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# MAPPING PROPERTIES OF BASIC HYPERGEOMETRIC FUNCTIONS

ÁRPÁD BARICZ AND ANBHU SWAMINATHAN

*Dedicated to Professor Tibor K. Pogány*
*on the occasion of his 60th birthday*

*Abstract.* It is known that the ratio of Gaussian hypergeometric functions can be represented by means of $g$-fractions. In this work, the ratio of $q$-hypergeometric functions are represented by means of $g$-fractions that lead to certain results on $q$-starlikeness of the $q$-hypergeometric functions defined on the open unit disk. Corresponding results for the $q$-convex case are also obtained.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, that is, $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. The Gaussian hypergeometric function (see [4]) $F(a,b;c;\cdot)={}_2F_1(a,b;c;\cdot)$ is defined by

$$F(a,b;c;z)=\sum_{n\geqslant 0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $a$, $b$ are complex numbers and $c\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$, and $(\lambda)_n$ is the Pochhammer symbol defined by $(\lambda)_n=\lambda(\lambda+1)\ldots(\lambda+n-1)$, $(\lambda)_0=1$, $n\in\mathbb{N}$. In the sequel, the parameters $a$, $b$ and $c$ will be treated as real parameters unless otherwise specified. The $g$-fraction expansion for the ratio of two hypergeometric functions is given in [18], see also [27, p. 337–339], as

$$\frac{F(a+1,b;c;z)}{F(a,b;c;z)}=\cfrac{1}{1-\cfrac{(1-g_0)g_1z}{1-\cfrac{(1-g_1)g_2z}{1-\cfrac{(1-g_2)g_3z}{1-\ldots}}}}. \tag{1.1}$$

It is known that $0\leqslant g_n\leqslant 1$ holds only when $-1\leqslant a\leqslant c$ and $0\leqslant b\leqslant c$. For $0\leqslant g_n\leqslant 1$, the continued fraction that appears on the right side of (1.1) is called a $g$-fraction. We refer to [18] for further details on this and various other ratios of Gaussian hypergeometric functions. Note that the continued fractions which are ratios of Gauss hypergeometric functions can be studied in fact as Stieltjes functions. Since the reciprocal of Stieltjes functions are in the class of Pick functions [6, 7], it will be of interest to study

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the conditions under which the ratios of various hypergeometric type functions are in the class of Pick functions. For example, the reciprocal function $1/f$ of the function

$$f(z)=\frac{\log\Gamma(z+1)}{z\operatorname{Log} z},\quad z\in\mathbb{C}\setminus(-\infty,0].$$

which is the holomorphic extension of the function defined on the positive real axis is a Pick function [7]. More information on the study of Pick functions can be found in [8].

We also recall that the sequence $\{a_k\}_{k\geqslant 0}$ of non-negative real numbers with $a_0=1$, is called a Hausdorff moment sequence if there is a probability measure (or positive Borel measure) $\mu$ on $[0,1]$ such that

$$a_k=\int_0^1 t^k d\mu(t),\quad k\geqslant 0 \qquad \text{or, equivalently}\quad F(z)=\sum_{k\geqslant 0}a_k z^k=\int_0^1\frac{d\mu(t)}{1-tz}.$$

Note that such $F$ is analytic in the slit domain $\mathbb{C}\setminus[1,\infty)$ and also belongs to the set of Pick functions. For the study of such moment problems, we suggest the interested readers to refer [3, 17, 27]. In particular, we note that Pick functions were studied by Nevanlinna using a moment problem [3], see also [17].

In this paper we collect the following lemma which is quite useful for further discussion.

LEMMA 1.1. [27] *For a real sequence $\{a_n\}_{n\geqslant 0}$, the followings are equivalent:*

(1) $\{a_n\}_{n\geqslant 0}$ *is totally monotone sequence, i.e.,*
$$\Delta^k a_n=\sum_{j=0}^k(-1)^j\binom{k}{j}a_{n+j}\geqslant 0 \text{ for all } k,n\geqslant 0.$$

(2) $\{a_n\}_{n\geqslant 0}$ *is a Hausdorff moment sequence, i.e., there exists a positive Borel measure* $\mu$ *on* $[0,1]$ *with* $a_n=\int_0^1 t^n d\mu(t)$ *for all* $n\geqslant 0$.

(3) *The power series* $\sum_{n\geqslant 0}a_n z^n$ *is analytic in* $\mathbb{D}$ *and has the analytic continuation for all* $z\in\mathbb{C}\setminus[1,\infty)$

$$\int_0^1\frac{1}{1-tz}d\mu(t)=\cfrac{a_0}{1-\cfrac{(1-g_0)g_1z}{1-\cfrac{(1-g_1)g_2z}{1-\cfrac{(1-g_2)g_3z}{1-\dots}}}},$$

*where* $g_n\in[0,1]$ *for all* $n\geqslant 0$.

