Sensor fault detection and isolation for nonlinear systems based on a sliding mode observer

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Abstract: In this paper, a sensor fault detection and isolation scheme for nonlinear systems is considered. A nonlinear diffeomorphism is introduced to explore the system structure and a simple filter is presented to 'transform' the sensor fault into a pseudo-actuator fault scenario. A sliding mode observer is designed to reconstruct the sensor fault precisely if the system does not experience any uncertainty, and to estimate the sensor fault when uncertainty exists. The reconstruction and estimation signals are based only on available information and thus can be implemented on-line. Finally a mass-spring system is used to illustrate the approach.

1 Introduction

Faults are classified, according to their physical locations, into system faults, actuator faults and sensor faults. In the past few decades, the study of fault detection and isolation (FDI) has made many significant advances [8, 5, 23, 4, 15]. Compared with actuators, sensors are passive elements in the sense that they only provide operational information about the system, and do not affect the system behaviour directly, and thus have been less studied when compared with the study of actuator FDI. With the development of modern technology, however, more and more automation systems have been employed in industrial processes and daily life. Obviously, autonomous systems, where the human operator is frequently removed from the loop, are more dependent on the increasing numbers of sensors to acquire system information. This, in turn, makes systems more vulnerable to faults in sensors. The potential for faults in the sensors becomes even more critical when they are applied to the automatic control of a system, where the effects of malfunctions may be devastating [7].

There have been many approaches developed for FDI. Among them, observer-based schemes are particularly effective and have been widely studied. Some control approaches – for instance, sliding mode techniques [4], modern differential geometric approaches [14] and adaptive control [23] have been successfully incorporated within an observer-based FDI framework. Systems with parametric uncertainty [23] or unknown inputs [19] have also been considered. Usually adaptive schemes are only powerful for overcoming linear parametric uncertainty whilst modern geometric approaches require the system to satisfy strong geometric conditions with strong limitations on the structure of the uncertainty. Sliding mode techniques however have good robustness and are completely insensitive to so-called matched uncertainty [3, 18]. Furthermore, it has been shown that sliding mode techniques can be used to deal with structural uncertainty [21]. The features of the sliding mode technique make it possible to reconstruct the faults precisely in certain situations [4, 20]. Therefore the application of sliding mode ideas to dynamic systems offers good potential in the field of FDI. In most of the work mentioned above, the focus is on actuator FDI. This paper will consider sensor fault FDI.

Sensor faults are incorrect readings due to malfunctions in the sensor components or transducers, such as broken wires, resulting in the loss of effectiveness, or more subtly, unknown biases at the sensor outputs as a result of poor calibration or even unexpected changes in the dynamic characteristics of the transducers. Since the signals from sensors often carry the most important information in automated/feedback control systems, the state of health of the sensors is therefore very important for the reliable operation of the entire system. This has motivated the study of sensor FDI. Sensor redundancy [1] is an obvious solution, where multiple sensors are installed to measure the same quantity. The main problem of this approach is the extra equipment and maintenance costs and the additional space required to accommodate the equipment. In [22], an isolation scheme for sensor faults is proposed using an adaptive estimator. A sensor fault FDI strategy for a linear discrete time system was discussed in [11] using a structural vector-based approach. By using sliding mode techniques, continuous time systems were considered in [4, 16, 17] where it is required that the systems are linear. However, most real systems are more accurately modelled by nonlinear equations. It is well known that one approach for dealing with nonlinear systems is to linearize around some operating point by using approximation techniques [12, 2]. However, the linear system obtained in this way is valid only in a neighbourhood of the operating point and tends to suffer from poor detection or high false alarm rates due to the error of approximation. Furthermore, when a large region of the state space is required to be considered, the linearization method may not be sufficient. Therefore it is necessary to study nonlinear systems.

In this paper, sensor FDI is studied for a class of nonlinear systems with uncertainty. The sensor fault considered in this paper is modelled as an additive fault. A diffeomorphism is first used to explore the system structure and no approximation is employed. By designing an appropriate filter, the sensor fault can be modelled as a pseudo-actuator fault. Then, using the transformed system structure and the characteristics of the designed filter, a sliding mode observer is presented to reconstruct the sensor fault precisely if no uncertainty exists in the system. A sensor fault estimation scheme is also proposed when the system is affected by uncertainty, in which case the estimation error depends on the bound on the uncertainty. The reconstruction/estimation schemes which are proposed can be implemented online. It is not required that the system is linear/linearizable, and the minimum phase limitation required in [4, 20] is removed. Therefore, this work is applicable to a wide-class of systems. Finally a simulation example is introduced to show the effectiveness of the approach.

Notation: For a square symmetric matrix A, $\lambda_{min}(A)$ ($\lambda_{\max}(A)$) denotes the minimum (maximum) eigenvalue, and $A > 0$ represents a symmetric positive definite matrix. For a matrix $A > 0$, $A^{\frac{1}{2}}$ denotes a symmetric positive definite matrix such that $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$. I_n represents an nth order unit matrix. $\mathcal{R}^+$ represents the set of nonnegative real numbers. For a smooth vector field $f(x,u) : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^n$ and a mapping $h(x) : \mathcal{R}^n \mapsto \mathcal{R}^p$, the symbol $L_f h$ denotes the Lie derivative of $h(\cdot)$ along the vector field f defined by $L_{f(x,u)}h(x) := \frac{\partial h}{\partial x} f(x,u)$. More generally $L_f^r h$ denotes the rth order Lie derivative. Finally, $\| \cdot \|$ denotes the Euclidean norm or its induced norm.

