Robust fault reconstruction in uncertain linear systems using multiple sliding mode observers in cascade

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***Abstract*—In observer-based fault reconstruction, one of the necessary conditions is that the first Markov parameter from the fault to the output must be full rank. This paper seeks to relax that requirement by using multiple sliding mode observers in cascade. Signals from an observer are used as the output of a fictitious system whose input is the fault. Another observer is then designed and implemented for the fictitious system. This process is repeated until the first Markov parameter of the fictitious system with respect to the fault is full rank. The result is that robust fault reconstruction can be carried out for a wider class of systems compared to other works that also seek to relax the requirement of a full rank first Markov parameter. In addition, this paper has also investigated and presented the necessary and sufficient conditions as easily testable conditions and also the precise number of observers required. A simulation example verifies the effectiveness of the scheme.**

***Index Terms*—sliding mode observer, robust fault reconstruction**

I. Introduction

FAULT reconstruction is an important area of research activity. A fault is deemed to occur when the system being monitored is subject to an abnormal condition, such as a malfunction [6]. The purpose of a fault reconstruction scheme is to estimate the fault so that its shape and magnitude can be understood and precise corrective action can be taken. However, most fault reconstruction schemes are designed about a model which does not perfectly represent the system – since some dynamics are either unknown or do not fit exactly into the framework of the model. These dynamics are usually represented as a class of (unknown) disturbances within the model. The disturbances corrupt the reconstruction signals, and could produce nonzero reconstructions when there are no faults, or worse, mask the effect of a fault. Therefore, schemes need to be designed so that the reconstruction is robust to disturbances. Edwards *et al.*[8] used a sliding mode observer to reconstruct faults, with no explicit consideration of the disturbances or uncertainty. Tan & Edwards [25] built on the work in [8] and presented a design algorithm for the observer, using Linear Matrix Inequalities (LMIs) [4], such that the $\mathcal{L}_2$ gain from the disturbances to the fault reconstruction is minimized. Saif & Guan [22] aggregated the faults and disturbances to form a new 'fault' vector and used a linear unknown input observer to reconstruct the new 'fault' vector. A necessary condition in [8], [25], [22] is that the first Markov parameter of the system connecting the fault to the output must be full rank. This limits the class of systems to which the schemes in [8], [25], [22] are applicable.

Recently, there have been developments in fault reconstruction for systems whose first Markov parameter is not full rank. Floquet & Barbot [10], [9] transformed the system into an 'output information' form such that existing techniques can be implemented to reconstruct the faults. Higher order sliding mode schemes have also been suggested [3], [7], [13]. The work in [13] uses the concept of 'strong observability' together with higher order sliding mode observers. Strong observability has also been exploited in [3] using a hierarchy of observers. Chen & Saif [7] used a bank of high-order sliding-mode differentiators to differentiate the outputs and estimate the faults from the output derivatives [7]. Floquet *et.al* [11], [12] suggest the use of exact differentiators to generate derivatives of the measurements to 'create' additional outputs to circumvent relative degree assumptions. However all the work in [10], [9], [7], [12], [3], [13] does not consider disturbances or uncertainty – unless the faults and disturbances are augmented and treated as 'unknown inputs' in which case the number of disturbances plus faults must not exceed the number of outputs. This results in stronger constraints which must be satisfied, and hence a smaller class of systems for which the results are applicable. Ng *et al.*[20] extended the work of Tan & Edwards [25] to relax the requirement of a full rank first Markov parameter by exploiting two sliding mode observers in cascade; signals from the first observer were considered as outputs of a 'fictitious' second system which has a first Markov parameter of full rank; then using the results in [25], a second sliding mode observer is designed based on the fictitious system to reconstruct the fault.

This paper builds on the work of [20] i.e. using multiple cascaded observers in cascade, however the observer that is used in this paper exploits a supertwisting structure [19] which will give a higher degree of accuracy for the fault estimation. The use of sliding mode observers in cascade for unknown input estimation is not new: see for example [23], [26], [15], [2]. However the work in [15] assumes full state measurement, whilst [2], [26] do not consider any external disturbances. Although [23] considers faults and uncertainties, they are aggregated and are both treated as unknown inputs – this introduces considerable conservatism since from the

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perspective of fault detection, it is less important to directly estimate the disturbances/uncertainty. In this paper the faults and disturbances are treated differently. Using similar techniques as in [20], signals from an observer are used as outputs of a fictitious system; the next observer is designed for the fictitious system and the signals from this observer are used as outputs of another fictitious system. The process is repeated until a fictitious system whose (first) Markov parameter is full rank is obtained. The technique in [25] is then used on the (final) fictitious system to robustly reconstruct the fault. This results in a robust fault reconstruction applicable to a wider class of systems than in [20]. The final fictitious system is found to be in the same framework as [25] which minimizes the $\mathcal{L}_2$ gain from the disturbances to the fault reconstruction (without reconstructing the disturbances); this enables the algorithm to be applicable for systems which has less outputs less than the sum of faults and disturbance channels (which cannot be achieved in [10], [9], [7]). Also, it is found that the design of previous observers do not affect the sliding motion of the final observer, which implies that the $\mathcal{L}_2$ gain from the disturbances to the fault reconstruction is affected only by the design of the final observer. Furthermore, necessary and sufficient conditions are investigated and presented in terms of the original system matrices so that the designer can determine at the outset whether the method is applicable or not. The results in this paper also indicate precisely the required number of cascaded observers. This identification of the class of systems for which the approach is applicable, is lacking in [10], [9], [7].

This paper is organized as follows; section II describes the fault reconstruction algorithm, section III investigates and presents the necessary and sufficient conditions, section IV shows a simulation example to validate the theory in this paper, and finally section V draws some conclusions. Throughout the paper, a superscript will be used to represent the recursion level in the cascade; for example X^i indicates that X is a parameter for observer i. To raise a variable to a power, it will be placed in brackets first; for example $(X)^i$ means that the variable X is raised to the power of i.

II. The robust fault reconstruction scheme

Consider a system represented in state-space as follows

$$\dot{x}^1 = A^1x^1 + M^1f^1 + Q^1\xi^1, y^1 = C^1x^1 \tag{1}$$

where $x^1 \in \mathbb{R}^{n^1}$ are the states, $y^1 \in \mathbb{R}^p$ are the outputs and $f^1 \in \mathbb{R}^q$ are unknown faults – for example actuator faults. The signals $\xi^1 \in \mathbb{R}^h$ are disturbances present in the system, such as nonlinearities, unmodelled dynamics or uncertainties.

Assume without loss of generality that $rank(M^1) = q, rank(C^1) = p$ and $rank(C^1M^1) = \bar{r}^1 \leq q$, which implies that $\bar{r}^1 \leq \min\{p, q\}$. Since $rank(C^1) = p$, then C^1 can be written without loss of generality in the form $C^1 = \begin{bmatrix} 0 & I_p \end{bmatrix}$. The signal ξ^1 is assumed to be smooth and an upper bound on its bandwidth is assumed known.

Remark 1: The assumption that a bound on the frequency content of the disturbances is known, is common in the applications literature. This sort of information has been used in the development of models of practical engineering systems such as satellites [5] and ships [16] and for process control [18] for example (where typically the disturbances are assumed to be low frequency in character). Insight from the underlying physics is usually employed to decide on the meaningful frequency range of the disturbance. ♯

From the bandwidth assumption it is possible to write

$$\xi^1 = \Omega(s)\xi^k \tag{2}$$

where $\Omega(s)$ represents a known filter with low-pass characteristics of appropriate bandwidth and ξ^k is a bounded unknown signal. As in other frequency domain based paradigms such as $\mathcal{H}_\infty$ and μ-synthesis, $\Omega(s)$ can be viewed as a 'weighting function' [28]. The frequency information about the disturbance associated with $\Omega(s)$ will then be incorporated into the observer design. Furthermore it is assumed that ξ^1, together with an appropriate number of its derivatives are bounded. Specific details pertaining to the weighting function $\Omega(s)$ will be given in the next section. Also the first derivative of f^1 is assumed to be bounded by a known constant. This assumption is not restrictive as it only implies that f^1 cannot be an abrupt step which is easy to detect; slow incipient faults are much more difficult to detect [6]. *The objective is to reconstruct f^1 whilst minimizing the effects of ξ^1 on the fault reconstruction.* If $\bar{r}^1 = q$ then the single-observer method in [25] can be used. However, if $\bar{r}^1 < q$, then an alternative approach is required. In this situation, this paper proposes the cascade observer scheme shown in Figure 1. The next subsection describes the fault reconstruction algorithm and a systematic way of designing the components in Figure 1.

A. Design algorithm

Firstly partition the matrices from (1) as

$$A^1 = \begin{bmatrix} A_1^1 & A_2^1 \\ A_3^1 & A_4^1 \end{bmatrix}, \; M^1 = \begin{bmatrix} M_1^1 \\ M_2^1 \end{bmatrix}, \; Q^1 = \begin{bmatrix} Q_1^1 \\ Q_2^1 \end{bmatrix} \begin{matrix} \updownarrow n^1-p \\ \updownarrow p \end{matrix}$$

where A_1^1 is square. Since by assumption $C^1 = \begin{bmatrix} 0 & I_p \end{bmatrix}$ and $rank(C^1M^1) = \bar{r}^1$, then it follows that $rank(M_2^1) = \bar{r}^1$. In the representation above, Q^1 has no particular structure. Set the index variable $i = 1$ and enter the following algorithm:

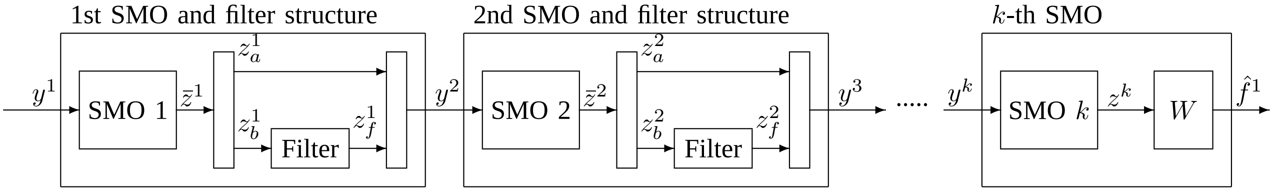

Fig. 1. The proposed scheme formed from a cascaded observer/filter structure

1) **Check algorithm termination**
Consider the generic uncertain faulty system

$$\dot{x}^i = A^i x^i + M^i f^i + Q^i \xi^i, y^i = C^i x^i \tag{3}$$

and define $\bar{r}^i := rank(C^i M^i)$. If $rank(C^i M^i) < rank(M^i)$ and $i = n^1$, then the method in this paper cannot be used to reconstruct the faults (the justification of this will be given in Theorem 1 in the sequel) and terminate the algorithm.