Note that the equivalence of (2) and (3) is stated in [27, Theorem 69.2] whereas that of (1) and (2) is stated in [27, Theorem 71.1]. The main objective of this work is to identify some members of the class of Pick functions. By virtue of Lemma 1.1, it is clear that finding $g$-sequence of a function will lead to finding members of the class of Pick functions. Thus, in this paper we are interested in the characterization of the ratios of basic hypergeometric functions so that the corresponding continued fractions lead to the geometric properties of Pick functions. The paper is organized as follows. In the next section we present some new results on the quotients of basic hypergeometric functions by following some ideas of Küstner [18] on Gaussian hypergeometric functions, while in section 3 we study the generalized starlikeness (introduced by Ismail et al. [14]) and convexity of basic hypergeometric functions by using the results of section 2. These results complement the main results from [1, 14, 18]. The last section is devoted for concluding remarks.

## 2. Some ratios of basic hypergeometric functions

The basic hypergeometric function $\phi$ of Heine is defined as

$$\phi(a,b;c;q,z) = {}_2\phi_1(a,b;c;q,z) = \sum_{n\geqslant 0} \frac{(a,q)_n(b,q)_n}{(c,q)_n(q,q)_n} z^n,$$

where the $q$-shifted factorials are

$$(\sigma,q)_n = \prod_{j=1}^{n}(1-\sigma q^{j-1}), \qquad n>0, \qquad (\sigma,q)_0 = 1,$$

and it is assumed that $0<q<1$. Note that for $q \nearrow 1$ the expression $(q^a,q)_n/(1-q)^n$ tends to $(a)_n = a(a+1)\ldots(a+n-1)$, and thus the basic hypergeometric series reduces to the well-known Gaussian hypergeometric function. More precisely, we have

$$\lim_{q\nearrow 1} \phi(q^a,q^b;q^c;q,z) = F(a,b;c;z) = \sum_{n\geqslant 0} \frac{(a)_n(b)_n}{(c)_n n!} z^n.$$

Now we focus on the problem proposed at the end of the introduction. To start with, the following result given in [13] (see also [20, p. 320]) is stated as a preliminary result.

LEMMA 2.1. [13] *For the basic hypergeometric function* $\phi(a,b;c;q,z)$ *we have*

$$\frac{\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)} = \cfrac{1}{1-\cfrac{d_1 z}{1-\cfrac{d_2 z}{1-\cfrac{d_3 z}{1-\ldots}}}},$$

*where* $d_n = d_n(a,b,c,q)$ *is given by*

$$d_n = \begin{cases} q^k \dfrac{(1-aq^k)(b-cq^k)}{(1-cq^{2k})(1-cq^{2k+1})} & \text{for} \quad n = 2k+1 \\[2ex] q^k \dfrac{(1-bq^{k+1})(a-cq^{k+1})}{(1-cq^{2k+1})(1-cq^{2k+2})} & \text{for} \quad n = 2k+2 \end{cases}, \tag{2.1}$$

*where* $k \in \{0,1,\ldots\}$.

Note that the continued fraction relation given above means, by definition, that at $z = 0$ the difference of the analytic function on the left hand side and of the $n$th approximant of the continued fraction on the right hand side has a zero of order at least $n$ or, in the terminating case, if $d_{n+1}$ and subsequent coefficients vanish, that this difference is zero.

For the interested reader on this result and various other continued fraction expansions, [11, 13, 14, 15, 27] and references therein may be useful. In particular, the above continued fraction can be obtained as a limiting case of a continued fraction available in [11, p. 488].

Now, we present an application of Lemma 2.1 by using the idea of [18, Theorem 1.5].

THEOREM 2.1. *Let $q \in (0,1)$ and $a,b,c > 0$ be such that $cq \leqslant a \leqslant 1$ and $cq^2 \leqslant bq \leqslant 1$. Then, for a non-decreasing function $\mu_0 : [0,1] \to [0,1]$ with $\mu_0(1) - \mu_0(0) = 1$ we have*

$$\frac{\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)} = \int_0^1 \frac{1}{1-tz} d\mu_0(t) \qquad \textit{for all} \qquad z \in \mathbb{C} \setminus [1,\infty). \tag{2.2}$$

*Proof.* Let us consider the sequence $\{\sigma_n\}_{n \geqslant 0}$, defined by $\sigma_0 = 1$ and for all $n \geqslant 1$

$$\sigma_{2n-1} = q^{n-1} \frac{a - cq^n}{1 - cq^{2n-1}}, \qquad \sigma_{2n} = q^n \frac{b - cq^n}{1 - cq^{2n}}.$$

