Optimal Control of Stochastic Partial Differential Equations in Banach Spaces

Rafael Antonio Serrano Perdomo

Thesis submitted for the degree of
DOCTOR OF PHILOSOPHY

Department of Mathematics
University of York
Heslington
York, YO10 5DD

September 2010

Abstract

In this thesis we study optimal control problems in Banach spaces for stochastic partial differential equations. We investigate two different approaches. In the first part we study Hamilton-Jacobi-Bellman equations (HJB) in Banach spaces associated with optimal feedback control of a class of non-autonomous semilinear stochastic evolution equations driven by additive noise. We prove the existence and uniqueness of mild solutions to HJB equations using the smoothing property of the transition evolution operator associated with the linearized stochastic equation. In the second part we study an optimal relaxed control problem for a class of autonomous semilinear stochastic stochastic PDEs on Banach spaces driven by multiplicative noise. The state equation is controlled through the nonlinear part of the drift coefficient and satisfies a dissipative-type condition with respect to the state variable. The main tools of our study are the factorization method for stochastic convolutions in UMD type-2 Banach spaces and certain compactness properties of the factorization operator and of the class of Young measures on Suslin metrisable control sets.

To Juan Manuel

Acknowledgements

First of all I would like to express my sincere gratitude to my PhD supervisor Zdzislaw Brzezniak for his advice, support and patience, and for introducing me to the fascinating field of stochastic partial differential equations. I have learned a great deal of mathematics under his supervision and I am deeply grateful to him for that.

I am indebted to the Dorothy Hodgkin Postgraduate Award scheme for funding my research and to the staff at the Department of Mathematics for their help throughout my PhD studies.

I thank my parents who have also provided for me financially and encouraged me to concentrate on my studies. They have always respected my choices in life and have stood by me offering support, guidance and love. I also thank my brother Andres and my sister Susana who always made sure that I had one less thing to worry about.

I owe my deepest gratitude to my wife Laura for her love, understanding and unconditional support. During my studies there have been high times and low times, and she had endured the bad times with me and enjoyed the good times with me. Without her help and encouragement, this study would not have been completed.

Finally, I am most grateful to my son Juan Manuel, for being my constant inspiration and the ultimate reason for finishing this project.

Author's declaration

I declare that all the material in the thesis is my own work unless stated otherwise.

Signed:

Date:

Contents

Chapter 1

Introduction

This thesis is concerned with optimal control problems in Banach spaces for stochastic partial differential equations (SPDEs). For instance, consider the control problem of minimizing a cost functional of the form

$$J(X,u) = \mathbb{E}\left[\int_0^T h(s, X(s,\zeta_1), \dots, X(s,\zeta_n), u(s))\, ds + \varphi(X(T))\right] \tag{1.1}$$

where $\zeta_1, \dots, \zeta_n$ are fixed points distributed over the interval $(0,1)$, $u(\cdot)$ is a control process with values in a separable metric space M, and $X(\cdot)$ is solution (in some sense) to the following controlled SPDE of *reaction-diffusion* with zero-Dirichlet boundary conditions and driven by multiplicative space-time white noise

$$\begin{aligned} \frac{\partial X}{\partial t}(t,\xi) + \frac{\partial^2 X}{\partial \xi^2}(t,\xi) &= f(t,\xi,X(t,\xi),u(t)) + g(X(t,\xi))\,\frac{\partial w}{\partial t}(t,\xi), && \text{in } [0,T]\times\mathcal{O} \\ X(t,\cdot) &= 0, && \text{on } (0,T]\times\partial\mathcal{O} \\ X(0,\cdot) &= x_0(\cdot), \end{aligned} \tag{1.2}$$

Equations of the form (1.2) appear when modelling the concentration (or density) of a certain substance subject to random perturbations, and cost functionals of the form (1.1) can be used to regulate the behaviour of such quantity on the fixed points $\zeta_1, \dots, \zeta_n$. Clearly, such cost functionals are well-defined only if $X(t,\xi)$ is continuous with respect to the space variable ξ. Therefore, we need to guarantee that the trajectories of the solution to the evolution equation induced by the controlled stochastic PDE take values in the Banach space $\mathcal{C}([0,1])$ of real-valued continuous functions on $[0,1]$. In fact, it is well-known that if the reaction term

$f : [0,T] \times (0,1) \times \mathbb{R} \times M \to \mathbb{R}$ is continuous and satisfies a one-sided polynomial growth condition of the form

$$f(t,\xi,x+y,u)\text{sgn}x \leq -k_1 |x| + k_2 |y|^m + \eta(t,u), \quad t \in [0,T], \ \ \xi \in (0,1), \ \ x,y \in \mathbb{R}, \ \ u \in M \tag{1.3}$$

where $m \geq 1$, $k_1 \in \mathbb{R}$, $k_2 \geq 0$ and $\eta : [0,T] \times M \to [0,+\infty]$ is a measurable mapping (possibly lower semi-continuous with respect to u) then the Nemytskii operator induced by f satisfies a dissipative-type condition on the space $\mathcal{C}([0,1])$.

Since real-valued stochastic integration theory extends directly to processes with values in Hilbert spaces, the controlled SPDE (1.2) is usually modelled in the Hilbert space of square-integrable functions $L^2(0,1)$. However, as the above considerations suggest, one should study the controlled evolution equation in a Banach space, like above on $\mathcal{C}([0,1])$, or in other cases on $L^p(0,1)$ with $p > 2$, rather than the Hilbert space $L^2(0,1)$.

The primary objective of this thesis is two fold. First, we revisit the Hamilton-Jacobi-Bellman (HJB) equations associated with optimal feedback control of the following non-autonomous stochastic evolution equation on a Banach space $\mathbf{E}$ driven by additive noise

$$\begin{aligned} dX(t) + A(t)X(t)\,dt &= F(t,X(t),u(t))\,dt + G(t)\,dW(t) \\ X(0) &= x_0 \in \mathbf{E}. \end{aligned} \tag{1.4}$$

Here $\{-A(t)\}_{t\in[0,T]}$ is the generator of an evolution family on the Banach space $\mathbf{E}$, $\{G(t)\}_{t\in[0,T]}$ are (possibly unbounded) linear operators from a separable Hilbert space $\mathbf{H}$ into $\mathbf{E}$, and $W(\cdot)$ is a $\mathbf{H}-$cylindrical Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

It is well known, thanks to Bellman's Dynamic programming principle, that solving the associated HJB equation can be useful to provide the verification and synthesis of optimal control strategies. This has been done extensively in the Hilbert space for autonomous controlled evolution equations see e.g. [Goz96, Cer99, DPZ02]. We will prove, using the the smoothing property of the transition evolution family associated with the linearized version of the non-autonomous equation (1.4), that the result on existence and uniqueness of mild solutions to the HJB equation can be easily generalized to the Banach space setting.

Secondly, we will employ relaxation methods to study the optimal control of stochastic evolution equations in Banach spaces with dissipative nonlinearities. It is well known that when no special conditions on the the dependence of the non-linear term F with respect to the control variable are assumed, to prove existence of an optimal control it is necessary to extend the original control system to one that allows for control policies whose instantaneous values are probability measures on the control set. Such control policies are known as *relaxed controls.*

This technique of measure-valued convexification of nonlinear systems has a long story starting with L.C.Young [You42, You69] and J.Warga [War62, War72] and their work on variational problems and existence of optimal controls for finite-dimensional systems. The use of relaxed controls in the context of evolution equations in Banach spaces was initiated by Ahmed [Ahm83] and Papageorgiou [Pap89a, Pap89b] (see also [AP90]) who considered controls that take values in a Polish space. More recently, optimal relaxed control of PDEs has been studied by Lou in [Lou03, Lou07] also with Polish control set.

Under more general topological assumptions on the control set, Fattorini also employed relaxed controls in [Fat94a] and [Fat94b] (se also [Fat99]) but at the cost of working with merely finitely additive measures instead of $\sigma-$additive measures.

In the stochastic case, relaxed control of finite-dimensional stochastic systems goes back to Fleming and Nisio [Fle80, FN84]. Their approach was followed extensively in [EKHNJP87] and [HL90], where the control problem was recast as a martingale problem. The study of relaxed control for stochastic PDEs seems to have been initiated by Nagase and Nisio in [NN90] and continued by Zhou in [Zho92], where a class of semilinear stochastic PDEs controlled through the coefficients of an elliptic operator and driven by a $d-$dimensional Wiener process is considered, and in [Zho93], where controls are allowed to be space-dependent and the diffusion term is a first-order differential operator driven by a one-dimensional Wiener process.

In [GS94], using the semigroup approach, Gatarek and Sobczyk extended some of the results described above to Hilbert space-valued controlled diffusions driven by a trace-class noise. The main idea of their approach is to show compactness of the space of admissible relaxed control policies by the factorization method introduced by Da Prato, Kwapien and Zabczyk (see [DPKZ87]). Later, in [Sri00] Sritharan studied optimal relaxed control of stochastic Navier-Stokes equations with monotone nonlinearities and Lusin metrisable control set. More recently, Cutland and Grzesiak combined relaxed controls with nonstandard analysis techniques in [CG05, CG07] to study existence of optimal controls for 3 and 2-dimensional stochastic Navier-Stokes equations respectively.

Here we will consider a control system and use methods that are similar to those of [GS94]. However, our approach allows to consider controlled processes with values in a larger class of state spaces, which permits to study running costs that are not necessarily well-defined in a Hilbert-space framework. Moreover, we consider controlled equations driven by cylindrical Wiener process, which includes the case of space-time white noise in one dimension, and with a drift coefficient that satisfies a dissipative-type condition with respect to the state-variable. In addition, the control set is assumed only metrisable and Suslin.

Let us briefly describe the contents of this thesis. In Chapter 1 we review some basic results on Gaussian measures and stochastic integration in Banach spaces. In Chapter 2 we recall the smoothing property of Ornstein-Uhlenbeck transition evolution families associated with the solutions of non-autonomous stochastic Cauchy problems and use this to extend the existence and uniqueness of mild solutions to HJB equations to the Banach space case.

Chapter 3 comprises the main result of this thesis. We start by recalling the notion of stochastic relaxed control and its connection with random Young measures, we define the stable topology and review some relatively recent results on (flexible) tightness criteria for relative compactness in this topology. Next, we introduce the factorization operator as the negative fractional power of a certain abstract parabolic operator associated with a Cauchy problem on UMD spaces and some of its smoothing and compactness properties. Then, we review some basics results on the factorization method for stochastic convolutions in UMD type-2 Banach spaces. Finally, we reformulate the control problem as a relaxed control problem in the weak stochastic sense and prove existence of optimal weak relaxed controls for a class of dissipative stochastic PDEs, and we illustrate this result with examples that cover a class of stochastic reaction-diffusion equations (driven by space-time multiplicative white noise in dimension 1) and also include the case of space-dependant control.

Notation. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$. For $m \in \mathbb{N}$ and $p \in [1, \infty]$, $W^{m,p}(\mathcal{O})$ will denote the usual Sobolev space, and for $s \in \mathbb{R}$, $H^{s,p}(\mathcal{O})$ will denote the space defined as

$$H^{s,p}(\mathcal{O}) := \begin{cases} W^{m,p}(\mathcal{O}), & \text{if } \ m \in \mathbb{N}; \\ \left[W^{k,p}(\mathcal{O}), W^{m,p}(\mathcal{O})\right]_\delta, & \text{if } \ s \in (0,\infty) \setminus \mathbb{N}, \end{cases}$$

where $[\cdot,\cdot]_\delta$ denotes complex interpolation and $k, m \in \mathbb{N}$, $\delta \in (0,1)$ are chosen to satisfy $s = (1-\delta)k + \delta m$. For a comprehensive overview on Sobolev spaces and (complex) interpolation we refer the reader to [Ama95] or [Tri78].

Chapter 2

Preliminaries

2.1 Gaussian measures on Banach spaces, Cameron-Martin formula and smoothing property

Throughout, $\mathbf{E}$ denotes a real Banach space, $\mathbf{E}^*$ denotes its continuous dual and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{E}$ and $\mathbf{E}^*$. $\mathcal{B}(\mathbf{E})$ will denote the Borel $\sigma-$algebra on $\mathbf{E}$.

Definition 2.1. A Radon measure μ on $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ is called *Gaussian* (resp. *centred Gaussian*) if, for any linear functional $x^* \in \mathbf{E}^*$ the image measure $\mu \circ (x^*)^{-1}$ is a Gaussian (resp. centred Gaussian) measure on $\mathbb{R}$.

If μ is a centred Gaussian measure on $\mathbf{E}$, there exists an unique bounded linear operator $Q \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$, called the *covariance operator* of μ, such that for all $x^*, y^* \in \mathbf{E}^*$ we have

$$\langle Qx^*, y^* \rangle = \int_{\mathbf{E}} \langle x, x^* \rangle \langle x, y^* \rangle \; \mu(dx).$$

(see e.g. [Bog98]). Notice that Q is *positive* in the sense that

$$\langle Qx^*, x^* \rangle \geq 0, \;\; \forall x^* \in \mathbf{E}^*,$$

and *symmetric* in the sense that

$$\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle \,, \quad \forall x^*, y^* \in \mathbf{E}^*.$$

The Fourier transform $\hat{\mu}$ of μ is given by

$$\hat{\mu}(x^*) = \exp\Big(-\frac{1}{2}\langle Qx^*, x^*\rangle\Big), \quad x^* \in \mathbf{E}^*,$$

This identity implies that two centred Gaussian measures are equal whenever their covariance operators are equal.

For any $Q \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$ positive and symmetric, the bilinear form on $Q(\mathbf{E}^*)$ given by

$$[Qx^*, Qy^*] := \langle Qx^*, y^*\rangle\,, \quad x^*, y^* \in \mathbf{E}^*. \tag{2.1}$$

is a well-defined inner product on $Q(\mathbf{E}^*)$. The Hilbert space completion of $Q(\mathbf{E}^*)$ with respect to this inner product will be denoted by H_Q. The inclusion mapping from $Q(\mathbf{E}^*)$ into $\mathbf{E}$ is continuous with respect to the inner product $[\cdot,\cdot]_{H_Q}$ and extends uniquely to a bounded linear injection $i_Q : H_Q \hookrightarrow \mathbf{E}$.

Definition 2.2. The pair (i_Q, H_Q) is called the *reproducing kernel Hilbert space* (RKHS) associated with Q.

It can be easily shown that the adjoint operator $i_Q^* : \mathbf{E}^* \to H_Q$ satisfies $i_Q^* x^* = Qx^*$ for all $x^* \in \mathbf{E}^*$. Therefore, Q admits the factorization

$$Q = i_Q \circ i_Q^*.$$

This factorization immediately implies that Q is weak*-to-weakly continuous and that H_Q is separable if $\mathbf{E}$ is separable. Whenever it is convenient, we will identify H_Q with its image $i_Q(H_Q)$ in $\mathbf{E}$.

Proposition 2.3 ([vN98], Proposition 1.1)**.** *Let $Q, \tilde{Q} \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$ be two positive symmetric operators. Then, for the corresponding reproducing kernel Hilbert spaces we have $H_Q \subset H_{\tilde{Q}}$ (as subsets of $\mathbf{E}$) if and only if there exist a constant $K > 0$ such that*

$$\langle Qx^*, x^*\rangle \le K\langle \tilde{Q}x^*, x^*\rangle, \quad \forall x^* \in \mathbf{E}^*.$$

If Q is the covariance operator of a Gaussian measure μ on $\mathbf{E}$, instead of H_Q (resp. i_Q) we will use the notation H_μ (resp. i_μ). In this case we can introduce a linear isometry from H_μ into $L^2(\mathbf{E}, \mu)$ as follows: first observe that $\langle x^*, \cdot\rangle \in L^2(\mathbf{E}, \mu)$ for every linear functional $x^* \in \mathbf{E}^*$ and that we have

$$\mathbb{E}^\mu \left|\langle x^*, \cdot\rangle\right|^2 = \int_{\mathbf{E}} \left|\langle x, x^*\rangle\right|^2 \, \mu(dx) = \langle Qx^*, x^*\rangle\,, \quad x^* \in \mathbf{E}^*. \tag{2.2}$$

Here $\mathbb{E}^\mu$ denotes the expectation on the probability space $(\mathbf{E}, \mathcal{B}(\mathbf{E}), \mu)$. Since Q is injective as an operator from $\mathbf{E}^*$ into $Q(\mathbf{E}^*)$, the linear map

$$Q(\mathbf{E}^*) \ni Q(x^*) \mapsto \langle x^*, \cdot \rangle \in L^2(\mathbf{E}, \mu) \tag{2.3}$$

is well-defined and is an isometry in view of (2.2). We define the mapping

$$\phi_\mu : H_\mu \to L^2(\mathbf{E}, \mu) \tag{2.4}$$

as the unique extension of the isometry (2.3) to H_μ. This isometry has the property that, for each $h \in H_\mu$, $\phi_\mu(h)$ is a $\mathcal{N}(0, |h|^2_{H_\mu})$ random variable. Indeed, for $h \in H_\mu$ fixed, if $(x_n^*)_n$ is a sequence in $\mathbf{E}^*$ such that $Qx_n^* \to h$ in H_μ, then

$$\langle x_n^*, \cdot \rangle = \phi_\mu(Qx_n^*) \to \phi_\mu(h), \quad \text{in } L^2(\mathbf{E}, \mu)$$

and this implies, in particular, that $\mathbb{E}^\mu[e^{i\lambda\langle x_n^*, \cdot\rangle}] \to \mathbb{E}^\mu[e^{i\lambda\phi_\mu(h)}]$, $\forall \lambda \in \mathbb{R}$. Since $\langle x_n^*, \cdot \rangle$ is normally distributed with mean 0 and variance $|Qx_n^*|^2_{H_\mu}$, we have

$$\mathbb{E}^\mu\big[e^{i\lambda\langle x_n^*, \cdot\rangle}\big] = \exp\Big(-\frac{\lambda^2}{2}\,|Qx_n^*|^2_{H_\mu}\Big), \quad \lambda \in \mathbb{R},$$

and by dominated convergence, taking the limit as $n \to \infty$ we get

$$\mathbb{E}^\mu\big[e^{i\lambda\phi_\mu(h)}\big] = \exp\Big(-\frac{\lambda^2}{2}\,|h|^2_{H_\mu}\Big), \quad \lambda \in \mathbb{R},$$

which implies that $\phi_\mu(h)$ is a $\mathcal{N}(0, |h|^2_{H_\mu})$-distributed random variable.

Finally, for each $h \in H_\mu$ we denote by μ^h the image of the measure μ under the translation $z \mapsto z + h$, that is,

$$\mu^h(A) := \mu(A - h), \quad A \in \mathcal{B}(\mathbf{E}).$$

We call μ^h the *shift of the measure* μ *by the vector* h.

With the above definitions we can now formulate the Cameron-Martin formula (for the proof see [Bog98]),

Theorem 2.4 (Cameron-Martin formula)**.** *Let μ be a centred Gaussian measure on $\mathbf{E}$ with covariance operator $Q \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$ and let (i_μ, H_μ) denote the RKHS associated with μ. Then, for* *any $h \in H_\mu$, the measure μ^h is absolutely continuous with respect to μ and we have*

$$\frac{d\mu^h}{d\mu} = \rho_h, \ \ \mu - a.s.$$

with

$$\rho_h := \exp\Big(\phi_\mu(h) - \frac{1}{2}\,|h|^2_{H_\mu}\Big), \quad h \in H_\mu.$$

For $\varphi \in B_b(\mathbf{E})$ fixed, we define the mapping $\psi : \mathbf{E} \to \mathbb{R}$

$$\psi(x) = \int_{\mathbf{E}} \varphi(x+z)\,\mu(dz), \quad x \in \mathbf{E}, \tag{2.5}$$

The following regularizing property was proved in the seminal paper by Leonard Gross on Potential theory in Hilbert spaces [Gro67, Proposition 9] using directly the definition of Fréchet derivative. Here we present an alternative proof based on Gâteaux differentiability.

Recall that $\psi : \mathbf{E} \to \mathbb{R}$ is Fréchet differentiable at $x \in \mathbf{E}$ in the direction of H_μ if there exists an element of H_μ^*, denoted by $D_{H_\mu}\psi(x)$, such that

$$\lim_{\substack{y \in H_\mu \\ y \to 0}} \frac{\big|\psi(x+y) - \psi(x) - \big(D_{H_\mu}\psi(x)\big)\,(y)\big|}{|y|_{H_\mu}} = 0.$$

Proposition 2.5. *The map $\psi : \mathbf{E} \to \mathbb{R}$ is infinitely Fréchet differentiable in the direction of H_μ. The first Fréchet derivative of ψ at $x \in \mathbf{E}$ in the direction of $y \in H_\mu$ is given by*

$$\big(D_{H_\mu}\psi(x)\big)\,(y) = \int_{\mathbf{E}} \varphi(x+z)\,\phi_\mu(y)(z)\,\mu(dz), \tag{2.6}$$

and the second Fréchet derivative of ψ at $x \in \mathbf{E}$ in the directions $y_1, y_2 \in H_\mu$ is given by

$$\Big(D^2_{H_\mu}\psi(x)\Big)\,(y_1,y_2) = -\psi(x)\,[y_1,y_2]_{H_\mu} + \int_{\mathbf{E}} \varphi(x+z)\,\phi_\mu(y_1)(z)\,\phi_\mu(y_2)(z)\,\mu(dz). \tag{2.7}$$

In particular, we have the estimates

$$\big\|D_{H_\mu}\psi(x)\big\|_{H_\mu^*} \le |\varphi|_0\,, \tag{2.8}$$

$$\Big\|D^2_{H_\mu}\psi(x)\Big\|_{\mathcal{L}(H_\mu,H_\mu^*)} \le 2\,|\varphi|_0\,. \tag{2.9}$$

Proof. Let us prove first that ψ is Gâteaux differentiable in the direction of H_μ, i.e. that for all

$x \in \mathbf{E}$ and $y \in H_\mu$, the mapping

$$\mathbb{R} \ni \alpha \mapsto \psi(x+\alpha y) \in \mathbb{R}$$

is differentiable at $\alpha = 0$. Let $x \in \mathbf{E}$ and $y \in H_\mu$ be fixed and let $\alpha \in \mathbb{R}$. Observe that by the Cameron-Martin formula, we have

$$\psi(x+\alpha y) = \int_{\mathbf{E}} \varphi(x+z)\, \mu^{\alpha y}(dz) = \int_{\mathbf{E}} \varphi(x+z)\rho_{\alpha y}(z)\, \mu(dz).$$

Since $\phi_\mu(\alpha y) = \alpha\phi_\mu(y)$ in $L^2(\mathbf{E}, \mu)$ observe that the random variable

$$\rho_{\alpha y} = \exp\Big(\alpha\phi_\mu(y) - \frac{1}{2}\,|\alpha y|^2_{H_\mu}\Big)$$

is defined on a set $\widehat{\mathbf{E}} = \widehat{\mathbf{E}}(y)$ of full measure which depends only on y, for all $\alpha \in \mathbb{R}$. Thus, the mapping

$$g : \mathbb{R} \times \mathbf{E} \ni (\alpha, z) \mapsto g(\alpha, z) := \rho_{\alpha y}(z) \in \mathbb{R} \tag{2.10}$$

is well-defined and measurable. Moreover, for $\varepsilon > 0$ fixed we have the following estimate for all $|\alpha_0| < \varepsilon,\ z \in \widehat{\mathbf{E}}$,

$$\begin{aligned} \left|\frac{\partial g}{\partial \alpha}(\alpha_0, z)\right| &= \rho(\alpha_0 y, z)\left|\phi_\mu(y)(z) - \alpha_0\,|y|^2_{H_\mu}\right| \\ &\le \exp(\varepsilon\,|\phi_\mu(y)(z)|)\left(|\phi_\mu(y)(z)| + \varepsilon\,|y|^2_{H_\mu}\right). \end{aligned} \tag{2.11}$$

We know $\phi_\mu(y)$ is Gaussian random variable with moment generating function

$$\mathbb{E}^\mu\big[e^{\lambda\phi_\mu(y)}\big] = \exp\Big(\frac{\lambda^2}{2}\,|y|^2_{H_\mu}\Big), \quad \lambda \in \mathbb{R}.$$

This implies, in particular, that $\exp(\varepsilon\,|\phi_\mu(y)|)$ belongs to $L^2(\mathbf{E}, \mu)$. Since $\phi_\mu(y) \in L^2(\mathbf{E}, \mu)$, by Hölder's inequality the RHS in (2.11) belongs to $L^1(\mathbf{E}, \mu)$. Thus, the function g satisfies the conditions of Lemma A.1 in the Appendix with $T = \mathbf{E}$ and $T_1 = \widehat{\mathbf{E}}$ which allows us to differentiate with respect to α under the integral sign and obtain that the Gâteaux derivative of ψ

at x in the direction of y is given by

$$
\begin{aligned}
(d_{H_\mu}\psi(x))(y) &= \frac{d}{d\alpha}\Big|_{\alpha=0}\psi(x+\alpha y)\\
&= \int_{\mathbf{E}} \varphi(x+z)\left[\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\rho_{\alpha y}(z)\right]\,\mu(dz),\\
&= \int_{\mathbf{E}} \varphi(x+z)\phi_\mu(y)(z)\,\mu(dz),
\end{aligned}
$$

as well as the following estimate

$$\left\|d_{H_\mu}\psi(x)\right\|_{\mathcal{L}(H_\mu,\mathbb{R})} \leq |\varphi|_0 .$$

In turn this implies that the Gateaux derivative $d\psi : H_\mu \to \mathcal{L}(H_\mu, \mathbb{R})$ is continuous and uniformly bounded. Since ψ is also continuous and uniformly bounded on H_μ, by Theorem 3 in [Aro76, Ch 2, Section 1] we conclude that ψ is Fréchet differentiable in the direction of H_μ and (2.6) follows.

For the second-order Gâteaux derivative, if $y_1, y_2 \in H_\mu$ and $\alpha \in \mathbb{R}$ we have

$$
\begin{aligned}
(d_{H_\mu}\psi(x+\alpha y_2))(y_1) &= \int_{\mathbf{E}} \varphi(x+\alpha y_2+z)\phi_\mu(y_1)(z)\,\mu(dz)\\
&= \int_{\mathbf{E}} \varphi(x+\xi)\phi_\mu(y_1)(\xi-\alpha y_2)\,\mu^{\alpha y_2}(d\xi)\\
&= \int_{\mathbf{E}} \varphi(x+\xi)\phi_\mu(y_1)(\xi-\alpha y_2)\,\rho_{\alpha y_2}(\xi)\,\mu(d\xi)
\end{aligned}
$$

where we have used again the Cameron-Martin formula and the change of variable $\xi = z + \alpha y_2$ whose push-forward measure with respect with μ is given by $\mu^{\alpha y_2}$.

