z_t v = t^r v$ as required. □

So, returning to our Lemma, if $\theta \in \mathrm{Hom}_{SL}(X, Y)$ and $g \in GL$ then $g = z_t h$ where $h \in SL$ and

$$\begin{aligned} \theta(gx) &= \theta(z_t hx) \\ &= \theta(t^r hx) \\ &= t^r \theta(hx) \\ &= t^r h\theta(x) \\ &= z_t h\theta(x) \\ &= g\theta(x). \end{aligned}$$

□

LEMMA 3.3.4 *If X and Y are polynomial modules of degree r then*

$$X \cong Y \text{ as } GL\text{-modules} \Leftrightarrow X \cong Y \text{ as } SL\text{-modules.}$$

Proof. ‘ $\Rightarrow$’ If $\phi : X \to Y$ is a GL-module isomorphism then certainly it is an SL-module isomorphism.
‘ $\Leftarrow$’ Suppose $\phi : X \to Y$ is an SL-module isomorphism. Then

$$\phi \in \mathrm{Hom}_{SL}(X, Y) = \mathrm{Hom}_{GL}(X, Y)$$

by Lemma 3.3.2, so ϕ is a GL-module map and is one-to-one and onto and thus is an isomorphism. □

Having clarified the relationship between GL and SL we now give the following formula which gives a way of working with characters of modules.

THEOREM 3.3.5 *The Clebsch-Gordon Formula[16, Chapter 22 Exercise 7] Let $G = SL_2(\mathbb{C})$ and let V_r be an irreducible G-module of dimension $r+1$. So, $V_0 \cong k, V_1 \cong E, V_2 \cong S^r E, ...$*
Take $r \leq s$, then $V_r \otimes V_s \cong V_{r+s} \oplus V_{r+s-2} \oplus \ldots \oplus V_{s-r}$. Now take $\chi(r) = \operatorname{ch} S^r E = x^r + x^{(r-2)} + \ldots + x^{-r}$, then we have

$$\chi(r)\chi(s) = \chi(r+s) + \chi(r+s-2) + \ldots + \chi(s-r).$$

EXAMPLE 3.3.6 Let $p = 5$,

$$\begin{aligned}\operatorname{ch}(T(4) \otimes T(3)) &= \operatorname{ch} T(4)\ \operatorname{ch} T(3)\\ &= \operatorname{ch}(S^4 E)\ \operatorname{ch}(S^3 E)\\ &= \chi(4)\chi(3)\\ &= \chi(7) + \chi(5) + \chi(3) + \chi(1).\end{aligned}$$

REMARK 3.3.7 (i) Let $a \geq b$, then

$$\chi(a,b) = \chi(1,1)\chi(a-b,0) = \operatorname{ch}(S^{a-b}E \otimes (\Lambda^2 E)^{\otimes b}),$$

(ii) $\operatorname{ch} T(r) = \chi(r)$ for $r \leq p-1$,
(iii) $\operatorname{ch} T(p-1+m) = \chi(p-1+m) + \chi(p-1-m)$ where $p-1+m \leq 2p-2$.

EXAMPLE 3.3.8 Let $p = 5$,
(i)

$$\begin{aligned}\operatorname{ch} T(7) &= \operatorname{ch} T(p-1+3)\\ &= \operatorname{ch} S^{p-1+3}E + \operatorname{ch} S^{p-1-3}E\\ &= \operatorname{ch} S^7 E + \operatorname{ch} S^1 E\\ &= \chi(7) + \chi(1).\end{aligned}$$

(ii)

$$\begin{aligned}\operatorname{ch} T(5) &= \operatorname{ch} T(p)\\ &= \operatorname{ch}(S^p E) + \operatorname{ch}(S^{p-2}E)\\ &= \operatorname{ch} S^5 E + \operatorname{ch} S^3 E\\ &= \chi(5) + \chi(3).\end{aligned}$$

Having gathered this information we are now able to prove Theorem 3.3.1.

Proof of Theorem 3.3.1 part(i) where $0 \leq r \leq p-1$.
$A(2,r)$ has indecomposable tilting comodules $T(r-\alpha,\alpha)$ where, for r even

$0 \leq \alpha \leq \frac{r}{2}$, and for r odd $0 \leq \alpha \leq \frac{r-1}{2}$. Moreover, as SL_2-modules, $T(a,b)$ has character $\chi(a-b)$, and thus we have that $\operatorname{ch} T(r-\alpha,\alpha) = \chi(r-2\alpha)$.

We know from Theorem 3.2.7, that $D_{2,p}(r) = \sum_{\iota \leq \lambda_1 \leq r} \operatorname{cf}(S^{\lambda_1}E \otimes S^{\lambda_2}E)$ for $0 \leq r \leq p-1$. So, for r even, take $\lambda_1 = \lambda_2 = \frac{r}{2}$. Then

$$\begin{aligned}
\operatorname{ch}(S^{\lambda_1}E \otimes S^{\lambda_2}E) &= \operatorname{ch}(S^{\frac{r}{2}}E \otimes S^{\frac{r}{2}}E) \\
&= \chi(\frac{r}{2})\chi(\frac{r}{2}) \\
&= \chi(r) + \chi(r-2) + \chi(r-4) + \ldots + \chi(0) \\
&= \sum_{\alpha=0}^{\frac{r}{2}} \chi(r-2\alpha) \\
&= \sum_{\alpha=0}^{\frac{r}{2}} \operatorname{ch} T(r-\alpha,\alpha).
\end{aligned}$$

Now, using Remark 2.3.7 (ii) and our work on tilting modules, we have that two tilting modules are isomorphic if and only if they have the same characters. Therefore $S^{\lambda_1}E \otimes S^{\lambda_2}E \cong \sum_{\alpha=0}^{\frac{r}{2}} \operatorname{ch} T(r-\alpha,\alpha)$ and hence

$$\operatorname{cf} T(r-\alpha,\alpha) \subseteq \operatorname{cf}(S^{\lambda_1}E \otimes S^{\lambda_2}E) \subset D_{2,p}(r),$$

for $0 \leq \alpha \leq \frac{r}{2}$.
Similarly, for r odd, we have $\operatorname{cf} T(r-\alpha,\alpha) \subseteq \operatorname{cf}(S^{\lambda_1}E \otimes S^{\lambda_2}E) \subset D_{2,p}(r)$, for $0 \leq \alpha \leq \frac{r-1}{2}$, where $\lambda_1 = \frac{r+1}{2}$ and $\lambda_2 = \frac{r-1}{2}$.
So $D_{2,p}(r)$ contains the coefficient space of all the tilting comodules of $A(2,r)$ and thus, for $0 \leq r \leq p-1$ we have

$$A(2,r) \subseteq D_{2,p}(r) \text{ and so } A(2,r) = D_{2,p}(r).$$

□

3.4 Proof of Theorem 3.1.1 part (ii)

We first study the characters of the tilting modules we wish to find. We express these characters in a way which will allow us to show that they occur in the coefficient space of a certain truncated module.

THEOREM 3.4.1 *Notation: For $p \leq r \leq t$, when r is even, write $r = p-1+j$. When r is odd write $r = p-1+i$.*
We know t is always even, thus for r even, we have $p+1 \leq r \leq t \Rightarrow\ p+1 \leq$

$p-1+j \leq t \Rightarrow\ 2 \leq j \leq p-1$.
For r odd, then $p \leq r \leq t-1 \Rightarrow\ p \leq p-1+i \leq t-1 \Rightarrow\ 1 \leq i \leq p-2$.

The characters of the tilting comodules of $A(2,r)$ for $p \leq r \leq t$ are as follows;
For r even;

$$\operatorname{ch} T(r-\alpha,\alpha) = \begin{cases} \chi(r-j+(j-2\alpha)) + \chi(r-j-(j-2\alpha)) & 0 \leq \alpha \leq \frac{j-2}{2} \\ \chi(r-2\alpha) & \frac{j}{2} \leq \alpha \leq \frac{r}{2} \end{cases}$$

For r odd;

$$\operatorname{ch} T(r-\alpha,\alpha) = \begin{cases} \chi(r-i+(i-2\alpha)) + \chi(r-i-(i-2\alpha)) & 0 \leq \alpha \leq \frac{i-1}{2} \\ \chi(r-2\alpha) & \frac{i+1}{2} \leq \alpha \leq \frac{r}{2} \end{cases}$$

Proof. Recall:
(i) $A(n,r)$ has tilting comodules $T(r-\alpha,\alpha)$ where $0 \leq \alpha \leq \frac{r}{2}$ for r even, and $0 \leq \alpha \leq \frac{r-1}{2}$ for r odd.
(ii) The $\operatorname{ch} T(a,b) = \operatorname{ch} T(a-b,0) = \operatorname{ch} T(a-b)$ as an SL_2-module. Hence

$$\operatorname{ch} T(a,b) = \begin{cases} \chi(a-b) & \text{for } 0 \leq a-b \leq p-1 \\ \chi(p-1+m) + \chi(p-1-m) & a-b \geq p, \quad p-1+m = a-b \end{cases}$$

1) r even
We first want to show $\operatorname{ch} T(r-\alpha,\alpha) = \chi(r-2\alpha)$ for $\frac{j}{2} \leq \alpha \leq \frac{r}{2}$. Here $a-b = r-2\alpha$ and so we require $0 \leq r-2\alpha \leq p-1$. Well, if $0 \leq r-2\alpha$ then $2\alpha \leq r$ which implies $\alpha \leq \frac{r}{2}$. Moreover if $r-2\alpha \leq p-1$ then with $r = p-1+j$ we have $p-1+j-2\alpha \leq p-1$ which implies $\alpha \geq \frac{j}{2}$. So as stated $\frac{j}{2} \leq \alpha \leq \frac{r}{2}$.

We next wish to show $\operatorname{ch} T(r-\alpha,\alpha) = \chi(r-j+(j-2\alpha)) + \chi(r-j-(j-2\alpha))$ for $0 \leq \alpha \leq \frac{j-2}{2}$. Again $a-b = r-2\alpha$ and so we require $r-2\alpha \geq p$, and so with $r = p-1+j$ we have $p-1+j-2\alpha \geq p$ which implies $\alpha \leq \frac{j-1}{2}$. However $\frac{j-1}{2}$ is odd and thus we must have $\alpha \leq \frac{j-2}{2}$ as stated.

So

$$\begin{aligned} \operatorname{ch} T(r-\alpha,\alpha) &= \operatorname{ch} T(r-2\alpha,0) \\ &= \operatorname{ch} T(r-2\alpha) \ \text{ as an } SL\text{-module} \\ &= \chi(p-1+m) + \chi(p-1-m) \\ &= \chi(r-j+m) + \chi(r-j-m) \\ &= \chi(r-j+(j-2\alpha)) + \chi(r-j-(j-2\alpha)) \end{aligned}$$

as $m = a-b-p+1 = r-2\alpha-p+1 = j-2\alpha$.

2) r odd
We first want to show $\operatorname{ch} T(r-\alpha,\alpha) = \chi(r-2\alpha)$ for $\frac{i+1}{2} \leq \alpha \leq \frac{r-1}{2}$. Here $a-b = r-2\alpha$ and so we require $0 \leq r-2\alpha \leq p-1$. Well, if $0 \leq r-2\alpha$ then $2\alpha \leq r$ which implies $\alpha \leq \frac{r}{2}$. However as we are considering when r is odd, then we must have $\alpha \leq \frac{r-1}{2}$. Moreover if $r-2\alpha \leq p-1$ then with $r = p-1+i$ we have $p-1+i-2\alpha \leq p-1$ which implies $\alpha \geq \frac{i}{2}$, and again as i is odd, then we require $\alpha \geq \frac{i+1}{2}$. So as stated $\frac{i+1}{2} \leq \alpha \leq \frac{r-1}{2}$.

We next wish to show $\operatorname{ch} T(r-\alpha,\alpha) = \chi(r-i+(i-2\alpha)) + \chi(r-i-(i-2\alpha))$ for $0 \leq \alpha \leq \frac{i-2}{2}$. Again $a-b = r-2\alpha$ and so we require $r-2\alpha \geq p$, and so with $r = p-1+i$ we have $p-1+i-2\alpha \geq p$ which implies $\alpha \leq \frac{i-1}{2}$ as stated. So

$$\begin{aligned}
\operatorname{ch} T(r-\alpha,\alpha) &= \operatorname{ch} T(r-2\alpha,0) \\
&= \operatorname{ch} T(r-2\alpha) \ \text{ as an } SL\text{-module} \\
&= \chi(p-1+m) + \chi(p-1-m) \\
&= \chi(r-i+m) + \chi(r-i-m) \\
&= \chi(r-i+(i-2\alpha)) + \chi(r-i-(i-2\alpha))
\end{aligned}$$

as $m = a-b-p+1 = r-2\alpha-p+1 = i-2\alpha$. □

We now move on to showing that the coefficient spaces of the tilting modules occur in the coefficient spaces of the truncated modules.

CALCULATION 3.4.2 Finding 'most' of $A(2,r)$'s tilting comodules for r even

For $0 \leq r \leq p-1$ we were able to show directly that $D_{2,p}(r)$ contains the coefficient spaces of the tilting modules of $A(n,r)$ by equating the characters of the tilting modules to those modules whose coefficient spaces sum to give the $D_{2,p}(r)$.
For $p \leq r \leq t$ we use the same method of equating characters but due to the structure of $D_{2,p}(r)$ for $p \leq r \leq t$ it is slightly more complex. We have to 'find' the tilting modules in two stages.

In this section we show how for $p \leq r \leq t$ and r even we can find most of the tilting modules of $A(n,r)$. We then do the same for r odd, and then in the next section we show how to 'find' the remaining tilting modules, again for r even and r odd.

Recall

$$D_{2,p}(r) = \sum_{\frac{r}{2} \leq \lambda_1 \leq p-1} \operatorname{cf}(S^{\lambda_1}E \otimes S^{\lambda_2}E) + \sum_{p \leq \lambda_1 \leq r} \operatorname{cf}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1})$$

for $p \leq r \leq t$ with r even.

THEOREM 3.4.3 *Let $r = p - 1 + j$ with $p \leq r \leq t$, then*

$$\begin{aligned} \mathrm{ch}\,(S^{\frac{r}{2}}E \otimes S^{\frac{r}{2}}E) = {} & \mathrm{ch}\,T(r,0) + \mathrm{ch}\,T(r-1,1) + \cdots + \mathrm{ch}\,T(r - \frac{j-2}{2}, \frac{j-2}{2}) \\ & + \mathrm{ch}\,T(r - \frac{j}{2}, \frac{j}{2}) \\ & + \mathrm{ch}\,T(r-(j-1), j-1) + \ldots + \mathrm{ch}\,T(\frac{r}{2}, \frac{r}{2}) \end{aligned}$$

Proof. $\mathrm{ch}\,(S^{\frac{r}{2}}E \otimes S^{\frac{r}{2}}E) = \chi(\frac{r}{2})\chi(\frac{r}{2}) = \chi(r) + \chi(r-2) + \chi(r-4) + \ldots + \chi(0)$

We now rewrite this sum using j, where, as before, $r = p - 1 + j$ and $2 \leq j \leq p-1$. In this way we can compare the $\mathrm{ch}\,(S^{\frac{r}{2}}E \otimes S^{\frac{r}{2}}E)$ with the characters of the tilting modules in the form we found them in the previous section.

So,

$$\begin{aligned} \mathrm{ch}\,(S^{\frac{r}{2}}E \otimes S^{\frac{r}{2}}E) & = \chi(r) + \chi(r-2) + \chi(r-4) + \ldots + \chi(4) + \chi(2) + \chi(0) \\ & = \overbrace{\chi(r-j+j)}^{1} + \overbrace{\chi(r-j+(j-2))}^{2} + \overbrace{\chi(r-j+(j-4))}^{3} \\ & + \ldots + \overbrace{\chi(r-j+2)}^{4} + \overbrace{\chi(r-j)}^{5} + \overbrace{\chi(r-j-2)}^{4} + \ldots \\ & + \overbrace{\chi(r-j-(j-4))}^{3} + \overbrace{\chi(r-j-(j-2))}^{2} + \overbrace{\chi(r-j-j)}^{1} \\ & + \overbrace{\chi(r-j-(j+2))}^{6} + \overbrace{\chi(r-j-(j+4))}^{7} + \ldots \\ & + \overbrace{\chi(r-j-(p-1))}^{8} \end{aligned}$$

We now rearrange this sum so that we are pairing some of these characters off:

$$\begin{aligned} \mathrm{ch}\,(S^{\frac{r}{2}}E \otimes S^{\frac{r}{2}}E) & = \overbrace{\chi(r-j+j) + \chi(r-j-j)}^{1} \\ & + \overbrace{\chi(r-j+(j-2)) + \chi(r-j-(j-2))}^{2} + \\ & \overbrace{\chi(r-j+(j-4)) + \chi(r-j-(j-4))}^{3} + \ldots \\ & + \overbrace{\chi(r-j+2) + \chi(r-j-2)}^{4} + \overbrace{\chi(r-j)}^{5} \\ & + \overbrace{\chi(r-j-(j+2))}^{6} + \overbrace{\chi(r-j-(j+4))}^{7} + \ldots \\ & + \overbrace{\chi(r-j-(p-1))}^{8} \end{aligned}$$

Then we compare these characters with those found for the tilting modules in

Theorem 3.4.1 and we have;

$$\begin{aligned}\mathrm{ch}\,(S^{\frac{r}{2}}E\otimes S^{\frac{r}{2}}E) &= \overbrace{\mathrm{ch}\,T(r,0)}^{1}+\overbrace{\mathrm{ch}\,T(r-1,1)}^{2}+\overbrace{\mathrm{ch}\,T(r-2,2)}^{3}+\dots\\ &+\overbrace{\mathrm{ch}\,T(r-\frac{j-2}{2},\frac{j-2}{2})}^{4}+\overbrace{\mathrm{ch}\;\;T(r-\frac{j}{2},\frac{j}{2})}^{5}\\ &+\overbrace{\mathrm{ch}\,T(r-(j-1),j-1)}^{6}\\ &+\overbrace{\mathrm{ch}\,T(r-(j-2),j-2)}^{7}+\dots+\overbrace{\mathrm{ch}\,T(\frac{r}{2},\frac{r}{2})}^{8}.\end{aligned}$$

To ensure this is clear we check the following;

i) $\mathrm{ch}\,T(r-\frac{j}{2},\frac{j}{2})=\chi(r-j)$.
Well, $\mathrm{ch}\,T(r-\frac{j}{2},\frac{j}{2})=\mathrm{ch}\,T(r-j,0)=\mathrm{ch}\,T(r-j)=\chi(r-j)$ as $r-j=p-1+j-j=p-1\leq p-1$.

ii) $\mathrm{ch}\,T(r-(j-1),j-1)=\chi(r-j-(j+2))$.
Well, $\mathrm{ch}\,T(r-(j-1),j-1)=\mathrm{ch}\,T(r-(2j-2),0)=\mathrm{ch}\,T(r-(2j-2))=\chi(r-(2j-2))$ as $r-2j+2=p-1+j-2j+2=p-1-j\leq p-1$ as $j\geq 2$.
Hence $\mathrm{ch}\,T(r-(j-1),j-1)=\chi(r-j-(j+2))$.

iii) $\mathrm{ch}\,T(\frac{r}{2},\frac{r}{2})=\chi(r-j-(p-1))$.
Well, $\mathrm{ch}\,T(\frac{r}{2},\frac{r}{2})=\mathrm{ch}\,T(0)=\chi(0)=\chi(r-j-(p-1))$ as $r=p-1+j$. □

So we see that by considering the character of the module $S^{\frac{r}{2}}E\otimes S^{\frac{r}{2}}E$ we can equate it to the character of certain tilting modules of the $A(2,r)$. Hence the coefficient spaces of these tilting modules are contained within the $D_{2,p}(r)$ of corresponding degree. As stated above we will show later how the remaining tilting modules can be 'found'.

EXAMPLE 3.4.4 $p=7$, $r=8$ thus $j=2$

$$\begin{aligned}\mathrm{ch}\,(S^{\frac{r}{2}}E\otimes S^{\frac{r}{2}}E) &= \mathrm{ch}\,(S^4E\otimes S^4E)\\ &=\chi(4)\chi(4)\\ &=\chi(8)+\chi(6)+\chi(4)+\chi(2)+\chi(0)\\ &=\chi(r-j+j)+\chi(r-j-(j-2))+\chi(r-j-2)\\ &\quad+\chi(r-j-(j+2))+\chi(r-j-(p-1))\\ &=\chi(r-j+j)+\chi(r-j)+\chi(r-j-j)\\ &\quad+\chi(r-j+(j+2))+\chi(r-j-(p-1))\end{aligned}$$

$A(2,8)$ has tilting modules $T(8,0),T(7,1),T(6,2),T(5,3),T(4,4)$, and;

$\mathrm{ch}\,T(8,0)=\chi(8)+\chi(4)=\chi(r-j+j)+\chi(r-j-j)$

$\mathrm{ch}\, T(7,1) = \chi(6) = \chi(r-j)$
$\mathrm{ch}\, T(6,2) = \chi(4) = \chi(r-j-2)$
$\mathrm{ch}\, T(5,3) = \chi(2) = \chi(r-j-(j+2))$
$\mathrm{ch}\, T(4,4) = \chi(0) = \chi(r-j-(p-1))$

So,

$$\begin{aligned}\mathrm{ch}\,(S^4E \otimes S^4E) &= \chi(r-j+j) + \chi(r-j) + \chi(r-j-j) \\ &\quad + \chi(r-j+(j+2)) + \chi(r-j-(p-1)) \\ &= \chi(r-j+j) + \chi(r-j-j) + \chi(r-j) \\ &\quad + \chi(r-j+(j+2)) + \chi(r-j-(p-1)) \\ &= \mathrm{ch}\ T(8,0) + \mathrm{ch}\ T(7,1) + \mathrm{ch}\ T(5,3) + \mathrm{ch}\ T(4,4).\end{aligned}$$

We most now repeat the process for r odd.

CALCULATION 3.4.5 Finding 'most' of $A(2,r)$'s tilting modules, for r odd

THEOREM 3.4.6 *Let $r = p-1+i$ with $p \leq r \leq t$, then*

$$\begin{aligned}\mathrm{ch}\,(S^{\frac{r+1}{2}}E \otimes S^{\frac{r-1}{2}}E) = \mathrm{ch}\, T(r,0) + \mathrm{ch}\, T(r-1,1) + \ldots + \\ \mathrm{ch}\, T(r-\frac{i-1}{2}, \frac{i-2}{2}) + \mathrm{ch}\, T(r-(i+1), i+1) \\ + \ldots + \mathrm{ch}\, T(\frac{r+1}{2}, \frac{r-1}{2})\end{aligned}$$

Proof. We follow a similar method as in the previous section, but for r odd. Take $\lambda_1 = \frac{r+1}{2}$, then

$$\mathrm{ch}\,(S^{\frac{r+1}{2}}E \otimes S^{\frac{r-1}{2}}E) = \chi(\frac{r+1}{2})\chi(\frac{r-1}{2}) = \chi(r) + \chi(r-2) + \ldots + \chi(1)$$

Now rewrite this sum using i, where $r = p-1+i$, for $1 \leq i \leq p-2$. So,

$$\begin{aligned}\mathrm{ch}\,(S^{\frac{r+1}{2}}E \otimes S^{\frac{r-1}{2}}E) = \;& \chi(r) + \chi(r-2) + \chi(r-4) + \ldots + \chi(3) + \chi(1) \\ = \;& \chi(r-i+i) + \chi(r-i+(i-2)) + \chi(r-i+(i-4)) \\ & + \ldots + \chi(r-i+1) + \chi(r-i-1) + \ldots + \\ & \chi(r-i-(i-4)) + \chi(r-i-(i-2)) + \\ & \chi(r-i-i) + \chi(r-i-(i+2)) + \chi(r-i-(i+4)) \\ & + \ldots + \chi(r-i-(p-2)).\end{aligned}$$

Again rearrange this sum, pairing off certain characters, and compare with the characters of the tilting modules found in Theorem 3.4.1;

$$\begin{aligned}\mathrm{ch}\,(S^{\frac{r+1}{2}}E\otimes S^{\frac{r-1}{2}}E) = \;& \chi(r-i+i)+\chi(r-i-i) \;+\; \chi(r-i+(i-2)) \\ &+\chi(r-i-(i-2)) \;+\; \chi(r-i+(i-4)) \\ &+\chi(r-i-(i-4))+\ldots \;+\chi(r-i+1) \\ &+\chi(r-i-1) \;+\chi(r-i-(i+2)) \;+ \\ &\chi(r-i-(i+4))+\ldots+\chi(r-i-(p-2)).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{ch}\,(S^{\frac{r+1}{2}}E\otimes S^{\frac{r-1}{2}}E) \;&= \mathrm{ch}\,T(r,0)+\mathrm{ch}\;T(r-1,1)+\mathrm{ch}\,T(r-2,2)+\ldots \\ &+\mathrm{ch}\,T(r-\tfrac{i-1}{2},\tfrac{i+1}{2}) \\ &+\mathrm{ch}\,T(r-(i+1),i+1)+\ldots+\mathrm{ch}\,T(\tfrac{r+1}{2},\tfrac{r-1}{2}).\end{aligned}$$

To ensure this is clear, we check the following;

i) $\mathrm{ch}\,T(r-(i+1),i+1)=\chi(r-i-(i+2))$.
Well, $\mathrm{ch}\,T(r-(i+1),i+1) = \mathrm{ch}\,T(r-(2i+2),0) = \mathrm{ch}\,T(r-(2i+2)) = \chi(r-(2i+2))$ as $p-1+i-2i-2 = p-3-i < p-1$ and hence $\mathrm{ch}\,T(r-(i+1),i+1)) = \chi(r-i-(i+2))$.

ii) $\mathrm{ch}\,T(\frac{r+1}{2},\frac{r-1}{2})=\chi(r-i-(p-2))$.
Well, $\mathrm{ch}\,T(\frac{r+1}{2},\frac{r-1}{2})=\mathrm{ch}\,T(1)=\chi(1)=\chi(r-i-(p-1))$ as $r-i-p+2=1$.

So as in the r even case, the character of $S^{\frac{r+1}{2}}E\otimes S^{\frac{r-1}{2}}E$ corresponds to the character of certain tilting modules of $A(n,r)$. Thus the coefficient spaces of these tilting modules are contained in $D_{2,p}(r)$. □

We then find the remaining tilting modules.

CALCULATION 3.4.7 Finding the coefficient spaces of the remaining tilting comodules of $A(2,r)$ in $D_{2,p}(r)$ for $p\le r\le t$, r even and odd.

If we look back at the list of characters of tilting modules which equate to the $\mathrm{ch}\,(S^{\frac{r}{2}}E\otimes S^{\frac{r}{2}}E)$ and $\mathrm{ch}\,(S^{\frac{r+1}{2}}E\otimes S^{\frac{r-1}{2}}E)$ respectively, then we can make the following list of 'missing' tilting modules, along with their characters;

For r even;

$$\mathrm{ch}\,T(r-\frac{j+2}{2},\frac{j+2}{2})=\chi(r-j-2)$$

$$\vdots$$

$$\mathrm{ch}\,T(r-j,j)=\chi(r-j-j)$$

and for r odd;

$$\mathrm{ch}\;T(r-\frac{i+1}{2},\frac{i+1}{2})=\chi(r-i-1)$$

$$\vdots$$

$$\operatorname{ch} T(r-i,i) = \chi(r-i-i)$$

Recall

$$D_{2,p}(r) = \sum_{\iota \leq \lambda_1 \leq p-1} \operatorname{cf}(S^{\lambda_1}E \otimes S^{\lambda_2}E) + \overbrace{\sum_{p \leq \lambda_1 \leq r} \operatorname{cf}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E \otimes \Lambda^2 E^{\otimes \lambda_1 - p+1})}^{\text{Section 2}}$$

for $p \leq r \leq t$ and call the second part of this sum Section 2.

THEOREM 3.4.8 *For r even*

$$\operatorname{ch}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) = \operatorname{ch} T(r - \frac{j+2}{2}, \frac{j+2}{2}) + \ldots + \operatorname{ch} T(r-j,j)$$

where $t - \lambda_1 + \lambda_2 = r - j - 2$.

For r odd

$$\operatorname{ch}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) = \operatorname{ch} T(r - \frac{i+1}{2}, \frac{i+1}{2}) + \ldots + \operatorname{ch} T(r-i,i)$$

where $t - \lambda_1 + \lambda_2 = r - i - 1$.

Proof. We start with the following claim

CLAIM 3.4.9 There exists a λ_1 and λ_2 such that;
(i) $t - \lambda_1 + \lambda_2 = r - i - 1$ for r odd
(ii) $t - \lambda_1 + \lambda_2 = r - j - 2$ for r even.

Proof. (i) r odd
Suppose $t - \lambda_1 + \lambda_2 = r - i - 1$
$\Rightarrow t - \lambda_1 + r - \lambda_1 = r - i - 1$
$\Rightarrow 2p - 2 - \lambda_1 + p - 1 + i - \lambda_1 = p - 1 + i - i - 1$
$\Rightarrow 3p - 3 - 2\lambda_1 + i = p - 2$
$\Rightarrow 2p - 1 - 2\lambda_1 + i = 0$
$\Rightarrow p + p - 1 + i - 2\lambda_1 = 0$
$\Rightarrow p + r = 2\lambda_1$
$\Rightarrow \frac{p+r}{2} = \lambda_1$.

• Now check that $p \leq \frac{p+r}{2} \leq r$ as we know $p \leq \lambda_1 \leq r$.
In fact, because in this case $i = 3, \ldots, t-p$ and $r = p - 1 + i$ then we have $p + 2 \leq r \leq t$, rather than $p \leq r \leq t$.

So, let us take $r = p + 2$.
Then $\frac{p+r}{2} = \frac{p+p+2}{2} = \frac{2p+2}{2} = p + 1$ and indeed $p \leq p + 1 \leq p + 2$ if and only if $p \leq p + 1 \leq r$ which holds if and only if $p \leq \frac{p+r}{2} \leq r$ as required.

Now take $r = t$, then we require $p \leq \frac{p+t}{2} \leq t$
$\Leftrightarrow p \leq \frac{p+2p-2}{2} \leq 2p - 2$
$\Leftrightarrow p \leq \frac{3p-2}{2} \leq 2p - 2$
$\Leftrightarrow 2p \leq 3p - 2 \leq 4p - 4$
$\Leftrightarrow 2 \leq p$ which is obviously true!

• Now we need to find λ_2.
Well, we have $t - \lambda_1 + \lambda_2 = r - i - 1$

$$\begin{aligned}
\Rightarrow \lambda_2 &= r - i - 1 - t + \lambda_1 \\
&= p - 1 + i - i - 1 - t + \lambda_1 \\
&= p - 2 - t + \lambda_1 \\
&= p - 2 - (2p - 2) + \lambda_1 \\
&= p - 2 - 2p + 2 + \lambda_1 \\
&= -p + \lambda_1 \\
&= \lambda_1 - p \\
&= \tfrac{p+r}{2} - p \\
&= \tfrac{r-p}{2}.
\end{aligned}$$

And with $\lambda_2 = \lambda_1 - p$ and $\lambda_1 \geq p$ then $\lambda_1 \geq \lambda_2$.

(ii) r even
Suppose $t - \lambda_1 + \lambda_2 = r - j - 2$
$\Rightarrow t - \lambda_1 + r - \lambda_1 = r - j - 2$
$\Rightarrow 2p - 2 - \lambda_1 + p - 1 - j - \lambda_1 = p - 1 + j - j - 2$
$\Rightarrow 3p - 3 - 2\lambda_1 + j = p - 3$
$\Rightarrow p - 1 + j + 2p - 2 - 2\lambda_1 = p - 3$
$\Rightarrow r + p + 1 - 2\lambda_1 = 0$
$\Rightarrow r + p + 1 = 2\lambda_1$
$\Rightarrow \frac{r+p+1}{2} = \lambda_1$

• Now check that $p \leq \frac{r+p+1}{2} \leq r$ as we know that $p \leq \lambda_1 \leq r$.
In fact, because in this case $j = 4, \ldots, p - 1$ and $r = p - 1 + j = p + 3$ then we have $p + 3 \leq r \leq t$ rather than $p \leq r \leq t$.

So, let us take $r = p + 3$.
Then $\frac{r+p+1}{2} = \frac{p+3+p+1}{2} = \frac{2p+4}{2} = p + 2$ and indeed $p \leq p + 2 \leq p + 3$ of and only if $p \leq p + 2 \leq r$ which holds if and only if $p \leq \frac{r+p+1}{2} \leq r$ as required.

Now take $r = t$ then we require $p \leq \frac{t+p+1}{2} \leq t$

$\Leftrightarrow 2p \leq t+p+1 \leq 2t$
$\Leftrightarrow 2p \leq 2p-2+p_1 \leq 4p-4$
$\Leftrightarrow p \geq 3.$

When $p=2$ we can see $D_{2,p}(r) = A(2,r)$ by observation thus it is enough to take $p \geq 3$.

• Now we need to find λ_2.
Well, we have $t-\lambda_1+\lambda_2 = r-j-2$

$$
\begin{aligned}
\Rightarrow \lambda_2 &= r-j-2-t+\lambda_1 \\
&= p-1+j-j-2+\lambda_1 \\
&= p-3-t+\lambda_1 \\
&= p-3-(2p-2)+\lambda_1 \\
&= \lambda_1-p-1 \\
&= \lambda_1-(p-1) \\
&= \tfrac{r+p+1}{2}-(p+1) \\
&= \tfrac{r-p-1}{2}.
\end{aligned}
$$

And with $\lambda_2 = \lambda_1-(p+1)$ and $\lambda_1 \geq p$ then $\lambda_1 \geq \lambda_2$. □

Now that we know there exists such a λ_1 and λ_2, we can go about calculating $\operatorname{ch}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E)$ for both r even and r odd. Firstly, for r even

$$
\begin{aligned}
\operatorname{ch}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) &= \chi(t-\lambda_1)\chi(\lambda_2) \\
&= \chi(t-\lambda_1+\lambda_2)+\chi(t-\lambda_1+\lambda_2-2) \\
&\quad +\chi(t-\lambda_1+\lambda_2-4) \quad +\ldots+\chi((t-\lambda_1)-\lambda_2) \\
&= \chi(r-j-2)+\chi(r-j-4)+\chi(r-j-6) \\
&\quad +\ldots+\chi(r-j-j)
\end{aligned}
$$

N.B.
(a) $t-\lambda_1 \geq \lambda_2$ because we have $\lambda_2 = r-\lambda_1$ and $t-\lambda_1 \geq r-\lambda_1$ because here $p+2 \leq r \leq t$
(b) For proof of $t-\lambda_1-\lambda_2 = r-j-j$ see proof of Claim 3.4.10.

Comparing these characters with those of the 'missing' tilting modules, we see;

$$\operatorname{ch}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) = \operatorname{ch} T(r-\frac{j+2}{2}, \frac{j+2}{2}) + \ldots + \operatorname{ch} T(r-j, j).$$

Hence we have shown that the coefficient spaces of the 'remaining' tilting modules are contained in $\operatorname{cf}(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) \subseteq D_{2,p}(r)$. Thus $D_{2,p}(r) = A(2,r)$ for r even and $p \leq r \leq t$.

For r odd

$$\begin{aligned}
\mathrm{ch}\,(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) &= \chi(t-\lambda_1)\chi(\lambda_2)\\
&= \chi(t-\lambda_1+\lambda_2)+\chi(t-\lambda_1+\lambda_2-2)\\
&\quad +\chi(t-\lambda_1+\lambda_2-4)+\chi((t-\lambda_1)-\lambda_2))\\
&= \chi(r-i-1)+\chi(r-i-3)+\chi(r-i-5)\\
&\quad +\ldots+\chi(r-i-i)\\
&= \mathrm{ch}\ T(r-\tfrac{i+1}{2},\tfrac{i+1}{2})+\ldots+\mathrm{ch}\ T(r-i,i)
\end{aligned}$$

N.B. For proof of $t-\lambda_1-\lambda_2=r-i-i$ see proof of Claim 3.4.10.

Hence we have shown that the coefficient spaces of the 'remaining' tilting modules are contained in $\mathrm{cf}\,(S^{t-\lambda_1}E \otimes S^{\lambda_2}E) \subseteq D_{2,p}(r)$. Thus $D_{2,p}(r)=A(2,r)$ for r odd and $p \le r \le t$. □

CLAIM 3.4.10 With λ_1 and λ_2 as in Claim 3.4.9, then $t-\lambda_1-\lambda_2=r-i-i$ for r odd, and $t-\lambda_1-\lambda_2=r-j-j$ for r even.

Proof. (i) r odd

$$\begin{aligned}
t-\lambda_1-\lambda_2 &= t-\tfrac{r+p}{2}-\left(\tfrac{r+p}{2}-p\right)\\
&= 2p-2-r-p+p\\
&= 2p-2-r\\
&= 2p-2-(p-1+i)\\
&= 2p-2-p+1-i\\
&= p-1-i\\
&= p-1+i-i-i\\
&= r-i-i
\end{aligned}$$

(ii) r even

$$\begin{aligned}
t-\lambda_1-\lambda_2 &= t-\tfrac{r+p+1}{2}-\left(\tfrac{r+p+1}{2}-p-1\right)\\
&= t-r-p-1+p+1\\
&= 2p-2-r\\
&= 2p-2-(p-1+j)\\
&= p-1-j\\
&= p-1+j-j-j\\
&= r-j-j
\end{aligned}$$

□

3.5 Proof of Theorem 3.1.1 part (iii)

AIM: This final section proves a reflection property, which completes the proof of Theorem 3.1.1. The previous sections proved this theorem for $0 \le r \le t$, whilst this section proves it for $\frac{nt}{2} \le r \le nt$, which for $n=2$ is the range

$t \leq r \leq 2t$. We prove this theorem for general n as it will be used again when we consider $n \geq 3$ in later chapters.
We first define the antipode map σ.

DEFINITION 3.5.1 Let $k[G] = k\text{-sp}\{x_{ij}, \det^{-1}\}$. Then we define the antipode map

$$\sigma : k[G] \to k[G] \text{ such that } \sigma(f)(x) = f(x^{-1}).$$

Then $\sigma : D_{n,p}(r) \to D_{n,p}(r)$.

REMARK 3.5.2

$$\begin{aligned} \sigma(\det)(x) &= \det(x^{-1}) \\ &= \tfrac{1}{\det(x)} \\ &= \det^{-1}(x) \end{aligned}$$

We therefore have that $\sigma(\det) = \det^{-1}$.

We can now give our theorem for proving the reflection property.

THEOREM 3.5.3

$$D_{n,p}(r) = \sigma(D_{n,p}(nt - r))det^t$$

where $\frac{nt}{2} \leq r \leq nt$ *and* σ *is the antipode map.*

Thus

$$D_{n,p}(r) = A(n, nt - r)det^t$$

Proof. For any n and any prime p then

$$\begin{aligned} S(\bar{E}) &= \bar{S}^0E \oplus \bar{S}^1E \oplus \bar{S}^2E \oplus \ldots \oplus \bar{S}^tE \\ &= k \oplus E \oplus \ldots \oplus D^{\otimes p-1}. \end{aligned}$$

Let

$$\gamma : \bar{S}^jE \otimes \bar{S}^{t-j}E \to \bar{S}^tE$$

be a multiplication G-map. So

$$\gamma : \bar{S}^jE \otimes \bar{S}^{t-j}E \to D^{\otimes p-1}.$$

Now let $D^{\otimes p-1} = L$ so $\dim L = 1$. Then multiplying each side by L^* gives the map

$$\beta : (\bar{S}^jE \otimes L^*) \otimes \bar{S}^{t-j}E \to k$$

as $L \otimes L^* \cong k$.
With $V = \bar{S}^jE \otimes L^*$ and $W = \bar{S}^{t-j}E$ then V, W are G-modules and we have a bilinear form

$$\beta : V \otimes W \to k \text{ where } (v, w) \mapsto \beta(v \otimes w).$$

This is non-degenerate:
Let $V^{\perp} = \{w \in W \mid (v,w) = 0 \text{ for all } v \in V\}$. This is a G-submodule as for $w \in V^{\perp}$ and $g \in G$ then $(v, gw) = (g^{-1}v, w) = 0$ so $gw \in V^{\perp}$. Now, V is simple and thus either $V^{\perp} = 0$ or $V^{\perp} = V$. Well, we cannot have $V^{\perp} = V$ as then the multiplication map $\bar{S}^j E \otimes \bar{S}^{t-j} E \to \bar{S}^t E$ would be zero. However we can find $v \in \bar{S}^j E$ and $w \in \bar{S}^{t-j} E$ such that $vw \neq 0$.

CLAIM 3.5.4 This gives an isomorphism

$$\phi : \bar{S}^j E \otimes L^* \to (\bar{S}^{t-j} E)^* \text{ where } \phi(v)(w) = (v, w)$$

Proof. Assume $\beta : V \otimes W \to k$ a G-homomorphism, then $\beta(gv \otimes gw) = \beta(v \otimes w)$ and so we have $(gv, gw) = (v, w)$. If $\phi(v) = 0$ then $\phi(v)(w) = 0$ for all $w \in W$ and thus $(v, w) = 0$ for all $w \in W$ and hence $v = 0$.
Therefore ϕ is injective and so $\dim V = \dim \mathrm{Im}\phi \leq \dim W^* = \dim W$.
We have that $\dim W \leq \dim V$ and so $\dim V = \dim W$ and thus ϕ is a linear isomorphism.

Now $\phi(gv)(w) = (gv, w) = (g^{-1}gv, g^{-1}w) = (v, g^{-1}w)$
and $(g\phi(v))(w) = \phi(v)(g^{-1}w) = (v, g^{-1}w) \Rightarrow \phi$ an isomorphism of G-modules. □

Hence

$$\bar{S}^j E \otimes L^* \cong (\bar{S}^{t-j} E)^* \;\Rightarrow\; \bar{S}^j E \cong (\bar{S}^{t-j} E)^* \otimes L$$

Now suppose $\bar{S}^{t-j} E = V$ has basis $v_1, v_2, \ldots, v_n$ with coefficient functions f_{ij}, where $gv_i = \sum f_{ji}(g) v_j$. Then V^* has dual basis $\alpha_1, \alpha_2, \ldots, \alpha_n$ where

$$\begin{aligned}(g\alpha_i)(v_j) &= \alpha_i(g^{-1}v_j) \\ &= \alpha_i(\textstyle\sum_r f_{rj}(g^{-1}) v_r) \\ &= f_{ij}(g^{-1}).\end{aligned}$$

Hence $g\alpha_i = \sum_j f_{ij}(g^{-1})\alpha_j$. So, if

$$g\alpha_i = \sum F_{ji}(g)\alpha_j \text{ then } F_{ji}(g) = f_{ij}(g^{-1})$$

i.e. $F_{ji} = \sigma(f_{ij})$.
Hence

$$\mathrm{cf}\,(V^*) = k\text{-sp}\{F_{ij}\} = k\text{-sp}\{\sigma(f_{ij})\} = \sigma(\mathrm{cf}\,(V)).$$

So

$$\mathrm{cf}\,(\bar{S}^{t-j} E^*) = \sigma(\mathrm{cf}\,(\bar{S}^{t-j} E)).$$

Now suppose that $V_1 \times W_1 \to k$ and $V_2 \times W_2 \to k$ are non-singular bilinear forms. Then the product form $V_1 \otimes V_2 \times W_1 \otimes W_2 \to k$ is also non-singular as $(v_1 \otimes v_2, w_1 \otimes w_2) = (v_1, w_1)(v_2, w_2)$. It then follows from this that the product form $\mathrm{Tr}^{\lambda} E \otimes (D^{\otimes t})^* \times \mathrm{Tr}^{\mu} E \to k$ with $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (t - \lambda_1, t - \lambda_2, \ldots, t - \lambda_n)$ is non-singular.
Then by 3.5.4

$$(\mathrm{Tr}^{\lambda} E \otimes (D^{\otimes t})^*)^* \cong \mathrm{Tr}^{\mu} E$$

and thus

$$(\mathrm{Tr}^{\lambda} E) \otimes (D^{\otimes t})^* \cong (\mathrm{Tr}^{\bar{\mu}} E)^*$$

where $\bar{\mu} = (t - \lambda_n, t - \lambda_{n-1}, \ldots, t - \lambda_1)$. So

$$(\mathrm{Tr}^{\lambda} E \cong (\mathrm{Tr}^{\bar{\mu}} E)^* \otimes D^{\otimes t}$$

and thus

$$\mathrm{cf}\,(\mathrm{Tr}^{\lambda} E) = \mathrm{cf}\,((\mathrm{Tr}^{\bar{\mu}} E)^*)\mathrm{cf}\,(D^{\otimes t}) = \sigma(\mathrm{cf}\,(\mathrm{Tr}^{\bar{\mu}} E))\mathrm{cf}\,(D^{\otimes t}).$$

Now

$$\begin{aligned} D_{n,p}(r) &= \textstyle\sum_{|\lambda|=r,\ \lambda_1 \leq t} \mathrm{cf}\,(\mathrm{Tr}^{\lambda} E) \\ &= \textstyle\sum_{|\bar{\mu}|+|\lambda|=nt,\ \bar{\mu_1} \leq t} \sigma(\mathrm{cf}\,(\mathrm{Tr}^{\bar{\mu}} E))\mathrm{cf}\,(D^{\otimes t}) \\ &= \sigma(D_{n,p}(nt - r))(\det)^t \end{aligned}$$

□

Chapter 4

The case $n = 3$ for all primes p

4.1 The theorem and the method

AIM: In this section we first give some brief definitions which are necessary in giving our main theorem 4.1.3. In Section 1.2 we defined the coalgebra $A(n, r)$, and then in Chapter 3 we proved that for $n = 2$ the coefficient spaces of all the tilting modules of $A(2, r)$ were contained in the Doty Coalgebra $D_{2,p}(r)$. For the case $n = 3$ we follow a similar method, but instead of finding all tilting modules of $A(3, r)$ we instead show that all coefficient spaces of the tilting modules of $A(\pi, r)$ are contained in the Doty Coalgebra $D_{3,p}(r)$ where π is a suitable saturated set. We explain why this is true, drawing on our previous work on quasi-hereditary algebras, and show how we will go about proving this.

DEFINITION 4.1.1 Recall the dominance order defined in 1.3.2. Let $\Lambda^+(n, r)$ be the partitions of r into at most n parts. We say a subset π of $\Lambda^+(n, r)$ is saturated if whenever $\lambda \in \pi$ and $\mu \trianglelefteq \lambda$ then $\mu \in \pi$.

EXAMPLE 4.1.2 In $\Lambda^+(3, 5)$, the set $\pi = \{(3, 2, 0), (3, 1, 1), (2, 2, 1)\}$ is saturated.

THEOREM 4.1.3 *The Doty Coalgebras* $D_{3,p}(r) = A(\pi, r)$ *for*

$$\pi = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \mid \lambda \textit{ is a partition of } r \textit{ and } \lambda_1 \leq t = 3(p-1)\}$$

a saturated set, and $0 \leq r \leq 3p - 1$ *or* $6p - 8 \leq r \leq 3t$.

We now explain the role this saturated set can play in quasi-hereditary algebras to be able to explain how we will prove Theorem 4.1.3.

DEFINITION 4.1.4 In Definition 1.3.8 we stated that for S a finite-dimensional algebra, and a subset $\pi \subseteq \Lambda^+$, then $V \in \text{mod}(S)$ belongs to π if all composition factors of V belong to $\{L(\lambda) \mid \lambda \in \pi\}$. Now fix some saturated subset of Λ, π, and let $I(\pi)$ be the ideal of all elements x in S such that $xV = 0$ for every module V belonging to π.

THEOREM 4.1.5 *If S is a quasi-hereditary algebra with poset Λ and $\pi \subseteq \Lambda$ is a saturated set then we can form another quasi-hereditary algebra*

$$S(\pi) = S/I(\pi).$$

Proof. [4, Lemma A3.10]. □

REMARK 4.1.6 i) $\{L(\lambda) \mid \lambda \in \pi\}$ is a full set of simples for $S(\pi)$ and $S(\pi)$ is a quasi-hereditary algebra, (with partial ordering on π being restriction to π of partial ordering on Λ).
ii) We can prove that $I(\pi)$ is the set of all $x \in S$ such that $xT = 0$ for all $T = T(\lambda)$ with $\lambda \in \pi$.
iii) We can dualise this for coalgebras. So define $A(\pi)$ to be the span of all $\text{cf}\,(V)$ such that V belongs to π. Then $A(\pi)$ is a quasi-hereditary coalgebra with $A(\pi)^* = S(n,r)/I(\pi)$.
In fact $A(\pi) = k\text{-sp}\{\text{cf}\,(T(\lambda)) \mid \lambda \in \pi\}$.

Proof. i) See [4, Proposition A3.11].
ii) If V belongs to π, then embed V in an injective module I, where $O_\pi(I)$ is the largest submodule of I belonging to π. Then I has a good filtration, which implies $O_\pi(I)$ has a good filtration. Now, any module M with a good filtration has a resolution by tilting modules [4, Lemma A4.3]

$$0 \to T_r \to \cdots \to T_1 \to T_0 \to M \to 0$$

and so we can assume that if M belongs to π, then so do all the T_i.
Now, if x vanish on all T_i then $xM = 0$, so $xO_\pi(I) = 0$ so $xV = 0$.
iii) See [7, Corollary 7.3]. □

REMARK 4.1.7 The proof of Theorem 4.1.3 is split into the following sections;
i) $0 \leq r \leq p-1$
ii) $p \leq r \leq 2p-1$
iii) $2p \leq r \leq 3p-1$
iv) $6p-8 \leq r \leq 3t$.

In Section 3.5 we proved a reflection property for all n which completed the case for $n = 2$. The same property is used here for the $n = 3$ case. We prove Theorem 4.1.3 for all r up to $3p - 1$, i.e. for each degree r we need to ensure the coefficient spaces of the tilting modules of $A(\pi, r)$ are contained in $D_{3,p}(r)$. We do this by considering characters of these tilting modules and showing they arise in the characters of certain truncated modules of $D_{3,p}(r)$.

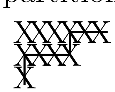
4.2 p-cores and core classes

AIM: Throughout the proof of our main theorem, the use of core classes plays a crucial role in clarifying whether we know that the coefficient spaces of the necessary tilting modules arise in the coefficient spaces of the truncated modules, and thus in this section we introduce p-cores and core classes. We start with p-cores, giving their definition and then showing how we calculate p-cores in general. We then define core classes, namely a block of partitions with the same p-core.

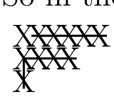
DEFINITION 4.2.1 Suppose we have a proper partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of r into $n = 3$ parts. Then we can draw this partition by placing λ_1 x's in row 1, λ_2 x's in row 2 and λ_3 x's in row 3. For example, with $\lambda = (6, 4, 1)$ then we can draw this as follows;

XXXXXX
XXXX
X

We call each of these x's a node.
The edge of this consists of the set of nodes of the partition which are on the right-hand edge and can be removed together whilst still leaving a proper partition. So in the above example the edge is as follows;



A rim p-hook consists of p nodes along the edge of the partition, and we then define the p-core to be what remains of the original partition once we have removed all possible rim p-hooks from it ensuring we leave a proper partition. So in the example we can remove two rim 5-hooks as follows;



and so what remains is simply
X
so in another way we can say that for $p = 5$, the partition $(6, 4, 1)$ has p-core $(1, 0, 0)$.

We can now define a core class.

DEFINITION 4.2.2 A core class consists of all partitions of r into n parts which have the same p-core.

REMARK 4.2.3 [21, Chapter 1, Section 1, Exercises 8b, 8c] Note that for some partitions the rim p-hooks can be removed in different ways whilst still leaving a proper partition. It is a theorem that it does not matter in which way the rim p-hooks are removed, as the result will always give the same p-core.

4.3 Classification of core classes for $p \leq r \leq 2p-1$

AIM: Having clarified this definition we shall now do some work on calculating p-cores for a general partition, as this will be needed later when we classify the core classes of partitions for each degree r. As shown above, we can see that depending on the partition, it may be possible to remove more than one rim p-hook, on the other hand it may not be possible to remove any rim p-hooks. For the moment however we shall consider the range $p \leq r \leq 2p-1$ and the cases where we can remove one rim p-hook from each partition, showing the general method for calculating a p-core. We then go on to classify the core classes for this range.

1) Consider first a partition of the form $(\lambda_1, 0, 0)$, so;
XXXX...X
then we can only remove a rim p-hook from this provided $\lambda_1 = r \geq p$, which is true for $p \leq r \leq 2p-1$ and thus the partition has p-core $(\lambda_1 - p, 0, 0)$. For example with $p = 5$ and $\lambda_1 = r = 6$ then the partition $(6,0,0)$ has p-core $(1,0,0)$;
XXXXXX

2) Now consider a partition of the form $(\lambda_1, \lambda_2, 0)$, so;
XXXX···X
X···X

i) We remove the rim p-hook by beginning at the top right hand corner and working back along the first row, and thus if $\lambda_1 - \lambda_2 \geq p$ then we can remove the rim p-hook without touching the second row, and hence this partition will have p-core $(\lambda_1 - p, \lambda_2, 0)$. For example with $p = 5$, $r = 7$ and $(\lambda_1, \lambda_2, 0) = (6, 1, 0)$ then $\lambda_1 - \lambda_2 = 5 = p$ and so $(6, 1, 0)$ has p-core $(\lambda_1 - p, \lambda_2, 0) = (1, 1, 0)$:
XXXXXX
X

ii) If $\lambda_1 - \lambda_2 < p$ then to remove the rim p-hook we must remove some nodes from the first row and some from the second. We therefore consider

$\lambda_1 - \lambda_2 + 1$ as this is the most we can remove from the first row before moving down to the second. Then what remains of the first row after removing what we can of the rim p-hook is $\lambda_1 - (\lambda_1 - \lambda_2 + 1) = \lambda_2 - 1$. We then have $p - (\lambda_1 - \lambda_2 + 1)$ remaining to remove from the second row. We therefore calculate $\lambda_2 - (p - \lambda_1 + \lambda_2 - 1) = \lambda_1 - p + 1$ to find out what remains of the second row once we have removed the rim p-hook. Thus the p-core here is $(\lambda_2 - 1, \lambda_1 - p + 1, 0)$.
For example with $r = 7$, $p = 5$ and $(\lambda_1, \lambda_2, 0) = (5, 2, 0)$ then we remove $\lambda_1 - \lambda_2 + 1 = 5 - 2 + 1 = 4$ from the first row, leaving $p - (\lambda_1 - \lambda_2 + 1) = 5 - 4 = 1$ to remove from the second row leaving p-core $(1, 1, 0)$:

XXXXX
XX

3) Finally consider a partition of the form $(\lambda_1, \lambda_2, \lambda_3)$, so;

XXXXXX$\cdots$XX
XXXX$\cdots$X
X$\cdots$X

i) As usual, if $\lambda_1 - \lambda_2 \geq p$ then we can remove the rim p-hook from the first row without affecting the second or third rows. In this case the partition will have p-core $(\lambda_1 - p, \lambda_2, \lambda_3)$. For example, with $r = 9$ and $p = 5$ then the partition $(7, 1, 1)$ has p-core $(2, 1, 1)$:

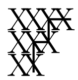
XXXXXXX
X
X

ii) Now suppose that $\lambda_1 - \lambda_2 < p$, so we remove $\lambda_1 - \lambda_2 + 1$ nodes from the first row, leaving $\lambda_2 - 1$. Then, as above, what remains of the rim p-hook to remove is $p - (\lambda_1 - \lambda_2 + 1)$, which we must now remove starting at the right hand side of the second row. Suppose that $\lambda_2 - \lambda_3 \geq p - (\lambda_1 - \lambda_2 + 1)$ then we can remove what remains of this rim p-hook from the second row without affecting the third row, and hence what remains of the second row is $\lambda_2 - (p - \lambda_1 + \lambda_2 - 1) = \lambda_1 - p + 1$ and so this partition has p-core $(\lambda_2 - 1, \lambda_1 - p + 1, \lambda_3)$. For example, with $r = 9$ and $p = 5$ then the partition $(5, 3, 1)$ has p-core $(\lambda_2 - 1, \lambda_1 - p + 1, \lambda_3) = (3 - 1, 5 - 5 + 1, 1) = (2, 1, 1)$:

XXXXX
XXX
X

iii) Finally, suppose $\lambda_1 - \lambda_2 < p$ and also $\lambda_2 - \lambda_3 < p - (\lambda_1 - \lambda_2 + 1)$ and thus to remove the rim p-hook we must use all three rows. So, we remove $\lambda_1 - \lambda_2 + 1$ from the first row, leaving $\lambda_2 - 1$ in that row, and $p - (\lambda_1 - \lambda_2 + 1)$ to remove from the second two rows. In this case, the most we can remove from the second row is $\lambda_2 - \lambda_3 + 1$ and so what remains in this second row is $\lambda_2 - (\lambda_2 - \lambda_3 + 1) = \lambda_3 - 1$, and calculating $p - (\lambda_1 - \lambda_2 + 1) - (\lambda_2 - \lambda_3 + 1) = p - \lambda_1 + \lambda_3 - 2$ gives what remains to take away from the third row, thus leaving $\lambda_3 - (p - \lambda_1 + \lambda_3 - 2) = \lambda_1 - p + 2$

in this third row. Hence this partition has p-core $(\lambda_2 - 1, \lambda_3 - 1, \lambda_1 - p + 2)$. For example with $r = 9$ and $p = 5$ then the partition $(4, 3, 2)$ has p-core $(3 - 1, 2 - 1, 4 - 5 + 2) = (2, 1, 1)$:



Note here that in case 3), all three examples have the same p-core, and hence we have a core class:

$(7, 1, 1)$
$(5, 3, 1)$
$(4, 3, 2)$

With $\lambda = (7, 1, 1)$, $\mu = (5, 3, 1)$ and $\sigma = (4, 3, 2)$ then we can see that $\lambda_1 \geq \mu_1 \geq \sigma_1$, $\lambda_1+\lambda_2 \geq \mu_1+\mu_2 \geq \sigma_1+\sigma_2$ and $\lambda_1+\lambda_2+\lambda_3 \geq \mu_1+\mu_2+\mu_3 \geq \sigma_1+\sigma_2+\sigma_3$ and thus $(7, 1, 1)$ is the highest or 'top' weight in this core class, $(5, 3, 1)$ is the 'middle' weight and $(4, 3, 2)$ is the lowest or 'bottom' weight. It is these core classes and the structure of the weights within them that will play a crucial role in solving Theorem 4.1.3.

