

A New Fixed Point Result and its Application to Existence Theorem for Nonconvex Hammerstein Type Integral Inclusions*

Hemant Kumar Pathak^a, Naseer Shahzad^{b,1}

^aSchool of Studies in Mathematics, Pt. Ravishankar Shukla University Raipur (C.G.) 492010, India

^bKing Abdulaziz University, Department of Mathematics, PO Box 80203, 21589 Jeddah, Saudi Arabia

Abstract. In this paper, a generalization of Nadler's fixed point theorem is presented for H^+ -type k -multi-valued weak contractive mappings. We consider a nonconvex Hammerstein type integral inclusion and prove an existence theorem by using an H^+ -type multi-valued weak contractive mapping.

Keywords and Phrases: Multi-valued contraction map, multi-valued weak contractive map, H^+ -type multi-valued weak contractive map, Hammerstein type integral inclusion, Fixed point.

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1. Introduction

In 1969, Nadler [16] proved a fixed point theorem for the set-valued contractions, which is of fundamental importance in nonlinear analysis. Inspired from the fixed point result of Nadler [16], the fixed point theory of set-valued contraction was further developed in different directions by many authors, in particular, by Reich [20, 21], Mizoguchi and Takahashi [15], Ćirić [3], Kaneko [9], Lim [13], Lami Dozo [14], Feng and Liu [5], Klim and Wardowski [10], Suzuki [22], Pathak and Shahzad [17, 18] and many others. For details, see [19]. An interesting application of a consequence of Nadler's fixed point theorem was given in Cernea [2]. For other applications of the same result see, for example, [4] [6], [7], [8], [12] and [19].

2. Preliminaries and Definitions

Let (X, d) be a metric space. Let $CB(X)$ and $C(X)$ denote the collection of all nonempty closed and bounded subsets of X and the collection of all compact subsets of X , respectively.

For $A, B \in CB(X)$, let

$$H(A, B) = \max \left\{ \rho(A, B), \rho(B, A) \right\},$$

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¹Corresponding author

E-mail addresses: hkpathak05@gmail.com (H.K. Pathak), nshahzad@kau.edu.sa (N. Shahzad).

$$H^+(A, B) = \frac{1}{2} \{ \rho(A, B) + \rho(B, A) \},$$

where $\rho(A, B) = \sup_{x \in A} d(x, B)$ and $d(x, B) = \inf_{y \in B} d(x, y)$. It is well known that H is a metric on $CB(X)$. Such a map H is called *Pompeiu-Hausdorff metric* induced by d .

A mapping $T : X \rightarrow CB(X)$ is said to be a

- *multi-valued contraction mapping* if there exists a fixed real number $k, 0 < k < 1$ such that

$$H(Tx, Ty) \leq k d(x, y), \quad (2.1)$$

for all $x, y \in X$.

- *multi-valued weak contractive mapping* if there exists a fixed real number $k, 0 < k < 1$ such that

$$H(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}, \quad (2.2)$$

for all $x, y \in X$.

- *multi-valued quasi-contraction mapping* if there exists a fixed real number $k, 0 < k < 1$ such that

$$H(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (2.3)$$

for all $x, y \in X$.

Proposition 2.1[18]. H^+ is a metric on $CB(X)$.

Notice that the two metrics H and H^+ are equivalent [11] since

$$\frac{1}{2}H(A, B) \leq H^+(A, B) \leq H(A, B).$$

In the light of this equivalence and referring to Kuratowski [11], we conclude that $(CB(X), H^+)$ is complete whenever (X, d) is complete. Indeed, it is a simple consequence of the completeness of the Hausdorff metric H . Moreover, $C(X)$ is a closed subspace of $(CB(X), H^+)$.

Notice also that $H^+ : CB(X) \times CB(X) \rightarrow \mathbf{R}$ is a continuous function. To see this, we observe that the inequality

$$H^+(A, B) \leq H^+(A, C) + H^+(C, B)$$

holds for any $A, B, C \in CB(X)$. Now pick any $(A_0, B_0) \in CB(X) \times CB(X)$. Then for a given $\epsilon > 0$, we can choose a positive number $\delta = \frac{\epsilon}{2}$ such that

$$|H^+(A, B) - H^+(A_0, B_0)| \leq H^+(A, A_0) + H^+(B_0, B) < \delta + \delta = 2\delta = \epsilon$$

whenever $H^+(A, A_0) < \delta, H^+(B_0, B) < \delta$. This shows that H^+ is continuous at (A_0, B_0) .

In [16], S. B. Nadler proved the following result, which he announced earlier.