If $y_1 = Qx_1^*$ for some $x_1^* \in \mathbf{E}^*$, from the definition of ϕ_μ it follows that

$$\phi_\mu(y_1)(\xi-\alpha y_2) = \langle x_1^*, \xi-\alpha y_2\rangle = \langle x_1^*, \xi\rangle - \alpha\,\langle x_1^*, y_2\rangle = \phi_\mu(y_1)(\xi) - \alpha[y_1, y_2]_{H_\mu}$$

in which case we have

$$(d_{H_\mu}\psi(x+\alpha y_2))(y_1) = \int_{\mathbf{E}} \varphi(x+\xi)\left(\phi_\mu(y_1)(\xi) - \alpha[y_1,y_2]_{H_\mu}\right)\rho_{\alpha y_2}(\xi)\,\mu(d\xi). \tag{2.12}$$

Since both sides of (2.12) are continuous in $y_1 \in H_\mu$ and $Q(\mathbf{E}^*)$ is dense in H_μ, the above equality holds for any $y_1 \in H_\mu$. In addition,

$$\frac{\partial}{\partial \alpha}\Big|_{\alpha=0} \left[\left(\phi_\mu(y_1)(\xi) - \alpha\,[y_1,y_2]_{H_\mu}\right)\rho(\alpha y_2,\xi)\right] = -\,[y_1,y_2]_{H_\mu} + \phi_\mu(y_1)(\xi)\phi_\mu(y_2)(\xi),$$

holds for all ξ in a subset of $\mathbf{E}$ with full measure that only depends on y_2. Therefore, we can apply again Lemma A.1 in the Appendix to obtain the second Gâteaux derivative of ψ at x in the direction o y_1 and y_2,

$$\begin{aligned}\left(d^2_{H_\mu}\psi(x)\right)(y_1,y_2) &= \frac{d}{d\alpha}\Big|_{\alpha=0}(d_{H_\mu}\psi(x+\alpha y_2))(y_1)\\ &= \int_{\mathbf{E}} \varphi(x+\xi)\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\left[\left(\phi_\mu(y_1)(\xi) - \alpha\,[y_1,y_2]_{H_\mu}\right)\rho_{\alpha y_2}(\xi)\right]\,\mu(d\xi)\\ &= \int_{\mathbf{E}} \varphi(x+\xi)\left(\phi_\mu(y_1)(\xi)\phi_\mu(y_2)(\xi) - [y_1,y_2]_{H_\mu}\right)\,\mu(d\xi)\end{aligned}$$

together with the following estimate

$$\left\|d^2_{H_\mu}\psi(x)\right\|_{\mathcal{L}(H_\mu,H_\mu^*)} \le 2\,|\varphi|_0\,,$$

for all $x \in \mathbf{E}$. By the same argument as above ψ is also twice Fréchet differentiable and (2.7) follows. □

By identifying H_μ with its dual H_μ^*, $D^2_{H_\mu}\psi(x)$ defines a linear bounded operator on H_μ. It is in fact a Hilbert-Schmidt operator as the following Lemma shows

Lemma 2.6. *For each $x \in \mathbf{E}$ we have $D^2_{H_\mu}\psi(x) \in \mathcal{T}_2(H_\mu)$ and*

$$\left\|D^2_{H_\mu}\psi(x)\right\|_{\mathcal{T}_2(H_\mu)} \le \sqrt{2}\,|\varphi|_0\,. \tag{2.13}$$

If $\varphi \in C^1_b(\mathbf{E})$ we have

$$\left\|D^2_{H_\mu}\psi(x)\right\|_{\mathcal{T}_2(H_\mu)} \le |\varphi|_1\,. \tag{2.14}$$

Proof. Let $(e_i)_i$ be an orthonormal basis of H_μ and let $x \in \mathbf{E}$ be fixed.
Let us prove first the case $\varphi \in C^1_b(\mathbf{E})$. By the same argument used in the proof of (2.6) one can derive

$$[D^2_{H_\mu}\psi(x)y_1,y_2] = \int_{\mathbf{E}} [D\varphi(x+z),y_1]_{H_\mu}\,\phi_\mu(y_2)(z)\,\mu(dz), \quad y_1,y_2 \in H_\mu.$$

Since the map ϕ_μ is an isometry from H_μ to $L^2(\mathbf{E},\mu)$, the random variables $\phi_\mu(e_k), k \in \mathbb{N}$, form a complete orthonormal system in $L^2(\mathbf{E},\mu)$ and by Parseval identity and dominated convergence we get

$$
\begin{aligned}
\left\| D^2_{H_\mu}\psi(x) \right\|^2_{\mathcal{J}_2(H_\mu)} &= \sum_{i=1}^{\infty} \left| D^2_{H_\mu}\psi(x)e_i \right|^2_{H_\mu} = \sum_{i,k=1}^{\infty} \left| [D^2_{H_\mu}\psi(x)e_i, e_k]_{H_\mu} \right|^2 \\
&= \sum_{i,k=1}^{\infty} \left| \left\langle [D\varphi(x+\cdot), e_i]_{H_\mu}, \phi_\mu(e_k) \right\rangle_{L^2(\mathbf{E},\mu)} \right|^2 \\
&= \sum_{i=1}^{\infty} \left\| [D\varphi(x+\cdot), e_i]_{H_\mu} \right\|^2_{L^2(\mathbf{E},\mu)} \\
&= \int_{\mathbf{E}} \sum_{i=1}^{\infty} \left| [D\varphi(x+z), e_i]_{H_\mu} \right|^2 \mu(dz) \\
&= \int_{\mathbf{E}} |D\varphi(x+z)|^2_{H_\mu} \mu(dz) \\
&\le |\varphi|_1^2
\end{aligned}
$$

and (2.14) follows. For the general case $\varphi \in B_b(\mathbf{E})$, we define the random variables

$$
\zeta_{i,k} := \begin{cases} \frac{1}{\sqrt{2}} \left(\phi_\mu(e_i)^2 - 1 \right), & \text{if } i = k, \\ \phi_\mu(e_i)\phi_\mu(e_k), & \text{if } i \neq k. \end{cases}
$$

Since $\phi_\mu(e_k), k \in \mathbb{N}$, are independent Gaussian random variables with mean 0 and variance 1, we get

$$
\langle \zeta_{i,k}, \zeta_{i',k'} \rangle_{L^2(\mathbf{E},\mu)} = 0, \quad \text{for } (i,k) \neq (i',k')
$$

and

$$
\begin{aligned}
\|\zeta_{i,k}\|^2_{L^2(\mathbf{E},\mu)} &= \mathbb{E}\zeta_{i,k}^2 = \mathbb{E}\left(\phi_\mu(e_i)^2 \phi_\mu(e_k)^2 \right) = 1, \quad i \neq k \\
\|\zeta_{i,i}\|^2_{L^2(\mathbf{E},\mu)} &= \mathbb{E}\zeta_{i,i}^2 = \frac{1}{2}\mathbb{E}\left(\phi_\mu(e_i)^4 - 2\phi_\mu(e_i)^2 + 1 \right) = \frac{1}{2}(3 - 2 + 1) = 1,
\end{aligned}
$$

i.e. the system $\{\zeta_{i,k} : i,k \in \mathbb{N}\}$ is orthonormal in $L^2(\mathbf{E},\mu)$. Recalling (2.7), for $i,k \in \mathbb{N}$ we have

$$
[D^2_{H_\mu}\psi(x)e_i, e_k]_{H_\mu} = \begin{cases} \sqrt{2} \langle \beta, \zeta_{i,i} \rangle_{L^2(\mathbf{E},\mu)}, & \text{if } i = k, \\ \langle \beta, \zeta_{i,k} \rangle_{L^2(\mathbf{E},\mu)}, & \text{if } i \neq k, \end{cases}
$$

where $\beta(z) := \varphi(x+z)$. Thus, from the Parseval identity and Bessel inequality it follows that

$$\begin{aligned}\left\|D^2_{H_\mu}\psi(x)\right\|^2_{\mathcal{T}_2(H_\mu)} &= \sum_{i=1}^{\infty}\left|D^2_{H_\mu}\psi(x)e_i\right|^2_{H_\mu} = \sum_{i,k=1}^{\infty}\left|[D^2_{H_\mu}\psi(x)e_i, e_k]_{H_\mu}\right|^2 \\ &= 2\sum_{i=1}^{\infty}\left|\langle\beta,\zeta_{i,i}\rangle_{L^2(\mathbf{E},\mu)}\right|^2 + \sum_{\substack{i,k=1\\ i\neq k}}^{\infty}\left|\langle\beta,\zeta_{i,k}\rangle_{L^2(\mathbf{E},\mu)}\right|^2 \\ &\leq 2\sum_{i,k=1}^{\infty}\left|\langle\beta,\zeta_{i,k}\rangle_{L^2(\mathbf{E},\mu)}\right|^2 \\ &\leq 2\left\|\beta\right\|^2_{L^2(\mathbf{E},\mu)} \\ &\leq 2\left|\varphi\right|^2_0 .\end{aligned}$$

□

2.2 Stochastic integration in Banach spaces

2.2.1 $\gamma-$radonifying operators

From this point onwards, $(\mathbf{H}, [\cdot,\cdot]_{\mathbf{H}})$ will denote a Hilbert space and $(\gamma_k)_k$ a sequence of real-valued standard Gaussian random variables. As before, $\mathbf{E}$ is a Banach space.

Definition 2.7. A bounded linear operator $R : \mathbf{H} \to \mathbf{E}$ is said to be $\gamma-$*radonifying* iff there exists an orthonormal basis $(e_k)_{k\geq 1}$ of $\mathbf{H}$ such that the sum $\sum_{k\geq 1}\gamma_k Re_k$ converges in $L^2(\Omega;\mathbf{E})$.

We denote by $\gamma(\mathbf{H},\mathbf{E})$ the class of $\gamma-$radonifying operators from $\mathbf{H}$ into $\mathbf{E}$, which can be proved to be a Banach space when equipped with the norm

$$\|R\|^2_{\gamma(\mathbf{H},\mathbf{E})} := \mathbb{E}\Big|\sum_{k\geq 1}\gamma_k Re_k\Big|^2_{\mathbf{E}}, \quad R\in\gamma(\mathbf{H},\mathbf{E}).$$

The above definition is independent of the choice of the orthonormal basis $(e_k)_{k\geq 1}$ of $\mathbf{H}$. Moreover, $\gamma(\mathbf{H},\mathbf{E})$ is continuously embedded into $\mathcal{L}(\mathbf{H},\mathbf{E})$ and is an operator ideal in the sense that if $\mathbf{H}'$ and $\mathbf{E}'$ are Hilbert and Banach spaces respectively such that $S_1 \in \mathcal{L}(\mathbf{H}',\mathbf{H})$ and $S_2 \in \mathcal{L}(\mathbf{E},\mathbf{E}')$ then $R\in\gamma(\mathbf{H},\mathbf{E})$ implies $S_2RS_1 \in \gamma(\mathbf{H}',\mathbf{E}')$ with

$$\|S_2RS_1\|_{\gamma(\mathbf{H}',\mathbf{E}')} \leq \|S_2\|_{\mathcal{L}(\mathbf{E},\mathbf{E}')}\|R\|_{\gamma(\mathbf{H},\mathbf{E})}\|S_1\|_{\mathcal{L}(\mathbf{H}',\mathbf{H})}$$

It can also be proved that $R \in \gamma(\mathbf{H}, \mathbf{E})$ iff RR^* is the covariance operator of a centered Gaussian measure on $\mathcal{B}(\mathbf{E})$, and if $\mathbf{E}$ is a Hilbert space, then $\gamma(\mathbf{H}, \mathbf{E})$ coincides with the space of Hilbert-Schmidt operators from $\mathbf{H}$ into $\mathbf{E}$ (see e.g. [vN08] and the references therein). There is also a very useful characterization of $\gamma-$radonifying operators if $\mathbf{E}$ is a L^p-space,

Lemma 2.8 ([vNVW08], Lemma 2.1)**.** *Let $(S, \mathfrak{A}, \rho)$ be a $\sigma-$finite measure space and let $p \in [1, \infty)$. Then, for an operator $R \in \mathcal{L}(\mathbf{H}, L^p(S))$ the following assertions are equivalent*

1. $R \in \gamma(\mathbf{H}, L^p(S))$,
2. *There exists a function $g \in L^p(S)$ such that for all $y \in \mathbf{H}$ we have*

$$|(Ry)(s)| \leq |y|_{\mathbf{H}} \cdot g(s), \quad \rho - a.e.\ s \in S.$$

In such situation, there exists a constant $c > 0$ such that $\|R\|_{\gamma(\mathbf{H}, L^p(S))} \leq c\, |g|_{L^p(S)}$.

2.2.2 Stochastic integration of operator-valued functions

In this section we briefly review some of the results from [vNW05a] on stochastic integration of deterministic operator valued functions with respect to a cylindrical Wiener process. For the rest of this section we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

Definition 2.9. A family $W(\cdot) = \{W(t)\}_{t \geq 0}$ of bounded linear operators from $\mathbf{H}$ into $L^2(\Omega; \mathbb{R})$ is called a $\mathbf{H}-$*cylindrical Wiener process* (with respect to the filtration $\mathbb{F}$) iff

(i) $\mathbb{E}\, W(t)y_1 W(t)y_2 = t[y_1, y_2]_{\mathbf{H}}$, for all $t \geq 0$ and $y_1, y_2 \in \mathbf{H}$,

(ii) for each $y \in \mathbf{H}$, the process $\{W(t)y\}_{t \geq 0}$ is a standard one-dimensional Wiener process with respect to $\mathbb{F}$.

Before we discuss the integral for $\mathcal{L}(\mathbf{H}, \mathbf{E})-$valued functions, we observe that we can integrate ceratin $\mathbf{H}-$valued functions with respect to a $\mathbf{H}-$cylindrical Wiener process $W(\cdot)$. For a step function of the form $\psi = \mathbf{1}_{(s,t]} y$ with $y \in \mathbf{H}$ we define

$$\int_0^T \psi(r)\, dW(r) := W(t)y - W(s)y.$$

This extends to arbitrary step functions ψ by linearity, and a standard computation shows that

$$\mathbb{E}\Big|\int_0^T \psi(r)\,dW(r)\Big|_{\mathbf{H}}^2 = \int_0^T |\psi(t)|_{\mathbf{H}}^2\,dt.$$

Since the set of step functions $L^2_{\text{step}}(0,T;\mathbf{H})$ is dense in $L^2(0,T;\mathbf{H})$, the map

$$I_T : L^2_{\text{step}}(0,T;\mathbf{H}) \ni \psi \mapsto \int_0^T \psi(t)\,dW(t) \in L^2(\Omega;\mathbf{H})$$

extends to an isometry from $L^2(0,T;\mathbf{H})$ into $L^2(\Omega;\mathbf{H})$. We now define the stochastic integral for certain $\mathcal{L}(\mathbf{H},\mathbf{E})-$valued functions with respect to $W(\cdot)$.

Definition 2.10. 1. A function $\Phi : (0,T) \to \mathcal{L}(\mathbf{H},\mathbf{E})$ is said to belong *scalarly* to $L^2(0,T;\mathbf{H})$ if the map $[0,T] \ni t \mapsto \Phi(t)^*x^* \in \mathbf{H}$ belongs to $L^2(0,T;\mathbf{H})$ for every $x^* \in \mathbf{E}^*$.

2. A function $\Phi : (0,T) \to \mathcal{L}(\mathbf{H},\mathbf{E})$ is said to be $\mathbf{H}-$*strongly measurable* if the mapping $[0,T] \ni t \mapsto \Phi(t)y \in \mathbf{E}$ is strongly measurable for all $y \in \mathbf{H}$.

Definition 2.11. A function $\Phi : (0,T) \to \mathcal{L}(\mathbf{H},\mathbf{E})$ is said to be *stochastically integrable* with respect to $W(\cdot)$ if it belongs scalarly to $L^2(0,T;\mathbf{H})$ for all measurable $A \subset (0,T)$ there exists a $Y_A \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbf{E})$ such that

$$\langle Y_A, x^*\rangle = \int_0^T \mathbf{1}_A(t)\Phi(t)^*x^*\,dW(t), \quad \mathbb{P}-\text{a.s., for all} x^* \in \mathbf{E}^*,$$

and we write

$$Y_A = \int_A \Phi(t)\,dW(t)$$

The $\mathbf{E}-$valued random variables Y_A are uniquely determined almost everywhere and Gaussian. In particular $Y_A \in L^p(\Omega;\mathbf{E})$ for all $p \geq 1$.

We collect some elementary properties of the stochastic integral that are immediate consequence of its definition. Let $\Phi, \Psi : (0,T) \to \mathcal{L}(\mathbf{H},\mathbf{E})$ be stochastically integrable with respect to $W(\cdot)$,

1. For all measurable subsets $B \subset (0,T)$ the function $\mathbf{1}_B\Phi$ is stochastically integrable with respect to $W(\cdot)$ and

$$\int_0^T \mathbf{1}_B(t)\Phi(t)\,dW(t) = \int_B \Phi(t)dW(t), \quad \mathbb{P}-\text{a.s.}$$

2. For all $a, b \in \mathbb{R}$ the map $a\Phi + b\Psi$ is stochastically integrable with respect to $W(\cdot)$ and

$$\int_0^T (a\Phi + b\Psi)(t)\, dW(t) = a \int_0^T a\Phi(t)\, dW(t) + b \int_0^T \Psi(t)\, dW(t), \quad \mathbb{P} - \text{a.s.}$$

3. For every real Banach space $\mathbf{E}'$ and $R \in \mathcal{L}(\mathbf{E}, \mathbf{E}')$ the function $R\Phi : (0, T) \to \mathcal{L}(\mathbf{H}, \mathbf{E}')$ is is stochastically integrable with respect to $W(\cdot)$ and

$$\int_0^T R\Phi(t)\, dW(t) = R \int_0^T \Phi(t) dW(t), \quad \mathbb{P} - \text{a.s.}$$

For a function $\Phi : (0, T) \to \mathcal{L}(\mathbf{H}, \mathbf{E})$ that belongs scalarly to $L^2(0, T; \mathbf{H})$ we define an operator $R_\Phi : L^2(0, T; \mathbf{H}) \to \mathbf{E}^{**}$ by

$$\langle x^*, R_\Phi f \rangle := \int_0^T [\Phi(t)^* x^*, f(t)]_{\mathbf{H}}\, dt, \quad f \in L^2(0, T; \mathbf{H}), \;\; x^* \in \mathbf{E}^*.$$

Observe that I_Φ is the adjoint of the operator $\mathbf{E}^* \ni x^* \mapsto \Phi(t)^* x^* \in L^2(0, T; \mathbf{H})$. If Φ is $\mathbf{H}-$strongly measurable then R_Φ maps $L^2(0, T; \mathbf{H})$ into $\mathbf{E}$. Indeed, this is clear for step functions Φ of the form $\sum_{k=1}^N \mathbf{1}_{A_k} y_k$ with the property that $[0, T] \ni t \mapsto \Phi(t) y_k$ is bounded on A_k, and the general case follows from the fact that these step functions are dense in $L^2(0, T; \mathbf{H})$.

The following theorem characterizes the class of stochastically integrable functions.

Theorem 2.12 ([vNW05a], Theorem 4.2)**.** *For a function $\Phi : (0, T) \to \mathcal{L}(\mathbf{H}, \mathbf{E})$ that belongs scalarly to $L^2(0, T; \mathbf{H})$ the following assertions are equivalent*

1. *Φ is stochastically integrable with respect to $W(\cdot)$;*

2. *There exists an $\mathbf{E}-$valued random variable Y and a weak*-sequentially dense linear subspace D of $\mathbf{E}^*$ such that for all $x^* \in D$ we have*

$$\langle Y, x^* \rangle = \int_0^T \Phi(t)^* x^*\, dW(t), \quad \mathbb{P} - a.s.$$

3. *There exists a centred Gaussian measure μ on $\mathbf{E}$ with covariance operator $Q \in \mathcal{L}(\mathbf{E}, \mathbf{E}^*)$ and a weak*-sequentially dense linear subspace D of $\mathbf{E}^*$ such that for all $x^* \in D$ we have*

$$\langle Qx^*, x^* \rangle = \int_0^T |\Phi(t)^* x^*|^2_{\mathbf{H}}\, dt;$$

4. *There exist a separable Hilbert space* $\mathfrak{H}$ *a linear bounded operator* $S \in \gamma(\mathfrak{H}, \mathbf{E})$, *and a weak*-sequentially dense linear subspace* D *of* $\mathbf{E}^*$ *such that for all* $x^* \in D$ *we have*

$$\int_0^T |\Phi(t)^* x^*|_{\mathbf{H}}^2 \, dt \leq |S^* x^*|_{\mathfrak{H}}^2$$

5. R_Φ *maps* $L^2(0, T; \mathbf{H})$ *into* $\mathbf{E}$ *and* $R_\phi \in \gamma(L^2(0, T; \mathbf{H}); \mathbf{E})$.

If these equivalent conditions hold then we may take $D = \mathbf{E}^*$. *Moreover, for all* $y \in \mathbf{H}$ *the function* $\Phi(\cdot)y$ *is both Pettis integrable and stochastic integrable with respect to* $W(\cdot)y$, *and we have the series representation*

$$\int_0^T \Phi(t) \, dW(t) = \sum_{n \geq 1} \int_0^T \Phi(t) e_n \, dW(t) e_n$$

where $(e_n)_{n \geq 1}$ *is any orthonormal basis for* $\mathbf{H}$. *The series converges* $\mathbb{P}-a.s$ *and in* $L^p(\Omega; \mathbf{E})$ *for all* $p \in [0, \infty)$. *The measure* μ *is the distribution of* $\int_0^T \Phi(t) \, dW(t)$ *and we have the isometry*

$$\mathbb{E} \Big| \int_0^T \Phi(t) \, dW(t) \Big|_{\mathbf{E}} = \| R_\Phi \|_{\gamma(L^2(0,T;\mathbf{H});\mathbf{E})}$$

Remark 2.13. In the last theorem, if Φ is strongly measurable (in particular, if $\mathbf{E}$ is separable), then it suffices to assume that D is weak*-dense.

We conclude this section by recalling a sufficient condition for stochastic integrability in spaces of type 2 (see [vNW05a, Theorem 4.7] and [vNW05b, Theorem 5.1]).

Definition 2.14. $\mathbf{E}$ is said to be of *type* 2 iff there exists $K_2 > 0$ such that

$$\mathbb{E} \Big| \sum_{i=1}^n \epsilon_i x_i \Big|_{\mathbf{E}}^2 \leq K_2 \sum_{i=1}^n |x_i|_{\mathbf{E}}^2 \tag{2.15}$$

for any finite sequence $\{x_i\}_{i=1}^n$ of elements of $\mathbf{E}$ and for any finite sequence $\{\epsilon_i\}_{i=1}^n$ of $\{-1, 1\}-$valued symmetric i.i.d. random variables.

Theorem 2.15. *Let* $\mathbf{E}$ *be a separable real Banach space of type 2. If* $\Phi : (0, T) \to \mathcal{L}(\mathbf{H}, \mathbf{E})$ *belongs scalarly to* $L^2(0, T; \mathbf{H})$, *for almost all* $t \in (0, T)$ *we have* $\Phi(t) \in \gamma(\mathbf{H}, \mathbf{E})$, *and*

$$\int_0^T \|\Phi(t)\|_{\gamma(\mathbf{H},\mathbf{E})}^2 \, dt < \infty$$

then Φ *is stochastically integrable with respect to* $W(\cdot)$ *and*

$$\mathbb{E}\Big|\int_0^T \Phi(t)\, dW(t)\Big|_{\mathbf{E}}^2 \leq K_2^2 \int_0^T \|\Phi(t)\|_{\gamma(\mathbf{H},\mathbf{E})}^2 \, dt.$$

Proof. See Theorem 5.1 in [vNW05b]. □

2.2.3 Stochastic integration in martingale type-2 Banach spaces

Finally, we outline the construction of the stochastic integral of operator-valued processes in martingale type-2 Banach spaces with respect to a cylindrical Wiener process (see e.g. [Det90, Brz97]. See also [Brz03] and the references therein). We need first some notation. For $h \in \mathbf{H}$ and $x \in \mathbf{E}$, $h \otimes x$ will denote the linear operator

$$(h \otimes x)y := [h, y]_{\mathbf{H}} x, \quad y \in \mathbf{H}.$$

For $p \geq 1$ and a Banach space $(V, |\cdot|_V)$, let $\mathcal{M}^p(0, T; V)$ denote the space of (classes of equivalences of) $\mathbb{F}-$progressively measurable processes $\Phi : [0, T] \times \Omega \to V$ such that

$$\|\Phi\|_{\mathcal{M}^p(0,T;V)}^p := \mathbb{E} \int_0^T |\Phi(t)|_V^p \, dt < \infty.$$

Notice that $\mathcal{M}^p(0, T; V)$ is a Banach space with the norm $\|\cdot\|_{\mathcal{M}^p(0,T;V)}$.

Definition 2.16. A process $\Phi(\cdot)$ with values in $\mathcal{L}(\mathbf{H}, \mathbf{E})$ is said to be *elementary* (with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$) if there exists a partition $0 = t_0 < t_1 \cdots < t_N = T$ of $[0, T]$ such that

$$\Phi(s) = \sum_{n=0}^{N-1} \sum_{k=1}^{K} \mathbf{1}_{[t_n, t_{n+1})}(s) e_k \otimes \xi_{kn}, \quad s \in [0, T].$$

where $(e_k)_k$ is an orthonormal basis of $\mathbf{H}$ and ξ_{kn} is a $\mathbf{E}-$valued $\mathcal{F}_{t_n}-$measurable random variable, for $n = 0, 1, \dots, N-1$. For such processes we define the *stochastic integral* as

$$I_T(\Phi) := \int_0^T \Phi(s)\, dW(s) := \sum_{n=0}^{N-1} \sum_{k=1}^{K} \left(W(t_{n+1})e_k - W(t_n)e_k\right) \xi_{kn}.$$

Definition 2.17. A Banach space $\mathbf{E}$ is said to be of *martingale type* 2 (and we write $\mathbf{E}$ is *M-type*

2) iff there exists a constant $C_2 > 0$ such that

$$\sup_n \mathbb{E}|M_n|_{\mathbf{E}}^2 \leq C_2 \sum_n \mathbb{E}|M_n - M_{n-1}|_{\mathbf{E}}^2 \tag{2.16}$$

for any $\mathbf{E}-$valued discrete martingale $\{M_n\}_{n\in\mathbb{N}}$ with $M_{-1} = 0$.

Example 2.18. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$. Then the Lebesgue spaces $L^p(\mathcal{O})$ are both type 2 and M-type 2, for $p \in [2, \infty)$.

Lemma 2.19. *Let* $\mathbf{E}$ *be a M-type 2 Banach space and let* $\Phi(\cdot)$ *be a* $\mathcal{L}(\mathbf{H}, \mathbf{E}) - valued$ *simple process. Then, the stochastic integral* $I_T(\Phi)$ *satisfies*

$$\mathbb{E}\left|I_T(\Phi)\right|_{\mathbf{E}}^2 \leq C_2 \mathbb{E} \int_0^T \|\Phi(s)\|_{\gamma(\mathbf{H},\mathbf{E})}^2 \, ds \tag{2.17}$$

Proof. The sequence $M_n := \sum_{i=0}^{n-1} \sum_{k=1}^{K} \left(W(t_{i+1})e_k - W(t_i)e_k\right) \xi_{ki}$ is a $\mathbf{E}-$valued martingale with respect to the filtration $\{\mathcal{F}_{t_n}\}_n$. Then, by the M-type 2 property we have

$$\begin{aligned}
\mathbb{E}\left|\int_0^T \Phi(s)\, dW(s)\right|_{\mathbf{E}}^2 &= \mathbb{E}\Big|\sum_{n=0}^{N-1} \sum_{k=1}^{K} \left(W(t_{n+1})e_k - W(t_n)e_k\right) \xi_{kn}\Big|_{\mathbf{E}}^2 \\
&\leq C_2 \sum_{n=0}^{N-1} \mathbb{E}\Big|\sum_{k=1}^{K} \left(W(t_{n+1})e_k - W(t_n)e_k\right) \xi_{kn}\Big|_{\mathbf{E}}^2 \\
&= C_2 \sum_{n=0}^{N-1} (t_{n+1} - t_n) \mathbb{E}\Big|\sum_{k=1}^{K} \eta_{kn}\xi_{kn}\Big|_{\mathbf{E}}^2
\end{aligned}$$

where

$$\eta_{kn} := \frac{W(t_{n+1})e_k - W(t_n)e_k}{\sqrt{t_{n+1} - t_n}}, \quad k = 1, \ldots, K, \;\; n = 0, \ldots, N-1.$$

Since for each n, ξ_{kn} is $\mathcal{F}_{t_n}-$measurable and η_{kn} is independent of $\mathcal{F}_{t_n}$, by Proposition 1.12 in

[DPZ92b] we have

$$\begin{aligned}
\mathbb{E}\Big|\sum_{k=1}^{K}\eta_{kn}\xi_{kn}\Big|_{\mathbf{E}}^{2} &= \mathbb{E}\left[\mathbb{E}\Big[\Big|\sum_{k=1}^{K}\eta_{kn}\xi_{kn}\Big|_{\mathbf{E}}^{2}\,\Big|\,\mathcal{F}_{t_n}\Big]\right] \\
&= \int_{\Omega}\int_{\Omega}\Big|\sum_{k=1}^{K}\eta_{kn}(\omega')\Phi(t_n,\omega)e_k\Big|_{\mathbf{E}}^{2}\,\mathbb{P}(d\omega')\,\mathbb{P}(d\omega) \\
&= \int_{\Omega}\|\Phi(t_n,\omega)\|_{\gamma(\mathbf{H},\mathbf{E})}^{2}\;\mathbb{P}(d\omega) \\
&= \mathbb{E}\,\|\Phi(t_n)\|_{\gamma(\mathbf{H},\mathbf{E})}^{2}
\end{aligned}$$

and (2.17) follows. □

Since the set of elementary processes is dense in $\mathcal{M}^2(0,T;\gamma(\mathbf{H},\mathbf{E}))$ (see e.g. [Nei78, Ch. 2, Lemma 18]) by (2.17) the linear mapping I_T extends to a bounded linear operator from $\mathcal{M}^2(0,T;\gamma(\mathbf{H},\mathbf{E}))$ into $L^2(\Omega;\mathbf{E})$. We denote this operator also by I_T.

Finally, for each $t\in[0,T]$ and $\Phi\in\mathcal{M}^2(0,T;\gamma(\mathbf{H},\mathbf{E}))$, we define

$$\int_0^t \Phi(s)\,dW(s) := I_T(\mathbf{1}_{[0,t)}\Phi).$$

The process $\int_0^t \Phi(s)\,dW(s),\ t\in[0,T]$, is a martingale with respect to $\mathbb{F}$. Moreover, we have the following Burkholder Inequality

Proposition 2.20. *Let $\mathbf{E}$ be a M-type 2 Banach space. Then, for any $p\in(0,+\infty)$ there exists a constant $C=C(p,\mathbf{E})$ such that for all $\Phi\in\mathcal{M}^2(0,T;\gamma(\mathbf{H},\mathbf{E}))$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\Big|\int_0^t\Phi(s)\,dW(s)\Big|_{\mathbf{E}}^{p}\right] \le \left(\tfrac{p}{p-1}\right)^{p} C\cdot\mathbb{E}\left[\left(\int_0^T\|\Phi(s)\|_{\gamma(\mathbf{H},\mathbf{E})}^{2}\,ds\right)^{p/2}\right]$$

Proof. See e.g. Theorem 2.4 in [Brz97]. □

Let $M(\cdot)$ be a $\mathbf{E}-$valued continuous martingale with respect to the filtration $\mathbb{F}=\{\mathcal{F}_t\}_{t\ge 0}$ and let $\langle\cdot,\cdot\rangle$ denote the duality between $\mathbf{E}$ and $\mathbf{E}^*$. The *cylindrical quadratic variation* of $M(\cdot)$, denoted by $[M]$, is defined as the (unique) cylindrical process (or linear random function) with values in $\mathcal{L}(\mathbf{E}^*,\mathbf{E})$ that is $\mathbb{F}-$adapted, increasing and satisfies

1. $[M](0)=0$

2. for arbitrary $x^*, y^* \in \mathbf{E}^*$, the real-valued process

$$\langle M(t), x^* \rangle \langle M(t), y^* \rangle - \langle [M](t) x^*, y^* \rangle , \quad t \geq 0$$

is a martingale with respect to $\mathbb{F}$.

For more details on this definition we refer to [Det90]. We will need the following version of the Martingale Representation Theorem in M-type 2 Banach spaces, see Theorem 2.4 in [Det90] (see also [Ond05]),

Theorem 2.21. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space and let* $\mathbf{E}$ *be a separable M-type 2 Banach space. Let* $M(\cdot)$ *be a* $\mathbf{E}-$*valued continuous square integrable* $\mathbb{F}-$*martingale with cylindrical quadratic variation process of the form*

$$[M](t) = \int_0^t g(s) \circ g(s)^* \, ds, \quad t \in [0, T],$$

where $g \in \mathcal{M}^2(0, T; \gamma(\mathbf{H}, \mathbf{E}))$. *Then, there exists a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *extension of* $(\Omega, \mathcal{F}, \mathbb{P})$, *and a* $\mathbf{H}-$*cylindrical Wiener process* $\{\tilde{W}(t)\}_{t \geq 0}$ *defined on* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, *such that*

$$M(t) = \int_0^t g(s) \, d\tilde{W}(s), \quad \tilde{\mathbb{P}} - \textit{a.s.}, \quad t \in [0, T].$$

Chapter 3

Ornstein-Uhlenbeck transition operators and mild solutions of Hamilton-Jacobi-Bellman equations in Banach spaces

Let $\mathbf{E}$ be a Banach space and let M be a separable metric space. Let $\mathbf{H}$ be a separable Hilbert space and let $W(\cdot)$ be a $\mathbf{H}-$cylindrical Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $T > 0$ be fixed and consider the finite-horizon control problem of minimizing a cost functional of the form

$$J(X, u) = \mathbb{E}\left[\int_0^T h(t, X(t), u(t))\, dt + \varphi(X(T))\right]$$

where $u(\cdot)$ is a $M-$valued control process and $X(\cdot)$ is solution to the controlled non-autonomous stochastic evolution equation (possibly the functional analytic formulation of a stochastic PDE)

$$\begin{aligned} dX(t) + A(t)X(t)\, dt &= F(t, X(t), u(t))\, dt + G(t)\, dW(t) \\ X(0) &= x_0 \in \mathbf{E} \end{aligned} \tag{3.1}$$

Here $\{-A(t)\}_{t\in[0,T]}$ is the generator of an evolution family on $\mathbf{E}$ and $\{G(t)\}_{t\in[0,T]}$ are (possibly unbounded) linear operators from $\mathbf{H}$ into $\mathbf{E}$. On the basis of Bellman's Dynamic Programming Principle, a well-known approach to the above control problem consists in showing the existence and uniqueness of solutions to the *Hamilton-Jacobi-Bellman* (HJB) equation on $[0,T]\times\mathbf{E}$,

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x)+L_t v(t,\cdot)(x)&=\mathcal{H}(t,x,D_x v(t,x)),\quad (t,x)\in[0,T]\times\mathbf{E}\\ v(T,x)&=\varphi(x)\end{aligned}\tag{3.2}$$

where

$$(L_t\phi)(x):=\langle -A(t)x,D_x\phi(x)\rangle+\frac{1}{2}\operatorname{Tr}_{\mathbf{H}}[G(t)^*D_x^2\phi(x)G(t)],\ \ x\in D(A(t)),\ \ \phi\in\mathcal{C}_b^2(\mathbf{E})$$

is the second-order operator associated with the linearized process (sometimes referred to as *Ornstein-Uhlenbeck* process)

$$\begin{aligned}dZ(t)+A(t)Z(t)\,dt&=G(t)\,dW(t)\\ Z(0)&=x\in\mathbf{E}\end{aligned}\tag{3.3}$$

and $\mathcal{H}:[0,T]\times\mathbf{E}\times\mathbf{E}^*\to\mathbb{R}$ is the *Hamiltonian* defined by

$$\mathcal{H}(t,x,p)=\sup_{u\in M}\left\{-\langle F(t,x,u),p\rangle-h(t,x,u)\right\},\quad t\in[0,T],\ x\in\mathbf{E},\ \ p\in\mathbf{E}^*.$$

Solving the HJB equation can be useful to provide the verification and the synthesis of optimal control strategies. This has been done extensively for stochastic optimal control problems governed by stochastic evolution equations in Hilbert spaces (see e.g. [Goz96, Cer99, DPZ02]). In this chapter we prove that the existence and uniqueness result of mild solutions to equation (3.2) using the transition evolution family associated with the process (3.3) can be easily generalized to the Banach space setting.

