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GODDARD SPACE FLIGHT CENTER
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ABSTRACT

This report investigates the accuracy with which the GEOS-C altitude may be estimated over long (7 day) and short (40 minute) orbital arcs. Over the long arc excellent agreement was attained between a simulation of the orbit determination process and a covariance analysis. Both approaches yielded RMS altitude errors of about 1.5 meters over the Caribbean calibration area and approximately 7.5 meters overall. The geopotential was identified as the largest error source. For the short arc, the covariance analysis revealed that the propagated altitude error is linearly dependent upon station survey component errors which are also the largest source of altitude errors. An Appendix contains the mathematics of covariance analysis as applied to orbit determination.

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LONG AND SHORT ARC ALTITUDE DETERMINATION FOR GEOS-C

I. INTRODUCTION

The spacecraft-borne radar altimeter has been recognized as an ideal instrument for determining the topography of the sea surface (Refs. 1, 2, 3). The sea surface topography has relevance to both geodesy and oceanography. It is relevant to geodesy because the mean sea level surface reflects the structure of the geopotential. The topography of the sea surface is relevant to oceanography since it is a manifestation of the dynamics of ocean circulation and the forces that shape the ocean surface such as tidal forces, wind stress, and storm surges.

The GEOS-C spacecraft, scheduled for launch in 1974, is the first satellite to be equipped with an altimeter which will be operated in a continuous mode over the world's oceans. The instrument will be capable of one meter precision. Hence, assuming that spacecraft altitude will be estimated perfectly and assuming that an in-flight calibration procedure will remove systematic errors in the instrument, the GEOS-C altimeter will in theory be capable of resolving the sea surface topography to within one meter. The extent to which this accuracy can be approached is determined by the effectiveness of the altitude determination of GEOS-C and the effectiveness of the in-flight calibration procedure. The fundamental question to be answered is that of the expected accuracy of GEOS-C altitude determination since the answer to this question strongly conditions the effectiveness of an in-flight calibration procedure.

In this paper the altitude determination accuracy for the GEOS-C orbit is studied from two different vantage points. Since the altimeter on board the GEOS-C spacecraft is capable of functioning in a global mode it is necessary to determine the GEOS-C altitude globally. The most convenient and desirable method of obtaining global estimates is to process long arcs of tracking data to estimate state at an epoch and then propagate this estimate along the length of the data arc. By means of both simulations and covariance analysis the accuracy of such an estimation procedure will be studied and the dominant error source identified. For the purpose of in-flight altimeter calibration, requirements for altitude determination accuracy are somewhat different. The calibration will occur in a region over the Carribbean where dense coverage from laser stations is expected. This implies that for the purpose of calibration it is sufficient to determine GEOS-C altitude only over the region where the calibration is occurring. One would expect that a short arc altitude determination procedure which would minimize effects of dynamical errors would be most effective in this case. Consequently the accuracy of a short arc altitude determination over the calibration region will be separately analyzed in a covariance mode.

II. LONG ARC ALTITUDE DETERMINATION ACCURACY FOR GEOS-C

A. Simulations Versus Covariance Analysis

In performing a simulation study for orbit determination the following procedure is usually followed. A nominal value of the state of a spacecraft is assumed at an epoch. A model for the geopotential field and models for other forces which may act on the spacecraft are defined. From this information a nominal orbit is obtained. Next assumptions are made concerning numbers and locations of tracking stations, data types and data acquisition rates. Then using purely geometrical considerations the correct or noiseless representation of the data is obtained. A random number generator is used to add stationary white noise with the appropriate standard deviation to the data. The procedure then is to introduce the simulated data into an orbit determination program (O.D.P.) and estimate the state of the spacecraft at epoch perhaps along with other parameters in the dynamic or measurement model. In order to be realistic, however, the models used in the O.D.P. should differ from the corresponding models used to generate the data. The differences will reflect an honest evaluation of the dynamic and measurement modeling errors to be expected in an orbit determination process. Finally the estimated state at epoch and the dynamic model in the O.D.P. are used to obtain an estimated orbit. The differences between the nominal orbit and the estimated orbit plotted as a function of time represent a typical realization of the error sequence of an orbit determination process.

For long arc studies a simulation procedure can be quite expensive in terms of human effort and computer time. Also a single simulation is not informative with regard to the relative importance of various error sources in degrading the quality of the least squares estimate. To obtain such information several simulations must be performed with an attendant multiplication of cost. It was in response to the prohibitive cost of performing an adequate simulation study of the GEOS-C long arc altitude determination problem that the necessary programming for a covariance analysis approach was developed.

The difference in approach between a simulation study and a covariance analysis can be described as follows: in a simulation, data are generated and a least squares adjustment process is actually performed. The estimated state is then compared to an assumed true state and conclusions are drawn. In a covariance analysis mode, the least squares adjustment process is postulated rather than actually performed and only its associated covariance matrix is computed.

The recently developed NAPCOV program was used to generate the covariance analysis results discussed in this paper. The program assumes as input a normal matrix for a set of parameters generated by an O.D.P. By manipulating rows and columns of the normal matrix the parameters are effectively divided into two categories, a "solve for" category and a "consider" category. Parameters in the solve for category are assumed to be adjusted in the postulated least squares process. Parameters in the consider category are assumed to influence the functional relationship between the observations and the solve for parameters but to be left unadjusted in the postulated least squares process. A consider category in a covariance analysis program is very important if reasonable results are to be obtained. In most cases the dominant source of error in a least squares orbit determination is not the noise on the data but the effect of misrepresented parameters in the dynamic and measurement models. Unless such parameters can be placed in a consider category the computed covariance matrix will reflect only the effect of data noise and a falsely optimistic estimate of the quality of the least squares process will be the result.