2 System Description and Analysis

Consider a nonlinear system described by

$$\dot{x}(t) = F(x(t), u(t)) + \Delta F(x(t)) \tag{1}$$

$$y(t) = h(x(t)) + Df_s(t), \qquad x_0 = x(0) \tag{2}$$

where $x \in \Omega \subset \mathcal{R}^n$ (and Ω is a neighbourhood of x_0), $u = \text{col}(u_1, u_2, \ldots, u_m) \in \mathcal{U} \in \mathcal{R}^m$, and $y = \text{col}(y_1, y_2, \ldots, y_p) \in \mathcal{R}^p$ are the state variables, inputs and outputs respectively where $\mathcal{U}$ is an admissible control set. $F(x, u)$ is a known smooth vector field in $\Omega \times \mathcal{U}$ and the known function $h : \Omega \mapsto \mathcal{R}^p$ is smooth; $D \in \mathcal{R}^{p\times q}$ $(q \leq p)$ is a known sensor fault distribution matrix which is full column rank; the unknown vector function $\Delta F(x(t))$ models all the uncertainties and disturbances affecting the system and $f_s(t) \in \mathcal{R}^q$ is a sensor fault satisfying

$$\|f_s(t)\| \leq \rho(t) \tag{3}$$

where $\rho(t)$ is a known continuous function. It is assumed that f_s is unknown and $f_s(t) = 0$ when there is no fault. Therefore the function $f_s(\cdot)$ is defined in $t \in \mathcal{R}^+$.

In this work, the fact that $\mathcal{U}$ is an admissible control set means that for any $u(t) \in \mathcal{U}$, the corresponding closed-loop system (1) has an unique solution lying in Ω.

Definition 1. *Consider system (1)–(2). The differential and algebraic equations*

$$\dot{x}(t) = F(x(t), u(t)) \tag{4}$$

$$y(t) = h(x(t)), \qquad x_0 = x(0) \tag{5}$$

are called the nominal system associated with (1)–(2).

For convenience, the nominal system (4)–(5) is also denoted by a pair $(F(x, u), h(x))$.

Definition 2. ([6]) *System (4)–(5) is said to be observable at* $(x_0, u_0) \in \Omega \times \mathcal{U}$ *if there exists a neighbourhood* $\mathcal{N}$ *of* (x_0, u_0) *in* $\Omega \times \mathcal{U}$ *and a set of nonnegative integer numbers* $\{r_1, r_2, \cdots, r_p\}$ *with* $\sum_{i=1}^{p} r_i = n$ *such that*

1) *for all* $(x, u) \in \mathcal{N}$
$$\frac{\partial}{\partial u_j} L^k_{F(x,u)} h_i(x) = 0 \tag{6}$$
for indices $i = 1, 2, \ldots, p$, $k = 0, 1, 2, \ldots, r_i - 1$ *and* $j = 1, 2, \ldots, m$;

2) *the* $p \times m$ *matrix* $M(x, u) := \{\frac{\partial}{\partial u_j} L^{r_i}_{F(x,u)} h_i(x)\}$ *has rank* p *in* (x_0, u_0)

Then, $\{r_1, r_2, \cdots, r_p\}$ *is called the observability index of system (4)–(5) at* (x_0, u_0). *Further, system (4)–(5) is said to be uniformly observable in* $\Omega \times \mathcal{U}$ *if for any* $(x_0, u_0) \in \Omega \times \mathcal{U}$, *the system is observable and the observability indices are fixed.*

This paper considers the problem of reconstructing (or estimating) the sensor faults $f_s(t)$ for system (1)–(2). A sliding mode observer will be established and then, based on the observer, a signal $\hat{f}$, which only depends on available information, will be given such that

i) $\hat{f}_s$ is a precise reconstruction of the sensor fault $f_s(t)$, i.e. $\lim_{t\to\infty} \|\hat{f}_s(t) - f_s(t)\| = 0$ if there is no uncertainty;

ii) $\|\hat{f}_s(t) - f_s(t)\| \leq \xi(t)$ if the system experiences some uncertainty, where $\xi(t)$ is the estimation error which usually depends on the bound on the uncertainty.

Assumption 1. The pair $(F(x, u), h(x))$ has an uniform observability index $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in the domain $\Omega \times \mathcal{U}$.

Construct a nonlinear transformation $T : x \mapsto z$ as follows:

$$\begin{aligned} z_{i1} &= h_i(x) && (7)\\ z_{i2} &= L_{F(x,u)} h_i(x) && (8)\\ &\vdots \\ z_{ir_i} &= L_{F(x,u)}^{r_i-1} h_i(x) && (9) \end{aligned}$$

where $z_i := \text{col}(z_{i1}, z_{i2}, \cdots, z_{ir_i})$ for $i = 1, 2, \ldots, p$ and $z := \text{col}(z_1, z_2, \cdots, z_p)$.

Remark 1. Under Assumption 1, it follows from Definition 2 that $M(x, u)$ has rank p in $\Omega \times \mathcal{U}$, implying all the z_i are independent of the control u, which combined with the restriction $\sum_{i=1}^{p} r_i = n$ means the corresponding Jacobian matrix of $T(x)$ is nonsingular. Therefore, (7)–(9) is a diffeomorphism in the domain Ω, and $z = \text{col}(z_1, z_2, \ldots, z_p)$ forms a new coordinate system which can be obtained by direct computation from (7)–(9).