3.1 Parabolic evolution families

Let $\{(A(t),D(A(t))),\ t\in[0,T]\}$ be a family of densely defined closed linear operators on a Banach space $\mathbf{E}$. For each $s\in[0,T]$, consider the following non-autonomous Cauchy problem

$$\begin{aligned}y'(t)+A(t)y(t)&=0,\quad t\in[s,T]\\ y(s)&=x\in\mathbf{E}.\end{aligned}\tag{3.4}$$

Definition 3.1. We say that $y\in\mathcal{C}((s,T];\mathbf{E})\cap\mathcal{C}^1((s,T];\mathbf{E})$ is a *classical* solution of (3.4) if $y(t)\in D(A(t))$ and $y'(t)+A(t)y(t)=0$ for all $t\in(s,T]$ and $y(s)=x$.

Definition 3.2. We say that $y \in \mathcal{C}^1([s,T];\mathbf{E})$ is a *strict* solution of (3.4) if $y(t) \in D(A(t))$ for all $t \in [s,T]$ and the equalities in (3.4) are satisfied.

Definition 3.3. In what follows we denote $\mathfrak{T} := \{(t,s) \in [0,T]^2 : s \leq t\}$. A family of bounded operators $\{S(t,s)\}_{(t,s)\in\mathfrak{T}}$ on $\mathbf{E}$ is called a *strongly continuous evolution family* if

1. $S(t,t) = I$, for all $t \in [0,T]$.
2. $S(t,s) = S(t,r)S(r,s)$ for all $0 \leq s \leq r \leq t \leq T$.
3. The mapping $\mathfrak{T} \ni (t,s) \mapsto S(t,s) \in \mathcal{L}(\mathbf{E})$ is strongly continuous.

We say that the family $\{S(t,s)\}_{(t,s)\in\mathfrak{T}}$ *solves* non-autonomous Cauchy problem (3.4) if there exist a family $(Y_s)_{s\in[0,T]}$ of dense subspaces of $\mathbf{E}$ such that for all $(s,t) \in \mathfrak{T}$ we have $S(t,s)Y_s \subset Y_t \subset D(A(t))$ and the function $y(t) := S(t,s)x$ is a strict solution of (3.4) for every $x \in Y_s$. In this case we say that $\{(-A(t), D(A(t))),\ t \in [0,T]\}$ generates the evolution family $\{S(t,s)\}_{(t,s)\in\mathfrak{T}}$.

We now briefly discuss the setting of Acquistapace and Terreni. We say that condition **(AT)** is satisfied if the following conditions hold

(AT1) *There exist constants* $w \in \mathbb{R},\ K \geq 0$ *and* $\phi \in (\frac{\pi}{2}, \pi)$ *such that*

$$\Sigma(\phi, w) := \{w\} \cup \{\lambda \in \mathbb{C} \setminus \{w\} : |\arg(\lambda - w)| \leq \phi\} \subset \rho(-A(t))$$

and for all $\lambda \in \Sigma(\phi, w)$ *and* $t \in [0,T]$,

$$\|(A(t) + \lambda I)^{-1}\|_{\mathcal{L}(\mathbf{E})} \leq \frac{K}{1 + |\lambda - w|}$$

(AT2) *There exist constants* $L \geq 0$ *and* $\mu, \nu \in (0,1)$ *with* $\mu + \nu > 1$ *such that for all* $\lambda \in \Sigma(\phi, 0)$ *and* $s, t \in [0,T]$,

$$\left\|(A(t) + wI)((A(t) + wI) + \lambda I)^{-1}[(A(t) + wI)^{-1} - (A(s) + wI)^{-1}]\right\|_{\mathcal{L}(\mathbf{E})} \leq L\frac{|t-s|^{\mu}}{(|\lambda| + 1)^{\nu}}.$$

If the assumption **(AT1)** is satisfied and the domains are constant i.e. $D(A(t)) = D(A(0))$ for all $t \in [0,T]$, and the map $[0,T] \ni t \mapsto A(t) \in \mathcal{L}(D(A(0)), \mathbf{E})$ is Hölder continuous with exponent η, then **(AT2)** is satisfied with $\mu = \eta$ and $\nu = 1$, see [AT87, Section 7]. In this case such conditions reduce to the theory of Sobolevskii and Tanabe for constant domains (see e.g. [Paz83, Tan79]).

Under the above assumptions we have the following well-known result (see [AT87, Theorems 6.1-6.4] and [Yag91, Theorem 2.1]).

Theorem 3.4. *If condition **(AT)** holds then there exists a unique strongly continuous evolution family $\{S(t,s)\}_{(t,s)\in\mathfrak{T}}$ that solves the non-autonomous Cauchy problem (3.4) with $Y_t = D(A(t))$ and for all $x \in \mathbf{E}$, the map $y(t) := S(t,s)x$ is a classical solution of (3.4). Moreover, $\{S(t,s)\}_{(t,s)\in\mathfrak{T}}$ is continuous on $0 \le s < t \le T$ and there exists a constant $C > 0$ such that for all $0 \le s < t \le T$ and $\theta \in [0,1]$,*

$$\left\|(A(t)+wI)^{\theta}S(t,s)\right\|_{\mathcal{L}(\mathbf{E})} \le C(t-s)^{-\theta}$$
$$\left\|S(t,s)-e^{-(t-s)A(s)}\right\|_{\mathcal{L}(\mathbf{E})} \le C(t-s)^{\mu+\nu-1}$$

Moreover, for all $\theta \in (0,\mu)$ and $x \in D\left((A(t)+wI)^{\theta}\right)$ we have

$$\left|S(t,s)(A(t)+wI)^{\theta}x\right|_{\mathbf{E}} \le C(\mu-\theta)^{-1}(t-s)^{-\theta}\,|x|_{\mathbf{E}}\,. \tag{3.5}$$

3.2 Ornstein-Uhlenbeck transition evolution families

Let $\{-A(t)\}_{t\in[0,T]}$ be the generator of an evolution family $\{S(t,s)\}_{(t,s)\in\mathfrak{T}}$ on $\mathbf{E}$ and let $\{G(t)\}_{t\in[0,T]}$ be closed operators from a constant domain $D(G) \subset \mathbf{H}$ into $\mathbf{E}$. We start this section by discussing the existence of mild solutions to the linearized version (3.3) of the controlled equation (3.1) with moving time origin $s \in [0,T]$ and initial data $x_0 \in \mathbf{E}$,

$$\begin{aligned} dZ(t) + A(t)Z(t)\,dt &= G(t)\,dW(t), \quad t \in [s,T], \\ Z(s) &= x_0 \in \mathbf{E}. \end{aligned} \tag{3.6}$$

We say that an $\mathbf{E}-$valued process $Z(\cdot)$ is a *mild solution* of problem (3.6) if for all $(t,s) \in \mathfrak{T}$ the mapping $S(t,s)G(s)$ has a continuous extension to a bounded operator from $\mathbf{H}$ into $\mathbf{E}$, which we will also denote by $S(t,s)G(s)$, such that the operator-valued function $(s,t) \ni r \mapsto S(t,r)G(r) \in \mathcal{L}(\mathbf{H},\mathbf{E})$ is stochastically integrable on the interval (s,t) and

$$Z(t) = S(t,s)x_0 + \int_s^t S(t,r)G(r)\,dW(r), \quad \mathbb{P}-\text{a.s}$$

We know from Theorem 2.12 that existence of a mild solution for (3.6) follows from the following condition

Assumption A.1. For each $(t,s) \in \mathfrak{T}$ the mapping $S(t,s)G(s) : D(G) \to \mathbf{E}$ extends to a bounded linear operator from $\mathbf{H}$ into $\mathbf{E}$, also denoted by $S(t,s)G(s)$, such that the positive symmetric operator $Q_{t,s} \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$ defined by

$$\langle Q_{t,s}x^*, y^* \rangle := \int_s^t \langle S(t,r)G(r)(S(t,r)G(r))^* x^*, y^* \rangle \, dr, \quad x^*, y^* \in \mathbf{E}^*. \tag{3.7}$$

is the covariance operator of a centred Gaussian measure $\mu_{t,s}$ on $\mathbf{E}$.

Example 3.5. Let $\mathbf{E}$ be a type-2 Banach space and suppose that for each $(t,s) \in \mathfrak{T}$ we have $S(t,s)G(s) \in \gamma(\mathbf{H}, \mathbf{E})$ and

$$\int_0^T \|S(t,s)G(s)\|^2_{\gamma(\mathbf{H},\mathbf{E})} \, dt < +\infty. \tag{3.8}$$

Then, by Theorem 2.15, Assumption A.1 holds.

Example 3.6. For each $t \in [0,T]$, let $\mathcal{A}_t$ denote the second order differential operator

$$(\mathcal{A}_t x)(\xi) := -a(t,\xi)\frac{d^2x}{d\xi^2}(\xi) + b(t,\xi)\frac{dx}{d\xi}(\xi) + c(t,\xi)x(\xi), \quad \xi \in (0,1)$$

where $a, b, c \in \mathcal{C}^\mu([0,T]; \mathcal{C}([0,1]))$ for some $\mu \in (\frac{1}{4}, 1]$, $a \in \mathcal{C}^\varepsilon([0,1]; \mathcal{C}([0,T]))$ for some $\varepsilon > 0$, and $\inf_{t\in[0,T],\xi\in[0,1]} a(t,\xi) > 0$. For $p \geq 2$ and $t \in [0,T]$, let $A_p(t)$ denote the realization in $L^p(0,1)$ of $\mathcal{A}_t$ with zero-Dirichlet boundary conditions,

$$\begin{aligned} D(A_p(t)) &:= H^{2,p}(0,1) \cap H_0^{1,p}(0,1), \\ A_p(t) &:= \mathcal{A}_t \end{aligned}$$

It is well-known that for w sufficiently large, the operator $A_p(\cdot) + wI$ satisfies **(AT)** with parameters μ and $\nu = 1$ (see e.g. [Tan79] or [AT87]). We will assume for simplicity that $w = 0$. Let $\{S_p(t,s)\}_{(t,s)\in\mathfrak{T}}$ denote the evolution family generated by $\{A_p(t)\}_{t\in[0,T]}$. Let $g \in L^1(0,T; L^\infty(0,1))$ be fixed and define, for almost every $t \in [0,T]$, the multiplication operators from $L^2(0,1)$ into $L^p(0,1)$ as follows

$$\begin{aligned} D(G(t)) &:= L^p(0,1) \subset L^2(0,1) \\ G(t)y &:= \{(0,1) \ni \xi \mapsto g(t,\xi)y(\xi) \in \mathbb{R}\}. \end{aligned}$$

The mapping $G(t)$ is not a bounded operator unless $p = 2$. However, as we will prove below, for each $(t,s) \in \mathfrak{T}$ the map $S_p(t,s)G(s)$ can be extended to bounded operator from $L^2(0,1)$ into

$L^p(0,1)$ that is $\gamma-$radonifying and satisfies (3.8). Since the space $L^p(0,1)$ has type-2 if $p \geq 2$, from Example 3.5 we conclude that Assumption **A.1** holds with $\mathbf{H} = L^2(0,1)$ and $\mathbf{E} = L^p(0,1)$ with $p \geq 2$.

Justification of Example 3.6. We show first that if $\sigma > \frac{1}{4}$ then $A_p(t)^{-\sigma}$ extends to a bounded operator from $L^2(0,1)$ into $L^p(0,1)$, which we also denote by $A_p(t)^{-\sigma}$, such that

$$A_p(t)^{-\sigma} \in \gamma\left(L^2(0,1), L^p(0,1)\right). \tag{3.9}$$

Let Δ_p denote the realization of $-\frac{d^2}{d\xi^2}$ in $L^p(0,1)$ with zero-Dirichlet boundary conditions. The functions $e_n(\xi) = \sqrt{2}\sin(n\pi\xi)$, $n \geq 1$, form an orthonormal basis of eigenfunctions for Δ_2 with eigenvalues $\lambda_n = (n\pi)^2$. If we endow $D(\Delta_2)$ with the equivalent Hilbert norm $|y|_{D(\Delta_2)} := |\Delta_2 y|_{L^2(0,1)}$, the functions $\lambda_n^{-1}e_n$ form an orthonormal basis for $D(\Delta_2)$. Let $(\gamma_n)_n$ be a Gaussian sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, we have

$$\begin{aligned}\mathbb{E}\Big|\sum_{n\geq 1}\gamma_n\lambda_n^{-1}e_n\Big|^2_{D(\Delta_p^{1-\sigma})} &= \mathbb{E}\Big|\sum_{n\geq 1}\gamma_n\lambda_n^{-1}\Delta_p^{1-\sigma}e_n\Big|^2_{L^p(0,1)} \\ &= \mathbb{E}\Big|\sum_{n\geq 1}\gamma_n(n\pi)^{-2\sigma}e_n\Big|^2_{L^p(0,1)}\end{aligned} \tag{3.10}$$

Now, by the Kahane-Khintchine inequality, there exist constants c_p and c_p' such that

$$
\begin{aligned}
\mathbb{E}\Big|\sum_{n=N}^{M}\gamma_n(n\pi)^{-2\sigma}e_n\Big|^2_{L^p(0,1)} &\leq c_p\left(\mathbb{E}\Big|\sum_{n=N}^{M}\gamma_n(n\pi)^{-2\sigma}e_n\Big|^p_{L^p(0,1)}\right)^{2/p}\\
&= c_p\left(\mathbb{E}\int_0^1\big|\sum_{n=N}^{M}\gamma_n(n\pi)^{-2\sigma}e_n(\xi)\big|^p\,d\xi\right)^{2/p}\\
&= c_p\left(\int_0^1\mathbb{E}\big|\sum_{n=N}^{M}\gamma_n(n\pi)^{-2\sigma}e_n(\xi)\big|^p\,d\xi\right)^{2/p}\\
&\leq c_p'\left(\int_0^1\Big(\mathbb{E}\big|\sum_{n=N}^{M}\gamma_n(n\pi)^{-2\sigma}e_n(\xi)\big|^2\Big)^{p/2}\,d\xi\right)^{2/p}\\
&= c_p'\left(\int_0^1\Big(\sum_{n=N}^{M}\big|(n\pi)^{-2\sigma}e_n(\xi)\big|^2\Big)^{p/2}\,d\xi\right)^{2/p}\\
&= c_p'\Big|\sum_{n=N}^{M}(n\pi)^{-4\sigma}e_n^2\Big|_{L^{p/2}(0,1)}\\
&\leq c_p'\sum_{n=N}^{M}(n\pi)^{-4\sigma}\,\big|e_n^2\big|_{L^{p/2}(0,1)}
\end{aligned}
$$

Since $|e_n^2|_{L^{p/2}(0,1)} = |e_n|^2_{L^p(0,1)} \leq 2$ for all $n \geq 1$, it follows

$$
\mathbb{E}\Big|\sum_{n=N}^{M}\gamma_n(n\pi)^{-2\sigma}e_n\Big|^2_{L^p(0,1)} \leq 2c_p'\sum_{n=N}^{M}(n\pi)^{-4\sigma}.
$$

The right-hand side of the last inequality tends to 0 as $N, M \to \infty$ since $\sigma > \frac{1}{4}$. Therefore, the right-hand side of (3.10) is finite, and it follows that the identity operator on $D(\Delta_2)$ extends to a continuous embedding $j : D(\Delta_2) \hookrightarrow D(\Delta_p^{1-\sigma})$ which is $\gamma-$radonifying.

By the results in [Tan79, Section 5.2], it follows that $\{A_p(t)A_p(s)^{-1} : s, t, \in [0,T]\}$ is uniformly bounded in $\mathcal{L}(L^p(0,1))$. In particular, this implies that $D(A_p(t)) = D(A_p(0))$ with equivalent norms uniformly in $t \in [0,T]$. Since $D(A_p(0)) = D(\Delta_p)$ with equivalent norms we conclude $D(A_p(t)) = D(\Delta_p)$ with equivalent norms uniformly in $t \in [0,T]$. Moreover, from the results in [DDH$^+$04] (see also [PS93]), by the $\varepsilon-$Hölder continuity assumption on the coefficients of $\mathcal{A}_t$ it follows that $A_p(t) \in \mathrm{BIP}(L^p(0,1), \phi)$ for some $\phi > 0$ (see the Appendix for the definition of the class BIP). Hence, by Theorem A.4 in the Appendix, we have

$$D(A_p(t)^{1-\sigma}) = [L^p(0,1), D(A_p(t))]_{1-\sigma} = [L^p(0,1), D(\Delta_p)]_{1-\sigma} = D(\Delta_p^{1-\sigma})$$

isomorphically, with equivalence in norm uniformly in $t \in [0,T]$. Therefore, by the ideal property of $\gamma\left(D(\Delta_2), D(\Delta_p^{1-\sigma})\right)$, we obtain

$$A_p(t)^{-\sigma} = A_p(t)^{1-\sigma} j A_2(t)^{-1} \in \gamma\left(L^2(0,1), L^p(0,1)\right)$$

with $\|A_p(t)^{-\sigma}\|_{\gamma(L^2(0,1),L^p(0,1))}$ uniformly bounded in $t \in [0,T]$.

Now, from (3.5) we know that if $\sigma \in (0,\mu)$ then the operator $S_p(t,s)A_p(s)^\sigma$ extends to a bounded operator $S_{p,\sigma}(t,s)$ on $L^p(0,1)$ with

$$\|S_{p,\sigma}(t,s)\|_{\mathcal{L}(L^p(0,1))} \leq C(\mu-\sigma)^{-1}(t-s)^{-\sigma}.$$

Hence, again by the ideal property of $\gamma-$radonifying operators, we conclude that if $\sigma \in (\frac{1}{4},\mu)$, for each $(t,s) \in \mathfrak{T}$ the linear mapping $S_p(t,s)G(s) = S_p(t,s)A_p(s)^\sigma A_p(s)^{-\sigma}G(s)$ has a continuous extension to a bounded operator from $L^2(0,1)$ into $L^p(0,1)$ that is $\gamma-$radonifying and satisfies (3.8). □

We now introduce the transition evolution operators associated with the linearized equation (3.3). Suppose that Assumptions **(AT)** and **A.1** are satisfied. Let $B_b(\mathbf{E})$ denote the set of Borel-measurable bounded real-valued functions on $\mathbf{E}$.

Definition 3.7. The *Ornstein-Uhlenbeck (OU) transition evolution operators* $\{P(s,t)\}_{(t,s)\in\mathfrak{T}}$ associated to equation (3.3) are defined by

$$[P(s,t)\varphi](x) := \int_{\mathbf{E}} \varphi(S(t,s)x+z)\,\mu_{t,s}(dz), \quad x \in \mathbf{E},\ \ \varphi \in B_b(\mathbf{E}), \quad (t,s) \in \mathfrak{T}$$

For each $(t,s) \in \mathfrak{T}$ let $(H_{t,s}, [\cdot,\cdot]_{H_{t,s}})$ denote the Reproducing Kernel Hilbert Space associated with the positive symmetric operator $Q_{t,s}$ defined by (3.7), and let $i_{t,s}$ denote the inclusion mapping from $H_{t,s}$ into $\mathbf{E}$.

Before we discuss the smoothing property of the OU transition operators, we present direct extensions of some results from [vN98, Section 1] on the relation between the spaces $H_{t,s}$ for different values of $s < t$. The first observation is the following algebraic relation between the operators $Q_{t,s}$, which is immediate from their definition

$$Q_{t,s} = Q_{t,r} + S(t,r)Q_{r,s}S(t,r)^*, \quad 0 \le s < r < t. \tag{3.11}$$

The following is a direct consequence of Proposition 2.3,

Proposition 3.8. *$H_{t,r} \subset H_{t,s}$ for all $0 \le s < r < t$.*

The last result, combined with the identity $S(t,r)Q_{r,s}S(t,r)^* = Q_{t,s} - Q_{t,r}$, implies that $S(t,r)$ maps the linear subspace $\mathrm{Range}\, Q_{r,s}S(t,r)^*$ of $H_{r,s}$ into $H_{t,s}$. The next result shows that we actually have $S(t,r)H_{r,s} \subset H_{t,s}$.

Theorem 3.9. *For all $0 \le s < r < t$ we have $S(t,r)H_{r,s} \subset H_{t,s}$. Moreover*

$$\|S(t,r)\|_{\mathcal{L}(H_{r,s},H_{t,s})} \le 1.$$

Proof. For all $x^* \in \mathbf{E}^*$ we have

$$\begin{aligned} |Q_{r,s}S(t,r)^*x^*|^2_{H_{r,s}} &= \langle Q_{r,s}S(t,r)^*x^*, S(t,r)^*x^*\rangle \\ &= \langle Q_{t,s}x^*, x^*\rangle - \langle Q_{t,r}x^*, x^*\rangle \\ &\le \langle Q_{t,s}x^*, x^*\rangle = |Q_{t,s}x^*|^2_{H_{t,s}}. \end{aligned} \tag{3.12}$$

Hence,

$$|\langle Q_{r,s}S(t,r)^*x^*, y^*\rangle| = \left|[Q_{r,s}S(t,r)^*x^*, Q_{r,s}y^*]_{H_{r,s}}\right| \le |Q_{t,s}x^*|_{H_{t,s}} |Q_{r,s}y^*|_{H_{r,s}}. \tag{3.13}$$

For $y^* \in \mathbf{E}^*$ fixed we define the linear functional $\psi_{y^*} : \mathrm{Range}\, Q_{t,s} \to \mathbb{R}$ by

$$\psi_{y^*}(Q_{t,s}x^*) := \langle Q_{r,s}S(t,r)^*x^*, y^*\rangle.$$

This is well-defined since, by (3.12), if $Q_{t,s}x^* = 0$ then $Q_{r,s}S(t,r)^*x^* = 0$. By (3.13) ψ_{y^*} extends to a bounded linear functional on $H_{t,s}$ with norm bounded by $|Q_{r,s}y^*|_{H_{r,s}}$. Identifying ψ_{y^*} with an element of $H_{t,s}$, for all $x \in \mathbf{E}^*$ we have

$$\langle \psi_{y^*}, x^*\rangle = [Q_{t,s}x^*, \psi_{y^*}]_{H_{t,s}} = \langle Q_{r,s}S(t,r)^*x^*, y^*\rangle = \langle S(t,r)Q_{r,s}y^*, x^*\rangle.$$

Therefore, $S(t,r)Q_{r,s}y^* = \psi_{y^*} \in H_{t,s}$ and $|S(t,r)Q_{r,s}y^*|_{H_{t,s}} \le |Q_{r,s}y^*|_{H_{r,s}}$, and the desired result follows. □

Next we characterize the equality of the Hilbert spaces $H_{t,r}$ and $H_{t,s}$ in terms of the restriction $S(t,r) \in \mathcal{L}(H_{r,s}, H_{t,s})$.

Theorem 3.10. *For all* $0 \le s < r < t$ *we have* $H_{t,s} = H_{t,r}$ *(as subsets of* $\mathbf{E}$*) if and only if* $\|S(t,r)\|_{\mathcal{L}(H_{r,s},H_{t,s})} < 1$.

Proof. We know already that $H_{t,r} \subset H_{t,s}$, so it remains to prove that $H_{t,s} \subset H_{t,r}$, if and only if $\|S(t,r)\|_{\mathcal{L}(H_{r,s},H_{t,s})} < 1$. First we assume that $\|S(t,r)\|_{\mathcal{L}(H_{r,s},H_{t,s})} < 1$. By Theorem 3.9, for $y^* \in \mathbf{E}^*$ we have $S(t,r)Q_{r,s}y^* \in H_{t,s}$, and it follows

$$
\begin{aligned}
[Q_{r,s}S(t,r)^*x^*, Q_{r,s}y^*]_{H_{r,s}} &= \langle S(t,r)^*x^*, Q_{r,s}y^* \rangle \\
&= \langle x^*, S(t,r)Q_{r,s}y^* \rangle \\
&= [Q_{t,s}x^*, S(t,r)Q_{r,s}y^*]_{H_{t,s}}.
\end{aligned} \tag{3.14}
$$

Hence

$$
\begin{aligned}
|Q_{r,s}S(t,r)^*x^*|_{H_{r,s}} &= \sup\left\{ [Q_{r,s}S(t,r)^*x^*, Q_{r,s}y^*]_{H_{r,s}} : y^* \in \mathbf{E}^*,\ |Q_{r,s}y^*|_{H_{r,s}} \le 1 \right\} \\
&= \sup\left\{ [Q_{t,s}x^*, S(t,r)Q_{r,s}y^*]_{H_{t,s}} : y^* \in \mathbf{E}^*,\ |Q_{r,s}y^*|_{H_{r,s}} \le 1 \right\} \\
&\le \|S(t,r)\|_{\mathcal{L}(H_{r,s},H_{t,s})} \cdot |Q_{t,s}x^*|_{H_{t,s}}.
\end{aligned}
$$

Using the last inequality, we get

$$
\begin{aligned}
\langle Q_{t,r}x^*, x^* \rangle = |Q_{t,r}x^*|^2_{H_{t,r}} &= \langle Q_{t,r}x^*, x^* \rangle \\
&= \langle Q_{t,s}x^*, x^* \rangle - \langle S(t,r)Q_{r,s}S(t,r)^*x^*, x^* \rangle \\
&= |Q_{t,s}x^*|^2_{H_{t,s}} - |Q_{r,s}S(t,r)^*x^*|^2_{H_{r,s}} \\
&\ge \left(1 - \|S(t,r)\|^2_{\mathcal{L}(H_{r,s},H_{t,s})}\right) |Q_{t,s}x^*|^2_{H_{t,s}} \\
&\ge \left(1 - \|S(t,r)\|^2_{\mathcal{L}(H_{r,s},H_{t,s})}\right) \langle Q_{t,s}x^*, x^* \rangle,
\end{aligned}
$$

and by Proposition 3.12, this gives the inclusion $H_{t,s} \subset H_{t,r}$. Conversely, assume that $H_{t,s} \subset H_{t,r}$. Then there exists $K > 0$ such that

$$
\langle Q_{t,s}x^*, x^* \rangle \le K \langle Q_{t,r}x^*, x^* \rangle = K \langle Q_{t,s}x^*.x^* \rangle - K \langle S(t,r)Q_{r,s}S(t,r)^*x, x^* \rangle, \quad \text{for all } x^* \in \mathbf{E}^*.
$$

Notice that $K > 1$ since $\langle Q_{t,r}x^*, x^* \rangle \le \langle Q_{t,s}x^*, x^* \rangle$ for all $x \in \mathbf{E}^*$. Then, the above inequality yields

$$
|Q_{r,s}S(t,r)^*x|^2_{H_{r,s}} \le (1 - K^{-1}) |Q_{t,s}x^*|^2_{H_{t,s}}, \quad \text{for all } x^* \in \mathbf{E}^*.
$$

Using (3.14) again we get

$$\begin{aligned}\left|[S(t,r)Q_{r,s}y^*, Q_{t,s}x^*]_{H_{t,s}}\right| &= \left|[Q_{r,s}y^*, Q_{r,s}S(t,r)^*x^*]_{H_{r,s}}\right| \\ &\le |Q_{r,s}y^*|_{H_{r,s}} \cdot |Q_{r,s}S(t,r)^*x^*|_{H_{r,s}} \\ &\le \sqrt{1-K^{-1}}\, |Q_{r,s}y^*|_{H_{r,s}}\, |Q_{t,s}x^*|_{H_{t,s}}\end{aligned}$$

which shows that $\|S(t,r)\|_{\mathcal{L}(H_{r,s},H_{t,s})} \le \sqrt{1-K^{-1}} < 1$. □

Finally, we establish the smoothing property of the OU transition operators. We need the following condition

Assumption A.2. For all $(t,s) \in \mathfrak{T}$ we have

$$\mathrm{Range}\, S(t,s) \subset H_{t,s} \tag{3.15}$$

If condition (3.15) holds we denote by $\Sigma(t,s)$ the map $S(t,s)$ regarded as an operator from $\mathbf{E}$ into $H_{t,s}$. By the Closed-Graph Theorem such operator is bounded, and we have $S(t,s) = i_{t,s} \circ \Sigma(t,s)$.