Not only is the covariance analysis mode of studying problems less expensive than the simulation mode, but it also provides more information. The output of the NAPCOV program displays a tabulation of the contribution to the uncertainty of each solve for parameter due to the uncertainty on each consider parameter and due to the uncertainty on the data. The resultant uncertainty for each solve for parameter is the root sum square of the individual contributions which we shall call the aliasing terms. The matrix of aliasing terms reveals the entire probability structure of the postulated estimator and it becomes immediately apparent which consider parameters have an important influence on the quality of the estimation of a given parameter and which do not. The mathematical details of covariance analysis as applied to orbit determination are supplied in the appendix.

Both a simulation study and a covariance analysis of the long arc altitude determination problem were performed. The simulation study was performed in order to provide a check on the correctness of the NAPCOV covariance analysis program. As mentioned previously the philosophies and assumptions of a simulation study and a covariance analysis are quite different. Nevertheless, we attempted to match the two studies as closely as possible so that if the programming were correct a certain statistical compatibility between the results could be expected.

B. The Simulation Study

The NAP-3 orbit determination program was used to generate laser data for a GEOS-C orbit. The tracking stations considered were Antigua, Bermuda, Canal Zone, Cape Kennedy, Grand Turk, Goddard, and Rosman. The tracking geometry for a typical GEOS-C pass is given by Figure 1. The tracking period was approximately seven days. A standard reference field was assumed and a random number generator was used to add white noise which a standard deviation of 10 cm to

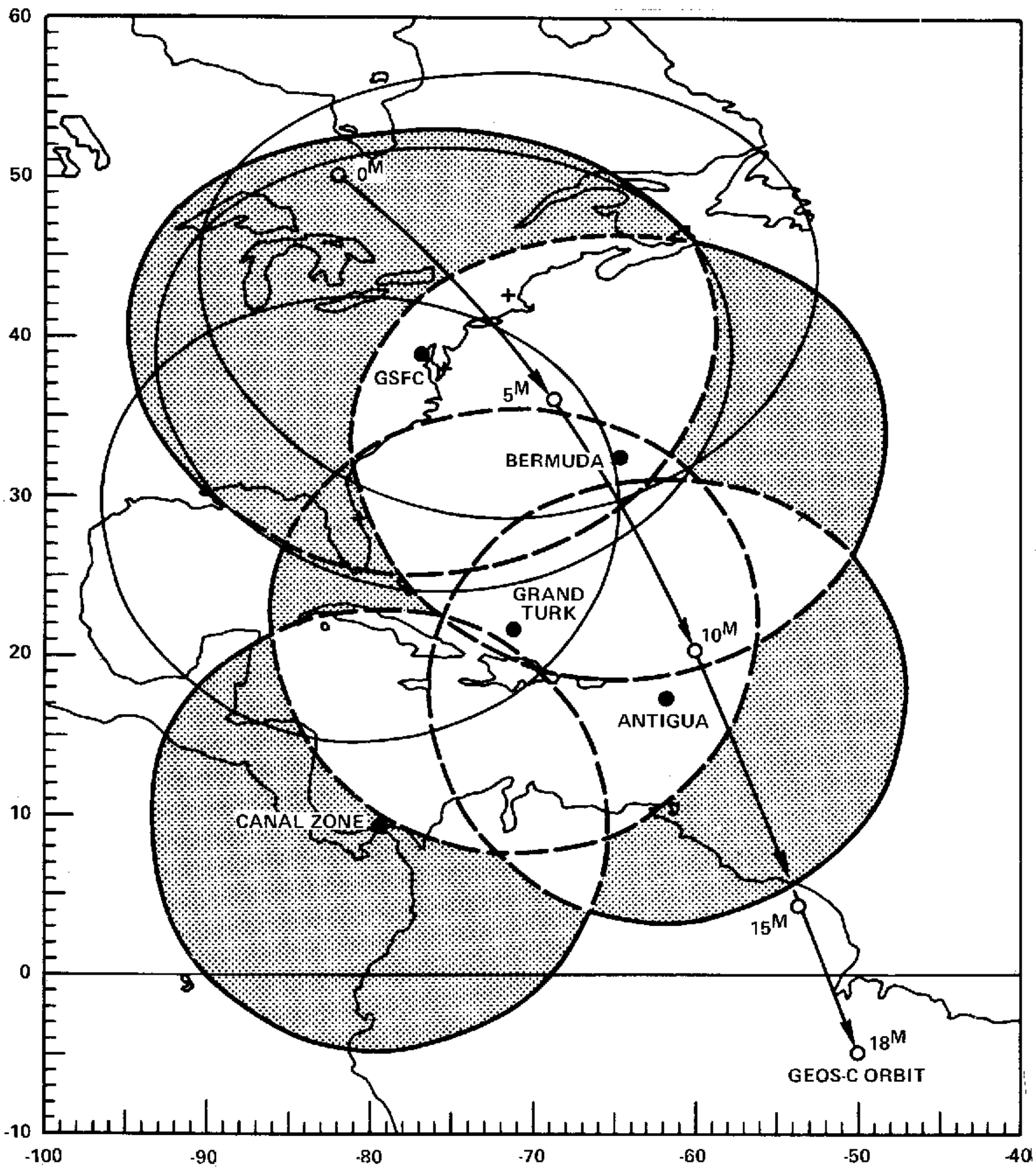

Figure 1. Visibility for GEOS-C

the data. The seven day arc was decomposed into one day mini-arcs and state and measurement biases were estimated for each mini-arc using the NAP-3 program. Measurement biases of ±10 cm and a total station location uncertainty of 8.5 meters were assumed as a priori estimates.