2) **Transform the system to achieve special structures in the fault and output matrices**
For the case when $i = 1$, define $\bar{M}_{11}^0 := M_1^1, \bar{M}_{12}^0 := M_2^1, m^1 := p, \bar{r}^0 := 0$, $\tilde{A}_{13}^0 := A_3^1, \tilde{A}_{11}^0 := A_1^1$, $\bar{A}_\Omega^0 = \alpha^0 = \bar{M}_{22}^0 = \phi$ where ϕ is the empty matrix.
Let $r^i := rank(\bar{M}_{12}^{i-1})$ and define two orthogonal matrices $T_2^i \in \mathbb{R}^{(q-\bar{r}^{i-1})\times(q-\bar{r}^{i-1})}$, $D^i \in \mathbb{R}^{m^i \times m^i}$ and $T_D^i := diag\left\{I_{n^i-p-(i-1)h}, (D^i)^{-1}\right\}$ such that

$$T_D^i \begin{bmatrix} \bar{M}_{11}^{i-1} \\ \bar{M}_{12}^{i-1} \end{bmatrix} (T_2^i)^{-1} = \begin{bmatrix} M_{11}^i & M_{12}^i \\ \hline 0 & 0 \\ 0 & M_{22}^i \end{bmatrix} \begin{matrix} \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i - r^i \\ \updownarrow r^i \end{matrix} \tag{4}$$

where $M_{22}^i \in \mathbb{R}^{r^i \times r^i}$ is invertible. Then define $T_1^i := T_{12}^i T_{11}^i$ where $T_{11}^i := diag\left\{I_{n^i-p}, (D^i)^{-1}, I_{p-m^i}\right\}$ and

$$T_{12}^i := \begin{bmatrix} I_{n^i-p} & T_{122}^i \\ 0 & T_{124}^i \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} T_{122}^i \\ T_{124}^i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -M_{12}^i (M_{22}^i)^{-1} & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{matrix} \\ \\ \updownarrow m^i - r^i \\ \updownarrow p - \bar{r}^{i-1} - m^i \\ \updownarrow r^i \\ \updownarrow \bar{r}^{i-1} \end{matrix}$$

Define

$$\tilde{A}_3^i := (D^i)^{-1} \tilde{A}_{13}^{i-1} = \begin{bmatrix} \tilde{A}_{31}^i \\ \tilde{A}_{32}^i \end{bmatrix} \begin{matrix} \updownarrow m^i - r^i \\ \updownarrow r^i \end{matrix} \tag{6}$$

$$\tilde{A}_1^i := \tilde{A}_{11}^{i-1} - M_{12}^i (M_{22}^i)^{-1} \tilde{A}_{32}^i \tag{7}$$

$$T_f^i := diag\left\{T_2^i, I_{\bar{r}^{i-1}}\right\} \tag{8}$$

Perform the transformations $x^i \mapsto T_1^i x^i$, $f^i \mapsto f^{i+1} := T_f^i f^i$ then A^i, M^i, C^i will be transformed into

$$A^i \mapsto \begin{bmatrix} A_1^i & A_2^i \\ A_3^i & A_4^i \end{bmatrix} = \left[\begin{array}{cc|c} \bar{A}_\Omega^{i-1} & 0 & \star \\ \star & \tilde{A}_1^i & \star \\ \hline \star & \tilde{A}_{31}^i & \star \\ \star & 0 & \star \\ \star & \star & \star \end{array}\right] \begin{matrix} \updownarrow (i-1)h \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i - r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{matrix} \tag{9}$$

$$M^i \mapsto \begin{bmatrix} M_1^i \\ M_2^i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ M_{11}^i & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & \bar{M}_{22}^i \end{bmatrix} \begin{matrix} \updownarrow (i-1)h \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i - r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{matrix} \tag{10}$$

$$C^i \mapsto \begin{bmatrix} 0 & C_2^i \end{bmatrix} \tag{11}$$

where

$$C_2^i = \begin{bmatrix} D^i & 0 \\ 0 & I_{p-m^i} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{matrix} \updownarrow m^i - r^i \\ \updownarrow p-\bar{r}^{i-1}-m^i \\ \updownarrow r^i \\ \updownarrow \bar{r}^{i-1} \end{matrix}$$

and $\bar{M}_{22}^i := diag\left\{M_{22}^i, \alpha^{i-1} \bar{M}_{22}^{i-1}\right\}$. It can be seen from the definition of $\bar{r}^i$ in step 1, M^i and $\bar{M}_{22}^i$ in (10), and C^i in (11) that

$$\bar{r}^i := \bar{r}^{i-1} + r^i \tag{12}$$

In this coordinate system Q^i has no specific structure. If $rank(C^i M^i) = rank(M^i)$ then go to step 7 and terminate the algorithm. Otherwise, go to the next step.

3) **Augment the system with the dynamics of the weight associated with the disturbance**
Assume that ξ^i is smooth resulting from the following stable system

$$\dot{\xi}^i = A_\Omega^i \xi^i + B_\Omega^i \xi^{i+1} \tag{13}$$

where $\xi^{i+1} \in \mathbb{R}^h$ and A_Ω^i, B_Ω^i are matrices to be chosen by the designer. In addition, assume that ξ^{i+1} is bounded. (The motivation and implication of this assumption, and a way to choose A_Ω^i and B_Ω^i will be discussed in Remark 2). Augment (13) with (3) to obtain the following system of order $\bar{n}^i := n^i + h$

$$\dot{\bar{x}}^i = \bar{A}^i \bar{x}^i + \bar{M}^i f^{i+1} + \bar{Q}^i \xi^{i+1}, y^i = \bar{C}^i \bar{x}^i \tag{14}$$

where $\bar{x}^i := col(\xi^i, x^i)$ and

$$\bar{M}^i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ M_{11}^i & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & \bar{M}_{22}^i \end{bmatrix}, \bar{Q}^i = \begin{bmatrix} B_\Omega^i \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \updownarrow h \\ \updownarrow (i-1)h \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i - r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{matrix}$$

$$\bar{A}^i = \left[\begin{array}{cc|c} \bar{A}_\Omega^i & 0 & 0 \\ \star & \tilde{A}_1^i & \star \\ \hline \bar{Q}_{21}^i & \tilde{A}_{31}^i & \star \\ \star & 0 & \star \\ \star & \star & \star \end{array}\right] \begin{matrix} \updownarrow ih \\ \updownarrow n^i-p-(i-1)h \\ \updownarrow m^i - r^i \\ \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow \bar{r}^i \end{matrix}$$

where $\bar{A}_\Omega^i := \begin{bmatrix} A_\Omega^i & 0 \\ \star & \bar{A}_\Omega^{i-1} \end{bmatrix}$.

4) **Transform the augmented system to achieve a special structure in the system matrix**
Define $m^{i+1} := rank(\tilde{A}_{31}^i)$. Let U_1^i and U_2^i be invertible matrices of dimensions $m^i - r^i$ and $n^i - p - (i-1)h$ respectively such that

$$U_1^i \tilde{A}_{31}^i (U_2^i)^{-1} = \begin{bmatrix} 0 & I_{m^{i+1}} \\ 0 & 0 \end{bmatrix} \tag{15}$$

$$U_1^i \bar{Q}_{21}^i = \begin{bmatrix} \bar{Q}_{211}^i \\ \bar{Q}_{212}^i \end{bmatrix} \begin{matrix} \updownarrow m^{i+1} \\ \updownarrow m^i - r^i - m^{i+1} \end{matrix}$$

where $\bar{Q}_{211}^i, \bar{Q}_{212}^i$ are general matrices with no particular structure. Also partition

$$U_2^i \tilde{A}_1^i (U_2^i)^{-1} = \begin{bmatrix} \tilde{A}_{11}^i & \tilde{A}_{12}^i \\ \tilde{A}_{13}^i & \tilde{A}_{14}^i \end{bmatrix} \begin{matrix} \updownarrow n^i-p-(i-1)h-m^{i+1} \\ \updownarrow m^{i+1} \end{matrix} \tag{16}$$

Introduce a transformation $\bar{x}^i \mapsto \bar{T}^i\bar{x}^i$ where $\bar{T}^i := \bar{T}_2^i\bar{T}_1^i$ with $\bar{T}_1^i := diag\left\{I_{ih}, U_2^i, U_1^i, I_{p+r^i-m^i}\right\}$ and

$$\bar{T}_2^i := \left[\begin{array}{cc|c} I_{ih} & 0 & 0 \\ \tilde{Q}^i & I_{n^i-p-(i-1)h} & 0 \\ \hline 0 & 0 & I_p \end{array}\right], \tilde{Q}^i := \begin{bmatrix} 0 \\ \bar{Q}_{211}^i \end{bmatrix} \quad (17)$$

Then $\bar{A}^i, \bar{M}^i, \bar{Q}^i, \bar{C}^i$ will respectively become

$$\begin{bmatrix} \bar{A}_1^i & \bar{A}_2^i \\ \bar{A}_3^i & \bar{A}_4^i \end{bmatrix} = \left[\begin{array}{ccc|c} \bar{A}_\Omega^i & 0 & 0 & \star \\ \star & \tilde{A}_{11}^i & \tilde{A}_{12}^i & \star \\ \star & \tilde{A}_{13}^i & \tilde{A}_{14}^i & \star \\ \hline 0 & 0 & I_{m^{i+1}} & \star \\ \star & 0 & 0 & \star \end{array}\right] \quad (18)$$

$$\begin{bmatrix} \bar{M}_1^i \\ \bar{M}_2^i \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ \bar{M}_{11}^i & 0 \\ \bar{M}_{12}^i & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & \bar{M}_{22}^i \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^i-p-m^{i+1}-(i-1)h \\ \updownarrow m^{i+1} \\ \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1}-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array} \quad (19)$$

$$\left[\begin{array}{c} \bar{Q}_1^i \\ \hline 0 \end{array}\right] \begin{array}{l} \updownarrow n^i-p+h \\ \updownarrow p \end{array}, \begin{bmatrix} 0 & \bar{C}_2^i \end{bmatrix} \quad (20)$$

where $\det(\bar{C}_2^i) \neq 0$. Partition $\bar{A}_3^i = \begin{bmatrix} \bar{A}_{31}^i \\ \bar{A}_{32}^i \end{bmatrix} \begin{array}{l} \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1} \end{array}$ which from (18) results in $\bar{A}_{31}^i = \begin{bmatrix} 0 & I_{m^{i+1}} \end{bmatrix}$.