Let $\phi_{t,s} : H_{t,s} \to L^2(\mathbf{E}, \mu_{t,s})$ be the unique bounded extension of the isometry

$$Q_{t,s}(\mathbf{E}^*) \ni Q_{t,s}x^* \mapsto \langle x^*, \cdot \rangle \in L^2(\mathbf{E}, \mu_{t,s}).$$

Let $C_b^\infty(\mathbf{E})$ denote the set of infinitely Fréchet-differentiable real-valued functions on $\mathbf{E}$. Using Proposition 2.5 together with the condition (3.15) we obtain the following

Theorem 3.11. *Let Assumptions **(AT)**, **A.1** and **A.2** be satisfied. Then the Ornstein-Uhlenbeck transition operators $\{P(s,t)\}_{(t,s)\in\mathfrak{T}}$ satisfy*

$$\varphi \in B_b(\mathbf{E}) \Rightarrow P(s,t)\varphi \in \mathcal{C}_b^\infty(\mathbf{E}).$$

The Fréchet derivative of the function $P(s,t)\varphi : \mathbf{E} \to \mathbb{R}$ at $x \in \mathbf{E}$ in the direction $y \in \mathbf{E}$ is given by

$$\langle DP(s,t)\varphi(x), y\rangle = \int_{\mathbf{E}} \varphi(S(t,s)x + z)\, \phi_{t,s}(\Sigma(t,s)y)(z)\, \mu_{t,s}(dz), \tag{3.16}$$

and the second Fréchet derivative of $P(s,t)\varphi$ at $x \in \mathbf{E}$ in the directions $y_1, y_2 \in \mathbf{E}$ is given by

$$\begin{aligned}\left\langle D^2 P(s,t)\varphi(x) y_1, y_2 \right\rangle = &-P(s,t)\varphi(x) \left[\Sigma(t,s)y_1, \Sigma(t,s)y_2\right]_{H_{t,s}} \\ &+ \int_{\mathbf{E}} \varphi(S(t,s)x + z)\, \phi_{t,s}(\Sigma(t,s)y_1)(z)\, \phi_{t,s}(\Sigma(t,s)y_2)(z)\, \mu_{t,s}(dz)\end{aligned}$$

In particular, we have the estimates

$$\|D_x P(s,t)\varphi(x)\|_{\mathbf{E}^*} \leq \|\Sigma(t,s)\|_{\mathcal{L}(\mathbf{E},H_{t,s})} |\varphi|_0 \tag{3.17}$$

$$\left\|D_x^2 P(s,t)\varphi(x)\right\|_{\mathcal{L}(\mathbf{E},\mathbf{E}^*)} \leq 2\,\|\Sigma(t,s)\|^2_{\mathcal{L}(\mathbf{E},H_{t,s})} |\varphi|_0 . \tag{3.18}$$

Remark 3.12. The condition (3.15) has a well-known control theoretic interpretation: for each $s \in [0,T]$ consider the nonhomogeneous Cauchy problem

$$\begin{aligned} y'(t) + A(t)y(t) &= G(t)u(t), \quad t \in [s,T], \\ y(s) &= x \in \mathbf{E}, \end{aligned} \tag{3.19}$$

with $u \in L^2(s,T;\mathbf{H})$. The *mild solution* of (3.19) is defined as

$$y^{x,u}(t) := S(t,s)x + \int_s^t S(t,r)G(r)u(r)\,dr, \quad t \in [s,T]. \tag{3.20}$$

We say that (3.19) is *null-controllable* in time t iff for all $x \in \mathbf{E}$ there exists a *control* $u \in L^2(s,t;\mathbf{H})$ such that $y^{x,u}(t) = 0$. Using the following characterization of the Hilbert spaces $H_{t,s}$,

$$H_{t,s} = \left\{ \int_t^s S(t,r)G(r)u(r)\,dr : u \in L^2(s,t;\mathbf{H}) \right\}, \quad (t,s) \in \mathfrak{T} \tag{3.21}$$

(see e.g. [vN01, Lemma 5.2]) it follows that (3.19) is null-controllable in time t if and only if condition (3.15) holds. In fact, we have

$$|x|_{H_{t,s}} = \inf \left\{ |u|_{L^2(s,t;\mathbf{H})} : u \in L^2(s,t;\mathbf{H}) \text{ and } \int_s^t S(t,r)G(r)u(r)\,dr = x \right\}. \tag{3.22}$$

That is, $|x|^2_{H_{t,s}}$ is the minimal energy needed to steer the control system (3.19) from 0 to x in time $t - s$.

Example 3.13. Suppose that for each $t \in [0,T]$ the map $G(t)$ is injective and for each $(t,s) \in \mathfrak{T}$ we have

$$\operatorname{Range} S(t,s) \subset \operatorname{Range} G(t).$$

Suppose also that for each $s \in [0,T]$ we have

$$\int_s^T \|G(t)^{-1}S(t,s)\|^2_{\mathcal{L}(\mathbf{E},\mathbf{H})}\,dt < +\infty.$$

Then Assumption A.2 holds. Indeed, let $x \in \mathbf{E}$ and $0 \le s < t \le T$, and define

$$u(r) := \frac{1}{t-s} G(r)^{-1} S(r,s)x, \quad r \in [s,t].$$

Then $u \in L^2(s,t;\mathbf{H})$ and we have

$$\int_s^t S(t,r)G(r)u(r)\,dr = \frac{1}{t-s}\int_s^t S(t,r)S(r,s)x\,dr = S(t,s)x$$

that is, $S(t,s)x \in H_{t,s}$ according to (3.21), and Assumption A.2 follows. Moreover, by (3.22), we have

$$|\Sigma(t,s)x|_{H_{t,s}} \le \frac{1}{t-s}\,|x|_{\mathbf{E}}\left(\int_s^t \|G(r)^{-1}S(r,s)\|^2_{\mathcal{L}(\mathbf{E},\mathbf{H})}\,dr\right)^{1/2}, \quad s < t \le T.$$

3.3 Hamilton-Jacobi-Bellman equations in Banach spaces

Let M be a separable metric space (the control set) and let $\{-A(t)\}_{t\in[0,T]}$ be the generator of an evolution family on a Banach space $\mathbf{E}$. Let $\mathbf{H}$ be a separable Hilbert space and let $\{G(t)\}_{t\in[0,T]}$ be a family of (possibly unbounded) linear operators from $\mathbf{H}$ into $\mathbf{E}$. We are now concerned with the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) + L_t v(t,\cdot)(x) &= \mathcal{H}(t,x,D_x v(t,x)), \quad (t,x) \in [0,T]\times\mathbf{E}, \\ v(T,x) &= \varphi(x), \end{aligned} \tag{3.23}$$

where for $t \in [0,T]$, L_t is the second-order differential operator

$$(L_t\phi)(x) := -\langle A(t)x, D_x\phi(x)\rangle + \frac{1}{2}\operatorname{Tr}_{\mathbf{H}}[G(t)^* D_x^2\phi(x) G(t)], \;\; x \in D(A(t)), \;\; \phi \in \mathcal{C}_b^2(\mathbf{E}).$$

and the function $\mathcal{H} : [0,T]\times\mathbf{E}\times\mathbf{E}^* \to \mathbb{R}$, called the *Hamiltonian*, is defined by

$$\mathcal{H}(t,x,p) = \sup_{u\in M}\left\{-\langle F(t,x,u),p\rangle - h(t,x,u)\right\}, \quad t \in [0,T],\; x \in \mathbf{E},\;\; p \in \mathbf{E}^*.$$

The Hamiltonian $\mathcal{H}$ is associated with the control problem of minimizing a cost functional of the form

$$J(X,u) = \mathbb{E}\left[\int_0^T h(t,X(t),u(t))\,dt + \varphi(X(T))\right]$$

where $h : [0,T]\times\mathbf{E}\times M \to [0,+\infty]$ is the *running cost* function, $\varphi : \mathbf{E}\to\mathbb{R}$ is the *final cost*, $u(\cdot)$ is a $M-$valued control process and $X(\cdot)$ is solution to the controlled non-autonomous stochastic evolution equation on $\mathbf{E}$

$$\begin{aligned} dX(t) + A(t)X(t)\,dt &= F(t,X(t),u(t))\,dt + G(t)\,dW(t) \\ X(0) &= x_0 \in \mathbf{E}. \end{aligned} \tag{3.24}$$

The approach we propose to the above optimal control problem is to use the associated OU-transition evolution operators $\{P(s,t)\}_{(t,s)\in\mathfrak{T}}$ to rewrite the HJB (3.23) in the integral form

$$v(t,x) = [P(t,T)\varphi](x) - \int_t^T [P(t,s)\mathcal{H}(s,\cdot,D_x v(s,\cdot))]\,(x)\,ds, \quad t\in[0,T],\ \ x\in\mathbf{E}. \tag{3.25}$$

Observe that the trace term in (3.23) may not be well-defined since $G(t)$ is not necessarily a bounded operator.

Definition 3.14. For $\alpha\in(0,1)$, we denote with $\mathfrak{S}_{T,\alpha}$ the set of bounded and measurable functions $v : [0,T]\times\mathbf{E}\to\mathbb{R}$ such that $v(t,\cdot)\in\mathcal{C}^1_b(\mathbf{E})$, for all $t\in[0,T)$, and the mapping

$$[0,T)\times\mathbf{E}\ni(t,x)\mapsto(T-t)^\alpha D_x v(t,x)\in\mathbf{E}^*$$

is bounded and measurable.

The space $\mathfrak{S}_{T,\alpha}$ is a Banach space endowed with the norm

$$\|v\|_{\mathfrak{S}_{T,\alpha}} := \sup_{t\in[0,T]}\|v(t,\cdot)\|_0 + \sup_{t\in[0,T]}(T-t)^\alpha\,\|D_x v(t,\cdot)\|_0\,.$$

Definition 3.15. We will say that a function $v : [0,T]\times\mathbf{E}\to\mathbb{R}$ is a *mild* solution of the HJB equation (3.23) if $v\in\mathfrak{S}_{T,\alpha}$ for some $\alpha\in(0,1)$, for each $(t,x)\in[0,T]\times\mathbf{E}$ the mapping

$$[t,T]\ni s\mapsto[P(t,s)\mathcal{H}(s,\cdot,D_x v(s,\cdot))](x)\in\mathbb{R}$$

is integrable and v satisfies (3.25).

Assumption A.3. There exists $\alpha \in (0,1)$ and $C > 0$ such that

$$\|\Sigma(t,s)\|_{\mathcal{L}(\mathbf{E},H_{t,s})} \leq C(t-s)^{-\alpha}, \quad 0 \leq s < t \leq T.$$

Example 3.16. Under the same assumptions of Example 3.13, assume further that there exists $\beta \in [0, \frac{1}{2})$ and $C > 0$ such that

$$\|G(t)^{-1}S(t,s)\|_{\mathcal{L}(\mathbf{E},\mathbf{H})} \leq C(t-s)^{-\beta}, \quad 0 \leq s < t \leq T.$$

Then Assumption A.3 holds with $\alpha = \beta + \frac{1}{2}$.

Assumption A.4. For all $(t,p) \in [0,T] \times \mathbf{E}^*$, the map

$$\mathbf{E} \ni x \mapsto \mathcal{H}(t,x,p) \in \mathbb{R}$$

is continuous and bounded, and there exists $C > 0$ such that

$$|\mathcal{H}(t,x,p) - \mathcal{H}(t,x,q)| \leq C\,|p-q|_{\mathbf{E}^*}\,, \quad t \in [0,T], \;\; x \in \mathbf{E}, \;\; p,q \in \mathbf{E}^*.$$

Assumption **A.4** holds, for instance, if F and h are uniformly bounded, F is locally uniformly continuous in $x \in \mathbf{E}$, uniformly with respect to $u \in M$ and h is uniformly continuous in $x \in \mathbf{E}$ uniformly with respect to $u \in M$ (see e.g. the proof of Theorem 10.1 in [FS06, Chapter II] or [BCD97, Chapter III, Lemma 2.11]).

The following result is a direct generalization of Theorem 9.3 in [Zab99] (see also [Mas05] and [DPZ02]) to the non-autonomous case.

Theorem 3.17. *Suppose Assumptions **(AT)** and **A.1-A.4** hold true and $\varphi \in \mathcal{C}_b(\mathbf{E})$. Then there exists a unique mild solution to equation (3.23).*

Proof. For any $v \in \mathfrak{S}_{T,\alpha}$ we define the function $\gamma(v)$ by

$$\gamma(v)(t,x) := [P(t,T)\varphi](x) - \int_t^T [P(t,s)\mathcal{H}(s,\cdot,D_x v(s,\cdot))]\,(x)\,ds, \quad (t,x) \in [0,T] \times \mathbf{E}.$$

By Theorem 3.11, estimate (3.17) and Assumptions A.3-A.4, it follows that $\gamma(v)$ belongs to $\mathfrak{S}_{T,\alpha}$. We will show that γ is a strict contraction on $\mathfrak{S}_{T,\alpha}$ when endowed with the equivalent norm

$$\|v\|_{\beta,\mathfrak{S}_{T,\alpha}} := \sup_{t\in[0,T]} \exp(-\beta(T-t))\,[\|v(t,\cdot)\|_0 + (T-t)^\alpha\,\|D_x v(t,\cdot)\|_0]$$

with $\beta > 0$ to be specified below. Let $v_1, v_2 \in \mathfrak{S}_{T,\alpha}$. Using again Assumptions A.3-A.4 and estimate (3.17), we obtain

$$\begin{aligned}
|\gamma(v_1)(t,x) - \gamma(v_2)(t,x)| &\leq \int_t^T \left| \left[P(t,s) \left(\mathcal{H}(s,\cdot,D_x v_1(s,\cdot)) - \mathcal{H}(s,\cdot,D_x v_2(s,\cdot)) \right) \right](x) \right| ds \\
&\leq C \int_t^T |D_x v_1(s,x) - D_x v_2(s,x)|_{\mathbf{E}^*} \, ds \\
&\leq C \int_t^T (T-s)^{-\alpha} \exp(\beta(T-s)) \, \|v_1 - v_2\|_{\beta,\mathfrak{S}_{T,\alpha}} \, ds.
\end{aligned}$$

We estimate the last integral

$$\begin{aligned}
\int_t^T (T-s)^{-\alpha} \exp(\beta(T-s)) \, ds &= (T-t)^{1-\alpha} \int_0^1 r^{-\alpha} \exp(\beta(T-t)r) \, dr \\
&= (T-t)^{1-\alpha} \left[\int_0^\varepsilon r^{-\alpha} \exp(\beta(T-t)r) \, dr + \int_\varepsilon^1 r^{-\alpha} \exp(\beta(T-t)r) \, dr \right] \\
&\leq (T-t)^{1-\alpha} \left[\frac{\varepsilon^{1-\alpha}}{1-\alpha} \exp(\beta(T-t)\varepsilon) + \varepsilon^{-\alpha}(1-\varepsilon) \exp(\beta(T-t)) \right]
\end{aligned}$$

and obtain

$$\begin{aligned}
&\exp(-\beta(T-t)) \, \|\gamma(v_1)(t,\cdot) - \gamma(v_2)(t,\cdot)\|_0 \\
&\qquad \leq C(T-t)^{1-\alpha} \left[\frac{\varepsilon^{1-\alpha}}{1-\alpha} \exp(\beta(T-t)(\varepsilon-1)) + \varepsilon^{-\alpha}(1-\varepsilon) \right] \|v_1 - v_2\|_{\beta,\mathfrak{S}_{T,\alpha}}.
\end{aligned}$$

We choose $\varepsilon_1 \in (0,1)$ such that

$$CT\varepsilon_1^{-\alpha}(1-\varepsilon_1) < \frac{1}{5}$$

and $\beta_1 = \beta_1(\varepsilon_1)$ satisfying

$$\sup_{t\in[0,T]} \frac{C[(T-t)\varepsilon_1]^{1-\alpha}}{1-\alpha} \exp(\beta(T-t)(\varepsilon_1-1)) < \frac{1}{5}.$$

Thus, if $\varepsilon \in (\varepsilon_1, 1)$ and $\beta > \beta_1(\varepsilon)$, we have

$$\sup_{t\in[0,T]} \exp(-\beta(T-t)) \, \|\gamma(v_1)(t,\cdot) - \gamma(v_2)(t,\cdot)\|_0 \leq \frac{2}{5} \|v_1 - v_2\|_{\beta,\mathfrak{S}_{T,\alpha}}.$$

Now, using again Assumption A.3 and estimate (3.17) it follows

$$\begin{aligned}
&|D_x\gamma(v_1)(t,x) - D_x\gamma(v_2)(t,x)|_{\mathbf{E}^*} \\
&\qquad \le \int_t^T |D_xP(t,s)\left[\mathcal{H}(s,\cdot,D_xv_1(s,\cdot)) - \mathcal{H}(s,\cdot,D_xv_2(s,\cdot))\right](x)|_{\mathbf{E}^*}\,ds \\
&\qquad \le C\int_t^T (s-t)^{-\alpha}\,|\mathcal{H}(s,\cdot,D_xv_1(s,\cdot)) - \mathcal{H}(s,\cdot,D_xv_2(s,\cdot))|\,ds \\
&\qquad \le C^2\int_t^T (s-t)^{-\alpha}\,|D_xv_1(s,x) - D_xv_2(s,x)|_{\mathbf{E}^*}\,ds \\
&\qquad \le C^2\int_t^T (s-t)^{-\alpha}(T-s)^{-\alpha}\exp(\beta(T-s))\,\|v_1-v_2\|_{\beta,\mathfrak{S}_{T,\alpha}}\,ds.
\end{aligned}$$

For the last integral we have

$$\begin{aligned}
&\int_t^T (s-t)^{-\alpha}(T-s)^{-\alpha}\exp(\beta(T-s))\,ds \\
&\quad = (T-t)^{1-2\alpha}\left[\int_0^\varepsilon (1-r)^{-\alpha}r^{-\alpha}\exp(\beta(T-t)r)\,dr\right. \\
&\qquad\qquad\qquad \left. + \int_\varepsilon^1 (1-r)^{-\alpha}r^{-\alpha}\exp(\beta(T-t)r)\,dr\right] \\
&\quad \le (T-t)^{1-2\alpha}\left[\frac{(1-\varepsilon)^{-\alpha}\varepsilon^{1-\alpha}}{1-\alpha}\exp(\beta(T-t)\varepsilon) + \frac{(1-\varepsilon)^{1-\alpha}\varepsilon^{-\alpha}}{1-\alpha}\exp(\beta(T-t))\right]
\end{aligned}$$

and it follows

$$\begin{aligned}
&\exp(-\beta(T-t))(T-t)^\alpha\,\|D_x\gamma(v_1)(t,\cdot) - D_x\gamma(v_2)(t,\cdot)\|_0 \\
&\quad \le C^2(T-t)^{1-\alpha}\left[\frac{(1-\varepsilon)^{-\alpha}\varepsilon^{1-\alpha}}{1-\alpha}\exp(\beta(T-t)(\varepsilon-1) + \frac{(1-\varepsilon)^{1-\alpha}\varepsilon^{-\alpha}}{1-\alpha}\right]\|v_1-v_2\|_{\beta,\mathfrak{S}_{T,\alpha}}.
\end{aligned}$$

We choose $\varepsilon_2 \in (0,1)$ such that

$$C^2T^{1-\alpha}\frac{(1-\varepsilon_2)^{1-\alpha}\varepsilon_2^{-\alpha}}{1-\alpha} < \frac{1}{5}$$

and then $\beta_2 = \beta_2(\varepsilon_2) > 0$ such that

$$\sup_{t\in[0,T]} C^2(T-t)^{1-\alpha}\frac{(1-\varepsilon_2)^{-\alpha}\varepsilon_2^{1-\alpha}}{1-\alpha}\exp(\beta_2(T-t)(\varepsilon_2-1) < \frac{1}{5}.$$

Thus, for $\varepsilon \in (\varepsilon_2, 1)$ and $\beta > \beta_2(\varepsilon)$,

$$\sup_{t\in[0,T]} \exp(-\beta(T-t))(T-t)^\alpha \left\| D_x\gamma(v_1)(t,\cdot) - D_x\gamma(v_2)(t,\cdot) \right\|_0 \leq \frac{2}{5} \left\| v_1 - v_2 \right\|_{\beta,\mathfrak{S}_{T,\alpha}}.$$

We conclude that for $\varepsilon \in (\max\{\varepsilon_1, \varepsilon_2\}, 1)$ and $\beta > \max\{\beta_1(\varepsilon), \beta_2(\varepsilon)\}$ we have

$$\left\| \gamma(v_1) - \gamma(v_2) \right\|_{\beta,\mathfrak{S}_{T,\alpha}} \leq \frac{4}{5} \left\| \gamma(v_1) - \gamma(v_2) \right\|_{\beta,\mathfrak{S}_{T,\alpha}}$$

and the desired result follows from the Banach fixed point Theorem. □

Chapter 4

Existence of optimal relaxed controls for stochastic evolution equations in Banach spaces with dissipative nonlinearities

4.1 Relaxed controls and Young measures

We start by recalling the definition of stochastic relaxed control and its connection with random Young measures. Throughout, M denotes a Hausdorff topological space (the control set), $\mathcal{B}(M)$ denotes the Borel $\sigma-$algebra on M and $\mathcal{P}(M)$ denotes the set of probability measures on $\mathcal{B}(M)$ endowed with the $\sigma-$algebra generated by the projection maps

$$\pi_C : \mathcal{P}(M) \ni q \mapsto q(C) \in [0,1], \quad C \in \mathcal{B}(M).$$

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A $\mathcal{P}(M)-$valued process $\{q_t\}_{t\geq 0}$ is called a *stochastic relaxed control* (or relaxed control process) on M if and only if the map

$$[0,T] \times \Omega \ni (t,\omega) \mapsto q_t(\omega,\cdot) \in \mathcal{P}(M)$$

is measurable. In other words, a stochastic relaxed control is a measurable $\mathcal{P}(M)-$valued process.

Definition 4.2. Let l denote the Lebesgue measure on $[0,T]$ and let λ be a bounded nonnegative $\sigma-$additive measure on $\mathcal{B}\left(M \times [0,T]\right)$. We say that λ is a *Young measure* on M if and only if

λ satisfies

$$\lambda(M \times D) = l(D), \quad \text{for all } D \in \mathcal{B}([0,T]). \tag{4.1}$$

We denote by $\mathcal{Y}(0,T;M)$, or simply $\mathcal{Y}$, the set of Young measures on M.

Lemma 4.3 (Disintegration of 'random' Young measures). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let M be a Radon space. Let $\lambda : \Omega \to \mathcal{Y}(0,T;M)$ be such that, for every $J \in \mathcal{B}(M \times [0,T])$, the mapping*

$$\Omega \ni \omega \mapsto \lambda(w)(J) \in [0,T]$$

is measurable. Then there exists a stochastic relaxed control $\{q_t\}_{t\geq 0}$ on M such that for $\mathbb{P}-$a.e. $\omega \in \Omega$ we have

$$\lambda(\omega, C \times D) = \int_D q_t(\omega, C)\, dt, \quad \textit{for all} \quad C \in \mathcal{B}(M),\ D \in \mathcal{B}([0,T]). \tag{4.2}$$

Proof. Define the measure μ on $\mathcal{B}(M) \otimes \mathcal{B}([0,T]) \otimes \mathcal{F}$ by

$$\mu(du, dt, d\omega) := \lambda(\omega)(du, dt)\, \mathbb{P}(d\omega),$$

that is,

$$\mu(C \times D \times E) = \mathbb{E}\left[1_E \lambda(C \times D)\right], \quad C \in \mathcal{B}(M),\ D \in \mathcal{B}([0,T]),\ E \in \mathcal{F}. \tag{4.3}$$

Notice that the marginals of μ on $\mathcal{F} \otimes \mathcal{B}([0,T])$ coincide with the product measure $d\mathbb{P} \otimes dt$. Hence, as M is a Radon space, by the Disintegration Theorem (cf. existence of conditional probabilities, see e.g. [Val73]), there exists a mapping

$$\tilde{q} : [0,T] \times \Omega \times \mathcal{B}(M) \to [0,1]$$

satisfying

$$\mu(C \times J) = \int_J \tilde{q}(t, \omega, C)\, d\mathbb{P} \otimes dt, \quad C \in \mathcal{B}(M),\ J \in \mathcal{F} \otimes B([0,T]), \tag{4.4}$$

and such that for every $C \in \mathcal{B}(M)$, the mapping

$$[0,T] \times \Omega \ni (t, \omega) \mapsto \tilde{q}(t, \omega, C) \in [0,1]$$

is measurable and, for almost every $(t, \omega) \in [0,T] \times \Omega$, $\tilde{q}(t, \omega, \cdot)$ is a Borel probability measure on $\mathcal{B}(M)$. Therefore

$$q : [0,T] \times \Omega \ni (t,\omega) \to \tilde{q}(t,\omega,\cdot) \in \mathcal{P}(M)$$

is a stochastic relaxed control. Moreover, by (4.3), (4.4) and Fubini's Theorem we have

$$\int_E \lambda(\omega)(C \times D)\,\mathbb{P}(d\omega) = \int_E \int_D q(t,\omega)(C)\,dt\,\mathbb{P}(d\omega)$$

for every $E \in \mathcal{F}$ and $C \in \mathcal{B}(M),\ D \in \mathcal{B}([0,T])$, and (4.2) follows. □

Remark 4.4. We will frequently denote the disintegration (4.2) by $\lambda(du, dt) = q_t(du)\,dt$.

4.1.1 Stable topology and tightness criteria

Definition 4.5. The *stable topology* on $\mathcal{Y}(0,T;M)$ is the weakest topology on $\mathcal{Y}(0,T;M)$ for which the mappings

$$\mathcal{Y}(0,T;M) \ni \lambda \mapsto \int_D \int_M f(u)\,\lambda(du,dt) \in \mathbb{R}$$

are continuous, for every $D \in \mathcal{B}([0,T])$ and $f \in \mathcal{C}_b(M)$.

The stable topology was studied under the name of *ws*-topology in [Sch75]. There it was proved that if M is separable and metrisable, then the stable topology coincides with the topology induced by the *narrow topology*. The case of M Polish (i.e. separable and completely metrisable) was studied in [JM81]. A comprehensive overview on the stable topology for a more general class of Young measures under more general topological conditions on M can be found in [CRdFV04].

Remark 4.6. It can be proved (see e.g. Remark 3.20 in [Cra02]) that if M is separable and metrisable, then $\lambda : \Omega \to \mathcal{Y}(0,T;M)$ is measurable with respect to the Borel $\sigma-$algebra generated by the stable topology iff for every $J \in \mathcal{B}(M \times [0,T])$ the mapping

$$\Omega \ni \omega \mapsto \lambda(w)(J) \in [0,T]$$

is measurable. This will justify addressing the maps considered in Lemma 4.2 as random Young measures.

A class of topological spaces that will be particularly useful for our purposes is that of Suslin space.

Definition 4.7. A Hausdorff topological space M is said to be *Suslin* if there exist a Polish space S and a continuous mapping $\varphi : S \to M$ such that $\varphi(S) = M$.

Remark 4.8. If M is Suslin then M is separable and Radon, see e.g. [Sch73, Chapter II]. In particular, Lemma 4.3 applies.

We will be mainly interested in Young measures on metrisable Suslin control sets. This class of Young measures has been studied in [Bal01] and [RdF03].

Proposition 4.9. *Let M be metrisable (resp. metrisable Suslin). Then $\mathcal{Y}(0, T; M)$ endowed with the stable topology is also metrisable (resp. metrisable Suslin).*

Proof. For the metrisability part, see Proposition 2.3.1 in [CRdFV04]. For the Suslin part, see Proposition 2.3.3 in [CRdFV04]. □

The notion of tightness for Young measures that we will use has been introduced by Valadier [Val90] (see also [Cra02]). Recall that a set-valued function $[0, T] \ni t \mapsto K_t \subset M$ is said to be *measurable* if and only if

$$\{t \in [0, T] : K_t \cap U \neq \emptyset\} \in \mathcal{B}([0, T])$$

for every open set $U \subset M$.

Definition 4.10. We say that a set $\mathfrak{J} \subset \mathcal{Y}(0, T; M)$ is *flexibly tight* if, for each $\varepsilon > 0$, there exists a measurable set-valued mapping $[0, T] \ni t \mapsto K_t \subset M$ such that K_t is compact for all $t \in [0, T]$ and

$$\sup_{\lambda \in \mathfrak{J}} \int_0^T \int_M \mathbf{1}_{K_t^c}(u)\, \lambda(du, dt) < \varepsilon.$$

In order to give a characterization of flexible tightness we need the notion of an inf-compact function,

Definition 4.11. A function $\eta : M \to [0, +\infty]$ is called *inf-compact* iff the level sets

$$\{\eta \leq R\} = \{u \in M : \eta(u) \leq R\}$$

are compact for all $R \geq 0$.

Observe that, since M is Hausdorff, for every inf-compact function η the level sets $\{\eta \leq R\}$ are closed. Therefore, every inf-compact function is lower semi-continuous and hence Borel-measurable (see e.g. [Kal02]).

Example 4.12. Let $(V, |\cdot|_V)$ be a reflexive Banach space *compactly* embedded into another Banach space $(M, |\cdot|_M)$, and let $a : \mathbb{R}^+ \to \mathbb{R}^+$ be strictly increasing and continuous. Then the map $\eta : M \to [0, +\infty]$ defined by

$$\eta(u) := \begin{cases} a(|u|_V), & \text{if } u \in V \\ +\infty, & \text{else.} \end{cases}$$

is inf-compact.

Proof of Example 4.12. Since $a(\cdot)$ is increasing, we only need to show that the closed unit ball D in V is compact in M. Let $(u_n)_n$ be a sequence in D. Since the embedding $V \hookrightarrow M$ is compact, there exist a subsequence, which we again denote by $(u_n)_n$, and $u \in M$ such that $u_n \to u$ in M as $n \to \infty$. Hence, if C is a constant such that $|v|_M \leq C\,|v|_V$, $v \in V$, and $\varepsilon > 0$ is fixed we can find $\bar{m} \in \mathbb{N}$ such that

$$|u_n - u|_M < \frac{\varepsilon}{1+C}, \quad \forall n \geq \bar{m}. \tag{4.5}$$

Now, since V is reflexive, by the Banach-Alaoglu Theorem there exists a further subsequence, again denoted by $(u_n)_n$, and $\bar{u} \in V$ such that $u_n \to \bar{u}$ weakly in V as $n \to \infty$. In particular, this implies

$$\bar{u} \in \overline{\{u_{\bar{m}}, u_{\bar{m}+1}, \ldots\}}^w \subset \overline{\mathrm{co}\{u_{\bar{m}}, u_{\bar{m}+1}, \ldots\}}^w$$

where $\mathrm{co}(\cdot)$ and $\overline{\cdot}^w$ denote the convex hull and weak-closure in V respectively. By Mazur Theorem (see e.g. [Meg98, Theorem 2.5.16]), we have

$$\overline{\mathrm{co}\{u_{\bar{m}}, u_{\bar{m}+1}, \ldots\}}^w = \overline{\mathrm{co}\{u_{\bar{m}}, u_{\bar{m}+1}, \ldots\}}.$$

Hence, there exist an integer $\bar{N} \geq 1$ and $\{\alpha_0, \ldots, \alpha_{\bar{N}}\}$ with $\alpha_i \geq 0$, $\sum_{i=0}^{\bar{N}} \alpha_i = 1$, such that

$$\Big|\sum_{i=0}^{\bar{N}} \alpha_i u_{\bar{m}+i} - \bar{u}\Big|_V < \frac{\varepsilon}{1+C}. \tag{4.6}$$

By (4.5) and (4.6) it follows that

$$
\begin{aligned}
|u - \bar{u}|_M &\le \Big|u - \sum_{i=1}^{\bar{N}} \alpha_i u_{\bar{m}+i}\Big|_M + \Big|\sum_{i=0}^{\bar{N}} \alpha_i u_{\bar{m}+i} - \bar{u}\Big|_M \\
&\le \Big|\sum_{i=0}^{\bar{N}} \alpha_i (u - u_{\bar{m}+i})\Big|_M + C\Big|\sum_{i=0}^{\bar{N}} \alpha_i u_{\bar{m}+i} - \bar{u}\Big|_V \\
&\le \sum_{i=0}^{\bar{N}} \alpha_i \left|u - u_{\bar{m}+i}\right|_M + \frac{C\varepsilon}{1+C} \\
&< \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we infer that $u = \bar{u} \in V$. Therefore, D is sequentially compact in M, and the desired result follows. □

Theorem 4.13 (Equivalence Theorem for flexible tightness). *Let $\mathfrak{J} \subset \mathcal{Y}(0, T; M)$. Then the following conditions are equivalent*

(a) *$\mathfrak{J}$ is flexibly tight*

(b) *There exists a measurable function $\eta : [0, T] \times M \to [0, +\infty]$ such that $\eta(t, \cdot)$ is inf-compact for all $t \in [0, T]$ and*

$$
\sup_{\lambda \in \mathfrak{J}} \int_0^T \int_M \eta(t, u)\, \lambda(du, dt) < +\infty.
$$

Proof. See e.g. [Bal00, Definition 3.3] □

Theorem 4.14 (Prohorov criterion for relative compactness). *Let M be a metrisable Suslin space. Then every flexibly tight subset of $\mathcal{Y}(0, T; M)$ is sequentially relatively compact in the stable topology.*

Proof. See [CRdFV04, Theorem 4.3.5] □

Lemma 4.15. *Let M be a metrisable Suslin space and let*

$$
h : [0, T] \times M \to [-\infty, +\infty]
$$

be a measurable function such that $h(t, \cdot)$ is lower semi-continuous for every $t \in [0, T]$ and satisfies one of the two following conditions

1. *$|h(t, u)| \le \gamma(t)$, a.e. $t \in [0, T]$, for some $\gamma \in L^1(0, T; \mathbb{R})$,*

2. $h \geq 0$.