The 24 geopotential coefficients which are dominant for the GEOS-C orbit were perturbed by 5% of their reference values and left undisturbed by the least squares adjustments. The 5% figure was obtained by computing the weighted R.S.S. difference between the Smithsonian Astrophysical Observatory 69, SAO69, field and the Goddard Earth Model 5, GEM5, field. The entire seven days of data were then processed by NAP-3 to estimate the epoch vector of the GEOS-C orbit. The estimated values of bias and station location and the perturbed geopotential coefficients were assumed for this computer run and were not adjusted in the least squares process. The perturbed reference field was utilized to propagate the estimated state vector to seven days after epoch. The time history of the true state of GEOS-C was obtained by propagating the true state at epoch by means of the unperturbed reference field. The dotted line of Figure 2 is a plot of the difference in altitude of these two propagated states versus time. It should represent a realization of the altitude resolution of the GEOS-C orbit when the various assumptions implicit in the simulation are satisfied. The differences reflected a sinusoidal perturbation whose period was that of the GEOS-C orbit (102 min.) and whose amplitude was modulated by an observability pattern which was repeated approximately every 13 hours. Hence it is sufficient to view the altitude errors during any 13 hour interval in order to understand the behavior of the differences for the entire seven day period. Figure 2 displays the altitude error for the first 15 hours after epoch. In the calibration areas the rms error was about 1.5 meters and the rms error over the entire seven day arc was approximately 7.5 meters.

C. Covariance Analysis

A least squares process was postulated which assumed that data from the seven laser stations mentioned in the simulation were collected for seven days. The state of the satellite at epoch was placed in the solve for mode. Station location and bias parameters along with 14 of the dominant geopotential terms were placed in the consider mode. Standard deviations of 10 cm for the bias terms and 5 meters for each station location parameter was assumed. The standard deviations of the geopotential coefficients were set at 3.3% of their reference values thus making the 5% perturbations of the simulation precisely 50% critical values. The resultant covariance matrix was found to be very poorly conditioned. The altitude determination of the satellite was good but the observability in the along track cross track plane was poor. It was felt that without elaborate numerical precautions the propagation of such a matrix could lead to spurious results.

