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NEW MECHANIZATION EQUATIONS FOR AIDED INERTIAL NAVIGATION SYSTEMS

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16. Abstract

Inertial navigation equations are developed which use area navigation (RNAV) waypoints and runway references as coordinate centers. The formulation is designed for aided inertial navigation systems and gives a high numerical accuracy through all phases of flight. A new formulation of the error equations for inertial navigation systems is also presented. This new formulation reduces numerical calculations in the use of Kalman filters for aided inertial navigation systems.

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SUMMARY

This report describes several improvements in "state-of-the-art" software which were made during the development of the RAINPAL* concept and in subsequent extensions of the concept. One of the improvements is to formulate the navigation equations in Earth-fixed coordinates, with the coordinate center and axes chosen so that

1. minimal auxiliary calculations are required for pilot displays and

2. high numerical accuracy is retained with single-precision, fixed-point arithmetic.

Use has been made of area navigation (RNAV) waypoints, in addition to runway reference points, as coordinate centers during the various phases of flight.

Another improvement is in the formulation of the variational equations which are used in the navaid data processing algorithm (a modified Kalman filter with a square-root implementation). Simplifications in these equations result with the use of a new concept which mathematically ties the variational equation reference to the platform. An added advantage of this new approach is that accelerometer measurements do not occur in the Jacobian matrix of the acceleration with respect to the state.

Detailed discussion of the specifics of the RAINPAL software are presented as well as the extensions of the RAINPAL concept to other phases of flight.

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* RAINPAL is an acronym for "recursive-aided-inertial-navigation-for-precision-approach-and-landing." The RAINPAL system concept has been validated through flight tests in a recent NASA Ames inhouse and contractual effort.

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ACKNOWLEDGMENTS

The high-accuracy and advanced software embodied in the experimental navigation system called RAINPAL resulted from the NASA/Ames philosophy of seeking out and developing advanced concepts for modern aviation systems. The NASA/Ames team responsible for the conceptual design, development, and flight test validation was drawn from the Guidance and Navigation Branch, the Systems Analysis Branch, and the Avionics Research Branch of the Flight and Systems Research Division. Team members contributing to the mathematical formulation, simulation, and onboard software for the overall effort were Messrs. Leonard A. McGee, Gerald L. Smith, Thomas M. Carson, Daniel M. Hegarty, and Robert B. Merrick. The authors wish to express appreciation to these individuals and to the other team members whose support helped to make this report possible.

I. INTRODUCTION

An attractive concept for an all-weather aircraft navigation system which can potentially satisfy the requirements for all phases of flight is to use measurements from external navigation aids in conjunction with an inertial navigation system (INS). The excellent high-accuracy, short-term characteristics of the INS provide a reference for filtering the noise in the navigation aid measurements. The position information from appropriate navigation aids provides a good reference for removing the drift characteristics of the INS. Current "state of the art" hardware for INS, navigation aids and onboard computers is potentially applicable for such an aided inertial navigation system. The feasibility of this concept for approach and landing operations has been demonstrated in recent NASA/Ames flight tests of an experimental aided inertial navigation system called RAINPAL.* This system was developed in a joint NASA/Ames and contractual effort aimed at the definition, formulation and validation of new or advanced navigation techniques which are compatible with the computational capabilities of "off the shelf" onboard computers.

Kalman filter theory provided the basis for the onboard computer algorithms used in combining the INS and navigation aid data. The square-root formulation of this filter[1,2] which incorporates random forcing functions in the INS error model forms part of the RAINPAL software. Other advanced techniques tested in the RAINPAL software included the solution of the navigation equations in a runway-referenced (Earth-fixed) coordinate frame and a new formulation of the error state vector for inertial navigation systems.

* RAINPAL is an acronym for "recursive-aided-inertial-navigation-for-precision-approach-and-landing."

Previous reports[3, 4, 5, 6] have described both simulation results and flight test results associated with the overall effort. The mathematical formulations and software details of the experimental system have not been presented in these reports.

This report begins (Section III) with a description of a hardware configuration and the software operation envisioned for an efficient implementation of an aided inertial navigation system. The configuration used in the flight test implementation is also presented in this section. The runway-referenced navigation equations used in the experimental software and the extensions required for obtaining software for a global navigation system are presented in Section IV. The software compensation for inertial measurement unit (IMU) anomalies such as bias, misalignment and scaling of the accelerometer measurements is included in this section. The philosophy behind the new formulation for the error state vector for inertial navigation systems and its relationship to traditional procedures is presented in Section V. The variational equations and approximations used in the experimental system are also summarized. The appendix gives a summary of vector identities and properties of orthogonal transformations which are used in the mathematical development.

II. NOTATION, DEFINITIONS AND CONSTANTS

2.1 Notational Conventions

Vector and matrix notation is used extensively throughout this report. Scalars and vectors will be denoted by lower case symbols, and any ambiguities of usage between the two will be resolved in the text. Components of vectors will be denoted by right superscripts. Coordinate reference frames for vectors are indicated by left subscripts on the vectors. For example,

$${}_c x \quad = \quad \text{position vector in the "c" frame, with components } {}_c x^1,\ {}_c x^2,\ {}_c x^3 .$$

Upper case symbols will denote matrices. In particular, the orthonormal 3 x 3 matrix for coordinate transformation from the "n" frame to the "c" frame is denoted by ${}_c T^n$. An example of such a transformation is

$${}_c x \quad = \quad {}_c T^n \, {}_n x \; .$$

The notation of " · " over a symbol will have its customary meaning of differentiation with respect to time. The " ^ " (hat) mark over a symbol means "estimated" or "computed" value of the symbolized quantity, while the " ~ " (tilde) indicates that the quantity is an error or small variation. For instance, if x is the true value of position, it may be written as the sum of the estimated position and the position error.

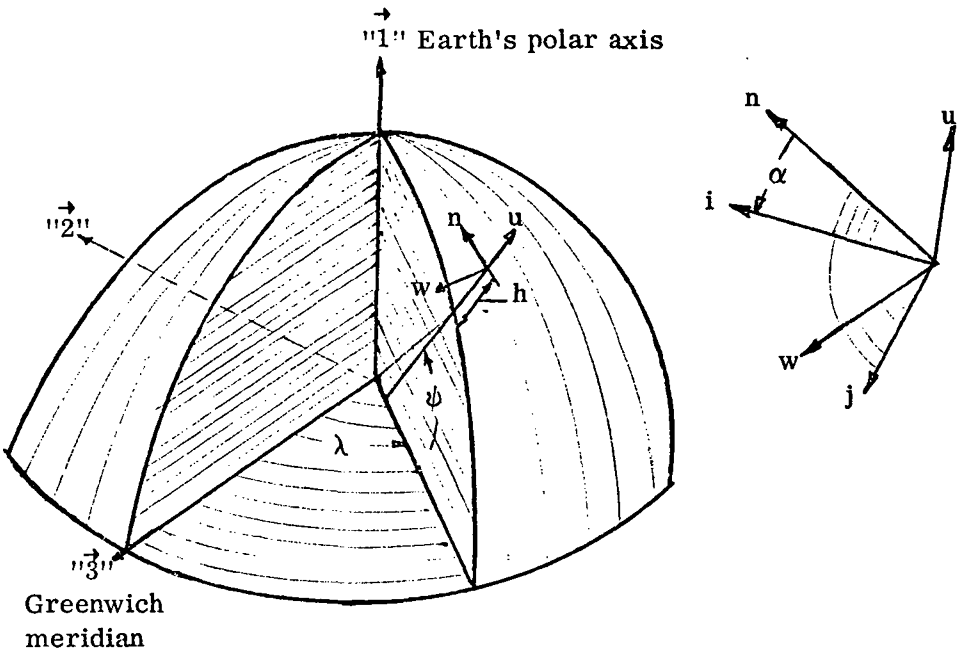
$$x \quad = \quad \hat{x} + \tilde{x} .$$

2.2 Reference Frames

All reference frames used in this report are right-handed orthonormal frames. The following lower case letters are used for their identification.

c	computer frame, fixed to the Earth at a runway or waypoint.
e	Earth-fixed equatorial frame with "1" axis along the Earth's polar axis and "3" axis in the equatorial plane at the Greenwich meridian.
i	inertial frame, non-rotating with respect to space.
ℓ	local level reference frame at the aircraft's position.
n	navigation frame, "tied" to the platform by a coordinate transformation.
p	platform frame, fixed to a stable platform with "1" axis along the #1 accelerometer axis.
r	runway frame, fixed to the Earth with "1" axis along the runway and "3" axis along the local vertical at the runway.

Figure 1 shows some geometrical relationships between the "e" frame and two local level reference frames. One of these "ℓ" frames is a north-pointing frame and the other is a "wander azimuth" reference. The computer and runway frames are also level frames, but these are fixed to the Earth and do not change orientation with changes in the aircraft's position.



Key

$\vec{1}, \vec{2}, \vec{3}$	basis vectors in Earth-fixed frame
n, w, u	basis vectors in local level north-pointing frame
i, j, u	basis vectors in wander azimuth frame
ψ	geodetic latitude
λ	longitude
α	wander azimuth
h	altitude

Figure 1. - Geometrical Relationships Between Earth-Fixed and Local Level Reference Frames

2.3 Definition of Symbols and Constants

Roman Symbols

a	equatorial radius of the Earth (6378.16 km)
b	polar radius of the Earth (6356.77 km)
C_a	accelerometer compensation matrix
C_g	gyro compensation matrix
c_{ij}	elements of the transformation between local velocity and craft rate
d_g	force-dependent gyro drift compensation
$d_x r^i$	differential operator defined by equation (5.28)
$d_y r^i$	differential operator defined by equation (5.29)
$d\hat{x}$	incremental state estimated by the filter
e	eccentricity of the oblate Earth model (.0818349)
e_3	unit basis vector, $e_3^T = (0 \quad 0 \quad 1)$
f	(1) specific force vector (compensated delta-velocities from accelerometers) (2) flattening of the oblate Earth model (.00335363)
f_a	measured specific force (raw delta-velocity data from the accelerometers)
f_b	accelerometer bias
F_n	Jacobian matrix of the state rates with respect to the random forcing functions
F_z	Jacobian matrix of the state rates with respect to the estimated state
g	gravitational acceleration
g_0	gravitational acceleration at the equator, zero altitude (.00978027 km/sec^2)
h	altitude (usually above mean sea level)
h_z	first-order altitude approximation from "c" frame position coordinates
I	3x3 identity matrix
M	Jacobian matrix of Earth central rotation with respect to position
n	white noise vector (random forcing function)
n_f	white noise on the accelerometer bias model

n_ω	white noise on the gyro drift model
r_a	apparent radius of curvature of the Earth
r_m	meridianal radius of curvature of the Earth
r_n	normal radius of curvature of the Earth
R	rotation matrix
t	time
T	matrix of transformation between coordinate reference frames
t_i	time at the beginning of a Kalman filter cycle
t_{ij}	elements of the transformation matrix, T
v	velocity of the aircraft
x	(1) position of the aircraft (2) state vector of the Kalman filter
x_e	east component of aircraft position
x_ℓ	level component of aircraft position
x_n	north component of aircraft position
y	measurement vector from the navigation aids
z	error state vector

Greek Symbols

α	wander azimuth angle, platform azimuth
α_r	azimuth of the runway or RNAV airway
β	rotation vector of tilts between the "c" and "n" frames
γ	Earth central angle between the "c" and "ℓ" frame origins
Γ	Jacobian matrix of gravity with respect to position
θ	rotation vector for Earth central angle between "c" and "ℓ" frames
θ_u	unit rotation vector, θ
λ	longitude
ν	rotation angle in the local level plane

ρ	rotation vector
τ_f	time constant for the correlated noise model of accelerometer bias
τ_k	duration of one Kalman cycle
τ_ω	time constant for the correlated noise model of gyro drift
φ	rotation vector of platform tilts
Φ	state transition matrix
Φ_n	state sensitivity to random forcing functions
ψ	geodetic latitude
ψ_r	geodetic latitude of the runway or waypoint reference center
ω	rotation rate of reference frame with respect to the Earth
ω_c	craft rate, desired platform rate
ω_d	drift rate between the "c" and "n" frames
ω_e	sidereal rate of rotation of the Earth (15.041067 deg/hr)
ω_g	platform command rate compensated for gyro scaling and misalignment
ω_{gs}	steady gyro drift rate
ω_ℓ	leveling control rate of the platform
ω_{max}	maximum leveling control rate
ω_t	total rate of rotation of the platform

III. AIDED INERTIAL NAVIGATION SYSTEMS

3.1 Overview

Figure 2 is a block diagram of the aided inertial navigation system configuration considered in this report. The main parts of the system are: (1) the inertial measurement unit (IMU) which consists of a stable platform* on which three integrating accelerometers sense the specific force vector acting on the case in the form of "delta-velocity" increments (actually, integrated specific force over a fixed time interval); (2) NAVAID receivers and transducers which provide the external "aiding" data; and (3) a digital computer, containing logic for IMU compensation, the navigation equations, and a modified Kalman filter for processing the external data.