Theorem 2.2. Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ a multi-valued contraction mapping. Then T has a fixed point.

In this paper, we intend to generalize this result by weakening the multi-valued contraction to an H^+ -type multi-valued weak contractive mapping. Our main result is summarized in Section 3. In Section 4, we consider a nonconvex Hammerstein type integral inclusion and prove an existence theorem by using an H^+ -type multi-valued weak contractive mapping.

3. Main results

We begin our discussion with the following definition.

Definition 3.1. Let (X, d) be a metric space. A multi-valued mapping $T : X \rightarrow \mathcal{CB}(X)$ is called H^+ -contraction if

(1) there exists a fixed real number k , $0 < k < 1$ such that

$$H^+(Tx, Ty) \leq kd(x, y) \text{ for every } x, y \in X,$$

(2) for every x in X , y in $T(x)$ and $\epsilon > 0$, there exists z in $T(y)$ such that

$$d(y, z) \leq H^+(T(y), T(x)) + \epsilon.$$

In [18], Pathak and Shahzad proved the following result.

Theorem 3.2. Every H^+ -type multi-valued contraction mapping $T : X \rightarrow CB(X)$ with Lipschitz constant $0 < k < 1$ has a fixed point.

We now introduce the following definition.

Definition 3.3. Let (X, d) be a metric space. A mapping $T : X \rightarrow CB(X)$ is called an H^+ -type multi-valued weak contractive mapping if the condition (2) holds and there exists a fixed real number k , $0 < k < 1$ such that

$$H^+(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}, \quad (3.1)$$

for all x, y in X .

Now we state and prove our main result.

Theorem 3.4. Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ an H^+ -type multi-valued weak k -contractive mapping with $0 < k < 1$. Then T has a fixed point.

Proof. Notice first that for each $A, B \in CB(X)$, $a \in A$ and $\alpha > 0$ with $H^+(A, B) < \alpha$, there exists $b \in B$ such that $\max\{d(a, b), d(a, Ta), d(b, Tb), \frac{1}{2}[d(a, Tb) + d(b, Ta)]\} < \alpha$. Now, let $L > 0$ be such that $k < L < 1$. Then

$$H^+(Tx, Ty) < L \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}, \quad (3.2)$$

for any $x, y \in X, x \neq y$.

Now we choose a sequence $\{x_n\}$ recursively in X in the following way. Let $x_0 \in X$ be arbitrary. Fix an element x_1 in Tx_0 . From (2) it follows that we can choose $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H^+(Tx_0, Tx_1) + \epsilon \quad (3.3)$$

In general, if x_n be chosen, then we choose $x_{n+1} \in Tx_n$ such that

$$d(x_n, x_{n+1}) \leq H^+(Tx_{n-1}, Tx_n) + \epsilon. \quad (3.4)$$

Set $\epsilon = (\frac{1}{\sqrt{L}} - 1)H^+(Tx_{n-1}, Tx_n)$. Then from (3.4), it follows that

$$d(x_n, x_{n+1}) \leq H^+(Tx_{n-1}, Tx_n) + (\frac{1}{\sqrt{L}} - 1)H^+(Tx_{n-1}, Tx_n) = \frac{1}{\sqrt{L}}H^+(Tx_{n-1}, Tx_n).$$

Thus, we have

$$\sqrt{L}d(x_n, x_{n+1}) \leq H^+(Tx_{n-1}, Tx_n) \tag{3.5}$$

for each $n \in \mathbf{N}$.

Thus, from (3.2) we have

$$\begin{aligned} \sqrt{L}d(x_n, x_{n+1}) &< L \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\quad [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]/2\} \\ &\leq (\sqrt{L})^2 \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})/2\} \\ &\leq (\sqrt{L})^2 \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]/2\} \\ &= (\sqrt{L})^2 \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned}$$

It follows that

$$d(x_n, x_{n+1}) < \sqrt{L} \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \tag{3.6}$$

for each $n \in \mathbf{N}$. Note that if $x_n = x_{n+1}$ for some $n \in \mathbf{N}$, then $x_n = x_{n+1} \in Tx_n$, that is, x_n is a fixed point of T and we are finished. So, we may assume that $d(x_{n+1}, x_n) > 0$ for each $n \in \mathbf{N}$. Suppose that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some $n \in \mathbf{N}$, then inequality (3.6) gives

$$d(x_n, x_{n+1}) < \sqrt{L}d(x_n, x_{n+1}),$$

a contradiction. So we must have $d(x_{n-1}, x_n) \geq d(x_n, x_{n+1})$ for each $n \in \mathbf{N}$. Hence, for all $n \in \mathbf{N}$, (3.6) yields