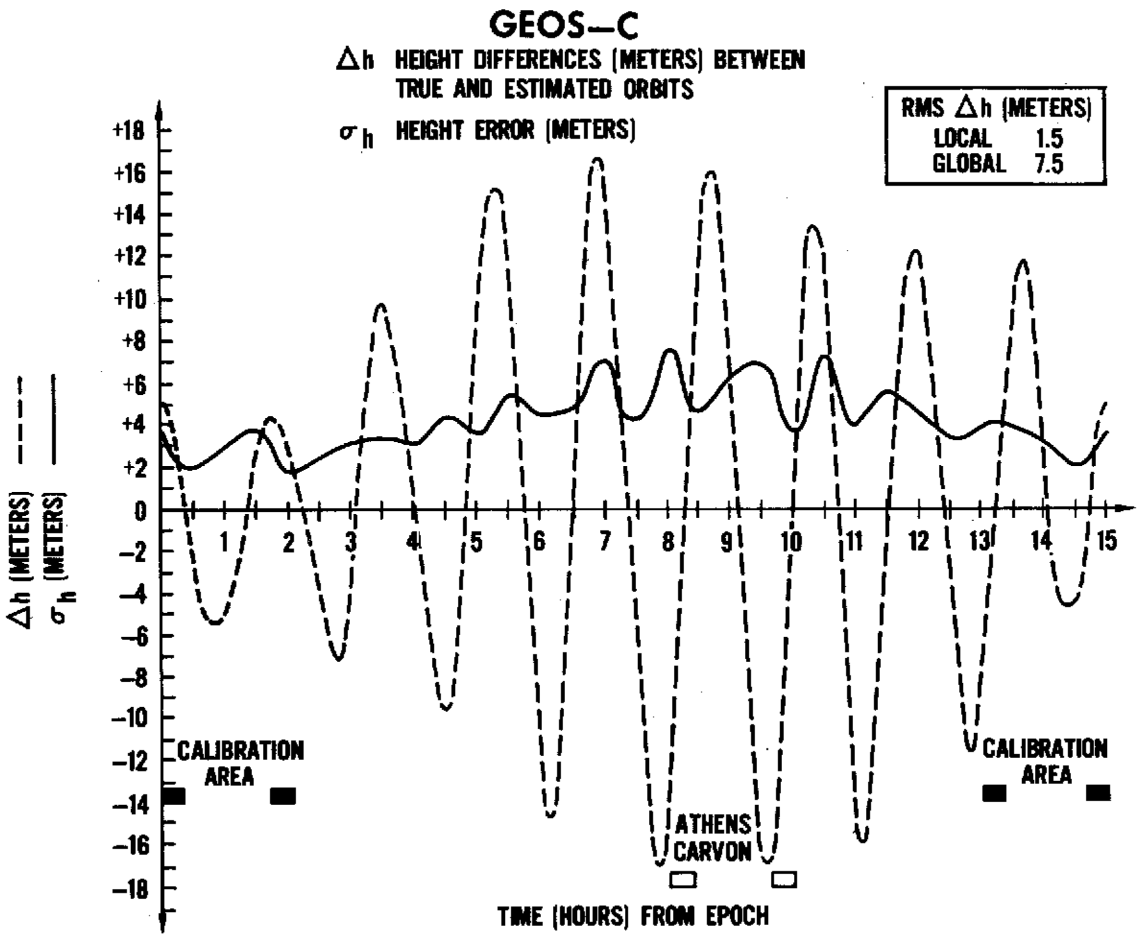

Figure 2. GEOS-C

The most convenient solution to the problem was the one adopted. We changed the terms of the problem, dropping Rosman as a tracking station and adding a postulated 3 days of data from laser stations at Athens and Canarvon. With the addition of these stations the observability of along track and cross track components are improved and a better conditioned covariance matrix is the result. This covariance matrix was propagated up to 15 hours after epoch and the standard deviation of the altitude component was computed at various time points and displayed as the solid line on Figure 2. One would expect such a curve to show a periodic disturbance of approximate period one half the orbital period and with an amplitude modulated by an observability pattern of approximately 13 hours duration. This in fact is what is seen. The results of the error analysis suggest that the altitude of GEOS-C may be resolved with a standard deviation of approximately 1.7 meters in the calibration region and an altitude resolution of no worse than 8 meters standard deviation may be expected elsewhere.

The two curves are incompatible only in that portion of the arc where there is observability from the Athens and Canarvon stations and not from the Carribbean stations. This incompatibility is obviously traceable to the different tracking assumptions employed in the simulation and in the covariance analysis.

The alias matrix generated by NAPCOV revealed that almost all the altitude error was due to geopotential error. At epoch the altitude standard deviation is 3.8 meters. If all bias terms and station location parameters were known perfectly and if there were no noise on the data, the standard deviation would be 3.7 meters. This result suggests that the acquisition of no amount of additional data will significantly improve on the results shown on Figure 2 and that substantial improvements will only come from a more accurate geopotential model.

III. SHORT ARC ALTITUDE DETERMINATION

In order to insure that the GEOS-C radar altimeter achieves a precision level of one meter, calibration of the instrument is necessary. Such a calibration is performed while the satellite passes over an area where short dense bursts of highly accurate tracking data are available. Only short bursts are used since this prevents the geopotential from perturbing the orbit to any appreciable degree. For GEOS-C a typical calibration area is formed by Goddard Space Flight Center, Cape Kennedy, and Bermuda. Laser system ranging accuracy at the aforementioned stations is 10 cm noise and ±10 cm bias. Ranging measurements of the satellite are made for about eleven minutes as it passes northeastward over the calibration area. The average satellite altitude over this area is approximately 840 km.

For the purpose of the study, it was assumed that the altimeter actually will be calibrated in the following manner. As GEOS-C traverses over the Carribbean calibration area, its onboard altimeter measures the instantaneous satellite altitude. Simultaneously the three laser tracking stations measure range to GEOS-C. Ranging data are used in order to improve the spacecraft orbit and also estimate the spacecraft altitude. The computed altitude is compared with the measured altitude. From this comparison, systematic errors associated with the altimeter measurement can be removed. Thus instrument errors which may degrade altimeter accuracy during the global mode of operation are removed. The accuracy with which the altimeter determines sea surface topography and hence the geoid is dependent upon many factors. Among these are the proper representation of the errors affecting the measurement, e.g. instrumental errors, refraction, timing, dynamic lag, etc., and oceanographic effects, e.g. tides. Another factor is the orbit used for scaling the altimeter measurements by comparing the corrected altimeter measurements to satellite altitude computed with the spacecraft orbit.

Among the principal questions which this short arc portion of the report attempts to investigate include: How accurately can the GEOS-C altitude be estimated during a pass by the spacecraft over the calibration area? Can the dominant error sources be identified and measured?

For the 40 minute GEOS-C short arc, only the covariance analysis approach is considered. One approach suffices since during the long arc both the covariance analysis and simulation approaches yielded entirely compatible results. The entire short arc is analyzed using the NAP/NAPCOV program combination. The GEOS-C state vector is placed in the "solve for"* mode with the remaining 36 parameters in the "consider"† mode. The principal assumptions include:

(1) Orbit:

a 7221.0 kilometers	Ω 270.0 degrees
e 0.004	ω 360.0 degrees
i 65.0 degrees	M 40.7 degrees
815 km $\leq$ h $\leq$ 870 km	P = 102 minutes

(2) Tracking Stations: Goddard, Cape Kennedy, and Bermuda

(3) Error Sources:

(a) GEOS-C Orbit: 1.7 m total position and 1.7 cm/s total velocity

(b) Station Survey: varied through 10 cm, 1 m, 5 m, and 10 m in each component

(c) Laser Range Bias: 10 cm

(d) Geopotential: 5% of the 24 dominant terms

(4) Parameter Treatment:

(a) "Solve for": GEOS-C 6 state

(b) "Consider": 9 station survey components, 3 laser ranging biases, and 24 geopotential terms.