From the definition of $d_n$ in Lemma 2.1 it can be seen that $d_n = (1-\sigma_n)\sigma_{n-1}$ holds for all $n \geqslant 1$. Now, we let $g_n = 1 - \sigma_n$. This gives

$$\frac{\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)} = \cfrac{1}{1 - \cfrac{(1-g_0)g_1 z}{1 - \cfrac{(1-g_1)g_2 z}{1 - \cfrac{(1-g_2)g_3 z}{1 - \dots}}}}$$

and by the hypothesis of the theorem we have $cq^{2n-1} \leqslant aq^{n-1} \leqslant 1$ and $cq^{2n} \leqslant bq^n \leqslant 1$ for all $n \geqslant 1$, and thus $0 \leqslant g_n \leqslant 1$ for all $n \geqslant 1$, and clearly $0 \leqslant g_0 \leqslant 1$. In this case, the above given continued fraction is called infinite $g$-fraction. Now, according to Lemma 1.1, that is, [27, p. 263, Theorem 69.2], the coefficients in the power series expansion at $z=0$ of the analytic function in the left hand side of the above continued fraction are the Hausdorff moments of a non-decreasing function on $(0,1)$, with infinitely many points of increase (with the total increase of $1$). Thus, there exists a function $\mu_0 : [0,1] \to [0,1]$ that satisfies $0 = \mu_0(0) \leqslant \mu_0(s) \leqslant \mu_0(t) \leqslant \mu_0(1) = 1$ for $0 < s < t < 1$ and its range containing infinitely many points such that

$$\frac{\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)} = \int_0^1 \frac{1}{1-tz} d\mu_0(t) \qquad \text{for all} \qquad z \in \mathbb{C} \setminus [1,\infty),$$

by analytic continuation, with $\mu_0(t) = \mu_0(a,b;c;q,t)$, and the integral being in the sense of Riemann-Stieltjes. $\square$

REMARK 2.1. By substituting $a$ by $q^a$, $b$ by $q^b$ and $c$ by $q^c$, and tending with $q$ to $1^-$, Theorem 2.1 becomes the following: if $c+1 \geqslant a > 0$ and $c+1 \geqslant b > 0$, then there exists a non-decreasing function $\mu_0 : [0,1] \to [0,1]$ with $\mu_0(1) - \mu_0(0) = 1$ such that

$$\frac{F(a,b+1;c+1;z)}{F(a,b;c;z)} = \int_0^1 \frac{1}{1-tz} d\mu_0(t) \qquad \text{for all} \qquad z \in \mathbb{C} \setminus [1,\infty).$$

This result is a natural companion to the results given by Küstner [18, Theorem 1.5].

Moreover, if we substitute $a$ by $aq$ in Theorem 2.1 and then we change there $a$ by $q^a$, $b$ by $q^b$ and $c$ by $q^c$, and tend with $q$ to $1^-$, we obtain the following result: if $c \geqslant a > 0$ and $c+1 \geqslant b > 0$, then there exists a non-decreasing function $\mu_0^* : [0,1] \to [0,1]$ with $\mu_0^*(1) - \mu_0^*(0) = 1$ such that

$$\frac{F(a+1,b+1;c+1;z)}{F(a+1,b;c;z)} = \int_0^1 \frac{1}{1-tz} d\mu_0^*(t) \qquad \text{for all} \qquad z \in \mathbb{C} \setminus [1,\infty).$$

This is a result given by Küstner [18, Theorem 1.5] for $c \geqslant a \geqslant -1$ and $c \geqslant b > 0$.

Finally, we would like to mention that taking into account the above discussion, our Theorem 2.1 is actually a slight modification of a result obtained recently by Agrawal and Sahoo [1, Theorem 2.3].

The following result is an immediate consequence of the above theorem.

COROLLARY 2.1. *Under the hypothesis of Theorem 2.1, $\phi(a,b;c;q,z) \neq 0$, for all $z \in \mathbb{C}$.*

The following result on the geometric properties of an analytic function $f$ is useful for further discussion.

LEMMA 2.2. [18, 21] *Let $\mu : [0,1] \to [0,1]$ be non-decreasing with $\mu(1) - \mu(0) = 1$. Then the function*

$$z \mapsto \int_0^1 \frac{z}{1-tz} d\mu(t)$$

*is analytic in the cut-plane $\mathbb{C} \setminus [1,\infty)$ and maps both the open unit disk $\mathbb{D}$ and the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ univalently onto domains that are convex in the direction of the imaginary axis. If further $\int_0^1 t d\mu(t) > 0$ then the same holds for*

$$z \mapsto \int_0^1 \frac{1}{1-tz} d\mu(t).$$

Here, by a domain convex in the direction of the imaginary axis we mean that every line parallel to the imaginary axis has either connected or empty intersection with the corresponding domain. For details regarding this and related geometric properties, we suggest the interested readers to refer [9].