The raw accelerometer data is corrected for biases, alignment errors, and scale factor calibrations in the IMU compensation section of the computer logic. This compensated data is used with the computed gravity vector and coriolis force to obtain the aircraft's acceleration which is then integrated in the navigation equation logic to form current position and velocity estimates. The navigation equation logic also calculates the desired platform leveling commands, which are modified in the IMU compensation logic to account for gyro torquer scale factors, gyro axis misalignment, and gyro drifts. The resulting signal feeds the gyros which, in turn, cause the platform drive systems to follow.

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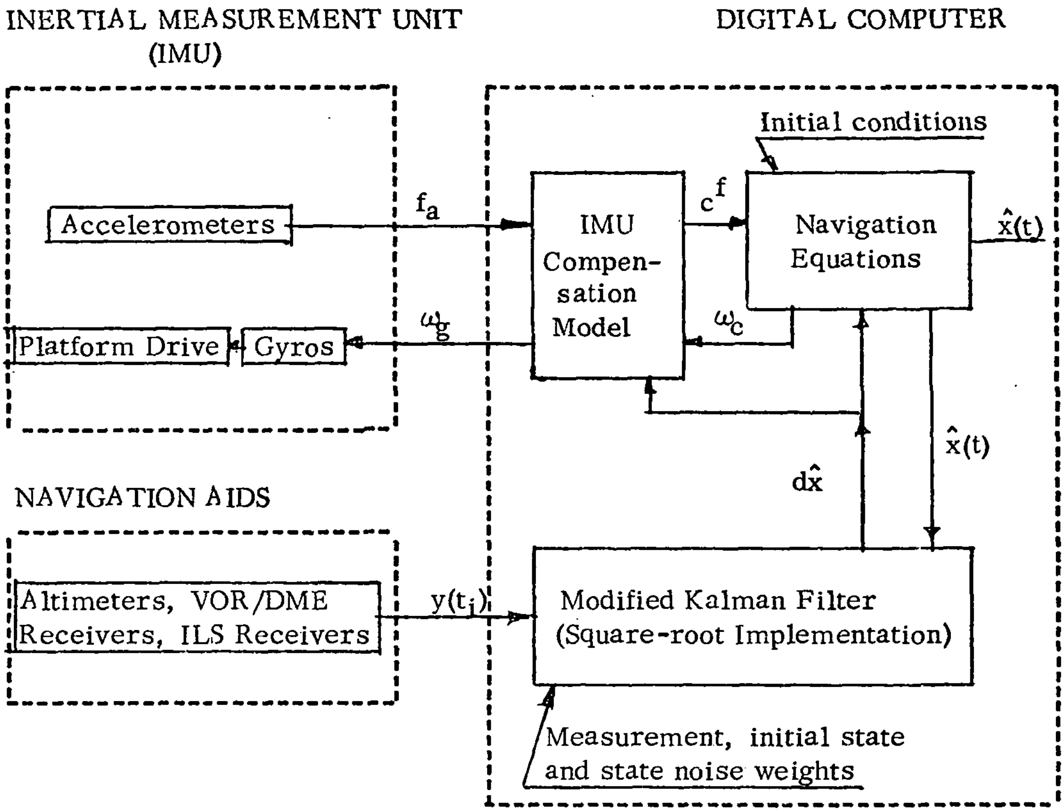
* Aided inertial navigation systems can incorporate strapped-down platforms too, but the discussion is restricted to stable platform implementations in this report.



Key:

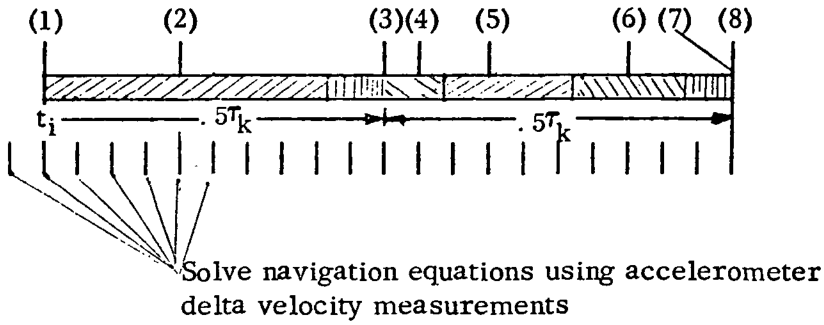
f_a = digitized accelerometer data
c^f = compensated accelerometer data in computer frame
ω_c = desired platform rate
ω_g = compensated platform command rate
$\hat{x}(t)$ = estimated state
$d\hat{x}(t)$ = incremental state estimated by the filter
$y(t_i)$ = vector of measurements from the navigation aids

Figure 2. - Block Diagram of an Aided Inertial Navigation System

Figure 3 illustrates a sequence of computer calculations suitable for an aided INS such as that considered in this report. At discrete times, measurements from the navigation aids (e.g., barometric altimeter, VOR/DME, and ILS receivers, etc.) are keyed into the Kalman filter section of the computer. The current state estimate, $\hat{x}$, is also keyed into the filter at the same time. The Kalman filter forms measurement residuals by computing estimates of each of the measured quantities based on $\hat{x}$ and subtracting these from the actual measurements. The filter calculates an incremental state, $d\hat{x}$, which is an estimate of the error in the state estimate, $\hat{x}$, based on the residuals, the measurement error model and the INS error model. The elements (components) of the vector, $d\hat{x}$, typically include position, velocity, "tilt" angles, and perhaps an assortment of biases (e.g., accelerometer bias, gyro drift, measurement biases). At the end of the Kalman filter cycle, the computed incremental state vector is added to the estimated state vector in the navigation equations, the measurement model, and the IMU compensation logic. The accelerometer data is processed at a high rate, so the estimated state is nearly continuous. The computer time required for the filter calculations depends on the complexity of the error model used in the filter and upon the amount of external data to be processed by the filter. A data rate much lower than that used for the accelerometer data is usually more than adequate for effective aiding, and the filter need be cycled at only a relatively modest rate.

3.2 The RAINPAL System

The RAINPAL configuration is like that shown in Figure 2, except that the INS, a Litton LTN-51, is used strictly as a source of accelerometer data. A separate computer (an SDS 920) performs all the aided inertial navigation computations. The SDS 920 computer estimates platform tilts and takes them into account in the navigation calculations, but does not torque the platform. This torquing is commanded by the LTN-51 computer on its own, operating as an independent free inertial system. The following data is transferred from



Events

(1) Save state estimate and measurements

(2) Compute incremental state and update square-root covariance for the measurements at time $t_i + \tau_k$

(3) Save state estimate at the midpoint of the filter cycle

(4) Compute the state transition matrix for one filter cycle using the state estimate at the midpoint

(5) Propagate the incremental state and the square-root covariance to the end of the filter cycle time $t_i + \tau_k$

(6) Reduce the square-root covariance to triangular form

(7) At time $t = t_i + \tau_k$ add the incremental state to the state estimate

(8) Start next filter cycle

Figure 3.- Sequence of Computer Calculations for One Kalman Filter Cycle

the LTN-51 to the SDS 920 computer:

1. digitized accelerometer data at the repetition rate of 20 Hz,
2. aircraft attitude with respect to the platform on request,
3. serial digital data giving latitude, longitude, true heading, and north-south and east-west components of velocity.

The accelerometer data is obtained with sufficient resolution to permit RAINPAL to navigate "free inertial" with accuracy equal to that of the LTN-51. Attitude data is used in RAINPAL only to apply second-order corrections to certain of the external aiding data, a function which does not require high precision. Of the serial digital data, only true heading is used in RAINPAL (for initialization), but the other quantities serve as an indication of the performance of the independent LTN-51.

The RAINPAL software processes accelerometer data at a 10 Hz rate. The error model used has 11 state variables (three positions, three velocities, three "tilts," plus barometric altimeter and vertical accelerometer biases). The resulting computer time (on the SDS 920) is such that the filter can be cycled at a maximum frequency of .5 Hz.

IV. NAVIGATION EQUATIONS AND IMU COMPENSATION

4.1 General Considerations

The coordinate frame selected for the navigation equations has a significant effect on the calculation time and numerical precision required by a navigation system, both in the internal calculations and in the provision of appropriate external information for pilot or autopilot usage. In the internal calculations, we have the requirement for processing accelerometer and navigation aid measurements and for calculating platform commands. The external calculations carry a requirement for providing appropriate displays such as:

(a) the position of the aircraft expressed in the global coordinates (geodetic latitude, longitude and altitude) and in relative coordinates (distance to/from en route waypoints or runway threshold, cross-track error and altitude) and

(b) the velocity of the aircraft with respect to the ground, expressed in a local reference (north-south velocity, east-west velocity and altitude rate).

Commercially available inertial navigation systems[7,8] generally use a wander azimuth[9] coordinate reference for velocity calculations. This is a local level reference and north-south and east-west velocities are calculated from wander-azimuth components of velocity by a coordinate transformation. The position is calculated and retained in a transformation from an Earth-fixed equatorial reference frame (in which one basis vector lies along the earth's polar axis and another is in the equator and plane of the Greenwich meridian) to the local level reference at the aircraft. The latitude and longitude are calculated by the use of inverse trigonometric functions from selected elements of the transformation. The altitude and altitude rate are not generally obtained from the INS. The commercially available inertial navigation systems have a relative position output capability in the area navigation (RNAV) reference frame consisting of distance to/from a waypoint, cross-track error and altitude. Great circle paths between waypoints define the desired RNAV airway.

The wander azimuth reference frame has several disadvantages when we consider its usage in an aided inertial navigation system for all phases of flight. These are:

(1) The position is normally retained in global (rather than relative) coordinates. The high numerical accuracy required during the landing phase necessitates added calculations in fixed-point computers, such as double-precision arithmetic or automatic scaling.

(2) The internally carried components of velocity (and position) require numerous calculations before they are appropriate for pilot display or autopilot usage.

(3) Processing of navigation aid measurements is not as efficient as desired, since these measurements are usually relative-type quantities such as range and bearing to VORTAC stations.