Since $L_{F(x,u)}^{j} h_i(x)$ is independent of u for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r_i - 1$, it follows by direct computation that for $i = 1, 2, \ldots, p$

$$\begin{aligned} \dot{z}_{i1} &= \tfrac{\partial h_i}{\partial x} F(x,u) = L_{F(x,u)} h_i(x) = z_{i2}\\ \dot{z}_{i2} &= \frac{\partial\big(L_{F(x,u)} h_i(x)\big)}{\partial x} F(x,u) = L_{F(x,u)}^{2} h_i(x) = z_{i3}\\ &\vdots \\ \dot{z}_{ir_{i-1}} &= L_{F(x,u)}^{r_i-1} h_i(x) = z_{ir_i}\\ \dot{z}_{ir_i} &= L_{F(x,u)}^{r_i} h_i(x) \end{aligned}$$

Therefore, in the new coordinates z defined by (7)–(9), the system (1)–(2) has the following form

$$\begin{aligned} \dot{z} &= Az + B\Phi(z,u) + \Psi(z) && (10)\\ y &= Cz + Df_s(t) && (11) \end{aligned}$$

where $A = \text{diag}\{A_1, \ldots, A_p\}$, $B = \text{diag}\{B_1, \ldots,\ B_p\}$ and $C = \text{diag}\{C_1, \ldots, C_p\}$ where $A_i \in \mathcal{R}^{r_i \times r_i}$, $B_i \in \mathcal{R}^{r_i \times 1}$ and $C_i \in \mathcal{R}^{1 \times r_i}$ for $i = 1, 2, \ldots, p$ are defined by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{12}$$

and

$$\Phi(z,u) := \begin{bmatrix} \phi_1(z,u)\\ \phi_2(z,u)\\ \vdots\\ \phi_p(z,u) \end{bmatrix} := \begin{bmatrix} L_{F(x,u)}^{r_1} h_1(x)\\ L_{F(x,u)}^{r_2} h_2(x)\\ \vdots\\ L_{F(x,u)}^{r_p} h_p(x) \end{bmatrix}_{x=T^{-1}(z)} \tag{13}$$

$$\Psi(z) := \begin{bmatrix} \psi_1(z)\\ \psi_2(z)\\ \vdots\\ \psi_p(z) \end{bmatrix} := \left[\frac{\partial T(x)}{\partial x} \Delta F(x)\right]_{x=T^{-1}(z)} \tag{14}$$

where $\phi_i : T(\Omega) \times \mathcal{U} \mapsto \mathcal{R}$ and $\psi_i : T(\Omega) \times \mathcal{U} \mapsto \mathcal{R}^{r_i}$ for $i = 1, 2, \ldots, p$.

Remark 2. It should be noted that system (10)–(11) is still a nonlinear system but possesses a structure which is convenient for the later analysis. In this paper it is not required that the system (1)–(2) is linearizable. It is also not required that the nonlinear function $\Phi(z,u)$ can be expressed as a function of u and y (in comparison with the work in [13]). Also there is no approximation employed above and this makes the transformations valid in the whole domain Ω instead of just a small neighbourhood of x_0 as in [2, 12].

Choose the constants $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{ir_i}$ such that all the roots of the polynomial algebraic equations

$$\lambda^{r_i} + \alpha_{i(r_i-1)}\lambda^{r_i-1} + \cdots + \alpha_{i1}\lambda + \alpha_{i0} = 0 \tag{15}$$

lie in the left-half plane for $i = 1, 2, \ldots, p$. Then, from (12), it follows that $(A - B\Lambda)$ is stable where

$$\Lambda = \text{diag}\{\Lambda_1, \Lambda_2, \cdots, \Lambda_p\} \tag{16}$$

with $\Lambda_i \in \mathcal{R}^{1\times r_i}$ defined by

$$\Lambda_i = [\alpha_{i0}\ \alpha_{i1} \cdots\ \alpha_{i(r_i-1)}] \tag{17}$$

which satisfy (15) for $i = 1, 2, \ldots, p$.

Assumption 2. The nonlinear function $\Phi(\cdot)$ in (13) can be expressed as

$$\Phi(z,u) = -\Lambda z + \Gamma(z,u) \tag{18}$$

where for any z, $\hat{z} \in T(\Omega)$ and $u \in \mathcal{U}$

$$\|\Gamma(z,u) - \Gamma(\hat{z},u)\| \le \mathcal{L}(u)\|z - \hat{z}\| \tag{19}$$

where $\mathcal{L}(\cdot)$ is a continuous function defined on $\mathcal{U}$.

Remark 3. Assumption 2 is a limitation on the nonlinear term $\Phi(\cdot)$. If the Jacobian matrix of $F(x,u)$ in (1), evaluated at (x_0,u_0) $(u_0 \in \mathcal{U})$ is stable, then Assumption 2 is likely to be satisfied in a neighbourhood of (x_0,u_0).

3 Main Results

In this section, the main results will be presented. The special case when $\Delta F(x,u) = 0$ is considered first, and the study of the uncertain system (1)–(2) when $\Delta F(x,u) \neq 0$ follows.

It is assumed Assumption 1 is true. Then, from the analysis in Section 2, it follows that in the new coordinates z defined by the diffeomorphism (7)–(9), system (1)–(2) can be described by (10)–(11). For system (10)–(11), the following linear filter is introduced

$$\dot{z}_a = A_a z_a + B_a y \tag{20}$$

where $z_a \in \mathcal{R}^p$ is the filter state, $A_a \in \mathcal{R}^{p\times p}$ and $B_a \in \mathcal{R}^{p\times p}$ are constant matrices which are design parameters to be defined later; y is the output of system (10)–(11). The matrix A_a must be Hurwitz, but for simplicity in the subsequent analysis it will be assumed that A_a is symmetric negative definite. This is not a stringent assumption since A_a is a design parameter. Then, under Assumption 2, the following augmented system can be obtained

$$\dot{z} = (A - B\Lambda)z + B\Gamma(z,u) + \Psi(z) \tag{21}$$

$$\dot{z}_a = B_a C z + A_a z_a + B_a D f_s(t) \tag{22}$$

$$y_a = C_a z_a \tag{23}$$

where $z \in T(\Omega) \subset \mathcal{R}^n$, $C_a \in \mathcal{R}^{p\times p}$ is orthogonal (where one simple choice is to let $C_a = I_p$), $\Gamma(\cdot)$ is determined by (18) and finally $\Psi(\cdot)$ is defined in (14) and involves the uncertainty.