DEFINITION 4.3.1 We call a partition self-titled if it is its own p-core. In other words it is impossible to remove a rim p-hook and still leave a proper partition.

PROPOSITION 4.3.2 *Let λ be a non self-titled partition of r into ≤ 3 parts. Then there are three possible types of core classes for the range $p \leq r \leq 2p - 1$. They are as follows;*
For $p \leq r \leq 2p - 1$, where $\lambda_1 - \lambda_2 \geq p$ and $\lambda_1 \leq 2p - 3$;
$(\lambda_1, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)$
$(\lambda_3 + p - 2, \lambda_1 - p + 1, \lambda_2 + 1)$

For $r = 2p - 1$ with $\lambda_1 + p > t$ and $\lambda_1 - p = p - 1$ we have;
$(\lambda_1, \lambda_2, \lambda_3) = (2p - 1, 0, 0)$
$(\lambda_1 - p, \lambda_1 - p, 1) = (p - 1, p - 1, 1)$

For $r = 2p - 2, 2p - 1$ with $\lambda_1 + p > t$ and $\lambda_1 - p = p - 2$;
$(\lambda_1, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)$
which for $r = 2p - 2$ is the core class
$(2p - 2, 0, 0)$
$(p - 1, p - 1, 0)$
and for $r = 2p - 1$ is the core class
$(2p - 2, 1, 0)$
$(p, p - 1, 0)$

Proof. Take a general p-core (μ_1, μ_2, μ_3), then there are three possible ways of attaching a rim p-hook to this p-core. Firstly by adding the rim p-hook to the first row, then adding it starting on the second row, and finally by adding it starting on the third row. We calculate the new partitions formed in each of these cases.

Adding to the first row gives a new partition $\eta = (\mu_1 + p, \mu_2, \mu_3)$, to the second row gives the partition $\phi = (p + \mu_2 - 1, \mu_1 + 1, \mu_3)$, and adding to the third row gives the partition $\psi = (\mu_3 + p - 2, \mu_1 + 1, \mu_2 + 1)$. Then letting $(\lambda_1, \lambda_2, \lambda_3) = (\mu_1 + p, \mu_2, \mu_3)$ gives the core class above consisting of three partitions.
We must now check that each of these new partitions is indeed a proper partition. Clearly η is a proper partition as $\mu_1 + p > \mu_2 \geq \mu_3$. Next consider ϕ, well this is a proper partition unless $p + \mu_2 + 1 < \mu_1 + 1$, i.e. if $p - 2 < \mu_1 - \mu_2$. Thus we would require $p - 1 \leq \mu_1 - \mu_2 \leq \mu_1$, but as we know that $\mu_1 \leq p - 1$ then the case where this is not a proper partition is when $\mu_1 = p - 1 = \lambda_1 - p$. Thus we have $\lambda_1 = 2p - 1$ and so must have that $r = 2p - 1$ giving the core class above consisting of two partitions where $r = 2p - 1$.
Finally consider ψ, well this is a proper partition unless $\mu_3 + p - 2 < \mu_1 + 1$ thus implying that $p - 3 < \mu_1 - \mu_3 \leq \mu_1$. So this would not be a proper partition when $\mu_1 = \lambda_1 - p \geq p - 2$ which requires $\lambda_1 \geq 2p - 2$ and thus $r = 2p - 2,\ 2p - 1$ and these give the core classes above consisting of two partitions where $r = 2p - 2,\ 2p - 1$. This also shows that we only have the core class consisting of three weights when $\lambda_1 = \mu_1 + p \leq 2p - 3$.

The final thing we need to check is that these partitions can only be ordered in the way shown in the above proposition. Consider the following [25] ;
Define $\alpha_1 = (1, -1, 0, \ldots, 0)$
$\alpha_2 = (0, 1, -1, 0, \ldots, 0)$
$\vdots$
$\alpha_{n-1} = (0, 0, \ldots, 1, -1)$
Then for two partitions λ and μ, we have $\lambda \geq \mu$ if and only if $\lambda - \mu = \sum_{i=1}^{n} a_i \alpha_i$ with $a_i \geq 0$.
We apply this now to the case $n = 3$. Firstly $\eta - \phi = (\mu_1 - \mu_2 + 1, \mu_2 - \mu_1 + 1, 0) = (\mu_1 - \mu_2 + 1)(1, -1, 0)$, secondly $\phi - \psi = (\mu_2 - \mu_3 - 1, 0, \mu_3 - \mu_2 + 1) = (\mu_2 - \mu_3 - 1)(1, 0, -1)$. □

It now remains to find the partitions in the range $p \leq r \leq 2p - 1$ that are self-titled.

PROPOSITION 4.3.3 *The self-titled partitions can be broken down into the following four cases;*
i) $\lambda_1 - \lambda_2 = p - 1$ *and* $\lambda_2 + 1 < p$ *with* $\lambda_3 \leq \lambda_2$*;*
ii) $\lambda_1 - \lambda_2 = p - \xi$, $\lambda_2 + 1 < \xi$ *and* $\lambda_3 = 0$ *where* $\lambda_1 \leq p - 3$ *and* $\xi > 2$*;*
iii) $\lambda_1 - \lambda_2 = p - \xi$, $\lambda_2 + 2 < \xi$ *and* $\lambda_3 \neq 0$ *where* $\lambda_1 \leq p - 3$ *and* $\xi > 2$*;*
iv) $\lambda_1 - \lambda_2 = p - \xi$ *and* $\lambda_2 - \lambda_3 = \xi - 2$ *where* $\xi > 1$.

Proof. i) With $\lambda_1 - \lambda_2 = p - 1$ then removing a rim p-hook from the first row would leave the p-core $(\lambda_2 - 1, \lambda_2, \lambda_3)$ which is not a proper partition. Moreover due to being restricted to the range $p \leq r \leq 2p - 1$ then we are unable to remove a rim p-hook from the second and third rows. This is equivalent to saying $\lambda_2 - \lambda_3 + \lambda_3 + 1 = \lambda_2 + 1 < p$.

ii) Now suppose $\lambda_1 - \lambda_2 = p - \xi$ where $\xi \geq 3$. Assume firstly that $\lambda_3 = 0$, then if we were able to remove a rim p-hook it would be from the first two rows only, so with $\lambda_1 - \lambda_2 = p - \xi$ then we could remove $p - \xi + 1$ from the first row, leaving $p - (p - \xi + 1) = \xi - 1$ to remove from the second row, however if $\lambda_2 + 1 < \xi \;\Rightarrow\; \lambda_2 < \xi - 1$ then there are not enough nodes to remove the remainder of the rim p-hook.
We must also state why $\lambda_1 \leq p - 3$ in this case. Well, take $\lambda_1 = p - 2$ then $p - 2 - \lambda_2 = p - \xi$ and so $\lambda_2 = \xi - 2$, which fits into the final case. Indeed if $\lambda_1 \geq p - 2$ then we can find a λ_2 such that $\lambda_2 = \xi - 2$. However, if $\lambda_1 \leq p - 3$ then $\lambda_2 \leq \xi - 3$.

iii) Suppose again that $\lambda_1 - \lambda_2 = p - \xi$ but in this case $\lambda_3 \neq 0$, then again we could, remove $p - \xi + 1$ from the first row, and we have remaining $\xi - 1$ to remove from the second two rows. However, if $\lambda_2 - \lambda_3 + \lambda_3 + 2 = \lambda_2 + 2 < \xi$ then $\lambda_2 + 1 < \xi - 1$ and there are not enough nodes to remove the remainder of the rim p-hook.

iv) As above, with $\lambda_1 - \lambda_2 = p - \xi$ then we can remove $p - \xi + 1$ from the first row, and so need to remove $\xi - 1$ from the remaining two rows. In a similar way to part i) if $\lambda_2 - \lambda_3 = \xi - 2$ then after removing the rim p-hook we will not leave a proper partition. □

We can split these self-titled partitions into three cases depending on λ_1 and λ_3.

COROLLARY 4.3.4 *The self-titled partitions given in the previous proposition each relate to one of the following three cases as shown;*
i) $\lambda_1 - \lambda_2 = p - 1$ *if and only if* $\lambda_1 - \lambda_3 > p - 2$.
ii) • *With* $\lambda_2 + 1 < \xi$ *and* $\lambda_3 = 0$, *where* $\xi \geq 3$, *then* $\lambda_1 - \lambda_2 = p - \xi$ *if and ony if* $\lambda_1 = \lambda_1 - \lambda_3 < p - 2$.
• *With* $\lambda_2 + 2 < \xi$ *and* $\lambda_3 \neq 0$, *where* $\xi \geq 3$ *then* $\lambda_1 - \lambda_2 = p - \xi$ *if and only if* $\lambda_1 - \lambda_3 < p - 2$.
iii) With $\xi \geq 2$, *then* $\lambda_1 - \lambda_2 = p - \xi$ *and* $\lambda_2 - \lambda_3 = \xi - 2$ *if and only if* $\lambda_1 - \lambda_3 = p - 2$.

This way of defining self-titled partitions will be used in the proof of Theorem 4.1.3, which is why we have introduced this now.
Proof. i) '$\Rightarrow$' If $\lambda_1 - \lambda_2 = p - 1$ and $\lambda_2 \geq \lambda_3$ then $\lambda_1 - \lambda_3 \geq p - 1 > p - 2$.
'$\Leftarrow$' We have $\lambda_1 - \lambda_3 > p - 2$ where λ is self-titled. Now suppose for a contradiction that $\lambda_1 - \lambda_2 \neq p - 1$. Well if $\lambda_1 - \lambda_2 \geq p$ then we can remove a

p-edge from the first row and thus λ is not self-titled. If, on the other hand $\lambda_1 - \lambda_2 \leq p - \xi$, where $\xi > 1$ then to ensure we have a self-titled partition we require either $\lambda_3 = 0$ and $\lambda_2 + 1 < \xi$, or $\lambda_3 \neq 0$ and $\lambda_2 + \lambda_3 + 1 < \xi$. But these are the cases in part ii) which we will prove imply $\lambda_1 - \lambda_3 < p - 2$. Thus we have a contradiction and so $\lambda_1 - \lambda_2 = p - 1$.

ii) '$\Rightarrow$' Firstly take the case where $\lambda_1 - \lambda_2 = p - \xi$ and $\lambda_2 + 1 < \xi$ where $\lambda_3 = 0$. Then $\lambda_1 - \lambda_3 = \lambda_1 = p - \xi + \lambda_2$. Suppose for a contradiction that $p - \xi + \lambda_2 \geq p - 2$, then $\lambda_2 \geq \xi - 2$ and $\lambda_1 = \lambda_1 - \lambda_3 \geq p - \xi + \xi - 2 = p - 2$ and thus $p - \xi \geq p - 2$ thus giving that $\xi \leq 2$, which is a contradiction as we have $\xi \geq 3$. Hence it must be that $p - \xi + \lambda_2 < p - 2$ implying $\lambda_1 < p - 2$ and thus $\lambda_1 - \lambda_3 < p - 2$ as required.

Now take the case where $\lambda_3 \neq 0$ and hence we have $\lambda_1 - \lambda_2 = p - \xi$ and $\lambda_2 + 2 < \xi$. Suppose for a contradiction that $\lambda_1 - \lambda_3 \geq p - 2$. Then $p-2 \leq \lambda_1 - \lambda_3 = \lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 = p - \xi + \lambda_2 - \lambda_3 < p - \xi + \xi - 2 - \lambda_3 = p - 2 - \lambda_3$. So, we have that $p - 2 < p - 2 - \lambda_3$ which is not true. Hence $\lambda_1 - \lambda_2 < p - 2$.

'$\Leftarrow$' Suppose $\lambda_1 - \lambda_3 < p - 2$, then $\lambda_1 < p - 2 + \lambda_3$ which implies that $\lambda_1 - \lambda_2 < p - 2 + \lambda_3 - \lambda_2$. We have that $\lambda_2 \geq \lambda_3$ and hence $\lambda_1 - \lambda_2 < p - 2 = p - \xi$ with $\xi > 2$. Moreover as we have a self-titled partition, then if $\lambda_3 = 0$ we require $\lambda_2 + 1 < \xi$, and if $\lambda_3 \neq 0$ then we require $\lambda_2 + 2 \leq \xi$.

iii) '$\Rightarrow$' If $\lambda_1 - \lambda_2 = p - \xi$ and $\lambda_2 - \lambda_3 = \xi - 2$ then $\lambda_1 - \lambda_3 = \lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 = p - \xi + \xi - 2 = p - 2$.
'$\Leftarrow$' Assume $\lambda_1 - \lambda_3 = p - 2$, then as $\lambda_2 \geq \lambda_3$ then $\lambda_1 - \lambda_2 \leq p - 2$ hence $\lambda_1 - \lambda_2 = p - \xi$ with $\xi \geq 2$. Moreover $\lambda_2 - \lambda_3 = p - 2 + \lambda_2 - \lambda_1 = p - 2 + \xi - p = \xi - 2$ as required. □

4.4 Classification of core classes for $2p \leq r \leq 3p - 1$

AIM: Looking back to 4.3.2 we can see that we have one type of core class consisting of 3 partitions, which we shall call a 3-set and two types of core classes consisting of 2 partitions, which we shall call 2-sets, and then with Corollary 4.3.4 we also have three types of self-titled partitions. Each of these self-titled partitions and core classes has a different p-core, so we shall take each separately and show how a rim p-hook can be added on in each case to preserve the p-core but create a new core class in the range $2p \leq r \leq 3p - 1$.

1) The 3-set.
We use the letters T, M and B to signify the partitions which are at the top, the middle and the bottom of the core class respectively. In this case we have the core class

$$\begin{aligned}(\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) &= (\lambda_{1T}, \lambda_{2T}, \lambda_{3T})\\ (\lambda_{1M}, \lambda_{2M}, \lambda_{3M}) &= (\lambda_{2T} + p - 1, \lambda_{1T} - p + 1, \lambda_{3T})\\ (\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) &= (\lambda_{3T} + p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)\end{aligned}$$

where $\lambda_{1T} - \lambda_{2T} \geq p$, and the core class has p-core $(\lambda_{1T} - p, \lambda_{2T}, \lambda_{3T})$.
Now consider a general partition λ where we need to add on a rim p-hook whilst maintaining the above p-core. What are the possible ways of attaching this p-core? Well, as stated in the proof of Proposition 4.3.2, we can add it to the top row, add it starting on the middle row and then moving up to the top row, or add it starting on the bottom row and then move up to the middle row and top row. So there are three possible options. Thus for each of the above partitions in the 3-set we shall find out the resulting partition when each of the above options is completed.

i) Adding to the top row:

T) Then $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$
M) $(\lambda_{1M}, \lambda_{2M}, \lambda_{3M})$ becomes $(\lambda_{1M} + p, \lambda_{2M}, \lambda_{3M}) = (\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T}) = (\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B})$ becomes $(\lambda_{1B}+p, \lambda_{2B}, \lambda_{3B}) = (\lambda_{3T}+2p-2, \lambda_{1T}-p+1, \lambda_{2T}+1)$

ii) Adding to the middle row:

T) $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$, this does not affect the top row as $\lambda_{1T} - \lambda_{2T} \geq p$ and hence $\lambda_{1T} \geq \lambda_{2T} + p$.
M) $(\lambda_{1M}, \lambda_{2M}, \lambda_{3M})$ becomes $(\lambda_{1M} + p - (\lambda_{1M} + 1) + \lambda_{2M}, \lambda_{1M} + 1, \lambda_{3M})$
$= (\lambda_{2M} + p - 1, \lambda_{1M} + 1, \lambda_{3M})$
$= (\lambda_{1T} - p + 1 + p - 1, \lambda_{2T} + p - 1 + 1, \lambda_{3T})$
$= (\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B})$ becomes $(\lambda_{1B} + p - (\lambda_{1B} + 1) + \lambda_{2B}, \lambda_{1B} + 1, \lambda_{3B})$
$= (\lambda_{2B} + p - 1, \lambda_{1B} + 1, \lambda_{3B})$
$= (\lambda_{1T} - p + 1 + p - 1, \lambda_{3T} + p - 2 + 1, \lambda_{2T} + 1)$
$= (\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$

iii) Adding to the bottom row:

T) $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{2T} + p - (\lambda_{2T} + 1) + \lambda_{3T}, \lambda_{2T} + 1)$

$= (\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$.
M) $(\lambda_{1M}, \lambda_{2M}, \lambda_{3M})$ becomes
$(\lambda_{1M} + p - (\lambda_{1M} + 1) - (\lambda_{2M} + 1) + \lambda_{2M} + \lambda_{3M}, \lambda_{1M} + 1, \lambda_{2M} + 1)$
$= (\lambda_{3M} + p - 2, \lambda_{1M} + 1, \lambda_{2M} + 1)$
$= (\lambda_{3T} + p - 2, \lambda_{2T} + p - 1 + 1, \lambda_{1T} - p + 1 + 1)$
$= (\lambda_{3T} + p - 2, \lambda_{2T} + p, \lambda_{1T} - p + 2)$
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B})$ becomes $(\lambda_{1B} + p - (\lambda_{1B} + 1) - (\lambda_{2B} + 1) + \lambda_{2B} + \lambda_{3B}, \lambda_{1B} + 1, \lambda_{2B} + 1)$
$= (\lambda_{3B} + p - 2, \lambda_{1B} + 1, \lambda_{2B} + 1)$
$= (\lambda_{2T} + 1 + p - 2, \lambda_{3T} + p - 2 + 1, \lambda_{1T} - p + 2)$
$= (\lambda_{2T} + p - 1, \lambda_{3T} + p - 1, \lambda_{1T} - p + 2)$

Looking at each of these results we can see that the partition we get from iT), namely $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$, gives the new top weight, followed by the partition we get from iM) and then the partition we get from iB). The next highest weight will be that from iiT), namely $(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$, which is also the result we get from iiM), and then the partition we get from iiB), namely $(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$, will be the fifth weight which is also the result for iiiT). Looking at iiiM), i.e. the partition $(\lambda_{3T} + p - 2, \lambda_{2T} + p, \lambda_{1T} - p + 2)$, we see that as $\lambda_{2T} \geq \lambda_{3T}$, then $\lambda_{2T} + p \geq \lambda_{3T} + p$ and thus $\lambda_{2T} + p \geq \lambda_{3T} + p - 2$ and therefore iiiM) is not a proper partition and we can discount this from our results. Finally we have that the partition from iiiB), namely $(\lambda_{2T} + p - 1, \lambda_{3T} + p - 1, \lambda_{1T} - p + 2)$, will be the final weight in our new core class for $2p \leq r \leq 3p - 1$.
We therefore have that the 3-set from the range $p \leq r \leq 2p - 1$ which has p-core $(\lambda_{1T} - p, \lambda_{2T}, \lambda_{3T})$ becomes the following 6-set in the range $2p \leq r \leq 3p - 1$;

$(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$
$(\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$
$(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)$
$(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$
$(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$
$(\lambda_{2T} + p - 1, \lambda_{3T} + p - 1, \lambda_{1T} - p + 2)$

The final thing we need to understand is why specifically we have the core class in this order, so why have we stated that the highest weight is $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$ and the lowest weight $(\lambda_{2T} + p - 1, \lambda_{3T} + p - 1, \lambda_{1T} - p + 2)$, and why do all the other partitions sit as they do in the block? From Definition 1.3.2 we know that for two weights $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$, that λ has a higher weight than μ if;
i) $\lambda_1 \geq \mu_1$
ii) $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$

iii) $\lambda_1 + \lambda_2 + \lambda_3 \geq \mu_1 + \mu_2 + \mu_3$

Now clearly $\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3$ as they both sum to give the same degree r so it is unnecessary to check iii) in each case. However, we must prove parts i) and ii) to show why the core class is structured as it is. We do this now by numbering the weights 1 to 6 then proving $1 \trianglerighteq 2$, $2 \trianglerighteq 3$, etc.
$1 \trianglerighteq 2$
i) $\lambda_{1T} - \lambda_{2T} \geq p$ and so $\lambda_{1T} + p \geq \lambda_{2T} + 2p - 1$.
ii) $(\lambda_{2T} + 2p - 1) + (\lambda_{1T} - p + 1) = (\lambda_{1T} + p) + \lambda_{2T}$
$2 \trianglerighteq 3$
i) $\lambda_{2T} \geq \lambda_{3T}$ so $\lambda_{2T} + 2p - 1 > \lambda_{3T} + 2p - 2$
ii) $\lambda_{1T} - p + 1 = \lambda_{1T} - p + 1$ so if i) holds then ii) holds.
$3 \trianglerighteq 4$
i) $\lambda_{1T} \leq 2p - 3$ so $\lambda_{3T} + 2p - 2 \geq 2p - 2 > 2p - 3 \geq \lambda_{1T}$
ii) If $\lambda_{1T} + p - 1 \geq \lambda_{2T} + p$ then $\lambda_{1T} - \lambda_{2T} \geq 2p - 1$ which is a contradiction, so in fact $\lambda_{2T} + p > \lambda_{1T} - p + 1$ so for 3>4 we could actually write the weights either way round, we choose as we have, simply because $\lambda_{3T} + 2p - 2 > \lambda_{1T}$
$4 \trianglerighteq 5$
i) Clear.
ii) $\lambda_{2T} \geq \lambda_{3T}$ so $\lambda_{1T} + \lambda_{2T} + p > \lambda_{1T} + \lambda_{3T} + p - 1$.
$5 \trianglerighteq 6$
i) $\lambda_{1T} - \lambda_{2T} \geq p$ and so $\lambda_{1T} > \lambda_{2T} + p - 1$.
ii) As i) holds then so does ii).

Thus we have clarified why the core class appears in the order it does.

2) The self-titled partition where $\lambda_1 - \lambda_3 < p - 2$.
We are beginning with a partition $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ where $\lambda_{1T} - \lambda_{3T} < p - 2$, and as before we can add a p-hook on in three ways, by adding it to the top row, adding it on starting on the second row, or adding it on starting on the third row. Note that as this partition has its own p-core then we need to preserve the structure of λ_T when adding on the p-hook.
i) Adding to the top row: $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$.
ii) Adding to the middle row: $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p - (\lambda_{1T} + 1) + \lambda_{2T}, \lambda_{1T} + 1, \lambda_{3T}) = (\lambda_{2T} + p - 1, \lambda_{1T} + 1, \lambda_{3T})$.
iii) Adding to the bottom row: $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p - (\lambda_{2T} + 1) - (\lambda_{1T} + 1) + \lambda_{2T} + \lambda_{3T}, \lambda_{1T} + 1, \lambda_{2T} + 1) = (\lambda_{3T} + p - 2, \lambda_{1T} + 1, \lambda_{2T} + 1)$

As each of these new weights are different then we can conclude that the self-titled partition from $p \leq r \leq 2p - 1$ where $\lambda_{1T} - \lambda_{3T} < p - 2$ becomes the following 3-set in the range $2p \leq r \leq 3p - 1$;
$(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$

$(\lambda_{2T} + p - 1, \lambda_{1T} + 1, \lambda_{3T})$
$(\lambda_{3T} + p - 2, \lambda_{1T} + 1, \lambda_{2T} + 1)$.

It now remains to prove why the core class appears in the above order, just as we did in 1). So ordering the core class 1,2,3, we prove $1 \trianglerighteq 2$, $2 \trianglerighteq 3$.
$1 \trianglerighteq 2$
i) $\lambda_{1T} - \lambda_{2T} \geq p$ and so $\lambda_{1T} + p > \lambda_{2T} + p - 1$.
ii) $(\lambda_{2T} + p - 1) + (\lambda_{1T} + 1) = \lambda_{1T} + p + \lambda_{2T}$
$2 \trianglerighteq 3$
i) Clearly holds as $\lambda_{2T} \geq \lambda_{3T}$.
ii) Holds as i) does.

3) The self-titled partition where $\lambda_1 - \lambda_3 > p - 2$.
We are beginning with a partition $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ where $\lambda_{1T} - \lambda_{3T} > p - 2$, and as before we can add a p-hook on in three ways, by adding it to the top row, adding it on starting on the second row, or adding it on starting on the third row. Note that as this partition has its own p-core then we need to preserve the structure of λ_T when adding on the p-hook.
i) Adding to the top row: $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$.
ii) Adding to the middle row: We know from Corollary 4.3.4 that if $\lambda_{1T} - \lambda_{3T} > p - 2$ then $\lambda_{1T} - \lambda_{2T} = p - 1$, and thus when adding p onto the second row we get that $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T}) = (\lambda_{1T}, \lambda_{1T} + 1, \lambda_{3T})$ which is clearly not a proper partition.
iii) Adding to the bottom row: Again we have $\lambda_{1T} - \lambda_{2T} = p - 1$ and thus if we start by adding on to the third row, then whatever remains to be added on to the second row will always fit without affecting the first row and hence $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{2T} + p - (\lambda_{2T} + 1) + \lambda_{3T}, \lambda_{2T} + 1) = (\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$

We can discount the weight in ii) and as the remaining two weights are different then we can conclude that the self-titled partition from $p \leq r \leq 2p - 1$ where $\lambda_{1T} - \lambda_{3T} > p - 2$ becomes the following 2-set in the range $2p \leq r \leq 3p - 1$;
$(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$
$(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$.

By now numbering the top weight 1 and the bottom weight 2, we must now explain why we have this ordering of the core class.
$1 \trianglerighteq 2$
i) Clear.
ii) $\lambda_{2T} \geq \lambda_{3T}$ and so $\lambda_{1T} + p + \lambda_{2T} > \lambda_{1T} + \lambda_{3T} + p - 1$.

4) The self-titled partition where $\lambda_1 - \lambda_3 = p - 2$.
So we are beginning with a partition $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ where $\lambda_{1T} - \lambda_{3T} = p - 2$, and as before we can add a p-hook on in three ways, by adding it to the top row, adding it on starting on the second row, or adding it on starting on the third row. Note that as this partition has its own p-core then we need to preserve the structure of λ_T when adding on the p-hook.
i) Adding to the top row: $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$.
ii) Adding to the middle row: $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T} + p - (\lambda_{1T} + 1) + \lambda_{2T}, \lambda_{1T} + 1, \lambda_{3T}) = (\lambda_{2T} + p - 1, \lambda_{1T} + 1, \lambda_{3T})$.
iii) Adding to the bottom row: We have that $\lambda_{1T} - \lambda_{3T} = p - 2$ and thus we have two more nodes to add on to get it up to a p-hook. Well, one of these will be used as we step up from the third row, to the second row, and thus the final node will be added on to the end of the second row. We therefore have that $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{1T} + 1, \lambda_{2T} + 1)$ which is clearly not a proper partition.

We can discount this final weight and as the remaining two new weights are different then we can conclude that the self-titled partition from $p \leq r \leq 2p-1$ where $\lambda_{1T} - \lambda_{3T} = p - 2$ becomes the following 2-set in the range $2p \leq r \leq 3p - 1$;
$(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$
$(\lambda_{2T} + p - 1, \lambda_{1T} + 1, \lambda_{3T})$.

Numbering the core class 1 and 2, we must now explain why we have this ordering of the core class.
$1 \trianglerighteq 2$
i) $\lambda_{1T} \geq \lambda_{2T}$ and thus $\lambda_{1T} + p > \lambda_{2T} + p - 1$.
ii) $(\lambda_{2T} + p - 1) + (\lambda_{1T} + 1) = \lambda_{1T} + p + \lambda_{2T}$.

5) The 2-set where $\lambda_{1T} + p > t$ and $\lambda_{1T} - p = p - 2$
In this case we have the following core class;

$$(\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) = (\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$$
$$(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (\lambda_{2T} + p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$$

So as usual we need to take these two partitions and see what happens when we add a p-hook first to the top row, then the middle row, and finally the bottom row.
i) Adding to the top row:
T) It is a requirement in any partition μ that $\mu_1 \leq t$ and thus as in this case $\lambda_{1T} + p > t$ then we cannot add a p-hook to the first row.
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B})$ becomes $(\lambda_{1B} + p, \lambda_{2B}, \lambda_{3B})$

$= (\lambda_{2T} + p - 1 + p, \lambda_{1T} - p + 1, \lambda_{3T})$
$= (\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$
ii) Adding to the middle row:
T) $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T})$ as here $\lambda_{1T} = 2p-2$ and $\lambda_{2T} = 0$ or 1 and hence adding a p-hook to the second row will not affect the first row.
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B})$ becomes $(\lambda_{1B} + p - (\lambda_{1B} + 1) + \lambda_{2B}, \lambda_{1B} + 1, \lambda_{3B})$
$= (\lambda_{2B} + p - 1, \lambda_{1B} + 1, \lambda_{3B})$
$= (\lambda_{1T} - p + 1 + p - 1, \lambda_{2T} + p, \lambda_{3T})$
$= (\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$
iii) Adding to the bottom row:
T) $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ becomes $(\lambda_{1T}, \lambda_{2T}+p-(\lambda_{2T}+1)+\lambda_{3T}, \lambda_{2T}+1) = (\lambda_{1T}, \lambda_{3T}+p-1, \lambda_{2T}+1)$ and as $\lambda_T = (2p-2, 1, 0)$ for $r = 2p-1$ and $\lambda_T = (2p-2, 0, 0)$ for $r = 2p-2$ then we have that $(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1) = (\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)$.
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (p-1, p-1, 0)$ for $r = 2p-2$ and thus we would result in $(p-1, p-1, p)$ which is not a proper partition. For $r = 2p-1$, $\lambda_B = (p, p-1, 0)$ and so result in the partition $(p, p-1, p)$ which again is not a proper partition. We therefore have that iB) gives the new top weight followed by iiT) which is the same as iiB). Then we have iiiT) as the iiiB) have been discounted. Thus the 2 set where $\lambda_{1T} - p < p - 1$ for $p \leq r \leq 2p-1$ becomes the following 3-set for $2p \leq r \leq 3p-1$;
$(\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$
$(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$
$(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)$.

We must now explain why we have this ordering of the core class, by numbering the core class 1 to 3.
$1 \trianglerighteq 2$
i) $\lambda_{2T} + 2p - 1 \geq 2p - 1 > 2p - 2 > 2p - 1$ and $\lambda_{1T} = 2p - 1,\ 2p - 2$
ii) $(\lambda_{2T} + 2p - 1) + (\lambda_{1T} - p + 1) = \lambda_{1T} + \lambda_{2T} + p$.
$2 \trianglerighteq 3$
i) $\lambda_{3T} = 0$ so $\lambda_{3T} + 2p - 2 = 2p - 2 \leq \lambda_{1T} = 2p - 1,\ 2p - 2$.
ii) $\lambda_{2T} + p \geq p \geq \lambda_{1T} - p + 1$ as $\lambda_{1T} = 2p - 1,\ 2p - 2$.

6) The 2-set where $\lambda_{1T} + p > t$ and $\lambda_{1T} - p = p - 1$, so for $r = 2p - 1$
In this case we have the following core class;

$$(\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) = (\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) = (2p-1, 0, 0)$$
$$(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (\lambda_{1T} - p, \lambda_{1T} - p, \lambda_{3T}) = (p-1, p-1, 1)$$

So as usual we need to take these two partitions and see what happens when

we add a p-hook first to the top row, then the middle row, and finally the bottom row.
i) Adding to the top row:
T) It is a requirement in any partition μ that $\mu_1 \leq t$ and thus as in this case $\lambda_{1T} + p > t$ then we cannot add a p-hook to the first row.
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (p-1, p-1, 1)$ becomes $(2p-1, p-1, 1) = (\lambda_{1T}, \lambda_{1T}-p, 1)$
ii) Adding to the middle row:
T) $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) = (2p-1, 0, 0)$ becomes $(2p-1, p, 0) = (\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T})$
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (p-1, p-1, 1)$ becomes $(2p-2, p, 1) = (\lambda_{1T}-1, \lambda_{2T}+p, 1)$
iii) Adding to the bottom row:
T) $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) = (2p-1, 0, 0)$ becomes $(2p-1, p-1, 1) = (\lambda_{1T}, \lambda_{1T}-p, 1)$.
B) $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (p-1, p-1, 1)$ becomes $(p-1, p, p)$ which is not a proper partition.
We therefore have that iiT) $(\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T})$ gives the new top weight followed by iT) $(\lambda_{1T}, \lambda_{1T}-p, 1)$ as $\lambda_{2T}+p = p > p-1 = \lambda_{1T}-p$. We have that iiiT) is the same as iB) and as iiiB) has been discounted then we are left with iiB) as the new lowest weight. Thus the 2-set where $\lambda_{1T}-p = p-1$ for $p \leq r \leq 2p-1$ becomes the following 3-set for $2p \leq r \leq 3p-1$;
$(\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{1T}, \lambda_{1T}-p, 1)$
$(\lambda_{1T}-1, p, 1)$.

It remains to show why we having this ordering on the core class, again by numbering the core classes 1 to 3.
$1 \trianglerighteq 2$
i) Clear.
ii) $\lambda_{1T} = 2p-1$, $\lambda_{2T} = 0$ so $\lambda_{2T}+p = p > p-1 = \lambda_{1T}-p$.
$2 \trianglerighteq 3$
i) Clear.
ii) $\lambda_{1T} + \lambda_{1T} + p = 3p-2 = 2p-1-1+p = \lambda_{1T}-1+p$.

7) Finally, for $2p \leq r \leq 3p-1$ we will also have the new self-titled partitions. As they follow the normal rules for a self-titled partition then they will be of the same form as those in the range $p \leq r \leq 2p-1$, and so using Proposition 4.2.19 and Corollary 4.2.20 we have $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1+\lambda_2+\lambda_3 = r$ such that;
i) $\lambda_1 - \lambda_3 > p-2$.
ii) $\lambda_1 - \lambda_3 < p-2$.
iii) $\lambda_1 - \lambda_3 = p-2$

4.5 Classification of core classes for $3p \le r \le 4p-1$

AIM: Although we are only proving Theorem 4.1.3 up to $r = 3p-1$ we still continue our classification of p-core for the range $3p \le r \le 4p-1$. This will help us in Chapter 5 when we conjecture that $D_{3,p}(r) \neq A(\pi, r)$ for $3p \le r \le 6p-9$. Using the above classification we can see that we have a 6-set, three 3-sets, two 2-sets and the three new types of self-titled partitions, each of which has a different p-core. We will not go into the same detail as for the range $p \le r \le 3p-1$, but by applying the same method we will give the new core classes formed for the range $3p \le r \le 4p-1$.

1) The 6-set from the 3-set.
In this case we have the core class;

$$\begin{aligned}
(\lambda_{11}, \lambda_{21}, \lambda_{31}) &= (\lambda_{1T}+p, \lambda_{2T}, \lambda_{3T}) \\
(\lambda_{12}, \lambda_{22}, \lambda_{32}) &= (\lambda_{2T}+2p-1, \lambda_{1T}-p+1, \lambda_{3T}) \\
(\lambda_{13}, \lambda_{23}, \lambda_{33}) &= (\lambda_{3T}+2p-2, \lambda_{1T}-p+1, \lambda_{2T}+1) \\
(\lambda_{14}, \lambda_{24}, \lambda_{34}) &= (\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T}) \\
(\lambda_{15}, \lambda_{25}, \lambda_{35}) &= (\lambda_{1T}, \lambda_{3T}+p-1, \lambda_{2T}+1) \\
(\lambda_{16}, \lambda_{26}, \lambda_{36}) &= (\lambda_{2T}+p-1, \lambda_{3T}+p-1, \lambda_{1T}-p+2)
\end{aligned}$$

where $\lambda_{1T} - \lambda_{2T} \ge p$ and each has p-core $(\lambda_{1T}-p, \lambda_{2T}, \lambda_{3T})$.
Following the method for the previous ranges, we result in a new core class for $3p \le r \le 4p-1$ consisting of 7 weights;

$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{3T}+p-1, \lambda_{2T}+1)$
$(\lambda_{2T}+2p-1, \lambda_{1T}+1, \lambda_{3T})$
$(\lambda_{2T}+2p-1, \lambda_{3T}+p-1, \lambda_{1T}-p+2)$
$(\lambda_{3T}+2p-2, \lambda_{1T}+1, \lambda_{2T}+1)$
$(\lambda_{3T}+2p-2, \lambda_{2T}+p, \lambda_{1T}-p+2)$
$(\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T}+p)$

2) The 3-set from the self-titled where $\lambda_{1T} - \lambda_{3T} < p-2$.
In this case we have the following core class for the range $2p \le r \le 3p-1$;
$(\lambda_{1T}+p, \lambda_{2T}, \lambda_{3T})$
$(\lambda_{2T}+p-1, \lambda_{1T}+1, \lambda_{3T})$
$(\lambda_{3T}+p-2, \lambda_{1T}+1, \lambda_{2T}+1)$
which has p-core $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$. We again work through each one adding on the p-hook to each row to form the new core class for the range $3p \le r \le 4p-1$;

$(\lambda_{1T}+2p,\lambda_{2T},\lambda_{3T})$
$(\lambda_{2T}+2p-1,\lambda_{1T}+1,\lambda_{3T})$
$(\lambda_{3T}+2p-2,\lambda_{1T}+1,\lambda_{2T}+1)$
$(\lambda_{1T}+p,\lambda_{2T}+p,\lambda_{3T})$
$(\lambda_{1T}+p,\lambda_{3T}+p-1,\lambda_{2T}+1)$
$(\lambda_{2T}+p-1,\lambda_{3T}+p-1,\lambda_{1T}+2)$.

3) The 2-set from the self-titled where $\lambda_{1T}-\lambda_{3T}>p-2$.
In this case, for $2p\leq r\leq 3p-1$ we have the following core class;
$(\lambda_{1T}+p,\lambda_{2T},\lambda_{3T})$
$(\lambda_{1T},\lambda_{3T}+p-1,\lambda_{2T}+1)$
and as usual we now add p-hooks to each one to find the new core class for $3p\leq r\leq 4p-1$;

$(\lambda_{1T}+p,\lambda_{2T}+p,\lambda_{3T})$
$(\lambda_{1T}+p,\lambda_{3T}+p-1,\lambda_{2T}+1)$
$(\lambda_{3T}+2p-2,\lambda_{1T}+1,\lambda_{2T}+1)$

4) The 2-set from the self-titled where $\lambda_{1T}-\lambda_{3T}=p-2$
In this case we have the following core class for $2p\leq r\leq 3p-1$;
$(\lambda_{1T}+p,\lambda_{2T},\lambda_{3T})$
$(\lambda_{2T}+p-1,\lambda_{1T}+1,\lambda_{3T})$
and we now add p-hooks to each of these partitions to find the new core class formed for the range $3p\leq r\leq 4p-1$;

$(\lambda_{2T}+2p-1,\lambda_{1T}+1,\lambda_{3T})$
$(\lambda_{1T}+p,\lambda_{2T}+p,\lambda_{3T})$
$(\lambda_{3T}+2p-2,\lambda_{1T}+1,\lambda_{2T}+1)$

5) The 2-set from the 3-set where $\lambda_{1T}+p>t$ and $\lambda_{1T}-p=p-2$.
In this instance we have the following core class for the range $r=3p-1,3p-2$;
$(\lambda_{2T}+2p-1,\lambda_{1T}-p+1,\lambda_{3T})$
$(\lambda_{1T},\lambda_{2T}+p,\lambda_{3T})$
$(\lambda_{3T}+2p-2,\lambda_{1T}-p+1,\lambda_{2T}+1)$

We now add p-hooks to these partitions to find the new core class for $r=4p-1,4p-2$.

$(\lambda_{2T}+2p-1,\lambda_{1T}+1,\lambda_{3T})$
$(\lambda_{3T}+2p-2,\lambda_{2T}+p,\lambda_{1T}-p+2)$

6) The 2-set from the 3-set where $\lambda_{1T} + p > t$ and $\lambda_{1T} - p = p - 1$.
In this instance we have the following core class for the range $r = 3p - 1$;
$(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$
$(\lambda_{1T}, \lambda_{1T} - p, 1)$
$(\lambda_{1T} - 1, p, 1)$

We now add p-hooks to these partitions to find the new core class for $r = 4p-1$.

$(\lambda_{1T}, \lambda_{1T}, 1)$
$(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T} + p)$

7)i) The new self-titled where $2p - 2 > \lambda_1 - \lambda_3 > p - 2$.
This is exactly the same proof as for number 3) in the range $2p \leq r \leq 3p - 1$

7)ii) The new self-titled where $\lambda_1 - \lambda_3 < p - 2$.
This is the same proof as for 2) in the range $2p \leq r \leq 3p - 1$.

7)iii) The new self-titled where $\lambda_1 - \lambda_3 = p - 2$.
This is the same proof as for 4) in the range $2p \leq r \leq 3p - 1$.

7)iv) The new self-titled where $\lambda_1 - \lambda_3 = 2p - 2$.
We show here that these cannot be made into a new partition.

CLAIM 4.5.1

$$\lambda_1 - \lambda_3 = 2p - 2 \Leftrightarrow \lambda_1 - \lambda_2 = p - 1 \text{ and } \lambda_2 - \lambda_3 = p - 1.$$

Suppose this is true and we go about trying to add p-hooks to this partition. Well, we know that $\lambda_1 \geq 2p - 2$ and thus $\lambda_1 + p \geq 3p - 2 > 3p - 3 = t$ so adding to the top row would produce a partition out of the range required. As $\lambda_1 - \lambda_2 = p - 1$ then adding to the middle row would give the partition $(\lambda_1, \lambda_1 + 1, \lambda_3)$ which is not a proper partition. Finally, as $\lambda_2 - \lambda_3 = p - 1$ then adding to the bottom row would give the partition $(\lambda_1, \lambda_2, \lambda_2 + 1)$ which again is not a proper partition. Hence no new partitions can be formed.

Proof. of Claim
'$\Rightarrow$'
Suppose for a contradiction that $\lambda_1 - \lambda_3 = 2p - 2$ but $\lambda_1 - \lambda_2 = p - 1 + \eta$ and $\lambda_2 - \lambda_3 = p - 1 - \eta$ where $\eta \geq 1$ then we can remove $p - 1 + \eta + 1 = p + \eta > p$ from the first row and hence the partition is not self-titled and we have a contradiction.
On the other hand, suppose $\lambda_1 - \lambda_3 = 2p - 2$ but $\lambda_1 - \lambda_2 = p - 1 - \eta$ and $\lambda_2 - \lambda_3 = p - 1 + \eta$ where $\eta \geq 1$ then we can remove $p - 1 - \eta + 1 = p - \eta$

from row one and as $p-1+\eta > \eta$ then we can remove the remainder of the p-hook from the second row and so the partition is not self-titled which is a contradiction. Hence we must have $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = p-1$.
'$\Leftarrow$'
If $\lambda_1-\lambda_2 = p-1 = \lambda_2-\lambda_3$ then $\lambda_1-\lambda_3 = \lambda_1-\lambda_2+\lambda_2-\lambda_3 = p-1+p-1 = 2p-2$ as required. □

7)v) The case where $\lambda_1 - \lambda_3 > 2p-2$. We show here that there exists no self-titled partitions where $\lambda_1 - \lambda_3 > 2p-2$. Recall from Corollary 4.3.4 that for $\lambda_1-\lambda_3 > p-2$ then $\lambda_1-\lambda_2 = p-1$. Then $\lambda_2-\lambda_3 > \lambda_1-p+1+2p-1-\lambda_1 = p$ and so we can always remove a p-hook from the second row and the partition is not self-titled.

8) Finally for this range we have the new self-titled partitions.

CLAIM 4.5.2 For the range $3p \leq r \leq 4p-1$ there are no self-titled partitions where $\lambda_1 - \lambda_3 < p-2$.

Proof. Suppose for a contradiction there does exist a self-titled partition where $\lambda_1 - \lambda_3 < p-2$ then from Corollary 4.3.4 we know that we have either;
i) $\lambda_1 - \lambda_2 = p - \xi$ and $\lambda_2 + 1 < \xi$ when $\lambda_3 = 0$
ii) $\lambda_1 - \lambda_2 = p - \xi$ and $\lambda_2 + 2 < \xi$ when $\lambda_3 \neq 0$.

i) If this holds then $r = \lambda_1+\lambda_2+\lambda_3 = \lambda_1+\lambda_2 = 2\lambda_1-p+\xi < 2p-2-p+\xi = p-2+\xi$, so to ensure that $r \geq 3p$ we require $p-2+\xi > 3p$ implying $\xi \geq 2p+2$. Thus we have that $\lambda_1 - \lambda_2 = p - \xi \leq p-(2p+2) = -p-2$ which is clearly a contradiction. Hence there are no self-titled partitions of type i).

ii) Now suppose this holds, then $r = \lambda_1+\lambda_2+\lambda_3 < (p-\xi+\lambda_2)+(\xi-2)+\lambda_3 < (p-2)+(\xi-2)+\lambda_3 = p-4+\xi+\lambda_3$. So to ensure $r \geq 3p$ we need $p-4+\xi+\lambda_3 > 3p$ so $\lambda_3 > 2p+4-\xi$. Now we know that $2 \leq \xi \leq p$ so $\lambda_3 > p+4$, moreover if $2 \leq \xi \leq p$ then $0 \leq \lambda_1 - \lambda_2 \leq p-2$ and hence $\lambda_2 \leq \lambda_1 \leq \lambda_2+p-2$ giving that $\lambda_2 = p-2 < \lambda_3$ which is a contradiction, so there are no self-titled partitions of type ii) □

We therefore have only the following type of self-titled partitions;
i) Those where $\lambda_1 - \lambda_3 > p-2$, for example $p = 7$, $r = 3p = 21$, $\lambda = (11, 5, 5)$.
ii) Those where $\lambda_1 - \lambda_3 = p-2$, for example $p = 11$, $r = 3p+3 = 36$, $\lambda = (17, 8, 8)$.

4.6 Classification of core classes for $4p \leq r \leq 5p-5$

AIM: Finally we repeat the process to find the core classes for $4p \leq r \leq 5p-5$. The reason we need go no further than $r = 5p-5$ is because the 'halfway' point $\frac{3t}{2} \leq 5p-5$, and after $r = \frac{3t}{2}$ we can apply the reflection property shown in Section 3.5. From above we can see that we have a 7-set, a 6-set, two 3-sets and four 2-sets, and the two types of new self-titled partitions. Note however that two of the 2-sets are for the cases $r = 4p-1,\ 4p-2$ and thus for our new range will be for $r = 5p-1,\ 5p-2$ which is greater than $5p-5$ and thus we do not need to worry about these particular core classes.

1) The 7-set from 6-set from the 3-set.
In this case we have the following core class;

$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{3T}+p-1, \lambda_{2T}+1)$
$(\lambda_{2T}+2p-1, \lambda_{1T}+1, \lambda_{3T})$
$(\lambda_{2T}+2p-1, \lambda_{3T}+p-1, \lambda_{1T}-p+2)$
$(\lambda_{3T}+2p-2, \lambda_{1T}+1, \lambda_{2T}+1)$
$(\lambda_{3T}+2p-2, \lambda_{2T}+p, \lambda_{1T}-p+2)$
$(\lambda_{1T}, \lambda_{2T}+p, \lambda_{3T}+p)$.

We now add p-hooks to these partitions to find the new core class for $4p-1 \leq r \leq 5p-5$;

$(\lambda_{1T}+p, \lambda_{2T}+2p, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{3T}+2p-1, \lambda_{2T}+1)$
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T}+p)$
$(\lambda_{2T}+2p-1, \lambda_{3T}+2p-1, \lambda_{1T}-p+2)$
$(\lambda_{2T}+2p-1, \lambda_{1T}+1, \lambda_{3T}+p)$
$(\lambda_{3T}+2p-2, \lambda_{1T}+1, \lambda_{2T}+p+1)$

2) The 6-set from the 3-set from the self-titled where $\lambda_{1T}-\lambda_{3T} < p-2$
In this case we have the core class;
$(\lambda_{1T}+2p, \lambda_{2T}, \lambda_{3T})$
$(\lambda_{2T}+2p-1, \lambda_{1T}+1, \lambda_{3T})$
$(\lambda_{3T}+2p-2, \lambda_{1T}+1, \lambda_{2T}+1)$
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{3T}+p-1, \lambda_{2T}+1)$
$(\lambda_{2T}+p-1, \lambda_{3T}+p-1, \lambda_{1T}+2)$

and we now go about adding on p-hooks to each partition to find the new core class for $4p-1 \leq r \leq 5p-5$;

$(\lambda_{1T}+2p, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{1T}+2p, \lambda_{3T}+p-1, \lambda_{2T}+1)$
$(\lambda_{2T}+2p-1, \lambda_{1T}+p+1, \lambda_{3T})$
$(\lambda_{2T}+2p-1, \lambda_{3T}+p-1, \lambda_{1T}+2)$
$(\lambda_{3T}+2p-2, \lambda_{1T}+p+1, \lambda_{2T}+1)$
$(\lambda_{3T}+2p-2, \lambda_{2T}+p, \lambda_{1T}+2)$
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T}+p)$

3) The 3-set from the 2-set from the self-titled where $\lambda_{1T}-\lambda_{3T} > p-2$
In this case we have the following core class from $3p \leq r \leq 4p-1$;
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{3T}+p-1, \lambda_{2T}+1)$
$(\lambda_{3T}+2p-2, \lambda_{1T}+1, \lambda_{2T}+1)$

which for the range $4p \leq r \leq 5p-5$ gives the new core class;

$(\lambda_{1T}+p, \lambda_{3T}+2p-1, \lambda_{2T}+1)$
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T}+p)$.

4) The 3-set from the 2-set from the self-titled where $\lambda_{1T}-\lambda_{3T} = p-2$.
In this case we have the following core class for $3p \leq r \leq 4p-1$;
$(\lambda_{2T}+2p-1, \lambda_{1T}+1, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T})$
$(\lambda_{3T}+2p-2, \lambda_{1T}+1, \lambda_{2T}+1)$

which gives the new core class for $4p \leq r \leq 5p-5$;

$(\lambda_{2T}+2p-1, \lambda_{1T}+p+1, \lambda_{3T})$
$(\lambda_{1T}+p, \lambda_{2T}+p, \lambda_{3T}+p)$

5) and 6) are not important as they are the cases where $r = 5p-2,\ 5p-1 > 5p-5 \geq \frac{3t}{2}$ so are out of the range we need to consider as any degree above $r = \frac{3t}{2}$ we shall prove with the reflection property in Section 3.5.

7)i) The 2-set from the self-titled where $2p-2 > \lambda_1 - \lambda_3 > p-2$.
This is the same proof as for 3) from the range $3p \leq r \leq 4p-1$
7ii) The 3-set from the self-titled where $\lambda_1 - \lambda_3 < p-2$.
This is the same proof as for 2) from the range $3p \leq r \leq 4p-1$.
7)iii) The 2-set from the self-titled where $\lambda_1 - \lambda_3 = p-2$.

This is the same proof as for 4) from the range $3p \leq r \leq 4p-1$.

8)i) The new self-titled where $2p-2 > \lambda_1 - \lambda_3 > p-2$.
This is the same proof as for 3) from the range $2p \leq r \leq 3p-1$.
8)ii) The new self-titled where $\lambda_1 - \lambda_3 = p-2$.
This is the same proof as for 4) from the range $2p \leq r \leq 3p-1$.

9) What remains now is the new-self-titled in this range.

CLAIM 4.6.1 The only self-titled partitions that exist in the range $4p \leq r \leq 5p-5$ are where $\lambda_1 - \lambda_3 > 2p-2$. For example, $p = 7$, $r = 5p-5 = 30$ and $\lambda = (16, 10, 4)$.

Proof. i) First we shall prove that there exists no self-titled partitions where $p-2 < \lambda_1 - \lambda_3 \leq 2p-2$. From Corollary 4.3.4 we know that if such a partition did exist then $\lambda_1 - \lambda_2 = p-1$, and thus $0 \leq \lambda_2 - \lambda_3 \leq p-2$. Let $\lambda_2 - \lambda_3 = \mu$, then for the self-titled partition to exist we would require $\lambda_3 < p - (\mu+1)$, as then we cannot remove the rim p-hook from the third row, and so we can say that $r = \lambda_1 + \lambda_2 + \lambda_3 < (p-1+\mu+(p-\mu-1)) + (\mu+(p-\mu+1)) + (p-\mu+1) = 4p-\mu-4 < 4p$, which is a contradiction as we require $4p \leq r \leq 5p-5$.

ii) We now prove there exists no self-titled partitions where $\lambda_1 - \lambda_3 = p-2$. Suppose there exists a partition of this kind, then from Corollary 4.3.4 we know that $\lambda_1 - \lambda_2 = p - \xi$ and $\lambda_2 - \lambda_3 = \xi - 2$ and of course we must have that $(\lambda_2 - \lambda_3) + \lambda_3 + 1 \leq p-1$ i.e. that $\lambda_2 + 1 \leq p-1$. Therefore we have that $\lambda_3 \leq p-\xi$ and so $r = \lambda_1 + \lambda_2 + \lambda_3 \leq (p-\xi+\xi-2+p-\xi) + (\xi-2+p-\xi) + (p-\xi) = 4p-4-2\xi < 4p$ which is a contradiction as we require $4p \leq r \leq 5p-5$.

iii) Finally we prove that there exists no self-titled partition where $\lambda_1 - \lambda_3 < p-2$. So suppose there is and that $\lambda_3 = 0$ then from 4.3.4 we know that $\lambda_1 - \lambda_3 = p-\xi$ and $\lambda_2 + 1 < \xi$, then $r = \lambda_1 + \lambda_2 < (p-2) + (p-2-p+\xi) = p-4+\xi$. So for $r \geq 4p$ then we require at least for $p-4+\xi > 4p$ which only occurs if $\xi \geq 3p+4$ which implies $\lambda_1 - \lambda_2 \leq -2p-4$ which is clearly a contradiction. If on the other hand we have $\lambda_3 \neq 0$ then $\lambda_1 - \lambda_2 = p - \xi$, $\lambda_2 + 2 < \xi$ and $\lambda_3 \leq p-1$. Therefore $r = \lambda_1 + \lambda_2 + \lambda_3 < (\xi-2+p-\xi) + (\xi-2) + (p-1) = 2p+\xi-5$ and so for $r \geq 4p$ we require at least for $2p+\xi-5 > 4p$ which implies $\xi \geq 2p+5$, thus giving that $\lambda_1 - \lambda_2 \leq -p-5$ which is a contradiction. So there are no self-titled partitions in the range $4p \leq r \leq 5p-5$ such that $\lambda_1 - \lambda_3 < p-2$.

□

4.7 Classification of core classes for $0 \leq r \leq 5p-5$

Bringing all of the previous work together we can now classify the core classes for each degree $0 \leq r \leq 5p-5$.

• We first do this for the case $p \leq r \leq 2p-1$, in which we have the following four categories;
1) Self-titled partitions λ where
i) $\lambda_1 - \lambda_3 < p-2$,
ii) $\lambda_1 - \lambda_3 > p-2$,
iii) $\lambda_1 - \lambda_3 = p-2$.

2) A 3-set where $\lambda_1 - \lambda_2 \geq p$ and $\lambda_1 \leq 2p-3$
$(\lambda_1, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)$
$(\lambda_3 + p - 2, \lambda_1 - p + 1, \lambda_2 + 1)$

3) A 2-set where $\lambda_1 - \lambda_2 \geq p,\ \lambda_1 + p > t$ and $\lambda_1 - p < p - 1$
$(\lambda_1, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)$

This is $r = 2p-2$ with $\lambda = (r, 0, 0)$ and $r = 2p-1$ with $\lambda = (r-1, 1, 0)$.