A convenient reference frame for navigation during landing is one which is aligned with the runway and centered at the threshold. In this frame, the internally calculated variables are distance to threshold, cross-track error, altitude above threshold and their corresponding velocities. The selection of this reference frame therefore simplifies the external calculations, minimizing the calculation time and numerical errors during the approach and landing phases of flight where rapid and accurate navigation is required.

If we were to select an Earth-fixed reference frame which is aligned with the desired RNAV airway and centered at the waypoint, then distance to/from the waypoint, cross-track error and a vertical distance would be directly available along with their corresponding velocities. The latitude and longitude can also be output with computation by series approximations which are potentially simpler than the inverse trigonometric functions used in the wander-azimuth mechanization. A further advantage in selecting this computational reference frame over the wander-azimuth mechanization is that processing VOR/DME data from stations in the vicinity of the waypoint involves simpler calculations and higher numerical accuracy with only single-precision arithmetic.

The runway (or waypoint) type of coordinate frame is adopted for the calculations in the system described in this report. Coordinate center shifts which involve a translation and rotation of the coordinates as the aircraft proceeds from one waypoint reference to the next will be described in the development.

4.2 Navigation Equations

The vector equation for acceleration of the aircraft with respect to the Earth (in a reference frame which is rotating at $\omega_e + \omega$ with respect to inertial space) is[9,10]

$$\dot{v} = f + g - (2\omega_e + \omega) \times v \tag{4.1}$$

where

ω = rotation rate of the reference frame with respect to the Earth

ω_e = rotation rate of the Earth with respect to inertial space

f = specific force

g = gravity, including centripetal acceleration.

If we select a local tangent plane frame at the take-off runway to start and at discrete times use a center-shift plus a rotation of coordinates, then $\omega = 0$. The transformation which transforms vectors from the aircraft's local level "ℓ" frame to the computer "c" frame (runway or waypoint) is ${}_c T^{\ell}$. The gravity vector is represented very simply* in the "ℓ" frame. We may rewrite (4.1) more specifically by including the reference frame in which each vector is expressed (denoted by a left subscript).

$$ {}_c\dot{v} = {}_c T^{\ell}({}_\ell T^p \, {}_p f + {}_\ell g) - 2\, {}_c\omega_e \times {}_c v \tag{4.2} $$

The position vector with respect to the coordinate center satisfies

$$ {}_c\dot{x} = {}_c v. \tag{4.3} $$

The calculation of the quantities in (4.2) is simple except for the transformation, ${}_c T^{\ell}$. The specific force vector, ${}_p f$, is obtained from the accelerometer measurements as shown on Figure 2. The platform-to-level transformation, ${}_\ell T^p$, is part of the IMU compensation model covered in Section 4.4. The gravity vector, ${}_\ell g$, can be approximated by a scalar function of altitude and latitude which acts along the vertical direction in the "ℓ" frame. The Earth rate vector, ${}_c\omega_e$, has a constant magnitude of about 15 degrees per hour and lies along the Earth's polar axis direction. This vector is a constant in the "c" frame and changes only at coordinate center shifts. To obtain the ${}_c T^{\ell}$ transformation, we note that what is desired is a transformation from a level frame at the aircraft to a level frame at the coordinate center. Neglecting azimuth considerations for now, we can calculate ${}_c T^{\ell}$ by a single rotation in the plane formed by the local vertical vectors of the "c"

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* One of the state variables in the present RAINPAL system is the bias of the vertical accelerometer. The barometric altimeter measurements calibrate this bias continuously and, as a result, g variations have only insignificant effects on the accuracy of the system.

and "ℓ" frames. Calculation of ${}_c T^{\ell}$ is complicated by the fact that the Earth is not spherical. An approximate solution is to form a unit rotation vector, θ_u, defined by

$$\theta_u = {}_c x \times e_3 / |{}_c x \times e_3| \tag{4.4}$$

where $e_3^T = (0 \quad 0 \quad 1)$ and ${}_c x$ is the position vector in the computer frame (i.e., the aircraft's position with respect to the "c" frame origin). Let γ be the rotation angle (to be defined) and then define the vector, θ, by $\theta = \theta_u \sin\gamma$. Then the transformation from level to computer frame is

$${}_c T^{\ell} = I - (\theta x) + (\theta x)(\theta x)/(1 + \cos\gamma) . \tag{4.5}$$

We approach the problem of defining the rotation angle, γ, by allowing two radii of curvature for the Earth to approximate the oblate spheroid. One is for rotations about an axis which is parallel to the east-west direction at the center of the reference and the other is for rotations about an axis parallel to the north-south axis. The two radii are shown as functions of latitude on Figure 4. The formulation of the equations is given in Reference 9. The difference in radii is largest at zero latitude, whereas the change with latitude is largest at 45^o. The approximate radii of curvature may be obtained by a Taylor's series expansion of the exact expressions (see Figure 4),

$$r_m = r_m(0) + (3\, e^2 \sin\psi_r \cos\psi_r)\,(x_n/2) \tag{4.6}$$

and

$$r_n = r_n(0) + (e^2 \sin\psi_r \cos\psi_r)(x_n/2) , \tag{4.7}$$

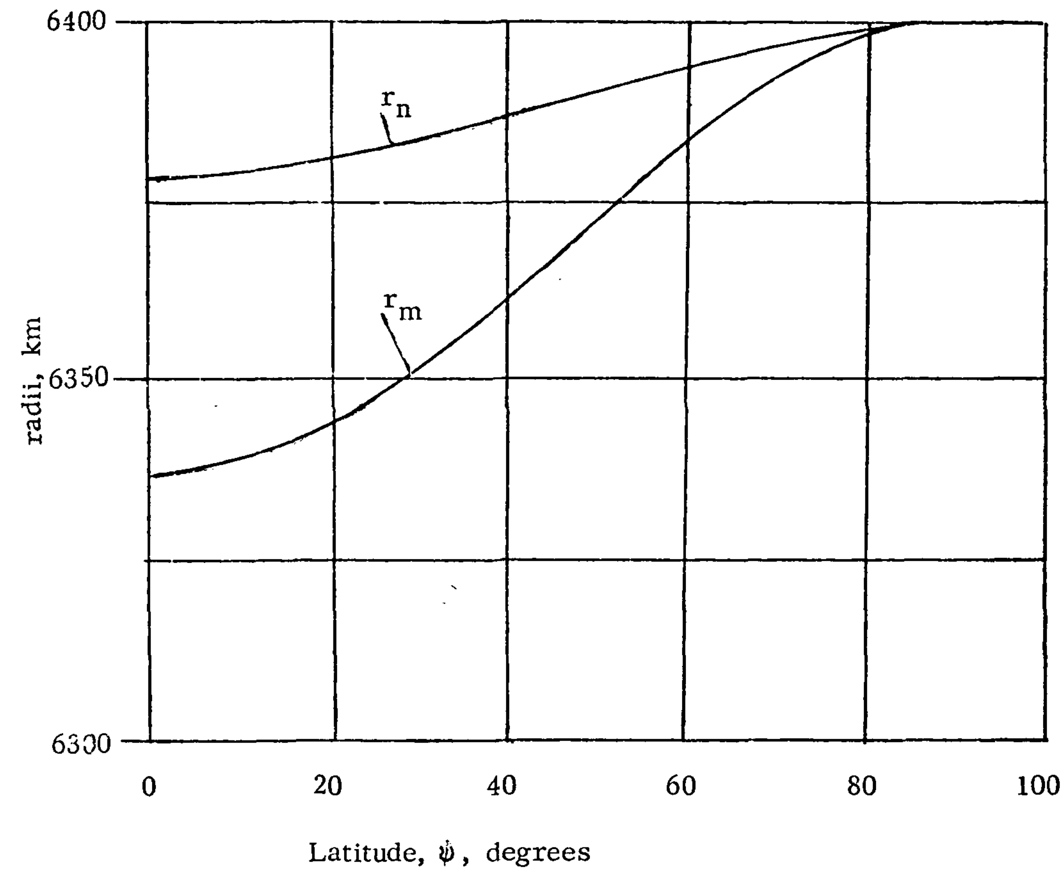
where

x_n = north component of aircraft position (${}_c x^1 \cos\alpha_r - {}_c x^2 \sin\alpha_r$, where α_r is the azimuth of the runway or RNAV airway),

e = eccentricity of the oblate Earth model,

ψ_r = latitude of the runway or waypoint,

$r_n(0), r_m(0)$ = radii of curvature at the runway or waypoint.



r_m = meridianal radius of curvature = $a(1 - e^2)/(1 - e^2 \sin^2 \psi)^{3/2}$

r_n = normal radius of curvature = $a/(1 - e^2 \sin^2 \psi)^{1/2}$

Figure 4. - Radii of Curvature Versus Latitude

The apparent radius of curvature, r_a, is calculated by the (empirically formulated) expression

$$r_a = 1/\sqrt{x_n^2/(x_\ell^2 r_n^2) + x_e^2/(x_\ell^2 r_m^2)} \tag{4.8}$$

where

$x_\ell^2 = ({}_c x^1)^2 + ({}_c x^2)^2$, square of the horizontal distance, and

$x_e^2 = x_\ell^2 - x_n^2$, square of the eastward position component.

The altitude may then be found using

$$h = h_z (1 - 0.5\, h_z / r_a) \tag{4.9}$$

where

$h_z = {}_c x^3 + {}_c x^T \, {}_c x \, / \, (2\, r_a)$, the first-order altitude.

The sine of the rotation angle is found using

$$\sin\gamma = x_\ell / (r_a + h) \, . \tag{4.10}$$

A numerical check of the approximations given in (4.9) and (4.10) shows errors smaller than one meter and 10 microradians, respectively, for distances less than 500 km from the coordinate center and altitudes below 20 km. The errors vary with direction of the aircraft from the runway or waypoint.

4.3 Platform Leveling Commands

The effects of many platform error sources can be minimized by keeping the platform in a near-level condition. The rotation rate (with respect to inertial space) required for accomplishing this is

$$\omega_\ell = \omega_e + \omega_c \tag{4.11}$$

where

ω_e = rotation rate of the Earth with respect to inertial space, and

ω_c = rotation rate of the craft with respect to the Earth.

In a local level reference frame the craft rate, ω_c, has two components in the level plane. The rotation rate about the vertical may be specified at any value for convenience. Normally it is taken as zero and the resulting mechanization is called a "wander-azimuth" mechanization, since the azimuth of the platform at any time depends on the path the aircraft took to arrive at its current location. For reasons to be discussed later, we will specify the component about the vertical such that the azimuth rate about the vertical in the runway (waypoint) frame is zero.

In the local level frame, the craft rate is related to velocity as follows.

$$\begin{bmatrix} {}_{\ell}\omega_c^1 \\ {}_{\ell}\omega_c^2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} {}_{\ell}v^1 \\ {}_{\ell}v^2 \end{bmatrix} \tag{4.12}$$

The elements, c_{ij}, are define by

$$c_{11} = -c_{22} = -2f({}_{\ell}t_{11}^e)({}_{\ell}t_{21}^e/a$$

$$c_{12} = -[1-f({}_{\ell}t_{31}^e)^2 + 2f({}_{\ell}t_{21}^e)^2 - h/a]/a$$

$$c_{21} = [1-f({}_{\ell}t_{31}^e)^2 + 2f({}_{\ell}t_{11}^e)^2 - h/a]/a$$

where f is the flattening of the oblate Earth model and a is the equatorial radius. The elements of the ${}_{\ell}T^e$ transformation have been denoted ${}_{\ell}t_{ij}^e$.

$${}_{\ell}T^e = {}_{\ell}T^c\,{}_{c}T^e \tag{4.13}$$

These elements can also be written as functions of latitude, ψ, and platform azimuth, α.

$${}_{\ell}t_{11}^e = \cos\psi\cos\alpha$$

$${}_{\ell}t_{21}^e = -\cos\psi\sin\alpha$$

$${}_{\ell}t_{31}^e = \sin\psi$$

A derivation of the coefficients, c_{ij}, and a discussion of the validity of the approximations are contained in Reference 10.