*"Solve for" means a parameter is actually estimated during least squares adjustment.
†"Consider" means a parameter is not estimated but its uncertainty is taken into account in computing the associated covariance matrix.

The station survey errors are parametrically varied through four values of increasing magnitude. The use of the unusually low 10 cm individual survey component errors is based on the condition that Goddard, Cape Kennedy, and Bermuda are referenced to a local topocentric coordinate system centered at Goddard rather than one referred to the center of the earth. An analysis of perturbations acting on the GEOS-C orbit which have amplitudes of 25 meters or more was made. Twenty-four geopotential coefficients were identified and subsequently used as the perturbative effect due to geopotential. These particular coefficients were termed "dominant."

Table 1 lists, as a function of station survey error, the propagated minimum and maximum GEOS-C altitude errors (σ_h) in meters and the total time span these errors remain under one meter. The one meter level corresponds to the anticipated precision of the GEOS-C radar altimeter. All errors are evaluated at epoch in the middle of the eleven minute tracking interval and then propagated for twenty minutes before and after epoch. The minimum error always occurs near epoch.

Table 1

Propagated GEOS-C Altitude Errors vs. Station Survey Error During a Calibration Triangle Pass

Station Survey Error (meters)		Propagated σ_h (meters)		Time (minutes)
Each Individual Component	Total	Minimum	Maximum	$\sigma_h < 1$ Meter
0.10	0.17	0.10	2.35	19
1.00	1.73	0.69	8.08	2
5.00	8.66	3.45	39.08	0
10.00	17.32	6.90	78.07	0

Except for the lowest value of survey error, one conclusion is immediately obvious: There is a strong linear relationship between propagated altitude error and station survey error. The minimum altitude error in this study was 0.7 of the individual survey error and 0.4 of the total error. The corresponding figures for the maximum altitude error are 8.0 and 4.5, respectively. These relationships are entirely expected. The total altitude error is the sum of the RSS of the error due to all the consider parameters and to the data noise. If, within the RSS, the error due to a certain subset of consider parameters is dominant, the error due to the remaining parameters is negligible and consequently the altitude error becomes a linear function of only the error in the dominant consider parameters.

The following Figure 3 depicts the time history of the GEOS-C altitude error for the 10 cm and 1 m individual survey error. The propagated error is minimized by the placement of epoch in the middle of the tracking interval. The general behavior of the errors is the same with minimum height error at the midpoint and maximum at the endpoints. The lower survey error (10 cm in each individual component) results in altitude errors less than the expected altimeter precision level of one meter for about half of the propagation interval.

The total GEOS-C altitude error is composed of the error due to the thirty-six consider parameters (nine station survey components, three range biases, and twenty-four potential terms) and to the data noise. Of the consider parameters, the survey components are by far the greatest error sources with the remaining contributing one to two orders of magnitude less. The data noise itself contributes virtually nothing. Table 2 is a decomposition of this total altitude error into its constituent parts. The error due to the data noise remains constant at slightly below the three millimeter level.

Table 2

Decomposition of Total GEOS-C Altitude Error

Station Survey Error (meters)		Total GEOS-C Altitude Error (meters) at Epoch	RSS (meters) of Error Due to the 36 Consider Parameters	Error (cm) Due to Data Noise
Each Individual Component	Total			
0.10	0.17	0.128	0.1278	.2835
1.00	1.73	0.858	0.8564	.2835
5.00	8.66	4.264	4.2560	.2835
10.00	17.32	8.526	8.5110	.2835