$$d(x_n, x_{n+1}) < c d(x_{n-1}, x_n), \tag{3.7}$$

where $c = \sqrt{L}$. Repeating the same argument n -times as in (3.7), we obtain

$$d(x_n, x_{n+1}) < c^n d(x_0, x_1). \tag{3.8}$$

It is obvious that $\{x_n\}$ is bounded. Indeed, for any $n \in \mathbf{N}$, we have

$$\begin{aligned} d(x_0, x_n) &\leq \sum_{i=0}^{n-1} d(x_i, x_{i+1}) < (1 + c + c^2 + \dots + c^{n-1})d(x_0, x_1) \\ &< (1 + c + c^2 + \dots)d(x_0, x_1) = \frac{1}{1-c}d(x_0, x_1) < \infty. \end{aligned}$$

Further, by virtue of (3.8), one may observe that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $\lim_{n \rightarrow \infty} x_n = u$. Assume that $u \notin Tu$, that is, $d(u, Tu) > 0$. Now using (3.2) we have

$$\begin{aligned} \frac{1}{2} \left\{ \rho(Tx_n, Tu) + \rho(Tu, Tx_n) \right\} &= H^+(Tx_n, Tu) \\ &< L \max\{d(x_n, u), d(x_n, Tx_n), d(u, Tu), [d(x_n, Tu) + d(u, Tx_n)]/2\} \\ &\leq L \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu), [d(x_n, Tu) + d(u, x_{n+1})]/2\}, \end{aligned}$$

it follows that

$$\frac{1}{2} \liminf_{n \rightarrow \infty} \left\{ \rho(Tx_n, Tu) + \rho(Tu, Tx_n) \right\} \leq L d(u, Tu).$$

Since $\lim_{n \rightarrow \infty} d(x_{n+1}, u) = 0$ exists, and

$$d(u, Tu) = \frac{1}{2} [d(u, Tu) + d(Tu, u)] \leq \frac{1}{2} [\rho(Tx_n, Tu) + \rho(Tu, Tx_n)] + d(x_{n+1}, u),$$

it follows that

$$\begin{aligned} d(u, Tu) &\leq \frac{1}{2} \liminf_{n \rightarrow \infty} [\rho(Tx_n, Tu) + \rho(Tu, Tx_n)] + \liminf_{n \rightarrow \infty} d(x_{n+1}, u) \\ &\leq L d(u, Tu) + \lim_{n \rightarrow \infty} d(x_{n+1}, u) = L d(u, Tu) < d(u, Tu), \end{aligned}$$

a contradiction. This implies that $d(u, Tu) = 0$, and, since Tu is closed, it must be the case that $u \in Tu$.

Notice that every multi-valued contraction mapping with respect to Pompeiu-Hausdorff metric H is an H^+ -type multi-valued weak contractive mapping but the converse implication need not be true. To see this, we have the following example:

Example 3.5. Let $X = [-2, 2]$ and $d : X \times X \rightarrow \mathbf{R}$ be a standard metric. Let $T : X \rightarrow CB(X)$ be defined by $Tx = \{\frac{x}{4}\}$, if $x \in [-1, 2]$ and $Tx = \{2\}$, otherwise. It is clear that if $x, y \in [-1, 2]$ or $x, y \in [-2, -1)$, then

$$H^+(Tx, Ty) \leq \frac{1}{4} d(x, y).$$

If $x \in [-1, 2]$ and $y \in [-2, -1)$, then we have

$$H^+(Tx, Ty) = \frac{1}{2} [|2 - \frac{x}{4}| + |2 - \frac{x}{4}|] = |2 - \frac{x}{4}| \leq 2 + \frac{1}{4} = \frac{3}{4} \cdot 3 \leq \frac{3}{4} \cdot \max\{d(y, Ty), d(x, Tx)\}.$$

It follows that

$$H^+(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$$

for all $x, y \in X$ and $k \in [\frac{3}{4}, 1)$. To check the condition (2), we consider the following cases:

Case 1. If $x \in [-2, -1)$, then for any $y \in Tx = \{2\}$, there exists $z \in Ty = \{\frac{1}{2}\}$ such that for any $\epsilon > 0$

$$d(y, z) = \frac{3}{2} \leq \frac{3}{2} + \epsilon = H^+(Ty, Tx) + \epsilon.$$

Case 2. If $x \in [-1, 2]$, then for any $y \in Tx = \{\frac{x}{4}\}$, there exists $z \in Ty = \{\frac{x}{16}\}$ such that for any $\epsilon > 0$

$$d(y, z) = \frac{3|x|}{16} \leq \frac{3|x|}{16} + \epsilon = H^+(Ty, Tx) + \epsilon.$$

Thus all the conditions of Theorem 3.4 are satisfied. Moreover, $0 \in T0 = \{0\}$ is a fixed point of T .