If $\lambda_n \to \lambda$ *stably in* $\mathcal{Y}(0,T;M)$, *then*

$$\int_0^T \int_M h(t,u)\,\lambda(du,dt) \leq \liminf_{n\to\infty} \int_0^T \int_M h(t,u)\,\lambda_n(du,dt).$$

Proof. If (1) holds, the result follows from Theorem 2.1.3–Part G in [CRdFV04]. If (2) holds, the result follows from Proposition 2.1.12–Part (d) in [CRdFV04]. □

The last two results will play an essential role in Section 4.3 in the proof of existence of stochastic optimal relaxed controls. They are, in fact, the main reasons why it suffices for our purposes to require that the control set M is only metrisable and Suslin, in contrast with the existing literature on stochastic relaxed controls. Indeed, Theorem 4.14 will be used to prove tightness of the laws of random Young measures (see Lemma 4.18 below) and Lemma 4.15 will be used to prove the lower semi-continuity of the relaxed cost functionals as well as Theorem 4.16 below which, in turn, will be crucial to pass to the limit in the proof of our main result.

Theorem 4.16. *Let* M *be a metrisable Suslin space. If* $\lambda_n \to \lambda$ *stably in* $\mathcal{Y}(0,T;M)$, *then for every* $f \in L^1(0,T;\mathcal{C}_b(M))$ *we have*

$$\lim_{n\to\infty} \int_0^T \int_M f(t,u)\,\lambda_n(du,dt) = \int_0^T \int_M f(t,u)\,\lambda(dt,du).$$

Proof. Use Lemma 4.15 with f and $-f$. □

We will need the following version of the so-called Fiber Product Lemma. For a measurable map $y : [0,T] \to M$, we denote by $\underline{\delta}_{y(\cdot)}(\cdot)$ the *degenerate Young measure* defined by

$$\underline{\delta}_{y(\cdot)}(du,dt) := \delta_{y(t)}(du)\,dt.$$

Lemma 4.17 (Fiber Product Lemma). *Let* $\mathbb{S}$ *and* M *be separable metric spaces and let*

$$y_n : [0,T] \to \mathbb{S}, \quad n \in \mathbb{N},$$

be a sequence of measurable mappings which converge pointwise to a mapping $y : [0,T] \to \mathbb{S}$. *Let* $\lambda_n \to \lambda$ *stably in* $\mathcal{Y}(0,T;M)$ *and consider the following sequence of Young measures on* $\mathbb{S} \times M$,

$$(\underline{\delta}_{y_n} \otimes \lambda_n)(dx,du,dt) := \delta_{y_n(t)}(dx)\,\lambda_n(du,dt), \;\; n \in \mathbb{N},$$

and

$$(\underline{\delta}_y \otimes \lambda)(dx, du, dt) := \delta_{y(t)}(dx)\, \lambda(du, dt).$$

Then $\underline{\delta}_{y_n} \otimes \lambda_n \to \underline{\delta}_y \otimes \lambda$ *stably in* $\mathcal{Y}(0, T; S \times M)$.

Proof. Proposition 1 in [Val94] implies that $\underline{\delta}_{y_n} \to \underline{\delta}_y$ stably in $\mathcal{Y}(0, T; \mathcal{S})$, and the result follows from Corollary 2.2.2 and Theorem 2.3.1 in [CRdF04]. □

Lemma 4.18. *Assume* M *is metrisable and Suslin. For each* $n \in \mathbb{N}$ *let* λ_n *be a random Young measure on* M *defined on a probability space* $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$. *Assume there exists a measurable function* $\eta : [0, T] \times M \to [0, +\infty]$ *such that* $\eta(t, \cdot)$ *is inf-compact for all* $t \in [0, T]$ *and*

$$\sup_{n \in \mathbb{N}} \mathbb{E}^{\mathbb{P}^n} \int_0^T \int_M \eta(t, u)\, \lambda_n(du, dt) < +\infty.$$

Then, the family of laws of $\{\lambda_n\}_{n \in \mathbb{N}}$ *is tight on* $\mathcal{Y}(0, T; M)$.

Proof. Let $R > 0$ such that $\mathbb{E}^{\mathbb{P}^n} \int_0^T \int_M \eta(t, u)\, \lambda_n(du, dt) \leq R$. For each $\varepsilon > 0$ define the set

$$K_\varepsilon := \left\{ \lambda \in \mathcal{Y} : \int_0^T \int_M \eta(t, u)\, \lambda(du, dt) \leq \frac{R}{\varepsilon} \right\}.$$

By Theorems 4.13 and 4.14, K_ε is relatively compact in the stable topology of $\mathcal{Y}(0, T; M)$, and by Chebyshev's inequality we have

$$\mathbb{P}^n \left(\lambda_n \in \mathcal{Y} \setminus \bar{K}_\varepsilon \right) \leq \mathbb{P}^n \left(\lambda_n \in \mathcal{Y} \setminus K_\varepsilon \right) \leq \frac{\varepsilon}{R}\, \mathbb{E}^{\mathbb{P}^n} \int_0^T \int_M \eta(t, u)\, \lambda_n(du, dt) \leq \varepsilon$$

and the tightness of the laws of $\{\lambda_n\}_{n \geq 1}$ follows. □

4.2 Stochastic convolutions in UMD type-2 Banach spaces

This section builds on the results on the factorization method for stochastic convolutions in UMD type 2 Banach spaces from [Brz97] and [BG99]. First, we recall the definition of the factorization operator as the negative fractional power of a certain abstract parabolic operator as well as some of its regularizing and compactness properties. Then, we review some basic properties of stochastic convolutions in M-type 2 Banach spaces.

In the sequel, $(\mathbf{E}, |\cdot|_{\mathbf{E}})$ will denote a Banach space and $T \in (0, +\infty)$ will be fixed. We start off by introducing the following Sobolev-type spaces,

$$W^{1,p}(0,T;\mathbf{E}) := \Big\{ y \in L^p(0,T;\mathbf{E}) : y' = \frac{dy}{dt} \in L^p(0,T;\mathbf{E}) \Big\}, \quad p > 1$$

where y' denotes the weak derivative, and

$$W_0^{1,p}(0,T;\mathbf{E}) := \{ y \in W^{1,p}(0,T;\mathbf{E}) : y(0) = 0 \}.$$

Observe that $y(0)$ is well defined for $y \in W^{1,p}(0,T;\mathbf{E})$ since by the Sobolev Embedding Theorem we have $W^{1,p}(0,T;\mathbf{E}) \subset \mathcal{C}([0,T];\mathbf{E})$, see e.g. [Tem01, Lemma 3.1.1].

Let A be a closed linear operator on $\mathbf{E}$ and let $D(A)$, the domain of A, be endowed with the graph norm. We define the abstract parabolic operator Λ_T on $L^p(0,T;\mathbf{E})$ through the formula

$$\begin{aligned} D(\Lambda_T) &:= W_0^{1,p}(0,T;\mathbf{E}) \cap L^p(0,T;D(A)), \\ \Lambda_T y &:= y' + A(y(\cdot)). \end{aligned} \tag{4.7}$$

Our aim is to define the factorization operator as the negative fractional powers of Λ_T. This definition relies on the closedness of the operator Λ_T, which will follow from the Dore-Venni Theorem, see [DV87]. This, however, requires further conditions on the Banach space $\mathbf{E}$ and the operator A.

Definition 4.19. A Banach space $\mathbf{E}$ is said to have the property of *unconditional martingale differences* (and we say that $\mathbf{E}$ is a *UMD* space) iff for some $p \in (1, \infty)$ there exists a constant $c \geq 0$ such that

$$\Big\| \sum_{k=0}^{n} \varepsilon_k (y_k - y_{k-1}) \Big\|_{L^p(\Omega,\mathcal{F},\mathbb{P};\mathbf{E})} \leq c \Big\| \sum_{k=0}^{n} (y_k - y_{k-1}) \Big\|_{L^p(\Omega,\mathcal{F},\mathbb{P};\mathbf{E})}$$

for all $n \in \mathbb{N}$, $\varepsilon_k \in \{\pm 1\}$ and all $\mathbf{E}-$valued discrete martingales $\{y_k\}_k$ with $y_{-1} = 0$.

Remark 4.20. A normed vector space $\mathbf{E}$ is said to be $\zeta-$*convex* iff there exists a symmetric, biconvex (i.e. convex in each component) function $\zeta : \mathbf{E}^2 \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and $\zeta(x,y) \leq |x+y|_{\mathbf{E}}$ for any $x, y \in \mathbf{E}$ with $|x|_{\mathbf{E}} = |y|_{\mathbf{E}} = 1$. Burkholder proved in [Bur81] that a Banach space is UMD iff it is $\zeta-$convex. Moreover, a necessary (see [Bur83]) and sufficient (see [Bou83]) condition for a Banach space $\mathbf{E}$ to be UMD is that the Hilbert transform is bounded on $L^p(\mathbb{R};\mathbf{E})$ for some $p \in (1, \infty)$.

Example 4.21. Hilbert spaces and the Lebesgue spaces $L^p(\mathcal{O})$, with $\mathcal{O}$ a bounded domain in $\mathbb{R}^d$ and $p \in (1, +\infty)$, are examples of UMD spaces, see e.g. [Ama95, Theorem 4.5.2].

Proposition 4.22 ([Brz97], Proposition 2.11)**.** *Let* $\mathbf{E}$ *be a UMD and type 2 Banach space. Then* $\mathbf{E}$ *is M-type 2.*

Our main Assumption on the operator A will be the following (see the Appendix for the definition of the class BIP),

Assumption A.1. $A \in \mathrm{BIP}^-(\frac{\pi}{2}, \mathbf{E})$.

In [PS90, Theorem 2] it was proved that if $A \in \mathrm{BIP}^-(\frac{\pi}{2}, \mathbf{E})$ then the operator $-A$ generates a (uniformly bounded) analytic C_0-semigroup $(S_t)_{t\geq 0}$ on $\mathbf{E}$. If furthermore $\mathbf{E}$ is a UMD space, by the Dore-Venni Theorem (see Theorems 2.1 and 3.2 in [DV87]) it follows that the parabolic operator Λ_T is positive on $L^p(0,T;\mathbf{E})$ and, in particular, admits the negative fractional powers $\Lambda_T^{-\alpha}$ for $\alpha \in (0,1]$. We have in fact the following formula

Proposition 4.23 ([Brz97], Theorem 3.1)**.** *Let* $\mathbf{E}$ *be a UMD Banach space and let Assumption A.1 be satisfied. Then, for any* $\alpha \in (0,1]$, $\Lambda_T^{-\alpha}$ *is a bounded linear operator on* $L^p(0,T;\mathbf{E})$*, and for* $\alpha \in (0,1]$ *we have*

$$\left(\Lambda_T^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} S_{t-r} f(r)\, dr, \quad t \in (0,T), \quad f \in L^p(0,T;\mathbf{E}). \tag{4.8}$$

The fractional powers $\Lambda_T^{-\alpha}$ also satisfy the following compactness property which will be crucial to infer tightness of a certain family of laws of processes in the proof of our main Theorem,

Theorem 4.24 ([BG99], Theorem 2.6)**.** *Under the same assumptions of Proposition 4.23, suppose further that* A^{-1} *is a compact operator (i.e. the embedding* $D(A) \hookrightarrow \mathbf{E}$ *is compact). Then, for any* $\alpha \in (0,1]$*, the operator* $\Lambda_T^{-\alpha}$ *is compact on* $L^p(0,T;\mathbf{E})$*.*

The following smoothing property of $\Lambda_T^{-\alpha}$ is a particular case of a more general regularizing result (see Lemma 3.3 in [Brz97]).

Lemma 4.25. *Under the same assumptions of Proposition 4.23, let* α *and* δ *be positive numbers satisfying*

$$\delta + \frac{1}{p} < \alpha \tag{4.9}$$

Then $\Lambda_T^{-\alpha} f \in \mathcal{C}([0,T]; D(A^\delta))$ *for all* $f \in L^p(0,T;\mathbf{E})$ *and* $\Lambda_T^{-\alpha}$ *is a bounded operator from* $L^p(0,T;\mathbf{E})$ *into* $\mathcal{C}([0,T]; D(A^\delta))$*.*

Using Theorem 4.24 and Lemma 4.25 one can prove the following,

Corollary 4.26 ([BG99], Corollary 2.8)**.** *Suppose the assumptions of Theorem 4.24 and Lemma 4.25 are satisfied. Then* $\Lambda_T^{-\alpha}$ *is a compact map from* $L^p(0,T;\mathbf{E})$ *into* $\mathcal{C}([0,T];D(A^\delta))$.

Remark 4.27. Since $T > 0$ is finite, it can be proved that the above results are still valid if $A + \nu I \in \mathrm{BIP}^-(\frac{\pi}{2},\mathbf{E})$ for some $\nu \geq 0$ (see e.g. [BG99] or [Ama95, Theorem 4.10.8]).

Example 4.28. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^d$ with boundary of class $\mathcal{C}^\infty$ and let $\mathcal{A}$ denote the second-order elliptic differential operator defined as

$$(\mathcal{A}x)(\xi) := -\sum_{i,j=1}^d a_{ij}(\xi)\frac{\partial^2 x}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^d b_i(\xi)\frac{\partial x}{\partial\xi_i} + c(\xi)x(\xi), \quad \xi \in \mathcal{O},$$

with $a_{ij} = a_{ji}$,

$$\sum_{i,j=1}^d a_{ij}(\xi)\lambda_i\lambda_j \geq C\,|\lambda|^2\,, \quad \lambda \in \mathbb{R}^d.$$

and $c, b_i, a_{ij} \in \mathcal{C}^\infty(\bar{\mathcal{O}})$, for some $C > 0$. Let $q \geq 2$ and let A_q denote the realization of $\mathcal{A}$ in $L^q(\mathcal{O})$, that is,

$$\begin{aligned} D(A_q) &:= H^{2,q}(\mathcal{O}) \cap H_0^{1,q}(\mathcal{O}), \\ A_q x &:= \mathcal{A}y. \end{aligned} \tag{4.10}$$

Then $A_q + \nu I \in \mathrm{BIP}^-(\frac{\pi}{2}, L^q(\mathcal{O}))$ for some $\nu \geq 0$, see e.g. [See71] (see also [PS93]). Other examples of differential operators satisfying such condition include realizations of higher order elliptic partial differential operators [See71] and the Stokes operator [GS91].

Finally, we recall some aspects of the factorization method for stochastic convolutions in UMD type-2 Banach spaces. Recall that, under Assumption **A.1**, the operator $-A$ generates an analytic C_0-semigroup $(S_t)_{t\geq 0}$ on $\mathbf{E}$.

Lemma 4.29 ([BG99], Lemma 3.7)**.** *Let* $\mathbf{E}$ *be a UMD type-2 Banach space and let Assumption A.1 be satisfied. Let* $p \geq 2$, $\sigma \in [0, \frac{1}{2})$ *and* $g(\cdot)$ *be a* $\mathcal{L}(\mathbf{H},\mathbf{E})-$*valued stochastic process satisfying*

$$A^{-\sigma}g(\cdot) \in \mathcal{M}^p\left(0,T;\gamma(\mathbf{H},\mathbf{E})\right). \tag{4.11}$$

Let $\alpha > 0$ *be such that* $\alpha + \sigma < \frac{1}{2}$. *Then, for every* $t \in [0,T]$ *the stochastic integral*

$$y(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}S_{t-r}g(r)\,dW(r), \tag{4.12}$$

exists and the process $y(\cdot)$ *satisfies*

$$\|y\|_{\mathcal{M}^p(0,T;\mathbf{E})} \leq C\,T^{\frac{1}{2}-\alpha-\sigma}\|A^{-\sigma}g\|_{\mathcal{M}^p(0,T;\gamma(\mathbf{H},\mathbf{E}))} \tag{4.13}$$

for some constant $C = C(\alpha, p, A, \mathbf{E})$, *independent of* $g(\cdot)$ *and* T. *In particular, the process* $y(\cdot)$ *has trajectories in* $L^p(0,T;\mathbf{E})$, $\mathbb{P}-$*a.s.*

Theorem 4.30 ([Brz97], Theorem 3.2)**.** *Under the same assumptions of Lemma 4.29, the stochastic convolution*

$$v(t) = \int_0^t S_{t-r}g(r)\,dW(r), \quad t \in [0,T], \tag{4.14}$$

is well-defined and there exists a modification $\tilde{v}(\cdot)$ *of* $v(\cdot)$ *such that* $\tilde{v}(\cdot) \in D(\Lambda_T^\alpha)$, $\mathbb{P}-$*a.s. and the following 'factorization formula' holds*

$$\tilde{v}(t) = (\Lambda_T^{-\alpha}y)(t), \quad \mathbb{P}-a.s., \quad t \in [0,T], \tag{4.15}$$

where $y(\cdot)$ *is the process defined in (4.12). Moreover,*

$$\mathbb{E}\|\tilde{v}(\cdot)\|^p_{D(\Lambda_T^\alpha)} \leq C^pT^{p(\frac{1}{2}-\alpha-\sigma)}\|A^{-\sigma}g\|_{\mathcal{M}^p(0,T;\gamma(\mathbf{H},\mathbf{E}))}.$$

Corollary 4.31. *Under the same assumptions of Theorem 4.30, let* δ *satisfy*

$$\delta + \sigma + \frac{1}{p} < \frac{1}{2}. \tag{4.16}$$

Then, there exists a stochastic process $\tilde{v}(\cdot)$ *satisfying*

$$\tilde{v}(t) = \int_0^t S_{t-r}g(r)\,dW(r), \quad \mathbb{P}-a.s., \quad t \in [0,T], \tag{4.17}$$

such that $\tilde{v}(\cdot) \in \mathcal{C}([0,T];D(A^\delta))$, $\mathbb{P}-$*a.s. and*

$$\mathbb{E}\|\tilde{v}(\cdot)\|^p_{\mathcal{C}([0,T];D(A^\delta))} \leq C_T\|A^{-\sigma}g\|^p_{\mathcal{M}^p(0,T;\gamma(\mathbf{H},\mathbf{E}))}$$

Proof. Follows from Theorem 4.30 and Lemma 4.25, by taking α such that

$$\delta + \frac{1}{p} < \alpha < \sigma - \frac{1}{2}.$$

□

Example 4.32. Let $\mathbf{H}$ be a separable Hilbert space and let $\mathcal{A}$ be the second order differential operator introduced in Example 4.28. Let A_q be the realization of $\mathcal{A} + \nu I$ on $L^q(\mathcal{O})$, where $\nu \geq 0$ is chosen such that $A_q \in \mathrm{BIP}^-(\frac{\pi}{2}, L^q(\mathcal{O}))$. Fix $q > d$ and σ satisfying

$$\frac{d}{2q} < \sigma < \frac{1}{2}. \tag{4.18}$$

Then, if $g \in \mathcal{M}^p\left(0, T; \mathcal{L}(\mathbf{H}, L^q(\mathcal{O}))\right)$ we have

$$A_q^{-\sigma} g \in \mathcal{M}^p\left(0, T; \gamma(\mathbf{H}, L^q(\mathcal{O}))\right). \tag{4.19}$$

Therefore, if $\alpha < \frac{1}{2} - \sigma$, the statements in Lemma 4.29, Theorem 4.30 and Corollary 4.31 apply.

Proof of (4.19). From (4.18) we have in particular $\sigma > 1/2q$. Hence, from [Tri78, Theorem 1.15.3], we have

$$D(A_q^\sigma) = [L^q(\mathcal{O}), D(A_q)]_\sigma = H_0^{2\sigma,q}(\mathcal{O})$$

isomorphically, and also by (4.18), we have

$$H_0^{2\sigma,q}(\mathcal{O}) \hookrightarrow \mathcal{C}_0(\bar{\mathcal{O}}).$$

Let $c_{\sigma,q} > 0$ be such that $|x|_{\mathcal{C}_0(\bar{\mathcal{O}})} \leq c_{\sigma,q} |x|_{H_0^{2\sigma,q}(\mathcal{O})}$, $x \in H_0^{2\sigma,q}(\mathcal{O})$. Then, for any $y \in \mathbf{H}$ and $t \in [0, T]$ we have

$$\begin{aligned} \left|A^{-\sigma} g(t) y\right|_{L^\infty(\mathcal{O})} &\leq c_{\sigma,q} \left|A^{-\sigma} g(t) y\right|_{D(A_q^\sigma)} \\ &\leq c_{\sigma,q} \left(1 + \|A^{-\sigma}\|_{\mathcal{L}(L^q(\mathcal{O}))}\right) |g(t)y|_{L^q(\mathcal{O})} \\ &\leq c_{\sigma,q} \left(1 + \|A^{-\sigma}\|_{\mathcal{L}(L^q(\mathcal{O}))}\right) \|g(t)\|_{\mathcal{L}(\mathbf{H}, L^q(\mathcal{O}))} |y|_{\mathbf{H}}. \end{aligned}$$

Hence, by Lemma 2.8, there exists $c' > 0$ such that

$$\left\|A_q^{-\sigma} g(t)\right\|_{\gamma(\mathbf{H}, L^q(\mathcal{O}))} \leq c' \|g(t)\|_{\mathcal{L}(\mathbf{H}, L^q(\mathcal{O}))}$$

and (4.19) follows. □

Example 4.33. Let A_q be again as above and let $\mathbf{H} = H^{\theta,2}(\mathcal{O})$ with $\theta > \frac{d}{2} - 1$. By Lemma 6.5 in [Brz97], if σ satisfies

$$\sigma > \frac{d}{4} - \frac{\theta}{2}$$

then $A_q^{-\sigma}$ extends to a bounded linear operator from $H^{\theta,2}(\mathcal{O})$ to $L^q(\mathcal{O})$, which we again denote by $A_q^{-\sigma}$, such that

$$A_q^{-\sigma} \in \gamma\left(H^{\theta,2}(\mathcal{O}), L^q(\mathcal{O})\right).$$

Hence, by the right-ideal property of $\gamma-$radonifying operators, the statements in Lemma 4.29, Theorem 4.30 and Corollary 4.31 hold true with the condition (4.11) now replaced by

$$g \in \mathcal{M}^p\big(0, T; \mathcal{L}(H^{\theta,2}(\mathcal{O}))\big).$$

Remark 4.34. Observe in Example 4.32 that, although we require $q > d$, the choice of σ is independent of the Hilbert space $\mathbf{H}$. In Example 4.33, however, the statement holds true for any value of q but the choice of σ depends on the Hilbert space $H^{\theta,2}(\mathcal{O})$.

The stochastic convolution process $v(\cdot)$ defined by (4.14) is frequently referred to as the *mild solution* to the stochastic Cauchy problem

$$\begin{aligned} dX(t) + AX(t) &= g(t)\,dW(t), \\ X(0) &= 0. \end{aligned} \tag{4.20}$$

The following result shows that, under some additional assumptions, the process $v(\cdot)$ is indeed a strict solution: let $D_A(\frac{1}{2}, 2)$ be the real interpolation space between $D(A)$ and $\mathbf{E}$ with parameters $(\frac{1}{2}, 2)$, that is,

$$D_A(\tfrac{1}{2}, 2) := \Big\{x \in \mathbf{E} : |x|^2_{D_A(\frac{1}{2},2)} = \int_0^1 |AS_t x|^2_{\mathbf{E}}\, dt < +\infty\Big\}.$$

Lemma 4.35. *Let $\mathbf{E}$ be a UMD type-2 Banach space and let Assumption A.1 be satisfied. Let $g \in \mathcal{M}^2\left(0, T; \gamma\left(\mathbf{H}, D_A(\frac{1}{2}, 2)\right)\right)$. Then the stochastic convolution process defined in (4.14) satisfies $v \in \mathcal{M}^2(0, T; D(A))$ and we have*

$$\mathbb{E}\int_0^T |v(s)|^2_{D(A)} ds \le C\mathbb{E}\int_0^T \|g(s)\|^2_{\gamma\left(\mathbf{H}, D_A(\frac{1}{2},2)\right)}\, ds \tag{4.21}$$

Moreover,

$$v(t) + \int_0^t Av(s)ds = \int_0^t g(s)\, dW(s), \qquad \mathbb{P}-a.s. \quad t \in [0, T] \tag{4.22}$$

Proof. In view of Burkholder's Inequality (Proposition 2.20), we have

$$\mathbb{E}|v(t)|^2_{D(A)} \le 4C_2\mathbb{E}\int_0^t \|S_{t-r}g(r)\|^2_{\gamma(\mathbf{H},D(A))}\, dr.$$

Hence, by Fubini Theorem

$$\begin{aligned}
\mathbb{E}\int_0^T |v(t)|^2_{D(A)}\,dt &\le 4C_2\mathbb{E}\int_0^T\int_0^t \|S_{t-r}g(r)\|^2_{\gamma(\mathbf{H},D(A))}\,dr\,dt \\
&= 4C_2\mathbb{E}\int_0^T\int_r^T \|S_{t-r}g(r)\|^2_{\gamma(\mathbf{H},D(A))}\,dt\,dr \\
&\le 4C_2\mathbb{E}\int_0^T \|g(r)\|^2_{\gamma\left(\mathbf{H},D_A(\frac{1}{2},2)\right)}\int_r^T \|S_{t-r}\|^2_{\mathcal{L}\left(D_A(\frac{1}{2},2),D(A)\right)}\,dt\,dr
\end{aligned}$$

By the definition of the norm in $D_A(\frac{1}{2},2)$ it follows that there exists a constant $C' > 0$ independent of t such that

$$|S_t x|_{D(A)} \le C' \, |x|_{D_A(\frac{1}{2},2)}\,, \quad x \in D_A(\tfrac{1}{2},2)$$

and the estimate (4.21) follows. To prove (4.22), observe that by (4.21) we can employ the stochastic Fubini Theorem (see e.g. [vNV06]) to obtain

$$\begin{aligned}
(-A)\int_0^t v(r)\,dr &= \int_0^t\int_0^r (-A)S_{r-s}g(s)\,dW(s)\,dr \\
&= \int_0^t\int_s^t (-A)S_{r-s}g(s)\,dr\,dW(s) \\
&= \int_0^t (S_{t-s}g(s) - g(s))\,dW(s) \\
&= y(t) - \int_0^t g(s)\,dW(s).
\end{aligned}$$

□

Theorem 4.36. *Let* $\mathbf{E}$ *be a UMD type-2 Banach space and let Assumption A.1 be satisfied. Let* $\xi \in L^2\left(\Omega, \mathcal{F}_0, \mathbb{P}; D_A(\frac{1}{2},2)\right)$, $g \in \mathcal{M}^2\left(0,T;\gamma(\mathbf{H}, D_A(\frac{1}{2},2))\right)$ *and* $f \in \mathcal{M}^2(0,T;D(A^\zeta))$ *for some* $\zeta \ge 0$. *Then the following conditions are equivalent,*

(i) $X(t) = S_t\xi + \int_0^t S_{t-r}f(r)\,dr + \int_0^t S_{t-r}g(r)\,dW(r),\ \mathbb{P}-\text{a.s.},\ t\in[0,T].$

(ii) $X(\cdot) \in \mathcal{M}^2(0,T;D(A))$ *and satisfies*

$$X(t) + \int_0^t AX(s)\,ds = \xi + \int_0^t f(s)\,ds + \int_0^t g(s)\,dW(s),\ \mathbb{P}-\text{a.s.},\ t\in[0,T].$$

Proof. Assume first that $X(\cdot)$ satisfies (i) and let $z(\cdot)$ denote the process defined by

$$z(t) := S_t\xi + \int_0^t S_{t-r}f(r)\,dr, \quad t \in [0,T].$$

$z(\cdot)$ is clearly $\mathbb{F}-$progressively measurable, and by Theorem 2.3 in [GS91] it is pathwise solution to the following Cauchy problem

$$\begin{aligned} z'(t) + Az(t) &= f(t), \quad t \in [0,T] \\ z(0) &= \xi \end{aligned} \tag{4.23}$$

with

$$\mathbb{E}\int_0^T |z(t)|^2_{D(A)} \leq C'\left[\mathbb{E}\int_0^T |f(t)|^2_{\mathbf{E}} + \mathbb{E}\,|\xi|^2_{D_A(\frac{1}{2},2)}\right]$$

for some constant $C' = C'(A, \mathbf{E})$ independent of f and T. By integrating (4.23) from 0 to t and using Lemma 4.35, we conclude that $X(\cdot)$ satisfies (ii). To prove the converse implication we consider the process $v(\cdot)$ defined by

$$v(t) := S_t\xi + \int_0^t S_{t-r}f(r)\,dr + \int_0^t S_{t-r}g(r)\,dW(r), \;\; t \in [0,T].$$

In view of Lemma 4.35, $v \in M^2(0,T;D(A))$ and

$$v(t) + \int_0^t Av(s)\,ds = \xi + \int_0^t g(s)\,dW(s) + \int_0^t f(s)\,ds, \quad \text{for a.e. } t \in [0,T].$$

Therefore it suffices to consider the case $f = 0$, $g = 0$ and $\xi = 0$. In other words, we need to show that if $u \in \mathcal{M}^2(0,T;D(A))$ and $u(t) + \int_0^t Au(s)ds = 0$ for a.e. $t \in [0,T]$ then $u(t) = 0$ for a.e. $t \in [0,T]$. Indeed, for $t \in [0,T]$ we have

$$\begin{aligned} \frac{d}{dr}[S_{t-r}u(r)] &= AS_{t-r}u(r) + S_{t-r}\frac{d}{dr}u(r) \\ &= AS_{t-r}u(r) + S_{t-r}(-A)u(r) = 0, \; r \in [0,t]. \end{aligned}$$

Hence $0 = S_tu(0) = S_{t-t}u(t) = u(t)$ for a.e. $t \in [0,T]$, and the desired result follows. □

4.3 Weak formulation of the optimal relaxed control problem and main existence result

Let $(B, |\cdot|_B)$ be a Banach space continuously embedded into $\mathbf{E}$ and let M be a metrisable control set. We are concerned with a control system consisting of a cost functional of the form

$$J(X,u) = \mathbb{E}\left[\int_0^T h(s, X(s), u(s))\,ds + \varphi(X(T))\right]. \tag{4.24}$$

and a controlled semilinear stochastic evolution equation of the form

$$\begin{aligned} dX(t) + AX(t)\,dt &= F(t, X(t), u(t))\,dt + G(t, X(t))\,dW(t), \quad t \in [0,T] \\ X(0) &= x_0 \in B \end{aligned} \tag{4.25}$$

where $W(\cdot)$ is a $\mathbf{H}-$cylindrical Wiener process, $F : [0,T] \times B \times M \to B$ and, for each $(t,x) \in [0,T] \times B$, $G(t,x)$ is a, possibly unbounded, linear mapping from $\mathbf{H}$ into $\mathbf{E}$. More precise conditions on the coefficients F and G and on the functions h and φ are given below.