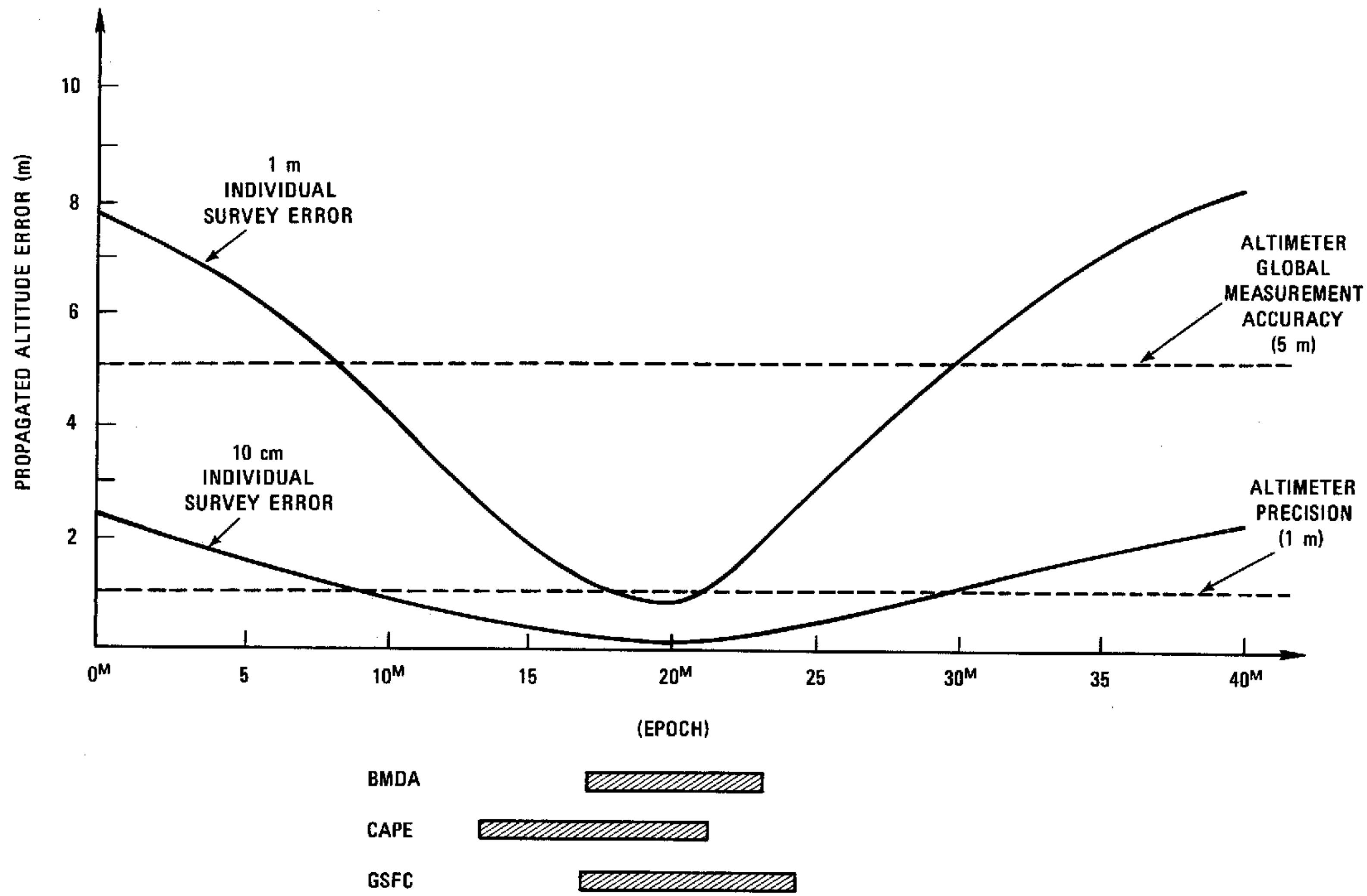

Figure 3. Propagated GEOS-C Altitude Error (m) vs. 10 cm and 1 m Individual Survey Error

Under the assumptions made during this short arc, the following conclusions are valid:

(1) It is feasible to obtain a GEOS-C altitude estimate accurate to approximately 10 cm provided total survey errors of about 20 cm are available. This altitude error can be controlled by a judicious choice of survey error. The resultant altitude error satisfies both altimeter precision and measurement accuracy constraints.

(2) The most degrading error sources clearly are survey errors. They contribute 80-90% of the error due to the consider parameters.

IV. CONCLUSIONS

The dominant error source in long (7 day) arc altitude determination of GEOS-C is the uncertaintity of the geopotential field. With the present accuracy of geopotential fields the altitude of GEOS-C can be determined to within one to two meters in the calibration region and seven to eight meters globally. The addition of more tracking stations will not significantly improve on these results. If further accuracy is desired an improved model for geopotential must be obtained.

With regard to altimeter calibration short arc determinations are adequate. The dominant error source in this case is survey error. Assuming that station location coordinates are known with an uncertainty of one meter, the altitude of GEOS-C can be determined with an error of less than one meter for approximately six minutes.

Both, simulations and covariance analysis were utilized to obtain these results. Compatibility between the two approaches is good and the covariance analysis provides added information which permits one to determine the dominant error sources in a given tracking situation.

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VI. APPENDIX

COVARIANCE ANALYSIS AS APPLIED TO ORBIT DETERMINATION

COMPUTING COVARIANCE MATRICES

Let $\tilde{y}(m)$ be an m dimensional vector consisting of the differences between the correct values of observations of a satellite and nominal values of the observations as determined from a nominal orbit. Also let $\tilde{z}(n)$ be an n dimensional vector of differences between actual and nominal values of the state of the satellite at an epoch and differences between actual and nominal values of parameters in the dynamic and measurement models whose associated uncertainties may limit our ability to estimate satellite state from the data. The sensitivity matrix c (m, n) is defined as that matrix whose element in the ith row and the jth column is the partial derivative of $\tilde{y}(i)$ with respect to $\tilde{z}(j)$. A first order Taylor series expansion of the functional relationship between $\tilde{y}$ and $\tilde{z}$ about the nominal value of $\tilde{z}$ yields

$$\tilde{y} = c\tilde{z} \tag{A-1}$$

An orbit determination program in processing observations y of $\tilde{y}$ to obtain a least square adjustment to $\tilde{z}$ computes a so-called normal matrix defined as

$$\eta(n, n) \equiv c^T wc \tag{A-2}$$

where w is a weighting matrix and is usually the inverse of the covariance matrix of the observations y of $\tilde{y}$. Once an orbit determination program computes and stores the normal matrix, a number of questions can be raised and answered at very little cost in terms of computation time.

The best estimate of the state of the satellite at epoch is obtained by performing a least squares adjustment of the state at epoch and all other parameters with which are associated significant uncertainties. But frequently this straightforward approach leads to severe core storage requirements. In practice some of the parameters in the dynamic and measurement models are estimated along with state and others are fixed at their nominal values and left unadjusted in the least squares process. In order to determine the consequences of estimating some parameters and ignoring others it is useful to compute the covariance matrix of such a least squares estimation procedure.