4) A 2-set where $\lambda_1 - \lambda_2 \geq p,\ \lambda_1 + p > t$ and $\lambda_1 - p = p - 1$
$(\lambda_1, \lambda_2, \lambda_3)$
$(\lambda_1 - p, \lambda_1 - p, 1)$

This is $r = 2p-1$ with $\lambda = (r, 0, 0)$.

• For $2p \leq r \leq 3p-1$ then the above categories form new core classes as follows;
1) A 3-set becomes a 6-set consisting of the core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)$
$(\lambda_3 + 2p - 2, \lambda_1 - p + 1, \lambda_2 + 1)$
$(\lambda_1, \lambda_2 + p, \lambda_3)$
$(\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_2 + p - 1, \lambda_3 + p - 1, \lambda_1 - p + 2)$

2) The self-titled partition where $\lambda_1 - \lambda_3 < p - 2$ becomes a 3-set consisting of the new core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_3 + p - 2, \lambda_1 + 1, \lambda_2 + 1)$

3) The self-titled partition where $\lambda_1 - \lambda_3 > p - 2$ becomes a 2-set consisting of the new core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)$

4) The self-titled where $\lambda_1 - \lambda_3 = p - 2$ becomes a 2-set consisting of the new core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$

5) A 2-set where $\lambda_1 - p < p - 1$ becomes a 3-set consisting of the new core class
$(\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)$
$(\lambda_1, \lambda_2 + p, \lambda_3)$
$(\lambda_3 + 2p - 2, \lambda_1 - p + 1, \lambda_2 + 1)$

6) A 2-set where $\lambda_1 - p = p - 1$ becomes a 3-set consisting of the new core class
$(\lambda_1, \lambda_2 + p, \lambda_3)$
$(\lambda_1, p - 1, 1)$
$(\lambda_1, p, 1)$

7) We have new self-titled partitions λ where
i) $\lambda_1 - \lambda_3 < p - 2$,
ii) $\lambda_1 - \lambda_3 > p - 2$,
iii) $\lambda_1 - \lambda_3 = p - 2$.

• For $3p \leq r \leq 4p - 1$ then the above categories form new core classes as follows;
1) The 6-set from the 3-set becomes a 7-set consisting of the new core class
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_2 + 2p - 1, \lambda_2 + p - 1, \lambda_1 - p + 2)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + 1)$
$(\lambda_3 + 2p - 2, \lambda_2 + p, \lambda_1 - p + 2)$

$(\lambda_1, \lambda_2 + p, \lambda_3 + p)$

2) The 3-set from the self-titled where $\lambda_1 - \lambda_2 < p - 2$ becomes a 6-set consisting of the new core class
$(\lambda_1 + 2p, \lambda_2, \lambda_3)$
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + 1)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_2 + p - 1, \lambda_2 + p - 1, \lambda_1 + 2)$

3) The 2-set from the self titled where $\lambda_1 - \lambda_3 > p - 2$ becomes a 3-set consisting of the new core class
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + 1)$

4) The 2-set from the self titled where $\lambda_1 - \lambda_3 = p - 2$ becomes a 3-set consisting of the new core class
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + 1)$

5) A 3-set from a 2-set where $\lambda_1 - p < p - 1$ becomes a 2-set consisting of the new core class
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_3 + 2p - 2, \lambda_2 + p, \lambda_1 - p + 2)$

6) A 3-set from a 2-set where $\lambda_1 - p = p - 1$ becomes a 2-set consisting of the new core class
$(\lambda_1, \lambda_1, 1)$
$(\lambda_1, \lambda_2 + p, \lambda_3 + p)$

7i) A new self-titled where $\lambda_1 - \lambda_3 < p - 2$ becomes a 3-set consisting of the new core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_3 + p - 2, \lambda_1 + 1, \lambda_2 + 1)$

7ii) A new self-titled where $2p - 2 > \lambda_1 - \lambda_3 > p - 2$ becomes a 2-set consisting of the new core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$

$(\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)$

7iii) A new self-titled where $\lambda_1 - \lambda_3 = p - 2$ becomes a 2-set consisting of the new core class
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$

8) We have new self-titled partitions λ where
i) $\lambda_1 - \lambda_3 > p - 2$
ii) $\lambda_1 - \lambda_3 = p - 2$

• For $4p \leq r \leq 5p - 5$ then the above categories form new core classes as follows;
1) The 7-set from the 6-set from the 3-set becomes a 6-set consisting of the new core class;
$(\lambda_1 + p, \lambda_2 + 2p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + 2p - 1, \lambda_2 + 1)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3 + p)$
$(\lambda_2 + 2p - 1, \lambda_3 + 2p - 1, \lambda_1 - p + 2)$
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3 + p)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + p_1)$

2) The 6-set from the 3-set from the self-titled where $\lambda_1 - \lambda_3 < p - 2$ becomes a 7-set consisting of the new core class;
$(\lambda_1 + 2p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + 2p, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_2 + 2p - 1, \lambda_1 + p + 1, \lambda_3)$
$(\lambda_2 + 2p - 1, \lambda_3 + p - 1, \lambda_1 + 2)$
$(\lambda_3 + 2p - 2, \lambda_1 + p + 1, \lambda_2 + 1)$
$(\lambda_3 + 2p - 2, \lambda_2 + p, \lambda_1 + 2)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3 + p)$

3) The 3-set from the 2-set from the self-titled where $\lambda_1 - \lambda_3 > p - 2$ becomes a 2-set consisting of the new core class;
$(\lambda_1 + p, \lambda_3 + 2p - 1, \lambda_2 + 1)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3 + p)$

4) The 3-set from the 2-set from the self-titled where $\lambda_1 - \lambda_3 = p - 2$ becomes a 2-set consisting of the new core class;
$(\lambda_2 + 2p - 1, \lambda_1 + p + 1, \lambda_3)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3 + p)$

5) and 6) are not important as they are the cases when $r = 5p-2, 5p-1$ which is out of the range we are considering.

7i) A 3-set from a new self-titled where $\lambda_1 - \lambda_3 < p-2$ becomes a 6-set consisting of the new core class;
$(\lambda_1 + 2p, \lambda_2, \lambda_3)$
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + 1)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_2 + p - 1, \lambda_2 + p - 1, \lambda_1 + 2)$

7ii) A 2-set from a new self-titled where $\lambda_1 - \lambda_3 = p-2$ becomes a 3-set consisting of the new core class;
$(\lambda_2 + 2p - 1, \lambda_1 + 1, \lambda_3)$
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + p - 1, \lambda_2 + 1)$

7iii) A 2-set from a new self-titled where $2p - 2 > \lambda_1 - \lambda_3 > p - 2$ becomes a 3-set consisting of the new core class;
$(\lambda_1 + p, \lambda_2 + p, \lambda_3)$
$(\lambda_1 + p, \lambda_3 + p - 1, \lambda_2 + 1)$
$(\lambda_3 + 2p - 2, \lambda_1 + 1, \lambda_2 + 1)$

8i) A new self-titled where $2p - 2\lambda_1 - \lambda_3 > p - 2$ becomes a 2-set consisting of the new core class;
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)$

8ii) A new self-titled where $\lambda_1 - \lambda_3 = p - 2$ becomes a 2-set consisting of the new core class;
$(\lambda_1 + p, \lambda_2, \lambda_3)$
$(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$

9) We have new self-titled partitions λ where $\lambda_1 - \lambda_3 > 2p - 2$.

EXAMPLE 4.7.1 To illustrate this we shall now give the results of the core classes for $p = 5$ and for all $0 \leq r \leq 5p - 5$. Each degree will be shown separately, along with each core class within that degree. The p-core of each core class will be given at the top of each core class, with self-titled partitions being marked ST.

• $0 \leq r \leq p-1$

deg 0

ST
0,0,0

deg 1

ST
1,0,0

deg 2

ST	ST
2,0,0	1,1,0

deg 3

ST	ST	ST
3,0,0	2,1,0	1,1,1

deg 4

ST	ST	ST	ST
4,0,0	3,1,0	2,2,0	2,1,1

• $p \leq r \leq 2p-1$

deg 5

Ø	ST	ST
5,0,0	3,2,0	2,2,1
4,1,0		
3,1,1		

deg 6

x	ST	ST	ST	ST
6,0,0	5,1,0	4,1,1	3,3,0	2,2,2
4,2,0				
3,2,1				

deg 7

xx	x x	ST	ST
7,0,0	6,1,0	5,1,1	4,2,1
4,3,0	5,2,0		
3,3,1	3,2,2		

deg 8

xxx	xx x	ST	x x x	ST
8,0,0	7,1,0	6,2,0	6,1,1	4,3,1
4,4,0	5,3,0		5,2,1	
	3,3,2		4,2,2	

deg 9

xxxx	xxx x	xx xx	xx x x	ST	ST
9,0,0	8,1,0	7,2,0	7,1,1	6,2,1	5,2,2
4,4,1	5,4,0	6,3,0	5,3,1		
		3,3,3	4,3,2		

• $2p \leq r \leq 3p - 1$

deg 10

Ø	xxx xx	ST	xx xx x	ST	ST
10,0,0	8,2,0	7,3,0	7,2,1	6,2,2	5,3,2
9,1,0	6,4,0		6,3,1		
8,1,1			4,3,3		
5,5,0					
5,4,1					
4,4,2					

deg 11

x	xxxxx x	xxxx x x	xxx xxx	ST	xx xx xx
11,0,0	10,1,0	9,1,1	8,3,0	7,3,1	7,2,2
9,2,0	5,4,2	5,5,1	7,4,0		6,3,2
8,2,1					5,3,3
6,5,0					
6,4,1					
4,4,3					

deg 12

xx	x x	xxxxx x x	xxxx xx x	ST	ST	ST
12,0,0	11,1,0	10,1,1	9,2,1	8,4,0	7,3,2	6,3,3
9,3,0	10,2,0	5,5,2	6,5,1			
8,3,1	8,2,2					
7,5,0	6,6,0					
7,4,1	6,4,2					
4,4,4	5,4,3					

deg 13

xx x	xxxxxx xx	x x x	xxx	xxxx xxx x	ST
12,1,0	11,2,0	11,1,1	9,4,0	9,3,1	7,3,3
10,3,0	6,4,3	10,2,1	8,5,0	7,5,1	
8,3,2		9,2,2	8,4,1		
7,6,0		6,6,1			
7,4,2		6,5,2			
5,4,4		5,5,3			

deg 14

xx xx	xx x x	xxxxxx xx x	xxx x	xxxxx xx xx	xxxx
12,2,0	12,1,1	11,2,1	10,4,0	10,2,2	9,5,0
11,3,0	10,3,1	6,5,3	8,6,0	6,6,2	9,4,1
8,3,3	9,3,2		8,4,2		8,5,1
7,7,0	7,6,1				
7,4,3	7,5,2				
6,4,4	5,5,4				

- $3p \leq r \leq 4p - 1$

deg 15

xxxxxxx xxx	xx xx x	xxx xx	xxxxxx xx xx	Ø	xxxxx xxx xx	ST
12,3,0	12,2,1	11,4,0	11,2,2	10,5,0	10,3,2	9,5,1
7,4,4	11,3,1	8,7,0	6,6,3	10,4,1	7,6,2	
	9,3,3	8,4,3		9,6,0		
	7,7,1			9,4,2		
	7,5,3			8,6,1		
	6,5,4			8,5,2		
				5,5,5		

deg 16

xxx xxx	xxxxxxx xxx x	xx xx xx	x	xxxxx x	xxxx x x
12,4,0	12,3,1	12,2,2	11,5,0	10,6,0	10,5,1
8,8,0	7,5,4	11,3,2	11,4,1	10,4,2	9,6,1
8,4,4		10,3,3	9,7,0		9,5,2
		7,7,2	9,4,3		
		7,6,3	8,7,1		
		6,6,4	8,5,3		
			6,5,5		

deg 17

xx	xxxxxxxx xxx xx	x x	xxxx xx x	xxxxxxx xxx xxx	xxxxx x x
12,5,0	12,3,2	11,6,0	11,5,1	11,3,3	10,6,1
12,4,1	7,6,4	11,4,2	9,7,1	7,7,3	10,5,2
9,8,0		10,7,0	9,5,3		9,6,2
9,4,4		10,4,3			
8,8,1		8,7,2			
8,5,4		8,6,3			
7,5,5		6,6,5			

deg 18

xx x	xxxx xxx x	xxxxxxxx xxx xxx	xxxxxxx xx	x x x	ST	xxx
12,6,0	12,5,1	12,3,3	11,7,0	11,6,1	10,6,2	9,9,0
12,4,2	9,8,1	7,7,4	11,4,3	11,5,2		8,5,5
10,8,0	9,5,4		8,7,3	10,7,1		
10,4,4				10,5,3		
8,8,2				9,7,2		
8,6,4				9,6,3		
7,6,5				6,6,6		

deg 19

xx xx	xx x x	xxxxxxx xx x	xxxxxx xx xx	xxx x	xxxx
12,7,0	12,6,1	11,7,1	11,6,2	10,9,0	9,9,1
12,4,3	12,5,2	11,5,3	10,7,2	8,6,5	9,5,5
11,8,0	10,8,1	9,7,3	10,6,3		
11,4,4	10,5,4				
8,8,3	9,8,2				
8,7,4	9,6,4				
7,7,5	7,6,6				

• $4p \leq r \leq 5p - 5$

deg 20

XXXXXXX XXX	XX XX X	XXXXX XXX XX	XXX XX	XXXXXX XX XX	∅
12,8,0	12,7,1	12,6,2	11,9,0	11,7,2	10,10,0
12,4,4	12,5,3	10,8,2	8,7,5	11,6,3	10,9,1
8,8,4	11,8,1	10,6,4		10,7,3	10,5,5
	11,5,4				9,9,2
	9,8,3				9,6,5
	9,7,4				8,6,6
	7,7,6				

4.8 Tilting Modules

AIM: We have clarified that to prove Theorem 4.1.3 it is necessary to find all tilting comodules of $A(\pi, r)$ for π a saturated set, which we do by comparing their characters with those of the truncated modules $\mathrm{Tr}^{\lambda} E$, whose coefficient spaces sum to form $D_{3,p}(r)$. These truncated modules are made up of tensor products of the modules $\bar{S}^{\lambda_i} E$, some of which are tilting, and some of which aren't. We therefore now study these modules, first stating when they are and are not tilting and then describing them just as we did for the $n = 2$ case. We also describe the coalgebras $D_{3,p}(r)$ and give an example when $p = 3$ and $r = 3$. From 4.8.8-4.8.15 we then use the information on the $\bar{S}^{\lambda_i} E$ to prove which of the truncated modules are tilting and which are not. This will then give us the tilting truncated modules we need to work with, to find the tilting comodules of $A(\pi, r)$.

THEOREM 4.8.1 *For $n = 3$ and all primes p the simple modules $\bar{S}^r E$ are tilting for $0 \leq r \leq p - 1$ and $2p - 2 \leq r \leq 3p - 3$. Analogously we have that the $\bar{S}^r E$ are not tilting for $p \leq r \leq 2p - 3$.*

Proof. Firstly, take the range $0 \leq r \leq p - 1$, then $\bar{S}^r E = S^r E = \nabla(r, 0, 0) = L(r, 0, 0) = \Delta(r, 0, 0)$ and thus $\bar{S}^r E = T(r, 0, 0)$. Similarly for the range $2p - 2 \leq r \leq 3p - 3$ we can use the multiplication map as described in Theorem 3.5.3

$$\bar{S}^r E \otimes \bar{S}^{t-r} E \to L$$

where $t = n(p - 1)$ and $L = \det^{\otimes p-1}$. This then induces the isomorphism

$$\bar{S}^{t-r} E \cong (\bar{S}^r E)^* \otimes L$$

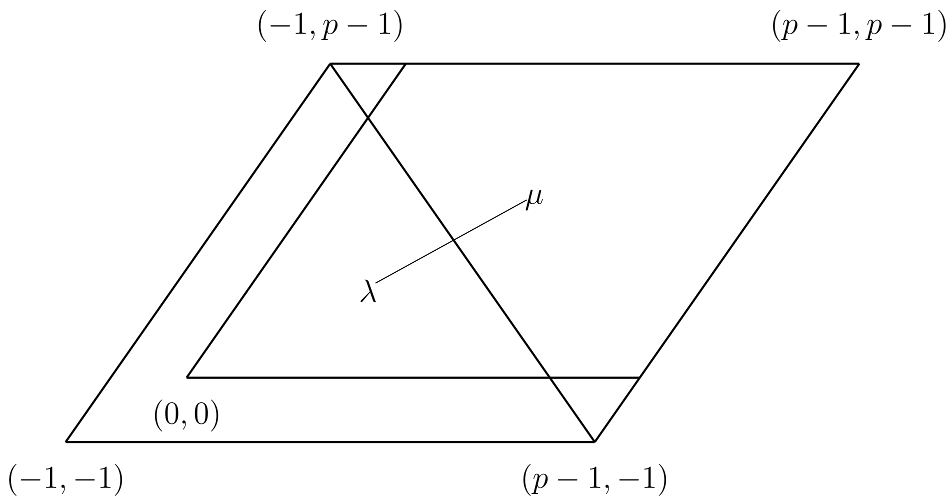
We know $\bar{S}^r E$ to be tilting and thus $(\bar{S}^r E)^*$ is tilting, moreover we have that L is tilting and hence $\bar{S}^{t-r} E$ is tilting.

If we now look at the range $p \leq r \leq 2p-3$ and consider the specific case $p = 3$ and $r = p$ then $\bar{S}^r E = L(2,1,0)$. This is tilting if $L(2,1,0) = \nabla(2,1,0)$ which, restricting to SL_3, would imply that $L(1,1) = \nabla(1,1)$. However $\nabla(1,1)$ is 8 dimensional, whilst we can see that $L(1,1)$ is 7 dimensional from the following short exact sequence:

$$0 \to L(1,0)^F \hookrightarrow \nabla(3,0) \to L(1,1) \to 0$$

and thus it is not possible for $L(2,1,0)$ to equal $\nabla(2,1,0)$ and hence $\bar{S}^p E$ is not tilting for $p = 3$.

Let us now prove the general result. Well for the range $p \leq r \leq 2p-3$ then for GL, $\bar{S}^r E = L(p-1,a,0)$ where $1 \leq a \leq p-2$. Then we have $L(p-1,a,0)|_{SL_3(k)} = L(p-1-a,a) = L(m,n)$ where $1 \leq m,n \leq p-2$. Now consider the following picture [14, Section 5.2].



For any weight in the bottom alcove, labeled λ, we have $\nabla(\lambda) = L(\lambda)$. However for weights in the top alcove, labeled μ, we have the short exact sequence

$$0 \to L(\mu) \to \nabla(\mu) \to L(\lambda) \to 0.$$

So, $\nabla(\mu) \neq L(\mu)$ and we see that the non tilting modules for GL_3 are those of the form $L(m,n,0)$ where $1 \leq m,n \leq p-2$, exactly the range we have above and hence we know that $\bar{S}^r E$ is not tilting for $p \leq r \leq 2p-3$. $\square$

We now give a detailed description of the $\bar{S}^r E$ whose tensor products form the truncated modules.

DESCRIPTION 4.8.2 Description of the $\bar{S}^r(E)$

i) For $0 \leq r \leq p-1$ we have $\bar{S}^r E = S^r E/\{0\} = S^r E$.
ii) We have $S^r E = k\text{-sp}\{e_1^a e_2^b e_3^c \mid a+b+c=r\}$ and let

$$I_r = k\text{-sp}\{e_1^a e_2^b e_3^c \mid a, b \text{ or } c \text{ is } \geq p\}.$$

Then for $p \leq r \leq 2p-2$ we have

$$\begin{aligned}\bar{S}^r E &= S^r E/I_r \\ &= k\text{-sp}\{e_1^a e_2^b e_3^c + I_r \mid a+b+c=r,\ a,b,c \leq p-1\} \\ &= L(p-1, r-(p-1), 0)\end{aligned}$$

iii) For $2p-2 \leq r \leq t$, with $S^r E$ and I_r as above then we have

$$\bar{S}^r E = L(p-1, p-1, r-(2p-2)).$$

Bringing this together we have that;

$$\bar{S}(E) = \bar{S}^0 E \oplus \bar{S}^1 E \oplus \ldots \oplus \bar{S}^{p-1} E \oplus \bar{S}^p E \oplus \ldots \oplus \bar{S}^{2p-2} E \oplus \bar{S}^{2p-1} E \oplus \ldots \oplus \bar{S}^{3p-3} E.$$

So therefore

$$\begin{aligned}\bar{S}(E) =& L(0,0,0) \oplus L(1,0,0) \oplus L(2,0,0) \oplus \ldots \oplus L(p-1,0,0) \oplus \\ & L(p-1,1,0) \oplus L(p-1,2,0) \oplus L(p-1,3,0) \oplus \ldots \oplus \\ & L(p-1,p-1,0) \oplus L(p-1,p-1,1) \oplus L(p-1,p-1,2) \\ & \oplus \ldots \oplus L(p-1,p-1,p-1),\end{aligned}$$

where the modules in the range $L(p-1,1,0), \ldots, L(p-1,p-2,0)$ are not tilting as $L(p-1,a,0) \neq \nabla(p-1,a,0)$ for $1 \leq a \leq p-2$.

EXAMPLE 4.8.3 When $n = 3$ and $p = 3$ then we have

$$\begin{aligned}\bar{S}(E) &= \bar{S}^0 E \oplus \bar{S}^1 E \oplus \bar{S}^2 E \oplus \bar{S}^3 E \oplus \bar{S}^4 E \oplus \bar{S}^5 E \oplus \bar{S}^6 E \\ &= L(0,0,0) \oplus L(1,0,0) \oplus L(2,0,0) \oplus L(2,1,0) \oplus L(2,2,0) \\ & \quad \oplus L(2,2,1) \oplus L(2,2,2)\end{aligned}$$

with $L(2,1,0)$ the only non-tilting summand.

We now give a description of the truncated modules and finally the Doty Coalgebras themselves.

DESCRIPTION 4.8.4 Description of the $D_{3,p}(r)$

We first consider $\mathrm{Tr}^{(\lambda_1,\lambda_2,\lambda_3)}E$ which is given as follows:

$$
\begin{array}{ll}
S^{\lambda_1}E \otimes S^{\lambda_2}E \otimes S^{\lambda_3}E & 0 \le \lambda_1, \lambda_2, \lambda_3 \le p-1 \\
L(p-1, \lambda_1-(p-1), 0) \otimes S^{\lambda_2}E \otimes S^{\lambda_3}E, & p \le \lambda_1 \le 2p-2,\ 0 \le \lambda_2, \lambda_3 \le p-1 \\
L(p-1, \lambda_1-(p-1), 0) \otimes L(p-1, \lambda_2-(p-1), 0) \otimes S^{\lambda_3}E, & p \le \lambda_1, \lambda_2 \le 2p-2,\ 0 \le \lambda_3 \le p-1 \\
L(p-1, \lambda_1-(p-1), 0) \otimes L(p-1, \lambda_2-(p-1), 0) \otimes L(p-1, \lambda_3-(p-1), 0), & p \le \lambda_1, \lambda_2, \lambda_3 \le 2p-2 \\
L(p-1, p-1, \lambda_1-(2p-2)) \otimes S^{\lambda_2}E \otimes S^{\lambda_3}E, & 2p-2 \le \lambda_1 \le t,\ 0 \le \lambda_2, \lambda_3 \le p-1 \\
L(p-1, p-1, \lambda_1-(2p-2)) \otimes L(p-1, \lambda_2-(p-1), 0) \otimes S^{\lambda_3}E, & 2p-2 \le \lambda_1 \le t,\ p \le \lambda_2 \le 2p-2, \\
 & 0 \le \lambda_3 \le p-1 \\
L(p-1, p-1, \lambda_1-(2p-2)) \otimes L(p-1, \lambda_2-(p-1), 0) \otimes L(p-1, \lambda_3-(p-1), 0), & 2p-2 \le \lambda_1 \le t,\ p \le \lambda_2, \lambda_3 \le 2p-2 \\
L(p-1, p-1, \lambda_1-(2p-2)) \otimes L(p-1, p-1, \lambda_2-(2p-2)) \otimes S^{\lambda_3}E, & 2p-2 \le \lambda_1, \lambda_2 \le t,\ 0 \le \lambda_3 \le p-1 \\
L(p-1, p-1, \lambda_1-(2p-2)) \otimes L(p-1, p-1, \lambda_2-(2p-2)) \otimes L(p-1, \lambda_3-(p-1), 0), & 2p-2 \le \lambda_1, \lambda_2 \le t,\ p \le \lambda_3 \le 2p-2 \\
L(p-1, p-1, \lambda_1-(2p-2)) \otimes L(p-1, p-1, \lambda_2-(2p-2)) \otimes L(p-1, p-1, \lambda_3-(2p-2)), & 2p-2 \le \lambda_1, \lambda_2, \lambda_3 \le t
\end{array}
$$

Recall that $D_{3,p}(r) = \sum_\lambda \mathrm{cf}\,(\mathrm{Tr}^{\lambda}E)$, where λ ranges over all proper partitions such that $|\lambda| = r$ and $\lambda_1 \le t$.

EXAMPLE 4.8.5 With $n = 3$ and $p = 3$ we have;

$$\begin{aligned}
D_{3,3}(0) = &\quad \text{cf}\,(S^0E \otimes S^0E \otimes S^0E)\\
D_{3,3}(1) = &\quad \text{cf}\,(S^1E \otimes S^0E \otimes S^0E)\\
D_{3,3}(2) = &\quad \text{cf}\,(S^2E \otimes S^0E \otimes S^0E) + \text{cf}\,(S^1E \otimes S^1E \otimes S^0E)\\
D_{3,3}(3) = &\quad \text{cf}\,(L(2,1,0) \otimes S^0E \otimes S^0E) + \text{cf}\,(S^2E \otimes S^1E \otimes S^0E)\\
&\quad +\text{cf}\,(S^1E \otimes S^1E \otimes S^1E)\\
D_{3,3}(4) = &\quad \text{cf}\,(L(2,2,0) \otimes S^0E \otimes S^0E) + \text{cf}\,(L(2,1,0) \otimes S^1E \otimes S^0E)\\
&\quad +\text{cf}\,(S^2E \otimes S^2E \otimes S^0E) + \text{cf}\,(S^2E \otimes S^1E \otimes S^1E)\\
D_{3,3}(5) = &\quad \text{cf}\,(L(2,2,1) \otimes S^0E \otimes S^0E) + \text{cf}\,(L(2,2,0) \otimes S^1E \otimes S^0E)\\
&\quad +\text{cf}\,(L(2,1,0) \otimes S^2E \otimes S^0E) + \text{cf}\,(L(2,1,0) \otimes S^1E \otimes S^1E)\\
&\quad +\text{cf}\,(S^2E \otimes S^2E \otimes S^1E)\\
D_{3,3}(6) = &\quad \text{cf}\,(L(2,2,2) \otimes S^0E \otimes S^0E) + \text{cf}\,(L(2,2,1) \otimes S^1E \otimes S^0E)\\
&\quad +\text{cf}\,(L(2,2,0) \otimes S^2E \otimes S^0E) + \text{cf}\,(L(2,2,0) \otimes S^1E \otimes S^1E)\\
&\quad +\text{cf}\,(L(2,1,0) \otimes L(2,1,0) \otimes S^0E) + \text{cf}\,(L(2,1,0) \otimes S^2E \otimes S^1E)\\
&\quad +\text{cf}\,(S^2E \otimes S^2E \otimes S^2E)
\end{aligned}$$

REMARK 4.8.6 We now understand when the $\bar{S}^rE$ are and are not tilting and how the coefficient spaces of these modules sit within the $D_{3,p}(r)$. It is now necessary to prove which of these truncated modules (whose coefficient spaces form the $D_{3,p}(r)$) are tilting, so we know which characters to calculate in the hope of equating them with the characters of the tilting modules of $A(\pi, r)$. The following theorems bring together a number of results which show which of the truncated modules are tilting.

We first consider the following remark which is necessary to prove this collection of theorems.

REMARK 4.8.7 [22, Section 5.1, Lemma 5.3.1] For S a finite dimensional algebra with blocks $B_1, \ldots, B_n$, and X a finitely generated S-module, then $X = X_1 \oplus X_2 \oplus \ldots \oplus X_n$ where each X_i belongs to block B_i. Then in the case where $S = S(n, r)$, each core class is a union of blocks so, with the core classes $\Lambda_1, \ldots \Lambda_m$ we get the decomposition $X = Y_1 \oplus \ldots \oplus Y_m$ where each Y_i belongs to the core class Λ_i.

We now begin our theorems, first looking at a specific truncated module for all p.

THEOREM 4.8.8

$$L(p-1, 1, 0) \otimes S^{p-2}E$$

is tilting for all primes p.

Proof.
i) For $p = 2$, $L(p-1,1,0) \otimes S^{p-2}E = L(1,1,0)$ which we know to be tilting by Theorem 4.8.1.
ii) Now let $p \geq 3$. Consider $\Delta(p-1,1,0) \otimes S^{p-2}E$ then we have the short exact sequence

$$0 \to L(p-2,1,1) \to \Delta(p-1,1,0) \to L(p-1,1,0) \to 0$$

giving the short exact sequence

$$0 \to L(p-2,1,1) \otimes S^{p-2}E \to \Delta(p-1,1,0) \otimes S^{p-2}E$$
$$\to L(p-1,1,0) \otimes S^{p-2}E \to 0.$$

We claim this splits. By using Pieri's Formula [21, Chapter 1, Section 5], we have

$$\begin{aligned}\text{ch } \Delta(p-1,1,0) \otimes S^{p-2}E &= s_{p-1,1} \cdot s_{p-2} \\ &= s_{2p-3,1} + s_{2p-4,2} + s_{2p-4,1,1} + s_{2p-5,3} + s_{2p-5,2,1} \\ &\quad + \ldots + s_{2p-p,p-2} + s_{2p-p,p-3,1} + s_{2p-(p+1),p-1} \\ &\quad + s_{2p-(p+1),p-2,1}.\end{aligned}$$

We also have

$$\begin{aligned}\text{ch } L(p-2,1,1) \otimes S^{p-2}E &= s_{p-2,1,1} \cdot s_{p-2} \\ &= s_{2p-4,1,1} + s_{2p-5,2,1} + s_{2p-6,3,1} + \ldots \\ &\quad + s_{2p-(p+1),p-2,1}.\end{aligned}$$

Hence

$$\begin{aligned}\text{ch } L(p-1,1,0) \otimes S^{p-2}E &= (s_{p-1,1} - s_{p-2,1,1})s_{p-2} \\ &= s_{2p-3,1} + s_{2p-4,2} + s_{2p-5,3} + \ldots + s_{p,p-2} + s_{p-1,p-1}.\end{aligned}$$

Then using Remark 4.8.7 we can write $\Delta(p-1,1,0) \otimes S^{p-2}E = M \oplus N$ where all composition factors of M have p-core from the set

$$M_{\text{core}} = \{(p-4,1,1),(p-5,2,1),(p-6,2,1),\ldots,(p-\xi,p-\xi,1),(p-1,p-2,1)\}$$

i.e. those with last entry 1, and all composition factors of N do not.
Then

$$\begin{aligned}\text{ch } M &= s_{2p-4,1,1} + s_{2p-5,2,1} + s_{2p-6,3,1} + \ldots + s_{p-1,p-2,1} \\ &= \text{ch } L(p-2,1,1) \otimes S^{p-2}E\end{aligned}$$

We also have that
$$L(p-2,1,1) \otimes S^{p-2}E \hookrightarrow M$$

by the definition of M. Hence by characters, $M = L(p-2,1,1) \otimes S^{p-2}E$. Thus

$$N \cong (\Delta(p-1,1,0) \otimes S^{p-2}E)/(L(p-2,1,1) \otimes S^{p-2}E)$$
$$\cong L(p-1,1,0) \otimes S^{p-2}E.$$

So $L(p-1,1,0) \otimes S^{p-2}E$ is isomorphic to a summand of a module with a Δ-filtration, thus it has a Δ-filtration [19, Page 211].
Run the same argument with $\nabla(p-1,1,0)$, then $\nabla(p-1,1,0) \otimes S^{p-2}E = M_1 \oplus N_1$ where all composition factors of M_1 have p-core from the set M_{core} above.
We now have

$$0 \to L(p-1,1,0) \otimes S^{p-2}E \xrightarrow{f} \nabla(p-1,1,0) \otimes S^{p-2}E$$

$$\xrightarrow{g} L(p-2,1,1) \otimes S^{p-2}E \to 0$$

and Ker $g = N_1 = L(p-1,1,0) \otimes S^{p-2}E$ hence $N_1 \subseteq L(p-1,1,0) \otimes S^{p-2}E$. Thus $L(p-1,1,0) \otimes S^{p-2}E$ is a direct summand of the module $\nabla(p-1,1,0) \otimes S^{p-2}E$ which has a ∇-filtration, thus $L(p-1,1,0) \otimes S^{p-2}E$ has a ∇-filtration and so has a good filtration. Hence $L(p-1,1,0) \otimes S^{p-2}E$ is tilting. □

We then move on to another specific truncated module for all p.

THEOREM 4.8.9 *The module*

$$L(p-1,2,0) \otimes S^{p-2}E$$

is tilting for all $p \geq 3$.

Proof. The structure of this proof follows that of the previous theorem.
i) When $p = 3$, $L(p-1,2,0) \otimes S^{p-2}E = L(2,2,0) \otimes S^1E$, and by Theorem 4.8.1 we know both $L(2,2,0)$ and S^1E are tilting, and hence the tensor product is also tilting.
ii) Now let $p \geq 5$. Consider $\Delta(p-1,2,0) \otimes S^{p-2}E$ then we have an exact sequence

$$0 \to L(p-2,2,1) \to \Delta(p-1,2,0) \to L(p-1,2,0) \to 0$$

giving an exact sequence

$$0 \to L(p-2,2,1) \otimes S^{p-2}E \to \Delta(p-1,2,0) \otimes S^{p-2}E$$

$$\to L(p-1,2,0) \otimes S^{p-2}E \to 0.$$

We claim this splits. We have

$$\begin{aligned}\text{ch }\ \Delta(p-1,2,0)\otimes S^{p-2}E &= s_{p-1,2}\cdot s_{p-2}\\ &= s_{2p-3,2}+s_{2p-4,3}+s_{2p-4,2,1}+s_{2p-5,4}+s_{2p-5,3,1}\\ &\quad +s_{2p-5,2,2}+\ldots+s_{2p-p,p-1}+s_{2p-p,p-2,1}\\ &\quad +s_{2p-p,p-3,2}+s_{2p-(p+1),p-1,1}+s_{p-1,p-2,2}\end{aligned}$$

Moreover, we have

$$\begin{aligned}\text{ch }\ L(p-2,2,1)\otimes S^{p-2}E &= s_{p-2,2,1}\cdot s_{p-2}\\ &= s_{2p-4,2,1}+s_{2p-5,3,1}+s_{2p-5,2,2}+\ldots+s_{2p-p,p-2,1}\\ &\quad +s_{2p-p,p-3,2}+s_{p-1,p-2,2}.\end{aligned}$$

Hence

$$\begin{aligned}\text{ch}\, L(p-1,2,0)\otimes S^{p-2}E &= (s_{p-1,2}-s_{p-2,2,1})s_{p-2}\\ &= s_{2p-3,2}+s_{2p-4,3}+s_{2p-5,4}+\ldots+s_{p,p-1}+s_{p-1,p-1,1}.\end{aligned}$$

Now, using Remark 4.8.7, write $\Delta(p-1,2,0)\otimes S^{p-2}E = M\oplus N$ where all composition factors of M have p-core of the form (α,β,γ) where $\gamma\geq 1$ and all composition factors of N do not, so have p-cores of the form $(a,b,0)$.
Then

$$\begin{aligned}\text{ch}\, M &= s_{2p-4,2,1}+s_{2p-5,3,1}+s_{2p-5,2,2}+\ldots+s_{2p-p,p-2,1}+s_{2p-p,p-3,2}+s_{p-1,p-2,2}\\ &= \text{ch}\, L(p-2,2,1)\otimes S^{p-2}E.\end{aligned}$$

We also have that $L(p-2,2,1)\otimes S^{p-2}E\hookrightarrow M$ by the definition of M. Hence $M = L(p-2,2,1)\otimes S^{p-2}E$. Thus

$$\begin{aligned}N &\cong (\Delta(p-1,2,0)\otimes S^{p-2}E)/(L(p-2,2,1)\otimes S^{p-2}E)\\ &\cong L(p-1,2,0)\otimes S^{p-2}E.\end{aligned}$$

So $L(p-1,2,0)\otimes S^{p-2}E$ is isomorphic to a summand of a module with a Δ-filtration, thus it has a Δ-filtration.
Run the same argument with $\nabla(p-1,2,0)$, then

$$\nabla(p-1,2,0)\otimes S^{p-2}E = M_1\oplus N_1$$

where all composition factors of M_1 have p-core of the form $(\alpha_1,\beta_1,\gamma_1)$ where $\gamma_1\geq 1$. We now have

$$0\to L(p-1,2,0)\otimes S^{p-2}E \xrightarrow{f} \nabla(p-1,2,0)\otimes S^{p-2}E \xrightarrow{g} L(p-2,2,1)\otimes S^{p-2}E\to 0$$

and Ker $g = N_1 = L(p-1,2,0)\otimes S^{p-2}E$ hence $N_1\subseteq L(p-1,2,0)\otimes S^{p-2}E$. Thus $L(p-1,2,0)\otimes S^{p-2}E$ is a direct summand of the module $\nabla(p-1,2,0)\otimes S^{p-2}E$ which has a ∇-filtration, thus $L(p-1,2,0)\otimes S^{p-2}E$ has a ∇-filtration and so has a good filtration. Hence $L(p-1,2,0)\otimes S^{p-2}E$ is tilting. □

The two previous theorems are then brought together to give a more general case.

THEOREM 4.8.10 *The module*

$$L(p-1,m,0)\otimes S^{p-2}E$$

is tilting for $1\leq m\leq p-2$ *and* $p\geq m$.

Proof. This again follows the same method as the previous two theorems.
i) Let $p=m+1$, then $L(p-1,m,0)\otimes S^{p-2}E=L(m,m,0)\otimes S^{m-1}E$ which we know to be tilting by Theorem 4.8.1.
ii) Now let $p\geq m$. Consider $\Delta(p-1,m,0)\otimes S^{p-2}E$ then we have

$$0\to L(p-2,m,1)\to\Delta(p-1,m,0)\to L(p-1,m,0)\to 0$$

giving

$$0\to L(p-2,m,1)\otimes S^{p-2}E\to\Delta(p-1,m,0)\otimes S^{p-2}E$$

$$\to L(p-1,m,0)\otimes S^{p-2}E\to 0.$$

We claim this splits, as using Pieri's Formula we have;
$\operatorname{ch}\Delta(p-1,m,0)\otimes S^{p-2}E=s_{p-1,m}\cdot s_{p-2}$

$$\begin{aligned}=&s_{2p-3,m}+s_{2p-4,m+1}+s_{2p-4,m,1}+s_{2p-5,m+2}+s_{2p-5,m+1,1}+s_{2p-5,m,2}+\dots\\&+s_{2p-(3+m),2m}+s_{2p-(3+m),2m-1,1}+s_{2p-(3+m),2m-2,2}+\dots\\&+s_{2p-(3+m),m,m}+s_{2p-(3+m+1),2m+1}+s_{2p-(3+m+1),2m,1}+s_{2p-(3+m+1),2m-1,2}\\&+\dots+s_{2p-(3+m+1),m+1,m}+\dots+s_{2p-(p+2-m),p-1}+s_{2p-(p+2-m),p-2,1}\\&+s_{2p-(p+2-m),p-3,2}+\dots+s_{2p-(p+2-m),p-(1+m),m}+s_{2p-(p+2-m+1),p-1,1}\\&+s_{2p-(p+2-m+1),p-2,2}+\dots+s_{2p-(p+2-m+1),p-m,m}+s_{2p-(p+2-m+1),p-2,2}+\dots\\&+s_{2p-(p+2-m+1),p-m,m}+s_{2p-(p+2-m+2),p-1,2}+s_{2p-(p+2-m+2),p-2,3}+\dots\\&+s_{2p-(p+2-m+2),p-(m-1),m}+\dots+s_{2p-(p+1),p-1,m-1}+s_{2p-(p+1),p-2,m}\end{aligned}$$

Moreover we have
$\operatorname{ch}L(p-2,m,1)\otimes S^{p-2}E$

$$\begin{aligned}=&s_{2p-4,m,1}+s_{2p-5,m+1,1}+s_{2p-5,m,2}+\dots+s_{2p-(3+m),2m-1,1}+s_{2p-(3+m),2m-2,2}\\&+\dots+s_{2p-(3+m),m,m}+s_{2p-(3+m+1),2m,1}+s_{2p-(3+m+1),2m-1,2}+\dots\\&+s_{2p-(3+m+1),m+1,m}+\dots+s_{2p-(p+2-m),p-2,1}+s_{2p-(p+2-m),p-3,2}+\dots\\&+s_{2p-(p+2-m),p-(1+m),m}+s_{2p-(p+2-m+1),p-2,2}+\dots+s_{2p-(p+2-m+1),p-m,m}\\&+s_{2p-(p+2-m+2),p-2,3}+\dots+s_{2p-(p+2-m+2),p-(m-1),m}+\dots+s_{2p-(p+1),p-2,m}.\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{ch} L(p-1,m,0)\otimes S^{p-2}E = & s_{2p-3,m} + s_{2p-4,m+1} + s_{2p-5,m+2} + \ldots + s_{2p-(3+m),2m} \\ & + s_{2p-(3+m+1),2m+1} + \ldots + s_{2p-(p+2-m),p-1} \\ & + s_{2p-(p+2-m+1),p-1,1} + s_{2p-(p+2-m+2),p-1,2} \\ & + \ldots + s_{2p-(p+1),p-1,m-1}.\end{aligned}$$

Now, using Remark 4.8.7, write $\Delta(p-1,m,0)\otimes S^{p-2}E = M \oplus N$ where all composition factors of M have p-core of the form (α,β,γ) where $\gamma \geq 1$ and all composition factors of N do not, so have p-cores of the form $(a,b,0)$.
Then $\operatorname{ch} M = \operatorname{ch} L(p-2,m,1)\otimes S^{p-2}E$. We also have that

$$L(p-2,m,1)\otimes S^{p-2}E \hookrightarrow M$$

by the definition of M. Hence $M = L(p-2,m,1)\otimes S^{p-2}E$. Thus

$$\begin{aligned}N &\cong (\Delta(p-1,m,0)\otimes S^{p-2}E)/(L(p-2,m,1)\otimes S^{p-2}E) \\ &\cong L(p-1,m,0)\otimes S^{p-2}E.\end{aligned}$$

So $L(p-1,m,0)\otimes S^{p-2}E$ is isomorphic to a summand of a module with a Δ-filtration, thus it has a Δ-filtration.
Run the same argument with $\nabla(p-1,m,0)$, then

$$\nabla(p-1,m,0)\otimes S^{p-2}E = M_1 \oplus N_1$$

where all composition factors of M_1 have p-core of the form $(\alpha_1,\beta_1,\gamma_1)$ where $\gamma_1 \geq 1$.
We now have a short exact sequence

$$0 \to L(p-1,m,0)\otimes S^{p-2}E \xrightarrow{f} \nabla(p-1,m,0)\otimes S^{p-2}E$$

$$\xrightarrow{g} L(p-2,m,1)\otimes S^{p-2}E \to 0$$

and $Ker\ g = N_1 = L(p-1,m,0)\otimes S^{p-2}E$ hence $N_1 \subseteq L(p-1,m,0)\otimes S^{p-2}E$. Thus $L(p-1,m,0)\otimes S^{p-2}E$ is a direct summand of the module $\nabla(p-1,m,0)\otimes S^{p-2}E$ which has a ∇-filtration, thus $L(p-1,m,0)\otimes S^{p-2}E$ has a ∇-filtration and so has a good filtration. Hence $L(p-1,m,0)\otimes S^{p-2}E$ is tilting. □

The previous theorem is then generalised further.

THEOREM 4.8.11 *The module*

$$L(p-1,m,0)\otimes \bar{S}^a E$$

is tilting for $1 \leq m \leq p-2$ *and* $p-2 \leq a \leq p-1$ *with* $p \geq m$.

Proof. We have $L(p-1,m,0)\otimes S^{p-2}E$ tilting, and as the tensor product of two tilting modules is again tilting then we have that $L(p-1,m,0)\otimes S^{p-2}E\otimes S^1E$ is also tilting.
We also have that $S^{p-1}E \mid S^{p-2}E\otimes S^1E$ thus implying $L(p-1,m,0)\otimes S^{p-1}E \mid L(p-1,m,0)\otimes S^{p-2}E\otimes S^1E$ and as the right hand side is tilting then so is the left hand side. $\square$

The following well known Proposition and Corollary are then needed to give the final two theorems on tilting truncated modules.

PROPOSITION 4.8.12 *For V any finite dimensional G-module then*

$$V \mid V\otimes V^*\otimes V$$

Proof. Define a G-homomorphism $V\otimes V^*\otimes V \xrightarrow{\pi} V$ such that $\pi(u\otimes\alpha\otimes v)=\alpha(v)u$ for $u,v\in V,\ \alpha\in\ V^*$. We wish to find the G-map $\phi: V\to V\otimes V^*\otimes V$ such that $\pi\circ\phi = id$.
Well, we have $V\otimes V^*\cong \mathrm{End}_k(V)$, and let $v_1,\ldots,v_n$ be a basis for V and $\alpha_1,\ldots,\alpha_n$ a dual basis for V^*, then $I=\sum v_i\otimes\alpha_i$ is an element in $V\otimes V^*$.
Let $\phi(v)=I\otimes v$, then $(\pi\circ\phi)(v)=\pi(\sum v_i\otimes\alpha_i\otimes v)=\sum v_i\alpha_i(v)=\sum\lambda_i v_i = v$ with $v=\lambda_1v_1+\ldots+\lambda_nv_n$.
Hence $\pi\circ\phi: V\to V$ is $\mathrm{id}: V\to V$, and so $V\otimes V^*\otimes V=\mathrm{Ker}(\pi)\oplus\mathrm{Im}(\phi)$ where $\mathrm{Im}(\phi)\cong V \ \Rightarrow\ V\mid V\otimes V^*\otimes V$. $\square$

COROLLARY 4.8.13 *Using this proposition we also have that if M is a tilting module and $X\otimes M$ and $Y\otimes M$ are tilting then $X\otimes Y\otimes M$ is also tilting.*

Proof. Since M is a direct summand of $M\otimes M^*\otimes M$ we have that $X\otimes Y\otimes M$ is a direct summand of

$$X\otimes Y\otimes M\otimes M\otimes M^*\cong (X\otimes M)\otimes(Y\otimes M)\otimes M^*.$$

Since $X\otimes M$, $Y\otimes M$ and M^* are all tilting then so is $(X\otimes M)\otimes(Y\otimes M)\otimes M^*$ and thus $X\otimes Y\otimes M$ is isomorphic to a direct summand of a tilting module and hence is itself a tilting module. $\square$

THEOREM 4.8.14 *The module*

$$L(p-1,m,0)\otimes L(p-1,m,0)\otimes \bar{S}^aE$$

is tilting for $1\le m\le p-2$ and $p-2\le a\le p-1$ with $p\ge m$.

Proof. For V any G-module, $V \mid V \otimes V \otimes V^*$, and hence $\bar{S}^a E \mid \bar{S}^a E \otimes \bar{S}^a E \otimes (\bar{S}^a E)^*$, and thus $L(p-1,m,0) \otimes L(p-1,m,0) \otimes \bar{S}^a E$ is a direct summand of

$$L(p-1,m,0) \otimes L(p-1,m,0) \otimes \bar{S}^a E \otimes \bar{S}^a E \otimes (\bar{S}^a E)^*.$$

We know $L(p-1,m,0) \otimes \bar{S}^a E$ is tilting from Theorem 4.3.8, thus the right hand side is tilting and thus the left hand side is tilting. □

THEOREM 4.8.15 *$L(p-1,m',0) \otimes L(p-1,m,0) \otimes \bar{S}^a E$ is tilting for $1 \leq m', m \leq p-2$ and $p-2 \leq a \leq p-1$ with $p \geq m$.*

Proof. $\bar{S}^a E \mid \bar{S}^a E \otimes \bar{S}^a E \otimes (\bar{S}^a E)^* \Rightarrow L(p-1,m',0) \otimes L(p-1,m,0) \otimes \bar{S}^a E \mid L(p-1,m',0) \otimes L(p-1,m,0) \otimes \bar{S}^a E \otimes \bar{S}^a E \otimes (\bar{S}^a E)^*$. We have $L(p-1,m',0) \otimes \bar{S}^a E$ tilting, and $L(p-1,m,0) \otimes \bar{S}^a E$ tilting, so the right hand side is tilting and thus the left hand side is tilting. □

4.9 Decomposition Numbers

AIM: Although we have stated that the core classes will play a crucial role in finding the necessary tilting modules, this is not all the information we need. We now give a short section on decomposition numbers, explaining why they are important and giving certain facts about them which we can use.

The previous section has given us a clear understanding of which of the truncated modules $\mathrm{Tr}^{\lambda} E$ are tilting and thus we know which we can consider calculating the character of. This is not the end of the story though. Suppose we calculate the character of a certain truncated module and the leading term in this character corresponds to the character of the highest weighted tilting module $T(t,0,0)$. Suppose moreover its character, when expressed as a sum of Schur functions, contains multiplicities of many other weights of tilting modules whose highest weight is less than that of $T(t,0,0)$. Without knowing the ∇-filtration multiplicities $(T(t,0,0) : \nabla(\mu,0,0))$ for $\mu < t$ we cannot be sure this tilting module $T(\mu,0,0)$ is in fact a composition factor of the truncated module in question, and thus do not know if $\mathrm{cf}\,(T(\mu,0,0)) \subseteq D_{3,p}(r)$. We need to therefore gather together information on these decomposition numbers, and we do this now. There is unfortunately not a clear understanding of these numbers for all values of n and p, but we do have the following facts, which together will be enough for our research.

FACT 4.9.1 [5, Corollary 3.8] For $r \leq n$ and $\lambda,\ \mu \in \Lambda^+(n,r)$ then

$$(T(\lambda) : \nabla(\mu)) = [\nabla(\mu') : L(\lambda)]$$

where μ' is the transpose of μ. So for example $(5,3,1)' = (3,2,2,1,1)$.

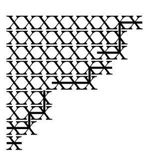
FACT 4.9.2 [22, Theorem 5.1.3] Let λ and μ be two partitions of r into n parts.
Then $(T(\lambda) : \nabla(\mu)) \neq 0 \;\Rightarrow\; \lambda$ and μ have the same p-core with $\mu \leq \lambda$.

We combine this information with our classification of core classes; only those tilting modules whose partitions are in the same core class may be composition factors of each other.

DEFINITIONS 4.9.3 [23, Sections 2 and 4] The Mullineux Bijection
The next fact uses the Mullineux Bijection so we do a little bit of background work before defining this bijection.

DEFINITION 4.9.4 i) The edge of a partition was defined in 4.2.1. In Mullineux's paper, he also defines a p-edge which consists of p-segments, all but at most one of which contains p nodes. The first p-segment comprises of either the first p nodes of the edge of the diagram starting at the uppermost right hand corner of the partition, or of the entire edge if its length is less than p. The next p-segment is obtained similarly starting in the row below that which contains the last node of the previous p-segment defined. This process continues until the final row is reached.
ii) For a partition P, $e(P)$ denotes the number of points in the p-edge of P, and $I(P)$ denotes the partition (possibly empty) obtained by removing the p-edge from P.

EXAMPLE 4.9.5 Let $p = 5$ and $P = (9, 8, 8, 7, 6, 3, 3, 2, 1)$



Then $e(P) = 16$ and $I(P) = (7, 7, 6, 5, 3, 2, 1)$

We now define a p-regular partition, with an example, which, combined with the above information, will allow us to define the Mullineux Bijection.

DEFINITION 4.9.6 A partition μ is p-singular if for some i

$$\mu_{i+1} = \mu_{i+2} = \ldots = \mu_{i+p} > 0.$$

Otherwise, μ is p-regular.

EXAMPLE 4.9.7 The partition $(6, 6, 5, 5, 5, 5, 1)$ is p-regular if and only if $p \geq 5$.

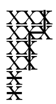
DEFINITION 4.9.8 The Mullineux Bijection
Given a partition P, let $I(P) = P'$, then we can continue removing the p-edge from P' to form $I(P')$, and so on until we have the empty partition. Suppose this takes α steps, then we have partitions $P_0, \ldots, P_\alpha$ where $P_\alpha = P$, $P_0 = \emptyset$, $P_{i-1} = I(P_i)$ for $i = 1, 2, \ldots, \alpha$.
Now set $a_i = e(P_i)$ and let r_i be the number of rows of P_i. Suppose also that P is p-regular, then so is each P_i ([23, Lemma 2.2]) and we define a sequence $(s_1, \ldots, s_\alpha)$ by the formula

$$s_i = a_i - r_i + \epsilon_i$$

where $\epsilon_i = 0$ if p divides a_i and is 1 otherwise.

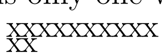
We now give an example of the Mullineux Bijection.

EXAMPLE 4.9.9 Let $p = 5$ and $P = (3, 3, 2, 2, 1, 1) = P_\alpha$, then $a_\alpha = 7$ and $r_\alpha = 6$, as there are 7 nodes in the p-edge removed from P and they are removed from 6 rows;



Then $I(P) = P' = P_{\alpha-1} = (2, 1, 1, 1)$ and so ${}_{\alpha-1} = 5$ and $r_{\alpha-1} = 4$, as there is only one p-edge that can be removed from the 4 rows.

After removing this p-edge we are left with the empty partition and so $P_0 = I(P') = \emptyset$ and therefore we know $\alpha = 2$. We can then go about reforming the new partition s from (s_1, s_2). We take s_1 to start with, where $s_1 = a_1 - r_1 + \epsilon_1 = 5 - 4 = 1$ so we have to put the $p = 5$ nodes of the p-edge of P_1 back in 1 row. There is only one way of doing this;
xxxxx
This is then built on with $s_2 = a_2 - r_2 + \epsilon_2 = 7 - 6 + 1 = 2$ so we must put back the 7 nodes of the p-edge of P_2 back in 2 rows, so that when we remove them, following the p-edge, what remains is the above partition xxxxx. There is only one way of doing this;



We therefore know that $\text{Mull}(P) = \text{Mull}(3, 3, 2, 2, 1, 1) = (10, 2)$ or $\tilde{P} = (10, 2)$.

We now give another fact we shall use in the proof of Theorem 4.1.3, which uses the Mullineux Bijection.

FACT 4.9.10 [2, Theorem 5.1] Let λ be some partition of r into n parts, and denote the transpose of λ by λ'. We say λ is restricted if $\lambda_i - \lambda_{i+1} < p$.

Then the injective module $I(\lambda) = T(\text{Mull}(\lambda'))$ provided λ is restricted and $\lambda_1 < (n-1)(p-1)$.

REMARK 4.9.11 If there exists a filtration $0 = V_0 < V_1 < V_2 < \ldots < V_m = V$ with $V_i/V_{i-1} \cong \nabla(\lambda_i)$ with $0 < i \leq m$ then we often represent this by saying V has filtration structure

$$\begin{array}{c}\nabla(\lambda_m)\\ \vdots \\ \nabla(\lambda_1)\end{array}$$

We follow this with an example of Fact 4.9.10.

EXAMPLE 4.9.12 Let $p = 5,\ r = 12$ and $\lambda = (6, 4, 2)$ then λ is restricted as;
i) $\lambda_1 - \lambda_2 = 6 - 4 = 2 < 5 = p$ and $\lambda_2 - \lambda_3 = 4 - 2 = 2 < 5 = p$
ii) $\lambda_1 = 6 < 8 = (3-1)(5-1) = (n-1)(p-1)$
Therefore $I(6, 4, 2) = T(\text{Mull}((6, 4, 2)'))$, so now we need to calculate $\text{Mull}((6, 4, 2)')$. Well $(6, 4, 2)' = (3, 3, 2, 2, 1, 1)$ and from Example 4.9.9 we know that $\text{Mull}(3, 3, 2, 2, 1, 1) = (10, 2)$ and thus $I(6, 4, 2) = T(10, 2)$.