Notice that the map T does not satisfy the assumptions of Theorem 2.2 and Theorem 3.2. Indeed, for $x = -1$ and $y \rightarrow -1$ from the left we have

$$H(T(-1), T(y)) = H^+(T(-1), T(y)) = 2 + \frac{1}{4} > k d(-1, y),$$

for all $k \in (0, 1)$.

We also notice that since

$$[d(x, Ty) + d(y, Tx)]/2 \leq \max\{d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, it follows that every weak contractive mapping is quasi-contraction.

Using the technique of the proof of Theorem 3.4, one can easily prove the following result.

Theorem 3.6. Let (X, d) be a complete metric space. Let $T : X \rightarrow CB(X)$ be a H^+ -type k -multi-valued quasi-contraction mapping with $0 < k < \frac{1}{2}$. Then, T has a fixed point.

Pathak and Shahzad [18] introduced the class of H^+ -type nonexpansive mappings

Definition 3.7. Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $T : X \rightarrow CB(X)$ is called H^+ -nonexpansive if

$$(1') \quad H^+(Tx, Ty) \leq \|x - y\| \text{ for every } x, y \in X,$$

(2') for every x in X, y in $T(x)$ and $\epsilon > 0$, there exists z in $T(y)$ such that

$$\|y - z\| \leq H^+(T(y), T(x)) + \epsilon.$$

Applying the main result of this section, we obtain the following result which plays a role in the next section.

Proposition 3.8.([18]). Let (X, d) be a complete metric space. Suppose that $T_i : X \rightarrow CB(X), i = 1, 2$, are two H^+ -type multi-valued contraction mappings with Lipschitz constant $L < 1$. Then if $Fix(T_1)$ and $Fix(T_2)$ denote the respective fixed point sets of T_1 and T_2 ,

$$H^+(Fix(T_1), Fix(T_2)) \leq \frac{1}{1 - \sqrt{L}} \sup_{x \in X} H^+(T_1x, T_2y).$$

4. Existence Theorem for Nonconvex Hammerstein Type Integral Inclusions

Let $0 < T < \infty$, $I := [0, T]$ and $\mathcal{L}(I)$ denote the σ -algebra of all Lebesgue measurable subsets of I . Let E be a real separable Banach space with the norm $\|\cdot\|$. Let $\mathcal{P}(E)$ denote the family of all nonempty subsets of E and $\mathcal{B}(E)$ the family of all Borel subsets of E .

In what follows, as usual, we denote by $C(I, E)$ the Banach space of all continuous functions $x(\cdot) : I \rightarrow E$ endowed with the norm $\|x(\cdot)\|_C = \sup_{t \in I} \|x(t)\|$. Consider the following integral equation

$$x(t) = \lambda(t) + \int_0^T k(t, s) g(t, s, u(s)) ds \text{ on } [0, T]. \quad (4.1)$$

Here λ, k and g are given functions, where $\lambda(\cdot) : I \rightarrow E$ is a function with Banach space value, $k : I \times I \rightarrow \mathbf{R}_+ = [0, \infty)$ is a positive real single-valued function, while $g : I \times I \times E \rightarrow E$ is a map. Let $p \in [1, \infty)$, $q \in [1, \infty)$, and let $r \in [1, \infty)$ be the conjugate exponent of q , that is $1/q + 1/r = 1$. Let $\|\cdot\|_p$ denote the p -norm of the space $L^p(I, E)$ and is defined by $\|u\|_p = (\int_0^T \|u(s)\|^p ds)^{1/p}$ for all $u \in L^p(I, E)$. Consider the Nemitsky operator associated to g, p, q and $G : L^p(I, E) \rightarrow L^q(I, E)$ given by

$$G(u) = g(t, s, u(s)) \text{ a.e. on } I.$$

Consider the linear integral operator of kernel $k, S : L^q(I, E) \rightarrow L^p(I, E)$ given by

$$S(u) = \lambda(t) + \int_0^T k(t, s)u(s)ds \text{ a.e. on } I.$$

Thus the Hammerstein type integral equation (4.1) is transformed into the form

$$x = SG(u), \quad u \in L^p(I, E) \text{ a.e. on } I \tag{4.1'}$$

$$u(t) \in F(t, V(x)(t)) \quad \text{a.e. } (I := [0, T]), \tag{4.2}$$

where $V : C(I, E) \rightarrow C(I, E)$ is a given mapping. In the sequel, we also use the following: For any $x \in E$, $\lambda \in C(I, E)$, $\sigma \in L^p(I, E)$, we define the set-valued maps $M_{\lambda, \sigma}(t) := F(t, V(x_{\sigma, \lambda})(t))$, $t \in I$, $T_\lambda(\sigma) := \{\psi(\cdot) \in L^p(I, E) : \psi(t) \in M_{\lambda, \sigma}(t) \text{ a.e. } (I)\}$.