It is observed that the sensor fault in system (1)–(2) has been transformed into a pseudo-actuator fault in system (21)–(23). Now, consider the following dynamical system

$$\dot{\hat{z}} = (A - B\Lambda)\hat{z} + B\Gamma(\hat{z}, u) \tag{24}$$
$$\dot{\hat{z}}_a = B_a C\hat{z} + A_a\hat{z}_a + \nu(t, y_a, \hat{y}_a) \tag{25}$$
$$\hat{y}_a = C_a\hat{z}_a \tag{26}$$

where

$$\nu := k(t)C_a^T \frac{y_a - \hat{y}_a}{\|y_a - \hat{y}_a\|} \tag{27}$$

and the scalar gain $k(t)$ is to be designed later.

Let $e(t) := z(t) - \hat{z}(t)$ and $e_a(t) := z_a(t) - \hat{z}_a(t)$. It follows from (21)–(23) and (24)–(26) that the error dynamics can be described by

$$\dot{e} = (A - B\Lambda)e + B\left(\Gamma(z, u) - \Gamma(\hat{z}, u)\right) + \Psi(z) \tag{28}$$
$$\dot{e}_a = B_a Ce + A_a e_a + B_a Df_s(t) - \nu(t, y_a, \hat{y}_a) \tag{29}$$

where $\Gamma(\cdot)$ is determined by (18), $\Psi(\cdot)$ is the uncertain term which is defined by (14) and $\nu(\cdot)$ is given by (27).

3.1 The nominal case

In this section, the special case $\Delta F \equiv 0$ is considered, which implies that the system under consideration does not experience any uncertainty. In this case, the corresponding augmented system is the same as (21)–(23) except $\Psi(\cdot) \equiv 0$ in (21), and thus the corresponding dynamical error system (28)–(29) is described by

$$\dot{e} = (A - B\Lambda)e + B\left(\Gamma(z, u) - \Gamma(\hat{z}, u)\right) \tag{30}$$
$$\dot{e}_a = B_a Ce + A_a e_a + B_a Df_s(t) - \nu(t, y_a, \hat{y}_a) \tag{31}$$

The objective now is to develop a condition under which (24)–(26) is a sliding mode observer of the system (21)–(23) with $\Psi(\cdot) \equiv 0$ in (21), and can be employed to reconstruct the fault signal $f_s(t)$.

From the analysis above, the following conclusion is ready to be presented:

Proposition 1 Suppose Assumption 2 holds. Then, system (30) is stable if there exists a matrix $P > 0$ such that

$$(A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 PBB^T P + \frac{1}{\varepsilon_1}(\mathcal{L}(u))^2 I_n < 0 \tag{32}$$

for all $u \in \mathcal{U}$ where ε_1 is a positive constant, Λ is defined by (16) and $\mathcal{L}(u)$ satisfies (19).

Proof: For system (30), consider a Lyapunov function candidate $V = e^T(t)Pe(t)$ where $P > 0$ is a solution for the matrix inequality (32). The time derivative of V along the trajectories of system (30) is given by

$$\dot{V}\mid_{(30)} \leq e^T(t)\left((A - B\Lambda)^T P + P(A - B\Lambda)\right)e(t) + 2e(t)^T PB\left(\Gamma(z, u) - \Gamma(\hat{z}, u)\right) \tag{33}$$

From the fact $2X^TY \le \varepsilon_1 X^TX + \frac{1}{\varepsilon_1}Y^TY$, it follows that

$$
\begin{aligned}
\dot{V}\mid_{(30)} &\le e^T(t)\left((A-B\Lambda)^TP + P(A-B\Lambda)\right)e(t) + \varepsilon_1(B^TPe(t))^TB^TPe(t)\\
&\quad + \frac{1}{\varepsilon_1}\left(\Gamma(z,u)-\Gamma(\hat{z},u)\right)^T\left(\Gamma(z,u)-\Gamma(\hat{z},u)\right)\\
&\le e^T(t)\left((A-B\Lambda)^TP + P(A-B\Lambda)\right)e(t) + \varepsilon_1 e^T(t)PBB^TPe(t) + \frac{1}{\varepsilon_1}(\mathcal{L}(u))^2\|z-\hat{z}\|^2\\
&= e^T\Big((A-B\Lambda)^TP + P(A-B\Lambda) + \varepsilon_1 PBB^TP + \frac{1}{\varepsilon_1}(\mathcal{L}(u))^2 I_n\Big)e
\end{aligned}
$$

where (19) is used to establish the 2nd inequality. Hence the conclusion follows from (32). △

It should be noted that:

- Proposition 1 implies that $e(t)$ is bounded, and thus

$$\sup_{0\le t<\infty}\{\|e(t)\|\} \le b \tag{34}$$

  for some finite positive scalar b;

- because of the scalar ε_1 in (32) which provides additional design freedom, without loss of generality it can be assumed that $P > I_n$ rather than just being positive definite.

Sliding mode design is typically composed of two stages: The first step is the establishment of the sliding surface such that the system under consideration has the desired performance when constrained to move on the sliding surface. The second is the development of a sliding mode controller/observer gain such that the system can be driven to the sliding surface and a sliding motion maintained thereafter. The subsequent study follows this procedure.