So why does this help us in our work? Well $I(\lambda)$ always has socle $L(\lambda)$ and we have that $L(\lambda) \subseteq \nabla(\lambda) \subseteq I(\lambda)$ and thus if $I(\lambda) = T(\text{Mull}(\lambda'))$ then $T(\text{Mull}(\lambda'))$ also has socle $L(\lambda)$, hence in constructing the filtration structure for $T(\text{Mull}(\lambda'))$ we always know that $\nabla(\lambda)$ is at the bottom. So in this example we have that $T(10, 2, 0)$ has filtration structure

$$\begin{array}{c}\nabla(10,2,0)\\ \vdots \\ \nabla(6,4,2)\end{array}$$

If we then refer back to the core classes of $r = 12$ for $p = 5$, then $(11, 1, 0)$ sits at the top of the following core class;

$$\begin{array}{l}11,1,0\\ 10,2,0\\ 8,2,2\\ 6,6,0\\ 6,4,2\\ 5,4,3\end{array}$$

Thus by this calculation, we know that $\nabla(5, 4, 3)$ is not in the filtration structure of $T(10, 2, 0)$, but $\nabla(6, 4, 2)$ is.

We now move onto another fact which we shall need in the proof of Theorem 4.1.3.

FACT 4.9.13 [10, 3.8][11, Section 6.3]
As stated previously, the full ∇-filtration multiplicities for $(T(\lambda) : \nabla(\mu))$ are not known, however, for $p = 2, 3$ and $0 \leq n \leq 13$ the results are known, and are contained in James' book 'The Representation Theory of the Symmetric Groups'. However the relevant numbers are given there as decomposition numbers of the Specht modules, thus it is necessary to understand how we get from one result to the other.
Let $S = S(n, r)$ for $n \geq r$ and $e = e^2 \in S$. Then $eSe = k\mathrm{Sym}(r)$ and we define the Schur functor $f : \mathrm{mod}(S) \to \mathrm{mod}(eSe)$ such that $f(V) = eV$ where $eV = \{ev \mid v \in V\}$. So when we apply this Schur functor to a $\nabla(\lambda)$ we get that $f\nabla(\lambda) = S^{\lambda}$ the Specht module, and applied to some $L(\mu)$ then $fL(\mu) = D^{\mathrm{Mull}(\mu)}$ where $D^{\mu} = \mathrm{hd}\ S^{\mu}$ [4, 4.4(5)].
Putting all this together, and using Fact 4.9.1 we get

$$\begin{aligned}(T(\lambda) : \nabla(\mu)) &= [\nabla(\mu') : L(\lambda)] \\ &= [f\nabla(\mu') : fL(\lambda)] \\ &= [S^{\mu'} : D^{\mathrm{Mull}(\lambda)}]\end{aligned}$$

Then we can refer to the tables of decomposition numbers given in James [17, Appendix].

EXAMPLE 4.9.14 Let $p = 3$ and $r = 6$, and suppose we wish to know the composition factors of the tilting module $T(4, 1, 1)$. Well, from our Classification of core classes, we have that $(4, 1, 1)$ sits third highest in the following core class;
$(6, 0, 0)$
$(5, 1, 0)$
$(4, 1, 1)$
$(3, 3, 0)$
$(3, 2, 1)$
$(2, 2, 2)$
Thus using Fact 4.9.2 we know that besides $\nabla(4, 1, 1)$, the only other possible modules in the ∇-filtration of $T(4, 1, 1)$ are $\nabla(3, 3, 0)$, $\nabla(3, 2, 1)$ and $\nabla(2, 2, 2)$. So we wish to find $(T(4, 1, 1) : \nabla(\mu))$ for $\mu \leq (4, 1, 1)$.

$$\begin{aligned}(T(4,1,1) : \nabla(\mu)) &= [\nabla(\mu') : L(4,1,1)] \\ &= [f\nabla(\mu') : fL(4,1,1)] \\ &= [S^{\mu'} : D^{\mathrm{Mull}(4,1,1)}]\end{aligned}$$

So let us calculate Mull(4, 1, 1), well $P_2 = (4, 1, 1)$, $a_2 = 5$, $r_2 = 3$, $P_1 = (1, 0, 0)$, $a_1 = 1$, $r_1 = 1$ and $P_0 = \emptyset$. So $s_1 = 1 - 1 + 1 = 1$ giving the partition consisting of one node, and $s_2 = 5 - 3 + 1 = 3$ and so we need to attach the 5 nodes to this single node over 3 rows. This is only possible by reforming the partition (4, 1, 1). So we need to consider $[S^{\mu'} : D^{\text{Mull}(4,1,1)}]$ and referring to James' book we see that $\mu' = (4, 1, 1),\ (3, 2, 1),\ (3, 1, 1, 1)$ and thus $\mu = (3, 1, 1, 1),\ (3, 2, 1),\ (4, 1, 1)$. Therefore the only composition factors of $T(4, 1, 1)$ are $\nabla(4, 1, 1)$ as expected and also $\nabla(3, 2, 1)$.

We now have one final fact to give which will help us in finding those tilting modules whose highest weight sits in the middle of its core class. To give the fact we first introduce the following theorem about horizontal h-cuts.

THEOREM 4.9.15 *Suppose $\lambda = (\lambda_1, \ldots)$ and $\mu = (\mu_1, \ldots)$ are partitions of r. We say that the partitions (λ, μ) admit a horizontal h-cut if we have $\lambda_1 + \ldots + \lambda_h = \mu_1 + \ldots \mu_h$. Put $\lambda^t(h) = (\lambda_1, \ldots, \lambda_h)$, $\lambda^b(h) = (\lambda_{h+1}, \lambda_{h+2}, \ldots)$ and $\mu^t(\lambda) = (\mu_1, \ldots, \mu_h)$, $\mu^b(h) = (\mu_{h+1}, \mu_{h+2}, \ldots)$, i.e. the top and bottom parts of λ and μ. Then*

$$(T(\lambda) : \nabla(\mu)) = (T(\lambda^t) : \nabla(\mu^t))(T(\lambda^b) : \nabla(\mu^b)).$$

Proof. Using [4, 4.2(9), 4.2(15)] we have the above for ordinary decomposition numbers. If we then apply the reciprocity formula of [4, 4.2(14)] it then applies to tilting modules. □

PROPOSITION 4.9.16 *Let λ and μ be two partitions of r into $n = 3$ parts. Then for the cases we are considering, where $0 \le r \le 3p - 1$, λ and μ always admit a horizontal h-cut.*

Proof. Clearly, in the range $0 \le r \le p - 1$ there are no weights in the middle of a core class as all weights are self-titled.
Now consider the range $p \le r \le 2p - 1$, where, from Proposition 4.3.2, we know we have only one 3-set to consider, which is of the form

$$\begin{aligned}&(\lambda_1, \lambda_2, \lambda_3)\\ &(\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)\\ &(\lambda_3 + p - 2, \lambda_1 - p + 1, \lambda_2 + 1)\end{aligned}$$

Then with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)$, because $\lambda_3 = \mu_3$ we can make a horizontal 2-cut.

Now consider the final range $2p \le r \le 3p - 1$, where, from Section 4.7, we know we have to consider three 3-sets and one 6-set. We shall look at the 3-sets first, which are as follows;

$$\begin{array}{l}(\lambda_1 + p, \lambda_2, \lambda_3)\\(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)\\(\lambda_3 + p - 2, \lambda_1 + 1, \lambda_2 + 1)\end{array}$$

$$\begin{array}{l}(\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)\\(\lambda_1, \lambda_2 + p, \lambda_3)\\(\lambda_3 + 2p - 2, \lambda_1 - p + 1, \lambda_2 + 1)\end{array}$$

$$\begin{array}{l}(\lambda_1, \lambda_2 + p, \lambda_3)\\(\lambda_1, p - 1, 1)\\(\lambda_1, p, 1)\end{array}$$

Now in the first two 3-sets here, we can see that the top and middle weights both have final entry λ_3, and hence they admit a horizontal 2-cut. Meanwhile, in the final 3-set, both the top and middle weight have first entry λ_1 and hence they admit a horizontal 1-cut. Finally we consider the 6-set in this range which is as follows;

$$\begin{array}{l}(\lambda_1 + p, \lambda_2, \lambda_3)\\(\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)\\(\lambda_3 + 2p - 2, \lambda_1 - p + 1, \lambda_2 + 1)\\(\lambda_1, \lambda_2 + p, \lambda_3)\\(\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)\\(\lambda_2 + p - 1, \lambda_3 + p - 1, \lambda_1 - p + 2)\end{array}$$

In this case, the first and second weight both have last entry λ_3 and thus these partitions admit a horizontal 2-cut. We shall show later that it is only then necessary to check the fourth and fifth weights in this core class, and indeed we can see these both have first entry λ_1 and thus they admit a horizontal 1-cut. □

We are now able to give our final fact.

FACT 4.9.17 Let λ and μ be partitions of r into at most 3 parts, with $\mu < \lambda$. Then in the cases given above

$$(T(\lambda) : \nabla(\mu)) \leq 1.$$

Proof. We have that λ and μ always admit a horizontal h-cut, and thus $(T(\lambda) : \nabla(\mu)) = (T(\lambda^t) : \nabla(\mu^t))(T(\lambda^b) : \nabla(\mu^b))$. There are two cases of this horizontal cut, the first being between the first and second row, in which case $(T(\lambda^t) : \nabla(\mu^t)) = 1$ as $T(\lambda^t)$ has highest weight $\lambda^t = \mu^t$. Hence $(T(\lambda) : \nabla(\mu)) = (T(\lambda^b) : \nabla(\mu^b)) \leq 1$ as these are now GL_2-modules [4, 3.4(3)].
The second case is where we make a horizontal cut between the second and third row, in which case $(T(\lambda^b) : \nabla(\mu^b)) = 1$ as $T(\lambda^b)$ has highest weight $\lambda^b = \mu^b$. Hence $(T(\lambda) : \nabla(\mu)) = (T(\lambda^t) : \nabla(\mu^t)) \leq 1$ as these are now GL_2-modules [4, 3.4(3)]. □

4.10 The method for the case $n = 3$

AIM: In this section we explain the method we use to prove the case for $n = 3$, namely that we are wishing to show that the coefficient spaces of all tilting modules of $A(\pi, r)$ for π a saturated set are contained in $D_{3,p}(r)$. By using the classification of the core classes in Section 4.7 we split this into those tilting modules whose weights are a) at the top of their core class or are self-titled, b) at the bottom of their core class, and c) in the middle of their core class, explaining in each case what we need to prove to show $\mathrm{cf}\,(T(\lambda)) \subseteq D_{3,p}(r)$ for $T(\lambda)$ a tilting module of $A(\pi, r)$. We then introduce Young's Rule and the Littlewood-Richardson Rule which we will apply to calculate the character of the truncated modules and find the tilting modules within them via the above methods a)-c).

a) Those tilting modules whose highest weights are either self-titled or are the highest in their core class.

Suppose we have a tilting module $T(\alpha)$ where this highest weight α is highest in its core class, then $T(\alpha)$ has filtration structure

$$\begin{matrix} \nabla(\alpha) \\ \nabla(\beta_1) \\ \vdots \\ \nabla(\beta_m) \end{matrix}$$

for $\alpha \geq \beta_1 \geq \ldots \geq \beta_m$.
So, provided there exists some tilting truncated module $\mathrm{Tr}^{\lambda} E$, such that when we express its character as the sum of Schur functions, the function s_α occurs as a coefficient, then we have that $\mathrm{cf}\,T(\alpha) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\lambda} E)$, as although there may exist other coefficients in this character expression which are of a higher weight than α they will sit in a different core class to α and by Fact 4.9.2 $[T(\mu) : \nabla(\alpha)] = 0$ for μ and α in different core classes with $\mu > \alpha$.

This is also true of self-titled weights and hence it is enough to show that the Schur function s_ν occurs in some $\mathrm{ch}\,\mathrm{Tr}^\lambda E$ for ν highest in its core class or self-titled. We will show later that these s_ν will always occur in the character of the truncated module $Tr^\lambda E$ where λ is minimal in its degree r.

b) Those tilting modules whose highest weights are at the bottom of their core class.

Suppose we have a core class where α is the highest weight in that core class and ω is the lowest. Suppose also that $\nabla(\omega)$ occurs as a section in the ∇-filtration of $T(\alpha)$, so

$$T(\alpha)= \begin{array}{c} \nabla(\alpha) \\ \hline \nabla(\beta) \\ \vdots \\ \hline \nabla(\omega) \end{array}$$

where $\omega < \beta < \alpha$.

As ω is the lowest weight in its core class then $T(\omega) = \nabla(\omega)$ as there is no weight lower which can sit under $\nabla(\omega)$ as a composition factor. Then in fact

$$T(\alpha) = \begin{array}{c} \nabla(\alpha) \\ \hline \vdots \\ \nabla(\beta) \\ \vdots \\ \hline T(\omega) \end{array}$$

Case 1:
Suppose we find both the $\mathrm{ch}\,T(\alpha)$ and the $\mathrm{ch}\,T(\omega)$ in some $\mathrm{ch}\,(\mathrm{Tr}^\lambda E)$. Then $\mathrm{cf}\,T(\omega) \subseteq \mathrm{cf}\,T(\alpha) \subseteq \mathrm{cf}\,(\mathrm{Tr}^\lambda E) \subseteq D_{3,p}(r)$. Hence we have the tilting module $T(\omega)$.

Case 2:
Suppose we cannot find the character of the tilting module $T(\alpha)$ in the character of some truncated module, moreover suppose $T(\omega)$ sits at the bottom of a number of tilting modules all whose p-cores are the same as that of α and ω, but are tilting modules whose coefficient spaces we cannot find in any $\mathrm{cf}\,(\mathrm{Tr}^\lambda E)$. If however, $\mathrm{ch}\,T(\omega)$ arises in some $\mathrm{ch}\,\mathrm{Tr}^\lambda E$ then we have $\mathrm{cf}\,T(\omega) \subseteq \mathrm{cf}\,(\mathrm{Tr}^\lambda E)$ as none of the tilting modules that $T(\omega)$ is a composition factor of can be found in the $\mathrm{Tr}^\lambda E$.
Thus in conclusion, if the character of all those tilting modules whose weight sits at the bottom of its core class arise at some point in the character of

some tilting $\mathrm{Tr}^{\lambda}E$, then whether we have found the coefficient spaces of the tilting modules above it in its core class or not, we know we have these tilting modules. We prove later that all these characters will arise.

c) The tilting modules whose highest weights are in the middle of their core class.

Suppose we have a certain core class whose highest weight is α and then there is a 'middle' weight μ. It may be that

$$\begin{array}{c} \underline{T(\alpha)} \\ \vdots \\ \overline{\underline{\nabla(\mu)}} \\ \vdots \end{array}$$

and thus the tilting module

$$T(\mu) = \begin{array}{c} \underline{\nabla(\mu)} \\ \vdots \end{array}$$

may arise as a section in the ∇-filtration of $T(\alpha)$. Suppose moreover there exists a tilting truncated module $\mathrm{Tr}^{\lambda}E$ where the character of $T(\alpha)$ and $T(\mu)$ arise with a multiplicity of m and n respectively. Using Fact 4.9.17 we know that $[T(\alpha) : \nabla(\mu)] = 0$ or 1, and thus we require that $n \geq m+1$, in fact we require that $\mathrm{ch}\, T(\mu)$ arises with a multiplicity of at least one higher than the sum of all the multiplicities of the $T(\alpha_i)$ where $\alpha_i > \mu$ and all α_i are in the same core class as μ.

For example, suppose $\mathrm{ch}\,\mathrm{Tr}^{\lambda}E = s_\alpha + 2s_\mu + \sum_{\sigma<\mu} a_\sigma s_\sigma$ with $\mu < \alpha$ and $\mathrm{ch}\, T(\alpha) = s_\alpha$ and $\mathrm{ch}\, T(\mu) = s_\mu$. Suppose also that $[T(\alpha) : \nabla(\mu)] = 1$ then $\mathrm{ch}\,\mathrm{Tr}^{\lambda}E = \mathrm{ch}\, T(\alpha) + \mathrm{ch}\, T(\mu) + \sum_{\sigma<\mu} a_\sigma \mathrm{ch}\, T(\sigma)$.

There is also a second option. Suppose there exists a truncated tilting module whose character has leading term s_μ where $\mathrm{ch}\, T(\mu) = s_\mu$ and μ is in the middle of its core class. As there is no higher weight in the character of that truncated module then there is no occurrence of a higher weighted tilting module which $T(\mu)$ may be a composition factor of. Hence $\mathrm{ch}\,\mathrm{Tr}^{\lambda}E = \mathrm{ch}\, T(\mu) + \sum_{\sigma<\mu} a_\sigma \mathrm{ch}\, T(\sigma)$, and we have the tilting module $T(\mu)$.

We will prove that for all tilting modules whose weight is in the middle of its core class, there exists either:
i) a truncated tilting module whose character contains that of this 'middle'

tilting module, with a multiplicity higher than the sum of all multiplicities of the tilting modules whose weights are higher and in the same core class.
ii) a truncated tilting module whose character has leading term equal to the highest weight of this 'middle' tilting module.

Now that we understand what we need to find it remains to identify the truncated modules whose coefficient spaces contain the coefficient spaces of the tilting modules. To do this it will sometimes be necessary to calculate the character of these truncated modules. This we did in Chapter 3 for the case $n = 2$, however for the case $n = 3$ the calculations are slightly more complicated and so we shall need to use both Young's Rule and the Littlewood-Richardson Rule which gives us a method for calculating these characters. We describe these now.

DEFINITIONS 4.10.1 All the information in this section can be found in G.D. James' book 'The Representation Theory of the Symmetric Groups' [17, Chapter 16]. Clearly there is a lot more detail contained in his work and it is unnecessary for this research to go into all of the background work that results in both Young's Rule and The Littlewood Richardson Rule. We shall go into it in as much detail as we feel necessary but the book can of course be consulted for further information.

DEFINITION 4.10.2 We call T a standard tableau if the numbers in the array increase along the rows and down the columns of T.

DEFINITION 4.10.3 A tableau T is called semistandard if the numbers in the array are non-decreasing along the rows and strictly increasing down the columns of T.

DEFINITION 4.10.4 A tableau T has type μ if for every i, the number i occurs μ_i times in T. For example
2211
1
is a $(4, 1)$-tableau of type $(3, 2)$.

DEFINITION 4.10.5 Let S_n be the symmetric group of degree n. Then for each partition μ of n we associate a Young Subgroup S_μ of S_n by taking

$$S_\mu = S_{\{1,2,\ldots,\mu_1\}} \times S_{\{\mu_1+1,\ldots,\mu_1+\mu_2\}} \times S_{\{\mu_1+\mu_2+1,\ldots,\mu_1+\mu_2+\mu_3\}} \times \ldots$$

For example with $n = 5$ and $\mu = (3, 1, 1)$ then $S_\mu = S_{\{1,2,3\}} \times S_{\{4\}} \times S_{\{5\}}$.

REMARK 4.10.6 [17, Chapter 4] The study of representations of S_n starts with the permutation module M^μ of S_n on the cosets of S_μ. The Specht module S^μ is a submodule of M^μ, and when the base field $k = \mathbb{Q}$ the different Specht modules, as μ varies over partitions of n, give all the ordinary irreducible representations of S_n.

Using the above definitions we are now able to define Young's Rule.

DEFINITION 4.10.7 Young's Rule [17, 14.1]
The multiplicity of $S^\lambda_\mathbb{Q}$ as a composition factor of $M^\mu_\mathbb{Q}$ is equal to the number of semi-standard λ-tableaux of type μ.

EXAMPLE 4.10.8 Consider $\mu = (3, 2, 2)$ then the semi-standard tableaux of this type are:

$$\begin{array}{l}1112233\end{array} \quad \begin{array}{l}111223\\3\end{array} \quad \begin{array}{l}11122\\33\end{array} \quad \begin{array}{l}111233\\2\end{array} \quad \begin{array}{l}11123\\23\end{array} \quad \begin{array}{l}11123\\2\\3\end{array} \quad \begin{array}{l}1112\\233\end{array} \quad \begin{array}{l}1112\\23\\3\end{array}$$

$$\begin{array}{l}11133\\22\end{array} \quad \begin{array}{l}1113\\223\end{array} \quad \begin{array}{l}1113\\22\\3\end{array} \quad \begin{array}{l}111\\223\\2\end{array} \quad \begin{array}{l}111\\33\\33\end{array}$$

Adjusting notation slightly, we then have that $s_3 s_2 s_2 = s_7 + 2s_{61} + 3s_{52} + 2s_{43} + s_{511} + 2s_{421} + s_{331} + s_{322}$ so for example the multiplicity of $S^{(6,1)}$ as a composition factor of $M^{(3,2,2)}$ is 2.

We now move onto the Littlewood Richardson Rule, first giving a definition and theorem which are necessary in defining it.

DEFINITION 4.10.9 Given a sequence of numbers, the quality of each term is determined as follows (each term is either good or bad);
i) All the 1s are good.
ii) An $i + 1$ is good if and only if the number of previous good i's is strictly greater than the number of previous good $(i + 1)$s.

EXAMPLE 4.10.10 Consider the sequence;

3	1	2	2	1	1	2	3
×	✓	✓	×	✓	✓	✓	✓

REMARK 4.10.11 It follows from Definition 4.10.9 that an $i + 1$ is bad if and only if the number of previous good i's equals the number of previous good $(i + 1)$s. Hence we have the following result.

THEOREM 4.10.12 *[17, 15.4] If a sequence contains m good $(i - 1)$'s in succession, then the next m i's in the sequence are all good.*

DEFINITION 4.10.13 The Littlewood-Richardson Rule
This rule is an algorithm for calculating $s_\lambda s_\mu$ where λ is a proper partition of $n-r$ and μ is a proper partition of r. The details of this rule can be found from pages 54-63 in James' book, however these details are unnecessary here and so what we give is the method for applying the rule.
First draw the diagram λ then add μ_1 1s, then μ_2 2s and so on, making sure that at each stage λ, together with the numbers which have been added, form a proper, semi-standard diagram. Then reject the result unless reading from right to left in successive rows, each i is preceded by more $(i-1)$'s than i's, thus ensuring that every term is good.

We shall now do two examples using this rule, the first being an example of Young's Rule with $\lambda = (\lambda_1, 0, 0)$, $\mu = (\mu_1, 0, 0)$, $\sigma = (\sigma_1, 0, 0)$, and the second being a more complex example using the Littlewood-Richardson Rule, with $\lambda = (\lambda_1, \lambda_2, 0)$, $\mu = (\mu_1, \mu_2, 0)$. We shall take information from the Classification of core classes in Section 4.7 and also from our description of how we go about finding the necessary tilting modules depending on where their weights sit in their respective core classes from Section 4.10. This will give a clear example of all future work and how we shall find the tilting modules needed to prove Theorem 4.1.3.

EXAMPLE 4.10.14 Let $n=3$, $p=5$ and $r=7$, then from the description of the $D_{n,p}(r)$ in 4.8.4 we have

$$\begin{aligned} D_{3,5}(7) = \ & \mathrm{cf}\,(\mathrm{Tr}^{\,7}E) + \mathrm{cf}\,(\mathrm{Tr}^{\,6,1}E) + \mathrm{cf}\,(\mathrm{Tr}^{\,5,2}E) + \mathrm{cf}\,(\mathrm{Tr}^{\,5,1,1}E) + \mathrm{cf}\,(\mathrm{Tr}^{\,4,3}E) \\ & + \mathrm{cf}\,(\mathrm{Tr}^{\,4,2,1}E) + \mathrm{cf}\,(\mathrm{Tr}^{\,3,3,1}E) + \mathrm{cf}\,(\mathrm{Tr}^{\,3,2,2}E) \\ = \ & \mathrm{cf}\,(L(4,3,0) \otimes S^0E \otimes S^0E) + \mathrm{cf}\,(L(4,2,0) \otimes S^1E \otimes S^0E) \\ & + \mathrm{cf}\,(L(4,1,0) \otimes S^2E \otimes S^0E) + \mathrm{cf}\,(L(4,1,0) \otimes S^1E \otimes S^1E) \\ & + \mathrm{cf}\,(S^4E \otimes S^3E \otimes S^0E) + \mathrm{cf}\,(S^4E \otimes S^2E \otimes S^1E) \\ & + \mathrm{cf}\,(S^3E \otimes S^3E \otimes S^1E) + \mathrm{cf}\,(S^3E \otimes S^2E \otimes S^2E) \end{aligned}$$

Now, for $r = 7 = p+2 < 2p-1 = 9$ we can refer to the Classification of core classes for $p \le r \le 2p-1$ to find the core classes in this case, where the partitions of $r=7$ into $n=3$ parts are as follows;

$$(7,0,0),\ (6,1,0),\ (5,2,0),\ (5,1,1),\ (4,3,0),\ (4,2,1),\ (3,3,1) \text{ and } (3,2,1).$$

For this range we have 3-sets where $\lambda_1 - \lambda_2 \ge p$ and $\lambda_1 \le 2p-3$, and as $7-0=7>5=p$ and $6-1=5=p$ then we have $(7,0,0)$ and $(6,1,0)$ sitting at the top of two different core classes. Using the information from Section 4.7 then we can complete these core classes as follows;

(7,0,0)	(6,1,0)
(4,3,0)	(5,2,0)
(3,3,1)	(3,2,2)

The only other core classes for this degree consist of self-titled partitions, and indeed as $5-1=4=p-1$ then $(5,1,1)$ is self-titled, and $(4,2,1)$ gives $\lambda_1-\lambda_2=4-2=2=p-3=p-\xi$ with $\lambda_2-\lambda_3=2-1=1=\xi-2$ and thus $(4,2,1)$ is also self-titled by Proposition 4.3.3. This accounts for all partitions of r into n parts.
We now use Section 4.10 which shows that for partitions at the top or bottom of their core class and those which are self-titled in this range we find their corresponding tilting modules by calculating the character of the lowest weighted truncated module in $D_{3,5}(7)$, and showing their weight arises at least once. The weights second highest in these core classes also occur in this truncated module as long as their weight arises with a higher multiplicity. We must therefore calculate the character of $\mathrm{Tr}^{(3,3,2)}E = S^3E\otimes S^2E\otimes S^2E$ as this is the lowest weighted tilting truncated module for this degree.
We now use the Young's Rule to calculate $s_\lambda s_\mu s_\sigma = s_3s_2s_2$, as shown in Example 4.10.8. We therefore have that

$$s_3s_2s_2 = s_7+2s_{61}+3s_{52}+s_{511}+2s_{43}+2s_{421}+s_{331}+s_{322},$$

and thus as the top weights s_7 and s_{61} arise at least once we have the corresponding tilting modules $T(7,0,0)$ and $T(6,1,0)$. Similarly, the bottom weights s_{331} and s_{322} arise once and so we have their corresponding tilting modules $T(3,3,1)$ and $T(3,2,2)$. Finally the self-titled weight s_{511} and s_{421} arise at least once giving their tilting modules $T(5,1,1)$ and $T(4,2,1)$. Looking at the second weights in the 3-sets, we have that s_{43} arises with multiplicity 2 which is greater than the multiplicity of s_7, the weight above it, and hence, by applying horizontal cuts and using Fact 4.9.17, we have the tilting module $T(4,3,0)$. In the same way, the weight s_{52} arises with multiplicity 3 which is greater than that of s_{61} and hence we have the tilting module $T(5,2,0)$. Thus the coefficient spaces of all of the tilting modules of $A(3,7)$ are contained in $\mathrm{cf}\,(\mathrm{Tr}^{(3,2,2)}E)$ which is itself contained in $D_{3,5}(7)$ and hence $D_{3,5}(7)=A(3,7)$ and so Theorem 4.1.3 is proven for this particular example.

We now move onto our second example which requires the use of the Littlewood-Richardson Rule.

EXAMPLE 4.10.15 Let $n=3$, $p=5$ and $r=14$ then

$$\mathrm{cf}\,(\mathrm{Tr}^{(8,6,0)}E) = \mathrm{cf}\,(L(4,4,0)\otimes L(4,2,0)\otimes S^0E)\subset D_{3,5}(14)$$

and it is the character of this truncated module that we shall calculate using the Littlewood-Richardson Rule. Recall from Theorem 4.8.1 that $L(4,4,0)$ is a tilting module, but $L(4,2,0)$ is not, and thus by [14, Page 18] we have

$$\mathrm{ch}\,(L(4,4,0)\otimes L(4,2,0)) = s_{44}(s_{42}-s_{321}).$$

We shall first calculate $s_{44} \cdot s_{42}$ and then take from this $s_{44} \cdot s_{321}$. So we begin by writing down s_{44} and then add on four 1s and two 2s for the s_{42}.

xxxx1111
xxxx22

Using the method from the previous remark we shall begin by moving down the 2s one by one, recalling that as the four 1s in the top row are good then both of the 2s will also be good.

xxxx1111 xxxx2 2	xxxx1111 xxxx 22

This is the most we can do with the 2s whilst still leaving all of the 1s on the top row, so we now move down one 1 and then begin to move down the two 2s. To ensure the resulting partition is semi-standard we cannot put a 1 under a 1, thus it must go on the third row. We therefore have three good 1s on the top row and so both 2s after this will also be good if left on the second row. Note that the 1 must come before the 2 to ensure the rows are non-decreasing.

xxxx111 xxxx22 1	xxxx111 xxxx2 12	xxxx111 xxxx 122

We now move down two 1s and then the two 2s. Again the 1s must go on the third row to ensure we have a semi-standard partition, and they must come before the 2s. Moreover there are enough good 1s on the top row to ensure both 2s will be good in the first partition where we do not move any 2s down.

xxxx11 xxxx22 11	xxxx11 xxxx2 112	xxxx11 xxxx 1122

This is the most we can do. If we were to move down three 1s, we would also have to move down at least one 2 as otherwise we will not have a proper partition. However, placing the three 1s before the 2 in the fourth row, means that reading from right to left in successive rows gives the sequence 122111, and thus the second two in this sequence is not good.

xxxx1
xxxx2
1112

Furthermore, we cannot go on to move down all four 1s as wherever we place the 2s we will not have a proper partition. For example

xxxx22
xxxx
1111

is not a proper partition and also the two 2s are bad as there are not two good 1s preceding them. We therefore have that

$$s_{44} \cdot s_{42} = s_{86} + s_{851} + s_{842} + s_{761} + s_{752} + s_{743} + s_{662} + s_{653} + s_{644}$$

We now go on to calculate $s_{44} \cdot s_{321}$ so again draw s_{44} first and then add on

s_{321} as shown;

xxxx111
xxxx22
3

We cannot place the 3 anywhere else except in the third row, so we start by moving the 2s which must sit before the 3 to ensure the result is semi-standard. Moreover, one 2 must always be left in the second row to ensure the 3 in the third row is good.

xxxx111
xxxx2
23

We now move one 1 and then move the 2s. The 1 must go down to the third row as it cannot sit below itself in the second row. The two good 1s left in the first row ensure the two 2s will also be good if left in the second row.

xxxx11	xxxx11
xxxx22	xxxx2
13	123

This is all we can do as if we move down another 1 then there will not be enough 1s in the first row to ensure both 2s are good. We thus have that

$$s_{44} \cdot s_{321} = s_{761} + s_{752} + s_{662} + s_{653}$$

and so subtracting this from $s_{44} \cdot s_{42}$ gives the final result

$$s_{44}(s_{42} - s_{321}) = s_{86} + s_{851} + s_{842} + s_{743}$$

and so we have calculated the character of the truncated module $\mathrm{Tr}^{(8,6,0)}E$ as a sum of Schur function.

REMARK 4.10.16 The Littlewood-Richardson Rule will be used when we are finding tilting modules in the range $2p \leq r \leq 3p-1$. By 4.7, we know that there are core classes in this range which consist of 6 weights, and thus to find those middle weights we must consider the multiplicity of the Schur function corresponding to their weight as it arises in the character of a tilting truncated module. The truncated modules we use require both Young's Rule and the Littlewood-Richardson Rule to calculate the multiplicities of these Schur functions. However, we first of all start with the range $0 \leq r \leq p-1$.

4.11 The proof for $0 \leq r \leq p-1$

AIM: The section contains a simple proof which resolves the above range.

THEOREM 4.11.1 *For* $0 \leq r \leq p-1$ *then* $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\lambda}E)$

Proof. For $0 \leq r \leq p-1$ then $L(\lambda) = \nabla(\lambda) = \Delta(\lambda) = T(\lambda)$ as for $r < p$, $S(n,r)$ is semisimple. Moreover each weight λ is self-titled and thus has its own core class. $\mathrm{Tr}^{\lambda}E = S^{\lambda_1}E \otimes S^{\lambda_2}E \otimes S^{\lambda_3}E$, where $\lambda_i \leq p-1$ and thus each $S^{\lambda_i}E$ is tilting. The Schur function s_λ arises in $\mathrm{ch}\,(\mathrm{Tr}^{\lambda}E)$ and as the weight λ is self-titled then $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\lambda}E)$ as required. □

4.12 The proof for $p \leq r \leq 2p-1$

AIM: In this section we shall prove that, when represented as Schur functions, the characters of the tilting modules whose weights are either self-titled or are highest or lowest in their core classes, always arise in the character of $\mathrm{Tr}^{\mu}E$ for μ the minimal element in $\Lambda^+(3,r)$. Moreover we shall prove that the coefficient space of the tilting modules whose weights are in the middle of their core class also arise in the coefficient space of the same truncated module, due to the Schur function corresponding to their weight arising with a higher multiplicity than that of the weight above them in their core class.

PROPOSITION 4.12.1 *For $p \leq r \leq 2p-1$, $\Lambda^+(3,r)$ has a unique minimal element μ and $\mu_1 < p-1$.*

Proof. Case 1: 3 divides r. In this case $\mu = (r/3, r/3, r/3)$ and as $r \leq 2p-1$ then $r/3 \leq 2p - 1/3 < p - 1/3$ and thus $r/3 \leq p-1$.
Case 2: 3 divides $r+1$. Here $\mu = ((r+1)/3, (r+1)/3, (r-2)/3)$ and so we require $(r+1)/3 \leq p-1$. Well $r \leq 2p-1$ and so $(r+1)/3 \leq 2p/3$ and we require $2p/3 \leq p-1$ which is true if and only if $3 \leq p$. For the case $p = 2$ we refer you to Chapter 5.
Case 3: 3 divides $r+2$. Here $\mu = ((r+2)/3, (r+2)/3-1, (r+2)/3-1)$ and so we require $(r+2)/3 \leq p-1$. As $r \leq 2p-1$ then $(r+2)/3 \leq (2p-1)/3 \leq p-1$ if and only if $2 \leq p$. □

THEOREM 4.12.2 *For $p \leq r \leq 2p-1$ and $|\lambda| = r$ then $\mathrm{Hom}_G(S^{\mu}E, \nabla(\lambda)) \neq 0$ where μ is the unique minimal element of $\Lambda^+(3,r)$, and hence $L(\lambda)$ is a composition factor of $S^{\mu}E$.*

Proof. First note that $(S^{\mu}E)^{\circ} \cong S^{\mu}E$ for $\mu_1 < p$ as each $S^{\mu_i}E$ is simple in this case.

Then, using the information given in Remark 1.1.11, we have that

$$\begin{aligned}\mathrm{Hom}_G(S^\mu E, \nabla(\lambda)) &= \mathrm{Hom}_G(\nabla(\lambda)^\circ, (S^\mu E)^\circ)\\ &\cong \mathrm{Hom}_G(\Delta(\lambda), (S^{\mu_1}E)^\circ \otimes (S^{\mu_2}E)^\circ \otimes (S^{\mu_3}E)^\circ)\\ &\cong \mathrm{Hom}_G(\Delta(\lambda), S^{\mu_1}E \otimes S^{\mu_2}E \otimes S^{\mu_3}E)\\ &\cong \mathrm{Hom}_G(\Delta(\lambda), S^\mu E)\\ &\cong \Delta(\lambda)^\mu\\ &\neq 0\end{aligned}$$

as by our knowledge about the characters of finite dimensional modules of complex Lie algebras by [16, Section 2.4] we have that μ is a weight of the Weyl module as $\mu \leq \lambda$.
There therefore exists a non-zero map $\phi : S^\mu E \to \nabla(\lambda)$ which implies $L(\lambda) \subseteq \mathrm{Im}(\phi)$ as $L(\lambda) = \mathrm{soc}_G\nabla(\lambda)$ and therefore $L(\lambda)$ is a composition factor of $\mathrm{Im}(\phi) = (S^\mu E)/\mathrm{Ker}(\phi)$ and thus $L(\lambda)$ is a composition factor of $S^\mu E$. □

The above proposition and theorem allow us to give the following corollary which resolves those tilting modules whose weights are highest in their core class or are self-titled.

COROLLARY 4.12.3 *If λ is maximal in its core class, or self-titled, then* $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(S^\mu E) = \mathrm{cf}\,(\mathrm{Tr}^{\,\mu} E)$ *where μ is the unique minimal element of* $\Lambda^+(3, r)$.

Proof. We have that $S^\mu E \cong T(v_1) \oplus \ldots \oplus T(v_m)$ and so as $L(\lambda)$ is a composition factor of $S^\mu E$ then $L(\lambda)$ is a composition factor of some $T(v_i)$. Thus $\lambda \leq v_i$ and λ and v_i are in the same core class, however λ is maximal in its core class and hence we must have $\lambda = v_i$. So, $T(\lambda)$ is a summand of $S^\mu E$ and therefore $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(S^\mu E)$. □

We now move on to the tilting modules whose weights are lowest in their core class.

THEOREM 4.12.4 *For $p \leq r \leq 2p - 1$ and λ lowest in its core class, then* $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(S^\mu E)$ *for μ minimal in* $\Lambda^+(3, r)$.

Proof. We have $\mathrm{Hom}_G(\nabla(\lambda), S^\mu E) \cong \nabla(\lambda)^\mu \neq 0$ as $\mu \leq \lambda$. Moreover for λ minimal in its core class we have $L(\lambda) = \nabla(\lambda) = \Delta(\lambda) = T(\lambda)$ and thus $\mathrm{cf}\,(T(\lambda)) = \mathrm{cf}\,(\nabla(\lambda)) \subseteq \mathrm{cf}\,(S^\mu E)$. □

We now look at the tilting modules whose weights are second highest in the core class of the 3-set.

THEOREM 4.12.5 *The tilting modules whose weights are in the middle of the core class consisting of three weights, arise in the truncated module* $\mathrm{Tr}^{\mu}E$ *for* μ *minimal in each* $\Lambda^{+}(3,r)$*, due to 4.10. Namely that the Schur function* s_{λ} *for each tilting module* $T(\lambda)$ *arises as a term in* $\mathrm{ch}\,(\mathrm{Tr}^{\mu}E)$ *with a higher multiplicity than the Schur function* s_{σ} *for* $\sigma > \lambda$ *in the same core class.*

Proof. To prove this we shall calculate the character of the truncated module $\mathrm{Tr}^{\mu}E$ and consider the multiplicities. Note that there are three cases to consider:
Case 1: For $3|r$ we use $\mathrm{Tr}^{(\frac{r}{3},\frac{r}{3},\frac{r}{3})}E = S^{\frac{r}{3}}E \otimes S^{\frac{r}{3}}E \otimes S^{\frac{r}{3}}E$ which we know to be tilting as $\frac{r}{3} < p-1$.
Case 2: For $3|r+1$ we use $\mathrm{Tr}^{(\frac{r+1}{3},\frac{r+1}{3},\frac{r+1}{3}-1)}E = S^{\frac{r+1}{3}}E \otimes S^{\frac{r+1}{3}}E \otimes S^{\frac{r+1}{3}-1}E$ which again is tilting as $\frac{r+1}{3} \leq p-1$.
Case 3: For $3|r+2$ we use $\mathrm{Tr}^{(\frac{r+2}{3},\frac{r+2}{3}-1,\frac{r+2}{3}-1)}E = S^{\frac{r+2}{3}}E \otimes S^{\frac{r+2}{3}-1}E \otimes S^{\frac{r+2}{3}-1}E$ which again is tilting as $\frac{r+2}{3} \leq p-1$.

Case 1: $3 \mid r$
Let us first do an example, so take $n=3$, $p=5$ and $r=9$, then we wish to calculate

$$\begin{aligned} \mathrm{ch}\,(\mathrm{Tr}^{333}E) = \;& S^3E \otimes S^3E \otimes S^3E \\ = \;& (s_3 \cdot s_3 \cdot)s_3 \\ = \;& (s_6 + s_{51} +_{42} + s_{33})s_3 \\ = \;& (s_9 + s_{81} + s_{72} + s_{63})+ \\ & (s_{81} + s_{72} + s_{711} + s_{63} + s_{621} + s_{54} + s_{531})+ \\ & (s_{72} + s_{63} + s_{621} + s_{54} + s_{1531} + s_{522} + s_{441} + s_{432})+ \\ & (s_{63} + s_{531} + s_{432} + s_{333}) \end{aligned}$$

We can display this result with multiplicities as follows;

$$\begin{array}{llll} 1s_9 & & & \\ 2s_{81} & & & \\ 3s_{72} & 1s_{711} & & \\ 4s_{63} & 2s_{621} & & \\ 2s_{54} & 3s_{531} & 1s_{522} & \\ & 1s_{441} & 2s_{432} & \\ & & & 1s_{333} \end{array}$$

THEOREM 4.12.6 The Schur function s_{λ} for all weights $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that $|\lambda| = r$ arises at least once in $\mathrm{ch}\,(\mathrm{Tr}^{\frac{r}{3},\frac{r}{3},\frac{r}{3}}E)$.

Proof. From Theorem 4.12.5 we have clarified that the truncated module we are using, $\mathrm{Tr}^{\mu}E$ where μ is minimal, is equal to $S^{\mu_1}E \otimes S^{\mu_2}E \otimes S^{\mu_3}E$ where each $S^{\mu_i}E$ is simple as $\mu_1 \leq p-1$. Hence, irrelevant of the fact we are now in a greater range for the degree r, we can still apply the proof of Theorem 4.12.2, to show that each weight λ arises at least once in the ch $(\mathrm{Tr}^{\frac{r}{3},\frac{r}{3},\frac{r}{3}}E)$. □

THEOREM 4.12.7 The character of the truncated module

$$\mathrm{ch}\,(\mathrm{Tr}^{\frac{r}{3},\frac{r}{3},\frac{r}{3}}E) = \sum a_\lambda s_\lambda \text{ where } a_\lambda = \begin{cases} \lambda_2 + 1 - \lambda_3 & \text{for } \lambda_2 \leq \frac{r}{3} \\ \lambda_1 + 1 - \lambda_2 & \text{for } \lambda_2 > \frac{r}{3} \end{cases}$$

Proof. 1) We shall first consider the case where $\lambda_2 \leq \frac{r}{3}$.
First recall that $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ where $|\lambda| = r$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Now, letting each $\frac{r}{3}$ be notated by 1s, 2s and 3s, then, by Young's Rule, a_λ is the number of ways we can form a semi-standard λ-tableau of type $(\frac{r}{3}, \frac{r}{3}, \frac{r}{3})$. Let us consider this proof for three cases; where $\lambda_1 \neq 0$ and $\lambda_2, \lambda_3 = 0$, where $\lambda_1, \lambda_2 \neq 0$ and $\lambda_3 = 0$, and finally where $\lambda_1, \lambda_2, \lambda_3 \neq 0$.

i) Let $\lambda = (\lambda_1, 0, 0)$, then we can form only one increasing row, where we first insert the 1s, followed by the 2s and finally the 3s. Thus $a_\lambda = 1 = \lambda_2 + 1 - \lambda_3$. For example, with $p = 7$ and $r = 9$ then we have $\lambda = (9, 0, 0)$ as
111222333.

ii) Let $\lambda = (\lambda_1, \lambda_2, 0)$, then to ensure we have a semi-standard tableau we first place all the 1s in the first row from left to right. As $\lambda_2 \leq \frac{r}{3}$ then we can fit at most λ_2 3s in the second row, so we have the following options;
Put all 2s in λ_2 and the remaining 2s and all 3s in λ_1 from left to right;
Put $\lambda_2 - 1$ 2s and then at the end of the row put one 3 in λ_2, then the remaining 2s and 3s in λ_1 from left to right;
Put $\lambda_2 - 2$ 2s and then at the end of the row put two 3s in λ_2, then the remaining 2s and 3s in λ_1 from left to right;
This continues until we reach the final option which is placing λ_2 3s in λ_2, and then all 2s and $\frac{r}{3} - \lambda_2$ 3s in λ_1 from left to right.
Then $a_\lambda = \lambda_2 + 1 = \lambda_2 + 1 - \lambda_3$.
For example with $p = 7$, $r = 9$ and $\lambda = (8, 1, 0)$ we have;

11122233
3

11122333
2

iii) Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. As before, all 1s must go first in the top row. Then to ensure we have a semi-standard tableau, and as $\lambda_3 \leq \frac{r}{3}$, the first λ_3 columns must be of the form;

1
2
3

There are then $\frac{r}{3} - \lambda_3$ 2s and 3s remaining to place in the first and second rows. Now as $\lambda_2 \leq \frac{r}{3}$ then $\lambda_1 - \lambda_2 \geq \lambda_1 - \frac{r}{3}$ but $\lambda_2 - \lambda_3 \leq \frac{r}{3} - \lambda_3$. The ways

of forming a semi-standard tableau are therefore dependent upon λ_2, and we have the following options;
Put $\lambda_2-\lambda_3$ 2s in λ_2 and the remaining 2s and all 3s in λ_1 from left to right;
Put $\lambda_2-\lambda_3-1$ 2s and then at the end of the row put one 3 in λ_2, then the remaining 2s and 3s in λ_1 from left to right;
Put $\lambda_2-\lambda_3-2$ 2s and then at the end of the row put two 3s in λ_2, then the remaining 2s and 3s in λ_1 from left to right;
This continues until we reach the final option which is placing $\lambda_2-\lambda_3$ 3s in λ_2, and then all 2s and $\frac{r}{3}-(\lambda_2-\lambda_3)$ 3s in λ_1 from left to right. Thus $a_\lambda=\lambda_2-\lambda_3+1$ as required.
For example, with $p=7$, $r=9$ and $\lambda=(6,2,1)$ we have

111233
22
3

111223
23
3

2) We now consider the case where $\lambda_2>\frac{r}{3}$.
If $\lambda_2>\frac{r}{3}$, then there are only two cases to consider, namely where $\lambda_1,\lambda_2\neq 0$ and $\lambda_3=0$, and then where $\lambda_1,\lambda_2,\lambda_3\neq 0$.
i) In the first case we have $\lambda=(\lambda_1,\lambda_2,0)$, and as usual, place all 1s in the first row from left to right. Now, as $\lambda_2>\frac{r}{3}$, then $\lambda_2-\frac{r}{3}>0$, and so we must place $\lambda_2-\frac{r}{3}$ 2s after the 1s in the first row, and underneath these 2s we must place $\lambda_2-\frac{r}{3}$ 3s at the end of the second row, to give strictly increasing columns. There therefore remains $\frac{r}{3}-(\lambda_2-\frac{r}{3})=\frac{2r}{3}-\lambda_2$ 2s and 3s to place in the tableau. Now, $\lambda_1<\frac{2r}{3}$ and so $\lambda_1-\lambda_2<\frac{2r}{3}-\lambda_2$ and so a_λ is dependent on $\lambda_1-\lambda_2$, and we have the following options;
Put $\lambda_1-\lambda_2$ 2s in λ_1 and the remaining 2s and all 3s in λ_2 from left to right;
Put $\lambda_1-\lambda_2-1$ 2s and then at the end of the row put one 3 in λ_1, then the remaining 2s and 3s in λ_2 from left to right;
Put $\lambda_1-\lambda_2-2$ 2s and then at the end of the row put two 3s in λ_1, then the remaining 2s and 3s in λ_2 from left to right;
This continues until we reach the final option which is placing $\lambda_1-\lambda_2$ 3s in λ_1, and then all 2s and $\frac{r}{3}-(\lambda_1-\lambda_2)$ 3s in λ_2 from left to right. Thus $a_\lambda=\lambda_1+1-\lambda_2$.
For example, with $p=7$, $r=9$ and $\lambda=(5,4,0)$ we have the options;

11122
2333

11123
2233

ii) For the case where $\lambda=(\lambda_1,\lambda_2,\lambda_3)$, as before, all 1s must go first in the top row. Then to ensure we have a semi-standard tableau, and as $\lambda_3\leq\frac{r}{3}$, the first λ_3 columns must be of the form;

1
2
3

Then again, as for the previous case, we must place $\lambda_2 - \frac{r}{3}$ 2s after the 1s in the first row, and below these place $\lambda_2 - \frac{r}{3}$ 3s at the end of the second row. There then remains $\frac{r}{3} - \lambda_3 - (\lambda_2 - \frac{r}{3}) = \frac{2r}{3} - \lambda_2 - \lambda_3$ 2s and 3s to place in the first and second rows. We can do this as follows;
Put $\lambda_1 - \lambda_2$ 2s in λ_1 and the remaining 2s and all 3s in λ_2 from left to right;
Put $\lambda_1 - \lambda_2 - 1$ 2s and then at the end of the row put one 3 in λ_1, then the remaining 2s and 3s in λ_2 from left to right;
Put $\lambda_1 - \lambda_2 - 2$ 2s and then at the end of the row put two 3s in λ_1, then the remaining 2s and 3s in λ_2 from left to right;
This continues until we reach the final option which is placing $\lambda_1 - \lambda_2$ 3s in λ_1, and then all 2s and $\frac{r}{3} - (\lambda_1 - \lambda_2)$ 3s in λ_2 from left to right. Thus $a_\lambda = \lambda_1 - \lambda_2 + 1$ as required.
For example, with $p = 7$, $r = 9$ and $\lambda = (4, 4, 1)$, we have the following option;

1112
2233
3

□

Case 2: $3 \mid r + 1$
Let us first do an example, so take $n = 3$, $p = 5$ and $r = 9$, then we wish to calculate

$$\begin{aligned} \mathrm{ch}\,(\mathrm{Tr}^{\,332}E) = & \quad S^3E \otimes S^3E \otimes S^2E \\ = & \quad (s_3 \cdot s_3 \cdot)s_2 \\ = & \quad (s_6 + s_{51} +_{42} + s_{33})s_2 \\ = & \quad (s_8 + s_{71} + s_{62}) + \\ & \quad (s_{71} + s_{62} + s_{611} + s_{53} + s_{521}) + \\ & \quad (s_{62} + s_{53} + s_{521} + s_{44} + s_{431} + s_{422}) + \\ & \quad (s_{53} + s_{431} + s_{332}) \end{aligned}$$

We can display this result with multiplicities as follows;

$$\begin{array}{lll} 1s_8 & & \\ 2s_{71} & & \\ 3s_{62} & 1s_{611} & \\ 3s_{53} & 2s_{521} & \\ 1s_{44} & 2s_{431} & 1s_{422} \\ & & 1s_{332} \end{array}$$

THEOREM 4.12.8 The character of the truncated module

$$\mathrm{ch}\,(\mathrm{Tr}^{\,\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3}-1}E) = \sum a_\lambda s_\lambda$$

where

$$a_\lambda = \begin{cases} \lambda_2 + 1 - \lambda_3 & \text{for } \lambda_2 \leq \frac{r+1}{3} - 1 \\ \lambda_1 + 1 - \lambda_2 & \text{for } \lambda_2 > \frac{r+1}{3} - 1 \end{cases}$$

Proof. The proof follows the same method as for case 1, except in this instance we replace $\frac{r}{3}$ by $\frac{r+1}{3} - 1$. $\square$

Case 3: $3 \mid r+2$
Let us first do an example, so take $n = 3$, $p = 5$ and $r = 7$, then we wish to calculate

$$\begin{aligned} \mathrm{ch}\,(\mathrm{Tr}^{\,322}E) = &\quad S^3E \otimes S^2E \otimes S^2E \\ = &\quad (s_3 \cdot s_2 \cdot) s_2 \\ = &\quad (s_5 + s_{41} + _{32}) s_2 \\ = &\quad (s_7 + s_{61} + s_{52}) + \\ &\quad (s_{61} + s_{52} + s_{511} + s_{43} + s_{421}) + \\ &\quad (s_{52} + s_{43} + s_{421} + s_{331} + s_{322}). \end{aligned}$$

We can display this result with multiplicities as follows;

$$\begin{array}{lll} 1s_7 & & \\ 2s_{61} & & \\ 3s_{52} & 1s_{511} & \\ 2s_{43} & 2s_{421} & \\ & 1s_{331} & 1s_{322} \end{array}$$

THEOREM 4.12.9 The character of the truncated module

$$\mathrm{ch}\,(\mathrm{Tr}^{\,\frac{r+2}{3}, \frac{r+2}{3}-1, \frac{r+2}{3}-1}E) = \sum a_\lambda s_\lambda$$

where

$$a_\lambda = \begin{cases} \lambda_2 + 1 - \lambda_3 & \text{for } \lambda_2 \leq \frac{r+2}{3} - 1 \\ \lambda_1 + 1 - \lambda_2 & \text{for } \lambda_2 > \frac{r+2}{3} - 1 \end{cases}$$

Proof. The proof follows the same method as for case 1, except in this instance we replace $\frac{r}{3}$ by $\frac{r+1}{3} - 2$. $\square$

So, how does this help us prove theorem 4.12.5? Well, letting the 'top' weight be λ and the 'middle' weight be μ, let us first assume that both λ_2 and $\mu_2 \leq \frac{r}{3}$ (or $\frac{r+1}{3} - 1$, $\frac{r+2}{3} - 1$ respectively), then $a_\lambda = \lambda_2 + 1 - \lambda_3$ and $a_\mu = \mu_2 + 1 - \mu_3$. Recalling from Section 4.7 that $\mu = (\mu_1, \mu_2, \mu_3) = (\lambda_2 + p - 1, \lambda_1 - p + 1, \lambda_3)$, we have that $a_\mu = (\lambda_1 - p + 1) + 1 - \lambda_3 = \lambda_1 - p + 2 - \lambda_3$, so we need to check that $\lambda_1 - p + 2 - \lambda_3 > \lambda_2 + 1 - \lambda_3$. Well, indeed this is true if and only if $\lambda_1 - p + 1 > \lambda_2$ which holds if and only if $\lambda_1 - \lambda_2 > p - 1$, which we know to be true. Thus the Schur function of the 'middle' weight arises with a higher multiplicity than that of the 'top' weight.
We now need to consider the case when both weights do not necessarily have second entry less than $\frac{r}{3}$. Well, it cannot be the case that a top weight can have second entry greater than $\frac{r}{3}$ as then $\lambda_1 - \lambda_2 < p$ which is a contradiction. So in all cases the top weight will arise with multiplicity $\lambda_2 + 1 - \lambda_3$. It is

possible however for a 'middle' weight, μ, to have second entry greater than $\frac{r}{3}$, and we know in this case that the multiplicity of $a_\mu = \mu_1 - \mu_2 + 1 = (\lambda_2 + p - 1) - (\lambda_1 - p + 1) + 1 = \lambda_2 - \lambda_1 + 2p - 1$. We therefore need to show that $a_\mu > a_\lambda$ i.e. that $\lambda_2 - \lambda_1 + 2p - 1 > \lambda_2 - \lambda_3 + 1$. This is true if and only if $\lambda_1 - \lambda_3 < 2p - 2$ which we know to be true, as by Proposition 4.3.2 we know that $\lambda_1 < 2p - 3$. Hence in all cases $a_\mu > a_\lambda$ and thus by 4.10 we have that the coefficient spaces of all tilting modules whose weights are second highest in the core class of the 3-set are contained in the coefficient space of the tilting truncated module $\mathrm{Tr}^\sigma E$ for σ minimal in the degree r. □

EXAMPLE 4.12.10 We now give a full example of the above proof for the case $n = 3$, $p = 5$ and $p \leq r \leq 2p - 1$. For each degree $p \leq r \leq 2p - 1$, the following tilting modules are needed;

$r = p = 5$
$T(5,0,0),\ T(4,1,0),\ T(3,2,0),\ T(3,1,1),\ T(2,2,1)$

$r = 6$
$T(6,0,0),\ T(5,1,0),\ T(4,2,0),\ T(4,1,1),\ T(3,3,0),\ T(3,2,1),\ T(2,2,2)$

$r = 7$
$T(7,0,0),\ T(6,1,0),\ T(5,2,0),\ T(5,1,1),\ T(4,3,0),\ T(4,2,1),\ T(3,3,1),$
$T(3,2,2)$

$r = 8$
$T(8,0,0),\ T(7,1,0),\ T(6,2,0),\ T(6,1,1),\ T(5,3,0),\ T(5,2,1),\ T(4,4,0),$
$T(4,3,1),\ T(4,2,2),\ T(3,3,2)$

$r = 2p - 1 = 9$
$T(9,0,0),\ T(8,1,0),\ T(7,2,0),\ T(7,1,1),\ T(6,3,0),\ T(6,2,1),\ T(5,4,0),$
$T(5,3,1),\ T(5,2,2),\ T(4,4,1),\ T(4,3,2),\ T(3,3,3)$

For each degree $p \leq r \leq 2p - 1$ the following partitions sit at the top of their core class;

$r = 5$ $(5,0,0)$
$r = 6$ $(6,0,0)$
$r = 7$ $(7,0,0),\ (6,1,0)$
$r = 8$ $(8,0,0),\ (7,1,0),\ (6,1,1)$
$r = 9$ $(9,0,0),\ (8,1,0),\ (7,2,0),\ (7,1,1)$

For each degree $p \leq r \leq 2p - 1$ the following partitions sit in the middle of their core class;

$r = 5$ $(4, 1, 0)$
$r = 6$ $(4, 2, 0)$
$r = 7$ $(4, 3, 0)$, $(5, 2, 0)$
$r = 8$ $(5, 3, 0)$, $(5, 2, 1)$
$r = 9$ $(6, 3, 0)$, $(5, 3, 1)$

For each degree $p \leq r \leq 2p - 1$ the following partitions sit at the bottom of their core class;

$r = 5$ $(3, 1, 1)$
$r = 6$ $(3, 2, 1)$
$r = 7$ $(3, 3, 1)$, $(3, 2, 2)$
$r = 8$ $(4, 4, 0)$, $(3, 3, 2)$, $(4, 2, 2)$
$r = 9$ $(4, 4, 1)$, $(5, 4, 0)$, $(3, 3, 3)$, $(4, 3, 2)$

For each degree $p \leq r \leq 2p - 1$ the following partitions are self-titled;

$r = 5$ $(3, 2, 0)$, $(2, 2, 1)$
$r = 6$ $(5, 1, 0)$, $(4, 1, 1)$, $(3, 3, 0)$, $(2, 2, 2)$
$r = 7$ $(5, 1, 1)$, $(4, 2, 1)$
$r = 8$ $(6, 2, 0)$, $(4, 3, 1)$
$r = 9$ $(6, 2, 1)$, $(5, 2, 2)$

Recall that the Schur functions corresponding to those weights at the top, the bottom and those that are self-titled must occur at least once in the character of a truncated tilting module, and those Schur functions corresponding to the middle weights must occur with a higher multiplicity than the weight above it. For each degree $p \leq r \leq 2p - 1$ we shall find $\mathrm{ch}\,(\mathrm{Tr}^{\lambda} E) = \mathrm{ch}\,(\bar{S}^{\lambda_1} E \otimes \bar{S}^{\lambda_2} E \otimes \bar{S}^{\lambda_3} E)$ for λ the lowest weight in that degree, and show how the multiplicities of the Schur functions occur in the necessary way.