In order to study problem (4.1)-(4.2) we introduce the following assumption.

Hypothesis 4.1. Let $F(\cdot, \cdot) : I \times E \rightarrow \mathcal{P}(E)$ be a set-valued map with nonempty closed values satisfying:

(H₁) The function $k : I \times I \rightarrow \mathbf{R}_+$ satisfies that $k(t, \cdot) \in L^r(I)$, and $t \rightarrow \|k(t, \cdot)\|_r \in L^p(I)$.

(H₂) The set-valued map $F(\cdot, \cdot)$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable.

(H₃) There exists $L(\cdot) \in L^1(I, \mathbf{R}_+)$ such that, for almost all $t \in I$, $F(t, \cdot)$ is $L(t)$ -Lipschitz in the sense that

$$H^+(F(t, x), F(t, y)) \leq L(t) \|x - y\| \tag{C1}$$

for all x, y in E , and for any $x, y \in X$, $w \in F(t, x)$ and any $\epsilon > 0$, there exists $z \in F(t, y)$ such that

$$\|w - z\|^p \leq H^+(F(t, x), F(t, y)) + \epsilon \tag{C2}$$

and $T_\lambda(\cdot)$ satisfies the condition: For any $\sigma \in L^p(I, E)$, $\sigma_1 \in T_\lambda(\sigma)$ and any given $\epsilon > 0$, there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_p \leq H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) + \epsilon. \tag{C3}$$

(H₄) The mappings $k : I \times I \rightarrow \mathbf{R}_+$, $g : I \times I \times E \rightarrow E$ are continuous, $V : C(I, E) \rightarrow C(I, E)$ and there exist constants $M_1, M_2, M_3 > 0$ such that

$$\|g(t, s, u_1) - g(t, s, u_2)\| \leq M_1 \|u_1 - u_2\|^p, \quad \forall u_1, u_2 \in E,$$

$$\|V(x_1)(t) - V(x_2)(t)\| \leq M_2 \|x_1(t) - x_2(t)\|, \quad \forall t \in I, \forall x_1, x_2 \in C(I, E),$$

and

$$|k(t, s)| \leq M_3 \quad \forall t, s \in I.$$

It is worth mentioning that the system (4.1)-(4.2) includes a large variety of differential inclusions and control systems.

Assume that U is an open bounded subset of \mathbf{R}^n (or Y , a subset of E homeomorphic to \mathbf{R}^n) and $U_T = (0, T] \times U$ for some fixed $T > 0$. We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is parabolic if there exists a constant $\theta > 0$ such that $\sum_{i,j=1}^n a^{ij}(t, x)\xi_i\xi_j \geq \theta|\xi|^2$ for all $(t, x) \in U_T, \xi \in \mathbf{R}^n$. The letter L denotes for each time t a second order partial differential operator, having either the divergence form $Lu = -\sum_{i,j=1}^n (a^{ij}(t, x)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(t, x)u_{x_i} + c(t, x)u$ or else the non-divergence form $Lu = -\sum_{i,j=1}^n a^{ij}(t, x)u_{x_i x_j} + \sum_{i=1}^n b^i(t, x)u_{x_i} + c(t, x)u$, for given coefficients a^{ij}, b^i, c ($i, j = 1, 2, \dots, n$).

A family $\{G(t) : t \in \mathbf{R}_+\}$ of bounded linear operators from X into E is a C_0 -semigroup (also called linear semigroup of class (C_0)) on X if

- (i) $G(0) =$ the identity operator, and $G(t + s) = G(t)G(s) \forall t, s \geq 0$;
- (ii) $G(\cdot)$ is strongly continuous in $t \in \mathbf{R}_+$;
- (iii) $\|G(t)\| \leq Me^{\omega t}$ for some $M > 0$, real ω and $t \in \mathbf{R}_+$.