5) **Implement observer i for the augmented system**
A sliding mode observer building on second order supertwisting ideas [17], [19] for (14) is

$$\dot{\hat{\bar{x}}}^i = \bar{A}^i\hat{\bar{x}}^i - \bar{G}_l^i\bar{e}_y^i + \bar{G}_n^i\bar{\nu}^i, \quad \bar{e}_y^i := \bar{C}^i\hat{\bar{x}}^i - y^i \quad (21)$$

where the matrices $\bar{G}_l^i, \bar{G}_n^i \in \mathbb{R}^{\bar{n}^i\times p}$ are to be designed. In particular, choose $\bar{G}_n^i$ as

$$\bar{G}_n^i = \begin{bmatrix} -\bar{L}^i \\ I_p \end{bmatrix} (\bar{C}_2^i)^{-1}, \ \bar{L}^i = \begin{bmatrix} \bar{L}_o^i & 0 \end{bmatrix} \quad (22)$$

where $\bar{L}_o^i \in \mathbb{R}^{(\bar{n}^i-p)\times m^{i+1}}$ is chosen such that $\bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i$ is stable. Partition component-wise the output estimation error as $\bar{e}_y^i = col\left\{\bar{e}_{y,1}^i, ..., \bar{e}_{y,p}^i\right\}$. As in [19] the term $\bar{\nu}^i := col\left\{\bar{\nu}_1^i, ..., \bar{\nu}_p^i\right\}$ is defined by

$$\bar{\nu}_j^i = -\psi_j^i \text{sign}(\bar{e}_{y,j}^i)|\bar{e}_{y,j}^i|^{\frac{1}{2}} + z_j^i, \quad j = 1, ..., p \quad (23)$$

$$\dot{z}_j^i = -\beta_j^i \text{ sign}(\bar{e}_{y,j}^i) - \gamma_j^i\bar{e}_{y,j}^i, \quad j = 1, ..., p \quad (24)$$

where ψ_j^i, β_j^i and γ_j^i are scalars to be selected by the designer. Define $\bar{e}^i := \hat{\bar{x}}^i - \bar{x}^i$ and combine (14) and (21) to obtain

$$\dot{\bar{e}}^i = (\bar{A}^i - \bar{G}_l^i\bar{C}^i)\bar{e}^i + \bar{G}_n^i\bar{\nu}^i - \bar{M}^i f^{i+1} - \bar{Q}^i\xi^{i+1} \quad (25)$$

Apply another change of coordinates associated with T_L^i to the triple (18) - (20) and $\bar{G}_n^i$ in (22) where

$$T_L^i := \begin{bmatrix} I_{\bar{n}^i-p} & \bar{L}^i \\ 0 & \bar{C}_2^i \end{bmatrix}$$

then $\bar{A}^i, \bar{M}^i, \bar{C}^i$ from (18) - (20) and $\bar{G}_n^i$ from (22) are respectively transformed to be

$$\begin{bmatrix} \mathcal{A}_{11}^i & \mathcal{A}_{12}^i \\ \mathcal{A}_{21} & \mathcal{A}_{22}^i \end{bmatrix}, \begin{bmatrix} \bar{M}_1^i \\ \bar{C}_2^i\bar{M}_2^i \end{bmatrix}, \begin{bmatrix} 0 & I_p \end{bmatrix}, \begin{bmatrix} 0 \\ I_p \end{bmatrix} \quad (26)$$

where $\mathcal{A}_{11}^i := \bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i, \mathcal{A}_{21}^i := \bar{C}_2^i\bar{A}_3^i$. The matrix $\bar{Q}^i$ retains the structure in (20) after the transformation. Define

$$T_L^i\bar{e}^i =: \begin{bmatrix} \bar{e}_1^i \\ \bar{e}_y^i \end{bmatrix}, \ T_L^i\bar{G}_l^i =: \begin{bmatrix} \mathcal{G}_1^i \\ \mathcal{G}_2^i \end{bmatrix} \begin{array}{l} \updownarrow \bar{n}^i-p \\ \updownarrow p \end{array} \quad (27)$$

and choose $\bar{G}_l^i$ so that $\mathcal{G}_1^i = \mathcal{A}_{12}^i$, $\mathcal{G}_2^i = \mathcal{A}_{22}^i + \mathcal{A}_s^i$ where $\mathcal{A}_s^i := diag\left\{\lambda_1^i, ..., \lambda_p^i\right\}$ and the scalars $\lambda_j^i > 0, j = 1, ..., p$. Partitioning (25) according to (26) - (27) results in

$$\dot{\bar{e}}_1^i = \mathcal{A}_{11}^i\bar{e}_1^i + \bar{M}_1^i f^{i+1} + \bar{Q}_1^i\xi^{i+1} \quad (28)$$

$$\dot{\bar{e}}_y^i = \mathcal{A}_{21}^i\bar{e}_1^i + \bar{C}_2^i\bar{M}_2^i f^{i+1} - \mathcal{A}_s^i\bar{e}_y^i + \bar{\nu}^i \quad (29)$$

where $\bar{M}_1^i$, $\bar{M}_2^i$ and $\bar{Q}_1^i$ are defined in (19) - (20). Equation (29) can be written as

$$\dot{\bar{e}}_y^i = \zeta^i - \mathcal{A}_s^i\bar{e}_y^i + \bar{\nu}^i \quad (30)$$

where $\zeta^i = \hat{G}(s)\begin{bmatrix} f^{i+1} \\ \xi^{i+1} \end{bmatrix}$ and

$$\hat{G}(s) := -\begin{bmatrix} \bar{C}_2^i\bar{M}_2^i & 0 \end{bmatrix} - \mathcal{A}_{21}^i\left(sI - \mathcal{A}_{11}^i\right)^{-1}\begin{bmatrix} \bar{M}_1^i & \bar{Q}_1^i \end{bmatrix}$$

It is obvious ζ^i and $\dot{\zeta}^i$ are bounded since $\mathcal{A}_{11}^i$ is stable and f^{i+1}, $\dot{f}^{i+1}$ and ξ^{i+1} are bounded by assumption. Let $\zeta^i = col\left\{\zeta_1^i, ..., \zeta_p^i\right\}$ and define $\hat{z}_j^i := z_j^i + \zeta_j^i$. Substitute (23) into (30) and combine with (24) to obtain

$$\dot{\bar{e}}_{y,j}^i = -\psi_j^i \text{ sign}(\bar{e}_{y,j}^i)|\bar{e}_{y,j}^i|^{\frac{1}{2}} - \lambda_j^i\bar{e}_{y,j}^i + \hat{z}_j^i \quad (31)$$

$$\dot{\hat{z}}_j^i = -\beta_j^i \text{ sign}(\bar{e}_{y,j}^i) - \gamma_j^i\bar{e}_{y,j}^i + \dot{\zeta}_j^i \quad (32)$$

where $j = 1, ..., p$. Define constants $d_j^i > |\dot{\zeta}_j^i|$ and choose the gains from (23) and (24) as

$$\psi_j^i > 2\sqrt{d_j^i}, \lambda_j^i > 0, \beta_j^i > d_j^i \quad (33)$$

$$\gamma_j^i > \frac{(\lambda_j^i)^2\left((\psi_j^i)^3 + \frac{5}{4}(\psi_j^i)^2 + \frac{5}{2}(\beta_j^i - d_j^i)\right)}{\psi_j^i(\beta_j^i - d_j^i)} \quad (34)$$

Then, it can be proved from Theorem 5 in [19] that if (33) - (34) are satisfied, a sliding motion will take place and force $\bar{e}_{y,j}^i = \dot{\bar{e}}_{y,j}^i = 0$ in finite time.

6) **Process the observer signals to obtain the output of a system for next observer $i + 1$**
Assume that a sliding motion has taken place, then (23) and (30) yields $z^i = -\zeta^i$ where $z^i := col\left\{z_1^i, ..., z_p^i\right\}$. Note that z^i is an available continuous signal since it is generated from $\bar{e}_{y,j}^i$ according to (24). Define $w^i := -e_1^i$ and partition (25) using (26) - (27) to obtain

$$\dot{w}^i = (\bar{A}_1^i + \bar{L}_o^i\bar{A}_{31}^i)w^i + \bar{M}_1^i f^{i+1} + \bar{Q}_1^i\xi^{i+1} \quad (35)$$

$$z^i = \bar{C}_2^i\bar{A}_3^i w^i + \bar{C}_2^i\bar{M}_2^i f^{i+1} \quad (36)$$

Define $\bar{z}^i := (\bar{C}_2^i)^{-1}z^i := \begin{bmatrix} z_a^i \\ z_b^i \end{bmatrix} \begin{array}{l} \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1} \end{array}$. Substituting for the partitions of $\bar{A}_3^i$ from step 4 and $\bar{M}_2^i$ from (19) into (36) results in

$$z_a^i = \begin{bmatrix} 0 & I_{m^{i+1}} \end{bmatrix} w^i \quad (37)$$

$$z_b^i = \bar{A}_{32}^i w^i + \begin{bmatrix} 0 & 0 \\ 0 & \bar{M}_{22}^i \end{bmatrix} f^{i+1} \quad (38)$$

Filter z_b^i in real-time to obtain z_f^i as follows:

$$\begin{aligned}\dot{z}_f^i &:= -\alpha^i z_f^i + \alpha^i z_b^i,\ \alpha^i \in \mathbb{R}_+ \\ &= -\alpha^i z_f^i + \alpha^i \bar{A}_{32}^i w^i + \begin{bmatrix} 0 & 0 \\ 0 & \alpha^i \bar{M}_{22}^i \end{bmatrix} f^{i+1} \end{aligned} \quad (39)$$

The purpose of filtering z_b^i will be discussed in Remark 2. Combine (35), (39) and (37) to obtain

$$\dot{x}^{i+1} = A^{i+1}x^{i+1} + M^{i+1}f^{i+1} + Q^{i+1}\xi^{i+1} \quad (40)$$
$$y^{i+1} = C^{i+1}x^{i+1} \quad (41)$$

where $x^{i+1} \in \mathbb{R}^{n^{i+1}}$ with $n^{i+1} := \bar{n}^i - m^{i+1}$ and

$$x^{i+1} := \begin{bmatrix} w^i \\ z_f^i \end{bmatrix},\ y^{i+1} := \begin{bmatrix} z_a^i \\ z_f^i \end{bmatrix},\ C^{i+1} := \begin{bmatrix} 0 & I_p \end{bmatrix} \quad (42)$$

By substituting (18) and (19) into (35) and (39), A^{i+1} and M^{i+1} can be expanded to be

$$A^{i+1} = \left[\begin{array}{cc|ccc} \bar{A}_\Omega^i & 0 & \star & 0 & 0 \\ \star & \tilde{A}_{11}^i & \star & 0 & 0 \\ \hline \star & \tilde{A}_{13}^i & \star & 0 & 0 \\ \star & 0 & \star & -\alpha^i I & 0 \\ \star & \star & \star & 0 & -\alpha^i I \end{array}\right] \begin{array}{l} \\ \\ \\ \updownarrow p-m^{i+1}-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array} \quad (43)$$

$$M^{i+1} = \left[\begin{array}{cc} 0 & 0 \\ \bar{M}_{11}^i & 0 \\ \hline \bar{M}_{12}^i & 0 \\ 0 & 0 \\ 0 & \alpha^i \bar{M}_{22}^i \end{array}\right] \begin{array}{l} \updownarrow ih \\ \updownarrow n^i-p-(i-1)h-m^{i+1} \\ \updownarrow m^{i+1} \\ \updownarrow p-m^{i+1}-\bar{r}^i \\ \updownarrow \bar{r}^i \end{array} \quad (44)$$

while Q^{i+1} has no specific structure. The structure of C^{i+1} in (42) is due to the structure of $\bar{A}_3^i$ in (18). Then increment the counter i by 1 and return to step 1.