Let $\tilde{z}$ be decomposed into two disjoint parameter sets as follows

$$\tilde{z} = \begin{bmatrix} \tilde{x}_1(n_1) \\ \\ \tilde{x}_2(n_2) \end{bmatrix} \tag{A-3}$$

where $\tilde{x}_1$, is a set of n_1, parameters which are to be estimated in a least squares process and $\tilde{x}_2$ is a set of n_2 parameters whose nominal values are left unadjusted by the least squares process but whose uncertainties are to be considered in computing the covariance matrix of the resulting estimator. Define a matrix $A(m, n_1)$ as a matrix whose element in the ith row and jth column is the partial derivative of $\tilde{y}(i)$ with respect to x_1 (j). Analogously define $B(m, n_2)$ as the matrix whose element in the ith row and jth column is the partial derivative of $\tilde{y}(i)$ with respect to x_2 (j). For future reference notice that the normal matrix η of $\tilde{z}$ as computed and stored by an orbit determination program and defined by Equation A-2 can be written as

$$\eta = \begin{bmatrix} A^T wA & A^T wB \\ \\ B^T wA & B^T wB \end{bmatrix} \tag{A-4}$$

Assume that there exists a priori estimates of $\tilde{x}_1$ and $\tilde{x}_2$ with properties

$$x_1' = \tilde{x}_1 + \alpha_1, \qquad E(\alpha_1) = \bar{0}, \qquad E(\alpha\alpha^T) = P_1$$

$$x_2' = \tilde{x}_2 + \alpha_2, \qquad E(\alpha_2) = \bar{0}, \qquad E(\alpha_2\alpha_2^T) = P_2$$

and assume that the observation vector y or $\tilde{y}$ has properties

$$y = \tilde{y} + \nu, \qquad E(\nu) = \bar{0}, \qquad E(\nu\nu^T) = w^{-1}$$

The least squares estimate of $\tilde{x}_1$ is obtained as the value of $\tilde{x}_1$ which minimizes the loss function

$$L(x_1) = (y - Ax_1 - Bx_2')^T w(y - Ax_1 - Bx_2') + (x_1' - x_1)^T P_1^{-1}(x_1' - x_1) \tag{A-5}$$

The resulting least squares estimator of $\tilde{x}_1$ is well known to be

$$\hat{x}_1 = (A^T wA + P_1^{-1})^{-1} \left[A^T w(y - Bx_2') + P_1^{-1} x_1' \right] \quad \text{(A-6)}$$

Define

$$P = \left[E \; (\hat{x}_1 - \tilde{x}_1) \; (\hat{x}_1 - \tilde{x}_1)^T \right] \quad \text{(A-7)}$$

A series of substitutions reveals that

$$\hat{x}_1 - \tilde{x}_1 = (A^T wA + P_1^{-1})^{-1} \; (- A^T wB\alpha_2 + A^T w\nu + P_1^{-1} \alpha_1) \quad \text{(A-8)}$$

Equation 8 yields

$$P = (A^T wA + P_1^{-1})^{-1} + (A^T wA + P_1^{-1})^{-1} \; A^T wBP_2 B^T wA(A^T wA + P_1^{-1})^{-1} \quad \text{(A-9)}$$

Notice that the right side of Equation 9 can be computed if one has a priori covariance matrices P_1 and P_2, and the upper right and upper left portions of the normal matrix. To determine the covariance matrix of an estimator which estimates some subset of $\tilde{z}$ other than $\tilde{x}_1$, all that is necessary is to permute the rows and columns of η in the appropriate fashion and proceed as before. Thus if one assumes that the normal matrix defined by Equation 2 is precomputed it becomes an easy matter to obtain the resultant covariance matrix when any subset of the $\tilde{z}$ parameters are estimated in a least squares sense and the rest are ignored.

THE ALIAS MATRIX

Assume that all the data has the same variance. Hence

$$w = (I\sigma_0^2)^{-1} \quad \text{(A-10)}$$

where σ_0^2 is the common variance of each data point. Also assume that the a priori estimates of the unadjusted parameters are independent. Under this assumption the covariance matrix P_2 of x_2' can be written as

$$P_2 = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix} \tag{A-11}$$

where σ_i^2 is the a priori variance of the ith unadjusted parameter. Also define a matrix $K(n_1, n_2)$ as

$$K = (A^T w A)^{-1} A^T w B \tag{A-12}$$

With these assumptions Equation 9 yields the following expression for the ith diagonal element of P

$$P(I, I) = \sum_{j=0}^{n_2} (\alpha_{i,j} \sigma_j)^2 \tag{A-13}$$

where $\alpha_{i,0}$ is the ith diagonal element of the matrix $(A^T A)^{-1}$ (this assumes that diagonal elements of the matrix P_1^{-1} are relatively small) and

$$\alpha_{i,j} = K(i, j), \quad j \geq 1 \tag{A-14}$$

The standard deviation of the ith estimated parameter is given by

$$\sigma_i = \left(\sum_{\alpha=0}^{n_2} (\beta_{i,j} \sigma_j)^2 \right)^{1/2} \tag{A-15}$$

where $\beta_{i,0} = \alpha_{i,0}^{1/2}$ and $\beta_{i,j} = K(i, j)$, $j \geq 1$. Define the error sensitivity matrix as

$$S = \{ \beta_{i,j} \}, \qquad i = 1, 2, \cdots n_1, \qquad j = 0, 1, \cdots n_2 \tag{A-16}$$

And finally define the Alias Matrix as

$$L = \bar{S}\bar{\sigma}^2 \tag{A-17}$$

where

$$\bar{\sigma}^2 = \begin{bmatrix} \sigma_0^2 & & & 0 \\ & \sigma_1^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{n_2}^2 \end{bmatrix} \tag{A-18}$$

The standard deviation of the ith estimated parameter is seen to be the root sum square of the terms in the ith row of the alias matrix. The elements in the first column of the alias matrix represent the RSS contribution to the standard deviation of each estimated parameter due to the data noise. The elements in the jth column, $j \geq 2$, represent the RSS contribution to the standard deviation of each estimated parameter due to the j - 1st unadjusted parameter.