• $r = 5$
Here $\lambda = (2, 2, 1)$ and

$$\begin{aligned}
\mathrm{ch}\,\mathrm{Tr}^{\lambda} E &= \mathrm{ch}\,(\bar{S}^2 E \otimes \bar{S}^2 E \otimes \bar{S}^1 E) \\
&= s_2 \cdot s_2 \cdot s_1 \\
&= (s_4 + s_{31} + s_{22}) \cdot s_1 \\
&= s_5 + s_{41} + s_{41} + s_{32} + s_{311} + s_{32} + s_{221} \\
&= s_5 + 2s_{41} + 2s_{32} + s_{311} + s_{221}.
\end{aligned}$$

As s_5, s_{32}, s_{311}, and s_{221} all occur and are either at the top or bottom of their core classes, or are self-titled then we have that the coefficient spaces of their corresponding tilting modules occur in cf $\mathrm{Tr}^{\lambda}E$. Moreover for $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T}) = (5,0,0)$ then $\lambda_M = (p-1, r-p+1, 0) = (4,1,0)$ and indeed s_{41} occurs with a higher multiplicity than s_5 in $\mathrm{ch}\,(\bar{S}^2E \otimes \bar{S}^2E \otimes \bar{S}^1E)$.

Hence $\mathrm{Tr}^{(2,2,1)}E = T(5,0,0) \oplus T(4,1,0) \oplus T(3,2,0) \oplus T(3,1,1) \oplus T(2,2,1)$.

• $r = 6$
Here $\lambda = (2,2,2)$ and

$$\begin{aligned}
\mathrm{ch}\,\mathrm{Tr}^{\lambda}E &= \mathrm{ch}\,(\bar{S}^2E \otimes \bar{S}^2E \otimes \bar{S}^2E)\\
&= s_2 \cdot s_2 \cdot s_2\\
&= (s_4 + s_{31} + s_{22}) \cdot s_2\\
&= s_6 + s_{51} + s_{42} + s_{51} + s_{42} + s_{411} + s_{33} + s_{321} + s_{42} + s_{321} + s_{222}\\
&= s_6 + 2s_{51} + 3s_{42} + s_{411} + s_{33} + 2s_{321} + s_{222}.
\end{aligned}$$

The issue here is $T(4,2,0)$ which is the only partition in the middle of its p-core. With $\lambda_T = (6,0,0)$ then $\lambda_M = (p-1, r-p+1, 0) = (4,2,0)$ and s_{42} occurs with a higher multiplicity than s_6. For all other weight, the corresponding Schur function occurs at least once and thus

$$\begin{aligned}
\mathrm{Tr}^{(2,2,2)}E =& T(6,0,0) \oplus T(5,1,0) \oplus T(4,2,0) \oplus T(4,1,1) \oplus T(3,3,0)\\
&\oplus T(3,2,1) \oplus T(2,2,2).
\end{aligned}$$

• $r = 7$
Here $\lambda = (3,2,2)$ and

$$\begin{aligned}
\mathrm{ch}\,\mathrm{Tr}^{\lambda}E =& \quad \mathrm{ch}\,(\bar{S}^3E \otimes \bar{S}^2E \otimes \bar{S}^2E)\\
=& \quad s_3 \cdot s_2 \cdot s_2\\
=& \quad (s_5 + s_{41} + s_{32}) \cdot s_2\\
=& \quad s_7 + s_{61} + s_{52} + s_{61} + s_{52} + s_{511} + s_{43} + s_{421} + s_{52} + s_{43} + s_{421}\\
& \quad + s_{331} + s_{322}\\
=& \quad s_7 + 2s_{61} + 3s_{52} + s_{511} + 2s_{43} + 2s_{421} + s_{331} + s_{322}.
\end{aligned}$$

The issues here are $T(4,3,0)$ and $T(5,2,0)$. With $\lambda_T = (r,0,0) = (7,0,0)$ then $\lambda_M = (4,3,0)$, and with $\lambda_T = (6,1,0) = (r-1,1,0)$ then $\lambda_M = (p+1-1, r-p-1+1, 0) = (5,2,0)$ and both s_{43} and s_{52} occur with a higher multiplicity than s_7 and s_{61} respectively. Hence

$$\begin{aligned}
\mathrm{Tr}^{(3,2,2)}E =& T(7,0,0) \oplus T(6,1,0) \oplus T(5,2,0) \oplus T(5,1,1) \oplus T(4,3,0)\\
&\oplus T(4,2,1) \oplus T(3,3,1) \oplus T(3,2,2).
\end{aligned}$$

• $r = 8$
Here $\lambda = (3,3,2)$ and

$$\begin{aligned}
\operatorname{ch}\operatorname{Tr}^{\lambda}E = \;& \operatorname{ch}(\bar{S}^3E \otimes \bar{S}^3E \otimes \bar{S}^2E)\\
=\;& s_3 \cdot s_3 \cdot s_2\\
=\;& (s_6 + s_{51} + s_{42} + s_{33})s_2\\
=\;& s_8 + s_{71} + s_{62} + s_{71} + s_{62} + s_{611} + s_{53} + s_{521} + s_{62} + s_{53} + s_{521}\\
& + s_{44} + s_{431} + s_{422} + s_{53} + s_{431} + s_{332}\\
=\;& s_8 + 2s_{71} + 3s_{62} + s_{611} + 3s_{53} + 2s_{521} + s_{44} + 2s_{431} + s_{422} + s_{332}.
\end{aligned}$$

Here the issues are $T(5,3,0)$ and $T(5,2,1)$. With $\lambda_T = (7,1,0) = (r-1,1,0)$ then $\lambda_M = (p+1-1, r-p-1+1, 0) = (5,3,0)$. With $\lambda_T = (6,1,1) = (r-\alpha, \alpha-\beta, \beta)$ the $\lambda_M = (p+\alpha-\beta-1, r-p-\alpha+1, \beta) = (5,2,1)$ and both s_{53} and s_{521} occur with a higher multiplicity than s_{71} and s_{611} respectively. Hence

$$\begin{aligned}
\operatorname{Tr}^{(3,3,2)}E = \;& T(8,0,0) \oplus T(7,1,0) \oplus T(6,2,0) \oplus T(6,1,1) \oplus T(5,3,0)\\
& \oplus T(5,2,1) \oplus T(4,4,0) \oplus T(4,3,1) \oplus T(4,2,2) \oplus (3,3,2).
\end{aligned}$$

• $r = 9$
Here $\lambda = (3,3,3)$ and

$$\begin{aligned}
\operatorname{ch}\operatorname{Tr}^{\lambda}E = \;& \operatorname{ch}(\bar{S}^3E \otimes \bar{S}^3E \otimes \bar{S}^3E)\\
=\;& s_3 \cdot s_3 \cdot s_3\\
=\;& (s_6 + s_{51} + s_{42} + s_{33})s_3\\
=\;& s_9 + s_{81} + s_{72} + s_{63} + s_{81} + s_{72} + s_{711} + s_{63} + s_{621} + s_{54} + s_{531}\\
& + s_{72} + s_{63} + s_{621} + s_{54} + s_{531} + s_{522} + s_{441} + s_{432} + s_{63} + s_{531}\\
& + s_{432} + s_{333}\\
=\;& s_9 + 2s_{81} + 3s_{72} + s_{711} + 4s_{63} + 2s_{621} + 2s_{54} + 3s_{531} + s_{522} + s_{441}\\
& + 2s_{432} + s_{333}.
\end{aligned}$$

The issues here are $T(6,3,0)$ and $T(5,3,1)$. With $\lambda_T = (7,2,0)$ then $\lambda_M = (6,3,0)$ and with $\lambda_T = (7,1,1)$ then $\lambda_M = (5,3,1)$, and both s_{63} and s_{531} occur with a higher multiplicity than s_{72} and s_{711} respectively. Hence

$$\begin{aligned}
\operatorname{Tr}^{(3,3,3)}E = \;& T(9,0,0) \oplus T(8,1,0) \oplus T(7,2,0) \oplus T(7,1,1) \oplus T(6,3,0)\\
& \oplus T(6,2,1) \oplus T(5,4,0) \oplus T(5,3,1) \oplus T(5,2,2) \oplus T(4,4,1)\\
& \oplus T(4,3,2) \oplus T(3,3,3).
\end{aligned}$$

Thus for each degree $p \le r \le 2p-1$, $\operatorname{cf} T(\mu) \subseteq \operatorname{cf}\operatorname{Tr}^{\lambda}E \subseteq D_{3,p}(r)$ for λ minimal in r and $T(\mu)$ a tilting module of $A(3,r)$, and hence $D_{3,5}(r) = A(3,r)$ for $p \le r \le 2p-1$.

4.13 The proof for $2p \leq r \leq 3p-1$

AIM: We now move on to the next range, following similar methods to the previous range and also using the Littlewood-Richardson Rule to prove Theorem 4.1.3. As before, the general method for the proof is the same, in that we need to show that the coefficient spaces of all tilting modules of $A(\pi, r)$, for π a saturated set, are contained within some tilting truncated module which is contained in $D_{3,p}(r)$. Again we do this by comparing characters, or more specifically multiplicities of Schur functions, dependent on where the weight of these tilting modules sit in their core classes.

For the range $2p \leq r \leq 3p-1$ we split the proof into a number of different parts.

• We first consider the range $2p \leq r \leq 3p-3$ and show;

i) That the cf $(T(\mu))$, for μ highest or lowest in its core classes or self-titled, is contained in cf $(\mathrm{Tr}^{\lambda} E)$ for λ the partition of minimal weight in r.

ii) The cf $(T(\sigma))$ for σ the second highest weight in the core class of the 3-set or the 6-set is also contained in cf $(\mathrm{Tr}^{\lambda} E)$ for λ minimal in r, due to 4.10 c).

iii) The cf $(T(\nu))$ for ν the third highest weight in the core class of the 6-set is contained in cf $(\mathrm{Tr}^{\lambda} E)$ for λ minimal, due to a rather complicated reason which will be explained at this point!

iv) The coefficient spaces of the tilting modules whose weights are fourth and fifth highest in the core class of the 6-set are contained in the coefficient space of the truncated module $\mathrm{Tr}^{\lambda} E = L(p-1, p-1, 0) \otimes \bar{S}^{\frac{r-(2p-2)}{2}} E \otimes \bar{S}^{\frac{r-(2p-2)}{2}} E$ for r even and in $\mathrm{Tr}^{\lambda} E = L(p-1, p-1, 0) \otimes \bar{S}^{\frac{r-(2p-2)+1}{2}} E \otimes \bar{S}^{\frac{r-(2p-2)-1}{2}} E$ for r odd.

• We then consider the case $r = 3p-2$ and show;

i) The cf $(T(\mu))$, for μ highest or lowest in its core classes or self-titled, is contained in cf $(\mathrm{Tr}^{\lambda} E)$ for $\mathrm{Tr}^{\lambda} E = L(p-1, 1, 0) \otimes S^{p-1} E \otimes S^{p-1} E$ which is the lowest weight for degree $r = 3p-2$. Moreover, the cf $(T(\sigma))$ for σ the second highest weight in the core class of the 6-set is also contained in cf $(\mathrm{Tr}^{\lambda} E)$.

ii) The cf $(T(\sigma))$ for σ the second highest weight in the core class of the 3-set which came from the self-titled weight λ where $\lambda_1 - \lambda_3 < p-2$ is contained in cf $(\mathrm{Tr}^{(p,p-1,p-1)} E) = $ cf $(L(p-1, 1, 0) \otimes S^{p-1} E \otimes S^{p-1} E)$.

iii) The cf $(T(\sigma))$ for σ the second highest weight in the core class of the 3-set where $\lambda_1 + p > t$ and $\lambda_1 - p < p-1$ is contained in cf $(\mathrm{Tr}^{\lambda} E) = $ cf $(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes \bar{S}^{p-2} E)$.

iv) For those weights sitting third, fourth and fifth in the core class of the 6-set we can use the same result as for the case $2p \leq r \leq 3p-3$.

• We then consider the case $r = 3p-1$ and show;

i) The cf $(T(\mu))$, for μ highest or lowest in its core classes or self-titled, is

contained in cf $(\mathrm{Tr}^{\lambda}E)$ for $\mathrm{Tr}^{\lambda}E = L(p-1,1,0)\otimes L(p-1,1,0)\otimes S^{p-1}E$ which is the lowest weight for degree $r = 3p-2$. Moreover, the cf $(T(\sigma))$ for σ the second highest weight in their core classes of the 6-set is also contained in cf $(\mathrm{Tr}^{\lambda}E)$.
ii) The cf $(T(\sigma))$ for σ the second highest weight in the core class of the 3-set which came from the self-titled weight λ where $\lambda_1 - \lambda_3 < p-2$ is contained in cf $(\mathrm{Tr}^{(p,p,p-1)}E) = $ cf $(L(p-1,1,0)\otimes L(p-1,1,0)\otimes S^{p-1}E)$.
iii) The cf $(T(\sigma))$ for σ the second highest weight in the core class of the 3-set where $\sigma_1 - p < p-1$ is contained in cf $(L(p-1,2,0)\otimes L(p-1,1,0)\otimes \bar{S}^{p-2}E$, whilst the cf $(T(\sigma))$ for σ the second highest weight in the core class of the 3-set where $\sigma_1 - p = p-1$ is contained in cf $(L(p-1,p-1,1)\otimes S^{\frac{p+1}{2}}E\otimes S^{\frac{p-1}{2}}E)$.
iv) For those weights sitting third, fourth and fifth in the core class of the 6-set we can use the same result as for the case $2p \le r \le 3p-3$.

Note that we use the results from Section 4.8 and Remark 2.4.3 (iii) to show that all these truncated modules are indeed tilting.

CALCULATION 4.13.1 The case $2p \le r \le 3p-3$

i) We consider the tilting modules whose weights are at the top or bottom of their p-core, or are self-titled, and show their coefficient spaces are contained in the coefficient space of $\mathrm{Tr}^{\lambda}E$ for λ minimal in r.

PROPOSITION 4.13.2 *For $2p \le r \le 3p-3$, $\Lambda^+(3,r)$ has a unique minimal element μ and $\mu_1 \le p-1$.*

Proof. Case 1) $3|r$
In this case $\mu = (\frac{r}{3},\frac{r}{3},\frac{r}{3})$ and as $r \le 3p-3$ then $\frac{3p-3}{3} \le p-1$ as required.

Case 2) $3|r+1$
In this case $\mu = (\frac{r+1}{3},\frac{r+1}{3},\frac{r+1}{3}-1)$ and so we check that $\frac{r+1}{3} \le p-1$. The greatest r can be such that 3 divides $r+1$ is $r = 3p-4$ and indeed $\frac{3p-4+1}{3} = p-1$ and thus $\frac{r+1}{3} \le p-1$, and thus $\frac{r+1}{3} - 1 < p-1$.

Case 3) $3|r+2$
In this case $\mu = (\frac{r+2}{3},\frac{r+2}{3}-1,\frac{r+2}{3}-1)$ and so we check that $\frac{r+2}{3} \le p-1$. Here the greatest r can be such that 3 divides $r+2$ is with $r = 3p-5$ and we have $\frac{3p-5+2}{3} = p-1$ and so $\frac{r+2}{3} \le p-1$ and $\frac{r+2}{3} - 1 < p-1$. □

THEOREM 4.13.3 *For $2p \le r \le 3p-1$ and $|\lambda| = r$ then $\mathrm{Hom}_G(S^{\mu}E, \nabla(\lambda)) \ne 0$ where μ is the unique minimal element of $\Lambda^+(3,r)$, and hence $L(\lambda)$ is a composition factor of $S^{\mu}E$.*

Proof. From Proposition 4.13.2 we have clarified that for the three cases $3|r$, $3|r+1$ and $3|r+2$, the truncated module we are using, $\mathrm{Tr}^{\mu}E$ where μ is minimal, is equal to $S^{\mu_1}E \otimes S^{\mu_2}E \otimes S^{\mu_3}E$ where each $S^{\mu_i}E$ is simple as $\mu_1 \leq p-1$. Hence, irrelevant of the fact we are now in a greater range for the degree r, we can still apply the proof of Theorem 4.12.2, to show that each $L(\lambda)$ is a composition factor of $S^{\mu}E$. □

COROLLARY 4.13.4 *If λ is maximal in its core class then $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(S^{\mu}E)$ where μ is the unique minimal element of $\Lambda^{+}(3,r)$.*

Proof. Same as for Corollary 4.12.3. □

THEOREM 4.13.5 *For $2p \leq r \leq 3p-1$ and λ lowest in its core class, then $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(S^{\mu}E)$ for μ minimal in $\Lambda^{+}(3,r)$.*

Proof. Same as for Theorem 4.12.4. □

ii) We now consider those tilting modules whose weights are second highest in their core classes of both the 3-set and the 6-set.

THEOREM 4.13.6 *The $\mathrm{cf}\,(T(\sigma))$ for σ the second highest weight in the core class of either the 6-set or the 3-set, are contained in $\mathrm{cf}\,(\mathrm{Tr}^{\mu}E)$ for μ minimal in each $\Lambda^{+}(3,r)$, due to Section 4.10. Namely that the Schur function s_{σ} for each tilting module $T(\sigma)$ arises as a term in $\mathrm{ch}\,(\mathrm{Tr}^{\mu}E)$ with a higher multiplicity than the Schur function s_{λ} for $\lambda > \sigma$ in the same core class.*

Proof. As we are again using the truncated module $\mathrm{Tr}^{\mu}E$ for μ minimal, where each $S^{\mu_i}E$ is simple due to $\mu_i \leq p-1$, then we can use the work from Section 4.12 despite being in a larger degree range for r, and thus we know that;

- The character of the truncated module $\mathrm{ch}\,(\mathrm{Tr}^{\frac{r}{3},\frac{r}{3},\frac{r}{3}}E) = \sum a_{\lambda}s_{\lambda}$
where $a_{\lambda} = \begin{cases} \lambda_2 + 1 - \lambda_3 & \text{for } \lambda_2 \leq \frac{r}{3} \\ \lambda_1 + 1 - \lambda_2 & \text{for } \lambda_2 > \frac{r}{3} \end{cases}$

- The character of the truncated module $\mathrm{ch}\,(\mathrm{Tr}^{\frac{r+1}{3},\frac{r+1}{3},\frac{r+1}{3}-1}E) = \sum a_{\lambda}s_{\lambda}$
where $a_{\lambda} = \begin{cases} \lambda_2 + 1 - \lambda_3 & \text{for } \lambda_2 \leq \frac{r+1}{3} - 1 \\ \lambda_1 + 1 - \lambda_2 & \text{for } \lambda_2 > \frac{r+1}{3} - 1 \end{cases}$

- The character of the truncated module $\mathrm{ch}\,(\mathrm{Tr}^{\frac{r+2}{3},\frac{r+2}{3}-1,\frac{r+2}{3}-1}E) = \sum a_{\lambda}s_{\lambda}$
where $a_{\lambda} = \begin{cases} \lambda_2 + 1 - \lambda_3 & \text{for } \lambda_2 \leq \frac{r+2}{3} - 1 \\ \lambda_1 + 1 - \lambda_2 & \text{for } \lambda_2 > \frac{r+2}{3} - 1 \end{cases}$

We therefore need to ensure for both the 3-set and the 6-set, that the multiplicity a_{σ} of the Schur function s_{σ} is greater than a_{λ}, where λ is the highest weight in its core class and σ is the second highest weight in its core class.

So, what do λ and σ look like? Well, let us first consider the 6-set. Referring back to Section 4.7, we have that $\lambda = (\lambda_1 + p, \lambda_2, \lambda_3)$ where the weight $(\lambda_1, \lambda_2, \lambda_3)$ was the highest weight in the 3-set for the range $p \leq r \leq 2p - 1$. We know that $\lambda_2 \leq \frac{r}{3}$ and thus $a_\lambda = \lambda_2 + 1 - \lambda_3$. The second weight $\sigma = (\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)$, and for the case where $\sigma_2 \leq \frac{r}{3}$, ($\frac{r+1}{3} - 1$ and $\frac{r+2}{3} - 1$ respectively), then $a_\sigma = (\lambda_1 - p + 1) + 1 - (\lambda_3)$. So we need to check that $a_\sigma > a_\lambda$ which is true if and only if $\lambda_1 - \lambda_2 > p - 1$ which we know to be true. If on the other hand, $\sigma_2 > \frac{r}{3}$, ($\frac{r+1}{3} - 1$ and $\frac{r+2}{3} - 1$ respectively) then $a_\sigma = \sigma_1 + 1 - \sigma_2 = (\lambda_2 + p - 1) + 1 - (\lambda_1 - p + 1)$, which is greater than $a_\lambda = \lambda_2 + 1 - \lambda_3$ if and only if $3p > \lambda_1 + 2 - \lambda_3$. Well, we know that $\lambda_1 \leq 2p - 3$ and thus $\lambda_1 + 2 - \lambda_3 \leq 2p - 1 - \lambda_3$, so it is enough to show that $3p > 2p - 1 - \lambda_3$ which is true if and only if $\lambda_2 > -p - 1$ which is of course true.

Now let us consider the 3-set. In this case the highest weight in the core class is $\mu = (\lambda_1 + p, \lambda_2, \lambda_3)$ where here $(\lambda_1, \lambda_2, \lambda_3)$ was a self-titled weight in the range $p \leq r_1 \leq 2p - 1$ such that $\lambda_1 - \lambda_3 < p - 2$. We now prove that $\mu_2 = \lambda_2 \leq \frac{r}{3}$.

Suppose for a contradiction that $\lambda_2 > \frac{r}{3}$, this holds if and only if $\lambda_1 + p + \lambda_3 < \frac{2r}{3}$, which holds if and only if $3\lambda_1 + 3\lambda_3 + 3p < 2r$. We have that $2p \leq r_2 \leq 3p - 3$ and so taking $r_2 = 3p - 3$ would give that $3\lambda_1 + 3\lambda_3 < -3$ which is clearly a contradiction. Hence we have that $\lambda_2 \leq \frac{r}{3}$, ($\frac{r+1}{3} - 1$ and $\frac{r+2}{3} - 1$ respectively), and thus $a_\lambda = \lambda_2 + 1 - \lambda_3$.

The second weight here is $\sigma = (\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$, and in the case where $\sigma_2 \leq \frac{r}{3}$ ($\frac{r+1}{3} - 1$ and $\frac{r+2}{3} - 1$ respectively) we have that $a_\sigma = (\lambda_1 + 1) + 1 - \lambda_3$, so we must check that $a_\sigma > a_\lambda$, which is indeed true as $\lambda_1 \geq \lambda_2$ and thus $\lambda_1 + 2 > \lambda_2 + 1$ and hence $\lambda_1 + 2 - \lambda_3 > \lambda_2 + 1 - \lambda_3$. If, on the other hand, $\sigma_2 > \frac{r}{3}$ ($\frac{r+1}{3} - 1$ and $\frac{r+2}{3} - 1$ respectively) then $a_\sigma = (\lambda_2 + p - 1) + 1 - (\lambda_1 + 1)$, and as $\lambda_1 - \lambda_2 < p - 2$ then we have that $a_\lambda = \lambda_2 + 1 - \lambda_3 < \lambda_2 + p - \lambda_1 - 1 = a_\sigma$ as required.

We therefore have that in $\mathrm{ch}\,(\mathrm{Tr}^{\,\mu} E)$ for μ minimal, the Schur function s_σ for σ the second highest in its core class always arises with a higher multiplicity than the Schur function s_λ for λ highest in its core class. Therefore by 4.10 c), we have that $\mathrm{cf}\,(T(\sigma)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,\mu} E)$. □

iii) We now consider those tilting modules whose weights are third highest in the core class of the 6-set. This tilting module arises as a composition factor of a tilting module of higher weight which we have already shown occurs. For notational purposes we shall call this 'third' weight $(\lambda_{31}, \lambda_{32}, \lambda_{33})$ and the weight above it $(\lambda_{21}, \lambda_{22}, \lambda_{23})$.

THEOREM 4.13.7 *The tilting module whose weight is third highest in the* *core class of the 6-set,* $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$ *has filtration structure*

$$\begin{array}{l}\nabla(\lambda_{31}, \lambda_{32}, \lambda_{32}) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

and $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ *has filtration structure*

$$\begin{array}{l}\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\tau_1, \tau_2, \tau_3) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

where $\nabla(\sigma_1, \sigma_2, \sigma_3) = \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$ *and* $\nabla(\tau_1, \tau_2, \tau_3) = \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43})$ *such that* $\lambda_{31} > \tau_1$ *but* $\lambda_{32} < \tau_2$.

Hence $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ *has filtration structure*

$$\begin{array}{c}\nabla(\lambda_{21}.\lambda_{22}, \lambda_{23}) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \oplus \nabla(\tau_1, \tau_2, \tau_3) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

and so we can also write this filtration structure of $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ *as*

$$\begin{array}{l}\nabla(\lambda_{21}, \lambda_{23}, \lambda_{23}) \\ \nabla(\tau_1, \tau_2, \tau_3) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

where

$$\begin{array}{l}\nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

is a non-split extension at the bottom of $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ *and so must be* $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$.

Hence we have that $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ *has filtration structure*

$$\begin{array}{l}\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \nabla(\tau_1, \tau_2, \tau_3) \\ T(\lambda_{31}, \lambda_{32}, \lambda_{33})\end{array}$$

And as we have already found the tilting module $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ in Theorem 4.13.6 then $\mathrm{cf}\,(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq \mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\lambda} E) \subseteq D_{3,p}(r)$ *for all primes p and λ minimal in degree r for $2p \leq r \leq 3p-3$.*

Proof of Theorem 4.13.7. We first show that $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$ has filtration structure

$$\begin{array}{l}\nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

and $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ has filtration structure

$$\begin{array}{l}\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\tau_1, \tau_2, \tau_3) \\ \nabla(\sigma_1, \sigma_2, \sigma_3)\end{array}$$

Let us first understand what these tilting modules look like. Well, for $p \leq r \leq 2p-1$, let $(\lambda_{1T}, \lambda_{2T}, \lambda_{3T})$ be the partition at the top of its p-core. Then by 4.7 we have $(\lambda_{1M}, \lambda_{2M}, \lambda_{3M}) = (\lambda_{2T} + p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$ and $(\lambda_{1B}, \lambda_{2B}, \lambda_{3B}) = (\lambda_{3T} + p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)$. Thus for $2p \leq r \leq 3p-3$ we have;

$(\lambda_{11}, \lambda_{12}, \lambda_{13}) = (\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T})$
$(\lambda_{21}, \lambda_{22}, \lambda_{23}) = (\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T})$
$(\lambda_{31}, \lambda_{32}, \lambda_{33}) = (\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)$
$(\lambda_{41}, \lambda_{42}, \lambda_{43}) = (\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$
$(\lambda_{51}, \lambda_{52}, \lambda_{53}) = (\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$
$(\lambda_{61}, \lambda_{62}, \lambda_{63}) = (\lambda_{2T} + p - 1, \lambda_{3T} + p - 1, \lambda_{1T} - p + 2)$

CLAIM 4.13.8 $\nabla(\sigma_1, \sigma_2, \sigma_3) = \nabla(\lambda_{51}\lambda_{52}, \lambda_{53})$

Proof of Claim 4.13.8. To prove this we use Fact 4.9.10 to show that $T(\lambda_{21}, \lambda_{22}, \lambda_{23}) \cong I(\lambda_{51}, \lambda_{52}, \lambda_{53})$ and hence $\nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$ sits at the bottom of the tilting module $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$. We consider this in stages by applying the Mullineux Bijection.
i) Considers where λ_T is equal to partitions of the form $(\lambda_{1T}, a, 0)$ where $a \geq 0$.
ii) Considers where λ_T is equal to partitions of the form $(\lambda_{1T}, a, 1)$ where $a \geq 1$.
iii) Considers where λ_T is equal to partitions of the form $(\lambda_{1T}, a, 2)$ where $a \geq 2$.

iv) Then looks to find the greatest that λ_{3T} can be which is found in Claim 4.13.9 to be $\frac{p-1}{3}$ when $3|p-1$ and $\frac{p-2}{3}$ when $3|p-2$. So we apply the Mullineux Bijection to these final two cases.
Having done this we then show that $\nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$ is a composition factor of $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$ by calculating $(T(\lambda_{31}, \lambda_{32}, \lambda_{33}) : \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53}))$.

We have $(\lambda_{51}, \lambda_{52}, \lambda_{53}) = (\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$. We now want to show that we can use Fact 4.9.10 and so check the following;
Firstly, $\lambda_{1T} \leq 2p - 3 < 2p - 2 = (n-1)(p-1)$.
Secondly,

$$\begin{aligned}
\lambda_{51} - \lambda_{52} &= \lambda_{1T} - \lambda_{3T} - p + 1 \leq 2p - 3 - \lambda_{3T} - p + 1 \\
&= p - \lambda_{3T} - 2 \\
&< p
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{52} - \lambda_{53} &= \lambda_{3T} + p - 1 - \lambda_{2T} - 1 \\
&< p
\end{aligned}$$

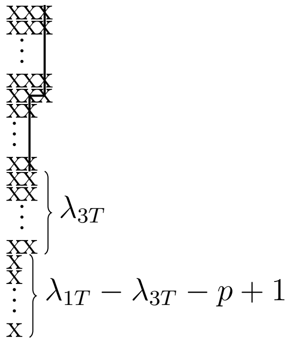
because $\lambda_{3T} - \lambda_{2T} - 2 < 0$ as $\lambda_{2T} \geq \lambda_{3T}$. We can therefore apply Fact 4.9.10 and say that

$$\begin{aligned}
I(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1) &\equiv T(\mathrm{Mull}(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)') \\
&\equiv T(\mathrm{Mull}(\underbrace{3, 3, \ldots, 3}_{\lambda_{2T}+1}, \underbrace{2, 2, \ldots, 2}_{\lambda_{3T}-\lambda_{2T}+p-2}, \underbrace{1, 1, \ldots, 1}_{\lambda_{1T}-\lambda_{3T}-p+1}))
\end{aligned}$$

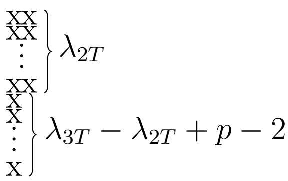
We therefore have to apply the Mullineux bijection to;

$$\left.\begin{array}{l} \mathrm{XXX} \\ \mathrm{XXX} \\ \vdots \\ \mathrm{XXX} \end{array}\right\} \lambda_{2T} + 1$$
$$\left.\begin{array}{l} \mathrm{XX} \\ \mathrm{XX} \\ \vdots \\ \mathrm{XX} \end{array}\right\} \lambda_{3T} - \lambda_{2T} + p - 2$$
$$\left.\begin{array}{l} \mathrm{X} \\ \mathrm{X} \\ \vdots \\ \mathrm{x} \end{array}\right\} \lambda_{1T} - \lambda_{3T} - p + 1$$

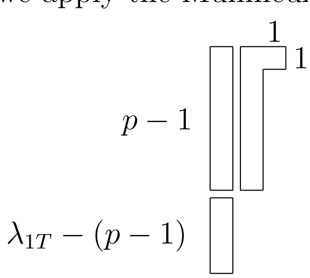
Well $(\lambda_{3T} - \lambda_{2T} + p - 2) - (p - (\lambda_{2T} + 2)) = \lambda_{3T}$, so what remains after removing a rim p-hook is;



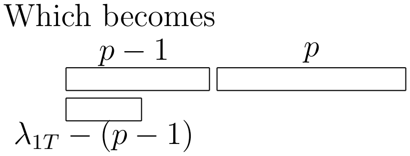
Now we know that $\lambda_{3T} < p$ as otherwise $\lambda_{1T} + \lambda_{2T} + \lambda_{3T} \geq 4p$ which is a contradiction as $2p \leq r \leq 3p - 3$. So when we move to the next row we take off $\lambda_{3T} + 1 + \mu$ where μ is some number of nodes from the final single line to create an edge of length at most p. We then move down each row removing the single x's.
Then what remains of the first column is $(\lambda_{2T}+1)+(\lambda_{3T}-\lambda_{2T}+p-2)-1 = \lambda_{3T}+p-2$, and what remains of the second column is $(\lambda_{2T}+1)-1 = \lambda_{2T}$.



We continue to remove Mullineux p-hooks until we result in the empty set, and then start rebuilding the partition using $s_i = a_i - r_i + \epsilon_i$. So we now go about looking at the different cases that arise when reconstructing the new partition dependent upon the values of λ_{2T} and λ_{3T}, and prove in every case that $I(\lambda_{51}, \lambda_{52}, \lambda_{53}) \cong T(\lambda_{21}, \lambda_{22}, \lambda_{23})$.
i)
• For $\lambda_{3T} + p - 1 = p - 1$ and $\lambda_{2T} + 1 = 1$, $\lambda_{3T} = 0$ then $\lambda_T = (\lambda_{1T}, 0, 0)$. So we apply the Mullineux Bijection to;



Which becomes

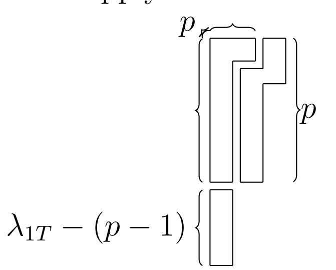


as $a - r + \epsilon = (p + \lambda_{1T} - p + 1) - (\lambda_{1T}) + 1 = 2$.
Then

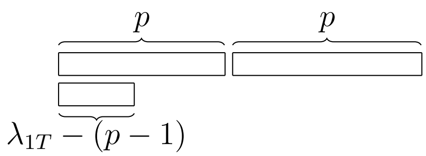
$$\begin{aligned} I(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1) &\cong T(2p-1, \lambda_{1T} - (p-1), 0) \\ &= T(\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T}) \\ &= T(\lambda_{21}, \lambda_{22}, \lambda_{23}) \end{aligned}$$

as required.

• For $\lambda_{3T}+p-1 = p-1$ and $\lambda_{2T}+1 = 2$, $\lambda_{3T} = 0$ then we have $\lambda_T = (\lambda_{1T}, 1, 0)$. So we apply the Mullineux bijection to;

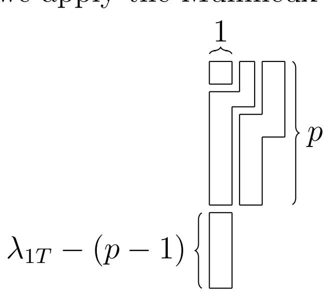


Which becomes;

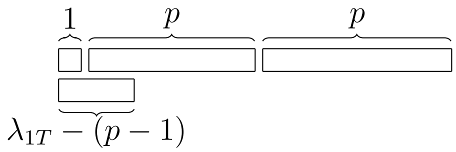


as $a_1 - r_1 + \epsilon_1 = (p + \lambda_{1T} - p + 1) - (\lambda_{1T}) + 1 = 2$ and $a_2 - r_2 + \epsilon_2 = p - (p-1) + 0 = 1$. Then $I(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1) \cong T(2p, \lambda_{1T} - (p-1), 0) = T(\lambda_{21}, \lambda_{22}, \lambda_{32})$ as required.

• For $\lambda_{3T} + p - 1 = p - 1$ and $\lambda_{2T} + 1 = 3$, $\lambda_{3T} = 0$ then $\lambda_T = (\lambda_{1T}, 2, 0)$. So we apply the Mullineux Bijection to;



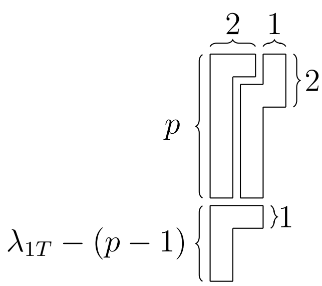
Which becomes;



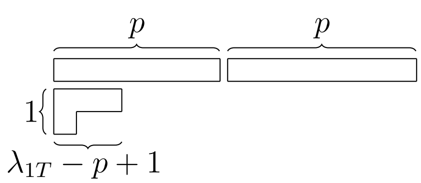
as $a_1-r_1+\epsilon_1 = (p+\lambda_{1T}-p+1)-(\lambda_{1T})+1 = 2$ and $a_2-r_2+\epsilon_2 = p-(p-1)+0 = 1$. Then $I(\lambda_{1T}, \lambda_{3T}+p-1, \lambda_{2T}) \cong T(2p-1, \lambda_{1T}-(p-1), 0) = T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ as required.

And so on for $\lambda_{3T}+p-1 = p-1$ and $\lambda_{2T}+1 = \alpha$, so with $\lambda_T = (\lambda_{1T}, \alpha-1, 0)$.

ii)
• For $\lambda_{3T}+p-1 = p,\ \lambda_{2T}+1 = 1$ and $\lambda_{3T} = 1$ then $\lambda_T = (\lambda_{1T}, 1, 1)$, so we apply the Mullineux Bijection to

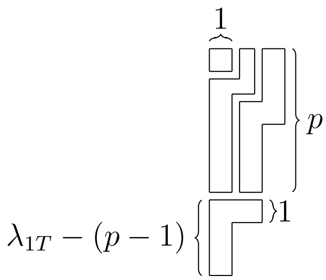


Which becomes;

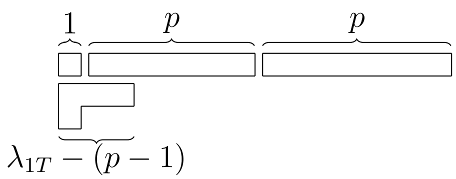


as $a_1 - r_1 + \epsilon_1 = (p + \lambda_{1T} - p + 1 + 1) - (\lambda_{1T}) + 1 = 3$ and $a_2 - r_2 + \epsilon_2 = p - (p-1) + 0 = 1$.
So $I(\lambda_{1T}, \lambda_{3T}+p-1, \lambda_{2T}+1) \cong T(2p, \lambda_{1T}-(p-1), 1) = T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ as required.

• For $\lambda_{3T}+p-1 = p,\ \lambda_{2T}+1 = 3$ and $\lambda_{3T} = 1$ then $\lambda_T = (\lambda_{1T}, 2, 1)$, and we apply the Mullineux Bijection to;



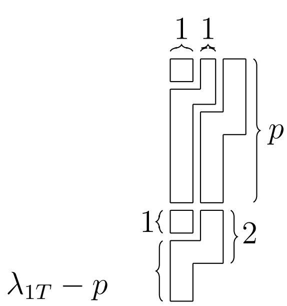
Which becomes;



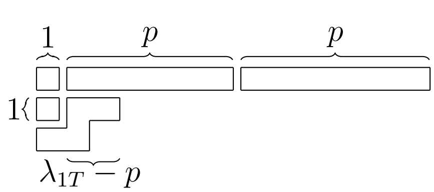
as $a_1 - r_1 + \epsilon_1 = (p + \lambda_{1T} - p + 1 + 1) - (\lambda_{1T}) + 1 = 3$ and $a_2 - r_2 + \epsilon_2 = p - (p-1) + 0 = 1$.
So $I(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1) \cong T(2p+1, \lambda_{1T} - (p-1), 1) = T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ as required.

And so on for $\lambda_{3T} + p - 1 = p$, $\lambda_{2T} + 1 = \alpha$ and $\lambda_{3T} = 1$, so with $(\lambda_{1T}, \alpha - 1, 1)$.

iii) For $\lambda_{3T} + p - 1 = p + 1$, $\lambda_{2T} + 1 = 3$ and $\lambda_{3T} = 2$, so we apply the Mullineux Bijection to;



which becomes;



as $a_1 - r_1 + \epsilon_1 = (p + \lambda_{1T} - p + 2) - (\lambda_{1T}) + 1 = 3$ and $a_2 - r_2 + \epsilon_2 = p + 1 - (p) + 1 = 2$. So $I(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1) \cong T(2p+1, \lambda_{1T} - p + 1, 2) = T(\lambda_{21}, \lambda_{22}, \lambda_{23})$

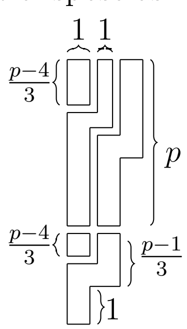
iv) To conclude this we need to understand what is the most λ_{3T} can be.

CLAIM 4.13.9 a) For $3 \mid p-1$, $0 \leq \lambda_{3T} \leq \frac{p-1}{3}$,
b) For $3 \mid p-2$, $0 \leq \lambda_{3T} \leq \frac{p-2}{3}$.

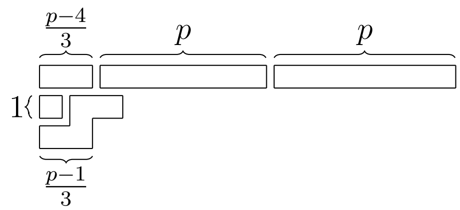
of Claim 4.13.9.
a) If $\lambda_{3T} = \frac{p-1}{3}$ then $\lambda_{2T} \geq \frac{p-1}{3}$ and as $\lambda_{1T} - \lambda_{2T} \geq p$ then $\lambda_{1T} \geq \frac{p-1}{3} + p$ and thus $\lambda_{1T} + \lambda_{2T} + \lambda_{3T} \geq 2p - 1$, so taking equalities throughout we have the largest r can be as $p \leq r \leq 2p - 1$.
On the other hand, if we took $\lambda_{3T} = \frac{p-1}{3} + 1$ then $\lambda_{2T} \geq \frac{p-1}{3} + 1$ and $\lambda_{1T} \geq \frac{p-1}{3} + 1 + p$ and therefore $\lambda_{1T} + \lambda_{2T} + \lambda_{3T} \geq 2p + 2$ which is a contradiction.

b) The proof is similar for $3 \mid p-2$. If $\lambda_{3T} = \frac{p-2}{3}$ then $\lambda_{2T} \geq \frac{p-2}{3}$ and as $\lambda_{1T} - \lambda_{2T} \geq p$ then $\lambda_{1T} \geq \frac{p-2}{3} + p$ and thus $\lambda_{1T} + \lambda_{2T} + \lambda_{3T} \geq 2p - 2$, so taking equalities we have the largest r can be.
On the other hand, if we took $\lambda_{3T} = \frac{p-2}{3} + 1$ then $\lambda_{2T} \geq \frac{p-2}{3} + 1$ and $\lambda_{1T} \geq \frac{p-2}{3} + 1 + p$ and therefore $\lambda_{1T} + \lambda_{2T} + \lambda_{3T} \geq 2p + 1$ which is a contradiction. □

So, returning to the proof of Claim 4.13.8, we have that the final case to consider is
a) $\lambda_{3T} + p - 1 = \frac{4p-4}{3}$ and $\lambda_{2T} + 1 = \frac{p+2}{3}$ so $r = 2p - 1$ and we have $(\frac{4p-1}{3}, \frac{4p-4}{3}, \frac{p+2}{3})$, then we can then remove Mullineux rim p-hooks from the transpose as follows;



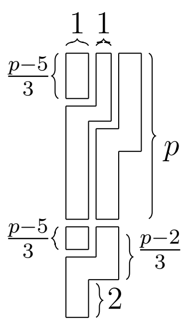
Then as $s_1 = (p + \frac{p-4}{3}) - (p + \frac{p-7}{3}) + 1 = 2$ and $s_2 = (p + \frac{p+5}{3}) - (p + \frac{p-1}{3}) + 1 = 3$ then rebuilding the partition we have



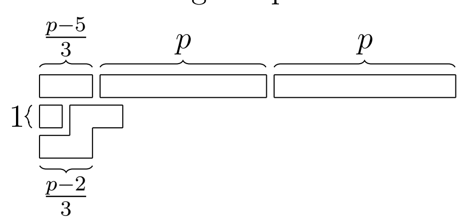
giving

$$\begin{aligned} I(\lambda_{1T}, \lambda_{3T}+p-1, \lambda_{2T}+1) &\cong T(2p+\tfrac{p-4}{3}, \tfrac{p-1}{3}+1, \tfrac{p-1}{3}) \\ &= T(\tfrac{p-1}{3}+2p-1, \tfrac{4p-1}{3}-p+1, \tfrac{p-1}{3}) \\ &= T(\lambda_{2T}+2p-1, \lambda_{1T}-p+1, \lambda_{3T}) \end{aligned}$$

b) $\lambda_{3T}+p-1 = \frac{4p-5}{3}$ and $\lambda_{2T}+1 = \frac{p+1}{3}$ so $r = 2p-1$ and we have $(\frac{4p+1}{3}, \frac{4p-5}{3}, \frac{p+1}{3})$. Then we can then remove Mullineux rim p-hooks from the transpose as follows;



Then as $s_1 = (p+\frac{p-5}{3})-(p+\frac{p-8}{3})+1 = 2$ and $s_2 = (p+\frac{p+4}{3})-(p+\frac{p+1}{3})+1 = 3$ then rebuilding the partition we have



giving

$$\begin{aligned} I(\lambda_{1T}, \lambda_{3T}+p-1, \lambda_{2T}+1) &\cong T(2p+\tfrac{p-6}{3}, \tfrac{p-2}{3}+1, \tfrac{p-2}{3}) \\ &= T(\tfrac{p-2}{3}+2p-1, \tfrac{4p-2}{3}-p+1, \tfrac{p-2}{3}) \\ &= T(\lambda_{2T}+2p-1, \lambda_{1T}-p+1, \lambda_{3T}) \end{aligned}$$

Thus $I(\lambda_{51}, \lambda_{52}, \lambda_{53}) \cong T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ and thus we know the tilting module $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ has filtration structure

$$\begin{array}{c} \nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \hline \vdots \\ \hline \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53}) \end{array}$$

Secondly we need to show that $\nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$ is a composition factor of $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$. Well, $T(\lambda_{31}, \lambda_{32}, \lambda_{33}) = (\lambda_{3T}+2p-2, \lambda_{1T}-p+1, \lambda_{2T}+1)$

and $\nabla(\lambda_{51}, \lambda_{52}, \lambda_{53}) = (\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1)$.
The $(T(\lambda_{31}, \lambda_{32}, \lambda_{33}) : \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53}))$
$= (T(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1) : \nabla(\lambda_{1T}, \lambda_{3T} + p - 1, \lambda_{2T} + 1))$
$= (T(\lambda_{3T}+2p-\lambda_{2T}-3, \lambda_{1T}-p-\lambda_{2T}, 0) : \nabla(\lambda_{1T}-\lambda_{2T}-1, \lambda_{3T}+p-\lambda_{2T}-2, 0))_{GL_3}$
$= (T(\lambda_{3T}-2p-\lambda_{2T}-3, \lambda_{1T}-p-\lambda_{2T}) : \nabla(\lambda_{1T}-\lambda_{2T}-1, \lambda_{3T}+p-\lambda_{2T}-2))_{GL_2}$
$= (T(\lambda_{3T}+2p-\lambda_{2T}-3-(\lambda_{1T}-p-\lambda_{2T})) : \nabla(\lambda_{1T}-\lambda_{2T}-1-(\lambda_{3T}+p-\lambda_{2T}-2)))_{SL_2}$
$= (T(\lambda_{3T} - \lambda_{1T} + 3p - 3) : \nabla(\lambda_{1T} - \lambda_{3T} - p + 1))_{SL_2}$

And from SL_2 we know $T(p-1+r)$ has filtration structure

$$\frac{\nabla(p-1+r)}{\nabla(p-1-r)}$$

Here $T(p-1+r) = T(p-1+(\lambda_{3T} - \lambda_{1T} + 2p - 2))$ and thus $T(p-1+r)$ has filtration structure

$$\begin{array}{c}\nabla(\lambda_{3T} - \lambda_{1T} + 3p - 3)\\ \nabla(\lambda_{1T} - \lambda_{3T} - p + 1)\end{array}$$

Therefore $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$ has filtration structure

$$\begin{array}{c}\nabla(\lambda_{31}, \lambda_{32}, \lambda_{33})\\ \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})\end{array}$$

and indeed $\nabla(\sigma_1, \sigma_2, \sigma_3) = \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$ □

CLAIM 4.13.10 The tilting module whose weight is second highest in the core class of the 6-set, $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$, has filtration structure

$$\begin{array}{c}\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23})\\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33})\\ \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43})\\ \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})\end{array}$$

Proof. We have $\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23})$ naturally at the top and Claim 4.13.8 proved that we have $\nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$ at the bottom. So we need to show that $(T(\lambda_{21}, \lambda_{22}, \lambda_{23}) : \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33})) \neq 0$ and $(T(\lambda_{21}, \lambda_{22}, \lambda_{23}) : \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43})) \neq 0$.

Well,
$(T(\lambda_{21}, \lambda_{22}, \lambda_{23}) : \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}))$

$= (T(\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T}) : \nabla(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1))_{GL_3}$
$= (T(\lambda_{2T} + 2p - 1 - \lambda_{1T} + p - 1, \lambda_{1T} - p + 1 - \lambda_{3T}) : \nabla(\lambda_{3T} + 2p - 2 - \lambda_{1T} + p - 1, \lambda_{1T} - p + 1 - \lambda_{2T} - 1))_{SL_3}$
$= (T(\lambda_{2T} - \lambda_{1T} + 3p - 2, \lambda_{1T} - \lambda_{3T} - p + 1) : \nabla(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p))_{SL_3}$
$= (T(\lambda_{2T} - \lambda_{1T} + 3p - 2 - \lambda_{1T} + \lambda_{3T} + p - 1) : \nabla(\lambda_{3T} - \lambda_{1T} + 3p - 3 - \lambda_{1T} + \lambda_{2T} + p))_{SL_2}$
$= (T(\lambda_{2T} + \lambda_{3T} - 2\lambda_{1T} + 4p - 3) : \nabla(\lambda_{2T} + \lambda_{3T} - 2\lambda_{1T} + 4p - 3))_{SL_2}$
$= 1$

Similarly,
$(T(\lambda_{21}, \lambda_{22}, \lambda_{23}) : \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43}))$
$= (T(\lambda_{2T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{3T}) : \nabla(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T}))_{GL_3}$
$= (T(\lambda_{2T} + 2p - 1 - \lambda_{3T}, \lambda_{1T} - p + 1 - \lambda_{3T}) : \nabla(\lambda_{1T} - \lambda_{3T}, \lambda_{2T} + p - \lambda_{3T}))_{GL_2}$
$= (T(\lambda_{2T} + 2p - 1 - \lambda_{3T} - \lambda_{1T} + p - 1 + \lambda_{3T}) : \nabla(\lambda_{1T} - \lambda_{3T} - \lambda_{2T} - p + \lambda_{3T}))_{SL_2}$
$= (T(\lambda_{2T} + 3p - 2 - \lambda_{1T}) : \nabla(\lambda_{1T} - \lambda_{2T} - p))_{SL_2}$

And $T(p - 1 + r) = T(p - 1 + (\lambda_{2T} + 2p - 1 - \lambda_{1T}))$ has filtration structure

$$\begin{gathered}\nabla(p - 1 + (\lambda_{2T} + 2p - 1 - \lambda_{1T}) \\ \nabla(p - 1 - (\lambda_{2T} + 2p - 1 - \lambda_{1T}))\end{gathered}$$

which we can write

$$\begin{gathered}\nabla(\lambda_{2T} + 3p - 2 - \lambda_{1T}) \\ \nabla(\lambda_{1T} - \lambda_{2T} - p).\end{gathered}$$

Thus $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ has filtration structure

$$\begin{gathered}\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43}) \\ \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})\end{gathered}$$

□

CLAIM 4.13.11 We can actually write the filtration structure of $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ as

$$\begin{gathered}\nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43}) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})\end{gathered}$$

Proof. Consider $T_0 = 0$, $T_1 \cong \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})$, $T_2/T_1 \cong \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43})$, $T_3/T_2 \cong \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33})$ and $T_4/T_3 \cong \nabla(\lambda_{21}, \lambda_{22}, \lambda_{23})$.
Then

$$\begin{array}{rcl} T_3/T_1 & \cong & \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ & & \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43}) \\ & & \\ & = & \nabla(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1) \\ & & \nabla(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T}) \end{array}$$

Now suppose $\lambda_{1T} - p + 1 \geq \lambda_{2T} + p \Rightarrow \lambda_{1T} - \lambda_{2T} \geq 2p - 1$. As $\lambda_{1T} + \lambda_{2T} + \lambda_{3T} = r$ for $p \leq r \leq 2p - 1$ then $\lambda_{1T} - \lambda_{2T} \not> 2p - 1$. So suppose $\lambda_{1T} - \lambda_{2T} = 2p - 1$, then $\lambda_T = (2p - 1, 0, 0)$ and $(\lambda_{1T} + p, \lambda_{2T}, \lambda_{3T}) = (3p - 1, 0, 0) > (t, 0, 0)$ so referring back to our classification of core classes, for $r = 2p - 1$ we have the core class
$(2p - 1, 0, 0)$
$(p - 1, p - 1, 0)$.

So, when finding the new partitions for $r = 3p - 1$ with p-core $(p - 1, 0, 0)$ we have only
$(2p - 1, p, 0)$
$(2p - 1, p - 1, 1)$
$(2p - 1, p, 1)$

and so in this case $(\lambda_{31}, \lambda_{32}, \lambda_{33})$ actually sits at the bottom of its core class, as we have only a 3-set, and thus is not relevant to the case we are at the moment considering. We can therefore assume that $\lambda_{1T} - \lambda_{2T} \neq 2p - 1$ and so $\lambda_{1T} - p + 1 < \lambda_{2T} + p$. However, as $\lambda_{2T} \geq \lambda_{3T}$ then $\lambda_{2T} + 1 > \lambda_{3T}$. Hence we in fact have $T_3/T_1 \cong \nabla(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1) \oplus \nabla(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T})$ and so we can write

$$\begin{array}{rcl} T_3/T_1 & \cong & \nabla(\lambda_{1T}, \lambda_{2T} + p, \lambda_{3T}) \\ & & \nabla(\lambda_{3T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{2T} + 1) \\ & & \\ & = & \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43}) \\ & & \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \end{array}$$

And so we have that $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ has filtration structure

$$\begin{array}{c} \nabla(\lambda_{21}, \lambda_{22}, \lambda_{23}) \\ \nabla(\lambda_{41}, \lambda_{42}, \lambda_{43}) \\ \nabla(\lambda_{31}, \lambda_{32}, \lambda_{33}) \\ \nabla(\lambda_{51}, \lambda_{52}, \lambda_{53}) \end{array}$$

□

CLAIM 4.13.12

$$\frac{\nabla(\lambda_{31},\lambda_{32},\lambda_{33})}{\nabla(\lambda_{51},\lambda_{52},\lambda_{53})}$$

is a non-split extension at the bottom of $T(\lambda_{21},\lambda_{22},\lambda_{23})$ and thus must be the tilting module $T(\lambda_{31},\lambda_{32},\lambda_{33})$.