Consider a sliding surface

$$\mathcal{S} = \{\mathrm{col}(e, e_a) \mid e_a = 0\} \tag{35}$$

Proposition 1 implies that the sliding mode dynamics of the error system (30)–(31) associated with the sliding surface (35) is stable. According to sliding mode theory, in order to guarantee the stability of the observer it is only required to prove that the error system can be driven to the sliding surface in finite time by choosing an appropriate gain $k(t)$ in (27). In view of this, the following conclusion is presented:

Proposition 2. If inequality (34) holds, then the error system (30)–(31) is driven to the sliding surface (35) if $k(\cdot)$ in (27) is chosen to satisfy

$$k(t) \ge \|B_aC\|b + \|B_aD\|\rho(t) + \eta \tag{36}$$

where η is a positive constant.

Proof: From equation (31), it follows that

$$e_a^T\dot{e}_a = e_a^TB_aCe + e_a^TA_ae_a + e_a^TB_aDf_s(t) - e_a^T\nu(t, y_a, \hat{y}_a)$$

Since $A_a < 0$ it follows that $e_a^TA_ae_a \le 0$. Since C_a is orthogonal,

$$\|y_a - \hat{y}_a\| = \sqrt{(C_ae_a)^TC_ae_a} = \|e_a\| \tag{37}$$

Then, from (34), (27) and (3)

$$
\begin{aligned}
e_a^T \dot{e}_a &\le e_a^T B_a C e + e_a^T B_a D f_s(t) - k(t) e_a^T C_a^T \frac{y_a - \hat{y}_a}{\|y_a - \hat{y}_a\|} \\
&\le \|e_a\| \, \|B_a C\| b + \|e_a\| \, \|B_a D\| \rho(t) - k(t) (C_a e_a)^T \frac{C_a e_a}{\|e_a\|} \\
&= \left(\|B_a C\| b + \|B_a D\| \rho(t) - k(t) \right) \|e_a\| \qquad (38)
\end{aligned}
$$

where (37) is used to obtain the 2nd inequality. Then, it follows from (38) and (36) that

$$
e_a^T \dot{e}_a \le -\eta \|e_a\|
$$

This means that the reachability condition is satisfied [18], and a sliding motion on S is attained in finite time. △

Remark 4. It should be stressed that the dynamics of the error system $e(t)$ in (28), which represents the reduced order sliding motion associated with the sliding surface (35), must be stable so that the term $B_a C e$ in equation (29) vanishes with time. This is very important and makes it possible to reconstruct/estimate the sensor fault.

Remark 5. It is tempting from (36) to select $B_a = 0$. However, this is not possible since if $B_a = 0$, it follows from (22) that the sensor fault term will also disappear and thus it cannot be reconstructed.

From sliding mode theory, Propositions 1 and 2 have shown that (24)–(26) is an observer of system (21)–(23) when $\Psi(\cdot) \equiv 0$ in (21). The objective is now to establish a reconstruction signal for the sensor fault $f_s(t)$ based on the sliding mode observer (24)–(26).

Since the fault distribution matrix D is assumed to be full column rank, there exists a nonsingular matrix $N \in \mathcal{R}^{p \times p}$ such that

$$
ND = \begin{bmatrix} 0_{(p-q) \times q} \\ D_1 \end{bmatrix} \qquad (39)
$$

where $D_1 \in \mathcal{R}^{q \times q}$ is nonsingular. The matrix N can be obtained from QR decomposition. Then, from the analysis above, it follows that a sliding motion takes place in finite time and during the sliding motion

$$
e_a = 0 \qquad \text{and} \qquad \dot{e}_a = 0
$$

and thus from (31)

$$
B_a C e + B_a D f_s(t) - \nu_{eq} = 0 \qquad (40)
$$

where ν_{eq} is the equivalent output error injection which plays the same role as the equivalent control in sliding mode control [3, 18]. The equivalent output injection signal represents the average behaviour of the discontinuous function ν defined by (27), which is necessary to maintain an ideal sliding motion.

In order to reconstruct the sensor fault, the design parameter B_a in filter (20) is chosen as $B_a = N$ where N is given by (39). It follows that

$$
B_a D = \begin{bmatrix} 0_{(p-q) \times q} \\ D_1 \end{bmatrix} \qquad (41)
$$

where D_1 is nonsingular. From (40) and (41)

$$
[\, 0_{q \times (p-q)} \quad I_q \,] B_a C e + D_1 f_s(t) - [\, 0_{q \times (p-q)} \quad I_q \,] \nu_{eq} = 0
$$

and since D_1 is nonsingular, it follows that

$$
\begin{aligned}
f_s(t) &= -[\, 0_{q \times (p-q)} \quad D_1^{-1} \,] (B_a C e - \nu_{eq}) \\
&= -\left[0_{q \times (p-q)} \; D_1^{-1} \right] B_a C e + \left[0_{q \times (p-q)} \; D_1^{-1} \right] \nu_{eq} \qquad (42)
\end{aligned}
$$

Now, it is required to recover the equivalent output error injection ν_{eq}. Considering the structure of $\nu(\cdot)$ in (27), it follows from [4] that by choosing an appropriate positive constant scalar σ, ν_{eq} can be approximated to any accuracy by

$$\nu_\sigma = k(t) C_a^T \frac{(y_a - \hat{y}_a)}{\|y_a - \hat{y}_a\| + \sigma} \tag{43}$$

where $k(\cdot)$ satisfies (36). Let

$$\hat{f}_s(t) := [\, 0_{q\times(p-q)} \quad D_1^{-1} \,]\, \nu_\sigma \tag{44}$$

where ν_σ is defined by (43) and D_1 is given by (39). Then from (42) and (44),

$$f_s(t) - \hat{f}_s(t) = -\,[\, 0_{q\times(p-q)} \quad D_1^{-1} \,]\, B_a C e + [\, 0_{q\times(p-q)} \quad D_1^{-1} \,]\, (\nu_{eq} - \nu_\sigma)$$

where $\lim_{t\to\infty} e(t) = 0$. Therefore, $\hat{f}_s$ defined by (44) is a reconstruction for the sensor fault $f_s(t)$ since $\|\nu_{eq} - \nu_\sigma\|$ can be made arbitrarily small by choice of σ.