The third component of ${}_\ell\omega_c$ is defined such that the azimuth rate with respect to the "c" reference is zero. If we let [see Eq. (A. 40)]

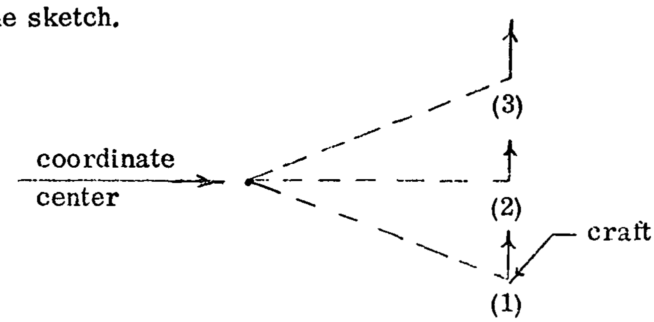
$$
\begin{aligned}
{}_\ell\dot{T}^c &= [-{}_\ell\omega_c \times]\,{}_\ell T^c = d/dt\,[R_1 R_2 R_3] \\
&= -[\dot{\theta}_1 u_1 + \dot{\theta}_2 R_1 u_2 + \dot{\theta}_3 R_1 R_2 u_3] \times {}_\ell T^c
\end{aligned}
\tag{4.14}
$$

then ${}_\ell\omega_c^3$ is selected to cause $\dot{\theta}_3$ (the azimuth rate) to vanish. Equation (4.14) can be used to select this value of ${}_\ell\omega_c^3$.

$$
{}_\ell\omega_c^3 = \tan\theta_1\,{}_\ell\omega_c^2 = [{}_\ell t^c_{23} / {}_\ell t^c_{33}]\,{}_\ell\omega_c^2 \tag{4.15}
$$

This quantity is very small compared to ${}_\ell\omega_c^2$ so long as the "c" reference is within 500 km of the craft (at which range $|\tan\theta_1| \leq .0786$). This particular platform command law will be called an "Earth-fixed azimuth" mechanization. It has the property that the azimuth on landing is the same as on take-off regardless of route, speed, and time, if the "c" frame is held at the same reference point throughout the flight.

The ${}_c T^\ell$ transformation (equation 4.5) has no rotation about the vertical axis (azimuth). It is necessary to maintain the proper azimuth in transforming the acceleration measurements from the platform reference to the runway or waypoint reference. As a result of the torquing about the vertical axis, the azimuth problem in the transformation, ${}_c T^p = {}_c T^\ell\,{}_\ell T^p$, is resolved by a fixed azimuth in ${}_\ell T^p$. This point may be clarified by considering the sketch.



With either the "Earth-fixed azimuth" or the "wander azimuth" mechanization the platform stays fixed with respect to the Earth when the craft is stationary. When the craft is moving, as shown in the sketch, there is no torque about the vertical axis in the level frame with the wander azimuth mechanization. As a result, the platform's orientation with respect to an Earth-fixed frame will "wander" and take on a value which is dependent on the path. With the "Earth-fixed azimuth" mechanization, the platform is torqued about the vertical in the level reference to maintain a zero rate about the vertical at the center of the Earth-fixed reference frame. The net result is that a simple rotation transformation transforms level to Earth-fixed coordinates. A constant rotation about the vertical transforms the platform reference to level axes (except for tilts).

4.4 IMU Compensation Model

A simple block diagram of the IMU compensation model is presented in Figure 5. The accelerometer outputs are compensated for known (or estimated by the Kalman filter) biases, scaling, and misalignment. The three-vector, ${}_{p}f$, of Figure 5 is the compensated specific force* in the platform reference frame. The tilt transformation, ${}_{\ell}T^{p}$, transforms the specific force vector to the local level reference frame. The tilt transformation obeys a differential equation which is updated in the box labeled "Tilt Update." The leveling control rate, ω_{ℓ}, drives the platform to a level condition when tilts are detected. This control law will be discussed in the next section. The IMU command rate is a compensated signal which drives the platform to a local level orientation and holds it there. The compensation includes scaling and misalignments (between platform and gyro input axes), compensation for steady drifts and drifts resulting from mass

* Accelerometer outputs are in units of velocity since the digitizing circuitry is a voltage/frequency converter and a counter. By dividing the output by the time interval of counting (a constant), the result is an average specific force. Since the time intervals are short (.1 sec or less), one may conceptually view the outputs as indications of instantaneous specific force.

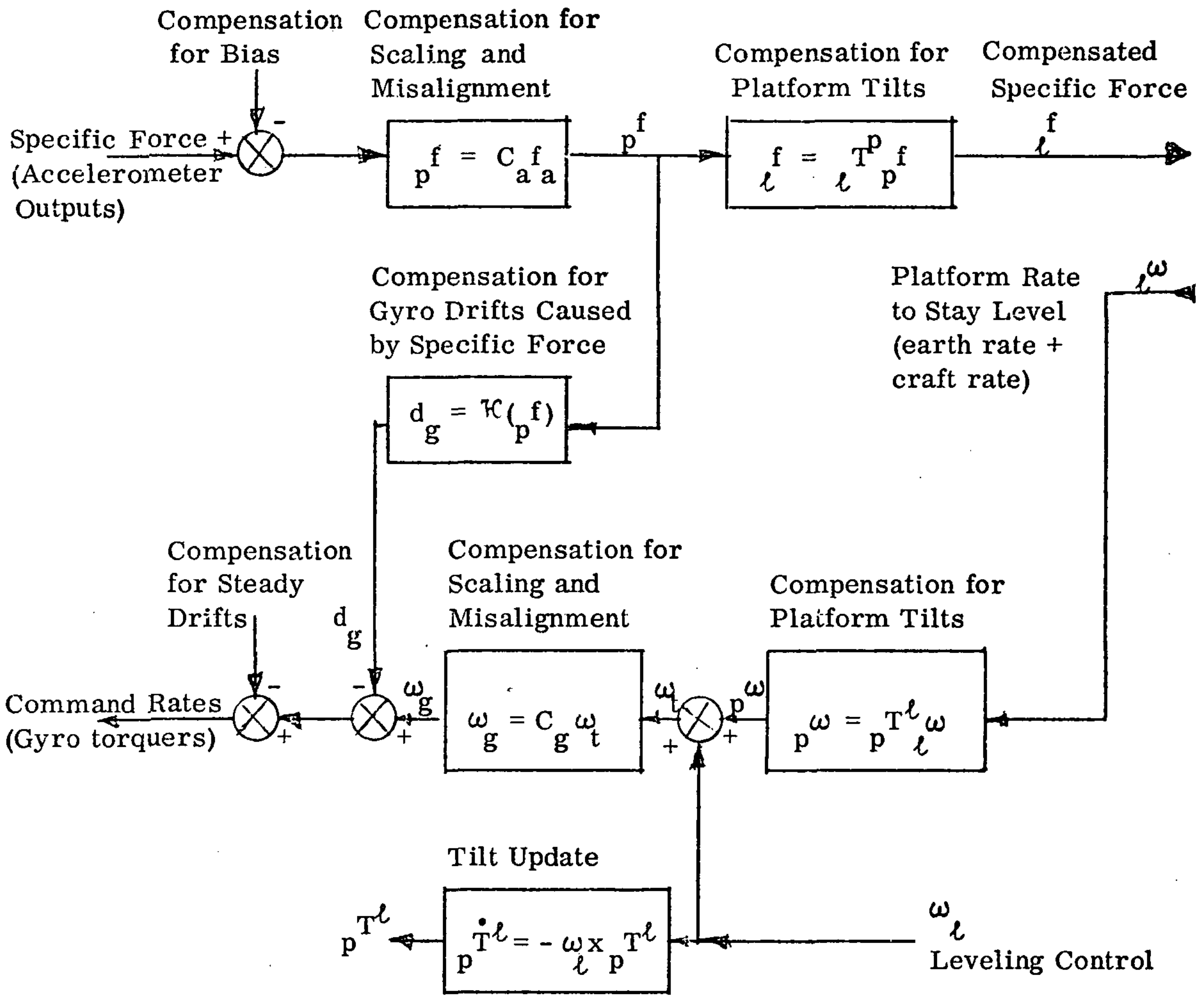


Figure 5. - Block Diagram of Software Compensation for IMU Anomalies

unbalance and anisoelastic effects. These latter drifts are functions of the specific force and may be compensated by an appropriate model.

Figure 5 shows a fair degree of sophistication in the software compensation of IMU anomalies. The amount needed in any specific situation is obviously dependent on the size of the hardware anomalies and the accuracy requirements of the application. In the flight test validation of the RAINPAL[5,6] navigator, an LTN-51 navigation system was used for accelerometer outputs as was mentioned earlier. The LTN-51 operated in its normal free-inertial mode and separate outputs of the platform accelerometers fed the RAINPAL software. Since the RAINPAL software could not drive the LTN-51 platform, no calculations of command rates were necessary. The accelerometers were compensated for biases and the tilt transformation, ${}_pT^{\ell}$, was used to compensate for the platform tilts. The transformation update was omitted since ${}_p\dot{T}^{\ell}$ is zero when one cannot torque the platform. The results obtained in these flight tests demonstrate the excellent performance of the LTN-51 INS hardware when used as part of an aided inertial navigation system.

4.5 Platform Leveling Control Logic

The software for compensation of IMU anomalies shown on Figure 5 includes a compensation for platform tilt with the transformation, ${}_pT^{\ell}$, such that it is not necessary to maintain the platform in an exact level condition. A leveling control rate, ω_{ℓ}, is used to drive the platform and the ${}_pT^{\ell}$ transformation to level as "tilts" are estimated by the filter. The control logic is entered at the beginning of each Kalman filter cycle. This logic implements the following steps.

1. Calculate the rotation vector, ρ, of a rotation matrix, R, (see equation A.44) such that $R\,{}_pT^{\ell}$ represents a transformation involving only a rotation about the vertical axis:

$$[I - (\rho x) + (\rho x)(\rho x)/(1 + \cos \sin^{-1}|\rho|)]\,{}_pT^{\ell} = \begin{bmatrix} \cos\nu & \sin\nu & 0 \\ -\sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$R(\rho, \sin^{-1}|\rho|)\,{}_pT^{\ell} = R(e_3, \nu) \tag{4.16}$$

The right-hand side of equation (4.16) is a rotation about the local vertical. The platform is not commanded with leveling torques on this axis.

2. Calculate the control rate, ω_ℓ, which will drive the platform to level in one filter cycle (period = τ_k).

$$\omega_\ell = \rho / \tau_k \tag{4.17}$$

3. Check whether the control rate is too large on any axis according to

$$\text{if } \left| \omega_\ell^i \right| > \omega_{max} \qquad \text{for } i = 1, 2 \tag{4.18}$$

$$\text{then set } \omega_\ell^j = \omega_\ell^j * \omega_{max} / \left| \omega_\ell^i \right| \quad \text{for } j = 1, 2 \tag{4.19}$$

Only two components of ω_ℓ need be checked, since the third is zero as a result of the control law of step (1) as will be shown.