Our approach is to associate the original control system (4.24)-(4.25) with a relaxed control system by extending the definitions of the nonlinear drift coefficient F and the running cost function as follows: we define the *relaxed coefficient* $\bar{F}$ through the formula

$$\bar{F}(t,x,\rho) := \int_M F(t,x,u)\,\rho(du), \quad \rho \in \mathcal{P}(M), \quad (t,x) \in [0,T] \times B, \tag{4.26}$$

whenever the map $M \ni u \mapsto F(t,x,u) \in B$ is Bochner-integrable with respect to $\rho \in \mathcal{P}(M)$. Similarly, we define the *relaxed running cost* function $\bar{h}$. With these notations, the controlled equation (4.25) in the relaxed control setting becomes, formally,

$$\begin{aligned} dX(t) + AX(t)\,dt &= \bar{F}(t, X(t), q_t)\,dt + G(t, X(t))\,dW(t), \quad t \in [0,T], \\ X(0) &= x_0 \in B, \end{aligned} \tag{4.27}$$

where $\{q_t\}_{t\geq 0}$ is a $\mathcal{P}(M)$-valued relaxed control process, and the associated relaxed cost functional is defined as

$$J(X,q) = \mathbb{E}\left[\int_0^T \bar{h}(s, X(s), q_s)\,ds + \varphi\left(X(T)\right)\right].$$

We will assume that the realization A_B of the operator A in B,

$$\begin{aligned} D(A_B) &:= \{x \in D(A) \cap B : Ax \in B\} \\ A_B &:= A|_{D(A_B)} \end{aligned}$$

is such that $-A_B$ generates a C_0-semigroup on B, also denoted by $(S_t)_{t\geq 0}$. We will only consider *mild* solutions to equation (4.27) i.e. solutions to the integral equation

$$X(t) = S_t x_0 + \int_0^t S_{t-r}\bar{F}(r, X(r), q_r)\, dr + \int_0^t S_{t-r}G(r, X(r))\, dW(r), \ \ t \in [0,T]. \quad (4.28)$$

Our aim is to study the existence of optimal controls for the above stochastic relaxed control system in the following weak formulation,

Definition 4.37. Let $x_0 \in B$ be fixed. A *weak admissible relaxed control* is a system

$$\pi = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, \{W(t)\}_{t\geq 0}, \{q_t\}_{t\geq 0}, \{X(t)\}_{t\geq 0}) \quad (4.29)$$

such that

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space endowed with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$

(ii) $\{W(t)\}_{t\geq 0}$ is a $\mathbf{H}-$cylindrical Wiener process with respect to $\mathbb{F}$

(iii) $\{q_t\}_{t\geq 0}$ is a $\mathbb{F}-$adapted $\mathcal{P}(M)$-valued relaxed control process

(iv) $\{X(t)\}_{t\geq 0}$ is a $\mathbb{F}-$adapted $B-$valued continuous process such that for all $t \in [0,T]$,

$$X(t) = S_t x_0 + \int_0^t S_{t-r}\bar{F}(r, X(r), q_r)\, dr + \int_0^t S_{t-r}G(r, X(r))\, dW(r), \ \mathbb{P}-\text{a.s.} \quad (4.30)$$

(v) The mapping

$$[0,T] \times \Omega \ni (t, \omega) \mapsto \bar{h}(t, X(t,\omega), q_t(\omega)) \in \mathbb{R}$$

belongs to $L^1([0,T] \times \Omega; \mathbb{R})$ and $\varphi(X(T)) \in L^1(\Omega; \mathbb{R})$.

The set of weak admissible relaxed controls will be denoted by $\bar{\mathcal{U}}^{\text{w}}_{\text{ad}}(x_0)$. Under this weak formulation, the relaxed cost functional is defined as

$$\bar{J}(\pi) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T \bar{h}(s, X^\pi(s), q^\pi_s)\, ds + \varphi\left(X^\pi(T)\right)\right], \quad \pi \in \bar{\mathcal{U}}^{\text{w}}_{\text{ad}}(x_0),$$

where $X^\pi(\cdot)$ is state-process corresponding to the weak admissible relaxed control π. The relaxed control problem **(RCP)** is to minimize $\bar{J}$ over $\bar{\mathfrak{U}}^{\rm w}_{\rm ad}(x_0)$ for $x_0 \in B$ fixed. Namely, we seek $\widetilde{\pi} \in \bar{\mathfrak{U}}^{\rm w}_{\rm ad}(x_0)$ such that

$$\bar{J}(\widetilde{\pi}) = \inf_{\pi \in \bar{\mathfrak{U}}^{\rm w}_{\rm ad}(x_0)} \bar{J}(\pi).$$

The following will be the main assumption on the Banach space B and the coefficient G.

Assumption A.2. There exist positive constants σ and δ such that $\sigma + \delta < \frac{1}{2}$ and

1. $D(A^\delta) \hookrightarrow B$
2. For all $(t,x) \in [0,T] \times B$, the linear map $A^{-\sigma}G(t,x)$ extends to a $\gamma-$radonifying operator from $\mathbf{H}$ into $\mathbf{E}$, also denoted by $A^{-\sigma}G(t,x)$, such that the map

$$[0,T] \times B \ni (t,x) \mapsto A^{-\sigma}G(t,x) \in \gamma(\mathbf{H},\mathbf{E})$$

is bounded, continuous with respect to $x \in B$ and measurable with respect to $t \in [0,T]$.

In order to formulate the main hypothesis on the drift coefficient $-A+F$ we need the notion of sub-differential of the norm. Recall that for $x, y \in B$ fixed, the map

$$\phi : \mathbb{R} \ni s \mapsto |x+sy|_B \in \mathbb{R}$$

is convex and therefore is right and left differentiable. Let $D_\pm|x|y$ denote the right/left derivative of ϕ at 0. Let B^* denote the dual of B and let $\langle \cdot,\cdot\rangle$ denote the duality pairing between B and B^*.

Definition 4.38. Let $x \in B$. The *sub-differential* $\partial|x|$ of $|x|$ is defined as

$$\partial|x|_B := \{x^* \in B^* : D_-|x|y \le \langle y, x^*\rangle \le D_+|x|y, \forall y \in B\}.$$

It can be proved, see e.g. [DPZ92b], that $\partial|x|$ is a nonempty, closed and convex set, and

$$\partial|x|_B = \{x^* \in B^* : \langle x, x^*\rangle = |x|_B \text{ and } |x^*|_{B^*} \le 1\}.$$

In particular, $\partial|0|_B$ is the unit ball in B^*. The following are the standing assumptions on the drift coefficient, the control set and the running and final cost functions,

Assumption A.3.

1. The control set M is a metrisable **Suslin** space.

2. The mapping $F : [0,T]\times B\times M \to B$ is continuous in every variable separately, uniformly with respect to $u \in M$.

3. There exist $k_1 \in \mathbb{R}, k_2 > 0,\ m \geq 1$ and a measurable function

$$\eta : [0,T] \times M \to [0,+\infty]$$

such that for each $t \in [0,T]$, the mapping $\eta(t,\cdot) : M \to [0,+\infty]$ is inf-compact and, for each $x \in D(A)$, $y \in B$ and $u \in M$ we have

$$\langle -A_B x + F(t, x+y, u), z^* \rangle \leq -k_1 |x|_B + k_2 |y|_B^m + \eta(t,u), \ \text{ for all } z^* \in \partial |x|_B. \tag{4.31}$$

4. The **running cost** function $h : [0,T] \times B \times M \to (-\infty, +\infty]$ is measurable in $t \in [0,T]$ and lower semi-continuous with respect to $(x,u) \in B \times M$.

5. There exist constants $C_1 \in \mathbb{R},\ C_2 > 0$ and

$$\gamma > \frac{2m}{1 - 2(\delta + \sigma)} \tag{4.32}$$

such that h satisfies the **coercivity** condition,

$$C_1 + C_2 \eta(t,u)^\gamma \leq h(t,x,u), \quad (t,x,u) \in [0,T] \times B \times M.$$

6. The **final cost** function $\varphi : B \to \mathbb{R}$ is uniformly continuous.

Notice that if F satisfies the dissipative-type condition (4.31), since $\partial|0|_B$ coincides with the unit ball in B^*, by the Hahn-Banach Theorem the following estimate holds

$$|F(t,y,u)|_B \leq k_2 |y|_B^m + \eta(t,u), \quad t \in [0,T], \ y \in B, \ u \in M. \tag{4.33}$$

We now state our main result on existence of weak optimal relaxed controls under the above assumptions.

Theorem 4.39. *Let* $\mathbf{E}$ *be a separable UMD type-2 Banach space and let* $A + \nu I$ *satisfy Assumption A.1 for some* $\nu \geq 0$ *(see page 55). Suppose that* $(A+\nu I)^{-1}$ *is compact and that Assumptions*

A.2 and A.3 also hold. Let $x_0 \in B$ be such that

$$\inf_{\pi \in \mathcal{U}^{\mathrm{w}}_{\mathrm{ad}}(x_0)} \bar{J}(\pi) < +\infty.$$

Then **(RCP)** *admits a weak optimal relaxed control.*

For the proof of Theorem 4.39 we need the following important consequence of (4.31) in order to obtain a-priori estimates for weak admissible relaxed controls.

Lemma 4.40. *Suppose that $F : [0,T] \times B \times M \to B$ satisfies Assumption A.3–(2) and that there exists a UMD Banach space Y continuously embedded in B such that the part A_Y of the operator A_B in Y satisfies $A_Y \in \mathrm{BIP}^-(\frac{\pi}{2}, Y)$. Suppose that a function $z \in \mathcal{C}([0,T]; B)$ satisfies*

$$z(t) = \int_0^t S_{t-r} \bar{F}(r, z(r) + v(r), q_r)\, dr, \quad t \in [0,T].$$

for some $\mathcal{P}(M)-$valued relaxed control $\{q_t\}_{t \geq 0}$ with

$$\int_0^T \int_M \eta(t,u)^\gamma \, q_t(du)\, dt < \infty$$

and some $v \in L^\gamma(0,T;B)$ with $\gamma > 1$. Then, z satisfies the following estimate

$$|z(t)|_B \leq \int_0^t e^{-k_1(t-s)} \Big[k_2 |v(s)|_B^m + \int_M \eta(s,u)\, q_s(du) \Big] ds, \quad t \in [0,T]. \tag{4.34}$$

Proof. For $\lambda > 0$ large enough define $R_\lambda := \lambda(\lambda I + A_B)^{-1} \in \mathcal{L}(B)$, $z_\lambda(t) := R_\lambda z(t)$ and

$$\bar{f}_\lambda(t) := R_\lambda \bar{F}(t, z(t) + v(t), q_t), \quad t \in [0,T].$$

Then z_λ satisfies

$$z_\lambda(t) = \int_0^t S_{t-r} \bar{f}_\lambda(r)\, dr, \quad t \in [0,T].$$

Since $\|R_\lambda\|_{\mathcal{L}(B,Y)} \leq M$ for $\lambda > 0$ large, we have $\bar{f}_\lambda \in L^\gamma(0,T;Y)$. Hence, by the Dore-Venni Theorem (see Theorem 3.2 in [DV87]), $z_\lambda \in W^{1,\gamma}(0,T;Y) \cap L^\gamma(0,T;D(A_Y))$ and satisfies in the $Y-$sense,

$$\frac{dz_\lambda}{dt}(t) + A_Y z_\lambda(t) = \bar{F}(t, z_\lambda(t) + v(t), q_t) + \zeta_\lambda(t), \quad \text{for a.e. } t \in [0,T]$$

with

$$\zeta_\lambda(t) := \bar{f}_\lambda(t) - \bar{F}(t, z_\lambda(t) + v(t), q_t), \quad t \in [0,T].$$

Since Y is continuously embedded in B, the map $z_\lambda : [0,T] \to B$ is also a.e. differentiable and by Assumption A.3–(2) satisfies

$$\frac{d^-}{dt} |z_\lambda(t)|_B \leq -k_1 |z_\lambda(t)|_B + k_2 |v(t)|_B^m + \zeta_\lambda(t) + \int_M \eta(t,u)\, q_t(du), \quad t \in [0,T].$$

Using Gronwall's Lemma it follows that

$$|z_\lambda(t)|_B \leq \int_0^t e^{-k_1(t-s)} \Big[k_2 |v(s)|_B^m + \zeta_\lambda(s) + \int_M \eta(s,u)\, q_s(du) \Big] ds, \quad t \in [0,T]$$

and the result follows since $z_\lambda(t) \to z(t)$ for $t \in [0,T]$ and $\zeta_\lambda \to 0$ in $L^1(0,T;B)$. □

Remark 4.41. If $A \in \mathrm{BIP}^-(\frac{\pi}{2}, \mathbf{E})$ then $D(A^\delta) \simeq [\mathbf{E}, D(A)]_\delta$ is a UMD space (see e.g. [Ama95, Theorem 4.5.2]). Since the resolvent of A commutes with A^δ, it follows that the realization of A in $D(A^\delta)$ belongs to $\mathrm{BIP}^-(\frac{\pi}{2}, D(A^\delta))$. Hence, if Assumptions A.2-A.3 are also satisfied then Lemma 4.40 applies with $Y = D(A^\delta)$.

Lemma 4.42. *Suppose that Assumption A.3 is satisfied and the Banach space B is separable. Define*

$$\mathcal{Y}^\gamma(0,T;M) := \left\{ \lambda \in \mathcal{Y}(0,T;M) : \int_0^T \int_M \eta(t,u)^\gamma\, \lambda(du,dt) < +\infty \right\}.$$

Then, for each $t \in [0,T]$, the mapping $\Gamma_t : \mathcal{C}([0,T];B) \times \mathcal{Y}^\gamma(0,T;M) \to B$ defined by

$$\Gamma_t(y,\lambda) := \int_0^t \int_M S_{t-r} F(r, y(r), u)\, \lambda(du,dr) \tag{4.35}$$

is Borel-measurable.

Proof. We fix $t \in [0,T]$ and prove first that the mapping $\Gamma_t(y,\cdot)$ is weakly-measurable for $y \in \mathcal{C}([0,T];B)$ fixed. Observe that for $x^* \in B^*$ fixed and $\lambda \in \mathcal{Y}^\gamma(0,T;M)$ we have

$$\begin{aligned}\langle \Gamma_t(y,\lambda), x^* \rangle &= \left\langle \int_0^t \int_M S_{t-r} F(r, y(r), u)\, \lambda(du,dr), x^* \right\rangle \\ &= \int_0^t \int_M \big\langle S_{t-r} F(r, y(r), u), x^* \big\rangle\, \lambda(du,dr).\end{aligned}$$

For each $N \in \mathbb{N}$ define

$$\phi_N(\lambda) := \int_0^t \int_M \min\{N, \langle S_{t-r}F(r,y(r),u), x^*\rangle\} \, \lambda(du,dr), \quad \lambda \in \mathcal{Y}^\gamma(0,T;M).$$

By Assumption A.3–(2), the integrand in the above expression is bounded and continuous with respect to $u \in M$. Therefore, by Lemma 4.16 ϕ_N is continuous for each $N \in \mathbb{N}$, and by the monotone convergence Theorem, $\phi_N(\lambda) \to \langle \Gamma_t(y,\lambda), x^*\rangle$ as $N \to \infty$ for all $\lambda \in \mathcal{Y}^\gamma(0,T;M)$. Hence, $\langle \Gamma_t(y,\cdot), x^*\rangle$ is measurable, i.e. $\Gamma_t(y,\cdot)$ is weakly-measurable. Since B is separable, by the Pettis measurability Theorem (see [Sho97, Theorem 3.1.1]), $\Gamma_t(y,\cdot)$ is also measurable.

Now, we prove that for $\lambda \in \mathcal{Y}^\gamma(0,T;M)$ fixed, the map $\Gamma_t(\cdot,\lambda)$ is continuous. Let $y_n \to y$ in $\mathcal{C}([0,T];B)$. Then, by Assumption A.3–(2) we have

$$\begin{aligned} &\left|S_{t-r}F(r,y(r),u) - S_{t-r}F(r,y_n(r),u)\right|_B \\ &\qquad \le \|S_{t-r}\|_{\mathcal{L}(B)} \left|F(r,y(r),u) - F(r,y_n(r),u)\right|_B \to 0 \end{aligned}$$

as $n \to \infty$ for all $(r,u) \in [0,t] \times M$. Moreover, for $\rho > 0$ fixed there exists $\bar{n} \in \mathbb{N}$ such that

$$\sup_{r\in[0,T]} \left|y_n(r) - y(r)\right|_B < \rho \quad \forall n \ge \bar{N}.$$

Set $\rho' := \rho \vee \max_{1\le n\le \bar{N}-1} \sup_{r\in[0,T]} |y(r) - y_n(r)|_B$. Then

$$\sup_{r\in[0,T]} \left|y_n(r) - y(r)\right|_B < \rho', \quad \forall n \in \mathbb{N}$$

and by (4.33), we have

$$\begin{aligned} &\left|S_{t-r}F(r,y(r),u) - S_{t-r}F(r,y_n(r),u)\right|_B \\ &\qquad \le \|S_{t-r}\|_{\mathcal{L}(B)} \cdot \left[k_2 \left(2^{m-1}\rho'^m + (2^{m-1}+1)\,\|y(\cdot)\|^m_{\mathcal{C}([0,T];B)}\right) + \eta(r,u)\right]. \end{aligned}$$

As η belongs to $L^1([0,T] \times M; \lambda)$, so does the RHS of the above inequality. Therefore, by Lebesgue's dominated convergence Theorem we have

$$\begin{aligned} &|\Gamma_t(y,\lambda) - \Gamma_t(y_n,\lambda)|_B \\ &\qquad \le \int_0^t \int_M |S_{t-r}F(r,y(r),u) - S_{t-r}F(r,y_n(r),u)|_B \, \lambda(du,dr) \to 0 \end{aligned}$$

as $n \to \infty$, that is, $\Gamma_t(\cdot,\lambda)$ is continuous. Since $\mathcal{Y}^\gamma(0,T;M)$ is separable and metrisable, by

Lemma 1.2.3 in [CRdFV04] it follows that Γ_t is jointly measurable. □

Lemma 4.43. *Let $-A$ be the generator of a C_0-semigroup $(S_t)_{t\geq 0}$ on a Banach space B such that $0 \in \rho(A)$ and let $f \in L^1(0,T;B)$. Then the function $z \in \mathcal{C}([0,T];B)$ defined by*

$$z(t) := \int_0^t S_{t-r} f(r)\, dr, \quad t \in [0,T],$$

satisfies

$$A^{-1}z(t) + \int_0^t z(s)\, ds = \int_0^t A^{-1} f(s), ds, \quad t \in [0,T].$$

Proof. From the identity

$$-A \int_0^t S_r x\, dr = S_t x - x$$

we have

$$\int_0^t S_r x\, dr + A^{-1} S_t x = A^{-1} x$$

and it follows that

$$\begin{aligned}
\int_0^t z(r)\, dr &= \int_0^t \int_0^r S_{r-s} f(s)\, ds\, dr \\
&= \int_0^t \int_s^t S_{r-s} f(s)\, dr\, ds \\
&= \int_0^t \int_0^{t-s} S_u f(s)\, du\, ds \\
&= \int_0^t A^{-1} f(s)\, ds - \int_0^t A^{-1} S_{t-s} f(s)\, ds \\
&= \int_0^t A^{-1} f(s)\, ds - A^{-1} z(t).
\end{aligned}$$

□

Proof of Theorem 4.39. Since we can write $-A + F = -(A + \nu I) + F + \nu I$, by Remark 4.27 we can assume without loss of generality that $\nu = 0$. Let

$$\pi_n = \left(\Omega^n, \mathcal{F}^n, \mathbb{P}^n, \mathbb{F}_n, \{W_n(t)\}_{t\geq 0}, \{q_t^n\}_{t\geq 0}, \{X_n(t)\}_{t\geq 0}\right), \ \ n \in \mathbb{N},$$

be a minimizing sequence of weak admissible relaxed controls, that is,

$$\lim_{n\to\infty} \bar{J}(\pi_n) = \inf_{\pi\in\mathcal{U}^{\mathrm{w}}_{\mathrm{ad}}(x)} \bar{J}(\pi).$$

From this and Assumption A.3-(5) it follows that there exists $R > 0$ such that

$$\mathbb{E}^n \int_0^T \int_M \eta(t,u)^\gamma \, q^n_t(du)\, dt \le -\frac{C_1}{C_2} + \frac{1}{C_2}\mathbb{E}^n \int_0^T \int_M h(t, X_n(t), u)\, q^n_t(du)\, dt \le R \tag{4.36}$$

for all $n \in \mathbb{N}$.

STEP 1. Set $p = \frac{\gamma}{m}$. Then, by (4.32) $p > 2$ and we can find α such that

$$\delta + \frac{1}{p} < \alpha < \frac{1}{2} - \sigma. \tag{4.37}$$

For each $n \in \mathbb{N}$ define the process

$$y_n(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} S_{t-r} G(r, X_n(r))\, dW_n(r), \quad t \in [0,T]. \tag{4.38}$$

Since, by Assumption A.2 the mapping $A^{-\sigma}G : [0,\infty) \times B \times M \to \gamma(\mathbf{H}, \mathbf{E})$ is bounded, from Lemma 4.29 we have

$$\sup_{n\ge 1} \mathbb{E}^n \, |y_n(\cdot)|^p_{L^p(0,T;\mathbf{E})} < +\infty. \tag{4.39}$$

Then, by Chebyshev's inequality, the processes $\{y_n(\cdot)\}_{n\in\mathbb{N}}$ are uniformly bounded in probability on $L^p(0,T;\mathbf{E})$. Since A^{-1} is compact, it follows from Assumption A.2–(1), (4.37) and Corollary 4.26 that $\Lambda_T^{-\alpha}$ is a compact operator from $L^p(0,T;\mathbf{E})$ into $\mathcal{C}([0,T];B)$. Hence, the family of laws of the processes

$$v_n := \Lambda_T^{-\alpha} y_n, \quad n \in \mathbb{N},$$

is tight on $\mathcal{C}([0,T];B)$. Now, for each $n \in \mathbb{N}$ set

$$f_n(t) := \bar{F}(t, X_n(t), q^n_t), \quad t \in [0,T], \ \ n \in \mathbb{N},$$

and $z_n := \Lambda_T^{-1} f_n$, i.e.

$$z_n(t) = \int_0^t S_{t-r} \bar{F}(r, X_n(r), q^n_r)\, dr, \quad t \in [0,T]. \tag{4.40}$$

Then, by Theorem 4.30, we have

$$X_n(t) = S_t x_0 + z_n(t) + v_n(t), \ \ \mathbb{P}^n - \text{a.s. } \ t \in [0,T]. \tag{4.41}$$

Applying Lemma 4.40 to the process $z_n(\cdot)$ and (4.41) we obtain the estimate

$$|z_n(t)|_B \leq \int_0^t e^{-k_1(t-s)} \Big[k_2 |S_t x_0 + v_n(s)|_B^m + \int_M \eta(s,u)\, q_s^n(du) \Big] ds, \ \ t \in [0,T]. \tag{4.42}$$

Moreover, by (4.39) and Lemma 4.25, we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \left[\sup_{t\in[0,T]} |v_n(t)|_B^\zeta \right] \leq \sup_{n\in\mathbb{N}} \mathbb{E}^n \left[\sup_{t\in[0,T]} |v_n(t)|_{D(A^\delta)}^\zeta \right] < +\infty, \ \ \forall \zeta \geq p. \tag{4.43}$$

Using (4.36), (4.42) and (4.43) with $\zeta = m^2 p$ we get

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \left[\sup_{t\in[0,T]} |z_n(t)|_B^{mp} \right] < +\infty.$$

This, in conjunction with (4.41) and (4.43) with $\zeta = mp$, yields

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \left[\sup_{t\in[0,T]} |X_n(t)|_B^{mp} \right] < +\infty. \tag{4.44}$$

Thus, by (4.33), (4.36) and (4.44), the processes $\{f_n(\cdot)\}_{n\in\mathbb{N}}$ satisfy

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n \, |f_n(\cdot)|_{L^p(0,T;\mathbf{E})}^p < +\infty.$$

This implies, again by Chebyshev's inequality, that the sequence of processes $\{f_n(\cdot)\}_{n\in\mathbb{N}}$ is uniformly bounded in probability on $L^p(0,T;\mathbf{E})$. By compactness of the operator Λ_T^{-1} and Corollary 4.26, it follows that the family of the laws of $z_n = \Lambda_T^{-1} f_n$, $n \in \mathbb{N}$, is tight on $\mathcal{C}([0,T];B)$. By (4.41) we conclude that the family of laws of the processes $X_n, \ n \in \mathbb{N}$, is also tight on $\mathcal{C}([0,T];B)$.

Now, for each $n \in \mathbb{N}$ we define a random Young measure λ_n on $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ by the following formula

$$\lambda_n(du, dt) := q_t^n(du)\, dt. \tag{4.45}$$

By (4.36) and Lemma 4.18 the family of laws of $\{\lambda_n\}_{n\in\mathbb{N}}$ is tight on $\mathcal{Y}(0,T;M)$. Hence, by Prohorov's Theorem, there exist a probability measure μ on $\mathcal{C}([0,T];B)^3 \times \mathcal{Y}(0,T;M)$ and a subsequence of $\{X_n, z_n, v_n, \lambda_n\}_{n\in\mathbb{N}}$, which we still denote by $\{X_n, z_n, v_n, \lambda_n\}_{n\in\mathbb{N}}$, such that

$$\text{law}(X_n, z_n, v_n, \lambda_n) \to \mu, \quad \text{weakly as } n \to \infty. \tag{4.46}$$

STEP 2. Since the space $\mathcal{C}([0,T]; B)^3 \times \mathcal{Y}(0,T;M)$ is separable and metrisable, using Dudley's generalization of the Skorohod Representation Theorem (see e.g. Theorem 4.30 in [Kal02]) we ensure the existence of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of random variables $\{\tilde{X}_n, \tilde{z}_n, \tilde{v}_n, \tilde{\lambda}_n\}_{n\in\mathbb{N}}$ with values in $\mathcal{C}([0,T]; B)^3 \times \mathcal{Y}(0,T;M)$, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, such that

$$(\tilde{X}_n, \tilde{z}_n, \tilde{v}_n, \tilde{\lambda}_n) \stackrel{d}{=} (X_n, z_n, v_n, \lambda_n), \quad \text{for all } n \in \mathbb{N}, \tag{4.47}$$

and, on the same stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a $\mathcal{C}([0,T]; B)^3 \times \mathcal{Y}(0,T;M)-$valued random variable $\left(\tilde{X}, \tilde{z}, \tilde{v}, \tilde{\lambda}\right)$ such that

$$(\tilde{X}_n, \tilde{z}_n, \tilde{v}_n) \to (\tilde{X}, \tilde{z}, \tilde{v}), \quad \text{in } \mathcal{C}([0,T]; B)^3, \ \tilde{\mathbb{P}} - \text{a.s.} \tag{4.48}$$

and

$$\tilde{\lambda}_n \to \tilde{\lambda}, \quad \text{stably in } \mathcal{Y}(0,T;M), \ \tilde{\mathbb{P}} - \text{a.s.} \tag{4.49}$$

For each $t \in [0,T]$, let π_t denote the evaluation map $\mathcal{C}([0,T]; B) \ni z \mapsto z(t) \in B$, and let $\Phi_t : \mathcal{C}([0,T]; B)^2 \times \mathcal{Y}^\gamma(0,T;M) \to B$ be the map defined by

$$\Phi_t(x, z, \lambda) := \Gamma_t(x, \lambda) - \pi_t(z), \quad (x, z) \in \mathcal{C}([0,T]; B)^2, \ \lambda \in \mathcal{Y}^\gamma(0,T;M),$$

with Γ_t as in (4.35). By Lemma 4.42, the map Φ_t is measurable. Hence, by (4.40) and (4.47), for each $t \in [0,T]$ we have

$$\tilde{z}_n(t) = \int_0^t \int_M S_{t-r} F(r, \tilde{X}_n(r), u)\, \tilde{\lambda}_n(du, dr), \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{4.50}$$

A similar argument used with (4.41) and (4.47) yields

$$\tilde{X}_n(t) = S_t x_0 + \tilde{z}_n(t) + \tilde{v}_n(t), \quad \tilde{\mathbb{P}} - \text{a.s.}, \quad t \in [0,T]. \tag{4.51}$$

Moreover, by Lemma 4.3, for each $n \in \mathbb{N}$ there exists a relaxed control process $\{\tilde{q}^n_t\}_{t\geq 0}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\tilde{\lambda}_n(du, dt) = \tilde{q}^n_t(du)\, dt, \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{4.52}$$

Thus, we can rewrite (4.50) as

$$\tilde{z}_n(t) = \int_0^t S_{t-r}\bar{F}(r, \tilde{X}_n(r), \tilde{q}_r^n)\, dr, \quad \tilde{\mathbb{P}} - \text{a.s.}, \quad t \in [0,T]. \tag{4.53}$$

STEP 3. For each $n \in \mathbb{N}$, we define an $\mathbf{E}-$valued process $\tilde{M}_n(\cdot)$ by

$$\tilde{M}_n(t) := A^{-1}\tilde{X}_n(t) + \int_0^t \tilde{X}_n(r)\, dr - A^{-1}x_0 - \int_0^t A^{-1}\bar{F}(r, \tilde{X}_n(r), \tilde{q}_r^n)\, dr, \quad t \in [0,T], \tag{4.54}$$

and a filtration $\tilde{\mathbb{F}}_n := \{\tilde{\mathfrak{F}}_t^n\}_{t\in[0,T]}$ by

$$\tilde{\mathfrak{F}}_t^n := \sigma\{(\tilde{X}_n(s), \tilde{q}_s^n) : 0 \le s \le t\}, \quad t \in [0,T].$$

Lemma 4.44. *The process $\tilde{M}_n(\cdot)$ is a $\tilde{\mathbb{F}}_n-$martingale with cylindrical quadratic variation*

$$[\tilde{M}_n](t) = \int_0^t \tilde{Q}_n(s)\, ds, \quad t \in [0,T].$$

where $\tilde{Q}_n(t) := [A^{-1}G(t, \tilde{X}_n(t))] \circ [A^{-1}G(t, \tilde{X}_n(t))]^* \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$. Moreover, $\tilde{M}_n(\cdot)$ satisfies

$$\tilde{M}_n(t) = A^{-1}\tilde{v}_n(t) + \int_0^t \tilde{v}_n(s)\, ds, \quad \tilde{\mathbb{P}} - a.s., \quad t \in [0,T]. \tag{4.55}$$