7) **Reconstruct the fault robustly if the Markov parameter is full rank**
Set $k = i$. Since $rank(C^k M^k) = rank(M^k)$, M_{11}^k in (4) and (10) does not exist since $\bar{r}^k = q$. As a result, choose $T_2^k = I_{q-\bar{r}^{k-1}} \Rightarrow f^{k+1} = f^k$ (see step 2). Set $\bar{A}^k = A^k, \bar{M}^k = M^k, \bar{C}^k = C^k, \bar{Q}^k = Q^k, m^{k+1} = p - q$. Also define Q_1^k, Q_2^k to be respectively the top $n^k - p$ and bottom p rows of Q^k. Design $\bar{L}_o^k$ as follows: minimize γ with respect to the variables $R_{11} = R_{11}^T > 0, R_{12}, W_1, \gamma$ subject to:

$$\begin{bmatrix} R_{11}A_1^k + R_{12}A_3^k + (\star) & (\star) & (\star) \\ (R_{11}Q_1^k + R_{12}Q_2^k)^T & -\gamma I_h & 0 \\ (WA_3^k)^T & 0 & -\gamma I_q \end{bmatrix} < 0 \quad (45)$$

where $(\star)$ are terms that make the inequality (45) symmetric, $W := \begin{bmatrix} W_1 & (\bar{M}_{22}^k)^{-1} \end{bmatrix} (C_2^k)^{-1}$, $R_{12} := \begin{bmatrix} R_{121} & 0 \end{bmatrix}, R_{121} \in \mathbb{R}^{(n^k-p)\times(p-q)}$. Then calculate $\bar{L}_o^k = (R_{11})^{-1}R_{121}$. When sliding motion has occurred, reconstruct the fault using $\hat{f}^k := Wz^k$. From [25], $\hat{f}^k$ will reconstruct f^k and a function of ξ^k; the design of $\bar{L}_o^k$ and W_1 in this step minimizes the $\mathcal{L}_2$ gain from ξ^k to $\hat{f}^k$. The reconstruction of f^1 can be obtained from

$$\hat{f}^1 := (T_f^{k-1})^{-1}...(T_f^2)^{-1}(T_f^1)^{-1}\hat{f}^k \quad (46)$$

where T_f^i is defined in (8).

Remark 2: The purpose of the assumption that the (unknown) signal ξ^i is obtained as the output of the low pass filter in equation (13), and the subsequent filtering of the (known) signal z_b^i in (39), is to achieve the recursive formulation in (40) - (41) where the faults and disturbances appear in the 'state' equation. It should be noted that there is no 'physical' filtration of the disturbances: the filter in (13) only implies that ξ^i is smooth and can be considered to be the output of a low-pass filter $G^i(s) := (sI - A_\Omega^i)^{-1}B_\Omega^i$ driven by an unknown signal ξ^{i+1}. The choice of A_Ω^i and B_Ω^i is not unique. In this paper, first order linear filter realizations have been chosen, although higher order linear filters could equally well have been selected. The crucial decision is the choice of the filter bandwidth and not the particular choice of filter itself. The relationship between the filter pairs (A_Ω^i, B_Ω^i) and the original weighting function in (2) is $\Omega(s) = C_\Omega(sI - A_\Omega)^{-1}B_\Omega$ where $C_\Omega = [\ I_h \quad 0_{h\times(k-2)h}\ ]$ and

$$A_\Omega := \begin{bmatrix} A_\Omega^1 & B_\Omega^1 & 0 & \dots & 0 \\ 0 & A_\Omega^2 & B_\Omega^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_\Omega^{k-2} & B_\Omega^{k-2} \\ 0 & \dots & 0 & 0 & A_\Omega^{k-1} \end{bmatrix}, B_\Omega := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B_\Omega^{k-1} \end{bmatrix}$$

Modeling the characteristics of the exogenous disturbances using filters is the basis of all the $\mathcal{H}_\infty$ and μ-synthesis paradigms which are based on frequency domain assumptions on the uncertainty. There are also some parallels with the work of [24] in the sense that the uncertainty belongs to a restricted class of signals. In terms of fault estimation, it is the low frequency components that are important; for example slow incipient faults are the most difficult to identify [6]. To decouple these low frequency faults from low frequency disturbances is very important (and non-trivial). To choose reasonable values of (A_Ω^i, B_Ω^i), let the assumed bandwidth of ξ^i be ω_c^i, and choose $A_\Omega^i = -\kappa I_h, B_\Omega^i = \kappa I_h$ where $\kappa \in \mathbb{R}_+$. If κ is chosen to be much larger than ω_c^i, then $\xi^i \approx \xi^{i+1}$ and ultimately $\xi^k \approx \xi^1$. In step 7 of the algorithm, the effect of ξ^1 on $\hat{f}^k$ is formally minimized. ♯

Remark 3: The approach which has been proposed is similar to the so-called 'step-by-step' methods [1], [27], [2], [15]. As the number of cascade operations increases, in practice, the accuracy of the estimation which is achieved degrades [14]. However, as argued in [2], the use of the supertwisting structure gives optimal performance at each step at least, and obviates the need to approximate the equivalent injection signals via sigmoidal approximations or low pass filtering of discontinuous injection signals. ♯

Since $\bar{n}^i = n^i + h$ (step 3) and $n^{i+1} = \bar{n}^i - m^{i+1}$ (step 6), it can be shown that

$$n^{i+1} = n^i + h - m^{i+1} \Rightarrow n^i = (i-1)h - \Sigma_{j=2}^i m^j + n^1 \quad (47)$$

Theorem 1: If $rank(C^{n^1}M^{n^1}) < rank(M^{n^1})$ then the fault can never be fully reconstructed. ♯

Proof: From (9), it can be seen that $\tilde{A}_1^i$ has $n^i - (i-1)h - p$ rows and therefore $n^i - (i-1)h - p \geq 0$. Substituting for n^i from (47) results in

$$n^1 - \Sigma_{j=2}^i m^j - p \geq 0 \quad (48)$$

Since $m^{i+1} = rank(\tilde{A}^i_{31})$ and knowing that $\tilde{A}^i_{31}$ has $m^i - r^i$ rows (see step 4), it is obvious that $m^{i+1} \le m^i$ hence resulting in $0 \le m^i \le m^{i-1} \le ... \le m^2 \le m^1 = p$. It follows from (48) that $m^i = 0$ when $i > n^1$. From (4), it is clear that $r^i \le m^i$ and therefore $r^i = 0$ when $i > n^1$. Then from (12), $\bar{r}^i = \bar{r}^{n^1}$ when $i > n^1$ which results in $rank(C^i M^i) = rank(C^{n^1} M^{n^1})$ when $i > n^1$. This means that if observer n^1 cannot reconstruct f^1, then subsequent observers will not be able to either, and the scheme in this paper is not feasible. ■

Remark 4: Notice from the structure of A^{i+1} in (43), the matrix $\bar{L}^i_o$ appears only in the last p columns of A^i. From the structure of C^{i+1} in (42), it is clear that $\bar{L}^i_o$ affects only the p output states of x^{i+1}, and hence $\bar{L}^i_o$ will not affect the sliding motion of observer $i+1$ and also all subsequent observers. Also, it is obvious that $\bar{G}^i_l$ does not affect subsequent observers as it vanishes during sliding motion ($\bar{e}^i_y = 0$). As the fault reconstruction in step 7 is performed during sliding motion of observer k, it can therefore be concluded that the gains of previous observers ($\bar{L}^i_o, \mathcal{A}^i_s$ and subsequently $\bar{G}^i_l, \bar{G}^i_n$) can be arbitrarily designed as they will not affect the quality of the fault reconstruction, and only observer k needs to be designed as described in step 7. ♯

III. Existence Conditions

The method proposed in Section II is feasible if and only if the following are satisfied

A1. $rank(C^k M^k) = rank(M^k)$, for some $1 \le k \le n^1$.

A2. All observers have a stable sliding motion.

It is of interest to find existence conditions for the method proposed in this paper in terms of the original matrices A^1, M^1, C^1, so that it can be easily ascertained from the beginning whether the method proposed in this paper is applicable or not. To conveniently analyze the existence conditions, A^1, M^1, C^1 will be transformed into a special structure.

A. Overall coordinate transformation

In the following analysis, i is an integer $1 \le i \le k$ unless otherwise specified. To achieve a convenient representation of A^1, M^1, C^1, parts of the transformations T^i_1, T^i_2 and $\bar{T}^i$ (from steps 2 and 4 in the algorithm in Section II-A) will be used. However, some modifications need to be made to $T^i_1, T^i_2, \bar{T}^i$ as the structure that will be aimed for will be of different order from the original system. Notice that for each observer, the system undergoes two transformations; the first one involves T^i_1 and T^i_2 which transforms the state and fault respectively so that the structures of M^i and C^i in (10) - (11) are achieved; the second transformation involves $\bar{T}^i$, implemented on the augmented system to obtain the structure of $\bar{A}^i$ in (18). It can be seen from the process described in Section II-A that to get to the system for the next observer design, there is an augmentation of h states (step 3), followed by the removal of the bottom m^1 (or p) states due to the sliding motion, and finally the addition of $m^1 - m^i$ states to the bottom of the state vector to obtain the next intermediate system (step 6). To obtain the system for the i-th observer, this process is repeated $i-1$ times on the original system (of order n^1). In order to obtain the transformation matrices for the original system, the process needs to be reversed and applied $i-1$ times to T^i_1, T^i_2 and $\bar{T}^i$.

From T^i_1 remove (from T^i_{11} and T^i_{12} in step 2) the sub-blocks associated with the last $m^1 - m^i$ states (i.e. the last $m^1 - m^i$ columns together with the relevant rows to make T^i_{11} and T_{12} square and invertible). Then add m^1 states to the bottom of the state space, by augmenting the truncated T^i_{11}, T^i_{12} with I_{m^1}, and then remove the first h rows and columns. Repeat this process $i-1$ times. Define the first transformation to be applied to the state of the original system as $T^i_a := T^i_{a,2} T^i_{a,1}$ where $T^i_{a,1} := diag\left\{ I_{n^1 - \Sigma^i_{j=1} m^j}, (D^i)^{-1}, I_{\Sigma^{i-1}_{j=1} m^j} \right\}$ and

$$T^i_{a,2} := \begin{bmatrix} I_{n^1 - \Sigma^{i-1}_{j=1} m^j - r^i} & -\tilde{M}^i \\ 0 & I_{\Sigma^{i-1}_{j=1} m^j + r^i} \end{bmatrix}, \tilde{M}^i = \begin{bmatrix} M^i_{12}(M^i_{22})^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that for systems 1 to i, the number of potential faults remain as q. Therefore, the transformation for the fault applied to the original system is identical to T^i_f defined in step 2.