The next result deals with convexity in the direction of the imaginary axis of some quotients of basic hypergeometric functions. The idea of the proof of this interesting result is taken from [18, Theorem 1.5]. We note that, quite recently some similar results were obtained by Agrawal and Sahoo [1] via the same approach.

THEOREM 2.2. *Let $q\in(0,1)$ and $a,b,c>0$ be such that $cq\leqslant a\leqslant 1$ and $cq^2\leqslant bq\leqslant 1$. Then the functions*

$$z\mapsto \frac{\phi(a,qb;qc;q,z)}{\phi(a,b;c;q,z)},\quad z\mapsto \frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,b;c;q,z)},\quad z\mapsto \frac{\phi(aq,bq;cq^2;q,z)}{\phi(aq,b;cq;q,z)},$$

$$z\mapsto \frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)},\quad z\mapsto \frac{z\phi(a,qb;qc;q,z)}{\phi(a,b;c;q,z)},\quad z\mapsto \frac{z\phi(aq,bq;cq^2;q,z)}{\phi(a,b;c;q,z)},$$

$$z\mapsto \frac{z\phi(aq,bq;cq^2;q,z)}{\phi(aq,b;cq;q,z)},\quad z\mapsto \frac{z\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)}$$

*are analytic in $\mathbb{C}\setminus[1,\infty)$ and each function map both the open unit disk $\mathbb{D}$ and the half plane $\{z\in\mathbb{C}:\operatorname{Re} z<1\}$ univalently onto domains that are convex in the direction of the imaginary axis. Further, if $a\neq c$, then the same result holds for*

$$z\mapsto \frac{\phi(a,b;c;q,qz)}{\phi(a,b;c;q,z)}\quad \textit{and}\quad z\mapsto \frac{z\phi(a,b;c;q,qz)}{\phi(a,b;c;q,z)}.$$

*Proof.* By using Theorem 2.1 we obtain

$$\frac{\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)}=\int_0^1\frac{1}{1-tz}d\mu_0(t)\qquad\text{for all}\qquad z\in\mathbb{C}\setminus[1,\infty),$$

which gives

$$\frac{z\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)}=\int_0^1\frac{z}{1-tz}d\mu_0(t)\qquad\text{for all}\qquad z\in\mathbb{C}\setminus[1,\infty).$$

Now, let $\mu_1(t)=\displaystyle\int_0^t\frac{sd\mu_0(s)}{\lambda}$, where $\lambda>0$. By using $\mu_0(1)-\mu_0(0)=1$ we find that

$$\begin{aligned}1+\lambda\int_0^1\frac{z}{1-tz}d\mu_1(t)&=1+\lambda\int_0^1\frac{z}{1-tz}\frac{1}{\lambda}td\mu_0(t)\\&=\int_0^1\frac{1}{1-tz}d\mu_0(t)=\frac{\phi(a,qb;qc;q,z)}{\phi(a,b;c;q,z)}.\end{aligned}$$

Setting $\lambda=d_1$ as given in (2.1), we write

$$\frac{\phi(a,qb;qc;q,z)}{\phi(a,b;c;q,z)}=1+d_1\int_0^1\frac{z}{1-tz}d\mu_1(t),\quad z\in\mathbb{C}\setminus[1,\infty).\tag{2.3}$$

By using Heine's contiguous relation [10, p. 22]

$$\phi(aq,b;cq;q,z)-\phi(a,b;c;q,z)=z\frac{(1-b)(a-c)}{(1-c)(1-cq)}\phi(aq,bq;cq^2;q,z)$$

we obtain

$$d_1z\phi(aq,bq;cq^2;q,z)=\phi(a,bq;cq;q,z)-\phi(a,b;c;q,z).\tag{2.4}$$

Combining this with (2.3) we obtain that

$$d_1 z \frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,b;c;q,z)} = d_1 \int_0^1 \frac{z}{1-tz} d\mu_1(t), \quad z \in \mathbb{C} \setminus [1,\infty)$$

and hence

$$z \frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,b;c;q,z)} = \int_0^1 \frac{z}{1-tz} d\mu_1(t), \quad z \in \mathbb{C} \setminus [1,\infty), \tag{2.5}$$

where according to [18, Remark 3.2] the function $\mu_1 : [0,1] \to [0,1]$ is also non-decreasing with $\mu_1(1) - \mu_1(0) = 1$. On the other hand, from the continued fraction expansion of the ratio $\phi(a,bq;cq;q,z)/\phi(a,b;c;q,z)$ given in Theorem 2.1, we can write

$$\frac{\phi(aq,bq;cq^2;q,z)}{\phi(aq,b;cq;q,z)} = \cfrac{1}{1 - \cfrac{d_1 z}{1 - \cfrac{d_2 z}{1 - \cfrac{d_3 z}{1 - \ldots}}}}$$