Proof of Lemma 4.44. By Theorem 2.6.13 in [Paz83], for each $t \ge 0$ we have $A^{-1}S_t = S_tA^{-1}$. Therefore, the process $X_n(\cdot)$ satisfies

$$A^{-1}X_n(t) = S_tA^{-1}x_0 + \int_0^t S_{t-r}A^{-1}\bar{F}(r, X_n(r), q_r^n)\, dr + \int_0^t S_{t-r}A^{-1}G(r, X_n(r))\, dW_n(r).$$

Since $1 - \sigma > \frac{1}{2}$, we have (see e.g. [Tri78]),

$$\text{Range}(A^{-(1-\sigma)}) = D(A^{1-\sigma}) \subset D_A(\tfrac{1}{2}, 2).$$

Therefore, by the left-ideal property of the $\gamma-$radonifying operators, for each $(t,x) \in [0,T] \times D(A^\delta)$ we get

$$A^{-1}G(t,x) = A^{-(1-\sigma)}A^{-\sigma}G(t,x) \in \gamma\left(\mathbf{H}, D_A(\tfrac{1}{2}, 2)\right).$$

Similarly, we see that $A^{-1}x \in D(A^{1-\delta}) \subset D_A(\frac{1}{2}, 2)$. Thus, from Theorem 4.36, it follows

$$A^{-1}X_n(t)+\int_0^t X_n(s)\,ds = A^{-1}x_0+\int_0^t A^{-1}\bar{F}(r, X_n(r), q_r^n)\,dr+\int_0^t A^{-1}G(r, X_n(r))\,dW_n(r).$$

Then, for each $n \in \mathbb{N}$, the $\mathbf{E}-$valued he process $M_n(\cdot)$ defined as

$$M_n(t) := A^{-1}X_n(t) + \int_0^t X_n(s)\,ds - A^{-1}x_0 - \int_0^t A^{-1}\bar{F}(r, X_n(r), q_r^n)\,dr, \quad t \geq 0,$$

is a $\mathbb{F}_n-$martingale with cylindrical quadratic variation

$$[M_n](t) = \int_0^t Q_n(s)\,ds, \quad t \geq 0,$$

where $Q_n(t) = [A^{-1}G(t, X_n(t))] \circ [A^{-1}G(t, X_n(t))]^* \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$. Clearly, the process $M_n(\cdot)$ is adapted to the filtration generated by the process (X_n, q^n), which in turn is adapted to $\mathbb{F}_n$. Then, the first part of the Lemma follows since

$$(X_n, q^n) \stackrel{d}{=} (\tilde{X}_n, \tilde{q}^n).$$

Now, by (4.53) and Lemma 4.43 in the Appendix, the process $\tilde{z}_n(\cdot)$ satisfies

$$A^{-1}\tilde{z}_n(t) + \int_0^t \tilde{z}_n(s)\,ds = \int_0^t A^{-1}\tilde{f}_n(s)\,ds, \quad \tilde{\mathbb{P}} - \text{a.s.}, \quad t \in [0, T], \tag{4.56}$$

with

$$\tilde{f}_n(t) := \bar{F}(t, \tilde{X}_n(t), \tilde{q}_t^n), \quad t \in [0, T]. \tag{4.57}$$

Thus, using (4.51) in (4.56) we get

$$A^{-1}\tilde{X}_n(t) - A^{-1}S_tx_0 - A^{-1}\tilde{v}_n(t) + \int_0^t (\tilde{X}_n(r) - S_rx_0 - \tilde{v}_n(r))\,dr = \int_0^t A^{-1}\tilde{f}_n(s)\,ds. \tag{4.58}$$

From the identity (see e.g. [Paz83, Theorem 1.2.4]),

$$-A\int_0^t S_rx_0\,dr = S_tx_0 - x_0$$

we get

$$\int_0^t S_rx_0\,dr + A^{-1}S_tx_0 = A^{-1}x_0.$$

Using this in (4.58) and rearranging the terms we obtain

$$A^{-1}\tilde{X}_n(t) + \int_0^t \tilde{X}_n(r)\,dr = A^{-1}x_0 + \int_0^t A^{-1}\tilde{f}_n(s)\,ds + A^{-1}\tilde{v}_n(t) + \int_0^t \tilde{v}_n(r)\,dr, \quad \tilde{\mathbb{P}} - \text{a.s.},$$

for $t \in [0, T]$ and so, in view of (4.54) and (4.57), equality (4.55) follows. □

STEP 4. We now define a $\mathbf{E}-$valued process $\tilde{M}(\cdot)$ by the formula

$$\tilde{M}(t) := A^{-1}\tilde{v}(t) + \int_0^t \tilde{v}(s)\,ds, \quad t \in [0, T].$$

Observe that from (4.48) we have $\tilde{v}_n \to \tilde{v}$ in $\mathcal{C}([0, T]; \mathbf{E})$ $\tilde{\mathbb{P}}-$a.s. which combined with (4.55) yields

$$\tilde{M}_n \to \tilde{M}, \quad \text{in } \mathcal{C}([0, T]; \mathbf{E}), \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{4.59}$$

We use once again Lemma 4.3 to ensure existence of a relaxed control process $\{\tilde{q}_t\}_{t\geq 0}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\tilde{\lambda}(du, dt) = \tilde{q}_t(du)\,dt, \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{4.60}$$

We define the filtration $\tilde{\mathbb{F}} := \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$ by

$$\tilde{\mathcal{F}}_t := \sigma\{(\tilde{X}(s), \tilde{q}_s) : 0 \leq s \leq t\}, \quad t \in [0, T].$$

Let also $\tilde{g}(t) := G(t, \tilde{X}(t))$ and $\tilde{Q}(t) := [A^{-1}\tilde{g}(t)] \circ [A^{-1}\tilde{g}(t)]^* \in \mathcal{L}(\mathbf{E}^*, \mathbf{E})$, $t \in [0, T]$.

Lemma 4.45. *The process $\tilde{M}(\cdot)$ is a $\tilde{\mathbb{F}}-$ martingale with cylindrical quadratic variation*

$$[\tilde{M}](t) = \int_0^t \tilde{Q}(s)\,ds, \quad t \in [0, T].$$

Proof of Lemma 4.45. From (4.48) we have

$$\sup_{t\in[0,T]} \left|\tilde{X}_n(t) - \tilde{X}(t)\right|_B^2 \to 0, \quad \text{as } n \to \infty, \ \ \tilde{\mathbb{P}} - \text{a.s.} \tag{4.61}$$

Moreover, by (4.44), (4.47) and Fatou's Lemma it follows that

$$\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]} \left|\tilde{X}(t)\right|_B^{mp}\right] < +\infty. \tag{4.62}$$

Therefore, by (4.44) and Chebyshev's inequality, the random variables in (4.61) are uniformly integrable, and by [Kal02, Lemma 4.11] we have

$$\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\tilde{X}_n(t)-\tilde{X}(t)\big|_B^2\Big]\to 0,\quad \text{as } n\to\infty. \tag{4.63}$$

The same argument applied to (4.59) yields

$$\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big|\tilde{M}_n(t)-\tilde{M}(t)\big|_{\mathbf{E}}^2\Big]\to 0,\quad \text{as } n\to\infty. \tag{4.64}$$

This, in conjunction with Lemma 4.44, implies that for all $0<s<t$ and for all

$$\phi\in\mathcal{C}_b\left(\mathcal{C}(0,s;B)\times\mathcal{Y}(0,s;M)\right)$$

we have

$$0=\tilde{\mathbb{E}}\left[\big(\tilde{M}_n(t)-\tilde{M}_n(s)\big)\phi(\tilde{X}_n,\tilde{\lambda}_n)\right]\to\tilde{\mathbb{E}}\left[\big(\tilde{M}(t)-\tilde{M}(s)\big)\phi(\tilde{X},\tilde{\lambda})\right]$$

as $n\to\infty$, which implies that $\tilde{M}(\cdot)$ is a $\tilde{\mathbb{F}}-$martingale. Moreover, for all $x_1^*,x_2^*\in\mathbf{E}$ and $n\in\mathbb{N}$,

$$\begin{aligned}\tilde{\mathbb{E}}\Bigg[\Big(&\big\langle\tilde{M}_n(t),x_1^*\big\rangle\big\langle\tilde{M}_n(t),x_2^*\big\rangle-\big\langle\tilde{M}_n(s),x_1^*\big\rangle\big\langle\tilde{M}_n(s),x_2^*\big\rangle\\&-\int_s^t\big[\big(A^{-1}G(r,\tilde{X}_n(r))\big)^*x_1^*,\big(A^{-1}G(r,\tilde{X}_n(r))\big)^*x_1^*\big]_{\mathbf{H}}\,dr\Big)\phi(\tilde{X}_n,\tilde{\lambda}_n)\Bigg]=0.\end{aligned} \tag{4.65}$$

By (4.43) and (4.55), the first two terms inside the expectation in (4.65) are uniformly integrable, and so is the third term by Assumption A.2. Hence, by (4.63), (4.64) and the continuity of $A^{-1}G$, the limit of (4.65) as $n\to\infty$ yields

$$\begin{aligned}\tilde{\mathbb{E}}\Bigg[\Big(&\big\langle\tilde{M}(t),x_1^*\big\rangle\big\langle\tilde{M}(t),x_2^*\big\rangle-\big\langle\tilde{M}(s),x_1^*\big\rangle\big\langle\tilde{M}(s),x_2^*\big\rangle\\&-\int_s^t\big[\big(A^{-1}G(r,\tilde{X}(r))\big)^*x_1^*,\big(A^{-1}G(r,\tilde{X}(r))\big)^*x_1^*\big]_{\mathbf{H}}\,dr\Big)\phi(\tilde{X},\tilde{\lambda})\Bigg]=0,\end{aligned}$$

and Lemma 4.45 follows. □

STEP 5. We now identify the process $\tilde{X}(\cdot)$ as mild solution of the equation controlled by $\{\tilde{q}_t\}_{t\geq 0}$. Notice that the coercivity condition in Assumption A.3–(5), which we used to obtain the uniform estimates for the minimizing sequence, is again needed to pass to the limit as the nonlinear term F is not necessarily bounded with respect to the control variable.

Let $\tilde{f}(t) := \bar{F}(t, \tilde{X}(t), \tilde{q}_t)$, $t \in [0, T]$. Observe that, by (4.33), (4.57) and (4.62), $\tilde{f}_n, \tilde{f}$ belong to $L^2([0,T] \times \tilde{\Omega}; B)$. We claim first that

$$\tilde{f}_n \to \tilde{f}, \quad \text{weakly in } L^2([0,T] \times \tilde{\Omega}; \mathbf{E}). \tag{4.66}$$

Proof of (4.66). For each $n \in \mathbb{N}$, define $\hat{f}_n(t) := \bar{F}(t, \tilde{X}(t), \tilde{q}^n_t)$, $t \in [0, T]$. First, we will prove

$$\tilde{f}_n - \hat{f}_n \to 0, \quad \text{(strongly) in } L^2([0,T] \times \tilde{\Omega}; B). \tag{4.67}$$

Indeed, by Assumption A.3, we have

$$\begin{aligned} I_n(t) :&= \int_M \left| F(t, \tilde{X}_n(t), u) - F(t, \tilde{X}(t), u) \right|_B^2 \tilde{q}^n_t(du) \\ &\le \sup_{u \in M} \left| F(t, \tilde{X}_n(t), u) - F(t, \tilde{X}(t), u) \right|_B^2 \to 0 \end{aligned}$$

as $n \to \infty$ for $t \in [0, T]$, $\tilde{\mathbb{P}}-$a.s. From (4.33), (4.36), (4.44) and (4.62) we have

$$\sup_{n \in \mathbb{N}} \tilde{\mathbb{E}} \int_0^T |I_n(t)|^{p/2} \, dt < +\infty.$$

Hence, $\{I_n(\cdot)\}_{n \in \mathbb{N}}$ is uniformly integrable on $\tilde{\Omega} \times [0, T]$, and by [Kal02, Lemma 4.11] we have

$$\tilde{\mathbb{E}} \int_0^T \left| \tilde{f}_n(t) - \hat{f}_n(t) \right|_B^2 dt \le \tilde{\mathbb{E}} \int_0^T I_n(t) \, dt \to 0, \quad \text{as } n \to \infty,$$

and (4.67) follows. Now, we will prove that

$$\hat{f}_n \to \tilde{f}, \quad \text{weakly in } L^2([0,T] \times \tilde{\Omega}; \mathbf{E}). \tag{4.68}$$

Since $\mathbf{E}$ is separable and reflexive, the dual space $\mathbf{E}^*$ is also separable and, therefore, has the Radon-Nikodym property with respect to the product measure $dt \otimes d\mathbb{P}$ (see e.g. Sections III.2 and IV.2 in [DU77]) and so we have

$$L^2([0,T] \times \tilde{\Omega}; \mathbf{E})^* \simeq L^2([0,T] \times \tilde{\Omega}; \mathbf{E}^*).$$

Let $\psi \in L^2([0,T] \times \tilde{\Omega}; \mathbf{E}^*)$ be fixed, and observe that for each $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{\mathbb{E}} \int_0^T \langle \hat{f}_n(t), \psi(t) \rangle \, dt &= \tilde{\mathbb{E}} \int_0^T \left\langle \int_M F(t, \tilde{X}(t), u) \, \tilde{q}_t^n(du), \psi(t) \right\rangle \, dt \\ &= \tilde{\mathbb{E}} \int_0^T \int_M \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{q}_t^n(du) dt \\ &= \tilde{\mathbb{E}} \int_0^T \int_M \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}_n(du, dt).\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{E}$ and $\mathbf{E}^*$. Let $\varepsilon \in (0,1)$ be fixed and take $C_\varepsilon > \max\{\frac{R}{\varepsilon}, 1\}$ with R as in (4.36). Then, for this choice of C_ε, we have

$$\begin{aligned}\tilde{\mathbb{E}} \left[\tilde{\lambda}_n \Big(\{ \eta^{\gamma-2} > C_\varepsilon \} \Big) \right] &= \tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} \tilde{\lambda}_n(du, dt) \\ &\leq \frac{1}{C_\varepsilon} \tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} \eta(t,u)^{\gamma-2} \, \tilde{\lambda}_n(du, dt) \\ &< \varepsilon.\end{aligned}$$

We write

$$\begin{aligned}&\tilde{\mathbb{E}} \int_0^T \int_M \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}_n(du, dt) \\ &\quad = \tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} \leq C_\varepsilon\}} \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}_n(du, dt) \\ &\qquad + \tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}_n(du, dt)\end{aligned}$$

and observe first that by Lemma 4.16 we have $\tilde{\mathbb{P}}-$a.s.

$$\begin{aligned}&\int_{\{\eta(t,u)^{\gamma-2} \leq C_\varepsilon\}} \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}_n(du, dt) \\ &\qquad \to \int_{\{\eta(t,u)^{\gamma-2} \leq C_\varepsilon\}} \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}(du, dt)\end{aligned}$$

as $n \to \infty$ and that, by (4.33),

$$\begin{aligned}&\int_{\{\eta(t,u)^{\gamma-2} \leq C_\varepsilon\}} \langle F(t, \tilde{X}(t), u), \psi(t) \rangle \, \tilde{\lambda}_n(du, dt) \\ &\qquad \leq k_2 T \left(\| \tilde{X}(\cdot) \|^m_{\mathcal{C}([0,T];B)} + C_\varepsilon \right) |\psi(\cdot)|_{L^2(0,T;\mathbf{E}^*)}, \quad \tilde{\mathbb{P}} - \text{a.s.}\end{aligned}$$

Thus, using Lebesgue's dominated convergence Theorem we get

$$\tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} \leq C_\varepsilon\}} \left\langle F(t, \tilde{X}(t), u), \psi(t) \right\rangle \tilde{\lambda}_n(du, dt) \\ \to \tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} \leq C_\varepsilon\}} \left\langle F(t, \tilde{X}(t), u), \psi(t) \right\rangle \tilde{\lambda}(du, dt)$$

as $n \to \infty$. Now, for each $n \in \mathbb{N}$, define the measure μ_n on $\mathcal{B}(M) \otimes \mathcal{B}([0,T]) \otimes \mathcal{F}$ as

$$\mu_n(du, dt, d\omega) := \tilde{\lambda}_n(\omega)(du, dt)\tilde{\mathbb{P}}(d\omega).$$

Then, again by (4.33), for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \tilde{\mathbb{E}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} &\left| \left\langle F(t, \tilde{X}(t), u), \psi(t) \right\rangle \right| \tilde{\lambda}_n(du, dt) \\ &\leq \int_{\tilde{\Omega}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} \varphi(t)\, \mu_n(du, dt, d\omega) \\ &\quad + \int_{\tilde{\Omega}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} \eta(t,u)\, |\psi(t)|_{\mathbf{E}^*}\, \mu_n(du, dt, d\omega) \end{aligned}$$

with $\varphi := k_2 |\tilde{X}(\cdot)|_{\mathbf{E}}^m \, |\psi(\cdot)|_{\mathbf{E}^*} \in L^r([0,T] \times \tilde{\Omega})$ and $\frac{1}{2} + \frac{1}{p} = \frac{1}{r}$, since by (4.62) we have

$$|\tilde{X}(\cdot)|_B^m \in L^p([0,T] \times \tilde{\Omega}).$$

Thus, by Hölder's inequality we get

$$\begin{aligned} &\int_{\tilde{\Omega}} \int_{\{\eta(t,u)^{\gamma-2} > C_\varepsilon\}} \varphi(t)\, \mu_n(du, dt, d\omega) \\ &\quad \leq \left(\int_{\tilde{\Omega}} \int_0^T \int_M \varphi(t)^r \, \mu_n(du, dt, d\omega) \right)^{1/r} \cdot \left(\tilde{\mathbb{E}} \left[\tilde{\lambda}_n \left(\eta^{\gamma-2} > C_\varepsilon \right) \right] \right)^{1-1/r} \\ &\quad < \|\varphi\|_{L^r([0,T]\times\tilde{\Omega})}\, \varepsilon^{1-1/r} \end{aligned}$$

and

$$\begin{aligned}
&\int_{\tilde\Omega}\int_{\{\eta(t,u)^{\gamma-2}>C_\varepsilon\}} \eta(t,u)\,|\psi(t)|_{\mathbf{E}^*}\;\mu_n(du,dt,d\omega)\\
&\quad\le \|\psi\|_{L^2([0,T]\times\tilde\Omega;\mathbf{E}^*)}\left(\int_{\tilde\Omega}\int_{\{\eta(t,u)^{\gamma-2}>C_\varepsilon\}} \eta(t,u)^2\,\mu_n(du,dt,d\omega)\right)^{1/2}\\
&\quad= \|\psi\|_{L^2([0,T]\times\tilde\Omega;\mathbf{E}^*)}\left(\tilde{\mathbb{E}}\int_{\{\eta(t,u)^{\gamma-2}>C_\varepsilon\}} \frac{\eta(t,u)^\gamma}{\eta(t,u)^{\gamma-2}}\,\tilde\lambda_n(du,dt)\right)^{1/2}\\
&\quad\le \|\psi\|_{L^2([0,T]\times\tilde\Omega;\mathbf{E}^*)}\left(\frac{1}{C_\varepsilon}\tilde{\mathbb{E}}\int_{\{\eta(t,u)^{\gamma-2}>C_\varepsilon\}} \eta(t,u)^\gamma\,\tilde\lambda_n(du,dt)\right)^{1/2}\\
&\quad\le \|\psi\|_{L^2([0,T]\times\tilde\Omega;\mathbf{E}^*)}\left(\frac{R}{C_\varepsilon}\right)^{1/2}\\
&\quad< \|\psi\|_{L^2([0,T]\times\tilde\Omega;\mathbf{E}^*)}\,\varepsilon^{1/2},
\end{aligned}$$

and this holds uniformly with respect to $n\in\mathbb{N}$. Since $\eta(t,\cdot)$ is lower semi-continuous for all $t\in[0,T]$, by Lemma 4.15 and Fatou's lemma we have

$$\tilde{\mathbb{E}}\int_0^T\int_M \eta(t,u)^\gamma\,\tilde\lambda(du,dt)\le \liminf_{n\to\infty}\tilde{\mathbb{E}}\int_0^T\int_M \eta(t,u)^\gamma\,\tilde\lambda_n(du,dt)\le R.$$

Therefore, the same estimate holds for $\tilde\lambda$, that is,

$$\begin{aligned}
&\tilde{\mathbb{E}}\int_{\{\eta(t,u)^{\gamma-2}>C_\varepsilon\}}\left|\left\langle F(t,\tilde X(t),u),\psi(t)\right\rangle\right|\tilde\lambda(du,dt)\\
&\qquad\qquad\le \|\varphi\|_{L^r([0,T]\times\tilde\Omega)}\,\varepsilon^{1-1/r}+\|\psi\|_{L^2([0,T]\times\tilde\Omega;\mathbf{E}^*)}\,\varepsilon^{1/2}
\end{aligned}$$

and since $\varepsilon\in(0,1)$ is arbitrary, we conclude that

$$\tilde{\mathbb{E}}\int_0^T\int_M\left\langle F(t,\tilde X(t),u),\psi(t)\right\rangle\tilde\lambda_n(du,dt)\to\tilde{\mathbb{E}}\int_0^T\int_M\left\langle F(t,\tilde X(t),u),\psi(t)\right\rangle\tilde\lambda(du,dt)$$

as $n\to\infty$. Thus, (4.68) follows. Note that (4.68) in conjunction with (4.67) implies (4.66). □

STEP 6. We now claim that the process $\tilde M(\cdot)$ satisfies, for each $t\in[0,T]$,

$$\tilde M(t)=A^{-1}\tilde X(t)+\int_0^t\tilde X(s)\,ds-A^{-1}x_0-\int_0^t A^{-1}\bar F(s,\tilde X(s),\tilde q_s)\,ds,\ \ \tilde{\mathbb{P}}-\text{a.s.}\qquad(4.69)$$

Proof of (4.69). By (4.63) and (4.64), for any $\varepsilon>0$ there exists an integer $\bar m=\bar m(\varepsilon)\ge 1$ for which

$$\tilde{\mathbb{E}}\bigg[\sup_{t\in[0,T]}\big|\tilde{X}_n(t)-\tilde{X}(t)\big|_B^2+\big|\tilde{M}_n(t)-\tilde{M}(t)\big|_{\mathbf{E}}^2\bigg]<\varepsilon,\quad \forall n\geq\bar{m}. \tag{4.70}$$

From (4.66) we have

$$\tilde{f}\in\overline{\{\tilde{f}_{\bar{m}},\tilde{f}_{\bar{m}+1},\dots\}}^{w}\subset\overline{\mathrm{co}\{\tilde{f}_{\bar{m}},\tilde{f}_{\bar{m}+1},\dots\}}^{w}$$

where $\mathrm{co}(\cdot)$ and $\overline{\cdot}^{w}$ denote the convex hull and weak-closure in $L^2([0,T]\times\tilde{\Omega};\mathbf{E})$ respectively. By Mazur's theorem (see e.g. [Meg98, Theorem 2.5.16]),

$$\overline{\mathrm{co}\{\tilde{f}_{\bar{m}},\tilde{f}_{\bar{m}+1},\dots\}}^{w}=\overline{\mathrm{co}\{\tilde{f}_{\bar{m}},\tilde{f}_{\bar{m}+1},\dots\}}.$$

Therefore, there exist an integer $\bar{N}\geq 1$ and $\{\alpha_1,\dots,\alpha_{\bar{N}}\}$ with $\alpha_i\geq 0$, $\sum_{i=1}^{\bar{N}}\alpha_i=1$, such that

$$\Big\|\sum_{i=1}^{\bar{N}}\alpha_i\tilde{f}_{\bar{m}+i}-\tilde{f}\Big\|^2_{L^2([0,T]\times\tilde{\Omega};\mathbf{E})}<\varepsilon. \tag{4.71}$$

Let $t\in[0,T]$ be fixed. Using the $\alpha_i's$ and the definition of the process $\tilde{M}_{\bar{m}+i}$ in (4.54) we can write

$$A^{-1}x_0=\sum_{i=1}^{\bar{N}}\alpha_i\left[A^{-1}\tilde{X}_{\bar{m}+i}(t)+\int_0^t\tilde{X}_{\bar{m}+i}(s)\,ds-\int_0^t A^{-1}\tilde{f}_{\bar{m}+i}(s)\,ds-\tilde{M}_{m+i}(t)\right]$$

Thus, we have

$$\begin{aligned}
&\Big|\tilde{M}(t)-A^{-1}\tilde{X}(t)-\int_0^t\tilde{X}(s)\,ds+A^{-1}x+\int_0^t A^{-1}\tilde{f}(s)\,ds\Big|^2_{\mathbf{E}}\\
&\leq 4\Bigg(\Big|\tilde{M}(t)-\sum_{i=1}^{\bar{N}}\alpha_i\tilde{M}_{\bar{m}+i}(t)\Big|^2_{\mathbf{E}}+\Big|\sum_{i=1}^{\bar{N}}\alpha_i A^{-1}\tilde{X}_{m+i}(t)-A^{-1}\tilde{X}(t)\Big|^2_{\mathbf{E}}\\
&\quad+\Big|\sum_{i=1}^{\bar{N}}\alpha_i\int_0^t\tilde{X}_{m+i}(s)\,ds-\int_0^t\tilde{X}(s)\,ds\Big|^2_{\mathbf{E}}\\
&\quad+\Big|\int_0^t A^{-1}\tilde{f}(s)\,ds-\sum_{i=1}^{\bar{N}}\alpha_i\int_0^t A^{-1}\tilde{f}_{\bar{m}+i}(s)\,ds\Big|^2_{\mathbf{E}}\Bigg).
\end{aligned}$$

Then, by (4.70) and (4.71) it follows that

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big|\tilde{M}(t) - A^{-1}\tilde{X}(t) - \int_0^t \tilde{X}(s)\,ds + A^{-1}x + \int_0^t A^{-1}\tilde{f}(s)\,ds\Big|_{\mathbf{E}}^2 \\
&\quad \le 4\left(\sum_{i=1}^{\bar{N}} \alpha_i \tilde{\mathbb{E}}\Big|\tilde{M}(t) - \tilde{M}_{m+i}(t)\Big|_{\mathbf{E}}^2 + \sum_{i=1}^{\bar{N}} \alpha_i \left\|A^{-1}\right\|_{\mathcal{L}(\mathbf{E})}^2 \tilde{\mathbb{E}}\Big|\tilde{X}_{\bar{m}+i}(t) - \tilde{X}(t)\Big|_{\mathbf{E}}^2 \right. \\
&\qquad + \sum_{i=1}^{\bar{N}} \alpha_i \tilde{\mathbb{E}}\Big|\int_0^t (\tilde{X}_{\bar{m}+i}(s) - \tilde{X}(s))\,ds\Big|_{\mathbf{E}}^2 \\
&\qquad \left. + T\left\|A^{-1}\right\|_{\mathcal{L}(\mathbf{E})}^2 \tilde{\mathbb{E}}\int_0^T \Big|\tilde{f}(s) - \sum_{i=1}^{\bar{N}} \alpha_i \tilde{f}_{\bar{m}+i}(s)\Big|_{\mathbf{E}}^2 ds\right) \\
&\quad \le 4\left(1 + \left\|A^{-1}\right\|_{\mathcal{L}(\mathbf{E})}^2 + T + T\left\|A^{-1}\right\|_{\mathcal{L}(\mathbf{E})}^2\right)\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (4.69) follows. □

STEP 7. In view of Lemma 4.45 and (4.69), by the Martingale Representation Theorem 2.21 there exist an extension of the probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$, which we also denote $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$, and a $\mathbf{H}-$cylindrical Wiener process $\{\tilde{W}(t)\}_{t\ge 0}$ defined on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$, such that

$$\tilde{M}(t) = \int_0^t A^{-1}\tilde{g}(s)\,d\tilde{W}(s), \quad \tilde{\mathbb{P}} - \text{a.s.}, \quad t \in [0,T],$$

that is, for each $t \in [0,T]$, we have

$$A^{-1}\tilde{X}(t) + \int_0^t \tilde{X}(s)\,ds = A^{-1}x_0 + \int_0^t A^{-1}\bar{F}(r, \tilde{X}(r), \tilde{q}_r)\,dr + \int_0^t A^{-1}G(r, \tilde{X}(r))\,d\tilde{W}(r).$$

By the same argument used in step 3 (cf. Theorem 4.36),

$$A^{-1}\tilde{X}(t) = S_t A^{-1}x_0 + \int_0^t S_{t-r}A^{-1}\bar{F}(r, \tilde{X}(r), \tilde{q}_r)\,dr + \int_0^t S_{t-r}A^{-1}G(r, \tilde{X}(r))\,d\tilde{W}(r)$$

for each $t \in [0,T]$, obtaining finally

$$\tilde{X}(t) = S_t x_0 + \int_0^t S_{t-r}\bar{F}(r, \tilde{X}(r), \tilde{q}_r)\,dr + \int_0^t S_{t-r}G(r, \tilde{X}(r))\,d\tilde{W}(r), \ \mathbb{P} - \text{a.s.}$$

In other words, $\tilde{\pi} := (\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}}, \{\tilde{W}(t)\}_{t\ge 0}, \{\tilde{q}_t\}_{t\ge 0}, \{\tilde{X}(t)\}_{t\ge 0})$ is a weak admissible relaxed control. Lastly, by the Fiber Product Lemma 4.17 we have

$$\underline{\delta}_{\tilde{X}_n} \otimes \lambda_n \to \underline{\delta}_{\tilde{X}} \otimes \lambda, \text{ stably in } \mathcal{Y}(0,T;\mathbf{E} \times M), \quad \tilde{\mathbb{P}} - \text{a.s.}$$

Since $\mathbf{E} \times M$ is also a metrisable Suslin space, using Lemma 4.15 and Fatou's Lemma we get

$$\tilde{\mathbb{E}} \int_0^T \int_M h(t, \tilde{X}(t), u)\, \tilde{\lambda}(du, dt) \leq \liminf_{n\to\infty} \tilde{\mathbb{E}} \int_0^T \int_M h(t, \tilde{X}_n(t), u)\, \tilde{\lambda}_n(du, dt)$$

and since $(\tilde{X}_n, \tilde{\lambda}_n) \stackrel{d}{=} (X_n, \lambda_n)$ it follows that

$$\begin{aligned}
\bar{J}(\tilde{\pi}) &= \tilde{\mathbb{E}} \int_0^T \int_M h(t, \tilde{X}(t), u)\, \tilde{\lambda}(du, dt) + \tilde{\mathbb{E}}\varphi(\tilde{X}(T)) \\
&\leq \liminf_{n\to\infty} \mathbb{E}^n \int_0^T \int_M h(t, X_n(t), u)\, \lambda_n(du, dt) + \liminf_{n\to\infty} \mathbb{E}^n \varphi(X_n(T)) \\
&\leq \liminf_{n\to\infty} \left[\mathbb{E}^n \int_0^T \int_M h(t, X_n(t), u)\, \lambda_n(du, dt) + \mathbb{E}^n \varphi(X_n(T)) \right] \\
&= \inf_{\pi \in \mathcal{U}^{\mathrm{w}}_{\mathrm{ad}}(x_0)} \bar{J}(\pi),
\end{aligned}$$

that is, $\tilde{\pi}$ is a weak optimal relaxed control for **(RCP)**, and this concludes the proof of Theorem 4.39. □

Example 4.46 (Optimal relaxed control of stochastic PDEs of reaction-diffusion type with multiplicative noise)**.** Let M be a Suslin metrisable control set and let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with $\mathcal{C}^\infty$ boundary. Let

$$f : [0,T] \times \mathcal{O} \times \mathbb{R} \times M \to \mathbb{R}$$

be measurable in t, continuous in u and continuous in $(\xi, x) \in \mathcal{O} \times \mathbb{R}$ uniformly with respect to u. Assume further that f satisfies

$$f(t,\xi,x+y,u)\operatorname{sgn} x \leq -k_1\,|x| + k_2\,|y|^m + \eta(t,u), \quad (t,\xi) \in [0,T] \times \mathcal{O}, \;\; x,y \in \mathbb{R}, \; u \in M \tag{4.72}$$

for some constants $m \geq 1$ and $k_1, k_2 > 0$ and some measurable function

$$\eta : [0,T] \times M \to [0,+\infty]$$

such that $\eta(t,\cdot)$ is inf-compact for all $t \in [0,T]$. Let $\mathcal{A}$ be the second-order differential operator

$$(\mathcal{A}x)(\xi) := -\sum_{i,j=1}^{d} a_{ij}(\xi)\frac{\partial^2 x}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^{d} b_i(\xi)\frac{\partial x}{\partial\xi_i} + c(\xi)x(\xi), \quad \xi \in \mathcal{O},$$

with $a_{ij} = a_{ji}$, $\sum_{i,j=1}^{d} a_{ij}(\xi)\lambda_i\lambda_j \geq C\,|\lambda|^2, \forall\lambda \in \mathbb{R}^d$, and $c, b_i, a_{ij} \in \mathcal{C}^\infty(\bar{\mathcal{O}})$. Finally, let

$$g : [0,T] \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$$

be bounded, measurable in t and continuous in $(\xi, x) \in \mathcal{O} \times \mathbb{R}$, and consider the following controlled stochastic PDE on $[0,T] \times \mathcal{O}$,

$$\begin{aligned} \frac{\partial X}{\partial t}(t,\xi) + (\mathcal{A}X)(t,\xi) &= f(t,\xi,X(t,\xi),u(t)) + g(t,\xi,X(t,\xi))\,\frac{\partial w}{\partial t}(t,\xi), \quad \text{on } [0,T]\times\mathcal{O} \\ X(t,\xi) &= 0, \qquad t \in (0,T],\ \xi \in \partial\mathcal{O} \\ X(0,\cdot) &= x_0(\xi), \qquad \xi \in \mathcal{O} \end{aligned} \tag{4.73}$$

where $w(\cdot)$ is a nondegenerate noise with Cameron-Martin space

$$\mathbf{H} := \begin{cases} L^2(\mathcal{O}), & \text{if } d = 1, \\ H^{\theta,2}(\mathcal{O}) \text{ with } \theta \in \left(\frac{d-1}{2}, \frac{d}{2}\right), & \text{if } d \geq 2. \end{cases}$$

In concrete situations the quantity $X(t,\xi)$ represents the concentration, density or temperature of a certain substance and, as mentioned in the introduction, our aim is to study a running cost function that regulates this quantity at some fixed points $\zeta_1, \ldots, \zeta_n \in \mathcal{O}$. Therefore, we need the trajectories of the state process to take values in the space of continuous functions on the domain $\mathcal{O}$. In view of this, and the zero-boundary condition in (4.73), we take $B = \mathcal{C}_0(\bar{\mathcal{O}})$ as state space.