From $\bar{T}^i$ in (17), remove the first h rows and columns (because it is applied to the augmented system) and repeat the process that was applied to T^i_1. The second state transformation be applied to the original system is $T^i_b = diag\left\{ U^i_2, U^i_1, I_{\Sigma^{i-1}_{j=1} m^j + r^i} \right\}$. As the algorithm is exited at step 2 of the k-th iteration, it is clear that the coordinate transformation in step 2 is performed k times, whereas the transformation in step 4 is performed only $k-1$ times. For convenience of analysis in this section, the transformations T_b and T_a (steps 2 and 4 of the algorithm) are also performed on the k-th system.

Define $T^i_{ba} := T^i_b T^i_a$ and also the following matrices

$$T_x : = T^k_{ba} T^{k-1}_{ba} T^{k-2}_{ba} ... T^2_{ba} T^1_{ba} \tag{49}$$

$$T_f : = T^k_f T^{k-1}_f ... T^3_f T^2_f T^1_f \tag{50}$$

Then perform the change of coordinates such that $x^1 \mapsto T_x x^1$, $f^1 \mapsto f^k := T_f f^1$. By using the relationship in (4) and (6) - (7) when applying the transformation T^i_a, and (15) and (16) when applying the transformation T^i_b, the following structure for $A^1 \mapsto T_x A^1 (T_x)^{-1}$ is obtained:

$$\left[\begin{array}{cc|c|c|c|c} U^k_2 \tilde{A}^k_1 (U^k_2)^{-1} & \star & ... & \star & \star & \star \\ U^k_1 \tilde{A}^k_{31} (U^k_2)^{-1} & \star & ... & \star & \star & \star \\ \tilde{A}^k_{32} & \star & ... & \star & \star & \star \\ \hline 0 & J^k & ... & \star & \star & \star \\ 0 & 0 & ... & \star & \star & \star \\ \star & \star & ... & \star & \star & \star \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & ... & J^3 & \star & \star \\ 0 & 0 & ... & 0 & \star & \star \\ \star & \star & ... & \star & \star & \star \\ \hline 0 & 0 & ... & 0 & J^2 & \star \\ 0 & 0 & ... & 0 & 0 & \star \\ \star & \star & ... & \star & \star & \star \end{array}\right] \begin{array}{l} \updownarrow n^1 - \Sigma^k_{j=1} m^j \\ \updownarrow m^k - r^k \\ \updownarrow r^k \\ \updownarrow m^k \\ \updownarrow m^{k-1} - m^k - r^{k-1} \\ \updownarrow r^{k-1} \\ \vdots \\ \updownarrow m^3 \\ \updownarrow m^2 - m^3 - r^2 \\ \updownarrow r^2 \\ \updownarrow m^2 \\ \updownarrow m^1 - m^2 - r^1 \\ \updownarrow r^1 \end{array} \tag{51}$$

where $J^i := D^i diag\left\{ (U^i_1)^{-1}, I_{r^i} \right\}$. Then by using (4), M^1 is transformed to $T_x M^1 T_f^{-1}$ with the structure

$$
\left[\begin{array}{c|c|c|c|c|c}
U_2^k M_{11}^k & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & M_{22}^k & \ldots & 0 & 0 & 0 \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & M_{22}^3 & 0 & 0 \\ \hline
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & M_{22}^2 & 0 \\ \hline
0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & M_{22}^1
\end{array}\right]
\begin{array}{l}
\updownarrow n^1-\Sigma_{j=1}^k m^j \\
\updownarrow m^k-r^k \\
\updownarrow r^k \\
\vdots \\
\updownarrow m^3-r^3 \\
\updownarrow r^3 \\
\updownarrow m^2-r^2 \\
\updownarrow r^2 \\
\updownarrow m^1-r^1 \\
\updownarrow r^1
\end{array}
\qquad (52)
$$

where $rank(M_{11}^k) = q - \Sigma_{j=1}^k r^j$. Note that J^i, M_{22}^i and $\tilde{A}_1^k$ are square (which determine the column widths in (51) and (52)), and $\tilde{A}_1^k, \tilde{A}_{31}^k, \tilde{A}_{32}^k$ have no particular structure. Also

$$
C^1 \mapsto C^1 T_x^{-1} = \left[\begin{array}{cc} 0 & D^1 \end{array} \right], \ \det(D^1) \neq 0 \qquad (53)
$$

For ease of analysis, it is convenient to first perform a change of coordinates using the following:

Proposition 1: There exists a change of coordinates such that A^1 in (51) can be written as

$$
\left[\begin{array}{c|c|c|c|c|c}
U_2^k \tilde{A}_1^k (U_2^k)^{-1} & \star & \ldots & \star & \star & \star \\
U_1^k \tilde{A}_{31}^k (U_2^k)^{-1} & \star & \ldots & \star & \star & \star \\
\tilde{A}_{32}^k & \star & \ldots & \star & \star & \star \\ \hline
0 & J^k & \ldots & 0 & 0 & \star \\
0 & 0 & \ldots & 0 & 0 & \star \\
\star & \star & \ldots & \star & \star & \star \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & \ldots & J^3 & 0 & \star \\
0 & 0 & \ldots & 0 & 0 & \star \\
\star & \star & \ldots & \star & \star & \star \\ \hline
0 & 0 & \ldots & 0 & J^2 & \star \\
0 & 0 & \ldots & 0 & 0 & \star \\
\star & \star & \ldots & \star & \star & \star
\end{array}\right]
\begin{array}{l}
\updownarrow n^1-\Sigma_{j=1}^k m^j \\
\updownarrow m^k-r^k \\
\updownarrow r^k \\
\updownarrow m^k \\
\updownarrow m^{k-1}-m^k-r^{k-1} \\
\updownarrow r^{k-1} \\
\vdots \\
\updownarrow m^3 \\
\updownarrow m^2-m^3-r^2 \\
\updownarrow r^2 \\
\updownarrow m^2 \\
\updownarrow m^1-m^2-r^1 \\
\updownarrow r^1
\end{array}
\qquad (54)
$$

In this coordinate system, the structures of M^1 in (52) and C^1 from (53) remain unchanged.

Proof: Define a transformation matrix H^i ($0 \leq i < k$) with the structure

$$
\begin{bmatrix}
I_{n^1-\Sigma_{j=1}^{k-1} m^j} & 0 & 0 & 0 \\
0 & I_{\Sigma_{j=k-i+1}^{k-1} m^j} & \bar{E}^i & 0 \\
0 & 0 & I_{m^{k-i}} & 0 \\
0 & 0 & 0 & I_{\Sigma_{j=1}^{k-i-1} m^j}
\end{bmatrix}
\qquad (55)
$$

where $\bar{E}^i$ is

$$
\left[\begin{array}{cc}
-E_{1(i-1)}^1 (J^{k-i+1})^{-1} & 0 \\
-E_{1(i-1)}^2 (J^{k-i+1})^{-1} & 0 \\
0 & 0 \\
\vdots & \vdots \\
-E_{(i-1)(i-1)}^1 (J^{k-i+1})^{-1} & 0 \\
-E_{(i-1)(i-1)}^2 (J^{k-i+1})^{-1} & 0 \\
0 & 0
\end{array}\right]
\begin{array}{l}
\updownarrow m^k \\
\updownarrow m^{k-1}-m^k-r^{k-1} \\
\updownarrow r^{k-1} \\
\\
\updownarrow m^{k-i+2} \\
\updownarrow m^{k-i+1}-m^{k-i+2}-r^{k-i+1} \\
\updownarrow r^{k-i+1}
\end{array}
$$

where the elements E will be formally defined below. Define $\bar{H}^i := H^i H^{i-1} \ldots H^2 H^1$. It can be seen that $H^1 = I_{n^1}$. Then define $\breve{A}^i$ which has the structure

$$
\left[\begin{array}{c|c|c|c|c|c|c|c}
U_2^k \tilde{A}_1^k (U_2^k)^{-1} & \star & \ldots & \star & \star & \ldots & \star & \star \\
U_1^k \tilde{A}_{31}^k (U_2^k)^{-1} & \star & \ldots & \star & \star & \ldots & \star & \star \\ \hline
0 & J^k & \ldots & 0 & E_{1i}^1 & \ldots & \star & \star \\
0 & 0 & \ldots & 0 & E_{1i}^2 & \ldots & \star & \star \\
\star & \star & \ldots & \star & \star & \ldots & \star & \star \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
0 & 0 & \ldots & J^{k-i+1} & E_{ii}^1 & \ldots & \star & \star \\
0 & 0 & \ldots & 0 & E_{ii}^2 & \ldots & \star & \star \\
\star & \star & \ldots & \star & \star & \ldots & \star & \star \\ \hline
0 & 0 & \ldots & 0 & J^{k-i} & \ldots & \star & \star \\
0 & 0 & \ldots & 0 & 0 & \ldots & \star & \star \\
\star & \star & \ldots & \star & \star & \ldots & \star & \star \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
0 & 0 & \ldots & 0 & 0 & \ldots & J^2 & \star \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \star \\
\star & \star & \ldots & \star & \star & \ldots & \star & \star
\end{array}\right]
\qquad (56)
$$

First, perform the transformation $\bar{H}^1 A^1 (\bar{H}^1)^{-1}$ to obtain $\breve{A}^1 = A^1$ in (51) (because $H^1 = I_{n^1}$), from which E_{11}^1, E_{11}^2 can be obtained. Then, H^2 (and $\bar{H}^2$) can be calculated, and it can be shown that $\bar{H}^2 A^1 (\bar{H}^2)^{-1} = \breve{A}^2$. The matrices $E_{12}^1, E_{12}^2, E_{22}^1, E_{22}^2$ can then be obtained from $\breve{A}^2$, and H^3 (and $\bar{H}^3$) can be calculated to get $\bar{H}^3 A^1 (\bar{H}^3)^{-1} = \breve{A}^3$. Repeat the process until $\breve{A}^{k-1} := \bar{H}^{k-1} A^1 (\bar{H}^{k-1})^{-1}$ is obtained. It can be shown that $\breve{A}^{k-1}$ is identical to A^1 in (54). ■

From this canonical form, the following subsections seek to recast Conditions A1 and A2 in terms of the original system matrices A^1, M^1, C^1. The main results in the paper are summarized in the following theorems.