4. Insert ω_ℓ in appropriate cells for controlling the platform and the ${}_pT^\ell$ transformation.

In step (1) we see that a value of the rotation vector, ρ, is to be found to make (4.16) hold. An examination of the right hand side of (4.16) shows that the third row of $R(\rho, \sin^{-1}|\rho|)$ must be equal to the third column of ${}_pT^\ell$. The orthonormal properties of the matrices will then cause zeroes and unity to occur in the proper locations in the product. This constraint and the expansion of $R(\rho, \sin^{-1}\rho)$ yields the following equations.

$$\rho^2 + \rho^1\rho^3/(1 + \cos\sin^{-1}|\rho|) = {}_pt^\ell_{13} \tag{4.20}$$

$$-\rho^1 + \rho^2\rho^3/(1 + \cos\sin^{-1}|\rho|) = {}_pt^\ell_{23} \tag{4.21}$$

$$1 - \left[(\rho^1)^2 + (\rho^2)^2\right] / (1 + \cos\sin^{-1}|\rho|) = {}_pt^\ell_{33} \tag{4.22}$$

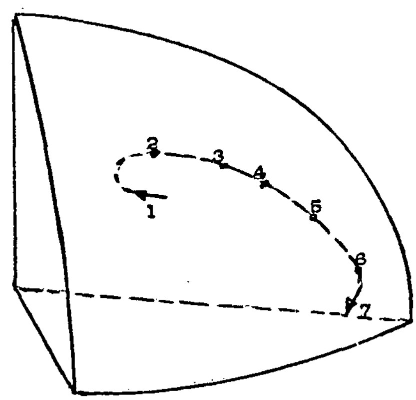
The solution of the simultaneous equations, (4.20) and (4.21) with $\rho^3 = 0$ yields

$$\rho = \begin{bmatrix} -t^{\ell}_{p\,23} \\ t^{\ell}_{p\,13} \\ 0 \end{bmatrix} \tag{4.23}$$

which is the solution of step (1) of the leveling logic for the rotation vector.

4.6 Coordinate Center Shifts

For en route applications the navigation system is initialized relative to the take-off runway coordinate system, waypoint #1 of the sketch.



Waypoints 2 through 6 define the en route RNAV path. The system switches to new waypoint-centered reference frames as the flight progresses. The spacing between waypoints must be less than 500 km for good free inertial performance. The system will insert waypoints as necessary to keep the coordinate centers within the limits. Waypoint #7 is the coordinate center at the landing runway.

The system switches coordinate centers when the next waypoint in the sequence is closer than the current waypoint. Prior to switching, it is assumed that the transformations relating waypoint "i" with "i+1" (i.e., the new "c" frame), ${}_{i+1}T^{i}$, and the coordinates of waypoint #i+1 in the "i"-centered reference, ${}_{i}x_{i+1}$, are available. Then the equations for shifting position and velocity between centers are

$$ {}_{i+1}x = {}_{i+1}T^{i} ({}_{i}x - {}_{i}x_{i+1}) \tag{4.24} $$

and

$$ {}_{i+1}v = {}_{i+1}T^{i} ({}_{i}v) \ . \tag{4.25} $$

A new transformation is required for ${}_{i+1}T^{p}$ in order to transform the accelerometer measurements to the new center. This is found from

$$ {}_{i+1}T^{p} = {}_{i+1}T^{i} \, {}_{i}T^{p} . \tag{4.26} $$

The transformation, ${}_{i+1}T^{\ell}$, is calculated using ${}_{i+1}x$ in the transformation described in section 4.2. A new transformation for ${}_{\ell}T^{p}$ is then calculated such that

$$ {}_{\ell}T^{p} = {}_{\ell}T^{i+1} \, {}_{i+1}T^{p} . \tag{4.27} $$

The coordinate center shift is completed by providing Earth rate in the "i+1" frame.

V. VARIATIONAL EQUATIONS

5.1 General Considerations

The traditional procedure[11,12] used in the development of linear variational (error) equations is to define an "ideal" INS as one which incorporates perfect components and a perfect set of navigation equations. When such a system is given the true initial conditions, it will maintain the true state (position and velocity) for an indefinite time period. For stable platform systems, the ideal INS includes a perfect platform which is kept level at the true location by the navigation equations. The dynamic equations (mathematical model) for the ideal system are formulated in a fashion similar to those of the actual error-prone system. The error equations are formed by subtracting the actual equations from the ideal equations and setting the result equal to the mathematical model of the various hardware anomalies and software approximations which cause the difference. In those situations where the model is nonlinear, a Taylor's Series expansion is performed and only the first order (linear) effects are retained in the error equations.

The primary use of the ideal INS in the above development is that it is an "errorless" navigation system. Another definition of an "errorless" INS is one where the actual hardware contains anomalies which are perfectly compensated by a perfect "inverse" model of the anomaly in the onboard computer. With reference to Figure 5, we have shown compensation for such anomalies as (1) bias, scaling and misalignment of accelerometers, (2) steady and specific-force-induced gyro drifts, and scaling and misalignment of the gyros, and (3) tilts of the platform. If this IMU compensation model were perfect and a perfect set of navigation equations were implemented with the true state in the computer, then we would have another "errorless" navigator.

It remains errorless even when the actual platform is tilted away from the vertical because these tilts (as well as the other platform anomalies) are exactly known and properly taken into account in the computer. With this philosophy of an "errorless" navigator, one tends to blame actual system errors on imperfect computer compensation rather than imperfect hardware. For example, platform tilts are not errors in themselves; rather, the error in the INS arises because of the difference between the true tilt and the estimate of tilt which is used in the computer's compensation.

If we use the above definition of the "errorless" INS in place of the "ideal" INS, we can develop the error equations in the manner described previously for the "ideal" INS. If we retain the same coordinate frames for the error equations in the transition between philosophies, the resulting error equations are the same. This "errorless" concept has the advantage of leading to a more straightforward derivation of the error equations, as will be seen in the sequel. Another advantage of this approach is that it lends itself more naturally to aided INS applications in which tilts are estimated using external data. This may be seen by referring to Figures 2 and 5. If the filter state vector ($d\hat{x}$ of Figure 2) contains IMU errors such as vertical accelerometer bias and tilts, then incremental estimates from the filter can be used to instantaneously change the computer compensation model of these IMU anomalies. The effects of the error in the actual navigation equations are generally reduced as external data is used to estimate such anomalies and the behavior of the actual system tends to approach that of the perfectly compensated "errorless" INS.

The error equations developed by either of these approaches contain forcing functions involving a product of tilt error and the measured specific force in the differential equation for the velocity error. These functions create some problems in implementing aided inertial systems since the measured specific force is not well behaved and can have rapid changes in accelerometer outputs caused by noise and vibration. The forcing functions arise in the error equations because of differences between the coordinate frame of the measurement and the coordinate frame for the computer calculation of the craft's acceleration (and velocity). Traditionally the "errorless" or "ideal" INS takes the computer reference frame as being correct (for the development of error equations) and blames the platform for these differences. We introduce instead the concept that the craft's acceleration and velocity should be calculated in a reference frame which is mathematically tied to the platform. The result is that the specific force is in the proper frame and tilt errors between the platform and computer frames do not exist. Instead, errors arise in the calculated acceleration because the gravitational attraction and coriolis force are not resolved into the proper reference frame. Errors also arise in the rate of change of position because the velocity reference frame is mathematically tied to the platform and the position reference frame is defined with respect to the Earth. Errors in knowledge of the velocity reference frame therefore cause position errors since the computed velocity cannot be precisely resolved into the position reference.

The development of the error equations in the sequel will use an "errorless" INS with the perfectly compensated IMU anomalies and perfect navigation equations. The navigation equations express the acceleration and velocity in the reference frame which is tied to the actual platform. In the errorless equations, the platform orientation is known and the velocity is transformed to the true position reference frame. Also the gravity and coriolis forces are transformed to the true reference frame for the acceleration. This new formulation has the advantage of removing the specific force from the error equations as was mentioned. Another advantage, as will be seen, is that the new coordinates for the velocity error and a new definition of "tilt" errors combine to render a simpler set of error equations than would otherwise result.

5.2 Equations for the Errorless Navigator

The errorless navigator's differential equation for velocity (i.e., the velocity-rate equation) is written in the "n" frame, which is mathematically tied to the real platform frame. The transformation, ${}_n\hat{T}^p$, connecting the platform ("p") frame and the "n" frame is a mathematical reality existing in the computer and is, by definition, errorless. The fact that we might introduce errors by trying to relate the "n" frame to the computer ("c") or local level ("ℓ") frames does not prevent us from properly writing errorless or actual equations in that frame for the purpose of deriving error equations. It is convenient to use an Earth-fixed frame for the computer frame, "c". If we define the ${}_n\hat{T}^p$ transformation as that which the actual navigation computer would use to transform from platform to computer frames, we can see that in the errorless situation, the "n" and "c" frames would be coincident. The errorless navigation equations are:

$$ {}_n\dot{v} = {}_n\hat{T}^p\,{}_pf + {}_nT^c\,{}_cT^\ell\,{}_\ell g - (2\,{}_nT^c\,{}_c\omega_e + {}_n\omega_d) \times {}_nv \tag{5.1} $$

and

$$ {}_c\dot{x} = {}_cT^n\,{}_nv \tag{5.2} $$

ω_d is the angular velocity with which the "n" frame drifts with respect to the "c" frame and, therefore, ${}_nT^c$ obeys the differential equation,

$$ {}_n\dot{T}^c = -\,{}_n\omega_d \times {}_nT^c \tag{5.3} $$

We will now explain the " ^ " (hat) notation which was introduced with ${}_n\hat{T}^p$. When the hat appears over a variable in the remainder of this section, that variable is the one used in the actual navigation system's computer to represent the corresponding errorless variable. The computer variable is generally in error because it is computed from "best estimate" (but imperfect) navigation information.

The ${}_n\hat{T}^p$ transformation is defined in the computer as the platform-to-computer-frame transformation, which may be formed as the product of the platform-to-level and level-to-computer (Earth-fixed) transformations.

$$ {}_n\hat{T}^p = {}_c\hat{T}^\ell \, {}_\ell\hat{T}^p \tag{5.4} $$

Note that the matrix product in (5.4) is errorless while the individual matrices are not necessarily so. That is, ${}_n\hat{T}^p \equiv {}_nT^p$, since the "n" frame is defined by means of the transformation. The individual matrices in the product may contain errors, because ${}_\ell\hat{T}^p$ cannot perfectly transform vectors from the real platform to the real level frame and ${}_c\hat{T}^\ell$ cannot be perfectly computed if position errors exist. Equation (5.4) implies

$$ {}_n\hat{T}^c = I \qquad \text{(the identity matrix)} \tag{5.5} $$

which, indeed, is the relationship used in the actual navigator. Now, to develop ω_d, we let

$$ {}_nT^c = {}_n\hat{T}^p \, {}_pT^c = {}_c\hat{T}^\ell \, {}_\ell\hat{T}^p \, {}_pT^i \, {}_iT^c \tag{5.6} $$

and differentiate with respect to time to put ${}_nT^c$ in correspondence with (5.3).

$$ {}_n\dot{T}^c = {}_c\dot{\hat{T}}^\ell \, {}_\ell\hat{T}^p \, {}_pT^c + {}_c\hat{T}^\ell \, {}_\ell\dot{\hat{T}}^p \, {}_pT^c + {}_n\hat{T}^p \, {}_p\dot{T}^i \, {}_iT^c + {}_n\hat{T}^p \, {}_pT^i \, {}_i\dot{T}^c $$

$$ = {}_n\hat{\omega}_c \times {}_nT^c + {}_n\hat{\omega}_\ell \times {}_nT^c - {}_n\omega_t \times {}_nT^c + {}_n\omega_e \times {}_nT^c \tag{5.7} $$