Let $\phi : [0,T] \times \mathcal{O} \times \mathbb{R} \times M \to \mathbb{R}_+$ be measurable and lower semi-continuous with respect to x and u. We will consider the running cost function defined by

$$h(t,x,u) := \sum_{i=1}^{n} \phi(t,\zeta_i,x(\zeta_i),u) + \eta(t,u)^\gamma, \quad t \in [0,T],\ x \in \mathcal{C}_0(\bar{\mathcal{O}}),\ u \in M \tag{4.74}$$

where $\zeta_1, \ldots, \zeta_n \in \mathcal{O}$ are fixed and $\gamma > 2$ is to be chosen below.

Theorem 4.47. *Let the constants q, σ and δ satisfy the following conditions*

1. *If* $d = 1$,
$$q > 2, \quad \frac{1}{4} < \sigma < \frac{1}{2} - \frac{1}{2q} \quad \textit{and} \quad \frac{1}{2q} < \delta < \frac{1}{2} - \sigma.$$
2. *If* $d \geq 2$,
$$2d < q \leq \frac{2d}{d - 2\theta}, \quad \frac{d}{q} < \sigma < \frac{1}{4}, \quad \textit{and} \quad \frac{d}{q} < \delta < \frac{1}{4}.$$

Assume also that γ satisfies condition (4.32). Then, if there exists $x_0 \in \mathcal{C}_0(\bar{\mathcal{O}})$ such that

$$\inf_{\pi \in \bar{\mathcal{U}}^{\mathrm{w}}_{\mathrm{ad}}(x_0)} \bar{J}(\pi) < +\infty,$$

the **(RCP)** *associated with (4.73) and the cost function (4.74) admits a weak optimal relaxed control.*

Proof. Let $\mathbf{E} = L^q(\mathcal{O})$ and let A_q denote the realization of $\mathcal{A}$ in $L^q(\mathcal{O})$. Then $A_q + \nu I$ satisfies Assumption A.1 for some $\nu \geq 0$ (see Example 4.28). Let us define the Nemytskii operator $F : [0,T] \times B \times M \to B$ by

$$F(t,x,u)(\xi) := f(t,\xi,x(\xi),u), \quad \xi \in \mathcal{O}, \ \ (t,x,u) \in [0,T] \times B \times M.$$

Let $x \in B$ and let $z^* \in \partial |x|_B$. Then

$$z^* = \begin{cases} \delta_{\xi_0}, & \text{if } x(\xi_0) = |x|_B \\ -\delta_{\xi_0}, & \text{if } x(\xi_0) = -|x|_B \end{cases} \tag{4.75}$$

for some $\xi_0 \in \mathcal{O}$ (see e.g. [DPZ92a]) and by condition (4.72), for each $y \in B$ and $u \in M$ we have

$$\begin{aligned} \langle F(t,x+y,u), z^* \rangle &= f\left(t, \xi_0, x(\xi_0) + y(\xi_0), u)\right) \operatorname{sgn} x(\xi_0) \\ &\leq -k_1 \left|x(\xi_0)\right| + k_2 \left|y(\xi_0)\right|^m + \eta(t,u) \\ &\leq -k_1 |x|_B + k_2 |y|_B^m + \eta(t,u). \end{aligned}$$

Moreover, we can find $k_0 \in \mathbb{R}$ such that the realization of $-\mathcal{A} + k_0 I$ in B is dissipative, i.e.

$$\langle (-\mathcal{A} + k_0 I)x, x^* \rangle \leq 0, \ \ x^* \in \partial |x|_B , \quad x \in B.$$

Hence, Assumption A.3 is satisfied. We now check that Assumption A.2–(2) also holds. Observe that by writing $-A_q + F = -(A_q + \nu I) + F + \nu I$, we can assume without loss of generality that

$\nu = 0$. Let us define the multiplication operator

$$(G(t,x)y)(\xi) := g(t,\xi,x(\xi))y(\xi), \quad \xi \in \mathcal{O}, \;\; y \in \mathbf{H}, \;\; x \in \mathcal{C}_0(\bar{\mathcal{O}}), \;\; t \in [0,T].$$

We consider first the case $d = 1$. Since g is bounded, for each $(t,x) \in [0,T] \times \mathcal{C}_0(\bar{\mathcal{O}})$ the map $\mathcal{O} \ni \xi \mapsto g(t,\xi,x(\xi)) \in \mathbb{R}$ belongs to $L^\infty(\mathcal{O})$. Therefore, by Hölder's inequality the map $G(t,x)$ is a bounded linear operator in $\mathbf{H} = L^2(\mathcal{O})$ and its operator norm is uniformly bounded from above by some constant independent of t and x. Moreover, as proved in Example 3.6, the map $A_q^{-\sigma}$ extends to a bounded linear operator from $L^2(\mathcal{O})$ to $L^q(\mathcal{O})$, also denoted by $A_q^{-\sigma}$, such that

$$A_q^{-\sigma} \in \gamma\left(L^2(\mathcal{O}), L^q(\mathcal{O})\right).$$

Then, by the right-ideal property of the $\gamma-$radonifying operators, Assumption A.2–(2) is satisfied.

In the case $d \geq 2$, the choice of the constants θ and q and the Sobolev Embedding Theorem imply that $\mathbf{H} = H^{\theta,2}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O})$. This combined again with Hölder's inequality implies that $G(t,x)$ is a bounded linear operator from $H^{\theta,2}(\mathcal{O})$ into $L^q(\mathcal{O})$, with operator norm again uniformly bounded from above by some constant independent of t and x. Since $\sigma > d/2q$, by the same argument used in Example 4.32 it follows that

$$A_q^{-\sigma} G(t,x) \in \gamma\big(H^{\theta,2}(\mathcal{O}), L^q(\mathcal{O})\big)$$

and that Assumption A.2–(2) holds. Since in both cases ($d = 1$ and $d \geq 2$) we have $\delta > d/2q$, as seen in Example 4.32, we have

$$D(A_q^\delta) = [L^q(\mathcal{O}), D(A_q)]_\delta = H_0^{2\delta,q}(\mathcal{O}) \hookrightarrow \mathcal{C}_0(\bar{\mathcal{O}})$$

and, therefore, Assumption A.2–(1) is satisfied too. Moreover, the last embedding is compact, which in turn implies that the embedding $D(A_q + \nu I) = D(A_q) \hookrightarrow L^q(\mathcal{O})$ is also compact, and the desired result follows from Theorem 4.39. □

Remark 4.48. Existence of weak optimal feedback controls for a similar cost functional and a similar class of dissipative stochastic PDEs has been recently proved in [Mas08a] and [Mas08b] using Backward SDEs and the associated Hamilton-Jacobi-Bellman equation. However, only the case of additive noise is considered and the nonlinear term is assumed to be bounded with respect to the control variable.

Example 4.49. The first example can be modified to allow the control process to be space-dependant. For instance, consider the controlled stochastic PDE on $[0,T]\times\mathcal{O}$,

$$\begin{aligned}\frac{\partial X}{\partial t}(t,\xi)+(\mathcal{A}X)(t,\xi)&=f(t,\xi,X(t,\xi),u(t,\xi))+g(t,\xi,X(t,\xi))\,\frac{\partial w}{\partial t}(t,\xi),\quad \text{on } [0,T]\times\mathcal{O}\\ X(t,\xi)&=0,\qquad t\in(0,T],\ \xi\in\partial\mathcal{O}\\ X(0,\cdot)&=x_0(\xi),\qquad \xi\in\mathcal{O}\end{aligned}\tag{4.76}$$

where

$$f:[0,T]\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$$

satisfies

$$f(t,\xi,x+y,u)\operatorname{sgn}x\le -k_1\,|x|+k_2\,|y|^m+a(t,|u|),\quad (t,\xi)\in[0,T]\times\mathcal{O},\ \ x,y,u\in\mathbb{R}\tag{4.77}$$

where $a:[0,T]\times\mathbb{R}_+\to\mathbb{R}_+$ is a measurable function such that $a(t,\cdot)$ is strictly increasing for each $t\in[0,T]$. Assume further that f is measurable in t, separately continuous in $(\xi,u)\in\mathcal{O}\times\mathbb{R}$ and continuous in $x\in\mathbb{R}$ uniformly with respect to ξ and u.

We take $M=\mathcal{C}(\bar{\mathcal{O}})$ as control set and fix $k,r>0$ such that $kr>d$, in which case the embedding $H^{k,r}(\mathcal{O})\hookrightarrow\mathcal{C}(\bar{\mathcal{O}})$ is compact. Hence, as seen in Example 4.12, the mapping

$$\eta:[0,T]\times M\to[0,+\infty]$$

defined as

$$\eta(t,u):=\begin{cases} a\big(t,c\,|u|_{H^{k,r}(\mathcal{O})}\big), & \text{if } u\in H^{k,r}(\mathcal{O})\\ +\infty, & \text{else}\end{cases}$$

satisfies

$$\eta(t,\cdot)\quad\text{is inf-compact, for each } t\in[0,T].$$

The constant $c>0$ in the definition of η is such that $|u|_{\mathcal{C}(\bar{\mathcal{O}})}\le c\,|u|_{H^{k,r}(\mathcal{O})}$, $u\in H^{k,r}(\mathcal{O})$. Finally, let

$$\phi:[0,T]\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}_+$$

be measurable and lower semi-continuous with respect to $x,u\in\mathbb{R}$, and define the running cost function

$$h(t,x,u):=\sum_{i=1}^{n}\phi(t,\zeta_i,x(\zeta_i),u(\zeta_i))+\eta(t,u)^\gamma,\quad t\in[0,T],\ x\in\mathcal{C}_0(\bar{\mathcal{O}}),\ u\in\mathcal{C}(\bar{\mathcal{O}})\tag{4.78}$$

where $\zeta_1,\dots,\zeta_n\in\mathcal{O}$ are fixed and $\gamma>2$ is chosen to satisfy condition (4.32). The Nemytskii operator $F : [0,T] \times B \times M \to B$ is now defined as

$$F(t,x,u)(\xi) := f(t,\xi,x(\xi),u(\xi)), \quad \xi \in \mathcal{O}, \ \ (t,x,u) \in [0,T] \times B \times M.$$

We see that Assumption A.3 is again satisfied since for each $(t,x) \in [0,T] \times B$ and $z^* \in \partial\,|x|_B$ as in (4.75), by condition (4.77), for all $y \in B$ and $u \in M$ we have

$$\begin{aligned} \langle F(t,x+y,u), z^* \rangle &= f\left(t,\xi_0,x(\xi_0)+y(\xi_0),u(\xi_0)\right) \operatorname{sgn} x(\xi_0) \\ &\le -k_1\,|x(\xi_0)| + k_2\,|y(\xi_0)|^m + a(t,|u(\xi_0)|) \\ &\le -k_1\,|x|_B + k_2\,|y|_B^m + \eta(t,u). \end{aligned}$$

Appendix A

A lemma on differentiation under the integral sign

Lemma A.1. *Let* $\mathbf{X}, \mathbf{Y}$ *be Banach spaces,* (T, μ) *a measurable space and* U *an open subset in* $\mathbf{X}$*. Let* $g : U \times T \to \mathbf{Y}$ *be a measurable function and assume that there exists* $T_1 \subset T$ *such that* $\mu(T \setminus T_1) = 0$ *and for all* $t \in T_1$,

$$U \ni x \mapsto g(x,t) \in \mathbf{Y}$$

is differentiable (resp. C^1*). We denote this derivative by* $\frac{\partial g}{\partial x} : U \times T \to \mathcal{L}(\mathbf{X}, \mathbf{Y})$*. Assume that, for every* $a \in U$*, the mapping*

$$T_1 \ni t \mapsto \frac{\partial g}{\partial x}(a,t) \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$$

is measurable and there exist $V_a \subset U$*, a neighborhood of* a*, and a measurable function* $k : T \to \mathbb{R}^+$ *such that* $\int_T k(t)\, \mu(dt) < \infty$ *and*

$$\left| \frac{\partial g}{\partial x}(x,t) \right| \le k(t), \quad \forall x \in V_a,\ t \in T_1.$$

Then $f : U \to \mathbf{Y}$ *defined by*

$$f(x) := \int_T g(x,t)\, \mu(dt), \quad x \in U$$

is differentiable (resp. C^1*) on* U *and*

$$\frac{df}{dx}(a) = \int_T \frac{\partial g}{\partial x}(a,t)\, \mu(dt), \quad \forall a \in U.$$

Fractional powers of positive operators. We briefly recall the definition of positive operator and a result on complex interpolation of domains of fractional powers of positive operators with bounded imaginary powers.

Definition A.2. Let A be a linear operator on a Banach space $\mathbf{E}$. We say that A is *positive* if it is closed, densely defined, $(-\infty, 0] \subset \rho(A)$ and there exists $C \geq 1$ such that

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(\mathbf{E})} \leq \frac{C}{1+\lambda}, \quad \text{for all } \lambda \geq 0.$$

It is well known that if A is a positive operator on $\mathbf{E}$, then A admits (not necessarily bounded) fractional powers A^z of any order $z \in \mathbb{C}$ (see e.g. [Ama95, Section 4.6]). In particular, for $|\mathrm{Re}\, z| \leq 1$, the fractional power A^z can be equivalently defined as the closure of the linear mapping

$$D(A) \ni x \mapsto \frac{\sin \pi z}{\pi z} \int_0^{+\infty} t^z (tI + A)^{-2} Ax \, dt \in \mathbf{E}, \tag{A.1}$$

see e.g. [Ama95, p. 153].

Definition A.3. The class $\mathrm{BIP}(\theta, \mathbf{E})$ of operators with *bounded imaginary powers* on $\mathbf{E}$ with parameter $\theta \in [0, \pi)$ is defined as the class of positive operators A on $\mathbf{E}$ with the property that $A^{is} \in \mathcal{L}(\mathbf{E})$ for all $s \in \mathbb{R}$ and there exists a constant $K > 0$ such that

$$\|A^{is}\|_{\mathcal{L}(\mathbf{E})} \leq K e^{\theta |s|}, \; s \in \mathbb{R}. \tag{A.2}$$

We denote $\mathrm{BIP}^{-}(\theta, \mathbf{E}) := \cup_{\sigma \in (0,\theta)} \mathrm{BIP}(\sigma, \mathbf{E})$. The proof of following well-known result can be found, for instance, in [Tri78, Theorem 1.15.3].

Theorem A.4. *Let $A \in \mathrm{BIP}(\sigma, \mathbf{E})$. Then for $0 \leq$ Re $\alpha <$ Re β we have*

$$[D(A^{\alpha}), D(A^{\beta})]_{\theta} = D(A^{(1-\theta)\alpha + \theta\beta}).$$

where $[\cdot, \cdot]_{\theta}$ denotes complex interpolation.

Bibliography

[Ahm83] N. U. Ahmed, *Properties of relaxed trajectories for a class of nonlinear evolution equations on a Banach space*, SIAM J. Control Optim. **21** (1983), no. 6, 953–967.

[Ama95] Herbert Amann, *Linear and quasilinear parabolic problems. Vol. I*, Monographs in Mathematics, vol. 89, Birkhäuser Boston Inc., Boston, MA, 1995, Abstract linear theory.

[AP90] Evgenios P. Avgerinos and Nikolaos S. Papageorgiou, *Optimal control and relaxation for a class of nonlinear distributed parameter systems*, Osaka J. Math. **27** (1990), no. 4, 745–767.

[Aro76] N. Aronszajn, *Differentiability of lipschitzian mappings between banach spaces*, Studia Mathematica **57** (1976), no. 2, 147–190.

[AT87] Paolo Acquistapace and Brunello Terreni, *A unified approach to abstract linear nonautonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova **78** (1987), 47–107.

[Bal00] Erik J. Balder, *Lectures on Young measure theory and its applications in economics*, Rend. Istit. Mat. Univ. Trieste **31** (2000), no. suppl. 1, 1–69, Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1997).

[Bal01] ______, *On ws-convergence of product measures*, Math. Oper. Res. **26** (2001), no. 3, 494–518.

[BCD97] Martino Bardi and Italo Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1997, With appendices by Maurizio Falcone and Pierpaolo Soravia.

[BG99] Zdzisław Brzeźniak and Dariusz Gątarek, *Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces*, Stochastic Process. Appl. **84** (1999), no. 2, 187–225.

[Bog98] Vladimir I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998.

[Bou83] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. **21** (1983), no. 2, 163–168.

[Brz97] Zdzisław Brzeźniak, *On stochastic convolution in Banach spaces and applications*, Stochastics Stochastics Rep. **61** (1997), no. 3-4.

[Brz03] ______, *Some remarks on Itô and Stratonovich integration in 2-smooth Banach spaces*, Probabilistic methods in fluids, World Sci. Publ., River Edge, NJ, 2003, pp. 48–69.

[Bur81] D. L. Burkholder, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probab. **9** (1981), no. 6, 997–1011.

[Bur83] ______, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 270–286.

[Cer99] Sandra Cerrai, *Differentiability of Markov semigroups for stochastic reaction-diffusion equations and applications to control*, Stochastic Process. Appl. **83** (1999), no. 1, 15–37.

[CG05] Nigel J. Cutland and Katarzyna Grzesiak, *Optimal control for 3D stochastic Navier-Stokes equations*, Stochastics **77** (2005), no. 5, 437–454.

[CG07] ______, *Optimal control for two-dimensional stochastic Navier-Stokes equations*, Appl. Math. Optim. **55** (2007), no. 1, 61–91.

[Cra02] Hans Crauel, *Random probability measures on Polish spaces*, Stochastics Monographs, vol. 11, Taylor & Francis, London, 2002.

[CRdF04] Charles Castaing and Paul Raynaud de Fitte, *On the fiber product of Young measures with application to a control problem with measures*, Advances in mathematical economics. Vol. 6, Adv. Math. Econ., vol. 6, Springer, Tokyo, 2004, pp. 1–38.

[CRdFV04] Charles Castaing, Paul Raynaud de Fitte, and Michel Valadier, *Young measures on topological spaces (with applications in control theory and probability theory)*, Mathematics and its Applications, vol. 571, Kluwer Academic Publishers, Dordrecht, 2004.

[DDH+04] Robert Denk, Giovanni Dore, Matthias Hieber, Jan Prüss, and Alberto Venni, *New thoughts on old results of R. T. Seeley*, Math. Ann. **328** (2004), no. 4, 545–583.

[Det90] Egbert Dettweiler, *Representation of Banach space valued martingales as stochastic integrals*, Probability in Banach spaces, 7 (Oberwolfach, 1988), Progr. Probab., vol. 21, Birkhäuser Boston, Boston, MA, 1990, pp. 43–62.

[DPKZ87] G. Da Prato, S. Kwapień, and J. Zabczyk, *Regularity of solutions of linear stochastic equations in Hilbert spaces*, Stochastics **23** (1987), no. 1, 1–23.

[DPZ92a] Giuseppe Da Prato and Jerzy Zabczyk, *Nonexplosion, boundedness, and ergodicity for stochastic semilinear equations*, J. Differential Equations **98** (1992), no. 1, 181–195.

[DPZ92b] ———, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.

[DPZ02] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.

[DU77] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[DV87] Giovanni Dore and Alberto Venni, *On the closedness of the sum of two closed operators*, Math. Z. **196** (1987), no. 2, 189–201.

[EKHNJP87] Nicole El Karoui, Du' Hůů Nguyen, and Monique Jeanblanc-Picqué, *Compactification methods in the control of degenerate diffusions: existence of an optimal control*, Stochastics **20** (1987), no. 3, 169–219.

[Fat94a] H. O. Fattorini, *Existence theory and the maximum principle for relaxed infinite-dimensional optimal control problems*, SIAM J. Control Optim. **32** (1994), no. 2, 311–331.

[Fat94b] ______, *Relaxation theorems, differential inclusions, and Filippov's theorem for relaxed controls in semilinear infinite-dimensional systems*, J. Differential Equations **112** (1994), no. 1, 131–153.

[Fat99] ______, *Infinite-dimensional optimization and control theory*, Encyclopedia of Mathematics and its Applications, vol. 62, Cambridge University Press, Cambridge, 1999.

[Fle80] Wendell H. Fleming, *Measure-valued processes in the control of partially-observable stochastic systems*, Appl. Math. Optim. **6** (1980), no. 3, 271–285.

[FN84] W. H. Fleming and M. Nisio, *On stochastic relaxed control for partially observed diffusions*, Nagoya Math. J. **93** (1984), 71–108.

[FS06] Wendell H. Fleming and H. Mete Soner, *Controlled Markov processes and viscosity solutions*, second ed., Stochastic Modelling and Applied Probability, vol. 25, Springer, New York, 2006.

[Goz96] Fausto Gozzi, *Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities*, J. Math. Anal. Appl. **198** (1996), no. 2, 399–443.

[Gro67] Leonard Gross, *Potential theory on Hilbert space*, J. Functional Analysis **1** (1967), 123–181.

[GS91] Yoshikazu Giga and Hermann Sohr, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal. **102** (1991), no. 1, 72–94.

[GS94] Dariusz Gątarek and Jarosław Sobczyk, *On the existence of optimal controls of Hilbert space-valued diffusions*, SIAM J. Control Optim. **32** (1994), no. 1, 170–175.

[HL90] U. G. Haussmann and J.-P. Lepeltier, *On the existence of optimal controls*, SIAM J. Control Optim. **28** (1990), no. 4, 851–902.

[JM81] Jean Jacod and Jean Mémin, *Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité*, Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French), Lecture Notes in Math., vol. 850, Springer, Berlin, 1981, pp. 529–546.

[Kal02] Olav Kallenberg, *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002.

[Lou03] Hongwei Lou, *Existence of optimal controls for semilinear parabolic equations without Cesari-type conditions*, Appl. Math. Optim. **47** (2003), no. 2, 121–142.

[Lou07] ———, *Existence and nonexistence results of an optimal control problem by using relaxed control*, SIAM J. Control Optim. **46** (2007), no. 6, 1923–1941.

[Mas05] Federica Masiero, *Semilinear Kolmogorov equations and applications to stochastic optimal control*, Appl. Math. Optim. **51** (2005), no. 2, 201–250.

[Mas08a] ———, *Stochastic optimal control for the stochastic heat equation with exponentially growing coefficients and with control and noise on a subdomain*, Stoch. Anal. Appl. **26** (2008), no. 4, 877–902.

[Mas08b] ———, *Stochastic optimal control problems and parabolic equations in Banach spaces*, SIAM J. Control Optim. **47** (2008), no. 1, 251–300.

[Meg98] Robert E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York, 1998.

[Nei78] A.L. Neidhardt, *Stochastic integrals in 2-uniformly smooth banach spaces*, Ph.D. thesis, University of Wisconsin, 1978.

[NN90] Noriaki Nagase and Makiko Nisio, *Optimal controls for stochastic partial differential equations*, SIAM J. Control Optim. **28** (1990), no. 1, 186–213.

[Ond05] Martin Ondreját, *Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces*, Czechoslovak Math. J. **55(130)** (2005), no. 4, 1003–1039.

[Pap89a] Nikolaos S. Papageorgiou, *Existence of optimal controls for nonlinear distributed parameter systems*, Funkcial. Ekvac. **32** (1989), no. 3, 429–437.

[Pap89b] ———, *Relaxation of infinite-dimensional control systems*, Nonlinear Anal. **13** (1989), no. 11, 1283–1293.

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[PS90] Jan Prüss and Hermann Sohr, *On operators with bounded imaginary powers in Banach spaces*, Math. Z. **203** (1990), no. 3, 429–452.

[PS93] ———, *Imaginary powers of elliptic second order differential operators in L^p-spaces*, Hiroshima Math. J. **23** (1993), no. 1, 161–192.

[RdF03] Paul Raynaud de Fitte, *Compactness criteria for the stable topology*, Bull. Polish Acad. Sci. Math. **51** (2003), no. 4, 343–363.

[Sch73] Laurent Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973, Tata Institute of Fundamental Research Studies in Mathematics, No. 6.

[Sch75] Manfred Schäl, *On dynamic programming: compactness of the space of policies*, Stochastic Processes Appl. **3** (1975), no. 4, 345–364.

[See71] Robert Seeley, *Norms and domains of the complex powers A_B^z*, Amer. J. Math. **93** (1971), 299–309.

[Sho97] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.

[Sri00] S. S. Sritharan, *Deterministic and stochastic control of Navier-Stokes equation with linear, monotone, and hyperviscosities*, Appl. Math. Optim. **41** (2000), no. 2, 255–308.

[Tan79] Hiroki Tanabe, *Equations of evolution*, Monographs and Studies in Mathematics, vol. 6, Pitman (Advanced Publishing Program), Boston, Mass., 1979, Translated from the Japanese by N. Mugibayashi and H. Haneda.

[Tem01] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1984 edition.

[Tri78] Hans Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam, 1978.

[Val73] Michel Valadier, *Désintégration d'une mesure sur un produit*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A33–A35.

[Val90] ______, *Young measures*, Methods of nonconvex analysis (Varenna, 1989), Lecture Notes in Math., vol. 1446, Springer, Berlin, 1990, pp. 152–188.

[Val94] ______, *A course on Young measures*, Rend. Istit. Mat. Univ. Trieste **26** (1994), no. suppl., 349–394, Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1993).

[vN98] J. M. A. M. van Neerven, *Nonsymmetric Ornstein-Uhlenbeck semigroups in Banach spaces*, J. Funct. Anal. **155** (1998), no. 2, 495–535.

[vN01] ______, *Uniqueness of invariant measures for the stochastic Cauchy problem in Banach spaces*, Recent advances in operator theory and related topics (Szeged, 1999), Oper. Theory Adv. Appl., vol. 127, Birkhäuser, Basel, 2001, pp. 491–517.

[vN08] ______, *Stochastic evolution equations*, ISEM Lecture Notes, 2008.

[vNV06] Jan van Neerven and Mark C. Veraar, *On the stochastic Fubini theorem in infinite dimensions*, Stochastic partial differential equations and applications—VII, Lect. Notes Pure Appl. Math., vol. 245, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 323–336.

[vNVW08] J. M. A. M. van Neerven, M. C. Veraar, and L. Weis, *Stochastic evolution equations in UMD Banach spaces*, J. Funct. Anal. **255** (2008), no. 4, 940–993.

[vNW05a] J. M. A. M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166** (2005), no. 2, 131–170.

[vNW05b] ______, *Weak limits and integrals of Gaussian covariances in Banach spaces*, Probab. Math. Statist. **25** (2005), no. 1, Acta Univ. Wratislav. No. 2784.

[War62] J. Warga, *Relaxed variational problems*, J. Math. Anal. Appl. **4** (1962), 111–128.

[War72] ______, *Optimal control of differential and functional equations*, Academic Press, New York, 1972.

[Yag91] Atsushi Yagi, *Abstract quasilinear evolution equations of parabolic type in Banach spaces*, Boll. Un. Mat. Ital. B (7) **5** (1991), no. 2, 341–368. MR MR1111127 (92h:47100)

[You42] L. C. Young, *Generalized surfaces in the calculus of variations*, Ann. of Math. (2) **43** (1942), 84–103.

[You69] ______, *Lectures on the calculus of variations and optimal control theory*, W. B. Saunders Co., Philadelphia, 1969.

[Zab99] J. Zabczyk, *Parabolic equations on Hilbert spaces*, Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998), Lecture Notes in Math., vol. 1715, Springer, Berlin, 1999, pp. 117–213.

[Zho92] Xun Yu Zhou, *On the existence of optimal relaxed controls of stochastic partial differential equations*, SIAM J. Control Optim. **30** (1992), no. 2, 247–261.

[Zho93] ______, *A class of semilinear stochastic partial differential equations and their controls: existence results*, Stochastic Process. Appl. **44** (1993), no. 1, 89–106.