where $d_n = d_n(aq,b,cq,q) = (1-g_{n-1})g_n$ satisfying $0 \leqslant g_n \leqslant 1$ for all $n \geqslant 1$. Note that for this we need the conditions $cq^{2n+1} \leqslant bq^n \leqslant 1$ and $cq^{2n} \leqslant aq^n \leqslant 1$ for all $n \geqslant 1$, however since $q \in (0,1)$ these are certainly satisfied because of the hypothesis of this theorem. Hence for the same conditions on $a$, $b$ and $c$ given in the hypothesis of the theorem, there exist a non-decreasing function $\mu_2 : [0,1] \to [0,1]$ with $\mu_2(1) - \mu_2(0) = 1$ for which we have

$$\frac{\phi(aq,bq;cq^2;q,z)}{\phi(aq,b;cq;q,z)} = \int_0^1 \frac{z}{1-tz} d\mu_2(t), \quad \text{for all } z \in \mathbb{C} \setminus [1,\infty).$$

On the other hand, observe that by using the contiguous relation (2.4) we obtain

$$\frac{\phi(a,bq;cq;q,z)}{\phi(a,b;c;q,z)} = \cfrac{1}{1 - d_1 \cfrac{z\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)}}$$

and thus the quotient

$$\frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)} = \frac{\phi(bq,aq;cq^2;q,z)}{\phi(bq,a;cq;q,z)} = \frac{\phi(\alpha,\beta q;\gamma q;q,z)}{\phi(\alpha,\beta;\gamma;q,z)},$$

where $\alpha = bq$, $\beta = a$ and $\gamma = cq$, can be rewritten as

$$\begin{aligned}\frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)} &= \cfrac{1}{1 - d_1(\alpha,\beta,\gamma) \cfrac{z\phi(\alpha q,\beta q;\gamma q^2;q,z)}{\phi(\alpha,\beta q;\gamma q;q,z)}} \\ &= \cfrac{1}{1 - d_2(a,b,c) \cfrac{z\phi(aq,bq^2;cq^3;q,z)}{\phi(aq,bq;cq^2;q,z)}}.\end{aligned}$$

Continuing in this way we obtain

$$\frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)} = \cfrac{1}{1-\cfrac{d_2 z}{1-\cfrac{d_3 z}{1-\cfrac{d_4 z}{1-\dots}}}},$$

where $d_n = (1-g_{n-1})g_n$ satisfying $0 \leqslant g_n \leqslant 1$ for all $n \geqslant 2$. Thus, there exists a non-decreasing function $\mu_3 : [0,1] \to [0,1]$ satisfying $\mu_3(1) - \mu_3(0) = 1$ such that

$$\frac{\phi(aq,bq;cq^2;q,z)}{\phi(a,bq;cq;q,z)} = \int_0^1 \frac{z}{1-tz} d\mu_3(t), \quad \text{for all } z \in \mathbb{C} \setminus [1,\infty). \tag{2.6}$$

Thus, we have verified that the functions $\mu_0$, $\mu_1$, $\mu_2$ and $\mu_3$ satisfy the conditions of Lemma 2.2, and thus the first eight functions in this theorem indeed satisfy the assertion. The last part of the theorem follows from (3.4) of Theorem 3.1, whenever $c \neq a$. □

We note that the first and fifth ratio given in Theorem 2.2 can be obtained as particular cases of some of the continued fractions given in [15]. In [15], $J$-fractions for some $q$-hypergeometric functions were given which are represented by analogous Stieltjes transforms related to a probability measure $\mu(t)$. See also [15, Theorem 4.1] for another continued fraction for the ratio $\phi(a,b;c;q,z)/\phi(aq,bq;cq;q,z)$. While introducing the associated Askey-Wilson polynomials in [16, p. 215], the ratios of two ${}_8\phi_7$ generalized basic hypergeometric polynomials (see [4, 10] for the details of this technical term) are given with a representation of a measure, which under certain limiting conditions are similar to the ratio given in (2.2) of Theorem 2.1. In particular, equation (4.18) in [16] is used to conjecture that the ${}_8\phi_7$ generalized basic hypergeometric function given in the denominator has no zeros for $z \in \mathbb{D}$. Note that a similar result for the limiting case leading to ${}_2\phi_1$ is proved in Corollary 2.1.