Theorem 2: Condition A1 is satisfied if and only if

$$
rank(\Xi^k) - rank(\Xi^{k-1}) = rank(M^1) \qquad (57)
$$

where $\Xi^i \in \mathbb{R}^{ip \times iq}$ ($0 \leq i \leq k$) is defined by

$$
\Xi^i = \begin{bmatrix}
\Pi^0 & 0 & \ldots & 0 \\
\Pi^1 & \Pi^0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Pi^{i-1} & \Pi^{i-2} & \ldots & \Pi^0
\end{bmatrix}
\qquad (58)
$$

where $\Pi^i := C^1 (A^1)^i M^1$. ♯

Theorem 3: Condition A2 is satisfied if and only if the triple (A^1, M^1, C^1) is minimum phase. ♯

The following subsections present constructive proofs of Theorems 2 and 3.

B. Proof of Theorem 2

Condition A1 is satisfied if and only if $\bar{r}^k = q$ which implies that $M_{11}^k = \phi$ (the empty matrix).

Let K^1 be the last m^1 columns of A^1 in (54) and define $A_o := A^1 - K^1 (C_2^1)^{-1} C^1$. Therefore A_o is identical to A^1 in

(54) except that the last m^1 columns of A_o are zero. It can then be shown that $C^1 A_o^{-1}$ can be expanded to be

$$\begin{array}{c} \begin{array}{cccccc} \overset{n^1-\Sigma_{j=1}^i m^j}{\leftrightarrow} & \overset{m^i}{\leftrightarrow} & \overset{m^{i-1}}{\leftrightarrow} & & \overset{m^2}{\leftrightarrow} & \overset{m^1}{\leftrightarrow} \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & I_{m^i} & 0 & \ldots & 0 & 0 \\ \star & \star & \star & \ldots & \star & 0 \end{array} \right] \begin{array}{l} \updownarrow p-m^i-\bar{r}^{i-1} \\ \updownarrow m^i \\ \updownarrow \bar{r}^{i-1} \end{array} \end{array} \quad (59)$$

where F^i is invertible, defined by $F^i := \bar{D}^1 \bar{D}^2 ... \bar{D}^{i-1} \bar{D}^i$ with $\bar{D}^j := diag\left\{I_{p-m^j-\bar{r}^{j-1}}, J^j, I_{\bar{r}^{j-1}}\right\}$

By multiplying $C^1 A_o^{i-1}$ with M^1 in (52) it can be shown that $C^1 A_o^{i-1} M^1 = F^i N^i$ where $N^i \in \mathbb{R}^{p \times q}$ is defined by

$$\begin{array}{c} \begin{array}{cccccc} \overset{q-\Sigma_{j=1}^i r^j}{\leftrightarrow} & \overset{r^i}{\leftrightarrow} & \overset{r^{i-1}}{\leftrightarrow} & & \overset{r^2}{\leftrightarrow} & \overset{r^1}{\leftrightarrow} \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & M_{22}^i & 0 & \ldots & 0 & 0 \\ \star & \star & \star & \ldots & \star & 0 \end{array} \right] \begin{array}{l} \updownarrow p-\bar{r}^i \\ \updownarrow r^i \\ \updownarrow \bar{r}^{i-1} \end{array} \end{array} \quad (60)$$

Proposition 2: For all positive integers $v > i$ the following matrix identity holds: $F^i N^i = F^v N^i$

Proof: It can be shown that

$$F^v N^i = \overbrace{\bar{D}^1 \bar{D}^2 ... \bar{D}^{i-1} \bar{D}^i}^{F^i} \bar{D}^{i+1} ... \bar{D}^v N^i = F^i \bar{D}^{i+1} ... \bar{D}^v N^i$$

From the definition of $\bar{D}^i$, it can be seen that pre-multiplying any matrix with $\bar{D}^i$ affects only the top $p - \bar{r}^{i-1}$ rows of the matrix. In addition, by knowing that $\bar{r}^{i+1} \geq \bar{r}^i$ (since $\bar{r}^{i+1} =: \bar{r}^i + r^{i+1}$) and that the top $p - \bar{r}^i$ rows of N^i are zero (see (60)), it can be concluded that $\bar{D}^{i+1} ... \bar{D}^v N^i = N^i$. Hence the proof is complete. ■

Define $\Pi_o^i := C^1 (A_o^1)^i M^1$ and

$$\Psi^i := \left[\begin{array}{cccc} \Pi_o^0 & 0 & \ldots & 0 \\ \Pi_o^1 & \Pi_o^0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_o^{i-1} & \Pi_o^{i-2} & \ldots & \Pi_o^0 \end{array} \right] \quad (61)$$

then the following result can be established:

Proposition 3: The matrix Ψ^i has rank $\Sigma_{j=1}^i (i+1-j) r^j$

Proof: It can be easily shown that

$$\Psi^i = \left[\begin{array}{cccc} F^1 N^1 & 0 & \ldots & 0 \\ F^2 N^2 & F^1 N^1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F^i N^i & F^{i-1} N^{i-1} & \ldots & F^1 N^1 \end{array} \right] \quad (62)$$

By using Proposition 2, Ψ^i in (62) is equivalent to $\Psi^i = diag\left\{F^i, F^i, ..., F^i, F^i\right\} N$ where

$$N := \left[\begin{array}{cccc} N^1 & 0 & \ldots & 0 \\ N^2 & N^1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N^i & N^{i-1} & \ldots & N^1 \end{array} \right]$$

By expanding N^i from (60), it follows that $rank(N^i) = r^i + 2r^{i-1} + 3r^{i-2} + ... + (i-1)r^2 + ir^1$. Since F^i is square and invertible, the proof is complete. ■

Proposition 4: Define $R^1 := -K^1 (C_2^1)^{-1}$. For any positive integer i the following identity holds

$$C^1 (A^1)^i = C^1 A_o^i - \Sigma_{h=1}^i C^1 (A)^{h-1} R^1 C^1 A_o^{i-h} \quad (63)$$

Proof: By straightforward induction. ■

Corollary 1: The matrices Ξ^i from (58) and Ψ^i from (61) have equal rank.

Proof: Define $\Pi_K^i := -C^1 K^i (C_2^1)^{-1}$ and the following matrix which by construction is square and invertible

$$\Phi^i := \left[\begin{array}{ccccc} I_p & 0 & 0 & \ldots & 0 \\ \Pi_K^1 & I_p & 0 & \ldots & 0 \\ \Pi_K^2 & \Pi_K^1 & I_p & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_K^{i-2} & \Pi_K^{i-3} & \Pi_K^{i-4} & \ldots & I_p \end{array} \right]$$

From Proposition 4, it is clear that $\Phi^i \Psi^i = \Xi^i$ and hence $rank(\Psi^i) = rank(\Xi^i)$ since Φ^i is square and invertible. ■

From Corollary 1 and Proposition 3, it is clear that $rank(\Xi^i) = \Sigma_{j=1}^i (i+1-j) r^j$. Then it follows:

$$\begin{aligned} & rank(\Xi^k) - rank(\Xi^{k-1}) \\ &= \Sigma_{j=1}^k (k+1-j) r^j - \Sigma_{j=1}^{k-1} (k-j) r^j \\ &= r^k + \Sigma_{j=1}^{k-1} (k+1-j) r^j - \Sigma_{j=1}^{k-1} (k-j) r^j \\ &= \Sigma_{j=1}^k r^j = \bar{r}^k \end{aligned} \quad (64)$$

Notice that the LHS of (64) is given in terms of the original system matrices A^1, M^1, C^1. Hence, Condition A1 can be re-cast in terms of the original system matrices as

$$rank(\Xi^k) - rank(\Xi^{k-1}) = rank(M^1) \quad (65)$$

From the algorithm in section II-A, note that for each iteration, one observer is needed. Furthermore, the algorithm is exited at the k-th iteration, which therefore implies that k observers are necessary and sufficient to reconstruct the fault. Hence, the results in this section also indicate precisely the number of observers that are required. Using the results of Theorem 1, the scheme in this paper can never reconstruct the faults when $rank(\Xi^{n^1}) - rank(\Xi^{n^1-1}) < rank(M^1)$ which results in $k \leq n^1$. Hence Theorem 2 is proven. □

The results of this section now enable the designer to systematically investigate the success of this scheme. The designer can construct Ξ^i and increment i systematically from 1 until $rank(\Xi^i) - rank(\Xi^{i-1}) = rank(M^1)$ is satisfied, and that value of i is set to be k. In addition, the user can also know the number of observers required, as well as when the scheme in this paper will fail.

C. Condition A2

Assume that A1 is already satisfied, i.e. $M_{11}^k = \phi$ (the empty matrix). Then from [25], observer k will have a stable sliding motion if and only if (A^k, M^k, C^k) is minimum phase.

Proposition 5: (A^k, M^k, C^k) is minimum phase if and only if (A^1, M^1, C^1) is minimum phase.

Proof: The invariant zeros of (A^k, M^k, C^k) are given by the values of s that make the following matrix pencil lose rank

$$P_{11}(s) := \left[\begin{array}{cc} sI - A^k & M^k \\ C^k & 0 \end{array} \right]$$

where $P_{11}(s)$ is commonly known as the Rosenbrock matrix of (A^k, M^k, C^k). Substitute for (A^k, M^k, C^k) from (9) - (11)

and $M_{11}^k = \phi$. Since $C_2^k, \bar{M}_{22}^k$ are square and invertible, then $P_{11}(s)$ loses rank if and only $P_{12}(s)$ loses rank, where

$$P_{12}(s) := \begin{bmatrix} sI - \bar{A}_{\Omega}^{k-1} & 0 \\ \star & sI - \tilde{A}_1^k \\ \star & -\tilde{A}_{31}^k \\ \star & 0 \end{bmatrix}$$

However, $\bar{A}_{\Omega}^{k-1}$ is stable, and hence the only possible unstable zeros of (A^k, M^k, C^k) are the unobservable modes of $(\tilde{A}_1^k, \tilde{A}_{31}^k)$.

Let $P_{21}(s)$ be the Rosenbrock matrix of (A^1, M^1, C^1). Then substitute for (A^1, M^1, C^1) from (51) - (53) into $P_{21}(s)$. Because J^i, M_{22}^i are nonsingular, and assuming that A1 is already satisfied ($M_{11}^k = \phi$), then it can be shown that $P_{21}(s)$ loses rank if and only if the following matrix pencil loses rank

$$P_{22}(s) = \begin{bmatrix} sI - U_2^k \tilde{A}_1^k (U_2^k)^{-1} \\ -U_1^k \tilde{A}_{31}^k (U_2^k)^{-1} \end{bmatrix} = \begin{bmatrix} U_2^k & 0 \\ 0 & U_1^k \end{bmatrix} \begin{bmatrix} sI - \tilde{A}_1^k \\ -\tilde{A}_{31}^k \end{bmatrix} (U_2^k)^{-1}$$

Since U_1^k and U_2^k are invertible, using the Popov-Hautus-Rosenbrock (PHR) rank test [21], the invariant zeros of (A^1, M^1, C^1) are the unobservable modes of $(\tilde{A}_1^k, \tilde{A}_{31}^k)$. It follows that (A^k, M^k, C^k) and (A^1, M^1, C^1) have the same unstable zeros. ■

From (35), the reduced order sliding motion matrix for the i-th observer ($i < k$) is $\bar{A}_1^i + L_o^i \bar{A}_{31}^i$. In order for the sliding motion matrix to be stable, it requires that $(\bar{A}_1^i, \bar{A}_{31}^i)$ be detectable.