Example 4.2. Set $k(t, \tau)g(t, \tau, u) = G(t - \tau)u, \Phi(x) = x, \lambda(t) = G(t)x_0$, where $\{G(t)\}_{t \geq 0}$ is a C_0 -semigroup with an infinitesimal generator A . Then a solution of system (4.1)-(4.2) represents a mild solution of

$$x'(t) \in Ax(t) + F(t, x(t)), \quad x(0) = x_0. \quad (4.3)$$

In particular, this problem includes control systems governed by parabolic partial differential equations as a special case. When $A = 0$, the relation (4.3) reduces to

$$x'(t) \in F(t, x(t)), \quad x(0) = x_0. \quad (5.4)$$

Denote

$$\Phi(u)(t) = \int_0^t k(t, \tau)g(t, \tau, u(\tau)) d\tau, \quad t \in I. \quad (4.5)$$

Then the integral inclusion system (4.1)-(4.2) reduces to the form

$$x(t) = \lambda(t) + \Phi(u)(t) \quad a.e. (I), \quad (S)$$

which may be written in more “compact” form as

$$u(t) \in F(t, V(\lambda + \Phi(u))(t)) \quad a.e. (I).$$

Now we recall the following:

Definition 4.3. A pair of functions (x, u) is called a solution pair of integral inclusion system (S) , if $x(\cdot) \in C(I, E), u(\cdot) \in L^p(I, E)$ and satisfy relation (S) .

For our further discussion, we denote by $S(\lambda)$ the solution set of (4.1) – (4.2).

For given $\alpha \in \mathbf{R}$ we denote by $L^p(I, E)$ the Banach space of all Bochner integrable functions $u(\cdot) : I \rightarrow E$ endowed with the norm

$$\|u(\cdot)\|_p = \left(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} \|u(t)\|^p dt \right)^{\frac{1}{p}},$$

where $m(t) = \int_0^t L(s) ds$, $t \in I$. For our further discussion, we denote $L = m(T)$.

Theorem 4.4. Let Hypothesis 4.1 be satisfied, let $\lambda(\cdot), \mu(\cdot) \in C(I, E)$ and let $v(\cdot) \in L^p(I, E)$ be such that

$$d(v(t), F(t, V(y)(t))) \leq p(t) \quad a.e. \quad (I),$$

where $p(\cdot) \in L^p(I, \mathbf{R}_+)$ and $y(t) = \mu(t) + \Phi(v)(t)$, $\forall t \in I$.

Then for every $\alpha > 1$, there exists $x(\cdot) \in S(\lambda)$ such that for every $t \in I$

$$\begin{aligned} \|x(t) - y(t)\| &\leq \|\lambda - \mu\|_C + M_1 M_3 e^{\alpha M_1 M_2 M_3 L} \left[\frac{1}{\alpha^{\frac{1}{2p}} (\alpha^{\frac{1}{2p}} - 1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C \right. \\ &\quad \left. + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}} - 1} \left(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \right)^{\frac{1}{p}} \right]^p. \end{aligned}$$

Proof. For $\lambda \in C(I, E)$ and $u \in L^p(I, E)$, define

$$x_{u,\lambda}(t) = \lambda(t) + \int_0^T k(t, s) g(t, s, u(s)) ds, \quad t \in I.$$

Let us consider that $\lambda \in C(I, E), \sigma \in L^p(I, E)$ and define the set-valued maps

$$M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t)), \quad t \in I, \quad (4.6)$$

$$T_\lambda(\sigma) := \{\psi(\cdot) \in L^p(I, E) : \psi(t) \in M_{\lambda,\sigma}(t) \quad a.e. \quad (I)\}. \quad (4.7)$$

Further, in view of condition (C3) of Hypothesis 4.1(H_3), $T_\lambda(\cdot)$ satisfies the condition: For any $\sigma \in L^p(I, E)$, $\sigma_1 \in T_\lambda(\sigma)$ and any given $\epsilon > 0$ there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_p \leq H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) + \epsilon. \quad (4.8)$$

Now we claim that $T_\lambda(\sigma)$ is nonempty, bounded and closed for every $\sigma \in L^p(I, E)$.

It is well known that the set-valued map $M_{\lambda,\sigma}(\cdot)$ is measurable. For example the map $t \rightarrow M_{\lambda,\sigma}(t)$ can be approximated by step functions and so we can apply Theorem III. 40 in [1]. As the values of F are closed, with the measurable selection theorem we infer that $M_{\lambda,\sigma}(\cdot)$ is nonempty.

Further, we note that the set $T_\lambda(\sigma)$ is bounded and closed. Indeed, if $\psi_n \in T_\lambda(\cdot)$ and $\|\psi_n - \psi\|_p \rightarrow 0$, then there exists a subsequence ψ_{n_k} such that $\psi_{n_k}(t) \rightarrow \psi(t)$ for a.e. $t \in I$ and we find that $\psi \in T_\lambda(\sigma)$.