Possession of the alias matrix reveals much of the probability structure of the postulated least squares estimator. With this information one can quickly determine which error sources are significant with regard to the estimation of a given parameter.

Propagating Covariance Matrices

Equation 9 provides the covariance matrix of the state $\tilde{x}_1$ at some specified epoch. In many cases it is important to determine how accurately the state can be determined at some time other than epoch. In order to do this correctly it is necessary to take into proper account uncertainties in dynamic parameters. These parameters may be in an estimated mode or in an unadjusted mode and to incorporate their effect one resorts to state transition matrices which presumably have been precomputed by an orbit determination program. Let $\tilde{x}_1$ (T) be the estimated state at time T. Assume as output from an orbit determination program the state transition matrices

$$\bar{v}_1(T) = \frac{\partial \tilde{x}_1(T)}{\partial \tilde{x}_1}, \qquad \bar{v}_2(T) = \frac{\partial \tilde{x}_1(T)}{\partial \tilde{x}_2} \tag{A-19}$$

If there are no dynamic parameters in the estimation vector $\tilde{x}_1$, the matrix $\bar{v}_1$ (T) takes on the particularly simple form,

$$\bar{v}_1(T) = \begin{bmatrix} \delta & 0 \\ 0 & I \end{bmatrix} \tag{A-20}$$

where δ is the six by six matrix defined as the partial derivative matrix of the state of the satellite at time T with respect to the state of the satellite at epoch. If dynamic parameters are included in the estimated state, the off diagonal matrices become non-zero and $\bar{v}_1$ (T) assumes a more complicated form. The matrix $\bar{v}_2$ (T) is the matrix of partial derivatives of the state $\tilde{x}_1$ (T) with respect to the unadjusted parameters $\tilde{x}_2$. If no dynamic parameters are in the unadjusted mode, $\bar{v}_2$ (T) is the null matrix. A first order Taylor series expansion of the function which describes the time evolution of the state $\tilde{x}_1$ (T) yields

$$\tilde{x}_1(T) = \bar{v}_1(T)\,\tilde{x}_1 + \bar{v}_2(T)\,\tilde{x}_2 \tag{A-21}$$

Substituting $\hat{x}_1$ as obtained from Equation 6 for $\tilde{x}_1$ and x_2' for $\tilde{x}_2$ provides the best estimate $\hat{x}_1$ (T), of $\tilde{x}_1$ (T)

$$\hat{x}_1(T) = \bar{v}_1(T)\,\hat{x}_1 + \bar{v}_2(T)\,x_2' \tag{A-22}$$

The covariance matrix of $\hat{x}_1$ (T) is given by

$$P(T) = \bar{v}_1(T)\,P\,\bar{v}_1^T(T) + \bar{v}_2(T)\,P_2\,\bar{v}_2^T(T) + \bar{v}_1(T)\,E[\hat{x}_1\,x_2'^{\,}]\,\bar{v}_2(T)$$
$$+ \bar{v}_2(T)\,E[x_2'\,\hat{x}_1^T]\,\bar{v}_1^T(T) \tag{A-23}$$

Equation 23 in conjunction with Equations 6 and 9 yields

$$P(T) = \bar{v}_1(T)\,(A^T w A + P_1^{-1})^{-1}\,\bar{v}_1^T(T) + \left[\bar{v}_1(T)\,(A^T w A + P_1^{-1})^{-1} A^T w B\right.$$
$$\left. - \bar{v}_2(T)\right] P_2 \left[\bar{v}_1(T)\,(A^T w A + P_1^{-1})^{-1} A^T w B - \bar{v}_2(T)\right]^T \tag{A-24}$$

Finally notice that in much the same fashion that Equation 9 was used to develop an alias matrix at epoch, Equation 24 can be utilized to develop an alias matrix for any time T.

REMARKS

If one possesses a functioning orbit determination program it becomes a relatively easy matter to add covariance analysis capability to the system. A computer program can be written which assumes as input a normal matrix and state transition matrices as generated by the orbit determination program. By permuting the rows and columns of the normal matrix and completing the matrix operations defined by Equation 9, the covariance matrix of a least square process which adjusts any subset of the parameters and ignores the rest can be computed. An alias matrix can be obtained and significant error sources can be identified. By utilizing the precomputed state transition matricies, the covariance matrix of the estimate of the state can be propagated from epoch to any other time. These operations are very simple and they consume little computer time.

Since the normal matrix and state transition matrices are computed once and permanently stored, it is possible to investigate a large number of possible estimation strategies. This can be done conveniently and cheaply. For many applications such a program is a useful and quickly developed addition to an orbit determination system.

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