## 3. Geometric properties of basic hypergeometric functions

The continued fraction expansion for the ratio of two basic hypergeometric functions that is obtained in the previous section can have some interesting applications. We exhibit one such application in this section. The following definition is given in [14]: an analytic function $f$ is said to belong to the class $PS_q$ of generalized starlike functions if for all $q \in (0,1)$ and $z \in \mathbb{D}$ we have

$$\left| \frac{z(D_q f)(z)}{f(z)} - \frac{1}{1-q} \right| \leqslant \frac{1}{1-q},$$

where

$$(D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \qquad z \neq 0, \qquad (D_q f)(0) = f'(0).$$

We note that for $q \nearrow 1$, the class of functions $PS_q$ reduces to $\mathscr{S}^*$, which is the class of functions that map the unit disk $\mathbb{D}$ conformally onto a domain which is starlike with respect to the origin. The function

$$G_q(z) = \frac{1-z}{1+qz} = 1 + \frac{1+q}{q}\sum_{n\geqslant 1}(-q)^n z^n$$

is a Carathéodory function which maps $\mathbb{D}$ onto the disk given by

$$\left|w - \frac{1}{1-q}\right| < \frac{1}{1-q}. \tag{3.1}$$

Moreover, the function $K_q$, which is the solution of the differential equation

$$\frac{z(D_qK_q)(z)}{K_q(z)} = G_q(z),$$

is extremal in the class $PS_q$. Note that the function $K_q$ plays the role in the class $PS_q$, equivalent to the role played by the Koebe function $z \mapsto z/(1-z)^2$ in the class $\mathscr{S}^*$. The next result gives sufficient and necessary conditions for a function to belong to the class $PS_q$.

LEMMA 3.1. *The necessary and sufficient condition for a function $f$ to be in the class $PS_q$ is $f(0) = 0,\ f'(0) = 1$ and $|f(qz)/f(z)| < 1$ for all $z \in \mathbb{D}$ and $q \in (0,1)$.*

For all the above results and further properties related to the class $PS_q$ we refer the interested reader to [14]. See also [25] for some continued fraction expansion related to $G_q$. In [24] certain conditions on the Taylor coefficients of $PS_q$ are discussed. We also note that recently in [1, 2, 26] the authors introduced the so-called $q$-close-to-convex functions and $q$-starlike functions of order $\alpha$ and obtained conditions under which the basic hypergeometric functions are in $q$-close-to-convex family, as well as determined the order of $q$-starlikeness of basic hypergeometric functions. As far as we know these are the only results available in the literature about the class $PS_q$ and thus it is worth to deduce some new results in this topic. Our first main result of this section is the following theorem.

THEOREM 3.1. *Let $a,b,c,q \in (0,1)$ satisfy the following conditions: $ab \leqslant c \leqslant a$, $c \leqslant b$ and*

$$4ac(1-c)^2(1-q)\,(b(1-q)+q(1-b)) \geqslant (q(1-b)(c-2a+ac)-(1-c)(1-q)(ab+c))^2. \tag{3.2}$$

*Then, for $|z| < 1$ we have $z\phi(a,b;c;q,z) \in PS_q$.*

*Proof.* Let us consider the notation $f(z) = z\phi(a,b;c;q;z)$. From Lemma 3.1, it is enough to prove that $|f(qz)| < |f(z)|$ for $|z| < 1$. We use the following contiguous relations given in [13] (see also [15])

$$z(1-a)(1-b)\phi(qa,qb;qc;q,z) = (1-c)[\phi(a,b;c;q,z) - \phi(a,b;c;q,qz)],$$

$$(1-a)(c-abz)\phi(qa,qb;qc;q,z) = a(1-c)\phi(a,b;c;q,z) - (a-c)\phi(a,qb;qc;q,z).$$

Some computations concerning the above two relations give the following identity

$$\frac{\phi(a,b;c;q,qz)}{\phi(a,b;c;q,z)} = 1 - \frac{a(1-b)z}{c-abz} + z\frac{(1-b)(a-c)\phi(a,qb;qc;q,z)}{(c-abz)(1-c)\phi(a,b;c;q,z)}. \tag{3.3}$$

From (3.3) and Theorem 2.1, we have

$$\frac{f(qz)}{f(z)} = \frac{q\phi(a,b;c;q,qz)}{\phi(a,b;c;q,z)} = q\left(1 - \frac{a(1-b)z}{c-abz} + z\frac{(1-b)}{(c-abz)}\frac{(a-c)}{(1-c)}\int_0^1 \frac{1}{1-tz}d\mu_0(t)\right), \tag{3.4}$$

where $\mu_0(t)$ is defined as in Theorem 2.1. To prove that $z\phi(a,b;c;q;z) \in PS_q$, we need to show that

$$\left| q\left(1 - \frac{a(1-b)z}{c-abz} + z\frac{(1-b)}{(c-abz)}\frac{(a-c)}{(1-c)}\int_0^1 \frac{1}{1-tz}d\mu_0(t)\right)\right| < 1. \tag{3.5}$$