Let $\sigma_1, \sigma_2 \in L^p(I, E)$ be given. Let $\psi_1 \in T_\lambda(\sigma_1)$ and let $\delta > 0$. Consider the following set-valued map:

$$\mathcal{G}(t) := M_{\lambda,\sigma_2}(t) \cap \left\{ z \in E : \|\psi_1(t) - z\|^p \leq M_1 M_2 M_3 L(t) \int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p ds + \delta \right\}.$$

By (C2), it follows that

$$\begin{aligned}
 d^p(\psi_1(t), M_{\lambda, \sigma_2}(t)) &\leq H^+\left(F(t, V(x_{\sigma_1, \lambda}(t))), F(t, V(x_{\sigma_2, \lambda}(t)))\right) + \epsilon \\
 &\leq L(t)\|V(x_{\sigma_1, \lambda}(t)) - V(x_{\sigma_2, \lambda}(t))\| + \epsilon \\
 &\leq M_2 L(t)\|x_{\sigma_1, \lambda}(t) - x_{\sigma_2, \lambda}(t)\| + \epsilon \\
 &\leq M_2 M_3 L(t) \int_0^T \|g(t, s, \sigma_1(s)) - g(t, s, \sigma_2(s))\| ds + \epsilon \\
 &\leq M_1 M_2 M_3 L(t) \int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p ds + \epsilon.
 \end{aligned}$$

Since ϵ is arbitrary, letting $\epsilon \rightarrow 0$, we deduce that $\mathcal{G}(\cdot)$ is nonempty bounded and has closed values.

Further, according to Proposition III.4 in [1], $\mathcal{G}(\cdot)$ is measurable.

Let $\psi_2(\cdot)$ be a measurable selector of $\mathcal{G}(\cdot)$. It follows that $\psi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned}
 \|\psi_1 - \psi_2\|_p^p &= \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} \|\psi_1(t) - \psi_2(t)\|^p dt \\
 &\leq \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} (M_1 M_2 M_3 L(t) \int_0^T \|\sigma_1(s) - \sigma_2(s)\|^p ds) dt \\
 &\quad + \delta \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} dt \\
 &\leq \frac{1}{\alpha} \|\sigma_1 - \sigma_2\|_p^p + \delta \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} dt.
 \end{aligned}$$

Since δ is arbitrary, so letting $\delta \rightarrow 0$ we deduce from the above inequality that

$$\|\psi_1 - \psi_2\|_p^p \leq \frac{1}{\alpha} \|\sigma_1 - \sigma_2\|_p^p$$

i.e.,

$$\|\psi_1 - \psi_2\|_p \leq \frac{1}{\alpha^{\frac{1}{p}}} \|\sigma_1 - \sigma_2\|_p.$$

This yields

$$d(\psi_1, T_\lambda(\sigma_2)) \leq \frac{1}{\alpha^{\frac{1}{p}}} \|\sigma_1 - \sigma_2\|_p.$$

Thus, we have

$$\rho(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) = \sup_{\psi_1 \in T_\lambda(\sigma_1)} d(\psi_1, T_\lambda(\sigma_2)) \leq \frac{1}{\alpha^{\frac{1}{p}}} \|\sigma_1 - \sigma_2\|_p. \quad (4.9)$$

Now replacing $\sigma_1(\cdot)$ with $\sigma_2(\cdot)$ and arguing as above, we obtain

$$\rho(T_\lambda(\sigma_2), T_\lambda(\sigma_1)) \leq \frac{1}{\alpha^{\frac{1}{p}}} \|\sigma_1 - \sigma_2\|_p. \quad (4.10)$$

Now adding (4.9) and (4.10) and dividing by 2, we obtain

$$\begin{aligned}
 H^+(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) &\leq \frac{1}{\alpha^{\frac{1}{p}}} \|\sigma_1 - \sigma_2\|_p \\
 &\leq \frac{1}{\alpha^{\frac{1}{p}}} \max\{\|\sigma_1 - \sigma_2\|_p, d(\sigma_1, T_\lambda(\sigma_1)), d(\sigma_2, T_\lambda(\sigma_2)), \\
 &\quad [d(\sigma_1, T_\lambda(\sigma_2)) + d(\sigma_2, T_\lambda(\sigma_1))]/2\}.
 \end{aligned}$$

Hence we conclude that $T_\lambda(\cdot)$ is an H^+ -type multi-valued weak contractive mapping on $L^p(I, E)$. Next, we consider the following set-valued maps

$$\begin{aligned}\tilde{F}(t, x) &:= F(t, x) + p(t), \\ \tilde{M}_{\lambda, \sigma}(t) &:= \tilde{F}(t, V(x_{\sigma, \lambda})(t)), \quad t \in I, \\ \tilde{T}_\lambda(\sigma) &:= \{\psi(\cdot) \in L^p(I, E) : \psi(t) \in \tilde{M}_{\lambda, \sigma}(t) \text{ a.e. } (I)\}.\end{aligned}$$

It is obvious that $\tilde{F}(\cdot, \cdot)$ satisfies Hypothesis 4.1.