From (5.3) and (5.7),

$$ \omega_d = -(\hat{\omega}_c + \hat{\omega}_\ell + \omega_e - \omega_t) \tag{5.8} $$

where

$\hat{\omega}_c$ = the computed craft rate, equations (4.12) and (4.14)

$\hat{\omega}_\ell$ = the rate used for leveling the platform (see Figure 5)

ω_e = the Earth's sidereal rate of rotation

ω_t = the platform's total rotation rate

The platform's total rate can be written in terms of rates commanded by the computer and errors in the model of the gyros' torquers (see Figure 5). It is beyond the scope of this report to develop ω_t as a function of all the individual gyro error sources. With reference to Figure 5, we will assume that

1. calibration of the gyro drifts due to mass unbalance and anisoelastic effects,
2. alignment of gyro input axes, and
3. scaling of the gyro torquers

has made these gyro error sources negligible. The only gyro error source which will be modeled here is a steady gyro drift rate, ω_{gs}, referred to the gyro input axes (see Figure 5).

$$_p\omega_t = {}_p\hat{T}^\ell \left({}_\ell\hat{T}^c \, {}_c\omega_e + {}_\ell\hat{\omega}_c \right) + C_g^{-1} \omega_{gs} + {}_p\hat{\omega}_\ell \tag{5.9}$$

Combining (5.8) with (5.9), we obtain another expression for ω_d.

$$_n\omega_d = - {}_n\hat{T}^p \left[{}_p\omega_e - {}_p\hat{T}^\ell \, {}_\ell\hat{T}^c \, {}_c\omega_e + C_g^{-1} \omega_{gs} \right] \tag{5.10}$$

We may notice that if the computed ${}_\ell\hat{T}^c$ and ${}_p\hat{T}^\ell$ transformations were errorless, the only contributor to a drift between the "c" and "n" frames would be the gyro drifts. The magnitude of ω_d will, therefore, generally be very small. If the "n" frame and "c" frame are closely aligned initially, they will stay that way over long time intervals.

5.3 Errors in the Actual Navigator

The differential equations implemented by the actual navigator are of the same form as those for the errorless navigator.

$$\dot{\hat{v}}_n = {}_n\hat{T}^p\,{}_p\hat{f} + {}_n\hat{T}^c\,{}_c\hat{T}^\ell\,{}_\ell\hat{g} - (2\,{}_n\hat{T}^c\,{}_c\omega_e) \times {}_n\hat{v} \qquad (5.11)$$

$$\dot{\hat{x}}_c = {}_c\hat{T}^n\,{}_n\hat{v} \qquad (5.12)$$

$${}_c\hat{T}^n = I \qquad (5.13)$$

As was noted before, the " ^ " notation symbolizes that the quantity to which it is applied is the actual (i.e., estimated) value of the quantity residing in the navigation system's computer. The ${}_n\hat{T}^p$ is errorless by definition, while the other "hatted" quantities may be in error. Earth's rate is "hatless" because we presume to know it perfectly (or much better than the other variables) in the actual navigation equations. We will use the "~" notation to denote "error" and will write the following type of relationships between the errorless variables and their counterparts in the actual navigation equations.

$${}_nv = {}_n\hat{v} + {}_n\tilde{v} \qquad \text{(for vectors)} \qquad (5.14)$$

$${}_cT^n = (I - \tilde{\beta}\times)\,{}_c\hat{T}^n \qquad \text{(for rotation matrices)}^{*} \qquad (5.15)$$

All of the errors of the actual navigator are represented in mathematical form as (5.14) or (5.15). If we assume accelerometer and gyro calibration is perfect except for biases and if we assume numerical approximation errors are negligible, the following quantities appropriately define the error state.

$${}_c\tilde{x}$$ = vector of position errors in the "c" frame

$${}_n\tilde{v}$$ = vector of velocity errors in the "n" frame

$${}_c\tilde{\beta}$$ = vector of "tilt" errors between the "c" and "n" frames

$$\tilde{f}_b$$ = vector of accelerometer bias errors

$$\tilde{\omega}_{gs}$$ = vector of steady gyro drift errors

---

* $(I - \tilde{\beta}\times)$ is a first-order approximation to $\left[I - \tilde{\beta}\times + (\tilde{\beta}\times)(\tilde{\beta}\times)/(1 + \cos\sin^{-1}|\tilde{\beta}|)\right]$ which is the general matrix (see appendix) for frame rotation about the rotation vector, $\tilde{\beta}$.

The "tilt" error vector, $\tilde{\beta}$, as it is defined here, is the only error state which is not generally familiar or readily recognizable. Tilts are usually defined between the level and platform frames. If we denote such tilts by the vector, $\tilde{\varphi}$, where

$$ {}_{p}T^{\ell} = (I - {}_{p}\tilde{\varphi}\, x)\, {}_{p}\hat{T}^{\ell} \tag{5.16} $$

we can relate $\tilde{\beta}$ to $\tilde{\varphi}$ as follows. Errors in the computer-to-level transformation are described by a rotation error vector, $\tilde{\theta}$.

$$ {}_{\ell}T^{c} = (I - {}_{\ell}\tilde{\theta}\, x)\, {}_{\ell}\hat{T}^{c} \tag{5.17} $$

By noting that $\tilde{\beta}$ is defined such that

$$ {}_{c}T^{n} = (I - {}_{c}\tilde{\beta}\, x)\, {}_{c}\hat{T}^{n} \tag{5.18} $$

and using the identity

$$ {}_{c}T^{n}\, {}_{n}T^{p} = {}_{c}T^{\ell}\, {}_{\ell}T^{p} \tag{5.19} $$

it can be seen that

$$ {}_{c}\tilde{\beta} = -\,({}_{c}\tilde{\theta} + {}_{c}\tilde{\varphi}) \tag{5.20} $$

The transformation ${}_{\ell}\hat{T}^{c}$ is calculated from the position vector in the Earth-fixed "c" frame. Hence, ${}_{c}\tilde{\theta}$ can be approximated as a linear combination of the position errors, ${}_{c}\tilde{x}$.

$$ {}_{c}\tilde{\theta} = M\, {}_{c}\tilde{x} \tag{5.21} $$

The matrix, M, which expresses the linear relationship will be defined later.

5.4 Linearization

The variational equations for the navigation equations are developed by differencing the errorless and actual navigation equations, then linearizing the result by omitting second and higher order terms. Considering first the position equations, (5.2) and (5.12), we develop a first-order linear differential equation for position error, using (5.14) and (5.15) to represent the error.

$$
\begin{aligned}
{}_c\dot{\tilde{x}} &= {}_c\dot{x} - {}_c\dot{\hat{x}} = {}_cT^n\,{}_nv - {}_c\hat{T}^n\,{}_n\hat{v} \\
&= (I - {}_c\tilde{\beta}\times)\,{}_c\hat{T}^n({}_n\hat{v} + {}_n\tilde{v}) - {}_c\hat{T}^n\,{}_n\hat{v} \\
&= {}_n\tilde{v} - {}_c\tilde{\beta}\times\,{}_c\hat{T}^n\,{}_n\hat{v} - {}_c\tilde{\beta}\times\,{}_c\hat{T}^n\,{}_n\tilde{v}
\end{aligned}
\qquad (5.22)
$$

By observing that ${}_c\tilde{\beta}$ and ${}_n\tilde{v}$ are small, we drop their product. Then, using (5.13), we may write the equation in its final desired form.

$$
{}_c\dot{\tilde{x}} = (I)\,{}_n\tilde{v} + ({}_n\hat{v}\times)\,{}_c\tilde{\beta} \qquad (5.23)
$$

Equation (5.23) is the vector differential equation for the position errors. It shows that the rate of change of position errors in the "c" frame is the velocity error in the "n" frame plus a term dependent on the magnitude of the estimated velocity and the rotational error vector, $\tilde{\beta}$, between the two frames.

Development of the rate of change of velocity error is formally identical to that of the position error, so the intermediate algebraic steps will be omitted. The result is

$$
\begin{aligned}
{}_n\dot{\tilde{v}} = {}& ({}_n\hat{T}^p)\,{}_p\tilde{f}_b - ({}_c\hat{g}\times)({}_c\tilde{\beta} + {}_c\tilde{\theta}) + ({}_c\hat{T}^\ell)\,{}_\ell\tilde{g} - (2\,{}_c\omega_e\times)\,{}_n\tilde{v} \\
& - \left[({}_n\hat{v}\times)(2\,{}_c\omega_e\times)\right]{}_c\tilde{\beta} + ({}_n\hat{v}\times)\,{}_n\omega_d\,.
\end{aligned}
\qquad (5.24)
$$

Equation (5.24) is not yet in its final explicit form in terms of the error state, since it includes the ${}_{\ell}T^{c}$ error, ${}_{c}\tilde{\theta}$, the gravity error, ${}_{\ell}\tilde{g}$, and the drift rate, ${}_{n}\omega_{d}$. But ${}_{c}\tilde{\theta}$ and ${}_{\ell}\tilde{g}$ can each be written as linear combinations of the position error, and ω_{d} can be expressed as a function of the tilt and gyro drift errors. Let us consider each of these terms.

For consideration of errors, the transformation from computer to level frames can be quite accurately computed in the vicinity of the Earth-fixed computer reference center by

$$ {}_{\ell}\hat{T}^{c} \cong (I - \hat{\theta} \times) \tag{5.25} $$

where

$$ \hat{\theta} = \frac{1}{r_a + h} \begin{bmatrix} -{}_{c}\hat{x}^{2} \\ {}_{c}\hat{x}^{1} \\ 0 \end{bmatrix} \tag{5.26} $$

and where r_a is the Earth's apparent radius of curvature and h is altitude. We can differentiate this representation with respect to position to obtain the matrix, M, of equation (5.21).

$$ M = \frac{1}{r_a + h} \begin{bmatrix} d_y r^{1} & -(1 - d_y r^{2}) & d_y r^{3} \\ (1 + d_x r^{1} & d_x r^{2} & d_x r^{3} \\ 0 & 0 & 0 \end{bmatrix} \tag{5.27} $$

In (5.27), we have made the substitutions

$$ d_x r^{i} = \frac{-\hat{x}^{1}}{r_a + h} \frac{\partial}{\partial \hat{x}^{i}} (r_a + h) \tag{5.28} $$

and

$$ d_y r^{i} = \frac{\hat{x}^{2}}{r_a + h} \frac{\partial}{\partial \hat{x}^{i}} (r_a + h) \tag{5.29} $$

For practical purposes, $d_x r^1$, $d_x r^2$, $d_y r^1$ and $d_y r^2$ are zero and the partial derivative of $(r_a + h)$ with respect to $\hat{x}^3$ is unity. These simplifications were used in the RAINPAL onboard implementation.

Reference 12 gives an evaluation of empirical, second order, and atandard gravity relationships. In the RAINPAL mechanization, the gravity vector is approximated by

$$
{}_{\ell}\hat{g} = \begin{bmatrix} 0 \\ 0 \\ g_0 (1 - 2 h/a + .00526 \sin^2 \psi) \end{bmatrix} . \tag{5.30}
$$

Whatever gravity model is used may be differentiated with respect to position to obtain the matrix, Γ, of

$$
{}_{\ell}\tilde{g} = \Gamma \, {}_{c}\tilde{x} . \tag{5.31}
$$

RAINPAL's error model considered variations with altitude only and for this case, equation (5.32) holds.

$$
\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2g_0/a \end{bmatrix} {}_{\ell}\hat{T}^{c} \tag{5.32}
$$

The coupling of vertical errors into velocity as indicated by equations (5.24) and (5.31) is small and probably could be completely omitted from any mechanization using barometric altimeter data to stabilize the vertical channel.