Proof. 1) Consider the case where $\lambda_{32}=\lambda_{33}$. We have $\mathrm{Tr}^{\lambda}E=\oplus I(\mu)^{(d_\mu)}$ where $d_\mu=\dim\ \mathrm{Hom}_G(L(\mu),\mathrm{Tr}^{\lambda}E)=\dim\ L(\mu)^{\lambda}$. Then taking λ to be the lowest weight for each degree r then we go back to our three cases:

i) $3|r$
Then $\lambda_T=(\frac{r}{3},\frac{r}{3},\frac{r}{3})$ and so
$\dim\ L(\lambda_{31},\lambda_{32},\lambda_{33})^{(\frac{r}{3},\frac{r}{3},\frac{r}{3})}$
$=\dim\ L(\lambda_{3T}+2p-1,\lambda_{1T}-p+1,\lambda_{1T}-p+1)^{(\eta,\eta,\eta)}$
$=\dim\ L(\lambda_{3T}+2p-2-\lambda_{1T}+p-1,0,0)^{(\zeta,\zeta,\zeta)}$
$=\dim\ L(\lambda_{3T}-\lambda_{1T}+3p-3,0)^{(0,0)}$

where $\eta=\frac{\lambda_{1T}+\lambda_{2T}+\lambda_{3T}+p}{3}$ and $\zeta=\frac{\lambda_{2T}+\lambda_{3T}-2\lambda_{1T}+4p-3}{3}$.
Now, we have $p\le r\le 2p-3 \Rightarrow p\le\lambda_{1T}-\lambda_{2T}\le 2p-3 \Rightarrow p\le \lambda_{1T}-\lambda_{3T}\le 2p-3$ as $\lambda_{2T}\ge\lambda_{3T}$. So, $\lambda_{1T}-\lambda_{3T}\le 2p-3 \Rightarrow \lambda_{3T}-\lambda_{1T}\ge 3-2p \Rightarrow \lambda_{3T}-\lambda_{1T}+3p-3\ge p$. Hence $L(\lambda_{3T}-\lambda_{1T}+3p-3,0)=L(\lambda_{3T}-\lambda_{1T}+2p-3,0)\otimes L(1,0)^F$.
$L_0(1,0)=E\downarrow_{SL_3}$ and has GL weights $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ thus giving SL weights $(1,0)$, $(-1,1)$, $(0,-1)$ and so $L_0(1,0)^F$ has weights $(p,0)$, $(-p,p)$ and $(0,-p)$.
Now, suppose (α_1,α_2) is a weight of $L(\lambda_{3T}-\lambda_{1T}+2p-3,0)$ then $L(\lambda_{3T}-\lambda_{1T}+3p-3)$ has weights $(\alpha_1,\alpha_2)+(p,0)$, $(\alpha_1,\alpha_2)+(-p,p)$ and $(\alpha_1,\alpha_2)+(0,-p)$.
Suppose
$(\alpha_1,\alpha_2)+(p,0)=(0,0)$ then $(\alpha_1,\alpha_2)=(-p,0)$
$(\alpha_1,\alpha_2)+(-p,p)=(0,0)$ then $(\alpha_1,\alpha_2)=(p,-p)$
$(\alpha_1,\alpha_2)+(0,-p)=(0,0)$ then $(\alpha_1,\alpha_2)=(0,p)$
This would imply $L(\lambda_{3T}-\lambda_{1T}+2p-3,0)$ has a weight of at least p. So suppose $\lambda_{3T}-\lambda_{1T}+2p-3\ge p \Rightarrow \lambda_{3T}-\lambda_{1T}\ge -p+3 \Rightarrow \lambda_{1T}-\lambda_{3T}\le p-3$ which is a contradiction as $p\le\lambda_1-\lambda_3\le 2p-3$.
Thus $L(\lambda_{3T}-\lambda_{1T}+3p-3,0)$ cannot have a weight $(0,0)$, so $\dim L(\lambda_{31},\lambda_{32},\lambda_{33})^{(\frac{r}{3},\frac{r}{3},\frac{r}{3})}=0$ and thus $L(\lambda_{31},\lambda_{32},\lambda_{33})$ does not occur in the socle of $\bar{S}^{\frac{r}{3}}E\otimes\bar{S}^{\frac{r}{3}}E\otimes\bar{S}^{\frac{r}{3}}E$ and so $\dim\ \mathrm{Hom}_G(L(\lambda_{31},\lambda_{32},\lambda_{33}),\bar{S}^{\frac{r}{3}}\otimes\bar{S}^{\frac{r}{3}}E\otimes\bar{S}^{\frac{r}{3}}E)=0$. Therefore $\nabla(\lambda_{31},\lambda_{32},\lambda_{33})$ does not embed in $\bar{S}^{\frac{r}{3}}\otimes\bar{S}^{\frac{r}{3}}E\otimes\bar{S}^{\frac{r}{3}}E$ and so there is a non-split extension

$$\frac{\nabla(\lambda_{31},\lambda_{32},\lambda_{33})}{\nabla(\lambda_{51},\lambda_{52},\lambda_{53})}$$

at the bottom of $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ which must be the tilting module $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$.

Hence cf $(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq$ cf $(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq$ cf $(\bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E) \subseteq D_{3,p}(r)$.

ii) $3|r+1$
The same method occurs, except here $\lambda_T = (\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3} - 1)$. So
dim $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3}-1)}$
$=$ dim $L(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{1T} - p + 1)^{(\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3}-1)}$
$=$ dim $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, 0)^{(0,1)}$.
So suppose
$(\alpha_1, \alpha_2) + (p, 0) = (0, 1)$ then $(\alpha_1, \alpha_2) = (-p, 1)$
$(\alpha_1, \alpha_2) + (-p, p) = (0, 1)$ then $(\alpha_1, \alpha_2) = (p, -p + 1)$
$(\alpha_1, \alpha_2) + (0, -p) = (0, 1)$ then $(\alpha_1, \alpha_2) = (0, p + 1)$.
So again $L(\lambda_{3T} - \lambda_{1T} + 2p - 3, 0)$ would need to have weight of at least p, which we know is not possible. Hence $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3}-1)} = 0$ and using the same method as for $3|\ r$ then cf $(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq$ cf $(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq$ cf $(\bar{S}^{\frac{r+1}{3}} \otimes \bar{S}^{\frac{r+1}{3}}E \otimes \bar{S}^{\frac{r+1}{3}-1}E) \subseteq D_{3,p}(r)$.

iii) $3|r+2$
Here $\lambda_T = (\frac{r+2}{3}, \frac{r+2}{3} - 1, \frac{r+2}{3} - 1)$. So
dim $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+2}{3}, \frac{r+2}{3}-1, \frac{r+2}{3}-1)} =$ dim $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, 0)^{(1,0)}$.
So suppose
$(\alpha_1, \alpha_2) + (p, 0) = (1, 0)$ then $(\alpha_1, \alpha_2) = (1 - p, 0)$
$(\alpha_1, \alpha_2) + (-p, p) = (1, 0)$ then $(\alpha_1, \alpha_2) = (p + 1, -p)$
$(\alpha_1, \alpha_2) + (0, -p) = (1, 0)$ then $(\alpha_1, \alpha_2) = (1, p)$.
So again $L(\lambda_{3T} - \lambda_{1T} + 2p - 3, 0)$ would need to have weight of at least p, which we know is not possible. Hence $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+2}{3}, \frac{r+2}{3}-1, \frac{r+2}{3}-1)} = 0$ and so cf $(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq$ cf $(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq$ cf $(\bar{S}^{\frac{r+2}{3}} \otimes \bar{S}^{\frac{r+2}{3}-1}E \otimes \bar{S}^{\frac{r+2}{3}-1}E) \subseteq D_{3,p}(r)$.

2) Consider the case where $\lambda_{32} > \lambda_{33}$
i) $3|r$
Then $\lambda_T = (\frac{r}{3}, \frac{r}{3}, \frac{r}{3})$ and so dim $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r}{3}, \frac{r}{3}, \frac{r}{3})}$
$=$ dim $L(\lambda_{3T} + 2p - 1, \lambda_{1T} - p + 1, \lambda_{2T} + 1)^{(\frac{\sum_{i=1}^{3}\lambda_{iT}+p}{3}, \frac{\sum_{i=1}^{3}\lambda_{iT}+p}{3}, \frac{\sum_{i=1}^{3}\lambda_{iT}+p}{3})}$
$=$ dim $L(\lambda_{3T} + 2p - 2 - \lambda_{1T} + p - 1, \lambda_{1T} - p + 1 - \lambda_{2T} + 1)^{(0,0)}$
$=$ dim $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p)^{(0,0)}$

Now, $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p)$
$= L(\lambda_{3T} - \lambda_{1T} + 2p - 3, \lambda_{1T} - \lambda_{2T} - p) \otimes L(1, 0)^F$, which has weights $(p, 0)$, $(-p, p)$ and $(0, -p)$.

Now, suppose (α_1, α_2) is a weight of $L(\lambda_{3T} - \lambda_{1T} + 2p - 3, \lambda_{1T} - \lambda_{2T} - p)$ then $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p)$ has weights $(\alpha_1, \alpha_2) + (p, 0)$, $(\alpha_1, \alpha_2) + (-p, p)$ and $(\alpha_1, \alpha_2) + (0, -p)$.
Suppose
$(\alpha_1, \alpha_2) + (p, 0) = (0, 0)$ then $(\alpha_1, \alpha_2) = (-p, 0)$
$(\alpha_1, \alpha_2) + (-p, p) = (0, 0)$ then $(\alpha_1, \alpha_2) = (p, -p)$
$(\alpha_1, \alpha_2) + (0, -p) = (0, 0)$ then $(\alpha_1, \alpha_2) = (0, p)$
This would imply $L(\lambda_{3T} - \lambda_{1T} + 2p - 3, \lambda_{1T} - \lambda_{2T} - p)$ has a weight of at least p. Now, we know that $\lambda_{3T} - \lambda_{1T} + 2p - 3 < p$, so what about $\lambda_{1T} - \lambda_{2T} - p$? Well $\lambda_{1T} - \lambda_{2T} - p \geq p$ if and only if $\lambda_{1T} \geq 2p + \lambda_{2T}$. However we know that $\lambda_{1T} \leq 2p - 4$ and thus we have a contradiction.
Thus $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p)$ cannot have a weight $(0, 0)$, so dim $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r}{3}, \frac{r}{3}, \frac{r}{3})} = 0$ and thus $L(\lambda_{31}, \lambda_{32}, \lambda_{33})$ does not occur in the socle of $\bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E$ and so dim $\mathrm{Hom}_G(L(\lambda_{31}, \lambda_{32}, \lambda_{33}), \bar{S}^{\frac{r}{3}} \otimes \bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E)$. Therefore $\nabla(\lambda_{31}, \lambda_{32}, \lambda_{33})$ does not embed in $\bar{S}^{\frac{r}{3}} \otimes \bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E$ and so there is a non-split extension

$$\frac{\nabla(\lambda_{31}, \lambda_{32}, \lambda_{33})}{\nabla(\lambda_{51}, \lambda_{52}, \lambda_{53})}$$

at the bottom of $T(\lambda_{21}, \lambda_{22}, \lambda_{23})$ which must be the tilting module $T(\lambda_{31}, \lambda_{32}, \lambda_{33})$.

Hence cf $(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq$ cf $(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq$ cf $(\bar{S}^{\frac{r}{3}} \otimes \bar{S}^{\frac{r}{3}}E \otimes \bar{S}^{\frac{r}{3}}E) \subseteq D_{3,p}(r)$.

ii) $3|r + 1$
The same method occurs, except here $\lambda_T = (\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3} - 1)$. So
dim $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3} - 1)}$
$=$ dim $L(\lambda_{3T} + 2p - 2, \lambda_{1T} - p + 1, \lambda_{2T} + 1)^{(\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3} - 1)}$
$=$ dim $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p)^{(0,1)}$.
So suppose
$(\alpha_1, \alpha_2) + (p, 0) = (0, 1) \;\Rightarrow\; (\alpha_1, \alpha_2) = (-p, 1)$
$(\alpha_1, \alpha_2) + (-p, p) = (0, 1) \;\Rightarrow\; (\alpha_1, \alpha_2) = (p, -p + 1)$
$(\alpha_1, \alpha_2) + (0, -p) = (0, 1) \;\Rightarrow\; (\alpha_1, \alpha_2) = (0, p + 1)$.
So again $L(\lambda_{3T} - \lambda_{1T} + 2p - 3, \lambda_{1T} - \lambda_{2T} - p)$ would need to have weight of at least p, which we know is not possible.
Hence $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+1}{3}, \frac{r+1}{3}, \frac{r+1}{3} - 1)} = 0$ and using the same method as for $3|r$ then cf $(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq$ cf $(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq$ cf $(\bar{S}^{\frac{r+1}{3}} \bar{S}^{\frac{r+1}{3}}E \otimes \bar{S}^{\frac{r+1}{3} - 1}E) \subseteq D_{3,p}(r)$.

iii) $3|r + 2$
Here $\lambda_T = (\frac{r+2}{3}, \frac{r+2}{3} - 1, \frac{r+2}{3} - 1)$. So dim $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+2}{3}, \frac{r+2}{3} - 1, \frac{r+2}{3} - 1)} =$

dim $L(\lambda_{3T} - \lambda_{1T} + 3p - 3, \lambda_{1T} - \lambda_{2T} - p)^{(1,0)}$.
So suppose
$(\alpha_1, \alpha_2) + (p, 0) = (1, 0) \;\Rightarrow\; (\alpha_1, \alpha_2) = (1 - p, 0)$
$(\alpha_1, \alpha_2) + (-p, p) = (1, 0) \;\Rightarrow\; (\alpha_1, \alpha_2) = (p + 1, -p)$
$(\alpha_1, \alpha_2) + (0, -p) = (1, 0) \;\Rightarrow\; (\alpha_1, \alpha_2) = (1, p)$.
So again $L(\lambda_{3T} - \lambda_{1T} + 2p - 3, 0)$ would need to have weight of at least p, which we know is not possible.
Hence $L(\lambda_{31}, \lambda_{32}, \lambda_{33})^{(\frac{r+2}{3}, \frac{r+2}{3}-1, \frac{r+2}{3}-1)} = 0$ and so
$\mathrm{cf}\,(T(\lambda_{31}, \lambda_{32}, \lambda_{33})) \subseteq \mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(\bar{S}^{\frac{r+2}{3}} \bar{S}^{\frac{r+2}{3}-1} E \otimes \bar{S}^{\frac{r+2}{3}-1} E)$
$\subseteq D_{3,p}(r)$. □

Thus in all cases we have that the coefficient space of the third highest tilting module in the core class of the 6-set for $2p \leq r \leq 3p$ is contained in the coefficient space of the lowest weighted tilting truncated module which is itself a subset of $D_{3,p}(r)$. □

iv) We now consider those tilting modules whose weights are fourth and fifth highest in the core class which is a 6-set, that which we have notated $(\lambda_{41}, \lambda_{42}, \lambda_{43})$ and $(\lambda_{51}, \lambda_{52}, \lambda_{53})$.

THEOREM 4.13.13 *The* cf $(T(\lambda_{41}, \lambda_{42}, \lambda_{43}))$ *and* cf $(T(\lambda_{51}, \lambda_{52}, \lambda_{53}))$ *are contained in* cf $(L(p-1, p-1, 0) \otimes S^{\frac{r-(2p-2)}{2}} E \otimes S^{\frac{r-(2p-2)}{2}} E) \subseteq D_{3,p}(r)$ *for r even.*
The cf $(T(\lambda_{41}, \lambda_{42}, \lambda_{43}))$ *and* cf $(T(\lambda_{51}, \lambda_{52}, \lambda_{53}))$ *are contained in*
cf $(L(p-1, p-1, 0) \otimes S^{\frac{r-(2p-2)+1}{2}} E \otimes S^{\frac{r-(2p-2)-1}{2}} E) \subseteq D_{3,p}(r)$ *for r odd.*

The case where r is even

EXAMPLE 4.13.14 Let $p = 7$ and $r = 18$, then we calculate

$$\begin{aligned}
\mathrm{ch}\,(\mathrm{Tr}^{\,12,3,3} E) &= \mathrm{ch}\,(L(6,6,0) \otimes S^3 E \otimes S^3 E) \\
&= (s_{66} \cdot s_3) \cdot s_3 \\
&= (s_{960} + s_{861} + s_{762} + s_{663}) s_3 \\
&= (s_{12,6} + s_{11,7} + s_{11,6,1} + s_{10,8} + s_{10,7,1} + s_{10,6,2} + s_{99} + s_{981} \\
&\quad + s_{972} + s_{963}) + (s_{11,6,1} + s_{10,7,1} + s_{10,6,2} + s_{981} + s_{972} + \\
&\quad s_{963} + s_{882} + s_{873} + s_{864}) + (s_{10,6,2} + s_{972} + s_{963} + s_{873} + \\
&\quad s_{864} + s_{774} + s_{765}) + (s_{963} + s_{864} + s_{765} + s_{666})
\end{aligned}$$

We can display this result, with multiplicities, as follows;

$1s_{12,6}$					
$1s_{11,7}$	$2s_{11,6,1}$				
$1s_{10,8}$	$2s_{10,7,1}$	$3s_{10,6,2}$			
$1s_{99}$	$2s_{981}$	$3s_{972}$	$4s_{963}$		
		$1s_{882}$	$2s_{873}$	$3s_{864}$	
				$1s_{774}$	$2s_{765}$
					$1s_{666}$

Proof. By 4.10 c), when proving this, it is enough to show that the Schur function corresponding to the weights which are fourth highest in the core class of the 6-set arise at least once in the character of the above truncated module, and that no Schur function corresponding to a weight which is higher in the same core class, also arises.

So let us first consider what a typical term s_σ looks like when it arises in the $\mathrm{ch}\,(L(p-1,p-1,0)\otimes S^\alpha E\otimes S^\alpha E)$, where $\alpha=\frac{r_2-(2p-2)}{2}$ and $2p\leq r_2\leq 3p-3$. Well if $\sigma=(\sigma_1,\sigma_2,\sigma_3)$, then using the Littlewood-Richardson Rule we have that $p-1\leq\sigma_1\leq p-1+2\alpha$, $p-1\leq\sigma_2\leq p-1+\alpha$ and $0\leq\sigma_3\leq 2\alpha$, where $\sigma_1+\sigma_2+\sigma_3=2p-2+2\alpha=r_2$. Now, let us consider a weight λ which is fourth highest in the core class of the 6-set. By Section 4.7 we know that $\lambda=(\lambda_1,\lambda_2+p,\lambda_3)$ where $(\lambda_1,\lambda_2,\lambda_3)$ was the highest weight in the core class of the 3-set for the range $p\leq r_1\leq 2p-1$. Now, clearly, we have that $\lambda_1+(\lambda_2+p)+\lambda_3=r_2$, so we need to check the following;

1) Is $p-1\leq\lambda_1\leq p-1+2\alpha$?
Well, clearly $\lambda_1\geq p$ as $p\leq r_1\leq 2p-3$, so indeed $\lambda_1\geq p-1$.
Now as $\lambda_1\leq r_1$, then it is enough to check that $r_1\leq p-1+2\alpha$ which is true if and only if $r_1\leq 2p-2$, which we know to be true as we are considering the range where $\lambda_1\leq 2p-3$.

2) Is $p-1\leq\lambda_2+p\leq p-1+\alpha$?
Well, $\lambda_2\geq 0$ and thus $\lambda_2+p\geq p>p-1$ as required.
So, is $\lambda_2+p\leq p-1+\alpha$?. Well this is true if and only if $\lambda_2\leq\frac{p-3}{2}$, so suppose for a contradiction that $\lambda_2\geq\frac{p-1}{2}$. As $\lambda_1-\lambda_2\geq p$, this would mean that $\lambda_1\geq\frac{3p-1}{2}$, and then we would have that $\lambda_1+\lambda_2\geq 2p-1$ which is a contradiction as $\lambda_1+\lambda_2+\lambda_3\leq 2p-3$, so indeed $\lambda_2+p\leq p-1+\alpha$.

3) Is $0\leq\lambda_3\leq 2\alpha$?
Clearly $\lambda_3\geq 0$. Now suppose for a contradiction that $\lambda_3\geq 2\alpha$ this holds if an only if $\lambda_2\geq 2\alpha$ and thus to ensure $\lambda_1-\lambda_2\geq p$ we would require $\lambda_1+\lambda_2+\lambda_3>r_1$, giving a contradiction, and thus $\lambda_3\leq p-1$.

We therefore have that all the Schur functions which correspond to the highest weight of the tilting modules whose weights are fourth highest in the core class of the six, will arise, at least once, in $\mathrm{ch}\,(L(p-1,p-1,0)\otimes S^{\alpha}E\otimes S^{\alpha}E)$. To ensure, then that the coefficient spaces of these tilting modules are contained in the coefficient space of this truncated module, we must check that no Schur functions corresponding to higher weights in the same core class also arise. Well, let us look at the top weight in the 6-set, namely $(\lambda_1+p,\lambda_2,\lambda_3)$. Note that $\lambda_2 < p-1$, as if $\lambda_2 \geq p$ then we would require $\lambda_1 \geq 2p$ and thus $r_1 \geq 3p$ which is not true. Therefore the Schur function corresponding to this weight cannot arise in the $\mathrm{ch}\,(L(p-1,p-1,0)\otimes S^{\alpha}E\otimes S^{\alpha}E)$, as we already have that for a typical term $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ in this character, $\sigma_2 \geq p-1$. For the same reason, the second and third weights cannot arise, these both have second entry λ_1-p+1 which, as $\lambda_1 \leq 2p-3$, then $\lambda_1-p+1 \leq p-2 < p-1$.

We now consider those weights which are fifth highest in the same core class, namely those of the form $(\lambda_1,\lambda_3+p-1,\lambda_2+1)$. Let us first show that the Schur function corresponding to this weight will actually arise, well, we have already clarified that $p-1 \leq \lambda_1 \leq p-1+2\alpha$. Next we must check that $p-1 \leq \lambda_3+p-1 \leq p-1+\alpha$ and that $0 \leq \lambda_2+1 \leq 2\alpha$. Well, for the former, as $\lambda_3 \geq 0$ then indeed $\lambda_3+p-1 \geq p-1$. Now suppose for a contradiction that $\lambda_3+p-1 > p-1+\alpha$ which occurs iff $\lambda_3 > \alpha$, then $\lambda_1+\lambda_2 < p-2+\alpha$ iff $\lambda_1 < p-2+\alpha-\lambda_2$. However, we know that $\lambda_1-\lambda_2 > p$ which occurs iff $\lambda_1 > p+\lambda_2$ and hence we would have $\lambda_2+p-1 < p-2+\alpha-\lambda_2$ which is true iff $\lambda_2 < \frac{\alpha-1}{2} < \alpha < \lambda_3$ which is a contradiction. Hence $\lambda_3+p-1 \leq p-1+\alpha$ as required. Now consider the latter, well clearly $\lambda_2+1 \geq 0$ as required. Now suppose for a contradiction that $\lambda_2+1 \geq 2\alpha$ which is true iff $\lambda_2 \geq 2\alpha-1$, however we know that $\lambda_1-\lambda_2 > p$ and thus to ensure $\lambda_1-\lambda_2 \geq p$ we would require $\lambda_1+\lambda_2+\lambda_3 > r_1$, giving a contradiction, and thus $\lambda_2+1 \leq 2\alpha$ as required. □

Now that we have clarified that the Schur functions corresponding to the fifth highest weights do indeed arise in $\mathrm{ch}\,(L(p-1,p-1,0)\otimes S^{\alpha}E\otimes S^{\alpha}E)$ it is now necessary to show that they arise with a higher multiplicity than the Schur functions which correspond to the weights which are fourth highest. We therefore make the following claim;

THEOREM 4.13.15 The character of the truncated module

$$\mathrm{ch}\,(\mathrm{Tr}^{\,2p-2,\alpha,\alpha}E)=\sum a_\lambda s_\lambda \text{ where } a_\lambda=\begin{cases}\lambda_3+1 & \text{for } \lambda_1 \geq p-1+\alpha\\ \lambda_1-\lambda_2+1 & \text{for } \lambda_1 < p-1+\alpha\end{cases}.$$

Here $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ such that $p-1 \leq \lambda_1 \leq p-1+2\alpha$, $p-1 \leq \lambda_2 \leq p-1+\alpha$ and $0 \leq \lambda_3 \leq 2\alpha$.

Proof. 1) Let us first consider the case where $\lambda_1 \geq p-1+\alpha$. Now, we know that any s_λ which arises in $\mathrm{ch}\,(L(p-1,p-1,0)\otimes S^{\alpha}E\otimes S^{\alpha}E)$ will be such that $\lambda_2 \geq p-1$ and thus there are two types of weights λ to consider, those were $\lambda_1, \lambda_2 \neq 0$ and $\lambda_3 = 0$, and those where $\lambda_1, \lambda_2, \lambda_3 \neq 0$. We begin with the former;

i) $\lambda = (\lambda_1, \lambda_2, 0)$.
Now, using x's for $L(p-1, p-1, 0)$ and 1s and 2s for the two $S^\alpha E$ we have that a_λ is the number of ways we can place the x's, 1s and 2s to form a semi-standard λ-tableau of type $(2p-2, \alpha, \alpha)$. Now, the $L(p-1, p-1, 0)$ must be placed first, so the first $p-1$ columns are filled with x's. This leaves α 1s and 2s to fill in. As we always need strictly increasing columns, then up to the end of λ_2 the columns must be of the form

1
2

We then just have the rest of the first row to fill in, which is increasing so we fill it first with the remaining 1s and then the remaining 2s. There is thus only one way to form λ so $a_\lambda = 1 = \lambda_3 + 1$.
For example, with $p = 7$, $r = 18$ and $\lambda = (11, 7, 0)$ we have;

xxxxxx11122
xxxxxx2

ii) $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.
As above, the first $p-1$ columns are filled with the x's, and again, as $\lambda_2 \geq p-1$ then the columns up to the end of λ_2 are of the form

1
2

It then remains to fill λ_3 and $\lambda_1 - \lambda_2$ with $\alpha - (\lambda_1 - (p-1)) = \alpha - \lambda_2 + p - 1$ 1s and 2s. Now $\lambda_1 \geq p - 1 + \alpha$ and thus $\lambda_2 + \lambda_3 \leq p - 1 + \alpha$ and so $\lambda_3 \leq \alpha - \lambda_2 + p - 1$. Therefore a_λ is dependent on λ_3 and we can fill the remainder as follows;
Put λ_3 2s in the third row and the remaining 1s and 2s in the first row from left to right;
Put one 1 and $\lambda_3 - 1$ 2s in the third row from left to right and then the remaining 1s and 2s in the first row from left to right;
Put two 1s and $\lambda_3 - 2$ 2s in the third row from left to right and then the remaining 1s and 2s in the first row from left to right;
This continues to the final option where we place λ_3 1s in the third row and the remaining 1s and 2s in the first row from left to right.
There are therefore $\lambda_3 + 1$ ways of forming λ and $a_\lambda = \lambda_3 + 1$ as required.

For example, with $p = 7$, $r = 18$ and $\lambda = (10, 6, 2)$ we have the following options;

xxxxxx1112
xxxxxx
22

xxxxxxx1122
xxxxxxx
12

xxxxxxx1222
xxxxxxx
11

2) The case where $\lambda_1 < p - 1 + \alpha$. As $\lambda_1 \leq p - 2 + \alpha$ then so is λ_2 and hence $\lambda_1 + \lambda_2 \leq 2p - 4 + 2\alpha$ and thus $\lambda_3 \geq 2$, hence $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1, \lambda_2, \lambda_3 \neq 0$. As in case 1), we must first fill the first $p - 1$ columns of the first two rows with x's. Then as $p - 1 \leq \lambda_2 \leq p - 2 + \alpha$, we have that the columns up to the end of the second row are of the form

1
2

It remains to fill the nodes in λ_3 and $\lambda_1 - \lambda_2$ with $\alpha - \lambda_2 + p - 1$ 1s and 2s. Now, with $\lambda_1 < \alpha + p - 1$ we have that $\lambda_3 > \alpha - \lambda_2 + p - 1$ but $\lambda_1 - \lambda_2 < \alpha - \lambda_2 + p - 1$, and thus a_λ is restricted by $\lambda_1 - \lambda_2$. We have the following options;
Put $\lambda_1 - \lambda_2$ 1s in the first row and the remaining 1s and 2s in the third row from left to right;
Put $\lambda_1 - \lambda_2 - 1$ 1s and one 2 in the first row from left to right and then the remaining 1s and 2s in the third row from left to right;
Put $\lambda_1 - \lambda_2 - 2$ 1s and two 2s in the first row from left to right and then the remaining 1s and 2s in the third row from left to right;
This continues to the final option where we place $\lambda_1 - \lambda_2$ 2s in the first row and the remaining 1s and 2s in the third row from left to right.
There are therefore a total of $\lambda_1 - \lambda_2 + 1$ ways to form λ and $a_\lambda = \lambda_1 - \lambda_2 + 1$ as required.

For example, with $p = 7$, $r = 18$ and $\lambda = (8, 7, 3)$ we have the following options;

xxxxxxx11
xxxxxxx2
122

xxxxxxx12
xxxxxxx2
112

□

We now go about showing that the Schur function corresponding to the fifth highest weight in the 6-set, always arises with a higher multiplicity than the Schur function corresponding to the fourth highest weight. The fourth weight is of the form $(\lambda_1, \lambda_2 + p, \lambda_3)$ whilst the fifth weight is of the form $(\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)$,

CLAIM 4.13.16 For both the fourth and fifth weight, the first entry, $\lambda_1 \geq p - 1 + \alpha$.

Proof. We shall first prove that $2\lambda_1 \geq r_2$, where $2p \leq r_2 \leq 3p-3$. Well, we know that $\lambda_1 - \lambda_2 \geq p$, so if λ_1 is the smallest it can be we have $\lambda_1 - \lambda_2 = p$ so $\lambda_2 = \lambda_1 - p$. Depending on the degree r_1, we will either have, for λ_1 smallest, that $\lambda_3 = \lambda_2$ or $\lambda_3 = \lambda_2 - 1$. Take the case where $\lambda_3 = \lambda_2$, then we have $\lambda_1+\lambda_2+\lambda_3 = r_1 \Leftrightarrow 3\lambda_1 - 2p = r_1 \Leftrightarrow 2\lambda_1 = r_2 + p - \lambda_1 \geq r_2$ as required. Taking the case where $\lambda_3 = \lambda_2 - 1$, then we have $\lambda_1 + \lambda_2 + \lambda_3 = r_1 \Leftrightarrow 3\lambda_1 - 2p - 1 = r_1 \Leftrightarrow 2\lambda_1 = r_2 + p - \lambda_1 + 1 > r_2$ as required.

Now, back to the claim. Assume for a contradiction that $\lambda_1 < p - 1 + \alpha$ which is true if and only if $\alpha = \frac{r_1 - p + 2}{2} > \lambda_1 - p + 1 \Leftrightarrow r_1 + p > 2\lambda_1 \Leftrightarrow r_2 > 2\lambda_1$, which we know to be false, and hence $\lambda_1 \geq p - 1 + \alpha$. □

So, we have that the Schur function s_σ corresponding to the fourth highest weight σ arises with multiplicity $\sigma_3 + 1 = \lambda_3 + 1$ whilst the Schur function s_τ corresponding to the fifth highest weight τ arises with multiplicity $\tau_3 + 1 = \lambda_2 + 2$. Clearly $\lambda_2 + 2 > \lambda_3 + 1$ and hence $a_\tau > a_\sigma$, and thus, using Section 4.10 c), we have that the coefficient spaces of the tilting modules whose weights are fifth highest in the core class of the 6-set arise in $\mathrm{cf}\,(\mathrm{Tr}^{\,2p-2,\alpha,\alpha}E)$.

The case where r is odd

EXAMPLE 4.13.17 Let $p = 7$ and $r = 17$, then we calculate

$$\begin{aligned}
\mathrm{ch}\,(\mathrm{Tr}^{\,12,3,3}E) &= \mathrm{ch}\,(L(6,6,0) \otimes S^3E \otimes S^2E) \\
&= (s_{66} \cdot s_3) \cdot s_2 \\
&= (s_{960} + s_{861} + s_{762} + s_{663})s_2 \\
&= s_{11,6} + s_{10,7} + s_{10,6,1} + s_{98} + s_{971} + s_{962} \\
&\quad + s_{10,6,1} + s_{971} + s_{962} + s_{881} + s_{872} + s_{863} \\
&\quad + s_{962} + s_{872} + s_{863} + s_{773} + s_{764} \\
&\quad + s_{863} + s_{764} + s_{665}
\end{aligned}$$

We can display this result, with multiplicities, as follows;

$1s_{11,6}$					
$1s_{10,7}$	$2s_{10,6,1}$				
$1s_{98}$	$2s_{971}$	$3s_{962}$			
	$1s_{881}$	$2s_{872}$	$3s_{863}$		
			$1s_{773}$	$2s_{764}$	
					$1s_{665}$

Proof. of Theorem 4.13.13
Following the proof for the case r even, we first consider what a typical term s_σ looks like when it arises in the $\mathrm{ch}\,(L(p-1,p-1,0) \otimes S^\alpha E \otimes S^\beta E)$, where $\alpha = \frac{r_2-(2p-2)+1}{2}$, $\beta = \frac{r_2-(2p-2)-1}{2}$ and $2p \leq r_2 \leq 3p-3$. Well if $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, then using the Littlewood-Richardson Rule we have that

$p-1 \leq \sigma_1 \leq p-1+\alpha+\beta$, $p-1 \leq \sigma_2 \leq p-1+\beta$ and $0 \leq \sigma_3 \leq \alpha+\beta$, where $\sigma_1+\sigma_2+\sigma_3 = 2p-2+\alpha+\beta = r_2$. Now, let us consider a weight λ which is fourth highest in the core class of the 6-set. By Section 4.7 we know that $\lambda = (\lambda_1, \lambda_2+p, \lambda_3)$ where $(\lambda_1, \lambda_2, \lambda_3)$ was the highest weight in the core class of the 3-set for the range $p \leq r_1 \leq 2p-1$. Now, clearly, we have that $\lambda_1 + (\lambda_2+p) + \lambda_3 = r_2$, so we need to check the following;

1) Is $p-1 \leq \lambda_1 \leq p-1+\alpha+\beta$?
Well, clearly $\lambda_1 \geq p$ as $p \leq r_1 \leq 2p-3$, so indeed $\lambda_1 \geq p-1$.
Now as $\lambda_1 \leq r_1$, then it is enough to check that $r_1 \leq p-1+\alpha+\beta$ which is true if and only if $r_1 \leq 2p-2$, which we know to be true as we are considering the range where $\lambda_1 \leq 2p-3$.

2) Is $p-1 \leq \lambda_2+p \leq p-1+\beta$?
Well, $\lambda_2 \geq 0$ and thus $\lambda_2+p \geq p > p-1$ as required.
So, is $\lambda_2+p \leq p-1+\beta$? Well this is true if and only if $\lambda_2 \leq \frac{p-4}{2}$, so suppose for a contradiction that $\lambda_2 \geq \frac{p-2}{2}$. As $\lambda_1 - \lambda_2 \geq p$, this would mean that $\lambda_1 \geq \frac{3p-2}{2}$, and then we would have that $\lambda_1+\lambda_2 \geq 2p-2$ which is a contradiction as $\lambda_1+\lambda_2+\lambda_3 \leq 2p-3$, so indeed $\lambda_2+p \leq p-1+\alpha$.

3) Is $0 \leq \lambda_3 \leq \alpha+\beta$?
Clearly $\lambda_3 \geq 0$. Now suppose for a contradiction that $\lambda_3 \geq \alpha+\beta$ this holds if an only if $\lambda_2 \geq \alpha+\beta$ and thus to ensure $\lambda_1-\lambda_2 \geq p$ we would require $\lambda_1+\lambda_2+\lambda_3 > r_1$, giving a contradiction, and thus $\lambda_3 \leq p-1$.

We therefore have that all the Schur functions which correspond to the highest weight of the tilting modules whose weights are fourth highest in the core class of the six, will arise, at least once, in $\mathrm{ch}\,(L(p-1,p-1,0) \otimes S^\alpha E \otimes S^\alpha E)$. To ensure then, that the coefficient spaces of these tilting modules are contained in the coefficient space of this truncated module, we must check that no Schur functions corresponding to higher weights in the same core class also arise. Well, let us look at the top weight in the 6-set, namely $(\lambda_1+p, \lambda_2, \lambda_3)$. Note that $\lambda_2 < p-1$, as if $\lambda_2 \geq p$ then we would require $\lambda_1 \geq 2p$ and thus $r_1 \geq 3p$ which is not true. Therefore the Schur function corresponding to this weight cannot arise in the $\mathrm{ch}\,(L(p-1,p-1,0) \otimes S^\alpha E \otimes S^\beta E)$, as we already have that for a typical term $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ in this character, $\sigma \geq p-1$. For the same reason, the second and third weights cannot arise, these both have second entry λ_1-p+1 which, as $\lambda_1 \leq 2p-3$, then $\lambda_1-p+1 \leq p-2 < p-1$.

We now consider those weights which are fifth highest in the same core class, namely those of the form $(\lambda_1, \lambda_3+p-1, \lambda_2+1)$. Let us first show that the Schur function corresponding to this weight will actually arise, well, we have already clarified that $p-1 \leq \lambda_1 \leq p-1+\alpha+\beta$. Next we must check that $p-1 \leq \lambda_3+p-1 \leq p-1+\beta$ and that $0 \leq \lambda_2+1 \leq \alpha+\beta$. Well, for the former, as $\lambda_3 \geq 0$ then indeed $\lambda_3+p-1 \geq p-1$. Now suppose for a contradiction that $\lambda_3+p-1 > p-1+\beta$ which occurs iff $\lambda_3 > \beta$, then $\lambda_1+\lambda_2 < p-2+\alpha$ iff $\lambda_1 < p-2+\alpha-\lambda_2$. However, we know that $\lambda_1-\lambda_2 > p$ which occurs iff $\lambda_1 > p+\lambda_2$ and hence we would have $\lambda_2+p < p-2+\alpha-\lambda_2$ which is true iff $\lambda_2 < \frac{\alpha-2}{2} < \beta < \lambda_3$ which is a contradiction. Hence $\lambda_3+p-1 \leq p-1+\beta$ as required. Now consider the latter, well clearly $\lambda_2+1 \geq 0$ as required. Now suppose for a contradiction that $\lambda_2+1 \geq \alpha+\beta$ which is true iff $\lambda_2 \geq \alpha+\beta-1$, however we know that $\lambda_1-\lambda_2 > p$ and thus to ensure $\lambda_1-\lambda_2 \geq p$ we would require $\lambda_1+\lambda_2+\lambda_3 > r_1$, giving a contradiction, and thus $\lambda_2+1 \leq 2\alpha$ as required.

Now that we have clarified that the Schur functions corresponding to the fifth highest weights do indeed arise in $\mathrm{ch}\,(L(p-1,p-1,0) \otimes S^{\alpha}E \otimes S^{\beta}E)$ it is now necessary to show that they arise with a higher multiplicity than the Schur functions which correspond to the weights which are fourth highest. We therefore make the following claim;

THEOREM 4.13.18 The character of the truncated module

$$\mathrm{ch}\,(\mathrm{Tr}^{\,2p-2,\alpha,\beta}E) = \sum a_\lambda s_\lambda \text{ where } a_\lambda = \begin{cases} \lambda_3+1 & \text{for } \lambda_1 \geq p-1+\alpha \\ \lambda_1-\lambda_2+1 & \text{for } \lambda_1 < p-1+\alpha \end{cases}.$$

Here $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ such that $p-1 \leq \lambda_2 \leq p-1+\alpha+\beta$, $p-1 \leq \lambda_2 \leq p-1+\beta$ and $0 \leq \lambda_3 \leq \alpha+\beta$.

Proof. The proof follows from that of the case where r is even, except we are now adding α 1s and β 2s. □

We now go about showing that the Schur function corresponding the fifth highest weight in the 6-set, always arises with a higher multiplicity than the Schur function corresponding to the fourth highest weight. The fourth weight is of the form $(\lambda_1, \lambda_2+p, \lambda_3)$ whilst the fifth weight is of the form $(\lambda_1, \lambda_3+p-1, \lambda_2+1)$,

CLAIM 4.13.19 For both the fourth and fifth weight, the first entry, $\lambda_1 \geq p-1+\alpha$.

Proof. This is the same proof as for the case r is even. □

So, we have that the Schur function s_σ corresponding to the fourth highest weight σ arises with multiplicity $\sigma_3+1 = \lambda_3+1$ whilst the Schur function s_τ corresponding to the fifth highest weight τ arises with multiplicity $\tau_3+1 = \lambda_2+2$. Clearly $\lambda_2+2 > \lambda_3+1$ and hence $a_\tau > a_\sigma$, and thus, using Section 4.10 c), we have that the coefficient spaces of the tilting modules whose weights are fifth highest in the core class of the 6-set arise in $\mathrm{cf}\,(\mathrm{Tr}^{\,2p-2,\alpha,\beta}E)$. □

CALCULATION 4.13.20 The case $r = 3p-2$
i) We first consider those weights which are highest or lowest in their core class, and which are self-titled. We then consider the weights which are second highest in the core class of the 6-set.

THEOREM 4.13.21 *The weights which are highest or lowest in their core class or are self-titled arise in the truncated module*

$$\mathrm{cf}\,(\mathrm{Tr}^{\,(p,p-1,p-1)}E) = L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E.$$

Moreover the weights which are second highest in the core class of the 6-set arise in the same truncated module due to 4.10 c), namely that the Schur function corresponding to their highest weight arises with a higher multiplicity than the Schur function corresponding to the weight that is highest in the same core class.

Proof. We first show that the coefficient space of all tilting modules whose weights are highest or lowest in their core class or are self-titled are contained in $\mathrm{cf}\,(\mathrm{Tr}^{\,(p,p,p-1)}E)$, by proving the following claim;

CLAIM 4.13.22 The Schur functions corresponding to all weights from $(3p-3,1,0)$ to $(p,p-1,p-1)$ arise at least once in the $\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$. Hence all weights in the degree $r = 3p-2$ arise at least once, and thus by 4.10 a) and b), we have the coefficient spaces of all tilting modules whose weights are highest or lowest in their core class, or are self-titled.

Proof. Consider a term s_σ in $\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$, what does $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ look like? Well, it is easy to see that $p \leq \sigma_1 \leq 3p-3$, as we can clearly place all nodes of both $S^{p-1}E$ in the top row, next to the $p-1$ nodes of $L(p-1,1,0)$. Moreover if $\sigma_1 \leq p-1$ then we would have $\sigma_2 > \sigma_1$ or $\sigma_3 > \sigma_1$, which we cannot have.
Now, consider σ_2. Well, as there is always at least one node in the second row, and we can place the remaining in the first row, then $\sigma_2 \geq 1$. On the other hand, we could distribute the remaining $2p-2$ nodes between the first two rows, whilst still following Young's Rule, such that $\sigma_1 = \frac{3p-1}{2}$ and $\sigma_2 = \frac{3p-3}{2}$. Hence $1 \leq \sigma_2 \leq \frac{3p-3}{2}$.
Finally consider σ_3, where clearly $\sigma_3 \geq 0$, and the greatest σ_3 can be is $p-1$, where we place all nodes of the second $S^{p-1}E$ in the third row. If we were to put p nodes or more in the third row, then either $\sigma_1 < \sigma_3$ or $\sigma_2 < \sigma_3$, which we cannot have. Thus $0 \leq \sigma_3 \leq p-1$.
Combining these restrictions together we have that $(p,p-1,p-1) \leq \sigma \leq (3p-3,1,0)$ as stated. Now, we know by Proposition 4.3.2 that for λ a weight in degree $r_1 = 2p-2$, with λ the highest weight in its core class consisting of

three weights, we have that $\lambda_1 \leq 2p-3$. When we then move up to degree $r_2 = 3p-2$ we have that the maximal weight is $(3p-3,1,0)$. Moreover it is clear that $(p,p-1,p-1)$ is the minimal weight in degree $r_2 = 3p-2$, and all weights in between fit into the boundaries given by σ. Hence the Schur functions corresponding to all weights in the degree $r = 3p-2$ arise at least once in $\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$, and thus by 4.10 a) and b), we have the coefficient spaces of all tilting modules whose weights are highest or lowest in their core class, or are self-titled. □

We now move onto showing that the coefficient spaces of the tilting modules whose weights are second highest in their core class are also contained in the same truncated module. To do this, we first prove the following claim;

CLAIM 4.13.23 For $\mu = (\mu_1, \mu_2, \mu_3)$ the highest weight in its core class, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ the second highest weight in its core class, we have that;
i) $\mu_1, \sigma_1 \geq 2p$
ii) $\mu_1 \leq p-3$ and $\sigma_1 \leq p-2$.

Proof. Well, $\mu = (\lambda_1 + p, \lambda_2, \lambda_3)$ and $\sigma = (\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is top of its core class in the range $p \leq r \leq 2p-1$, and hence $\lambda_1 - \lambda_2 \geq p$. So, let us first consider μ. Well, $\lambda_1 \geq p$ and hence $\mu_1 = \lambda_1 + p \geq 2p$ as required. Moreover, with $\lambda_1 - \lambda_2 \geq p$ then we have that $\lambda_2 \leq \lambda_1 - p \leq p-3$ as we have that $\lambda_1 \leq 2p-3$. Hence $\mu_2 = \lambda_2 \leq p-3$ as required.
Now consider σ. Well, using that $\lambda_1 \leq 2p-3$ we have that $\lambda_2 \geq 1$ and hence $\sigma_1 = \lambda_2 + 2p - 1 \geq 2p$ as required. Finally, with $\lambda_1 \leq 2p-3$ we have that $\sigma_2 = \lambda_1 - p + 1 \leq p-2$. □

We therefore know that when considering the multiplicity a_τ of a general Schur function s_τ in the $\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$ it is only necessary to consider the cases where $\tau_1 \geq 2p$ and $\tau_2 \leq p-2$. This brings us to the following theorem.

THEOREM 4.13.24 *If*

$$\tau \in \{(\tau_1, \tau_2, \tau_3) \mid (\tau_1, \tau_2, \tau_3) \leq (3p-3, 1, 0),\ \tau_1 \geq 2p \text{ and } \tau_2 \leq p-2\},$$

then, in the $\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$*, the Schur function* s_τ *arises with multiplicity* a_τ*, where* $a_\tau = \begin{cases} \tau_2 & \text{for } \tau_3 \leq 1 \\ \tau_2 - \tau_3 + 1 & \text{for } \tau_3 > 1. \end{cases}$

Proof. We have that

$$\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E) = (s_{p-1,1} - s_{p-2,1,1}) s_{p-1} s_{p-1},$$

and thus the multiplicity $a_\tau = a_\tau^1 - a_\tau^2$ where a_τ^1 is the number of semi-standard tableaux formed from $s_{p-1,1} s_{p-1} s_{p-1}$, and a_τ^2 is the number of semi-standard tableaux formed from $s_{p-2,1,1}s_{p-1}s_{p-1}$

1) The case where $\tau_3 \leq 1$.
i) Let us first look at the case where $\tau_3 = 0$, and follow the method through by using the example $p = 7$, $r = 19$ and $\tau = (15, 4, 0)$. So, we first draw $(p-1, 1)$ which we notate by x's, and then to this we must add $p-1$ 1s and $p-1$ 2s. As $\tau_2 \leq p-2$ then we can put at most $\tau_2 - 1$ of these 1s and 2s in the second row, the rest must go in the first row. We therefore have the following options;
Place no 1s and $\tau_2 - 1$ 2s in the second row, then the remaining nodes in the first row from left to right;
Place one 1 and $\tau_2 - 2$ 2s in the second row, then the remaining nodes in the first row from left to right;
We continue increasing the 1s in the second row until we reach the final option where we place $\tau_2 - 1$ 1s and no 2s in the second row, and the remaining nodes in the first row from left to right. This gives us $a^1_\tau = \tau_2 - 1 + 1 = \tau_2$ tableaux, which in our example are;

xxxxxx111111222
x222

xxxxxx111112222
x122

xxxxxx111122222
x112

xxxxxx111222222
x111

Now we need to find a^2_τ, however, for $\tau_3 = 0$ this is equal to zero as the tableaux formed from $s_{p-2,1,1}s_{p-1}s_{p-1}$ will have at least one node in the third row. Hence $a^2_\tau = 0$, and so $a_\tau = \tau_2$ as required.

ii) We now look at the case where $\tau_3 = 1$, and use the example $p = 7$, $r = 19$ and $\tau = (15, 3, 1)$. We again start by drawing $(p-1, 1)$ with x's, to which we must add $p-1$ 1s and $p-1$ 2s. Now one of these 1s or 2s must be placed in the third row as $\tau_3 = 1$, let us first choose to place a 1 in the third row. Then the remaining must be placed in the first and second rows such that the number of nodes in row two is equal to the number in τ_2. We can do this as follows;
Place no 1s and $\tau_2 - 1$ 2s in the second row, with the remaining in the first row from left to right;
Place one 1 and $\tau_2 - 2$ 2s in the second row, with the remaining in the first row from left to right;
This continues to the final option where we place no 2s and $\tau_2 - 1$ 1s in the second row and the remaining in the first row form left to right. This gives

a total of τ_2 tableaux. If we then place a 2 in the third row instead of a 1, then we follow the same method again to get a further τ_2 tableaux, and hence $a^1_\tau = 2\tau_2$. In our example, these are as follows;

xxxxxxx111112222
x22
1

xxxxxxx111122222
x12
1

xxxxxxx111222222
x11
1

xxxxxxx111111222
x22
2

xxxxxxx111112222
x12
2

xxxxxxx111122222
x11
2

We must now find a^2_τ, so we draw $(p-2,1,1)$ using x's and then have $p-1$ 1s and $p-1$ 2s to add to either row one or row two, such that the number of nodes in row two is equal to the number in τ_2. We therefore have the following options;
Place no 1s and $\tau_2 - 1$ 2s in the second row and the remaining in the first row from left to right;
Place one 1 and $\tau_2 - 2$ 2s in the second row and the remaining in the first row from left to right;
This continues to the final option where we place $\tau_2 - 2$ 1s and no 2s in the second row and the remaining in the first row from left to right. Thus $a^2_\tau = \tau_2$, and in our example these tableaux are as follows;

xxxxxx1111112222
x22
x

xxxxxx1111122222
x12
x

xxxxxx1111222222
x11
x

Thus we have that $a_\tau = a^1_\tau - a^2_\tau = (2\tau_2) - \tau_2 = \tau_2$ as required.

2) The case where $\tau_3 > 1$.
For this case we shall follow our work with the example $p = 7$, $r = 19$ and $\tau(14,3,2)$. Starting with a^1_τ, we again draw $(p-1,1)$ using x's to which we must add $p-1$ 1s and 2s. Now, as $\tau_3 \geq 2$, then from column two, to the end

of the third row we must place 1s in the second row, and 2s in the third row, to ensure we have strictly increasing columns. Now there is only one node left empty in the third row, so let us choose to put a 1 in this space. Then the remaining 1s and 2s must go in rows one and two such that row two has the same number of nodes as τ_2. We can therefore do this as follows;
Place no 1s and $\tau_2 - \tau_3$ 2s in row two and the remaining in row one;
Place one 1 and $\tau_2 - \tau_3 - 1$ 2s in row two and the remaining in row one;
This continues to the final option where we place $\tau_2 - \tau_3$ 1s and no 2s in the second row and the remaining in the first row. Thus we have $\tau_2 - \tau_3 + 1$ tableaux. If we then go back and place a 2 in the empty node in row three instead of a 1, then we can repeat the process again, getting another $\tau_2 - \tau_3 + 1$ tableaux, and thus $a^1_\tau = 2(\tau_2 - \tau_3 + 1)$. In our example these are as follows;

xxxxxx11112222
x12
12

xxxxxx11122222
x11
12

xxxxxx11111222
x12
22

xxxxxx11112222
x11
22

It now remains to find a^2_τ, so we draw $(p-2, 1, 1)$, and then fill rows two and three with 1s and 2s respectively, up to the end of the third row to ensure we have strictly increasing columns. The remaining 1s and 2s can then be placed in rows one and two as follows;
Place no 1s and $\tau_2 - \tau_3$ 2s in the second row and the remaining in the first row;
Place one 1 and $\tau_2 - \tau_3 - 1$ 2s in the second row and the remaining in the first row;
This continues to the final option where we place $\tau_2 - \tau_3$ 1s and no 2s in the second row and the remaining in the first row. We thus have that $a^2_\tau = \tau_2 - \tau + 1$, and our example gives these tableaux;

xxxxx111112222
x12
x2

xxxxx111122222
x11
x2

We therefore have that $a_\tau = a^1_\tau - a^2_\tau = 2(\tau_2 - \tau_3 + 1) - (\tau_2 - \tau_3 + 1) = \tau_2 - \tau_3 + 1$.
□

With this information we can now check that for μ the highest weight in its core class and σ the second highest weight, that $a_\sigma > a_\mu$. Well, $\mu = (\lambda_1 + p, \lambda_2, \lambda_3)$ and $\sigma = (\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)$, and so when $\mu_3 = \sigma_3 = \lambda_3 \leq 1$ then $a_\mu = \mu_2 = \lambda_2$ and $a_\sigma = \sigma_2 = \lambda_1 - p + 1$, and indeed, as $\lambda_1 - \lambda_2 \geq p$ then $\lambda_1 - p + 1 > \lambda_2$. On the other hand, when $\lambda_3 > 1$ then $a_\mu = \lambda_2 - \lambda_3 + 1$ and $a_\sigma = \lambda_1 - p + 1 - \lambda_3 + 1 = \lambda_1 - \lambda_3 - p + 2$, and again as $\lambda_1 - \lambda_2 \geq p$ then $\lambda_1 - \lambda_3 - p + 2 > \lambda_2 - \lambda_3 + 1$. Hence the coefficient spaces of the tilting modules whose weights are second highest in the core class of the 6-set are contained in $\mathrm{cf}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$. □

ii) Here we consider those tilting modules whose weights are second highest in the core class of the 3-set which came from the self-titled weight λ such that $\lambda_1 - \lambda_3 < p - 2$.

THEOREM 4.13.25 *For $r = 3p - 2$ where λ was a self-titled weight of degree $r = 2p - 2$ with $\lambda_1 - \lambda_3 < p - 2$, then*

$$\mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E).$$

Proof. From Section 4.7, we have that the 3-set formed from this self-titled weight λ with $\lambda_1 - \lambda_3 < p - 2$ is as follows;

$$\begin{aligned}&(\lambda_1 + p, \lambda_2, \lambda_3)\\&(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)\\&(\lambda_3 + p - 2, \lambda_1 + 1, \lambda_2 + 1)\end{aligned}$$

Now consider the tilting truncated module $L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E$, and let us first calculate

$$\begin{aligned}L(p-1,1,0) \otimes S^{p-1}E &= (s_{p-1,1} - s_{p-2,1,1})s_{p-1}\\&= s_{2p-2,1} + s_{2p-3,2} + s_{2p-4,3} + \ldots + s_{p,p-1} + s_{p-1,p-1,1}.\end{aligned}$$

Now, to ensure we have the tilting module in the middle of this core class we need to ensure that the Schur function corresponding to its highest weight arises with a higher multiplicity than the Schur function corresponding to the weight at the top of this core class, when we calculate $\mathrm{ch}\,(L(p-1,1,0) \otimes S^{p-1}E \otimes S^{p-1}E)$. Well, the only terms in the above sum which can be multiplied by another s_{p-1} such that the resulting Schur function has weight $(\lambda_1 + p, \lambda_2, \lambda_3)$, are those in the range

$$(2p - (\lambda_3 + 1), \lambda_3, 0), (2p - (\lambda_3 + 2), \lambda_3 + 1, 0), \ldots, (2p - (\lambda_2 + 1), \lambda_2, 0)$$

If the second entry is less than λ_3 then when we add on the remaining nodes we would not be able to form a partition with strictly increasing columns. If the second entry is greater than λ_2 then we have too many nodes in the second row.

On the other hand, the terms from the above sum which can be multiplied by another s_{p-1} such that the resulting Schur function has weight $(\lambda_2+p-1, \lambda_1+1, \lambda_3)$, are those in the range

$$(2p-(\lambda_3+1), \lambda_3, 0), (2p-(\lambda_3+2), \lambda_3+1, 0), \ldots, (2p-(\lambda_1+2), \lambda_1+1, 0)$$

If the second entry is less than λ_3 then when we add on the remaining nodes we would not be able to form a partition with strictly increasing columns. If the second entry is greater than λ_1+1 then we have too many nodes in the second row.
Thus, as $\lambda_1+1 > \lambda_2$ then the multiplicity of the Schur function corresponding the 'middle' weight is greater than the multiplicity of the Schur function corresponding to the top weight. Hence by 4.10 c) we have that $\mathrm{cf}\,(T(\lambda_2+p-1, \lambda_1+1, \lambda_3) \subseteq \mathrm{cf}\,(L(p-1, \lambda_1+1, 0) \otimes S^{\lambda_2}E \otimes S^{\lambda_3}E)$ as required.
□

iii) Here we consider those tilting modules whose weights are second highest in the core class of the 3-set where $\lambda_1 - p < p-1$

THEOREM 4.13.26 *For $r = 3p-2$ where $\lambda_1 + p > t$ and $\lambda_1 - p < p-1$ then* $\mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes S^{p-2}E)$.