Proposition 6: The undetectable modes (if any) for observer i are given by the undetectable modes of $(\tilde{A}_1^i, \tilde{A}_{31}^i)$.

Proof: The unobservable modes of observer i are the unobservable modes of $(\bar{A}_1^i, \bar{A}_{31}^i)$, which (from the PHR rank test) are given by the values of s that make the following matrix pencil lose rank

$$P_{31}^i(s) = \begin{bmatrix} sI - \bar{A}_1^i \\ -\bar{A}_{31}^i \end{bmatrix}$$

Substituting from (18) into $P_{31}^i(s)$, it is clear that $P_{31}^i(s)$ loses rank if and only if $P_{32}^i(s)$ loses rank, where

$$P_{32}^i(s) := \begin{bmatrix} sI - \bar{A}_{\Omega}^i & 0 \\ \star & sI - \tilde{A}_{11}^i \\ \star & -\tilde{A}_{13}^i \end{bmatrix}$$

However, $\bar{A}_{\Omega}^i$ is stable, hence values of $s \in \mathbb{C}_+$ at which $P_{31}^i(s)$ lose rank are the undetectable modes of $(\tilde{A}_{11}^i, \tilde{A}_{13}^i)$.

By carrying out the PHR rank test on $(\tilde{A}_1^i, \tilde{A}_{31}^i)$ and substituting from (15) and (16), it is clear that the unobservable modes of $(\tilde{A}_1^i, \tilde{A}_{31}^i)$ are the unobservable modes of $(\tilde{A}_{11}^i, \tilde{A}_{13}^i)$. Therefore the undetectable modes of observer i are the undetectable modes of $(\tilde{A}_1^i, \tilde{A}_{31}^i)$. ■

Proposition 7: The unobservable modes of $(\tilde{A}_1^i, \tilde{A}_{31}^i)$ are a subset of the unobservable modes of $(\tilde{A}_1^{i+1}, \tilde{A}_{31}^{i+1})$ when $i < k$.

Proof: From the proof of Proposition 6, $(\tilde{A}_1^i, \tilde{A}_{31}^i)$ and $(\tilde{A}_{11}^i, \tilde{A}_{13}^i)$ have the same unobservable modes. Define D_x^{i+1} to be the bottom r^{i+1} rows of $(D^{i+1})^{-1}$. From (6) - (7), it can be shown that

$$\begin{bmatrix} I & -M_{12}^{i+1}(M_{22}^{i+1})^{-1} D_x^{i+1} \\ 0 & D^{i+1} \end{bmatrix} \begin{bmatrix} sI - \tilde{A}_{11}^i \\ -\tilde{A}_{13}^i \end{bmatrix} = \begin{bmatrix} sI - \tilde{A}_1^{i+1} \\ -\tilde{A}_3^{i+1} \end{bmatrix} = \begin{bmatrix} sI - \tilde{A}_1^{i+1} \\ -\tilde{A}_{31}^{i+1} \\ -\tilde{A}_{32}^{i+1} \end{bmatrix} \quad (66)$$

Since $\det(D^{i+1}) \neq 0$, any unobservable modes of $(\tilde{A}_{11}^i, \tilde{A}_{13}^i)$ (or equivalently, the unobservable modes of $(\tilde{A}_1^i, \tilde{A}_{31}^i)$) will be a subset of the unobservable modes of $(\tilde{A}_1^{i+1}, \tilde{A}_{31}^{i+1})$. ■

If (A^1, M^1, C^1) is not minimum phase, then a stable sliding motion for observer k does not exist [25]. But, if (A^1, M^1, C^1) is minimum phase, then a stable sliding motion exists for observer k, and $(\tilde{A}_1^k, \tilde{A}_{31}^k)$ is detectable. Then from Proposition 7, $(\tilde{A}_1^i, \tilde{A}_{31}^i)$ is also detectable for $i < k$, which implies that stable sliding motions exist for all previous observers (Proposition 6). Hence, A2 is satisfied if and only if (A^1, M^1, C^1) is minimum phase and Theorem 3 is proven. □

IV. Design Example

The method proposed in this paper will now be demonstrated using a model of a 2-cart system shown in Figure 2.

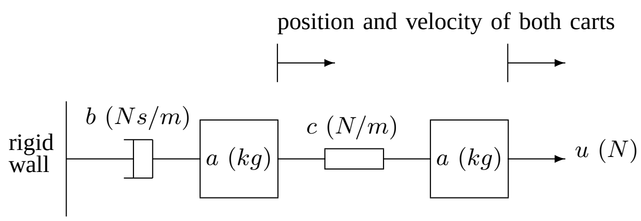

Fig. 2. The schematic diagram of the 2-cart system.

The first cart is connected to a rigid wall via a damper, and is connected to a second cart by a spring. An external force is then applied to the second cart via an actuator. Assume both carts have a nominal mass of $a = 1\ kg$, the damper has a nominal constant of $b_o = 2\ Ns/m$ and the spring has a nominal constant of $c_o = 1\ N/m$. Assume that the positions of both carts are measurable and the control input is the force command. Assume that the force on the second cart is achieved from the force command via an actuator modelled as a first order lag with a time constant $\tau = 0.2$. If the states are the force, velocity of the first cart, velocity of the second cart, position of the first cart and position of second cart, and if the actuator is faulty, then in the notation of (1), the matrices that describe the system are as follows: $C^1 = \begin{bmatrix} 0 & I_2 \end{bmatrix}$ and

$$A^1 = \begin{bmatrix} -\frac{1}{\tau} & 0 & 0 & 0 & 0 \\ 0 & -\frac{b}{a} & 0 & -\frac{c}{a} & \frac{c}{a} \\ \frac{1}{a} & 0 & 0 & \frac{c}{a} & -\frac{c}{a} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M^1 = \begin{bmatrix} \frac{1}{\tau} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Further suppose that the spring and damper constants are imprecisely known; their actual values can deviate respectively by $\pm 2\%$ and $\pm 10\%$ of their nominal known values. Therefore the state equation of the system becomes

$$\dot{x}^1 = (A^1 + \triangle A^1)x + M^1 f^1 \quad (67)$$

where $\triangle A^1$ is the discrepancy between the known matrix A^1 and its actual value. Notice that the 1st, 4th and 5th rows of the matrix A^1 do not contain any uncertainty due to the nature of the state equations. Hence, any parametric uncertainty will appear in the second and third and fourth rows of A^1. Equation (67) can be placed in the framework of (1) by writing

$$\triangle A^1 x^1 = \underbrace{\begin{bmatrix} 0_{1\times 2} \\ I_2 \\ 0_{2\times 2} \end{bmatrix}}_{Q^1} \underbrace{\begin{bmatrix} 0 & \Delta b & 0 & -\Delta c & \Delta c \\ 0 & 0 & 0 & \Delta c & -\Delta c \end{bmatrix}}_{E} x^1 \quad (68)$$

From (68), the disturbance $\xi^1 = Ex^1$ will be generated by the states x^1, which is in turn generated by the fault f^1. Notice that the method in [10] cannot be used on this system as there is no consideration of the disturbance ξ^1. If the signals f^1 and ξ^1 are augmented to form a new 'fault' vector, as in [22], this results in the new 'fault' vector having 3 components. The number of outputs in this system is only 2, resulting in a 'more faults than outputs' scenario, and hence the method in [10], [23] is still not applicable. In addition, it can be verified that $C^1M^1 = C^1A^1M^1 = 0_{2\times 1}, C^1(A^1)^2M^1 = \begin{bmatrix} 0 & 5 \end{bmatrix}^T$. Hence $rank(\Xi^2) - rank(\Xi^1) < rank(M^1)$, and the method in [20] will also not be applicable. However, it can be shown that $rank(\Xi^3) - rank(\Xi^2) = rank(M^1)$ (hence $k = 3$), hence the fault can be reconstructed using the method in this paper, specifically 3 observers in cascade. It can be established that $n^1 = 5, p = 2, q = 1, h = 2, \bar{r}^1 = 0$.

A. Design of observers

Performing the transformation for A^1, M^1, C^1, Q^1 given in step 2 in the algorithm, where appropriate values for T_1^1, T_2^1 are $T_1^1 = I_5,\ T_2^1 = 1$. It can be shown that $M_1^1 = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}^T$, $M_2^1 = 0_{2\times 1},\ M_{22}^1 = \alpha,\ C_2^1 = I_2$.

From (67) - (68), the disturbance is generated as $\xi^1 = E(sI - (A^1 + \Delta A^1))^{-1}M^1 f^1$. Since the bounds on Δb and Δc are known, bounds on the crossover frequencies for the transfer function $G_\xi(s) := E(sI - (A^1 + \Delta A^1)^{-1}M^1$ can be found from Bode diagrams. It was found that 5 rad/s comfortably upper bounds the crossover frequency of $G_\xi(s)$ and as a result of the high roll-off rate, at 10 rad/s, an approximate attenuation level of -80 dBs is attained for all possible variations of Δb and Δc. Consequently all the frequency content of ξ^1 will be below 10rad/s. In some situations where the disturbance ξ^1 represents a physical quantity, engineering judgement and practical experience can be used to define suitable bounds on the frequency content of the disturbances: see for example [5], [16], [18]. Hence the filter matrices that appropriately describe the characteristics of ξ^1 are chosen as $A_\Omega^1 = -\kappa I_2, B_\Omega^1 = \kappa I_2$, where $\kappa = 10 >> 5$. Note the choice of (A_Ω^1, B_Ω^1) is not unique. In this example, first order filter linear realizations have been chosen although higher order linear filters could equally well have been chosen resulting in a different (A_Ω^1, B_Ω^1) pair. The crucial decision is the choice of the filter bandwidth and not the particular choice of filter itself. Here choosing first order filter representations minimize the order $\bar{n}^1$. With this choice of (A_Ω^1, B_Ω^1) an augmented system of dimension $\bar{n}^1 = n^1 + h = 7$ is produced (as in (14)). It can be shown that $m^2 = 2$. Then, to obtain the structures in (18) - (20), a suitable transformation $\bar{T}^1$ is $\bar{T}^1 = I_7$.