The drift rate, ω_d, between the "c" and "n" frames was stated in equation (5.10). Using (5.18) in (5.10) with ${}_{c}\hat{T}^{n} = I$ gives

$$
\omega_d = + ({}_{c}\omega_e \times) \, {}_{c}\tilde{\beta} - ({}_{n}\hat{T}^{p} \, C_g^{-1}) \, \tilde{\omega}_{gs} \tag{5.33}
$$

In the RAINPAL equations, $\tilde{\omega}_{gs}$ was assumed to be described by

$$\dot{\tilde{\omega}}_{gs} = -\left(\frac{1}{\tau_\omega}\right) \tilde{\omega}_{gs} + n_\omega \quad . \tag{5.34}$$

The time constant, τ_ω, was set at 10 minutes and the magnitude of the white noise, n_ω, was set to give an "rms" drift rate of about 2 deg/hr.

Using equations (5.21), (5.31) and (5.33), we can re-write the differential equation for velocity error, (5.24), in its desired form.

$$\begin{aligned} {}_n\dot{\tilde{v}} = & \left[(-{}_c\hat{g}\,x)\,M + {}_c\hat{T}^{\ell}\Gamma \right] {}_c\tilde{x} + \left[-2\,{}_c\omega_e\,x \right] {}_n\tilde{v} \\ & + \left[(-{}_c\hat{g}\,x) - ({}_n\hat{v}\,x)({}_c\omega_e\,x) \right] {}_c\tilde{\beta} \\ & + \left[{}_n\hat{T}^p \right] {}_p\tilde{f}_b + \left[({}_n\hat{v}\,x)\,{}_n\hat{T}^p\,C_g^{-1} \right] \tilde{\omega}_{gs} \end{aligned} \tag{5.35}$$

The variational equation for $\tilde{\beta}$ can be derived by combining equations (5.3), (5.13) and (A.9) as follows.

$$\frac{d}{dt}\left({}_cT^n - {}_nT^c\right) = -2\dot{\tilde{\beta}}x = -\omega_d\,x\,(I - \tilde{\beta}\,x) - (I + \tilde{\beta}\,x)\,\omega_d\,x \tag{5.36}$$

$$\begin{aligned} \dot{\tilde{\beta}} &= \omega_d - (\tilde{\beta}\,x\,\omega_d)/2 \\ &\cong \omega_d = -({}_c\omega_e\,x)\,\tilde{\beta} + ({}_n\hat{T}^p C_g^{-1})\,\tilde{\omega}_{gs} \end{aligned} \tag{5.37}$$

The accelerometer bias was taken (in RAINPAL) as correlated noise.

$$\dot{\tilde{f}}_b = -\left(\frac{1}{\tau_f}\right) \tilde{f}_b + n_f \tag{5.38}$$

When the vertical accelerometer bias is included as a state variable in the RAINPAL mechanization, it is continuously compensated by the altimeter measurement.

5.5 Summary and Mechanization

The variational equations may be summarized in matrix form if we define a vector of errors, z.

$$
z = \begin{bmatrix} {}_c\tilde{x} \\ {}_n\tilde{v} \\ {}_c\tilde{\beta} \\ f_b \\ \tilde{\omega}_{gs} \end{bmatrix} \tag{5.39}
$$

We may then write the variational equations of the preceding section as a single equation.

$$
\dot{z} = F_z \, z + F_n \, n \tag{5.40}
$$

We utilize the equations of section 5.4 to identify the elements of F_z and F_n.

$$
F_z = \begin{bmatrix}
0 & [\,I\,] & [\,{}_n\hat{v}\times\,] & 0 & 0 \\
\left[(-{}_c\hat{g}\times)M + {}_c\hat{T}^{\ell}T\right] & \left[-2\,{}_c\omega_e\times\right] & \left[-{}_c\hat{g}\times - ({}_n\hat{v}\times)({}_c\omega_e\times)\right] & \left[{}_n\hat{T}^p\right] & \left[{}_n\hat{v}\times{}_n\hat{T}^p C_g^{-1}\right] \\
0 & 0 & \left[-{}_c\omega_e\times\right] & 0 & \left[{}_n\hat{T}^p C_g^{-1}\right] \\
0 & 0 & 0 & \left[-\frac{1}{\tau_f}I\right] & 0 \\
0 & 0 & 0 & 0 & \left[-\frac{1}{\tau_\omega}I\right]
\end{bmatrix} \tag{5.41}
$$

$$
F_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \qquad n = \begin{bmatrix} n_f \\ n_\omega \end{bmatrix} \tag{5.42}
$$

The individual elements of F_z and F_n shown here are each 3 x 3 matrices.

The general solution of equation (5.40) can be expressed in a simple form if n is approximated as a constant over a Kalman filter cycle of period τ_k. This solution is

$$z(t_i + \tau_k) = \Phi(t_i + \tau_k; t_i)\ z(t_i) + \Phi_n(t_i + \tau_k; t_i)\ n(t_i) \tag{5.43}$$

In (5.44) Φ is the transition matrix which obeys

$$\dot{\Phi} = F_z(t)\,\Phi \qquad \text{with} \qquad \Phi(t_i\,; t_i) = I \tag{5.44}$$

and Φ_n is the sensitivity to forcing functions which obeys

$$\dot{\Phi}_n = F_z(t)\,\Phi_n + F_n(t) \qquad \text{with} \qquad \Phi_n(t_i;t_i) = 0\ . \tag{5.45}$$

We assume the state-dependent elements of $F_z(t)$ may be approximated over the Kalman cycle as constants, with their value calculated from the state at the midpoint. Then by use of rectangular integration of (5.44) and (5.45)

$$\Phi(t_i + \tau_k; t_i) \cong I + F_z(t_i + \tau_k/2) * \tau_k \tag{5.46}$$

and

$$\Phi_n(t_i + \tau_k; t_i) \cong F_n(t_i + \tau_k/2) * \tau_k \tag{5.47}$$

As may be noted from (5.41) and (5.42) a large number of zeroes occur in the two matrices given by (5.46) and (5.47). To avoid wasting memory and calculation time, only the non-zero elements and non-unity elements are stored and calculated. Constant arrays containing appropriate indices allow calculations such as a prediction of the incremental state

$$z(t_i + \tau_k) = z(t_i) + F_z(t_i + \tau_k/2) * \tau_k * z(t_i) \tag{5.48}$$

without requiring multiplications by zeroes or storage of the complete $F_z \tau_k$ matrix. Equation (5.43) may also be obtained by use of rectangular integration of (5.40) with $n = 0$. In the RAINPAL implementation the transition matrix as such is neither calculated nor used.

VI. CONCLUDING REMARKS

The material presented in this report extends the navigation equations and variational equations of the RAINPAL formulation to an accurate all-weather aided inertial navigation system for all phases of aircraft operations. The development has been for stable platform-type inertial measurement units. A great deal of the work, however, applies directly to strapped-down IMU's as well. Some of the features incorporated in this preliminary design for new INS software include the following.

1. Relative position-type navigation is used for all phases of the mission so that high scaling gives good numerical accuracy with single-precision fixed-point arithmetic. In contrast, existing systems use an Earth-centered reference with which single precision results in a large truncation error.

2. Relative navigation is used with the coordinate frame and coordinate center selected to minimize output calculation. This feature can markedly reduce the real-time computer requirements of the navigator and therefore release computer time for other aircraft subsystems and displays.

3. The Earth-fixed azimuth platform drive holds the platform azimuth fixed with respect to the azimuth at the coordinate center. This feature reduces the complexity of the calculations and removes the drifting character of the "wander azimuth" mechanization.

4. The platform tilt control law and tilt compensation transformation eliminates the coupling between filtering and platform control. This feature should eliminate some difficulties in those areas where nonlinear effects have caused convergence problems in the past.

5. The new approach for development of the variational equations leads to a better separation and clearer understanding of the effects of INS errors. The resulting simpler set of error equations which do not contain the accelerometer measurement in the Jacobian matrix of acceleration with respect to the state is felt to be a significant advance in INS error modeling.

Due to the significance of these results, the incorporation of this improved software in a flight test prototype should be rapidly pursued. The potential benefits of a single system for the complete flight envelope is a very attractive cost-effective concept.

This report has focused on the navigation equation and variational equation software for improvements in aided inertial navigation systems. In the RAINPAL study effort, about 4K of memory was sufficient for aided inertial navigation, and time was available for other calculations. This system used a 1963 vintage computer (the SDS 920) which is slow by comparison to currently available computers. With typical "state of the art" computers and improved RAINPAL-type software, the total navigation function for all phases of flight will require about 25% real-time utilization. The percentage real-time utilization will reduce by another order of magnitude with the next generation of computers. Although there is certain justification for further software improvements, such improvements are not the road blocks to achieving the fast and accurate navigation systems of the future.

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APPENDIX

Vector and Matrix Relationships

A.1 Mathematical Operations for 3-Vectors

This appendix presents definitions of some frequently-used relationships between vectors and orthogonal (rotation) matrices.

The general 3-dimensional vector, a, is considered to be a column matrix and its transpose a row matrix.

$$a = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \qquad a^T = \begin{bmatrix} a^1 & a^2 & a^3 \end{bmatrix} \tag{A.1}$$

Thus, the dot or inner product, $a \cdot b$, is written $a^T b$ or $b^T a$. The outer product, ab^T, is defined by

$$ab^T = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \begin{bmatrix} b^1 & b^2 & b^3 \end{bmatrix} = \begin{bmatrix} a^1b^1 & a^1b^2 & a^1b^3 \\ a^2b^1 & a^2b^2 & a^2b^3 \\ a^3b^1 & a^3b^2 & a^3b^3 \end{bmatrix}. \tag{A.2}$$

The vector cross-product, $a \times b$, is often written as a skew-symmetric matrix, $(a\,x)$, operating on the vector, b.

$$a \times b = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix} \times \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} = \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} = (a\,x)\, b \tag{A.3}$$

The matrix, $(a\,x)$, pronounced "a-cross", has the property

$$(a\,x)^T = -(a\,x) \tag{A.4}$$

so that

$$(a \times b)^T = -b^T (a\,x)\,. \tag{A.5}$$

The vector triple cross-product is conveniently expressed as a difference between two vectors,

$$a \times (b \times c) = b\,(a \cdot c) - c\,(a \cdot b)\,, \tag{A.6}$$

which leads directly to a decomposition of the matrix product, $(a\,x)\,(b\,x)$,

$$(a\,x)\,(b\,x) = ba^T - (a^T b)\, I \tag{A.7}$$

and an alternative representation of the operation $(a \times b)\, x$.

$$(a \times b)\, x = ba^T - ab^T \tag{A.8}$$

Notice also that

$$(a\,x)\,(b\,x) \;-\; (b\,x)\,(a\,x) \;=\; (a \times b)\,x \quad . \tag{A.9}$$

A unit vector, u, may be defined from a by

$$u \;=\; a/\sqrt{a^T a} \tag{A.10}$$

from which we observe that $u^T u = 1$. As a special case of the triple cross-product, we have

$$(u\,x)\,(u\,x) \;=\; uu^T \;-\; I \tag{A.11}$$

and, since $(u\,x)\,u \;=\; 0$,

$$(u\,x)^n \;=\; \left\{ \begin{array}{ll} (-1)^{(n-1)/2}(u\,x) & \text{for n odd} \\ & \\ (-1)^{(n/2)}(\,I - uu^T) & \text{for n even} \end{array} \right\} \tag{A.12}$$

The matrix, $(I - uu^T)$, is of special interest since

$$(\,I \;-\; uu^T)^n \;=\; (\,I \;-\; uu^T), \tag{A.13}$$

which is a property of idempotent matrices. This matrix projects any vector, b, into the plane normal to u by subtracting off that component of b which lies along u.

$$(\,I - uu^T)\,b \;=\; b \;-\; u\,(u \cdot b) \tag{A.14}$$

Conversely, the matrix (uu^T) operating on b removes those components of b which do not lie along u.