Proof. We know here that $(\lambda_1, \lambda_2, \lambda_3) = (r_1, 0, 0) = (2p-2, 0, 0)$. Therefore for $r_2 = 3p-2$ then the weight highest in its core class is $\alpha = (\lambda_2+2p-1, \lambda_1-p+1, \lambda_3) = (2p-1, p-1, 0)$ and the weight second highest is $(\lambda_1, \lambda_2+p, \lambda_3) = (2p-2, p, 0)$. When calculating the character of $\mathrm{Tr}^{\,p,p,p-2}E$ it is therefore only necessary to consider weights who have last entry zero. We first find those in $s_{p-1,1} \cdot s_{p-2}$, namely $(2p-3, 1, 0), (2p-4, 2, 0), (2p-5, 3, 0), \ldots, (p-1, p-1, 0)$, and note that the negative part of this calculation $s_{p-2,1,1} \cdot s_{p-2}$ will give no weights with last entry zero. We know multiply each of these weights by $s_{p-1,1}$ and see how many times $(2p-1, p-1, 0)$ and $(2p-2, p, 0)$ occur in each.
$s_{p-1,1} \cdot s_{2p-3,1,0}$ gives weights $(3p-4, 2, 0), \ldots, (2p-2, p, 0)$ and each occur once in this.
$s_{p-1,1} \cdot s_{2p-4,2,0}$ gives weights $(3p-5, 3, 0), \ldots, (2p-3, p+1, 0)$ and each occur once in this.
$s_{p-1,1} \cdot s_{2p-5,3,0}$ gives weights $(3p-6, 4, 0), \ldots, (2p-4, p+2, 0)$ and each occur once in this.
$\vdots$
$s_{p-1,1} \cdot s_{\frac{3p-3}{2}, \frac{p-1}{2}, 0}$ gives weights $(\frac{5p-5}{2}, \frac{p+1}{2}, 0), \ldots, (\frac{3p-1}{2}, \frac{3p-3}{2}, 0)$ and each occur once in this.

$s_{p-1,1} \cdot s_{\frac{3p-5}{2},\frac{p+1}{2},0}$ gives weights $(\frac{5p-7}{2}, \frac{p+3}{2}, 0), \ldots, (\frac{3p-1}{2}, \frac{3p-3}{2}, 0)$ and each occur once in this.
⋮
$s_{p-1,1} \cdot s_{p,p-2,0}$ gives weights $(2p-1, p-1, 0), \ldots, (\frac{3p-1}{2}, \frac{3p-3}{2}, 0)$ and each occur once in this.
$s_{p-1,1} \cdot s_{p-1,p-1,0}$ gives weights $(2p-2, p, 0), \ldots, (\frac{3p-1}{2}, \frac{3p-3}{2}, 0)$ and only the second highest weight occurs once in this.

We therefore have that, in $\mathrm{ch}\,(\mathrm{Tr}^{\,p,p,p-2}E)$, the multiplicity of the Schur function corresponding to the second highest weight is $p-1$, whilst the multiplicity of the Schur function corresponding to the highest weight is only $p-2$. Thus by 4.10 c), the cf $(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq$ cf $(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes S^{p-2}E)$. □

iv) For the weights which are third, fourth and fifth highest in the core class of the 6-set, we can use the same proof as for the range $2p \leq r \leq 3p-3$ as the 6-sets for $r = 3p-2$ are formed in exactly the same way as for $2p \leq r \leq 3p-3$. The only difference is that there are a smaller number of these 6-sets due to there being a smaller number of 3-sets for $2p-2$ because of the case where $\lambda_T = (2p-2, 0, 0)$ as $\lambda_{1T} - p < p-1$ which forms a 3-set not a 6-set which was resolved in sections i) and ii). For the fourth and fifth highest weights we use the truncated module $L(p-1, p-1, 0) \otimes S^{\alpha}E \otimes S^{\beta}E$ as $r = 3p-2$ is odd, where $\alpha = \frac{p+1}{2}$ and $\beta = \frac{p-1}{2}$. We can then use the proof of Theorem 4.13.13 from the range $2p \leq r \leq 3p-3$ to show that the Schur functions corresponding to both the fourth and fifth weights arise at least once in the character of the above truncated module. Moreover, from the range $2p \leq r \leq 3p-3$ we also have the following result;

THEOREM 4.13.27 The character of the truncated module

$$\mathrm{ch}\,(\mathrm{Tr}^{\,2p-2,\alpha,\beta}E) = \sum a_\lambda s_\lambda \text{ where } a_\lambda = \begin{cases} \lambda_3 + 1 & \text{for } \lambda_1 \geq p-1+\alpha \\ \lambda_1 - \lambda_2 + 1 & \text{for } \lambda_1 < p-1+\alpha \end{cases}$$

Where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that $p-1 \leq \lambda_1 \leq p-1+\alpha+\beta$, $p-1 \leq \lambda_2 \leq p-1+\beta$ and $0 \leq \lambda_3 \leq \alpha + \beta$.

Proof. See proof of the same theorem 4.13.18 for the range $2p \leq r \leq 3p-3$, which follows the proof for the even case in Theorem 4.13.15. □

Let s_σ be the Schur function corresponding to the fourth highest weight, then $\sigma = (\lambda_1, \lambda_2 + p, \lambda_3)$ and let s_τ be the Schur function corresponding to the fifth highest weight, then $\tau = (\lambda_1, \lambda_3 + p - 1, \lambda_2 + 1)$. For the range $2p \leq r \leq 3p-3$ we had that $\sigma_1, \tau_1 \geq p-1+\alpha$, and we proved that in this case $a_\tau = \tau_3 + 1 = \lambda_2 + 2 > \lambda_3 + 1 = \sigma_3 + 1 = a_\sigma$. However for $r = 3p-1$ it may be possible for,

$\sigma_1 = \tau_1 = \lambda_1 < p - 1 + \alpha$. For example, with $p = 7$ and $r_2 = 19 = r_1 + p$, then $\lambda = (9, 2, 1)$ is a weight at the top of its core class as $9 - 2 = 7 = p$, and yet $\lambda_1 = 9 < p - 1 + \alpha = 10$, hence $\sigma_1 = \tau_1 < p - 1 + \alpha$.
So, in this case $a_\sigma = \sigma_1 - \sigma_2 + 1 = \lambda_1 - (\lambda_2 + p) + 1 = \lambda_1 - \lambda_2 - p + 1$ and $a_\tau = \tau_1 - \tau_2 + 1 = \lambda_1 - (\lambda_3 + p - 1) + 1 = \lambda_1 - \lambda_3 - p + 2$, and we need to check that $a_\tau > a_\sigma$. Well, we know that $\lambda_2 \geq \lambda_3$ and hence $\lambda_2 + p - 1 \geq \lambda_3 + p - 1 > \lambda_3 + p - 2$ which is true if and only if $\lambda_1 - \lambda_3 - p + 2 > \lambda_1 - \lambda_2 - p + 1$ and thus $a_\tau > a_\sigma$ as required.
Hence both $\mathrm{cf}\,(T(\sigma))$ and $\mathrm{cf}\,(T(\tau))$ are contained in $\mathrm{cf}\,(L(p-1, p-1, 0) \otimes S^\alpha E \otimes S^\beta E)$ where $\alpha = \frac{p+1}{2}$ and $\beta = \frac{p-1}{2}$.

CALCULATION 4.13.28 The case $r = 3p - 1$
i) We first consider those weights which are highest or lowest in their core class, and which are self-titled. We then consider the weights which are second highest in the core class of the 6-set.

THEOREM 4.13.29 *The weights which are highest or lowest in their core class or are self-titled arise in the truncated module*

$$\mathrm{cf}\,(\mathrm{Tr}^{(p,p,p-1)} E) = \mathrm{cf}\,(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes S^{p-1} E).$$

Moreover the weights which are second highest in the core class of the 6-set arise in the same truncated module due to 4.10 c), namely that the Schur function corresponding to their highest weight arises with a higher multiplicity than the Schur function corresponding to the weight that is highest in the same core class.

Proof. We first show that the coefficient space of all tilting modules whose weights are highest or lowest in their core class or are self-titled are contained in $\mathrm{cf}\,(\mathrm{Tr}^{(p,p,p-1)} E)$, by proving the following claim;

CLAIM 4.13.30 The Schur functions corresponding to all weights from $(3p-3, 2, 0)$ to $(p, p, p-1)$ arise at least once in the $\mathrm{ch}\,(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes S^{p-1} E)$. Hence all weights in the degree $r = 3p - 1$ arise at least once, and thus by 4.10 a) and b), we have the coefficient spaces of all tilting modules whose weights are highest or lowest in their core class, or are self-titled.

Proof. Consider a term s_σ in $\mathrm{ch}\,(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes S^{p-1} E)$, what does $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ look like? Well, it is easy to see that $p \leq \sigma_1 \leq 3p - 3$, as we can clearly place all nodes of $S^{p-1}E$ in the top row, next to the $2p-2$ nodes of both $L(p-1, 1, 0)$. Moreover if $\sigma_1 \leq p - 1$ then we would have $\sigma_2,\ \sigma_3 > \sigma_1$, which we cannot have.
Now, consider σ_2. Well, as there are always at least two nodes in the second row, and we can place the remaining in the first row, then $\sigma_2 \geq 2$. On the

other hand, we could distribute the remaining $2p-2$ nodes between the first two rows, whilst still following Young's Rule, such that $\sigma_1 = \sigma_2 = \frac{3p-1}{2}$. Hence $1 \leq \sigma_2 \leq \frac{3p-1}{2}$.
Finally consider σ_3, where clearly $\sigma_3 \geq 0$, and the greatest σ_3 can be is $p-1$, where we place all nodes of the $S^{p-1}E$ in the third row. If we were to put p nodes or more in the third row, then either $\sigma_1 < \sigma_3$ or $\sigma_2 < \sigma_3$, which we cannot have. Thus $0 \leq \sigma_3 \leq p-1$.
Combining these restrictions together we have that $(p,p,p-1) \leq \sigma \leq (3p-3,2,0)$ as stated. Now, we know by Proposition 4.3.2 that for λ a weight in degree $r_1 = 2p-1$, with λ the highest weight in its core class consisting of three weights, we have that $\lambda_1 \leq 2p-3$. When we then move up to degree $r_2 = 3p-1$ we have that the maximal weight is $(3p-3,2,0)$. Moreover it is clear that $(p,p,p-1)$ is the minimal weight in degree $r_2 = 3p-1$, and all weights in between fit into the boundaries given by σ. Hence the Schur functions corresponding to all weights in the degree $r = 3p-1$ arise at least once in $\mathrm{ch}\,(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E)$, and thus by 4.10 a) and b), we have the coefficient spaces of all tilting modules whose weights are highest or lowest in their core class, or are self-titled. □

We now move onto showing that the coefficient spaces of the tilting modules whose weights are second highest in their core class are also contained in the same truncated module. To do this, we first prove the following claim;

CLAIM 4.13.31 For $\mu = (\mu_1, \mu_2, \mu_3)$ the highest weight in its core class, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ the second highest weight in its core class, we have that;
i) $\mu_1,\ \sigma_1 \geq 2p$
ii) $\mu_1 \leq p-3$ and $\sigma_1 \leq p-2$.

Proof. Same proof as for the case $r = 3p-2$. □

We therefore know that when considering the multiplicity a_τ of a general Schur function s_τ in the $\mathrm{ch}\,(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E)$ it is only necessary to consider the cases where $\tau_1 \geq 2p$ and $\tau_2 \leq p-2$. This brings us to the following theorem.

THEOREM 4.13.32 *If*

$$\tau \in \{(\tau_1, \tau_2, \tau_3) \mid \tau \leq (3p-3,2,0),\ \tau_1 \geq 2p \text{ and } \tau_2 \leq p-2\},$$

then, in the $\mathrm{ch}\,(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E)$, *the Schur function* s_τ *arises with multiplicity* a_τ, *where* $a_\tau = \begin{cases} \tau_2 - 1 & \text{for } \tau_3 = 0 \\ \tau_2 - \tau_3 + 1 & \text{for } \tau_3 \neq 0 \end{cases}$

Proof. We have that

$$\mathrm{ch}\,(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E) = (s_{p-1,1} - s_{p-2,1,1})(s_{p-1,1} - s_{p-2,1,1})s_{p-1},$$

and thus the multiplicity $a_\tau = a_\tau^1 - a_\tau^2 + a_\tau^3$ where a_τ^1 is the number of semi-standard tableaux formed from $s_{p-1,1}s_{p-1,1}s_{p-1}$ which follow the Littlewood-Richardson Rule, a_τ^2 is twice the number of semi-standard tableaux formed from $s_{p-1,1}s_{p-2,1,1}s_{p-1}$ which follow the Littlewood-Richardson Rule, and a_τ^3 is the number of semi-standard tableaux formed from $s_{p-2,1,1}s_{p-2,1,1}s_{p-1}$ which follow the Littlewood-Richardson Rule.

1) The case where $\tau_3 = 0$.
With $\tau_3 = 0$ then $a_\tau^2 = a_\tau^3 = 0$ and thus $a_\tau = a_\tau^1$. So, when calculating $s_{p-1,1}s_{p-1,1}s_{p-1}$, we first draw $(p-1,1)$ which we notate by x's. To this we must add another $(p-1,1)$ which we notate by 1s and a 2. To follow the Littlewood-Richardson Rule and maintain a semi-standard tableau it is necessary to put the 2 in the second row. We then add the 1s to either the first or second row, such that the number of nodes in row two is less than or equal to those in $\tau_2 \geq 2$ and we can do this as follows;
Place no 1s in row two and $p-1$ 1s in row one;
Place one 1 in row two and $p-2$ 1s in row one;
And so on to the last option where we add $\tau_2 - 2$ 1s to row two and the remaining 1s to row one.
To each of these new tableaux formed we can then add the remaining $p-1$ nodes (notated by 4s) in one way, such that we have a τ-tableau. Hence $a_\tau = a_\tau^1 = \tau_2 - 2 + 1 = \tau_2 - 1$.

For example, let $p = 7$, $r = 20$ and $\tau = (16, 4, 0)$, then we have the following tableaux;

xxxxxx1111114444
x244

xxxxxx1111144444
x124

xxxxxx1111444444
x112

2) The case where $\tau_3 \neq 0$.
Due to the nature of the truncated module we are using it is necessary here to split this part into three cases, namely when $\tau_3 = 1$, when $\tau_3 = 2$ and when $\tau_3 \geq 3$. We start with;

i) The case where $\tau_3 = 1$.
In this case $a_\tau^3 = 0$ as any tableaux formed from $s_{p-2,1,1}s_{p-2,1,1}$ will have a minimum of two nodes in the third row. Thus $a_\tau = a_\tau^1 - a_\tau^2$. So let us first look at a_τ^1. We begin by drawing $(p-1,1)$ which we notate by x's, then to this we must add another $(p-1,1)$ notated by $p-1$ 1s and one 2. The 2 we can either place in the second or third row (if it were in the first then it would

not be 'good' as necessitated by the Littlewood-Richardson Rule). Let us first choose to place it in the second row, then we must now add the $p-1$ 1s, only one of which can be placed in third row, and the remaining $p-2$ 1s must be added to the first or second row, such that the number of nodes in the second row is less than or equal to the number in τ_2. We therefore have the following options;
Place no 1s in the second row and $p-2$ in the first row;
Place one 1 in the second row, and $p-3$ in the first row;
This continues until we reach the final option where we place $\tau_2 - 2$ 1s in the second row, and $p-2-(\tau_2-2)$ 1s in the first row.
To each of these we can then add the remaining $p-1$ 4s, in one way, to form a τ-tableau, and from this method we have $\tau_2 - 1$ tableaux.

For example with $p = 5$, $r = 14$ and $\tau = (10, 3, 1)$ we have the following tableaux;

xxxx111444
x24
1

xxxx114444
x12
1

Now, after placing the 2 in the second row, we could have then chosen to not put a 1 in the third row, so all 1s must be placed in the first or second rows again such that the number of nodes in the second row is less than or equal to the number in τ_2. So we have the options;
Put no 1s in the second row and $p-1$ in the first row;
Put one 1 in the second row and $p-2$ in the first row;
This continues to the last option where we put $\tau_2 - 2$ 1s in the second row and $p-2-(\tau_2-2)$ in the first row.
To each of these tableau we can then add the remaining $p-1$ 4s, in one way, to form a τ-tableau, and this method gives us another $\tau - 1$ tableaux. With the same example as above we get the following tableaux;

xxxx111144
x24
4

xxxx111444
x12
4

Now let us go back to when we placed the 2 in the second row, and instead, choose to place it in the third row. Then the remaining 1s must be added to the first or second row, again such that the number of nodes in the second row does not exceed those in τ_2. So we can;
Place non in the second row, and $p-1$ in the first row;
Place one in the second row, and $p-2$ in the first row;

And so on to the last option where we place $\tau_2 - 1$ 1s in the second row, and $p-1-(\tau_2-1)$ in the first row.
Again, to each of these tableaux we can then add the remaining $p-1$ 4s, in one way, to form a τ-tableau, and this method gives another τ_2 tableaux. For the example given above, these are

xxxx111144
x44
2

xxxx111444
x14
2

xxxx114444
x11
2

We have now covered all possible ways of forming a semi-standard τ-tableau from $s_{p-1,1}s_{p-1,1}s_{p-1}$ and we have that $a_\tau^1 = 2(\tau_2 - 1) + \tau_2$.

We now need to find a_τ^2 which is twice the number of semi-standard τ-tableaux formed from $s_{p-1,1}s_{p-2,1,1}s_{p-1}$ such that they obey the Littlewood-Richardson Rule. So, as before, we start by drawing $(p-1,1)$ which we notate by x's. To this we must add $(p-2,1,1)$ which we notate by $p-2$ 1s, one 2 and one 3. To ensure we follow the Littlewood-Richardson Rule, we must place the 3 in row three and the 2 in row two, and thus it remains to add the $p-2$ 1s, which we can add to either the first row or the second row, ensuring that the number of nodes in the second row does not exceed the number in τ_2. We thus have the following options;
Put no 1s in row two and $p-2$ in row one;
Put one 1 in row two and $p-3$ in row one;
This continues to the final option where we place $\tau_2 - 2$ 1s in row two and $p-2-(\tau_2-2)$ in row one.
To each of these tableaux we can then add the remaining $p-1$ 4s in one way to form a τ-tableau. Hence we have $\tau_2 - 1$ ways of forming such a tableau and thus $a_\tau^2 = 2(\tau_2 - 1)$. Referring back to our example, the tableaux formed here are as follows;

xxxx111444
x24
3

xxxx114444
x12
3

So we can now calculate $a_\tau = a_\tau^1 - a_\tau^2 = 2(\tau_2-1)+\tau_2-2(\tau_2-1)-\tau_2 = \tau_2-\tau_3+1$ as $\tau_3 = 1$.

ii) The case where $\tau_3 = 2$. The method for finding a_τ^1 follows that of the case where $\tau_3 = 1$. We draw $(p-1,1)$ and then choosing to place the 2 of

$(p-1,1)$ in the second row and one 1 in the third row gives $\tau_2 - 1$ tableaux, whilst placing again the 2 in the second row but putting no 1s in the third row gives another $\tau_2 - 1$ tableaux.
Consider the example $p = 5$, $r = 14$ and $\tau = (10, 2, 2)$, then these tableaux are as follows;

xxxx111444
x1
14

xxxx111144
x2
44

We then choose to place the 2 in the third row, and then add the remaining 1s, this time ensuring that not only are the number of nodes in the second row less than or equal to the number in τ_2, but that they are also greater than or equal to the number of nodes in $\tau_3 = 2$. If the number was less than τ_3 then to get a τ-tableau it would be necessary to place two of the $p-1$ 4s underneath each other, which is not possible in a standard tableau. So, we can first place no 1s in the third row, which gives the following options;
Put $\tau_3 - 1$ 1s in the second row and $p - 1 - (\tau_3 - 1)$ in the first row;
Put τ_3 1s in the second row and $p - 1 - \tau_3$ in the first row;
And so on until we reach the final option where we place $\tau_2 - 1$ 1s in the second row and $p - 1 - (\tau_2 - 1)$ in the first row.
To each of these tableaux there is then one possible way to add the remaining $p-1$ 4s such that we form a semi-standard τ-tableau. So this method gives $\tau_2 - 1 - (\tau_3 - 1) + 1 = \tau_2 - \tau_3 + 1$ options.
In the example $p = 5$, $r = 19$ and $\tau = (10, 2, 2)$ this gives the tableau;

xxxx111444
x1
24

We can then choose to place one 1 in the third row, and then attach the remaining 1s to rows one and two again so that the number of nodes in row two is greater than or equal to τ_3 and less than or equal to τ_2, this then gives another $\tau_2 - \tau_3 + 1$ options, which in our example gives the tableau;

xxxx114444
x1
12

Thus, bringing all our options together we have that $a^1_\tau = 2(\tau_2 - 1) + 2(\tau_2 - \tau_3 + 1)$.

We now move onto a^2_τ. As for the case $\tau_3 = 1$ we draw $(p-1,1)$ and then when adding $(p-2,1,1)$ to it we must place the 3 in the third row, and the two in the second row. It then remains to attach the $p-2$ 1s, which, if we choose to place non of these in the third row, again gives $\tau_2 - 1$ tableaux. In our example, this is the following tableau;

xxxx111444
x2
34

As $\tau_3 = 2$, we could also choose to place one of these 1s in the third row, before the 3, to ensure we have a semi-standard tableau, and then attach the remaining $p-3$ 1s to the first and second rows, giving another $\tau_2 - 1$ tableaux, which for our example is the following;

xxxx111444
x2
13

There is thus a total of $2(\tau_2 - 1)$ tableaux and hence $a_\tau^2 = 4(\tau_2 - 1)$.

It now remains to find a_τ^3. We first draw $(p-2, 1, 1)$ notated by x's, and then add to it another $(p-2, 1, 1)$ notated by $p-2$ 1s, one 2 and one 3. Again, to ensure we follow the Littlewood-Richardson Rule, it is necessary to place the 3 in the third row and the 2 in the second row. We then must attach the $p-2$ 1s to the first and second rows, we we can do as follows;
Add no 1s to the second row, and $p-2$ to the first row;
Add one 1 to the second row and $p-3$ to the first row;
This continues until we reach the final option where we add $\tau_2 - 2$ 1s to the second row and $p - 2 - (\tau_2 - 1)$ 1s to the first row. To each of these there is one way to add the remaining $p-1$ 4s such that we have a τ-tableau, and thus $a_\tau^3 = \tau_2 - 1$. Finishing our example, the tableau formed is as follows;

xxx1114444
x2
x3

We thus have that $a_\tau = (2(\tau_2 - 1) + 2(\tau_2 - \tau_3 + 1)) - (4(\tau_2 - 1)) + (\tau_2 - 1) = \tau_2 - 1 = \tau_2 - \tau_3 + 1$ as $\tau_3 = 2$.

iii) The case where $\tau_3 \geq 3$.
For this case we use the example $p = 7$, $r = 20$ and $\tau = (14, 3, 3)$. We follow a similar method to the previous case, so start with finding a_τ^1. As before, we draw $(p-1, 1)$ with x's, and first choose to place the 2 of the next $(p-1, 1)$ in the second row, and one of the 1s in the third row. We then add the remaining 1s such that the number of nodes in row two is less than or equal to those in τ_2 and greater than or equal to the number in τ_3. We can thus;
Place $\tau_3 - 2$ 1s in row two and $p - 2 - (\tau_3 - 2)$ in row one;
Place $\tau_3 - 1$ in row two and $p - 2 - (\tau_3 - 1)$ in row one;
And so on to the final option where we place $\tau_2 - 2$ in row two and $p - 2 - (\tau_2 - 2)$ in row one. To each of these there is one way to add the $p-1$ 4s to form a τ-tableau, and thus we have $\tau_2 - \tau_3 + 1$ tableaux. In our example, this gives;

xxxxxx11114444
x12
144

When we then choose to not place a 1 in the third row, then we add them again to rows one and two in the same way, thus giving another $\tau_2 - \tau_3 + 1$ options, which in our example is as follows;

xxxxxx11111444
x12
444

Finally, we can choose to place the 2 in the third row as opposed to the second row. From there we can add the 1s such that row two is restricted as above and row three has either one 1 in it, or no 1s at all. This gives a further $2(\tau_2 - \tau_3 + 1)$ tableaux, which in our example are as follows;

xxxxxx11144444
x11
124

xxxxxx11114444
x11
244

Thus in total we have that $a^1_\tau = 4(\tau_2 - \tau_3 + 1)$.

We now move on to a^2_τ, as before, we draw the $(p-1,1)$ with x's, and then add a $(p-2,1,1)$ where we must put the 3 in the third row and the 2 in the second row. With the $p-2$ 1s, we can then place either non of them in the third row, or just one of them in the third row. The number of 1s placed in the second row is again restricted by τ_2 and τ_3, and the remaining are placed in the first row. To each of these we can add the $p-1$ 4s in just one way to form a τ-tableau and thus there are $2(\tau_2 - \tau_3 + 1)$ of these tableaux. In our example these are as follows;

xxxxxx11114444
x12
344

xxxxxx11144444
x12
134

Thus we have that $a^2_\tau = 4(\tau_2 - \tau_3 + 1)$.

We now find a^3_τ, first drawing $(p-2,1,1)$ with x's, then adding another $(p-2,1,1)$ where we must put the 2 in the second row and the 3 in the third row. The remaining 1s are added to the first or second row such that the number in the second row is again restricted by τ_2 and τ_3. This gives that $a^3_\tau = \tau_2 - \tau_3 + 1$, which in our example is;

xxxxx111144444
x12
x34

We can then calculate $a_\tau = (4(\tau_2 - \tau_3 + 1)) - (4(\tau_2 - \tau_3 + 1)) + (\tau_2 - \tau_3 + 1) = \tau_2 - \tau_3 + 1$ as required. □

With this information we can now check that for μ the highest weight in its core class and σ the second highest weight, that $a_\sigma > a_\mu$. Well, $\mu = (\lambda_1 + p, \lambda_2, \lambda_3)$ and $\sigma = (\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3)$, and so when $\mu_3 = \sigma_3 = \lambda_3 = 0$ then $a_\mu = \lambda_2 - 1$ and $a_\sigma = \lambda_1 - p + 1 - 1 = \lambda_1 - p$, and indeed, as $\lambda_1 - \lambda_2 \geq p$ then $\lambda_1 - p > \lambda_2 - 1$. On the other hand, when $\lambda_3 \neq 0$ then $a_\mu = \lambda_2 - \lambda_3 + 1$ and $a_\sigma = \lambda_1 - p + 1 - \lambda_3 + 1 = \lambda_1 - \lambda_3 - p + 2$, and again as $\lambda_1 - \lambda_2 \geq p$ then $\lambda_1 - \lambda_3 - p + 2 > \lambda_2 - \lambda_3 + 1$. Hence the coefficient spaces of the tilting modules whose weights are second highest in the core class of the 6-set are contained in cf $(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E)$. □

ii) Here we consider those tilting modules whose weights are second highest in the core class of the 3-set which came from the self-titled weight λ such that $\lambda_1 - \lambda_3 < p - 2$.

THEOREM 4.13.33 *For $r = 3p - 1$ where λ was a self-titled weight of degree $r = 2p - 1$ with $\lambda_1 - \lambda_3 < p - 2$, then*

$$\mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E.$$

Proof. The proof is similar to that of the case $r = 3p-2$. From Section 4.7, we have that the 3-set formed from this self-titled weight λ with $\lambda_1 - \lambda_3 < p - 2$ is as follows;

$$\begin{aligned}&(\lambda_1 + p, \lambda_2, \lambda_3)\\&(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)\\&(\lambda_3 + p - 2, \lambda_1 + 1, \lambda_2 + 1)\end{aligned}$$

Now consider the tilting truncated module $L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E$, and let us first calculate $L(p-1,1,0) \otimes L(p-1,1,0) = (s_{p-1,1} - s_{p-2,1,1})(s_{p-1,1} - s_{p-2,1,1})$. Using the Littlewood-Richardson Rule we result in the following sum;

$$s_{2p-2,2,0} + s_{2p-2,1,1} + s_{2p-3,3,0} + s_{2p-4,4,0} + \ldots + s_{p,p,0} + s_{p-1,p-1,2}.$$

Now, to ensure we have the tilting module in the middle of this core class we need to ensure that the Schur function corresponding to its highest weight arises with a higher multiplicity than the Schur function corresponding to the weight at the top of this core class, when we calculate ch $(L(p-1,1,0) \otimes L(p-1,1,0) \otimes S^{p-1}E)$. Well, the only terms in the above sum which can be multiplied by another s_{p-1} such that the resulting Schur function has weight $(\lambda_1 + p, \lambda_2, \lambda_3)$, are those in the range

$$(2p - \lambda_3, \lambda_3, 0), (2p - (\lambda_3 + 1), \lambda_3 + 1, 0), \ldots, (2p - \lambda_2, \lambda_2, 0)$$

where $\lambda_2 \leq p - 3$ as $\lambda_1 \leq p - 3$ by Proposition 4.3.3. If the second entry is less than λ_3 then when we add on the remaining nodes we would not be able to form a partition with strictly increasing columns. If the second entry is greater than λ_2 then we have too many nodes in the second row. Note that

On the other hand, the terms from the above sum which can be multiplied by another s_{p-1} such that the resulting Schur function has weight $(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3)$, are those in the range

$$(2p - \lambda_3, \lambda_3, 0), (2p - (\lambda_3 + 1), \lambda_3 + 1, 0), \ldots, (2p - (\lambda_1 + 1), \lambda_1 + 1, 0)$$

where $\lambda_1 + 1 \leq p - 2$ as $\lambda_1 \leq p - 3$. If the second entry is less than λ_3 then when we add on the remaining nodes we would not be able to form a partition with strictly increasing columns. If the second entry is greater than $\lambda_1 + 1$ then we have too many nodes in the second row.
Thus, as $\lambda_1 + 1 > \lambda_2$ then the multiplicity of the Schur function corresponding the 'middle' weight is greater than the multiplicity of the Schur function corresponding to the top weight. Hence by 4.10 c) we have that $\mathrm{cf}\,(T(\lambda_2 + p - 1, \lambda_1 + 1, \lambda_3) \subseteq \mathrm{cf}\,(L(p-1, 1, 0) \otimes L(p-1, 1, 0) \otimes S^{p-1}E$ as required. □

iii) Here we consider those tilting modules whose weights are second highest in the core class of the 3-set where $\lambda_1 - p < p - 1$ and where $\lambda_1 - p = p - 1$.

THEOREM 4.13.34 *For $r = 3p - 1$ where $\lambda_1 + p > t$ and $\lambda_1 - p = p - 2$ then*

$$\mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(L(p-1, 2, 0) \otimes L(p-1, 1, 0) \otimes S^{p-2}E).$$

For $r = 3p - 1$ where $\lambda_1 + p > t$ and $\lambda_1 - p = p - 1$ then

$$\mathrm{cf}\,(T(\lambda_{21}, \lambda_{22}, \lambda_{23})) \subseteq \mathrm{cf}\,(L(p-1, p-1, 1) \otimes S^{\frac{p+1}{2}}E \otimes S^{\frac{p-1}{2}}E).$$

Proof. Let us first consider the case where $\lambda_1 - p = p - 2$, we know here that $(\lambda_1, \lambda_2, \lambda_3) = (r_1 - 1, 1, 0) = (2p - 2, 1, 0)$. Therefore for $r_2 = 3p - 1$ then the weight highest in its core class is $(\lambda_2 + 2p - 1, \lambda_1 - p + 1, \lambda_3) = (2p, p - 1, 0)$ and the weight second highest is $(\lambda_1, \lambda_2 + p, \lambda_3) = (2p - 2, p + 1, 0)$. When calculating the character of $\mathrm{Tr}^{\,p+1,p,p-2}E$ it is therefore only necessary to consider weights who have last entry zero. We first find those in $s_{p-1,1} \cdot s_{p-2}$, namely $(2p-3, 1, 0), (2p-4, 2, 0), (2p-5, 3, 0), \ldots, (p-1, p-1, 0)$, and note that the negative part of this calculation $s_{p-2,1,1} \cdot s_{p-2}$ will give no weights with last entry zero. We now multiply each of these weights by $s_{p-1,2}$ and see how many times $(2p, p-1, 0)$ and $(2p-2, p+1, 0)$ occur in each.
$s_{p-1,2} \cdot s_{2p-3,1,0}$ gives weights $(3p-4, 3, 0), \ldots, (2p-1, p, 0)$ and only the highest weight occurs once in this.

$s_{p-1,2} \cdot s_{2p-4,2,0}$ gives weights $(3p-5,4,0),\ldots,(2p-2,p+1,0)$ and each occur once in this.
$s_{p-1,2} \cdot s_{2p-5,3,0}$ gives weights $(3p-6,5,0),\ldots,(2p-3,p+2,0)$ and each occur once in this.
$\vdots$
$s_{p-1,2} \cdot s_{\frac{3p-5}{2},\frac{p+1}{2},0}$ gives weights $(\frac{5p-7}{2},\frac{p+5}{2},0),\ldots,(\frac{3p-1}{2},\frac{3p-1}{2},0)$ and each occur once in this.
$s_{p-1,2} \cdot s_{\frac{3p-7}{2},\frac{p+3}{2},0}$ gives weights $(\frac{5p-9}{2},\frac{p+7}{2},0),\ldots,(\frac{3p-1}{2},\frac{3p-1}{2},0)$ and each occur once in this.
$\vdots$
$s_{p-1,2} \cdot s_{p+1,p-3}$ gives weights $(2p,p-1,0),\ldots,(\frac{3p-1}{2},\frac{3p-1}{2},0)$ and each occur once in this.
$s_{p-1,2} \cdot s_{p,p-2,0}$ gives weights $(2p-1,p,0),\ldots,(\frac{3p-1}{2},\frac{3p-1}{2},0)$ and only the second occurs once in this.
$s_{p-1,2} \cdot s_{p-1,p-1,0}$ gives weights $(2p-2,p+1,0),\ldots,(\frac{3p-1}{2},\frac{3p-1}{2},0)$ and only the second highest weight occurs once in this.

We therefore have that, in $\mathrm{ch}\,(\mathrm{Tr}^{\,p+1,p,p-2}E)$, the Schur function corresponding to the second highest weight arises with multiplicity $p-2$, whilst the Schur function corresponding to the highest weight only arises with multiplicity $p-3$. Hence, by 4.10 c), the coefficient space of the tilting module whose weight is second highest in the core class of the 3-set with $\lambda_1 - p = p-2$ is contained in $\mathrm{cf}\,(\mathrm{Tr}^{\,p+1,p,p-2}E)$.

We now move onto the 3-set where $\lambda_1 - p = p-1$ where we know that $(\lambda_1,\lambda_2,\lambda_3) = (r,0,0) = (2p-1,0,0)$ and so the second highest weight in the 3-set for $r_2 = 3p-1$ is $(\lambda_1, \lambda_1 - p, 1) = (2p-1, p-1, 1)$, and indeed, the leading term in $\mathrm{ch}\,(L(p-1,p-1,1) \otimes S^{\frac{p+1}{2}}E \otimes S^{\frac{p-1}{2}}E)$ is $s_{2p-1,p-1,1}$. Thus by 4.10 c), the coefficient space of the tilting module whose weight is second highest in the core class of the 3-set with $\lambda_1 - p = p-1$ is contained in $\mathrm{cf}\,(L(p-1,p-1,1) \otimes S^{\frac{p+1}{2}}E \otimes S^{\frac{p-1}{2}}E)$. □

iv) For the weights which are third, fourth and fifth highest in the core class of the 6-set, we can use the same proof as for the range $2p \leq r \leq 3p-3$ as the 6-sets for $r = 3p-1$ are formed in exactly the same way as for $2p \leq r \leq 3p-3$, it is just that there are a smaller number of these 6-sets due to there being a smaller number of 3-sets for $2p-1$ because of the cases where $\lambda = (2p-1,0,0)$, $(2p-2,1,0)$ as $\lambda_{1T} - p = p-1$ which form 3-sets not 6-sets which were resolved in sections i)and ii). For the fourth and fifth highest weights we use the truncated module $L(p-1,p-1,0) \otimes S^{\alpha}E \otimes S^{\alpha}E$ as $r = 3p-1$ is even, where $\alpha = \frac{p+1}{2}$. We can then use the proof of the even part of Theorem 4.13.13 from the range $2p \leq r \leq 3p-3$, to show that the Schur functions corresponding to the fourth and fifth weights arise at least once in the character of the above truncated module. Moreover, from the range $2p \leq r \leq 3p-3$ we also have the following result;

THEOREM 4.13.35 *The character of the truncated module*

$$\mathrm{ch}\,(\mathrm{Tr}^{\,2p-2,\alpha,\alpha}E) = \sum a_\lambda s_\lambda \text{ where } a_\lambda = \begin{cases} \lambda_3+1 & \text{for } \lambda_1 \geq p-1+\alpha \\ \lambda_1-\lambda_2+1 & \text{for } \lambda_1 < p-1+\alpha \end{cases}$$

Where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *such that* $p-1 \leq \lambda_2 \leq p-1+2\alpha$, $p-1 \leq \lambda_2 \leq p-1+\alpha$ *and* $0 \leq \lambda_3 \leq 2\alpha$.

Proof. See proof of the same Theorem 4.13.15 for the range $2p \leq r \leq 3p-3$. □

Let s_σ be the Schur function corresponding to the fourth highest weight, then $\sigma = (\lambda_1, \lambda_2+p, \lambda_3)$ and let s_τ be the Schur function corresponding to the fifth highest weight, then $\tau = (\lambda_1, \lambda_3+p-1, \lambda_2+1)$. For the range $2p \leq r \leq 3p-3$ we had that $\sigma_1, \tau_1 \geq p-1+\alpha$, and we proved that in this case $a_\tau = \tau_3+1 = \lambda_2+2 > \lambda_3+1 = \sigma_3+1 = a_\sigma$. However for $r = 3p-1$ it may be possible for, $\sigma_1 = \tau_1 = \lambda_1 < p-1+\alpha$. For example, with $p = 7$ and $r_2 = 20 = r_1+p$, then $\lambda = (9,2,2)$ is a weight at the top of its core class as $9-2 = 7 = p$, and yet $\lambda_1 = 9 < p-1+\alpha = 10$, hence $\sigma_1 = \tau_1 < p-1+\alpha$.
So, in this case $a_\sigma = \sigma_1-\sigma_2+1 = \lambda_1-(\lambda_2+p)+1 = \lambda_1-\lambda_2-p+1$ and $a_\tau = \tau_1-\tau_2+1 = \lambda_1-(\lambda_3+p-1)+1 = \lambda_1-\lambda_3-p+2$, and we need to check that $a_\tau > a_\sigma$. Well, we know that $\lambda_2 \geq \lambda_3$ and hence $\lambda_2+p-1 \geq \lambda_3+p-1 > \lambda_3+p-2$ which is true if and only if $\lambda_1-\lambda_3-p+2 > \lambda_1-\lambda_2-p+1$ and thus $a_\tau > a_\sigma$ as required.
Hence both $\mathrm{cf}\,(T(\sigma))$ and $\mathrm{cf}\,(T(\tau))$ are contained in $\mathrm{cf}\,(L(p-1,p-1,0) \otimes S^\alpha E \otimes S^\alpha E)$ where $\alpha = \frac{p+1}{2}$.

REMARK 4.13.36 *We have therefore found that the coefficient spaces of all tilting modules of* $A(\pi, r)$ *are contained in some tilting truncated module of* $D_3(p,r)$ *and thus have proven Theorem 4.1.3 for the range* $0 \leq r \leq 3p-1$. *Applying the reflection property from Section 3.5 also proves* $D_{3,p}(r) = A(\pi, r)$ *for the range* $6p-8 \leq r \leq 3t$. *Hence, the Doty colagebras are quasi-hereditary and thus have finite global dimension for these ranges.*

Chapter 5

Further cases

AIM: This chapter looks at the range we did not prove $D_{3,p}(r) = A(\pi, r)$ in Chapter Four, namely $3p \leq r \leq 6p - 9$ for $n = 3$, stating why we believe $D_{3,p}(r) \neq A(\pi, r)$ for this range. We then move on to the case $n \geq 4$, again explaining why we conjecture $D_{n,p}(r) \neq A(\pi, r)$ except in certain cases. Finally we prove $D_{n,2}(r) = A(\pi, r)$ for all n and for π a suitable saturated set.

5.1 The case $n = 3$ for $3p \leq r \leq 6p - 9$

AIM: In Chapter Four we proved $D_{3,p}(r) = A(\pi, r)$ for $0 \leq r \leq 3p - 1$ and along with the reflection property in Section 3.5 we also have that $D_{3,p}(r) = A(\pi, r)$ for $6p - 8 \leq r \leq 3t$ where $t = n(p - 1)$. In this chapter we consider the 'middle' range $3p \leq r \leq 6p - 9$, and so begin with a conjecture that $D_{3,p}(r) \neq A(\pi, r)$ for this range. We consider examples for $p = 3$, 5 and 7, and show that we are unable to find the coefficient spaces of the tilting modules of each $A(\pi, r)$ respectively, via the methods we used to prove our theorem for the range $0 \leq r \leq 3p - 1$. We look at other methods we could use and explain why we have been unable to prove $D_{3,p}(r) = A(\pi, r)$ for $3p \leq r \leq 5p - 5$, and thus why we conjecture that $D_{3,p}(r) \neq A(\pi, r)$ for $3p \leq r \leq \frac{nt}{2}$ and thus for the reflection of this range, i.e. $\frac{nt}{2} \leq r \leq 6p - 9$.

CONJECTURE 5.1.1 *The Doty Coalgebras $D_{3,p}(r) \neq A(\pi, r)$ for $3p \leq r \leq 6p - 9$ and $\pi = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1 \leq t,\ |\lambda| = r\}$.*

5.1.1 The example $n = 3$, $p = 3$ and $r = 3p = 9$

It is interesting to point out, that for this example, $nt = 3(3p-3) = 9p-9 = 18$ and thus the 'halfway' point we would need to reach to then use the reflection property in Section 3.5 would be $r = 9 = 3p$. Let us first study the details of this example. There are eight partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ of r into n parts,

such that $\lambda_1 \leq n(p-1) = 6$, namely $(6,3,0),\ (6,2,1),\ (5,4,0),$
$(5,3,1),\ (5,2,2),\ (4,4,1),\ (4,3,2)$ and $(3,3,3)$, and thus there are eight tilting modules whose coefficient spaces we need to find, and eight truncated modules with which to find them. These truncated modules are as follows;
$\mathrm{Tr}^{630}E = L(2,2,2) \otimes L(2,1,0)$
$\mathrm{Tr}^{621}E = L(2,2,2) \otimes L(2,0,0) \otimes L(1,0,0)$
$\mathrm{Tr}^{540}E = L(2,2,1) \otimes L(2,2,0)$
$\mathrm{Tr}^{531}E = L(2,2,1) \otimes L(2,1,0) \otimes L(1,0,0)$
$\mathrm{Tr}^{522}E = L(2,2,1) \otimes L(2,0,0) \otimes L(2,0,0)$
$\mathrm{Tr}^{441}E = L(2,2,0) \otimes L(2,2,0)$
$\mathrm{Tr}^{432}E = L(2,2,0) \otimes L(2,1,0) \otimes L(2,0,0)$
$\mathrm{Tr}^{333}E = L(2,1,0) \otimes L(2,1,0) \otimes L(2,1,0)$

By Theorem 4.8.15 we know that $\mathrm{Tr}^{630}E$ and $\mathrm{Tr}^{333}E$ are not tilting, and thus we have six tilting truncated modules with which to work. It is next necessary to understand the core classes of the tilting modules. Well, each weight apart from $(5,3,1)$ has an empty p-core, whilst $(5,3,1)$ is self-titled, so we have two core classes. So, let us consider each tilting module separately.

$\mathrm{ch}\,(\mathrm{Tr}^{432}E)$ has leading term s_{630} and thus $\mathrm{cf}\,(T(6,3,0)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{432}E)$.
$\mathrm{ch}\,(\mathrm{Tr}^{522}E)$ has leading term s_{621} and thus $\mathrm{cf}\,(T(6,2,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{522}E)$.
$\mathrm{ch}\,(\mathrm{Tr}^{441}E)$ has leading term s_{540} and thus $\mathrm{cf}\,(T(5,4,0)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{441}E)$.
$\mathrm{ch}\,(\mathrm{Tr}^{531}E)$ has leading term s_{531} and thus $\mathrm{cf}\,(T(5,3,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{531}E)$.
$\mathrm{ch}\,(\mathrm{Tr}^{621}E)$ has leading term s_{522} and thus $\mathrm{cf}\,(T(5,2,2)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{621}E)$.
$\mathrm{ch}\,(\mathrm{Tr}^{540}E)$ has leading term s_{441} and thus $\mathrm{cf}\,(T(4,4,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{540}E)$.

The next tilting module to consider is $T(4,3,2)$. Unfortunately the Schur function s_{432} is not the leading term in the character of a tilting truncated module, so we need to look beyond this. As we are looking at $p=3$ we can use James' book to find the composition factors of all the tilting modules, as shown in Fact 4.9.13. These are shown below;
$T(6,3,0)$ has filtration structure

$$\begin{array}{c}\nabla(6,3,0)\\ \nabla(6,2,1)\\ \nabla(5,4,0)\\ \nabla(5,3,2)\\ \nabla(4,4,1)\\ \nabla(4,3,2)\end{array}$$

$T(6,2,1)$ has filtration structure

$$\begin{gathered}\nabla(6,2,1)\\ \nabla(5,2,2)\\ \nabla(4,4,1)\\ \nabla(4,3,2)\end{gathered}$$

$T(5,4,0)$ has filtration structure

$$\begin{gathered}\nabla(5,4,0)\\ \nabla(5,2,2)\\ \nabla(4,4,1)\\ \nabla(4,3,2)\end{gathered}$$

$T(5,2,2)$ has filtration structure

$$\begin{gathered}\nabla(5,2,2)\\ \nabla(4,3,2)\end{gathered}$$

$T(4,4,1)$ has filtration structure

$$\begin{gathered}\nabla(4,4,1)\\ \nabla(4,3,2)\end{gathered}$$

$T(4,3,2)$ has filtration structure

$$\begin{gathered}\nabla(4,3,2)\\ \nabla(3,3,3)\end{gathered}$$

So $\nabla(4,3,2)$ arises as a section in a ∇-filtration of every tilting module which is above it in its core class. We now consider each tilting truncated module, starting with $Tr^{621}E$ and look at the multiplicities of the weights in its character.
The $\operatorname{ch}(\operatorname{Tr}^{621}E) = s_{522} + s_{432}$, so the one $\nabla(4,3,2)$ is 'taken' by the $T(5,2,2)$.
The $\operatorname{ch}(\operatorname{Tr}^{540}E) = s_{441} + s_{432}$, so the one $\nabla(4,3,2)$ is 'taken' by the $T(4,4,1)$.
The $\operatorname{ch}(\operatorname{Tr}^{531}E) = s_{531} + s_{522} + s_{431} + 2s_{432}$, so the two $\nabla(4,3,2)$ are 'taken' by the $T(5,2,2)$and the $T(4,4,1)$.

The ch (Tr ${}^{522}E$) $= s_{621} + s_{531} + 2s_{522} + s_{441} + 2s_{432}$, so the two $\nabla(4, 3, 2)$ are 'taken' by the $T(6, 2, 1)$ and the $T(5, 2, 2)$.
The ch (Tr ${}^{441}E$) $= s_{540} + s_{531} + s_{522} + 2s_{441} + 2s_{432}$, so the two $\nabla(4, 3, 2)$ are 'taken' by the $T(5, 4, 0)$ and the $T(4, 4, 1)$.
The ch (Tr ${}^{432}E$) $= s_{630} + s_{621} + s_{540} + 2s_{531} + 2s_{522} + 2s_{441} + 3s_{432}$, so the three $\nabla(4, 3, 2)$ are 'taken' by the $T(6, 3, 0)$, the $T(5, 2, 2)$ and the $T(4, 4, 1)$.

Therefore, via these methods, we have been unable to find a tilting truncated module whose coefficient space contains the coefficient space of the tilting module $T(4, 3, 2)$. We must therefore now consider other methods. Well, if it were true that cf $(T(4, 3, 2)) \subseteq D_{3,3}(9)$ then it would be the case that $T(4, 3, 2)$ would be contained in the image of some $\phi(\mathrm{Tr}^{\lambda}E)$ where λ is one of the eight partitions given above and $\phi : \mathrm{Tr}^{\lambda}E \to D_{3,3}(9)$ is a homomorphism. Much detailed work in this area has found this is not the case. Other methods were considered and tested, and we were unable to show that cf $(T(4, 3, 2)) \subseteq D_{3,3}(9)$, and so we therefore conjecture that we cannot find this tilting module $T(4, 3, 2)$.

5.1.2 The example $n = 3$, $p = 5$ and $r = 3p = 15$

For this example we have the following core classes;

XXXXXXXX XXX	XX XX X	XXX XX	XXXXXX XX XX	Ø	XXXXX XXX XX	ST
12,3,0	12,2,1	11,4,0	11,2,2	10,5,0	10,3,2	9,5,1
7,4,4	11,3,1	8,7,0	6,6,3	10,4,1	7,6,2	
	9,3,3	8,4,3		9,6,0		
	7,7,1			9,4,2		
	7,5,3			8,6,1		
	6,5,4			8,5,2		
				5,5,5		

It can be shown that the coefficient spaces of all tilting modules whose highest weight is highest or lowest in its core class, or self-titled, will arise in the coefficient space of the tilting truncated module Tr ${}^{654}E$ which is the lowest weighted tilting truncated module. It therefore remains to find those tilting modules whose highest weights sit in the middle of their core class. We shall consider each one in turn, firstly looking at the 6-set. The Schur function of the second highest weight $(11, 3, 1)$ arises in ch (Tr ${}^{654}E$) with multiplicity 2, one higher than the multiplicity of the Schur function of $(12, 2, 1)$ and thus cf $(T(11, 3, 1)) \subseteq$ cf (Tr ${}^{654}E$). The tilting module with the third highest weight $T(9, 3, 3)$ we have due to the same method used in Theorem 4.13.7. The cf $(T(7, 7, 1)) \subseteq$ cf (Tr ${}^{933}E$) as in the character of Tr ${}^{933}E$, s_{771} is the highest

weight that arises in its core class. Finally, the cf $(T(7,4,3)) \subseteq$ cf $(\mathrm{Tr}^{11,3,1})$ as in the character of $\mathrm{Tr}^{11,3,1}E$, s_{743} is the highest weight in its core class to arise.
For the 3-set, we only need to find $T(8,7,0)$, and as s_{870} arises with a higher multiplicity than $s_{11,4,0}$ in ch $(\mathrm{Tr}^{753}E)$, then cf $(T(8,7,0)) \subseteq$ cf $(\mathrm{Tr}^{753}E)$.
The 7-set is where the problems arise. For the tilting module with the second highest weight, we have that cf $(T(10,4,1)) \subseteq$ cf $(\mathrm{Tr}^{933}E)$ as the leading term in the character of $\mathrm{Tr}^{933}E$ is indeed $s_{10,4,1}$. In a similar way, cf $(T(9,4,2)) \subseteq$ cf $(Tr^{10,4,1}E)$ as the leading term in ch $(\mathrm{Tr}^{10,4,1}E)$ is s_{942}. It remains to find $T(9,6,0)$, $T(8,6,1)$ and $T(8,5,2)$, and in a similar way to the previous example, we can show that when we calculate the character of each of the truncated modules, the highest weights of the tilting modules we wish to find never arise with a higher multiplicity than the sum of the multiplicities of the weights above them in their core class, and thus we cannot say that we have found the coefficient spaces of these tilting modules in $D_{3,5}(15)$. There is also the added difficulty in this and all other examples where $p \geq 5$ in that we do not have a full understanding of the composition factors of the tilting modules, as we did for $p = 3$ via James' book. Moreover in this case, we do not just have one tilting module to find, but three. The same continues in the following example.

5.1.3 The example $n = 3$, $p = 7$ and $r = 3p = 21$

In this example we have the following core classes;

∅	XXXXX XX	XXXX XXX	XXXX XX X	XXX XXX X	XXX XX XX	XXXXXXXXXXX XXXX
14,7,1	15,6,0	18,3,0	18,2,1	17,3,1	17,2,2	17,4,0
14,6,1	12,9,0	16,5,0	15,5,1	16,4,1	15,4,2	10,6,5
13,8,0	12,6,3	12,5,4	13,5,3	13,4,4	14,4,3	
13,6,2		11,10,0	11,9,1	10,10,1	10,9,2	
12,8,1		11,6,4	11,7,3	10,7,4	10,8,3	
12,7,2		9,6,6	8,7,6	9,7,5	8,8,5	
7,7,7						

XXXXXXXXX XXX XX	XXXXXXXX XXX XXX	XXXXXXX XXXXX XX	ST	ST
16,3,2	15,3,3	14,5,2	13,7,1	11,5,5
9,8,4	9,9,3	11,8,2		

As in the previous example all tilting modules whose highest weights are highest or lowest in their core class, or are self-titled can be found in the truncated module $\mathrm{Tr}^{876}E$, which is the lowest weighted, tilting truncated module for this degree. The tilting modules whose weights are second highest in the core class

of the 6-set arise in the same truncated module as their weight arises with a higher multiplicity than that of the weight at the top of their core class. The tilting modules whose weights are third highest in the core class of the 6-set arise due to the same method used in Theorem 4.13.7. The remaining tilting modules in the 6-sets arise in the following truncated modules;
$\mathrm{cf}\,(T(11,10,0)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,12,6,3}E)$
$\mathrm{cf}\,(T(11,6,4)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,16,4,1}E)$
$\mathrm{cf}\,(T(11,9,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,13,4,4}E)$
$\mathrm{cf}\,(T(11,7,3)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,15,4,2}E)$
$\mathrm{cf}\,(T(10,10,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,13,4,4}E)$
$\mathrm{cf}\,(T(10,7,4)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,16,4,1}E)$
$\mathrm{cf}\,(T(10,9,2)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,14,6,1}E)$
$\mathrm{cf}\,(T(10,8,3)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,15,4,2}E)$

For the 3-set, $\mathrm{cf}\,(T(12,9,0) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,975}E)$. Finally, for the 7-set we can resolve another two tilting modules; $\mathrm{cf}\,(T(14,6,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,13,4,4}E)$ and $\mathrm{cf}\,(T(13,6,2)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,14,6,1}E)$. What remains are the same tilting modules as for the previous example, those which are third, fifth and sixth highest in the 7-set, and again when we calculate the character of all tilting truncated modules we find that their highest weight does not arise with a higher multiplicity than the sum of the multiplicities of all the weights above them in their core class. We can therefore not say whether these tilting modules are contained in $D_{3,7}(21)$.

REMARK 5.1.2 If we then look back at Section 4.7, we can see that for the next range of r, namely $4p \leq r \leq 5p-5$, the same issue will arise. For this range we have two core classes which are 6-sets and again a core class which is a 7-set, thus there are many more tilting modules to find whose weights sit in the middle of their core class.

5.2 The case $n \geq 4$

AIM: Our understanding of what happens for $n \geq 4$ is not complete, however there are a number of points which can be made for this case. We firstly prove that $D_{n,p}(r) = A(\pi, r)$ for $0 \leq r \leq p-1$ and $nt-(p-1) \leq r \leq nt$. We then consider further ranges of r stating why we believe $D_{n,p}(r) \neq A(\pi, r)$ for $p \leq r \leq nt-p$ except in the case $n=4$ and $p=3$.

5.2.1 The range $0 \leq r \leq p-1$

THEOREM 5.2.1 *For* $0 \leq r \leq p-1$ *then* $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,\lambda}E)$

Proof. For $0 \leq r \leq p-1$ then $L(\lambda) = \nabla(\lambda) = \Delta(\lambda) = T(\lambda)$. Moreover each weight λ is self-titled and thus has its own core class. $\mathrm{Tr}^{\,\lambda}E = S^{\lambda_1}E \otimes S^{\lambda_2}E \otimes$

$S^{\lambda_3}E$, where $\lambda_i \leq p-1$ and thus each $S^{\lambda_i}E$ is tilting. The Schur function s_λ arises in $\mathrm{ch}\,(\mathrm{Tr}^{\,\lambda}E)$ and as the weight λ is self-titled then $\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\,\lambda}E)$ as required. □

COROLLARY 5.2.2 *Using the reflection property of Section 3.5, this also proves that $D_{n,p}(r) = A(\pi, r)$ for $nt-(p-1) \leq r \leq nt$.*

5.2.2 The range $p \leq r \leq 2p-1$

For each self-titled weight from $0 \leq r \leq p-1$ there are n possible ways to add on a new p-hook. Namely adding it on to the first row, then adding it on starting at the second row, and so forth until we add it on starting at the nth row. Thus there will be core classes consisting of n weights.