For the first observer, $\bar{L}_o^1$ was chosen so that $\lambda(\bar{A}_1^1 + \bar{L}_o^1\bar{A}_{31}^1) = \{-1, -2, -3, -4, -5\}$. Then $\mathcal{A}_s^1 = diag\{-3, -4\}$ was chosen yielding the following:

$$\bar{G}_l^1 = \begin{bmatrix} -370.848 & -32.160 \\ -77.886 & -349.233 \\ -0.359 & -0.068 \\ 45.291 & 4.903 \\ 7.754 & 35.883 \\ -3.686 & -0.728 \\ -1.397 & -1.313 \end{bmatrix}, \bar{G}_n^1 = \begin{bmatrix} 52.978 & 5.360 \\ 11.126 & 58.205 \\ 0.179 & 0.068 \\ -6.686 & -0.728 \\ -1.397 & -5.313 \\ 1.000 & 0 \\ 0 & 1.000 \end{bmatrix}$$

Since $p - m^2 = 0$, then α^1 does not exist. It follows that the parameters for the system associated with the second observer (with order $n^2 = \bar{n}^1 - m^2 = 5$ and the number of outputs $p = 2$) are A^2, M^2, Q^2 respectively being

$$\begin{bmatrix} -10 & 0 & 0 & -5.3601 & -52.9784 \\ 0 & -10 & 0 & -58.2056 & -11.1267 \\ 0 & 0 & -5 & -0.0680 & -0.1798 \\ 0 & 1 & -1 & 5.3134 & 1.3975 \\ 1 & 0 & 0 & 0.7283 & 4.6866 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10I_2 \\ 0_{3\times 2} \end{bmatrix}$$

It is clear that $C^2M^2 = 0$, and hence $\bar{r}^2 = 0$ which results in $r^2 = 0$. Then to obtain the structures of (9) - (11), suitable coordinate transformations T_1^2, T_2^2 are $T_1^2 = I_5,\ T_2^2 = 1$.

Here the matrices A_Ω^2, B_Ω^2 that describe ξ^2 are chosen as $A_\Omega^2 = -\kappa I_2, B_\Omega^2 = \kappa I_2$. The augmented system (14) can then be formed. It can be shown that $m^3 = 1$. To obtain the structure (18) - (20) as in step 4, a suitable transformation matrix $\bar{T}^2$ is

$$\begin{bmatrix} I_3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be seen that $\bar{A}_1^2$ is stable. Hence, a convenient choice is $\bar{L}_o^2 = 0$. Then choosing $\mathcal{A}_s^2 = diag\{-3, -4\}$ results in

$$\bar{G}_l^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -52.9784 & -5.3601 \\ -11.1267 & -58.2056 \\ -10.9469 & -58.1377 \\ 1.3975 & 9.3134 \\ 7.6866 & 0.7283 \end{bmatrix}, \ \bar{G}_n^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The filter scalar α^2 was chosen as 10. It follows that the system for observer 3 will be of order $n^3 = \bar{n}^2 - m^3 = 6$ and the number of outputs is $p = 2$. The matrices A^3, M^3, Q^3 respectively are

$$\begin{bmatrix} -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -5 & -5 & 0 \\ 0 & 0 & 10 & 0 & 0 & -10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 10I_2 \\ 0_{4\times 2} \end{bmatrix}$$

It is obvious that $rank(C^3 M^3) = rank(M^3)$, which confirms the initial check that three observers are necessary and sufficient to reconstruct the fault f^1. Finally, a sliding mode observer can be designed based on A^3, M^3, C^3, Q^3 using step 7 of the algorithm. It is clear that a choice of $D^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ places A^3, M^3 in the structure of (9) - (10). Choosing $A_s^3 = diag\{-3, -4\}$ and minimizing γ subject to (45) yielded $\gamma = 1.2097, W_1 = 0$ and

$$\bar{G}_l^3 = \begin{bmatrix} 0 & -17.4555 \\ 0 & 0 \\ 0 & 19.4717 \\ 0 & 0 \\ -2 & 0 \\ 0 & 10.0346 \end{bmatrix}, \ \bar{G}_n^3 = \begin{bmatrix} 0 & 2.9093 \\ 0 & 0 \\ 0 & 1.6035 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

B. Simulation results

For observer 1, the gains were chosen as $\psi_1^1 = \psi_2^1 = 2\sqrt{50}, \beta_1^1 = \beta_2^1 = 50, \gamma_1^1 = 197.5, \gamma_2^1 = 351.1$. For observers 2 and 3, the same gains were chosen. Firstly, the nominal uncertainty-free situation will be considered, where $\Delta b = 0, \Delta c = 0 \Rightarrow \Delta A^1 = 0 \Rightarrow \xi^1 = 0$. The left subfigure of Figure 3 shows the applied fault, and the right subfigure shows the reconstruction. It is clear that the reconstruction is a visually perfect replica of the fault, which shows that any degradation in accuracy due to the cascade observer scheme is not significant. The remaining simulations are associated with the presence of uncertainty: specifically when $\Delta b = 0.2$ and $\Delta c = 0.02$. The left subfigure of Figure 5 shows the disturbances ξ^1 that arise from the applied fault. The left subfigure of Figure 4 shows the fault reconstruction. The right subfigure of Figure 5 shows ξ^3 which is a fictitious signal obtained from ξ^1 by performing the operation $\xi^2 = \frac{1}{\kappa}\dot{\xi}^1 + \xi^1, \xi^3 = \frac{1}{\kappa}\dot{\xi}^2 + \xi^2$ (which is the reverse of the fictitious filtering of ξ^3 to obtain ξ^1 using $A_\Omega^1 = A_\Omega^2 = -\kappa I_2, B_\Omega^1 = B_\Omega^2 = \kappa I_2$) where $\kappa = 10$. It can be seen in Figure 5 that ξ^3 is almost identical to ξ^1 which implies the weighting function for the disturbance using the values of $A_\Omega^1 = A_\Omega^2 = -\kappa I_2, B_\Omega^1 = B_\Omega^2 = \kappa I_2$ is valid for this example. Although there is a slight degradation due to $\Delta b, \Delta c \neq 0$, the reconstruction is not severely affected by ξ^1 (which is significant – being more than 10% of the magnitude of the fault) because the fault reconstruction scheme has been designed to minimize the upper bound of the $\mathcal{L}_2$ gain from ξ^3 to $\hat{f}^1$ (where $\xi^3 \approx \xi^1$). Then, white noise of standard deviation 10^{-3} has been added to the sensors and the simulation repeated. The right subfigure of Figure 4 shows the fault reconstruction performance. It can be seen that although the fault reconstruction is noisy, the 'underlying signal' is a good approximation to the fault itself. This demonstrates that the fault reconstruction scheme can also cope with the effects of sensor noise, and is practical.

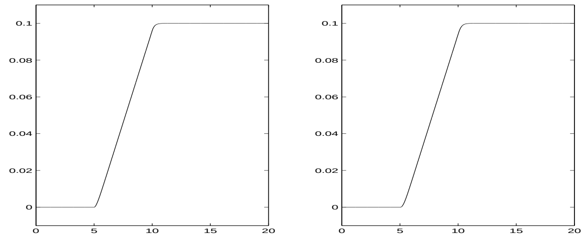

Fig. 3. The simulation where $\Delta b = \Delta c = 0$. The left subfigure is the fault applied to the actuator. The right subfigure is its reconstruction.

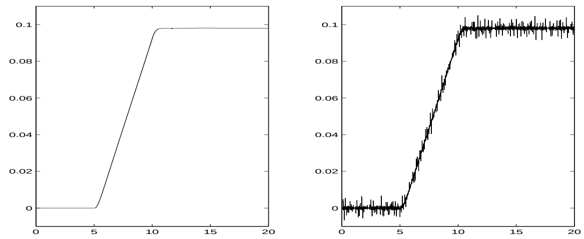

Fig. 4. The left subfigure is the fault reconstruction for $\Delta b = 0.2, \Delta c = 0.02$. The right subfigure is the reconstruction with sensor white noise.

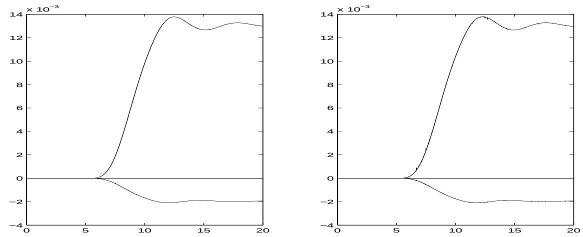

Fig. 5. The left subfigure shows the components of ξ^1. The right subfigure shows the fictitious signal ξ^3.

Additional designs and simulations have been performed, where the values of κ have been varied to investigate the effect of bandwidth choices on the performance of the fault reconstruction scheme. Figure 6 shows the fault reconstructions when $\kappa = 10^{-4}, 10^{-3}, 10^{-2}, 0.1, 1$ and 10. For these values of $\kappa = 10^{-4}, 10^{-3}, 10^{-2}, 0.1$ (all considerably smaller than 10), it can be verified that ξ^3 is not a good approximation of ξ^1, and the fault reconstruction is worse compared to the case when $\kappa = 10$ in Figure 4. It can be noted however, that the fault reconstruction improves as κ progressively moves towards 10. For the cases when $\kappa = 20, 50$ and 70, the quality of the fault reconstruction is indistinguishable from $\kappa = 10$. These simulation results confirm the claims in Remark 2.

V. Conclusion

This paper has presented a new scheme for robust fault reconstruction, using multiple observers in cascade. Signals from one observer are used as outputs of a fictitious system, and the next observer is designed based on the fictitious system. The novelty of this scheme is that it can reconstruct faults in a wider class of systems, compared to previous methods. In addition, the scheme is formulated into a framework which enables the minimization of disturbances on the fault

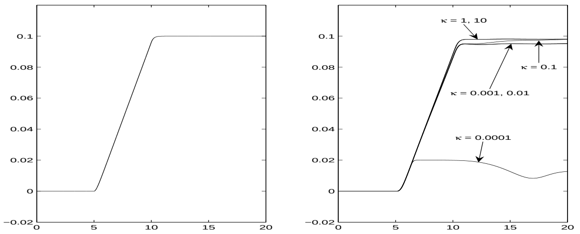

Fig. 6. The left subfigure is the fault applied to the actuator, the right subfigure is its reconstruction for various values of κ.

reconstruction. This is particularly useful in cases when the number of outputs is less than the number of disturbances and faults, a scenario that will render many other multiple observer methods inapplicable. Necessary and sufficient conditions, in terms of original system matrices, have been investigated. This enables the designer to immediately know if the scheme is applicable, something which is absent in some other multiple observer methods. In addition, the results in this paper also indicate precisely the number of observers in cascade that are required and sufficient. A simulation example verifies the effectiveness of the scheme.

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