Let $\phi \in T_\lambda(\sigma)$, $\delta > 0$ and define

$$\mathcal{G}_1(t) := \tilde{M}_{\lambda, \sigma}(t) \cap \left\{ z \in X : \|\phi(t) - z\|^p \leq M_2 L(t) \|\lambda - \mu\|_C^p + p(t) + \delta \right\}.$$

Using the same argument as used for the set valued map $\mathcal{G}(\cdot)$, we deduce that $\mathcal{G}_1(\cdot)$ is measurable with nonempty closed values.

Next, we prove the following estimate:

$$H^+(T_\lambda(\sigma), \tilde{T}_\mu(\sigma)) \leq \frac{1}{\alpha^{\frac{1}{p}} M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C + \left(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \right)^{\frac{1}{p}}. \quad (4.11)$$

Let $\psi(\cdot) \in \tilde{T}_\mu(\sigma)$. Then

$$\begin{aligned}\|\phi - \psi\|_p^p &= \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} \|\phi(t) - \psi(t)\|^p dt \\ &\leq \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} [M_2 L(t) \|\lambda - \mu\|_C^p + p(t) + \delta] dt \\ &\leq \|\lambda - \mu\|_C^p \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} M_2 L(t) dt \\ &\quad + \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt + \delta \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} dt \\ &\leq \frac{1}{\alpha M_1 M_3} \|\lambda - \mu\|_C^p + \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \\ &\quad + \delta \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} dt.\end{aligned}$$

Since δ is arbitrary, so letting $\delta \rightarrow 0$ we deduce from the above inequality that

$$\|\phi - \psi\|_p^p \leq \frac{1}{\alpha M_1 M_3} \|\lambda - \mu\|_C^p + \int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt.$$

Thus, by taking $\frac{1}{p}$ th power on both sides of the above inequality breaking the right hand side, one obtains (4.11).

Now applying Proposition 3.8 we obtain

$$\begin{aligned}H^+(Fix(T_\lambda), Fix(\tilde{T}_\mu)) &\leq \frac{1}{\alpha^{\frac{1}{2p}} (\alpha^{\frac{1}{2p}} - 1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C \\ &\quad + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}} - 1} \left(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \right)^{\frac{1}{p}}.\end{aligned}$$

Since $v(\cdot) \in \text{Fix}(\tilde{T}_\mu)$, it follows that there exists $u(\cdot) \in \text{Fix}(T_\lambda)$ such that

$$\|v - u\|_p \leq \frac{1}{\alpha^{\frac{1}{2p}} (\alpha^{\frac{1}{2p}} - 1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}} - 1} \left(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \right)^{\frac{1}{p}}. \quad (4.12)$$

We define

$$x(t) = \lambda(t) + \int_0^T k(t, s) g(t, s, u(s)) ds.$$

Then one has the following inequality:

$$\begin{aligned} \|x(t) - y(t)\| &\leq \|\lambda(t) - \mu(t)\| + M_1 M_3 \int_0^T \|u(s) - v(s)\|^p ds \\ &\leq \|\lambda - \mu\|_C + M_1 M_3 e^{\alpha M_1 M_2 M_3 L} \|u - v\|_p^p. \end{aligned}$$

Combining the last inequality with (4.12) we obtain

$$\begin{aligned} \|x(t) - y(t)\| &\leq \|\lambda - \mu\|_C + M_1 M_3 e^{\alpha M_1 M_2 M_3 L} \left[\frac{1}{\alpha^{\frac{1}{2p}} (\alpha^{\frac{1}{2p}} - 1) M_1^{\frac{1}{p}} M_3^{\frac{1}{p}}} \|\lambda - \mu\|_C \right. \\ &\quad \left. + \frac{\alpha^{\frac{1}{2p}}}{\alpha^{\frac{1}{2p}} - 1} \left(\int_0^T e^{-\alpha M_1 M_2 M_3 m(t)} p(t) dt \right)^{\frac{1}{p}} \right]^p. \end{aligned}$$

This completes the proof.

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