A.2 Rotation "Coordinate Transformation" Matrices

Let us now consider the 3 x 3 rotation matrix, R. The orthonormal property of R is

$$R^T R \;=\; RR^T \;=\; I \tag{A.15}$$

which is equivalent to $R^{-1} = R^T$. The effect of multiplying an arbitrary vector, a, by R is to rotate a, since the magnitude of a is unchanged by the operation.

$$b \;=\; R\,a \tag{A.16}$$

$$b^T b \;=\; a^T R^T R\,a \;=\; a^T a \tag{A.17}$$

Let us now consider the effect of rotation on the vector-matrix operations which have been described. If we use the notation, ${}_r a \;=\; R\,a$, ${}_r b \;=\; R\,b$, etc., it is apparent that

$$ {}_r a^T \, {}_r b = a^T b \qquad \text{(A. 18)} $$

and

$$ {}_r a \, {}_r b^T = R\,(ab^T)\,R^T \,. \qquad \text{(A. 19)} $$

In order that the vector cross-product be preserved under orthogonal transformation,

$$ {}_r a \times {}_r b = R\,(\,a \times b\,)\,, \qquad \text{(A. 20)} $$

we must have

$$ ({}_r a\, x) = R\,(a\, x)\,R^T \qquad \text{(A. 21)} $$

(an important result) for then

$$ {}_r a \times {}_r b = R\,(a\,x)\,RR^T b = R\,(a\,x)\,b = {}_r(a \times b)\,. \qquad \text{(A. 22)} $$

We will now state without derivation that the general rotation matrix can be written in the form

$$ R(u,\theta) = \cos\theta\; I + (1 - \cos\theta)\,uu^T - \sin\theta\,(u\,x) \qquad \text{(A. 23)} $$

or

$$ R(u,\theta) = \cos\theta\,(\,I - uu^T) + (uu^T) - \sin\theta\,(u\,x) \quad . \qquad \text{(A. 24)} $$

It is readily observed that this representation has the orthonormal property.

$$ R(u,\theta)\; R^T(u,\theta) = I \qquad \text{(A. 25)} $$

To show that any general rotation matrix may be put into the form, $R(u,\theta)$, we will present a means of solving for u and θ from a general R. The matrix,

$$ C = R^T - R\,, \qquad \text{(A. 26)} $$

is always skew-symmetric, even if R is not a rotation, since

$$ C^T = -C\,. \qquad \text{(A. 27)} $$

If the elements of C are c_{ij} , we can solve for u as the vector

$$ u = \frac{1}{\sqrt{c_{32}^2 + c_{13}^2 + c_{21}^2}} \begin{bmatrix} c_{32} \\ c_{13} \\ c_{21} \end{bmatrix} \qquad \text{(A. 28)} $$

which follows from the definition of the skew-symmetric matrix, u x , in (A. 3). The rotation angle, θ, is found from

$$ 2\sin\theta = \sqrt{c_{32}^2 + c_{13}^2 + c_{21}^2} \qquad \text{(A. 29)} $$

and $$\text{trace } R = 3\cos\theta + (1-\cos\theta) = 1 + 2\cos\theta \tag{A.30}$$

so that $$\tan\theta = \sqrt{c_{32}^2 + c_{13}^2 + c_{21}^2}\ /\ \text{trace } R\ . \tag{A.31}$$

The vector, u, is the eigenvector of R which corresponds to the eigenvalue of unity since

$$R\,u = u. \tag{A.32}$$

It may be of interest to note that if u is rotated by the orthogonal matrix, M, to render

$$_m u = M\,u \tag{A.33}$$

then $$R({}_m u, \theta) = M\,R(u,\theta)\,M^T\ , \tag{A.34}$$

this result following directly from the form of $R(u,\theta)$ and the earlier result that $({}_m u\, x) = M\,(u\, x)\,M^T$.

The "canonical" rotations about the references bases are special cases of $R(u,\theta)$. For example, if

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{A.35}$$

then $$R(u,\theta) = \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}. \tag{A.36}$$

A transformation of coordinate frames is frequently made up of three sequential canonical rotations in a given order. It is possible to duplicate the result with a single rotation through θ about u, where θ and u are obtained from the product matrix by the process described earlier. In this way θ and u may be expressed as functions of the canonical rotation angles. Another item of frequent interest is an expression for the angular velocity as a function of the Euler angles (canonical rotation angles) and their rates. Let us assume that

$$R = R(u_3,\theta_3)\, R(u_2,\theta_2)\, R(u_1,\theta_1) = R_3 R_2 R_1 \tag{A. 37}$$

where u_i and θ_i are the canonical axis and angle of the i-th rotation. It can be shown that there is always a vector, ω, such that the rate of change of any orthogonal matrix, R, may be written

$$\dot{R} = -\omega \times R . \tag{A. 38}$$

If $\dot{u} = 0$, $\omega = \dot{\theta} u$ and we have the special-case result,

$$\dot{R}(u,\theta) = -\dot{\theta}\, u \times R(u,\theta) . \tag{A. 39}$$

Applying this special result repeatedly to the derivative of R in (A. 37) we have

$$\begin{aligned} \dot{R} &= \dot{R}_3 R_2 R_1 + R_3 \dot{R}_2 R_1 + R_3 R_2 \dot{R}_1 \\ &= -\dot{\theta}_3 u_3 \times R - R_3 \dot{\theta}_2 u_2 \times R_2 R_1 - R_3 R_2 \dot{\theta}_1 u_1 \times R_1 \\ &= -\dot{\theta}_3 u_3 \times R - R_3 (\dot{\theta}_2 u_2 \times) R_3^T R - R_3 R_2 (\dot{\theta}_1 u_1 \times) R_2^T R_3^T R \\ &= -(\dot{\theta}_3 u_3 + \dot{\theta}_2 R_3 u_2 + \dot{\theta}_1 R_3 R_2 u_1) \times R \end{aligned} \tag{A. 40}$$

The vector sum in (A. 40) is equated to the angular velocity, ω, in rotated coordinates.

$${}_r\omega = \dot{\theta}_3 u_3 + \dot{\theta}_2 R_3 u_2 + \dot{\theta}_1 R_3 R_2 u_1 \tag{A. 41}$$

Its expression in non-rotated coordinates is found from $\omega = R^T {}_r\omega$.

$$\omega = \dot{\theta}_3 R_1^T R_2^T u_3 + \dot{\theta}_2 R_1^T u_2 + \dot{\theta}_1 u_1 \tag{A. 42}$$

Another form of the general rotation matrix, (A. 24), can be quite useful. If we define a vector, b, as

$$b = u \sin\theta \tag{A. 43}$$

the general rotation matrix, $R(b,\theta)$, can be written

$$R(b,\theta) = I - (b\times) + (b\times)(b\times)/(1+\cos\theta) . \tag{A. 44}$$

This form explicitly shows the error in approximating $R(b,\theta)$ by $(I - b\times)$ for small angles, when $(b\times)(b\times)$ is negligible to first order.

Another substitution leads to the "Euler Parameters." Define a vector, q, as

$$q = u \sin(\theta/2) \tag{A. 45}$$

Then, using double-angle identities for $\sin\theta$ and $\cos\theta$ in (A. 24), we obtain

$$R(q,\theta) = I + 2\,(q\,x)\,(q\,x) - 2\,(q\,x)\cos(\theta/2)$$

$$= I + 2\,(q\,x)\Big[(q\,x) - (I)\cos(\theta/2)\Big] \qquad (A.46)$$

By defining a fourth element of q to be $\cos(\theta/2)$, the vector of Euler's Parameters is completed.

$$\underline{q} = \begin{bmatrix} q \\ \cos(\theta/2) \end{bmatrix} \qquad (A.47)$$

Another form, then, for the rotation matrix can be written as a function of $\underline{q}$.

$$R(\underline{q}) = 2\Big[(q_4^2 - 1/2)(I) + (qq^T) - q_4(q\,x)\Big] \qquad (A.48)$$

A.3 <u>Derivative of the General Rotation Matrix</u>

We again consider the general rotation matrix in the form

$$R(u,\theta) = \cos\theta\,(I) + (1-\cos\theta)\,(uu^T) - \sin\theta\,(u\,x) \qquad (A.49)$$

where u is a unit column vector. Let the derivatives of u and θ with respect to any scalar (such as time) be denoted by $\dot{u}$ and $\dot{\theta}$, respectively. Note that

$$\dot{u}^T u = u^T \dot{u} = 0 \qquad (A.50)$$

since u is constrained to be a unit vector.

$$\dot{R}(u,\theta) = \Big[-\sin\theta\,(I - uu^T) - \cos\theta\,(u\,x)\Big]\dot{\theta} + (1-\cos\theta)(\dot{u}u^T + u\dot{u}^T)$$

$$- \sin\theta\,(\dot{u}\,x) \qquad (A.51)$$

Equation (A.51) may be put into more useful form by post-multiplying both sides by $R^T(u,\theta)\,R(u,\theta) = I$ and re-forming the matrix product, $\dot{R}(u,\theta)\,R^T(u,\theta)$, which will be shown to be skew-symmetric. Consider first only the coefficient of $\dot{\theta}$ in this product and leave the $\dot{u}$-terms until later.

$$\Big[-\sin\theta\,(I - uu^T) - \cos\theta\,(u\,x)\Big]R^T(u,\theta) = -(u\,x) \qquad (A.52)$$

This preliminary result is the same as (A.39). It was obtained algebraically using the following identities

$$(I - uu^T)\,(uu^T) = 0 \qquad (A.53)$$

$$(I - uu^T)\,(u\,x) = (u\,x) \qquad (A.54)$$

$$(u\,x)\,(u\,x) = (uu^T - I) \qquad (A.55)$$

$$(u\,x)\,(uu^T) = I \quad . \qquad (A.56)$$

Equation (A.55) is a special case of the vector triple cross-product, (A.6) and (A.7). Using the vector identity of (A.8), we now consider terms in u to complete the derivation of $\dot{R}(u,\theta)$. Omitting a great deal of intermediate algebra,

$$\left[(1 - \cos\theta)(u\dot{u}^T + \dot{u}u^T) - \sin\theta\,(\dot{u}\,x) \right] R^T(u,\theta) = (\cos\theta - 1)(\dot{u} \times u)\,x - \sin\theta\,\dot{u}\,x \,. \tag{A.57}$$

Combining equations (A.52) and (A.57) with (A.51) to form $\dot{R}(u,\theta)\,R^T(u,\theta)\,R(u,\theta)$ we find

$$\dot{R}(u,\theta) = -\,\omega \times R(u,\theta) \tag{A.58}$$

where

$$\omega = \dot{\theta}u + (1 - \cos\theta)(\dot{u} \times u) + \sin\theta\,\dot{u} \,. \tag{A.59}$$

It may be observed that u, $\dot{u} \times u$, and $\dot{u}$ are mutually orthogonal. It may also be of interest to consider the fact that

$$u^T\omega = \dot{\theta} \,. \tag{A.60}$$

Although the "dot" notation usually denotes differentiation with respect to time, the result given here holds for differentiation with respect to any other quantity.

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