For $|z| \leqslant r < 1$, by applying the triangle inequality to the left side of (3.5), it is enough to prove,

$$1 + \frac{a(1-b)r}{c-abr} + \frac{r(1-b)(a-c)}{(c-abr)(1-c)}\frac{1}{1-r} < \frac{1}{q}, \qquad 0 \leqslant r < 1.$$

This reduces to proving

$$Ar^2 + Br + C > 0, \qquad 0 \leqslant r < 1,$$

where

$$A = a(1-c)\Big(b(1-q) + q(1-b)\Big),$$

$$C = c(1-c)(1-q) \quad \text{and} \quad B = q(1-b)(c-2a+ac) - (1-c)(1-q)(ab+c).$$

Since $a > 0,\ c,b,q \in (0,1)$ we have that $A > 0$ and hence, we are left to prove $B^2 \leqslant 4AC$. This is nothing but (3.2) and the proof is complete. □

REMARK 3.1. After the substitutions $a \leftrightarrow q^a,\ b \leftrightarrow q^b$, and $c \leftrightarrow q^c$ the condition 3.2 in Theorem 3.1 becomes

$$4q^{a+c}(1-q^c)^2(1-q)\left(q^b(1-q) + q(1-q^b)\right)$$
$$\geqslant (q(1-q^b)(q^c - 2q^a + q^{a+c}) - (1-q^c)(1-q)(q^{a+b} + q^c))^2,$$

that is,

$$4q^{a+c}\left(q^b + q\frac{1-q^b}{1-q}\right) \geqslant \left(q\frac{1-q^b}{1-q^c}\frac{q^c - 2q^a + q^{a+c}}{1-q} - q^{a+b} - q^c\right)^2.$$

Thus, in the limiting case $q \nearrow 1$ Theorem 3.1 becomes the following: if $a,b,c>0$, $q\in(0,1)$ satisfy the following conditions $a+b\geqslant c\geqslant a$, $c\geqslant b$ and $4c^2(b+1)\geqslant (ab-2bc-2c)^2$, then for $|z|<1$ we have that $z\mapsto zF(a,b;c;z)$ is starlike. It is worth to mention that other conditions for the parameters $a,b,c$ on univalence, starlikeness and convexity of the normalized Gaussian hypergeometric functions were obtained by many authors, see for example [12, 19, 22, 23] and the references therein.

COROLLARY 3.1. *If $q\in(0,1)$ and $a,b,c$ satisfy the conditions as in Theorem 3.1 along with the condition $|c-ab|<2|c|$, then the function $z\mapsto z\phi(c/a,c/b;c;q,abz/c)$ is in $PS_q$.*

*Proof.* Let $h(z)=z\phi(c/a,c/b;c;q,abz/c)$. Now using

$$\phi(c/a,c/b;c;q,abz/c)=\frac{(z;q)_\infty}{(abz/c;q)_\infty}\phi(a,b;c;q,z)$$

given in [10, p.10], we get

$$\left|\frac{h(qz)}{h(z)}\right|=\left|\frac{qz\phi(c/a,c/b;c;q,abqz/c)}{z\phi(c/a,c/b;c;q,abz/c)}\right|=\left|\frac{1-abz/c}{1-z}\frac{z\phi(a,b;c;q,qz)}{\phi(a,b;c;q,z)}\right|.$$

Note that here $(\alpha,q)_\infty$ means $\lim_{n\to\infty}(\alpha,q)_n$. Now, applying Theorem 3.1 with the hypothesis $|c-ab|<2|c|$, we get that the right hand side of the above equality is bounded by 1 and Lemma 3.1 guarantees that $h$ is in $PS_q$. □

Now, by using the idea of the well-known Alexander duality between starlike and convex functions it is of interest to define the class of generalized convex functions $PC_q$. We note that if $q\nearrow 1$, then the class $PC_q$ reduces to the class of functions that map the open unit disk onto convex domain.

DEFINITION 3.1. *Let $PC_q$ be the class of functions defined by $f\in PC_q \Longleftrightarrow zD_q(f)(z)\in PS_q$.*

From the definition of $PC_q$, the following results are immediate.

THEOREM 3.2. *Let $a,b,c,q\in(0,1)$ satisfy the following conditions: $abq\leqslant c\leqslant a$, $c\leqslant b$, and*

$$\begin{aligned}&4acq(1-cq)^2(1-q)\left(b(1-q)+1-bq\right)\\ &\geqslant \left(q(1-bq)(c-2a+acq)-(1-cq)(1-q)(abq+c)\right)^2.\end{aligned}$$

*Then, we have that*

$$z\mapsto\frac{(1-c)(1-q)}{(1-a)(1-b)}\phi(a,b;c;q,z)\in PC_q.$$

COROLLARY 3.2. *Let $a$, $b$ and $c$ satisfy the hypothesis of Theorem 3.2. Then for $q\in(0,1)$, $\phi(q^a,q^b;q^c;q,z)\neq 0$, for all $z\in\mathbb{C}$.*

We note that in [14] the following result is obtained using the continued fraction results for a circular domain given in [27].