Remark 6. From (43) and (44), it is clear that the reconstruction signal $\hat{f}_s$ given by (44) is only dependent on y_a and $\hat{y}_a$ which can be obtained on-line. Therefore, the fault reconstruction scheme is convenient for real implementation.

Remark 7. The main task of this paper is to reconstruct/estimate the sensor fault f_s. From the structure of the system (21)–(23) it follows that the relative degree from f_s to the output y_a is one. Therefore, it is unnecessary to construct a high order sliding mode observer. However, a super-twisting structure ([10]) could be used componentwise in place of the unit vector injection signal.

3.2 Systems with uncertainty

Here, it is assumed $\Delta F(x(t)) \neq 0$. This means that the system under consideration is affected by uncertainties or disturbances. In this case, the corresponding dynamical error equation is given by (28)–(29).

Assumption 3. The uncertain function $\Psi(z)$ defined by (14) satisfies

$$\sqrt{\Psi(z)^T P \Psi(z)} \le \frac{1}{2} d, \qquad \forall z \in T(\Omega)$$

where the s.p.d. matrix P satisfies (32) with $P > I_n$ and d is a known constant.

Remark 8 Assumption 3 is a limitation on the magnitude of the uncertainty $\Psi(\cdot)$. It can be written as $\|P^{\frac{1}{2}}\Psi(z)\| \le \frac{1}{2}d$ which is just a special weighted norm for $\Psi(\cdot)$. Assumption 3 can therefore be interpreted as a requirement that the uncertainty $\Psi(\cdot)$ is bounded (in the special norm) and its bound is known. It is clear that Assumption 3 holds if $\Psi(\cdot)$ is bounded in the domain $T(\Omega)$.

Define

$$Q := (A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 P B B^T P + \tfrac{1}{\varepsilon_1}(\mathcal{L}(u))^2 I_n \tag{45}$$

where ε_1 is the positive constant associated with (32).

Proposition 3. Assume that the matrix inequality (32) is solvable for $P > 0$. Then under Assumptions 2 and 3, for any scalar $\varepsilon_2 > 0$ there exists a time T_1 such that for $t \ge T_1$, $e(t)$ will enter the set

$$\mathcal{B} = \left\{ e \mid e^T P e \le \left(\tfrac{d+\varepsilon_2}{\alpha}\right)^2 \right\} \tag{46}$$

and remains there for all subsequent time, where the positive constant $\alpha := -\lambda_{max}(P^{-1}Q)$.

Proof: Consider $V = e(t)^T Pe(t)$ as a potential Lyapunov function for system (28). Since (32) is solvable, the matrix Q defined by (45) is symmetric negative definite and so $\alpha := -\lambda_{max}(P^{-1}Q)$ is a positive quantity. By the same reasoning as in the proof of Proposition 1, it follows from (45) and Assumptions 2 and 3 that

$$\begin{aligned} \dot{V}\mid_{(28)} &\leq e^T(t)Qe(t) + 2e(t)^T P\Psi(z) \\ &= e^T(t)P^{1/2}P^{-1/2}QP^{-1/2}P^{1/2}e(t) + 2e(t)^T P^{1/2}P^{1/2}\Psi(z) \\ &\leq \lambda_{max}(P^{-1/2}QP^{-1/2})V + \sqrt{V}d \end{aligned} \tag{47}$$

since $V = e^T P^{1/2}P^{1/2}e = \|P^{1/2}e\|^2$. Also since $\lambda_{max}(P^{-1/2}QP^{-1/2}) = \lambda_{max}(P^{-1}Q)$ from the standard properties of eigenvalues, inequality (47) can be written as

$$\dot{V}\mid_{(28)} \leq \left(d - \alpha\sqrt{V}\right)\sqrt{V}$$

It follows that for any $\varepsilon_2 > 0$, if $e(t) \notin \mathcal{B}$, $d - \alpha\sqrt{V} < -\varepsilon_2$ and so

$$\dot{V}\mid_{(28)} \leq -\varepsilon_2\sqrt{V} \tag{48}$$

This implies that system (28) is uniformly ultimate bounded with respect to $\mathcal{B}$: i.e. $e(t)$ will enter the ball $\mathcal{B}$ defined in (46) after a finite time T_1 and remain in it thereafter. Hence the conclusion follows.

△

It should be noted that (31) is exactly the same as (29). Therefore, by the same reasoning as in Section 3.1, system (28)–(29) will be driven to the sliding surface (35) in finite time, and a sliding motion maintained on it, if the function ν is designed as in (27) and $k(\cdot)$ satisfies (36). The main difference is that in this case when uncertainty exists, the sliding motion is ultimately bounded instead of asymptotically stable. By combining Proposition 3, it follows that (24)–(26) is an approximate observer of system (21)–(23) when uncertainty is considered. Similar to the analysis in Section 3.1, it follows that (42) is true when a sliding motion takes place. It follows that

$$\hat{f}_s(t) = \begin{bmatrix} 0_{q\times(p-q)} & D_1^{-1} \end{bmatrix} \nu_\sigma \tag{49}$$

is an estimation of the sensor fault f_s where ν_σ is given by (43) and D_1 is defined by (39). From (42) and (49),

$$\|f_s - \hat{f}_s\| \leq \left\|D_1^{-1}\right\| \left(\|B_a\|\,\|C\|\,\|e(t)\| + \|\nu_{eq} - \nu_\sigma\|\right)$$