Now, the lowest weighted truncated module for each degree r is as follows;

For $n \mid r$: $(\frac{r+1}{n}, \frac{r+1}{n}, \ldots, \frac{r+1}{n})$

For $n \mid r+1$: $(\frac{r+1}{n}, \ldots, \frac{r+1}{n}, \frac{r+1}{n}-1)$

For $n \mid r+2$: $(\frac{r+2}{n}, \ldots, \frac{r+2}{n}, \frac{r+2}{n}-1, \frac{r+2}{n}-1)$

$\vdots$

For $n \mid r+(n-1)$: $(\frac{r+(n-1)}{n}, \frac{r+(n-1)}{n}-1, \ldots, \frac{r+(n-1)}{n}-1)$.

The greatest entry here is $\frac{r+(n-1)}{n} \leq \frac{2p-1+n-1}{n} = \frac{2p-2+n}{n}$. Now $\frac{2p-2+n}{n} \leq p-1$ if and only if $p \geq \frac{2n-2}{n-2}$. However $\frac{2n-2}{n-2} < 3$ so we would require $p \geq 3$, which we know to be true. The case $p=2$ is resolved in the following section. We therefore know that the lowest weighted truncated module is always tilting and then by the Littlewood-Richardson Rule, the $\mathrm{ch}\,(\mathrm{Tr}^{\,\lambda}E)$ for λ minimal gives the Schur function of each weight in the degree r at least once. We therefore have that the coefficient spaces of the tilting modules whose highest weights are highest or lowest in their core class, or self-titled, are contained in $\mathrm{cf}\,(\mathrm{Tr}^{\,\lambda}E)$ for λ minimal. Moreover, just as in the case $n=3$ the Schur function of the tilting module whose weight is second highest in its core class arises with a higher multiplicity than the Schur function of the weight above it. Thus we have the coefficient space of this tilting module aswell.

Thus, because of the structure of the case $n=4$, $p=3$ where there are at most three weights in a core class for the range $p \leq r \leq 2p-1$, we have that the coefficient spaces of all tilting modules in this range are contained in $\mathrm{cf}\,(\mathrm{Tr}^{\,\lambda}E)$ for λ minimal, i.e. $D_{4,3}(r) = A(\pi, r)$ for $p \leq r \leq 2p-1$. Moreover, using the reflection property of Section 3.5 we also have that $D_{4,3}(r) = A(\pi, r)$ for $nt-(2p-1) \leq r \leq nt$.

However for all other cases there will be the third highest weights, to the $n-1$ highest weights which it still remains to find.

EXAMPLE 5.2.3 Let $n = 4$ and $r = p = 5$, then we have the core classes;

Ø	ST	ST
(5,0,0,0)	(3,2,0,0)	(2,2,1,0)
(4,1,0,0)		
(3,1,1,0)		
(2,1,1,1)		

The lowest weighted truncated module is $\mathrm{Tr}^{2111}E$ whose character is

$$s_5 + 3s_{41} + 3s_{32} + 3s_{311} + 2s_{221} + s_{2111},$$

so as stated we have that $\mathrm{cf}\,(T(5,0,0,0))$, $\mathrm{cf}\,(T(3,2,0,0))$, $\mathrm{cf}\,(T(2,2,1,0))$, $\mathrm{cf}\,(T(2,1,1,1))$ and $\mathrm{cf}\,(T(4,1,0,0))$ are contained in $\mathrm{cf}\,(\mathrm{Tr}^{2111}E)$. It remains to find $T(3,1,1,0)$ but as the multiplicity of s_{311} is 3 and the multiplicities of s_{41} and s_5 sum to give 4 we cannot say that $\mathrm{cf}\,(T(3,1,1,1)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{2111}E)$. Moreover, for the remaining tilting truncated modules $\mathrm{Tr}^{41}E$, $\mathrm{Tr}^{32}E$, $\mathrm{Tr}^{311}E$ and $\mathrm{Tr}^{221}E$, the same occurs in that in their character the multiplicity of s_{311} is not greater than the sum of the multiplicities of the Schur functions of s_{41} and s_5.

This is just one example, and obviously as n increases so the number of tilting modules to find will be greater. It is also the case for $n \geq 4$, that we cannot necessarily use the crucial Fact 4.9.17, which we used for the case $n = 3$. Firstly, it may not be possible for two weights, λ and μ, in the same core class to admit a horizontal h-cut. However, more importantly, even if they do, we would then be in the case GL_{n-1}, and unless $n - 1 = 2$, or we can make more horizontal h-cuts to get back to GL_2, we cannot be sure that $(T(\lambda) : \nabla(\mu)) \leq 1$ where $\lambda > \mu$. Therefore the method given in 4.10 c), of ensuring the multiplicity of s_μ in strictly greater than that of s_λ, cannot be used. This makes it that much harder to find tilting modules whose weights are in the middle of their core class.

It is also the case that the number of tilting truncated modules decreases as n increases. We have

$$\begin{aligned}\bar{S}E = &L(0,\ldots,0) \oplus L(1,0,\ldots,0) \oplus \ldots \oplus L(p-1,0,\ldots,0)\\ &\oplus L(p-1,1,0,\ldots,0) \oplus L(p-1,2,0,\ldots,0) \oplus \ldots\\ &\oplus L(p-1,p-1,0,\ldots,0)\\ &\;\vdots\\ &\oplus L(p-1,\ldots,p-1,1,0,0) \oplus L(p-1,\ldots,p-1,2,0,0) \oplus \ldots\\ &\oplus L(p-1,\ldots,p-1,0) \oplus (p-1,\ldots,p-1,1) \oplus L(p-1,\ldots,p-1,2)\\ &\oplus \ldots \oplus L(p-1,\ldots,p-1,p-1)\end{aligned}$$

where the only tilting modules are those in the ranges

$$L(1, 0, \dots, 0), \dots, L(p-1, 0, \dots, 0)$$

and

$$L(p-1, \dots, p-1, 0), \dots, L(p-1, \dots, p-1, p-1).$$

This then greatly restricts the number of tilting truncated modules which we can work with, which plays a clear negative role for the range $r > t$.

5.2.3 The range $r > t$

Take the example $n = 4$, $p = 3$ and $r = 11$. The lowest weighted truncated module is

$$\operatorname{Tr}^{3332} E = L(2, 1, 0, 0) \otimes L(2, 1, 0, 0) \otimes L(2, 1, 0, 0) \otimes S^2 E.$$

Now $L(2, 1, 0, 0) \otimes S^2 E = (s_{21} - s_{111}) \cdot s_2 = s_{41} + s_{32} + s_{221} - s_{2111}$ is not tilting. Moreover when we then calculate $(s_{21} - s_{111})(s_{21} - s_{111})(s_{41} + s_{32} + s_{221} - s_{2111})$ we find negatives in the result and thus $\operatorname{Tr}^{3332} E$ is not tilting. The next lowest weight is

$$\operatorname{Tr}^{4322} E = L(2, 2, 0, 0) \otimes L(2, 1, 0, 0) \otimes S^2 E \otimes S^2 E$$

which again we can show is not tilting. This continues to the next truncated module

$$\operatorname{Tr}^{4331} E = L(2, 2, 0, 0) \otimes L(2, 1, 0, 0) \otimes L(2, 1, 0, 0) \otimes S^1 E$$

which again is not tilting. It is unnecessary to continue looking at the truncated modules as when considering the character of all other truncated modules, whether they be tilting or not, if s_μ is the leading term in the character of these truncated modules, then $\mu \leq (7, 4, 0, 0)$, and thus we cannot find the tilting modules whose highest weights are higher than μ, namely $T(8, 3, 0, 0)$, $T(8, 2, 1, 0)$ and $T(8, 1, 1, 1)$.

It is the combination of these issues that leads us to the conclusion that $D_{n,p}(r) \neq A(\pi, r)$ for $n \geq 4$ and $p \leq r \leq nt - p$ except for the case were $p = 3$.

5.3 The case $p = 2$ for all n

THEOREM 5.3.1 *For $p = 2$, and for all n, the Doty Coalgebras $D_{n,2}(r) = A(\pi, r)$ for $\pi = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \mid \lambda_1 \leq n(p-1) = n\}$ a saturated set and where $0 \leq r \leq nt = n^2$.*

Proof. For n arbitrary, $\bar{S}E = \bar{S}^0 E \oplus \bar{S}^1 E \oplus \ldots \oplus \bar{S}^n E$ where

$$\bar{S}^j E = k = L(\overbrace{1, 1, \ldots, 1}^{j}, \overbrace{0, \ldots, 0}^{n-j}).$$

Now,

$$\mathrm{dim} L(\overbrace{1, 1, \ldots, 1}^{j}, \overbrace{0, \ldots, 0}^{n-j}) = \mathrm{dim} \bar{S}^j E$$

and in this case $\bar{S}^j E$ has weights $\sum \epsilon_{i_1} + \cdots + \epsilon_{i_j}$ where $i_1 < i_2 < \cdots < i_j$ which we know to be the weights of $\Lambda^j E$ due to the basis of $\Lambda^j E$ consisting of the elements $e_{i_1} \wedge \cdots \wedge e_{i_j}$ and thus

$$\mathrm{dim} \bar{S}^j E = \mathrm{dim} \Lambda^j E = \mathrm{dim} \nabla(1, 1, \ldots, 1, 0, \ldots, 0)$$

and so

$$L(1, 1, \ldots, 1, 0, \ldots, 0) = \nabla(1, 1, \ldots, 1, 0, \ldots, 0).$$

Similarly

$$L(1, 1, \ldots, 1, 0, \ldots, 0) = \Delta(1, 1, \ldots, 1, 0, \ldots, 0)$$

and thus

$$L(1, 1, \ldots, 1, 0, \ldots, 0) = T(1, 1, \ldots, 1, 0, \ldots, 0).$$

Therefore all $\bar{S}^j E$ are tilting.

Now, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i = r$, than we can find the transpose of λ, which we shall call $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$, where $m \leq n$ as for $\lambda \in \pi$, $\lambda_1 \leq n$. For example, consider $\lambda = (4, 3, 1, 0)$ then $\lambda' = (3, 2, 2, 1)$. Then

$$\mathrm{Tr}^{\lambda'} E = \Lambda^{\lambda'} E = T(\lambda) \oplus [\oplus_{\mu < \lambda} T(\mu)^{(d_\mu)}]$$

and therefore

$$\mathrm{cf}\,(T(\lambda)) \subseteq \mathrm{cf}\,(\mathrm{Tr}^{\lambda'} E) \subseteq D_{n,2}(r)$$

so

$$A(\pi) \subseteq D_{n,2}(r).$$

□

Chapter 6

A family of quiver algebras of finite global dimension which are not quasi-hereditary

AIM: In Section 1.3 we defined quasi-hereditary algebras and stated that all quasi-hereditary algebras have finite global dimension, but that the reverse is not necessarily true. Whilst researching quasi-hereditary algebras we studied a paper by Dlab and Ringel [3], which described an 11-dimensional serial algebra of global dimension 4 which is not quasi-hereditary.
By considering this example and the structure of the algebra we were able to discover an infinite family of algebras of finite global dimension which are not quasi-hereditary. All of these algebras are quiver algebras and thus have a nice picture to go with each of them! This chapter defines this infinite family of quiver algebras.

6.1 Preliminaries

AIM: This section gives the information needed in defining our family of quiver algebras. We start with the definition of a quiver and quiver algebra. We then define idempotents and projective indecomposable modules, and show the correspondence between these projective indecomposable modules and the simple modules of a ring.

DEFINITION 6.1.1 A quiver Γ is a directed graph and is given by a set of vertices and a set of arrows between these vertices. An arrow α starts at the vertex $s(\alpha)$ and terminates at the vertex $t(\alpha)$.
A path in Γ consists of a sequence of arrows $\alpha_1\alpha_2\ldots\alpha_n$ with $n \geq 1$ such that the $t(a_i) = s(a_{i+1})$. We use the convention of concatenating paths from left

to right. For each vertex i we denote by e_i the trivial path which starts and terminates at i

DEFINITION 6.1.2 Let k be a field, and Γ a quiver. Then a quiver algebra $k\Gamma$ is a k-algebra with basis consisting of all paths (including trivial paths) in Γ, with multiplication given by concatenation of paths. We thus have that for two paths x and y then the product is equal to zero unless $t(x) = s(y)$.
This multiplication is associative.

EXAMPLE 6.1.3 If Γ is the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

then $k\Gamma = k\text{-sp}\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$.

Having defined quiver algebras, we now go on to consider idempotents.

The following definitions are given for finite-dimensional algebras.

DEFINITION 6.1.4 Let A be algebra. Then $e \in A$ is called an idempotent if $e^2 = e$. Moreover two idempotents are called orthogonal if $e_1e_2 = e_2e_1 = 0$. We then say an idempotent is primitive if $e = e^2 \neq 0$ and there is no expression $e = f + g$ where f and g are non zero idempotents such that $fg = gf = 0$.

We now go on to define projective indecomposable modules.

DEFINITION 6.1.5 Let A be an algebra such that ${}_AA = \oplus_{i=1}^{t} P_i$ for P_i indecomposable. Then these summands are called the principal indecomposable modules of A.

THEOREM 6.1.6 *[20, Chapter 1, 3.13] i) If ${}_AA \cong \oplus_{i\in I} P_i \cong \oplus_{j\in J} Q_j$ where the P_i and Q_j are all indecomposable, then there exists a bijection $\phi : I \to J$ such that $P_i = Q_{\phi(i)}$, for all $i \in I$.
ii) A finitely-generated indecomposable A-module M is projective if and only if $M \cong P_i$ for P_i a primitive indecomposable module of ${}_AA$. Hence every P_i is projective.*

We now describe the link between projective indecomposable modules and simple modules.

THEOREM 6.1.7 *[20, Chapter 1, 3.14] Let A be a finite dimensional algebra and e a primitive idempotent. Set $P = eA$, then P contains a unique maximal submodule, namely $eJ(A)$, where $J(A)$ is the radical of A.*

COROLLARY 6.1.8 *[20, Chapter 1, 3.15] There is a one-to-one correspondence between the isomorphism classes of the projective indecomposable modules of R and the isomorphism classes of the simple R-modules. This is given by $P = eA \mapsto eA/eJ(A)$.*

6.2 The family of quiver algebras

AIM: We first give the thereom we wish to prove, which describes our family of quiver algebras. We then break this thereom down into different values of r. For each case we will give an example of such a quiver algebra, showing why it has finite global dimension and why it is not quasi-hereditary. We will then generalise each case with proofs.

THEOREM 6.2.1 *For the following values of n and λ there exists a quiver algebra A of dimension n with λ simples, such that A has finite global dimension $m = (\lambda - 1)2$, but is not quasi-hereditary.*

No. of simples λ	*gl.dim* $m = (\lambda - 1)2$	*dim* n		
		$r = 1$	$r = 2$	$r = 3$
3	4	11	17	23
4	6	19	31	43
5	8	29	49	69
6	10	41	71	101
7	12	55	97	139
8	14	71	127	183
9	16	89	161	233
10	18	109	199	289
11	20	131	241	351

This is an infinite table where r is the number of paths between vertices $\lambda - 1$ and λ.

We make the following remark regarding quasi-hereditary algebras, which will be used when showing our family of quiver algebras are not quasi-hereditary.

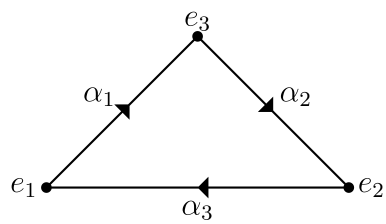
REMARK 6.2.2 Let A be a quasi-hereditary algebra with simples L_i, projective modules P_i and standard modules Δ_i where $1 \leq i \leq n$. If μ is maximal, then $P_\mu = \Delta_\mu$ as otherwise P_μ would have filtration structure

$$\begin{array}{c} \underline{\Delta_\mu} \\ \vdots \\ \Delta_x \end{array}$$

for $x > \mu$.

We now begin to define the family of quiver algebras, starting with the example given by Dlab and Ringel [3].

EXAMPLE 6.2.3 Let A be the quiver algebra



modulo $I =< \alpha_1\alpha_2\alpha_3\alpha_1,\ \alpha_3\alpha_1\alpha_2 >$.

Then $A = k\text{-sp}\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_1\alpha_2\alpha_3, \alpha_2\alpha_3\alpha_1\}$ is a serial algebra of dimension $n = 11$. This algebra has the 3 idempotents e_1, e_2, e_3, and with the projective indecomposables $P_i = e_iA$ then;

$P_1 = k\text{-sp}\{e_1, \alpha_1, \alpha_1\alpha_2, \alpha_1\alpha_2\alpha_3\}/I$
$P_2 = k\text{-sp}\{e_2, \alpha_2, \alpha_2\alpha_3, \alpha_2\alpha_3\alpha_1\}/I$
$P_3 = k\text{-sp}\{e_3, \alpha_3, \alpha_3\alpha_1\}/I$.

We can lay out the projectives as a table of paths starting at e_i, for each P_i;

P_1	P_2	P_3
1	2	3
2	3	1
3	1	2
1	2	

It is now necessary to prove that A has finite global dimension, and so find the projective dimension of each simple module. By Corollary 6.1.8 we have an isomorphism between the projective indecomposables and the simples, and indeed setting $L_i = \text{hd}(P_i)$, then we have three simple modules L_1, L_2, L_3. The fact that A is a serial algebra [3] means we can resolve these simple modules in the following way:

We can resolve L_1 as follows:

$$0 \to P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

and thus $\text{pdim}(L_1) = 3$.
We can resolve L_2 as follows:

$$0 \to P_3 \to P_2 \to L_2 \to 0$$

and thus $\text{pdim}(L_2) = 1$.
We can resolve L_3 as follows:

$$0 \to P_3 \to P_2 \to P_3 \to P_1 \to P_3 \to L_3 \to 0$$

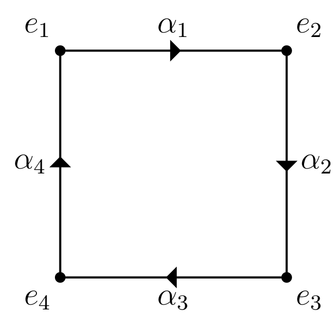
and thus $\text{pdim}(L_3) = 4$.

So A has global dimension $4 = (3-1)2 = (\lambda - 1)2$.

We now show why A is not quasi-hereditary. To do this we consider each possible way of ordering the simple modules, and show that none of these orderings give a way of filtering the projective modules P_i, by the standard modules Δ_i. Well, by Remark 6.2.2, we cannot take 1 or 2 to be maximal as $[P_1 : L_1] = 2$ and $[P_2 : L_2] = 2$. Our only choice then is to take 3 to be maximal. Now let us assume for a contradiction that A is quasi-hereditary. We have $[P_1 : L_3] = 1$ which implies $(P_1 : \Delta_3) = 1$ which implies Δ_3 embeds in P_1, thus giving that $\text{soc}(\Delta_3)$ embeds in P_1. However this is a contradiction as we can see from the above table that there is a L_1 sitting at the bottom of P_1. Hence A cannot be quasi-hereditary.

We now consider another example, but this time with four simple modules.

EXAMPLE 6.2.4 Let A be the quiver algebra



modulo $I = < \alpha_1\alpha_2\alpha_3\alpha_4\alpha_1,\ \alpha_2\alpha_3\alpha_4\alpha_1\alpha_2,\ \alpha_4\alpha_1\alpha_2\alpha_3 >$.

Then A has dimension $n = 19$. Moreover it has the 4 idempotents e_1, e_2, e_3, e_4, and again, laying out the projectives as paths starting at each e_i, we have the four projective modules;

P_1	P_2	P_3	P_4
1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3
1	2	3	

It is now necessary to prove that A has finite global dimension, and so we find the projective dimension of each simple module. We first show that A is serial, by showing each P_i is uniserial. Well, we have $J = k$-span of all non-trivial paths, and $P = eA$. Then $PJ^t = k$-span of all non-trivial paths starting at e of length $\geq t$, where $1 \leq t \leq n$. We then have that $\dim PJ^t/PJ^{t+1}$ is equal to the number of non-trivial paths of length precisely $t = 1$. Hence, by [1, Chapter II, Proposition 5], we have that each P_i is uniserial, and hence A is a serial algebra. We can therefore form the following minimal projective resolutions for each L_i as follows.

We can resolve L_1 as follows:

$$0 \to P_4 \to P_3 \to P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

and thus $\text{pdim}(L_1) = 5$.
We can resolve L_2 as follows:

$$0 \to P_4 \to P_3 \to P_3 \to P_2 \to L_2 \to 0$$

and thus $\text{pdim}(L_2) = 3$.
We can resolve L_3 as follows:

$$0 \to P_4 \to P_3 \to L_3 \to 0$$

and thus $\text{pdim}(L_3) = 1$.
We can resolve L_4 as follows:

$$0 \to P_4 \to P_3 \to P_4 \to P_2 \to P_4 \to P_1 \to P_4 \to L_4 \to 0$$

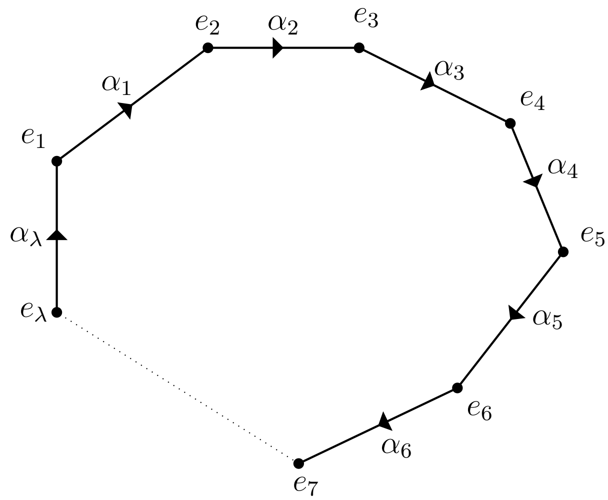
and thus $\text{pdim}(L_4) = 6$.

So A has global dimension $6 = (4-1)2 = (\lambda - 1)2$.

We now show why A is not quasi-hereditary. To do this we consider each possible way of ordering the simple modules, and show that none of these orderings give a way of filtering the projective modules P_i, by the standard modules Δ_i. By Remark 6.2.2, we can only take 4 to be maximal as $[P_i : L_i] = 2$ for $1 \leq i \leq 3$. So, choose 4 to be greatest, then assume A is quasi hereditary. We have $[P_1 : L_4] = 1$ and hence $(P_1 : \Delta_4) = 1$. We therefore have that Δ_4 embeds in P_1 and hence so does $\text{soc}(\Delta_4)$ thus giving that L_3 embeds in P_1. However this is a contradiction as we can see by the above table, where L_1 sits at the bottom of P_1, thus A is not quasi-hereditary.

We now bring these two examples together in the following theorem.

THEOREM 6.2.5 *Quiver algebras of the form*



modulo

$I = < \alpha_1\alpha_2 \ldots \alpha_\lambda\alpha_1, \alpha_2\alpha_3 \ldots \alpha_\lambda\alpha_1\alpha_2, \ldots, \alpha_{\lambda-2}\alpha_{\lambda-1}\alpha_\lambda\alpha_1 \ldots \alpha_{\lambda-2}, \alpha_\lambda\alpha_1 \ldots \alpha_{\lambda-1} >$

having λ simple modules and dimension n, are not quasi-hereditary, but have finite global dimension $m = (\lambda - 1)2$.

REMARK 6.2.6 In these cases we have one path between vertices $\lambda - 1$ and λ and thus $r = 1$.

Proof. We first prove that these algebras have finite global dimension and then prove that they are not quasi-hereditary. Let P_i be the projective modules where $i \in \{1, \ldots, \lambda\}$. Then we can display the projectives as follows;

P_1	P_2	P_3	$\ldots$	$P_{\lambda-1}$	P_λ
1	2	3		$\lambda - 1$	λ
2	3	4		λ	1
3	4	$\vdots$		1	2
4	$\vdots$	λ		2	$\vdots$
$\vdots$	λ	1		$\vdots$	$\lambda - 2$
λ	1	2		$\lambda - 2$	$\lambda - 1$
1	2	3		$\lambda - 1$	

We have that $\dim PJ^t/PJ^{t+1} = 1$ for $1 \leq t \leq n$ and thus each P_i is uniserial, and hence A is a serial algebra. We can therefore form the following minimal projective resolutions for each simple module. We start with L_1;

$$0 \to P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_4 \to P_3 \to P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

Then the $\text{pdim}(L_1) = 2(\lambda - 2) + 2 - 1 = 2\lambda - 3$.

We resolve L_2 as follows;

$$0 \to P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_5 \to P_4 \to P_4 \to P_3 \to P_3 \to P_2 \to L_2 \to 0$$

Then the $\text{pdim}(L_2) = 2(\lambda - 3) + 2 - 1 = 2\lambda - 5$.

We resolve L_3 as follows;

$$0 \to P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to P_5 \to P_4 \to P_4 \to P_3 \to L_3 \to 0$$

Then the $\text{pdim}(L_3) = 2(\lambda - 4) + 2 - 1 = 2\lambda - 7$. This continues for each L_i and we now show how we resolve $L_{\lambda-1}$;

$$0 \to P_\lambda \to P_{\lambda-1} \to e_{\lambda-1} \to 0$$

So $\text{pdim}(L_{\lambda-1}) = 2(\lambda - (\lambda - 1 + 1)) + 2 - 1 = 1$. Bringing this together, then in general we can resolve each L_i for $1 \leq i \leq \lambda - 1$ as follows;

$$0 \to P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_{i+2} \to P_{i+1} \to P_{i+1} \to P_i \to L_i \to 0$$

hence $\text{pdim}(L_i) = 2(\lambda - (i+1)) + 2 - 1 = 2\lambda - (2i + 1)$.

Finally we resolve L_λ as follows;

$$0 \to P_\lambda \to P_{\lambda-1} \to P_\lambda \to \ldots \to P_\lambda \to P_2 \to P_\lambda \to P_1 \to P_\lambda \to L_\lambda \to 0$$

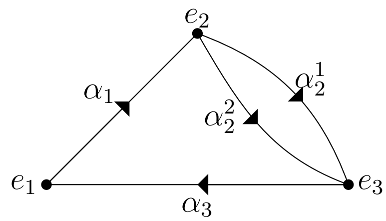
and $\text{pdim}(L_\lambda) = (\lambda - 1) + (\lambda - 1) + 1 - 1 = 2\lambda - 2$.

Hence the gl.dim $A = \text{pdim}(L_\lambda) = 2\lambda - 2$.

We now prove why these algebras are not quasi-hereditary, by considering all possible orderings on the simple modules. Again, by Remark 6.2.2, we must take λ to be maximal, and then let us assume that these algebras are quasi-hereditary. We have $[P_1 : L_\lambda] = 1$ which implies $(P_1 : \Delta_\lambda) = 1$ and thus Δ_λ embeds in P_1 and hence $\text{soc}(\Delta_\lambda)$ embeds in P_1 which is a contradiction, as seen from the above table of projectives. Hence these algebras are not quasi-hereditary. □

Having looked at the case $r = 1$ we now extend on the two examples given and so look at the case $r = 2$.

EXAMPLE 6.2.7 Let A be the quiver algebra



modulo $I =< \alpha_1\alpha_2^2\alpha_3\alpha_1,\ \alpha_3\alpha_1\alpha_2^1,\ \alpha_3\alpha_1\alpha_2^2 >$.

This algebra still has the 3 idempotents e_1, e_2, e_3, and 3 projective modules, and thus also 3 simple modules, just as in Example 6.2.3. However, its projectives, when layed out as a table of paths, are as follows;

P_1	P_2	P_3
1	2	3
2	3	1
3	1	2
1	2	
2	3	
3	1	
1	2	

Now, P_2 has a maximal submodule

$$M_2 = k\text{-sp}\{\alpha_2^1, \alpha_2^1\alpha_3, \alpha_2^1\alpha_3\alpha_1, \alpha_2^2, \alpha_2^2\alpha_3, \alpha_2^2\alpha_3\alpha_1\}.$$

Moreover $M_2 = V \oplus W$ where $V = k\text{-sp}\{\alpha_2^1, \alpha_2^1\alpha_3, \alpha_2^1\alpha_3\alpha_1\}$ and $W = k\text{-sp}\{\alpha_2^2, \alpha_2^2\alpha_3, \alpha_2^2\alpha_3\alpha_1\}$. Let $\phi : P_3 \to V$ such that $\phi(x) = \alpha_2^1 x$ for $x \in P_3$, then $V \cong P_3$. Similarly, with $\theta : P_3 \to W$ such that $\theta(y) = \alpha_2^2 y$ for $y \in P_3$, then $W \cong P_3$.
In a similar way, P_1 has submodule

$$N_1 = k\text{-sp}\{\alpha_1\alpha_2^1, \alpha_1\alpha_2^1\alpha_3, \alpha_1\alpha_2^1\alpha_3\alpha_1, \alpha_1\alpha_2^2, \alpha_1\alpha_2^2\alpha_3\}.$$

Moreover $N_1 = V_1 \oplus W_1$ where $V_1 = k\text{-sp}\{\alpha_1\alpha_2^1, \alpha_1\alpha_2^1\alpha_3, \alpha_1\alpha_2^1\alpha_3\alpha_1\}$ and $W = k\text{-sp}\{\alpha_1\alpha_2^2, \alpha_1\alpha_2^2\alpha_3\}$. In this case, letting $\phi : P_3 \to V_1$ such that $\phi(x) = \alpha_1\alpha_2^1 x$ for $x \in P_3$ gives $V_1 \cong P_3$. Hence, we can also display the above table in the following way;

P_1	P_2	P_3
1	2	3
2	$P_3 \oplus P_3$	1
$P_3 \oplus \begin{matrix} 3 \\ 1 \end{matrix}$		2

We then show how this algebra has finite global dimension by forming minimal resolutions for each of the simple modules. We start with L_1;

$$0 \to P_3 \oplus P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

so $\text{pdim}(L_1) = 3$.

We now resolve the simple module L_2;

$$0 \to P_3 \oplus P_3 \to P_2 \to L_2 \to 0$$

and so $\text{pdim}(L_2) = 1$.

Finally we resolve L_3;

$$0 \to P_3 \oplus P_3 \to P_2 \to P_3 \oplus P_3 \to P_1 \to P_3 \to L_3 \to 0$$

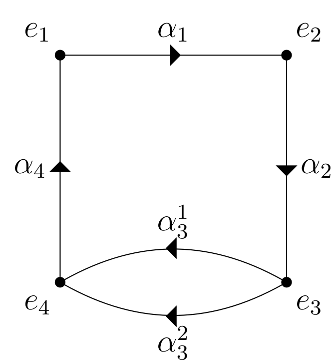
and thus $\text{pdim}(L_3) = 4$.

We therefore have that this algebra has finite global dimension

$$m = 4 = (\lambda - 1)2,$$

just as for Example 6.2.3. We now show that this algebra is not quasi-hereditary by considering all possible orderings on the simple modules. Well, again by Remark 6.2.2, the only possibility is to take 3 to be maximal as $[P_i : L_i] = 3$ for $i = 1, 2$. So, take 3 to be maximal, and assume A is quasi-hereditary. We have $[P_1 : L_3] = 2$ which implies $(P_1 : \Delta_3) = 2$ and hence $\Delta_3 \oplus \Delta_3$ embeds in P_1. This implies L_2 occurs twice in the socle of P_1 which we can see is a contradiction by the above table of projectives. Hence A is not quasi-hereditary.

EXAMPLE 6.2.8 We now extend on Example 6.2.4, and consider the quiver algebra



modulo $I =< \alpha_1\alpha_2\alpha_3^2\alpha_4\alpha_1,\ \alpha_2\alpha_3^2\alpha_4\alpha_1\alpha_2,\ \alpha_4\alpha_1\alpha_2\alpha_3^1,\ \alpha_4\alpha_1\alpha_2\alpha_3^2 >$.

Again, A has the 4 idempotents e_1, e_2, e_3, e_4, but now has projective modules as follows:

P_1	P_2	P_3	P_4
1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3
1	2	3	
2	3	4	
3	4	1	
4	1	2	
1	2	3	

Now, P_3 has a maximal submodule

$$M_3 = k\text{-sp}\{\alpha_3^1, \alpha_3^1\alpha_4, \alpha_3^1\alpha_4\alpha_1, \alpha_3^1\alpha_4\alpha_1\alpha_2, \alpha_3^2, \alpha_3^2\alpha_4, \alpha_3^2\alpha_4\alpha_1, \alpha_3^2\alpha_4\alpha_1\alpha_2\}.$$

Moreover $M_3 = V_3 \oplus W_3$ where

$$V_3 = k\text{-sp}\{\alpha_3^1, \alpha_3^1\alpha_4, \alpha_3^1\alpha_4\alpha_1, \alpha_3^1\alpha_4\alpha_1\alpha_2\}$$

and

$$W_3 = k\text{-sp}\{\alpha_3^2, \alpha_3^2\alpha_4, \alpha_3^2\alpha_4\alpha_1, \alpha_3^2\alpha_4\alpha_1\alpha_2\}.$$

Let $\phi : P_4 \to V_3$ such that $\phi(x) = \alpha_3^1 x$ for $x \in P_4$, then $V_3 \cong P_4$. Similarly, with $\theta : P_4 \to W_3$ such that $\theta(y) = \alpha_3^2 y$ for $y \in P_4$, then $W_3 \cong P_4$.
In a similar way, P_2 has submodule

$$N_2 = k\text{-sp}\{\alpha_2\alpha_3^1, \alpha_2\alpha_3^1\alpha_4, \alpha_2\alpha_3^1\alpha_4\alpha_1, \alpha_2\alpha_3^1\alpha_4\alpha_1\alpha_2, \alpha_2\alpha_3^2, \alpha_2\alpha_3^2\alpha_4, \alpha_2\alpha_3^2\alpha_4\alpha_1\}.$$

Moreover $N_2 = V_2 \oplus W_2$ where

$$V_2 = k\text{-sp}\{\alpha_2\alpha_3^1, \alpha_2\alpha_3^1\alpha_4, \alpha_2\alpha_3^1\alpha_4\alpha_1, \alpha_2\alpha_3^1\alpha_4\alpha_1\alpha_2\}$$

and

$$W = k\text{-sp}\{\alpha_2\alpha_3^2, \alpha_2\alpha_3^2\alpha_4, \alpha_2\alpha_3^2\alpha_4\alpha_1\}.$$

In this case, letting $\phi : P_4 \to V_2$ such that $\phi(x) = \alpha_2\alpha_3^1 x$ for $x \in P_4$ gives $V_2 \cong P_4$.
Finally, consider P_1, which has submodule

$$N_1 = k\text{-sp}\{\alpha_1\alpha_2\alpha_3^1, \alpha_1\alpha_2\alpha_3^1\alpha_4, \alpha_1\alpha_2\alpha_3^1\alpha_4\alpha_1, \alpha_1\alpha_2\alpha_3^1\alpha_4\alpha_1\alpha_2, \alpha_1\alpha_2\alpha_3^2, \alpha_1\alpha_2\alpha_3^2\alpha_4\}.$$

Moreover $N_1 = V_1 \oplus W_1$ where

$$V_1 = k\text{-sp}\{\alpha_1\alpha_2\alpha_3^1, \alpha_1\alpha_2\alpha_3^1\alpha_4, \alpha_1\alpha_2\alpha_3^1\alpha_4\alpha_1, \alpha_1\alpha_2\alpha_3^1\alpha_4\alpha_1\alpha_2\}$$

and

$$W_1 = k\text{-sp}\{\alpha_1\alpha_2\alpha_3^2, \alpha_1\alpha_2\alpha_3^2\alpha_4\}.$$

In this case, letting $\phi : P_4 \to V_1$ such that $\phi(x) = \alpha_1\alpha_2\alpha_3^2 x$ for $x \in P_4$ gives $V_1 \cong P_4$. Hence, we can also display the above table in the following way;

P_1	P_2	P_3	P_4
1	2	3	4
2	3	$P_4 \oplus P_4$	1
3	$P_4 \oplus \begin{smallmatrix}4\\1\\2\end{smallmatrix}$		2
$P_4 \oplus \begin{smallmatrix}4\\1\end{smallmatrix}$			3

It is now necessary to prove that A has finite global dimension, and so we find the projective dimension of each simple module.

We can resolve L_1 as follows:

$$0 \to P_4 \oplus P_4 \to P_3 \to P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

and thus $\text{pdim}(L_1) = 5$.
We can resolve L_2 as follows:

$$0 \to P_4 \oplus P_4 \to P_3 \to P_3 \to P_2 \to L_2 \to 0$$

and thus $\text{pdim}(L_2) = 3$.
We can resolve L_3 as follows:

$$0 \to P_4 \oplus P_4 \to P_3 \to L_3 \to 0$$

and thus $\text{pdim}(L_3) = 1$.
We can resolve L_4 as follows:

$$0 \to P_4 \oplus P_4 \to P_3 \to P_4 \oplus P_4 \to P_2 \to P_4 \oplus P_4 \to P_1 \to P_4 \to L_4 \to 0$$

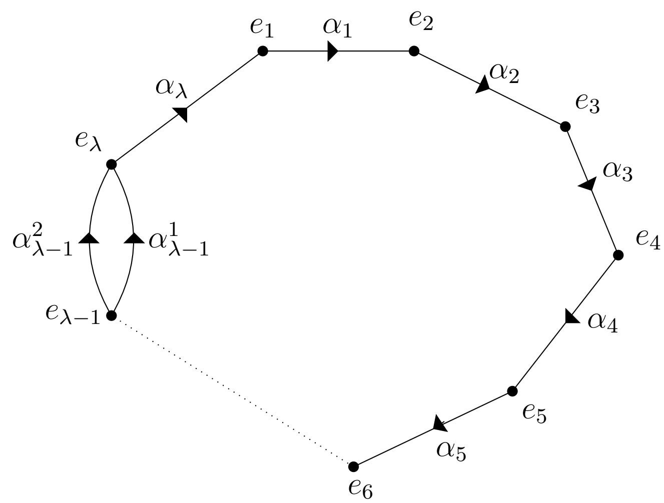
and thus $\text{pdim}(L_4) = 6$.

So A has global dimension $6 = (4-1)2 = (\lambda - 1)2$.

We now show why A is not quasi-hereditary. The only choice here is to take 4 to be maximal as $[P_i : L_i] = 3$ for $1 \le i \le 3$. Then assuming A is quasi-hereditary would imply that L_3 occurs twice in the socle of P_1 which we can see is not true by the above table of projectives. Hence we have a contradiction and so A is not quasi-hereditary.

We now combine these examples to give the following theorem.

THEOREM 6.2.9 *Quiver algebras of the form*



module

$$I =< \alpha_1 \dots \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1,\ \alpha_2 \dots \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1\alpha_2, \dots, \alpha_{\lambda-2}\alpha^2_{\lambda-1}\alpha_\lambda\alpha_1 \dots \alpha_{\lambda-2},$$

$$\alpha_\lambda\alpha_1 \dots \alpha^1_{\lambda-1}, \alpha_\lambda\alpha_1 \dots \alpha^2_{\lambda-1} >$$

having λ *simple modules and dimension* n *are not quasi-hereditary, but have finite global dimension* $m = (\lambda - 1)2$.

REMARK 6.2.10 In these cases we have two paths between vertices $\lambda - 1$ and λ and hence $r = 2$.

Proof. We first prove that these algebras have finite global dimension and then prove that they are not quasi-hereditary.
Let P_i be the projective modules where $i \in \{1, \dots, \lambda\}$. Then we can display

the projectives as follows;

P_1	P_2	P_3	P_4	P_5	$\dots$	$P_{\lambda-1}$	P_λ
1	2	3	4	5		$\lambda-1$	λ
2	3	4	5	6		λ	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		1	2
$\lambda-4$	$\lambda-3$	$\lambda-2$	$\lambda-1$	λ		2	3
$\lambda-3$	$\lambda-2$	$\lambda-1$	λ	1		$\vdots$	$\vdots$
$\lambda-2$	$\lambda-1$	λ	1	2		$\vdots$	$\lambda-3$
$\lambda-1$	λ	1	2	$\vdots$		$\vdots$	$\lambda-2$
λ	1	2	$\vdots$	λ		$\lambda-2$	$\lambda-1$
1	2	$\vdots$	λ	1		$\lambda-1$	
2	$\vdots$	λ	1	2		λ	
$\vdots$	λ	1	2	3		1	
λ	1	2	3	4		$\vdots$	
1	2	3	4	5		$\lambda-1$	

Now, $P_{\lambda-1}$ has maximal submodule

$$\begin{aligned} M_{\lambda-1} = \quad & k\text{-sp}\{\alpha^1_{\lambda-1}, \alpha^1_{\lambda-1}\alpha_\lambda, \alpha^1_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^1_{\lambda-1}\alpha_\lambda\alpha_1\cdots\alpha_{\lambda-2}, \\ & \alpha^2_{\lambda-1}, \alpha^2_{\lambda-1}\alpha_\lambda, \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1\cdots\alpha_{\lambda-2}\}. \end{aligned}$$

Moreover $M_{\lambda-1} = V_{\lambda-1} \oplus W_{\lambda-1}$ where

$$V_{\lambda-1} = k\text{-sp}\{\alpha^1_{\lambda-1}, \alpha^1_{\lambda-1}\alpha_\lambda, \alpha^1_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^1_{\lambda-1}\alpha_\lambda\alpha_1\cdots\alpha_{\lambda-2}\}$$

and

$$W_{\lambda-1} = k\text{-sp}\{\alpha^2_{\lambda-1}, \alpha^2_{\lambda-1}\alpha_\lambda, \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1\cdots\alpha_{\lambda-2}\}.$$

Now

$$P_\lambda = k\text{-sp}\{e_\lambda, \alpha_\lambda, \alpha_\lambda\alpha_1, \dots, \alpha_\lambda\alpha_1\alpha_2\cdots\alpha_{\lambda-2}\},$$

and thus if we define $\phi : P_\lambda \to V_{\lambda-1}$ such that $\phi(x) = e^1_{\lambda-1}x$ then $P_\lambda \cong V_{\lambda-1}$. Similarly, with $\theta : P_\lambda \to W_{\lambda-1}$ such that $\theta(x) = e^2_{\lambda-1}x$ then $P_\lambda \cong W_{\lambda-1}$.

Now consider P_i for $1 \le i \le \lambda - 2$. Then P_i has submodule

$$\begin{aligned} N_i = \quad & k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda, \dots, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda\cdots\alpha_{\lambda-2}, \\ & \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda, \dots, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda\cdots\alpha_{i-1}\} \end{aligned}$$

except in the case where $i = 1$ in which case

$$\begin{aligned} N_i = \quad & k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda\cdots\alpha_{\lambda-2}, \\ & \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda\}. \end{aligned}$$

We also have that $N_i = V_i \oplus W_i$ where

$$V_i = k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda\cdots\alpha_{\lambda-2}\}$$

and

$$W_i = k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda\cdots\alpha_{i-1}\}$$

except in the case $i = 1$ in which case

$$W_i = k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda\}.$$

Then with $\phi : P_\lambda \to V_i$ such that $\phi(x) = \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}x$ we have that $P_\lambda \cong V_i$, and thus, just as in the previous examples, we can also display the projectives as follows;

$$\begin{array}{cccccc}
P_1 & P_2 & P_3 & \ldots & P_{\lambda-1} & P_\lambda \\
 & & & & & \\
1 & 2 & 3 & \ldots & \lambda-1 & \lambda \\
2 & 3 & 4 & \ldots & P_\lambda \oplus P_\lambda & 1 \\
\vdots & \vdots & \vdots & \ldots & & 2 \\
\lambda-1 & \lambda-1 & \lambda-1 & \ldots & & \vdots \\
P_\lambda \oplus \begin{array}{c}\lambda\\1\end{array} & P_\lambda \oplus \begin{array}{c}\lambda\\1\\2\end{array} & P_\lambda \oplus \begin{array}{c}\lambda\\1\\2\\3\end{array} & \ldots & & \lambda-1
\end{array}$$

We now find minimal projective resolutions for each simple module L_i, starting with L_1 as follows;

$$0 \to P_\lambda \oplus P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_3 \to P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

then the $\text{pdim}(L_1) = 2(\lambda - 2) + 2 - 1 = 2\lambda - 3$.

We resolve L_2 as follows;

$$0 \to P_\lambda \oplus P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_3 \to P_3 \to P_2 \to L_2 \to 0$$

then the $\text{pdim}(L_2) = 2(\lambda - 3) + 2 - 1 = 2\lambda - 5$.

We resolve L_3 as follows;

$$0 \to P_\lambda \oplus P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_4 \to P_4 \to P_3 \to L_3 \to 0$$

then the pdim$(L_3) = 2(\lambda - 4) + 2 - 1 = 2\lambda - 7$.

This continues for each L_i and we now show how we resolve $L_{\lambda-1}$;

$$0 \to P_\lambda \oplus P_\lambda \to P_{\lambda-1} \to L_{\lambda-1} \to 0$$

so pdim$(L_{\lambda-1}) = 2(\lambda - (\lambda - 1 + 1)) + 2 - 1 = 1$.

Bringing this together, then in general we can resolve each L_i for $1 \leq i \leq \lambda - 1$ as follows;

$$0 \to P_\lambda \oplus P_\lambda \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_{i+1} \to P_{i+1} \to P_i \to L_i \to 0$$

hence pdim$(L_i) = 2(\lambda - (i+1)) + 2 - 1 = 2\lambda - (2i+1)$.

Finally we resolve L_λ as follows;

$$0 \to P_\lambda \oplus P_\lambda \to P_{\lambda-1} \to P_\lambda \oplus P_\lambda \to \ldots \to P_\lambda \oplus P_\lambda \to$$

$$P_2 \to P_\lambda \oplus P_\lambda \to P_1 \to P_\lambda \to e_\lambda \to 0$$

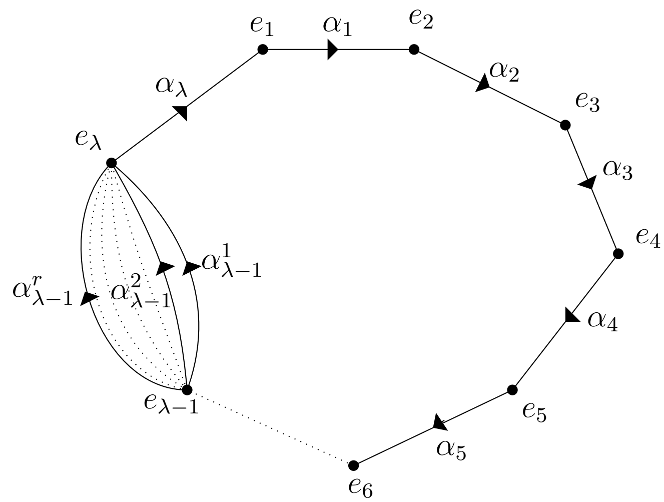
and pdim$(L_\lambda) = (\lambda - 1) + (\lambda - 1) + 1 - 1 = 2\lambda - 2$.

Hence the gl.dim $A =$pdim$(L_\lambda) = 2\lambda - 2$.

We now prove why these algebras are not quasi-hereditary, by considering all possible orderings on the simple modules. We know that the only possible option is to take λ to be maximal, as $[P_i : L_i] = 3$ for $1 \leq i \leq \lambda - 1$. So, take λ to be maximal, and then assume that these algebras are quasi-hereditary. So, with $[P_1 : L_\lambda] = 2$ we have $(P_1 : \Delta_\lambda) = 2$ which implies that $\Delta_\lambda \oplus \Delta_\lambda$ embeds in P_1 and hence $L_{\lambda-1} \oplus L_{\lambda-1}$ embeds in P_1 which we can see is not true by the above table of projectives. Hence these algebras are not quasi-hereditary. □

REMARK 6.2.11 Just as we added one extra path between vertex $\lambda - 1$ and vertex λ in examples 6.2.7 and 6.2.8, we can in fact add any number of paths between these two vertices. We can now give our final theorem, which covers all values of r.

THEOREM 6.2.12 *Quiver algebras of the form*



modulo

$$I =< \alpha_1 \dots \alpha^r_{\lambda-1}\alpha_\lambda\alpha_1,\ \alpha_2 \dots \alpha^r_{\lambda-1}\alpha_\lambda\alpha_1\alpha_2,\ \dots\ , \alpha_{\lambda-2}\alpha^r_{\lambda-1}\alpha_\lambda\alpha_1 \dots \alpha_{\lambda-2},\ \alpha_\lambda\alpha_1 \dots \alpha^1_{\lambda-1},$$
$$\alpha_\lambda\alpha_1 \dots \alpha^2_{\lambda-1},\ \dots\ , \alpha_\lambda\alpha_1 \dots \alpha^r_{\lambda-1} >$$

having λ *simple modules and dimension* n *are not quasi-hereditary, but have finite global dimension* $m = (\lambda - 1)2$.

Proof. We first prove that these algebras have finite global dimension and then prove that they are not quasi-hereditary.
Let P_i be the projective modules where $i \in \{1, \dots, \lambda\}$. Now, $P_{\lambda-1}$ has maximal submodule

$$\begin{aligned} M_{\lambda-1} = \quad & k\text{-sp}\{\alpha^1_{\lambda-1}, \alpha^1_{\lambda-1}\alpha_\lambda, \alpha^1_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^1_{\lambda-1}\alpha_\lambda\alpha_1 \cdots \alpha_{\lambda-2}, \\ & \alpha^2_{\lambda-1}, \alpha^2_{\lambda-1}\alpha_\lambda, \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^2_{\lambda-1}\alpha_\lambda\alpha_1 \cdots \alpha_{\lambda-2}, \dots, \\ & \alpha^r_{\lambda-1}, \alpha^r_{\lambda-1}\alpha_\lambda, \alpha^r_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^r_{\lambda-1}\alpha_\lambda\alpha_1 \cdots \alpha_{\lambda-2}\}. \end{aligned}$$

Moreover $M_{\lambda-1} = V^1_{\lambda-1} \oplus V^{\oplus}_{\lambda-1} \dots \oplus V^r_{\lambda-1}$ where

$$V^j_{\lambda-1} = k\text{-sp}\{\alpha^j_{\lambda-1}, \alpha^j_{\lambda-1}\alpha_\lambda, \alpha^j_{\lambda-1}\alpha_\lambda\alpha_1, \dots, \alpha^j_{\lambda-1}\alpha_\lambda\alpha_1 \cdots \alpha_{\lambda-2}\}.$$

Now

$$P_\lambda = k\text{-sp}\{e_\lambda, \alpha_\lambda, \alpha_\lambda\alpha_1, \dots, \alpha_\lambda\alpha_1\alpha_2 \cdots \alpha_{\lambda-2}\},$$

and thus if we define $\phi : P_\lambda \to V^j_{\lambda-1}$ such that $\phi(x) = e^j_{\lambda-1}x$ then $P_\lambda \cong V^j_{\lambda-1}$.

Now consider P_i for $1 \le i \le \lambda - 2$. Then P_i has submodule

$$\begin{aligned} N_i = \quad & k\text{-sp}\{\alpha_i\alpha_{i+1} \cdots \alpha^1_{\lambda-1}, \alpha_i\alpha_{i+1} \cdots \alpha^1_{\lambda-1}\alpha_\lambda, \dots, \alpha_i\alpha_{i+1} \cdots \alpha^1_{\lambda-1}\alpha_\lambda \cdots \alpha_{\lambda-2}, \\ & \alpha_i\alpha_{i+1} \cdots \alpha^2_{\lambda-1}, \alpha_i\alpha_{i+1} \cdots \alpha^2_{\lambda-1}\alpha_\lambda, \dots, \alpha_i\alpha_{i+1} \cdots \alpha^2_{\lambda-1}\alpha_\lambda \cdots \alpha_{\lambda-2}, \\ & \alpha_i\alpha_{i+1} \cdots \alpha^r_{\lambda-1}, \alpha_i\alpha_{i+1} \cdots \alpha^r_{\lambda-1}\alpha_\lambda, \dots, \alpha_i\alpha_{i+1} \cdots \alpha^r_{\lambda-1}\alpha_\lambda \cdots \alpha_{i-1}\} \end{aligned}$$

except in the case where $i = 1$ in which case

$$\begin{aligned} N_i = \quad & k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^1_{\lambda-1}\alpha_\lambda\cdots\alpha_{\lambda-2}, \\ & \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^2_{\lambda-1}\alpha_\lambda\cdots\alpha_{\lambda-2}, \\ & \alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}\alpha_\lambda\}. \end{aligned}$$

We also have that $N_i = V_i^1 \oplus V_i^2 \oplus \ldots \oplus V_i^{r-1} \oplus W$ where

$$V_i^j = k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^j_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^j_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^j_{\lambda-1}\alpha_\lambda\cdots\alpha_{\lambda-2}\}$$

and

$$W = k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}\alpha_\lambda, \ldots, \alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}\alpha_\lambda\cdots\alpha_{i-1}\}$$

except in the case $i = 1$ in which case

$$W = k\text{-sp}\{\alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}, \alpha_i\alpha_{i+1}\cdots\alpha^r_{\lambda-1}\alpha_\lambda\}.$$

Then with $\phi : P_\lambda \to V_i^j$ such that $\phi(x) = \alpha_i\alpha_{i+1}\cdots\alpha^j_{\lambda-1}x$ we have that $P_\lambda \cong V_i^j$, and thus, just as in the previous examples, we can also display the projectives as follows;

P_1	P_2	P_3	$\ldots$	$P_{\lambda-1}$	P_λ
1	2	3	$\ldots$	$\lambda-1$	λ
2	3	4	$\ldots$	$P_\lambda^{\oplus r}$	1
$\vdots$	$\vdots$	$\vdots$	$\ldots$		2
$\lambda-1$	$\lambda-1$	$\lambda-1$	$\ldots$		$\vdots$
$P_\lambda^{\oplus r-1} \oplus \begin{matrix}\lambda\\1\end{matrix}$	$P_\lambda^{\oplus r-1} \oplus \begin{matrix}\lambda\\1\\2\end{matrix}$	$P_\lambda^{\oplus r-1} \oplus \begin{matrix}\lambda\\1\\2\\3\end{matrix}$	$\ldots$		$\lambda-1$

We can now form the following minimal projective resolutions for each simple module L_i.
We resolve L_1 as follows;

$$0 \to P_\lambda^{\oplus r} \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_3 \to P_2 \to P_2 \to P_1 \to L_1 \to 0$$

then the $\text{pdim}(L_1) = 2(\lambda-2) + 2 - 1 = 2\lambda - 3$.

We resolve L_2 as follows;

$$0 \to P_\lambda^{\oplus r} \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_3 \to P_3 \to P_2 \to L_2 \to 0$$

then the $\text{pdim}(L_2) = 2(\lambda - 3) + 2 - 1 = 2\lambda - 5$.

We resolve L_3 as follows;

$$0 \to P_\lambda^{\oplus r} \to P_{\lambda-1} \to P_{\lambda-1} \to P_4 \to P_4 \to P_3 \to L_3 \to 0$$

then the $\text{pdim}(L_3) = 2(\lambda - 4) + 2 - 1 = 2\lambda - 7$.

This continues for each L_i and we now show how we resolve $L_{\lambda-1}$;

$$0 \to P_\lambda^{\oplus r} \to P_{\lambda-1} \to L_{\lambda-1} \to 0$$

so $\text{pdim}(L_{\lambda-1}) = 2(\lambda - (\lambda - 1 + 1)) + 2 - 1 = 1$.

Bringing this together, then in general we can resolve each L_i for $1 \leq i \leq \lambda - 1$ as follows;

$$0 \to P_\lambda^{\oplus r} \to P_{\lambda-1} \to P_{\lambda-1} \to \ldots \to P_{i+1} \to P_{i+1} \to P_i \to L_i \to 0$$

hence $\text{pdim}(L_i) = 2(\lambda - (i + 1)) + 2 - 1 = 2\lambda - (2i + 1)$.

Finally we resolve L_λ as follows;

$$0 \to P_\lambda^{\oplus r} \to P_{\lambda-1} \to P_\lambda^{\oplus r} \to \ldots \to P_\lambda^{\oplus r} \to P_2 \to P_\lambda^{\oplus r} \to P_1 \to P_\lambda \to L_\lambda \to 0$$

and $\text{pdim}(L_\lambda) = (\lambda - 1) + (\lambda - 1) + 1 - 1 = 2\lambda - 2$.

Hence the gl.dim $A = \text{pdim}(L_\lambda) = 2\lambda - 2$.

We now prove why these algebras are not quasi-hereditary, by considering all possible orderings of the simple modules. Well, as in previous cases, we can only consider taking λ to be maximal as $[P_i : L_i] = r$ for $1 \leq i \leq \lambda - 1$. So, let λ be maximal, and assume these algebras are quasi-hereditary, then $[P_1 : L_\lambda] = r$ implies $\Delta_\lambda^{\oplus r}$ embeds in P_1 which implies $L_{\lambda-1}^{\oplus r}$ embeds in P_1. We can see this is not true by the table of projectives above. Hence these algebras are not quasi-hereditary. □

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