THEOREM 3.3. *Let* $c > a,\ b < 1$, *and*

$$\sup_{|z|\leqslant r<1}\left\{\left|\frac{w(1-z)-az(1-b)}{c-abz}\right|+|w(1-\overline{z})|\right\}\leqslant\frac{1-q}{q},$$

*where*

$$w=\frac{(a-c)z(1-b)}{(c-abz)\left(1+a-2c+(c-1)(2\mathrm{Re}z)+(1-a)|z|^2\right)}.$$

*Then,* $z\phi(a,b;c;q,rz)\in PS_q$.

Theorem 3.1 is different from Theorem 3.3 in the sense that the conditions given in Theorem 3.1 are easier to formulate than the conditions given in Theorem 3.3. In order to obtain the results in Theorem 3.3, the authors of [14] have used the result of [27, p. 341] which is related to Theorem 11.1 of [27] for circular domain. In Theorem 3.1, instead of this result, a different idea of [27, p. 341] which is related to Theorem 69.2 of [27] is applied which leads to a different result. Note that Theorem 3.3 deals with a particular domain, whereas Theorem 3.1 has no restriction on the variable $z$. We also observe that Theorem 3.1 has $c < a$ whereas Theorem 3.3 has $c > a$ and hence these two results are different.

## 4. Concluding remarks

**A.** Theorem 3.1 is obtained using one ratio given in Theorem 2.1. Using other ratios in Theorem 2.2, many other interesting cases can be obtained. We restrict only to the single case, in order to show the application of the results in Section 2 and to avoid deviating from the main objective of the work. It is also worth to mention that Theorems 2.1 and 2.2 are natural companions of [1, Corollary 2.8], [1, Theorem 2.10] and [1, Theorem 2.13]. Moreover, we note that substituting $a$ by $q^a$, $b$ by $q^b$ and $c$ by $q^c$, and tending with $q$ to $1^-$, Theorem 2.2 offers many other results which complement those of Küstner [18, Theorem 1.5].

**B.** Replacing $z$ by real $x$ and writing $\Phi(a,b;c;q,z)$ as $\Phi(q^a,q^b;q^c;q;x)$ it can be seen that the ratio $\dfrac{\phi(q^a,q^{b+1};q^{c+1};q,x)}{\phi(q^a,q^b;q^c;q,x)}$ has the representation

$$\frac{\phi(q^a,q^{b+1};q^{c+1};q,x)}{\phi(q^a,q^b;q^c;q,x)}=\cfrac{1}{1-\cfrac{d_1x}{1-\cfrac{d_2x}{1-\cfrac{d_3x}{1-\dots}}}},\tag{4.1}$$

where $d_n = d_n(a,b,c,q)$ is given by

$$d_n = \begin{cases} q^{k+b}\dfrac{(1-q^{k+a})(1-q^{k+c-b+1})}{(1-cq^{2k})(1-cq^{2k+1})} & \text{for} \quad n = 2k+1, \\ q^{k+a}\dfrac{(1-bq^{k+1})(1-q^{k+c-a+1})}{(1-cq^{2k+1})(1-cq^{2k+2})} & \text{for} \quad n = 2k+2, \end{cases}$$

where $k \in \{0,1,2,\ldots\}$. In [5], for $a \geqslant c > 0$, $b > 0$, and $x, q \in (0,1)$ the Turán type inequality

$$\frac{\phi(q^{a+2},q^b;q^{c+2};q;x)}{\phi(q^{a+1},q^b;q^{c+1};q;x)} \geqslant \frac{\phi(q^{a+1},q^b;q^{c+1};q;x)}{\phi(q^a,q^b;q^c;q;x)}$$

is obtained. Note that the ratios given in either side of the inequality can have the continued fraction expansion given by (4.1). However, the continued fractions cannot be compared as they do not converge to the same limiting functions in general. Hence finding a suitable method relating these continued fractions would help in finding Turán type inequalities for various ratios given in Theorem 2.2.

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*Árpád Baricz*
*Department of Economics, Babeş-Bolyai University*
*400591 Cluj-Napoca, Romania*
*and*
*Institute of Applied Mathematics*
*John von Neumann Faculty of Informatics*
*Óbuda University*
*1034 Budapest, Hungary*
*e-mail:* bariczocsi@yahoo.com

*Anbhu Swaminathan*
*Department of Mathematics*
*Indian Institute of Technology Roorkee*
*247667 Roorkee, India*
*e-mail:* swamifma@iitr.ernet.in