Since ν_{eq} can be approximated by ν_σ to any accuracy by choosing an appropriate σ, it follows that for any $\epsilon > 0$ there exists a time T_2 such that for $T > T_2$

$$\|\nu_{eq} - \nu_\sigma\| < \tfrac{1}{\|D_1^{-1}\|}\epsilon \tag{50}$$

From the fact that C from (12) has the property $\|C\| = 1$ and $P > I_n$ it follows that $V = e^T Pe \geq \|e\|^2$ and

$$\mathcal{B} \subset \left\{ e \mid \|e\| < \tfrac{(d+\varepsilon_2)}{\alpha} \right\} \tag{51}$$

Hence by combining with Proposition 3, the sensor fault estimation error

$$\|f_s(t) - \hat{f}_s(t)\| \leq \left\|D_1^{-1}\right\|\,\|B_a\|\tfrac{d+\varepsilon_2}{\alpha} + \epsilon \tag{52}$$

for all $t > T := \max\{T_1, T_2\}$, where the scalars ε_2 and ϵ are arbitrary small positive constants. Clearly, from (52), the estimation error is closely connected with the uncertain bound d.

Remark 9. Sensor fault estimation has been considered in [4, 16, 17]. Slowly varying sensor faults are considered in [4] but more general ones are considered in [16, 17]. However, in these papers, only linear systems are considered. In [20], actuator fault reconstruction was developed but a minimum phase condition was required for the system. In this paper, the corresponding minimum phase limitation has been removed which makes the work applicable to a wider class of systems.

Remark 10. The sensor faults considered in this paper are modelled as an additive disturbance. Fault detection is concerned with identifying that some thing is wrong in the monitored system while fault isolation is the determination of which component is faulty. If the system is not affected by any uncertainty/disturbance, then a 'precise' reconstruction signal has been proposed in this paper, i.e. after some time the reconstruction signal can duplicate the fault precisely. In this situation it is clear to see which channel has the fault through the reconstruction signal. This implies that the solution of the isolation problem is inherent in the approach. If the system is subject to uncertainty, the results developed in this paper only represent an estimation of the fault. In this case, an appropriate threshold is required to be established for fault isolation. Its accuracy will be limited by the size of the bound on the uncertainty compared to the size of the fault signals to be detected.

In the following, an approach based on LMI techniques is presented to determine the design parameters. Suppose $\beta \in \mathcal{R}$ is such that $\mathcal{L}(u) \leq \beta$ for all $u \in \mathcal{U}$. Also suppose Λ has been chosen so that $(A - B\Lambda)$ is stable (clearly this is a necessary condition for (32) to have a positive definite solution for P). Consider the matrix inequality

$$(A - B\Lambda)^T P + P(A - B\Lambda) + PBB^T P + \beta^2 I_n + \frac{1}{\gamma} P < 0 \tag{53}$$

where $\gamma \in \mathcal{R}$ is a positive scalar. If (53) is satisfied for some matrix $P > 0$, then from the definition of Q in (45) where ε_1 is chosen as 1, it follows that

$$P^{-1/2} Q P^{-1/2} + \frac{1}{\gamma} I_n < 0$$

which implies $\lambda_{\max}(P^{-1}Q) < -\frac{1}{\gamma}$ and so $\gamma > -1/\lambda_{\max}(P^{-1}Q) = \frac{1}{\alpha}$. Consequently minimizing γ, subject to solving (53) for P, decreases the radius of the ultimate boundedness set $\mathcal{B}$ from (51). A plausible convex optimization problem is to minimize γ with respect to P and X subject to

$$\begin{bmatrix} (A - B\Lambda)^T P + P(A - B\Lambda) + \varepsilon_1 \beta^2 I_n + X & PB \\ B^T P & -\varepsilon_1 I_p \end{bmatrix} < 0 \tag{54}$$

$$I_n < P \tag{55}$$

$$P < \gamma X \tag{56}$$

where $X \in \mathcal{R}^{n \times n}$ is a s.p.d 'slack' variable. This is a well posed convex optimization problem and can be solved using LMI techniques. From the Schur complement, if (54) is satisfied then

$$(A - B\Lambda)^T P + P(A - B\Lambda) + PBB^T P + \beta^2 I_n < -X$$

and since from (56) $-X < -\frac{1}{\gamma} P$, it follows that (53) is satisfied.

4 Procedure for sensor fault reconstruction/estimation

Based on the analysis above, a design procedure is summarized as follows:

1. Check system (1)–(2) has uniform observability indices $\{r_1, r_2, \dots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in $\Omega \times \mathcal{U}$;

2. Find the diffeomorphism T defined by (7)–(9). Then compute the transformed system (10)–(11);
3. Choose constants α_{ij} for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r_i$ such that all the roots of the polynomials (15) lie in the left-half plane;
4. Check Assumption 2 and compute the functions $\Gamma(z, u)$ satisfying (18) and $\mathcal{L}(u)$ satisfying (19);
5. Choose $A_a < 0$, $B_a = N$ satisfying (39) and C_a orthogonal. Then, establish the filter (20) and the observer (24)–(26);
6. Using an LMI package, find the solution P of the matrix inequalities (54)–(56), then Q can be obtained from (45);
7. Choose the gain $k(\cdot)$ to satisfy (36) and establish the observer (24)–(26);
8. According to (44) compute the reconstruction/estimation signal $\hat{f}_s$. (The estimation error can be obtained from (52)).

If the system under consideration satisfies the conditions proposed in this paper, then the procedure described above can be employed to reconstruct/estimate the sensor fault signal.

5 Simulation example

Consider a mass-spring system with a hardening spring, linear viscous friction and an external force described by

$$M\ddot{x} + c\dot{x} + \mu x + \mu a^2 x^3 = u \tag{57}$$

where x denotes displacement from a reference position, M is the mass of the object sliding on a horizontal surface, μ is the spring constant, a represents a coefficient which is associated with the hardening properties of the spring and u is the control signal which represents an external force applied to the system (see, [9], pages 8-9). Let $z = \mathrm{col}(z_1, z_2) = (x, \dot{x})$. The system output is assumed to be $y = z_1$. The parameters are chosen as in ([9], pages 172-173). Then, the system is described in the form of (10)–(11) as follows:

$$\dot{z} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{(-z_1 - z_2 - z_1^3 + u)}_{\Phi(x,u)} + \Psi(z) \tag{58}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} z + Df_s(t) \tag{59}$$

where $D = 1$ and the term $Df_s(z)$ is the sensor fault added to illustrate the results developed in this paper. It is assumed that $\Psi(\cdot)$ includes all uncertainties present in the system and in this case is assumed to satisfy $|\Psi(z)| \le 0.1 \sin^2 y$. This function has been added to this paper to demonstrate the results which have been developed and is not a feature of [9]. The domain considered in this example is

$$\Omega = \{(z_1, z_2) \mid |z_1| < 0.44, z_2 \in \mathcal{R}\}$$

Let $\Lambda = [-1 \quad -1]$. It follows that Assumption 2 holds with $\Gamma(z, u) = -z_1^3 + u$ which satisfies (19) in Ω with $\mathcal{L}(u) = 0.5808$. Choose $\varepsilon_1 = \gamma = 1$. It follows that the LMIs (54)–(56) have a solution

$$P = \begin{bmatrix} 1.2047 & 0.2428 \\ 0.2428 & 1.2881 \end{bmatrix}$$

and $\beta = 0.6 > \mathcal{L}(u)$. Therefore, the conditions of Proposition 1 are satisfied in the domain Ω.

Choose $A_a = -1$. Obviously D_1 can be chosen as $D_1 = D = 1$ and $B_a = 1$. Then the filter is described by

$$\begin{aligned} \dot{x}_a &= -x_a + y \\ y_a &= x_a \end{aligned}$$

If $k(t)$ is chosen to satisfy (36), it follows that $\hat{f}_s = \nu_\sigma$ is a reconstruction for $f_s(t)$ if $\Psi(\cdot) = 0$ and an estimation of the fault $f_s(t)$ if $\Psi(\cdot) \neq 0$. The simulation in Figure 2 shows that the approach is effective. The middle figure shows that the reconstruction signal reproduces the fault faithfully if no uncertainty is present in the system and thus the sensor fault can be detected easily from the reconstruction. The lower figure considers the case when $\Psi = [0.6 \quad 0.6]^T \sin^2 y$. It shows that the estimation signal still reproduces the fault to a reasonable extent when uncertainty $\Psi(\cdot)$ is present. In the presence of uncertainty, it follows that for any $\varepsilon > 0$, there exists a time T_1 such that

$$\|f_s(t) - \hat{f}(t)\| \leq \frac{0.0444 + \varepsilon}{0.0554} + \varepsilon, \qquad t > T_1$$

In this case, an appropriate threshold is required to be established for fault detection. The error bound given above is conservative and the performance which is achieved by the scheme is considerably better.

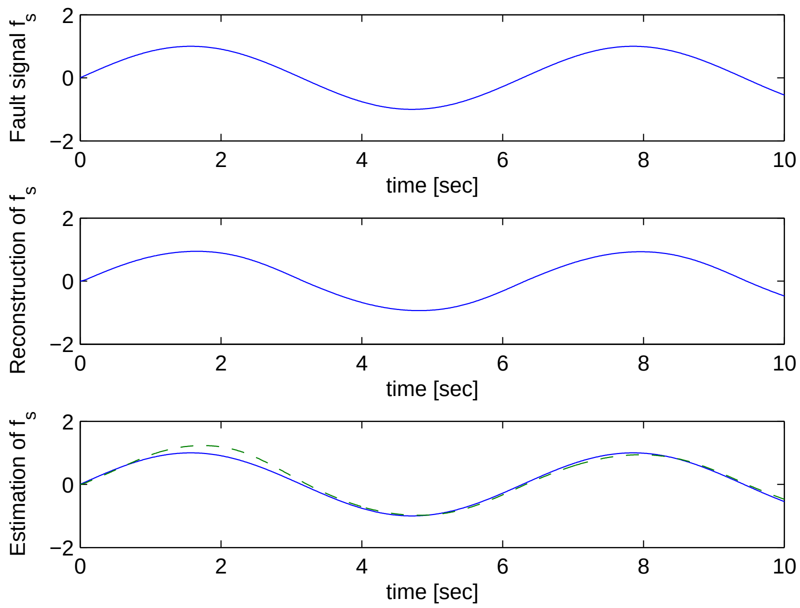

Figure 1: Sensor fault reconstruction/estimation for Mass-Spring system (58)–(59) (Upper: fault signal; Middle: reconstruction signal; Bottom: Estimation signal where the dashed line is the estimation signal and the solid line is the fault signal)

6 Conclusion

In this paper, a sliding mode observer has been considered for FDI in a class of nonlinear systems. First, the nonlinear system is transformed and an augmented system is established by designing a simple filter to process the outputs. A sliding mode observer is proposed for the augmented system to estimate the system states. Based on the observer, FDI schemes are presented for the system with/without uncertainty. The simulation example also shows how to use the reconstruction signal to detect the sensor fault.

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