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THE STATISTICS OF LASER RETURNS FROM CUBE-CORNER ARRAYS ON SATELLITES

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ABSTRACT

A method first presented by Goodman is used to derive an equation for the statistical effects associated with laser returns from satellites having retroreflecting arrays of cube corners. The effect of the distribution on the returns of a satellite-racking system is illustrated by a computation based on randomly generated numbers.

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1. INTRODUCTION

The retroreflecting arrays on certain satellites currently in orbit are examples of what Goodman (1965) calls "rough targets." He predicts that the laser returns from such objects will have a statistical variation; that is, their energies will vary significantly from the mean pulse energy calculated from the range equation, and this variation will occur in a random manner. Consequently, the energy will vary from one pulse to the next to a much greater extent then might be expected from the change in satellite range over the time between successive pulses.

This "scintillation," or random energy variation, from pulse to pulse occurs when the coherence length of the laser radiation is comparable to or exceeds a typical dimension of the satellite's retroreflecting array. The optical phase differences of waves reflected from individual cube corners are preserved, and a random interference pattern, similar to the speckle pattern commonly observed with CW lasers (Eaglesfield, 1967), is produced when these waves combine. In the case of the satellite-tracking laser, however, the receiver is far removed from the reflecting surface, and the individual bright and dark spots in the speckle pattern become very large by the time they reach the receiving telescope. In fact, the area of one of these spots is comparable to that of the aperture of the telescope, and hence there is hardly any aperture averaging. As a result, a large signal is obtained when a bright spot nearly fills the aperture, and a small one, when a dark spot takes its place.

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It is interesting to note that the diffraction spreading of the reflected beam has an important effect on the statistics of the returned signals. If the receiving telescope were close to the retroreflector, the bright spots would be much more closely spaced and the intensity variation would be averaged out over the aperture of the receiver. It is apparent, therefore, that the problem cannot be solved by using geometrical optics, even when the phase changes along the rays are considered. In this report, Goodman's analysis will be presented in a simplified form applicable to the special case of satellite tracking.

The statistical effect for a typical laser system is illustrated with an example computed by generating a sequence of random numbers.

2. THEORY

Goodman's derivation of the effects of target-induced scintillation is more general than is needed for the case of satellite returns. First, he considers both specular reflectors (each consisting of one cube corner) and rough reflectors (many cube corners). He analyzes both the case where the reflecting array totally intercepts the transmitted laser beam and that of partial interception. Further, he allows for two possibilities: that the receiving telescope can resolve the reflected energy distribution from the target, and that it cannot. The theoretical considerations are simplified when there is a restriction to earth satellites, because we need to consider only 1) rough reflectors that intercept a small fraction of the transmitted laser beam and 2) reflectors that are sufficiently distant to be unresolved by the receiving telescope.

It is helpful to consider first the physical conditions and assumptions on which the analysis is based. To begin with, the number of cube corners on the array will not enter the analysis specifically. We can assume that this number is large in the sense that the central-limit theorem (Arley and Buch, 1950) applies; i.e., the amplitude of the electric-field vector at a point in a plane in front of the retroreflector has a normal distribution. The polarization of this field is assumed linear. It is also assumed that since the relative positions of the individual cube corners do not change during the ~20-ns duration of the laser pulse, the distribution is stationary. A satellite with a speed of 7 km/s will move 140 μm in this time, but this motion will not significantly change the relative distances of the cube corners within the array. For the relative motion to be a significant fraction of an optical wavelength for the duration of a Q-switched pulse, the satellite rotation rate would have to be of the order of 1 per second. Another assumption is that the amplitude of the laser radiation does not fluctuate during the pulse; in other words, the pulse duration T is much smaller than the coherence time τ (Troup, 1972).

Using the scalar theory of diffraction (Huygens-Fresnel principle), Goodwin shows that the energy distribution of the return signals depends on M, the number of correlation cells intercepted by the receiving telescope. These cells are equivalent to the bright spots in the speckle pattern. The value of M depends on the satellite range, the laser wavelength, and the diameters of the retroreflector and the receiving telescope. Goodwin computes a relation between M and the normalized range $R\lambda/\pi\ell_0\ell_R$, where R is the satellite range, λ is the laser wavelength, and ℓ_0 and ℓ_R are the lengths of the sides of a square retroreflector and a square receiving telescope, respectively. For a satellite-tracking system, the normalized range is about 1.3 and the corresponding M from Goodwin's Figure 4 is 1. Since an exact calculation of M for the general case is very difficult, Goodwin made a few simplifying assumptions, several of which are undoubtedly valid when satellite ranges are of the order of a megameter. Two of them, however, might have some small effect: The first is that both the target and the receiver have square apertures, and the second, that the receiving aperture can be divided into correlation cells within each of which the energy is constant and statistically independent of that in any other correlation cell.

It seems that for the purpose at hand, the number of correlation cells can be determined more simply by considering how well the retroreflector is resolved by the receiving telescope. If it cannot be resolved at all – i.e., if the telescope sees it as a point source – then M = 1. If there is resolution, an approximate value of M can be determined as follows. Consider the observation of a retroreflector at a distance R from a receiving telescope of diameter D_T. The full width of the resolution angle of the telescope is approximately $2\lambda/D_T$ radians. Hence, intensity variations across the retroreflector must have diameters at least $2R\lambda/D_T$ in order to be resolved. If the retroreflector is D_R in diameter, it contains about $D_R/(2R\lambda/D_T)$ resolution elements. Equating this number of resolution elements to the number of correlation cells gives

$$M = \left[\frac{D_T D_R}{2R\lambda}\right] \quad , \tag{1}$$

where the brackets indicate that M must be a positive integer greater than zero. For $D_T = D_R = 0.5$ m, R = 0.2 Mm, and $\lambda = 694$ nm, we get M = 1. Thus, we will assume that the retroreflectors on satellites are unresolved by the laser systems that track them.

Since the retroreflector is unresolved, we neglect the spatial distribution of its reflected electric field at the receiving telescope. Only a time variation in the amplitude of this field will remain, and it will be the same at any point within the aperture of the receiving telescope. This E field is represented by the following equation:

$$E = X \cos \omega t + Y \sin \omega t = A (\cos \omega t + \phi) \quad , \tag{2}$$

where X and Y are electric-field phasors assumed to be of a single polarization, and ω is the optical frequency (2.713×10^{15} rad/s for a ruby laser). Both X and Y are random variables, normally distributed with zero mean values and equal standard deviations, $\sigma > 0$. The corresponding energy W is proportional to A^2, where

$$W = X^2 + Y^2 = A^2 \quad . \tag{3}$$

From the central-limit theorem, X and Y are independent variables with the following jointly normal density distribution*:

$$f_{X,Y}(x, y) = (2\pi)^{-1/2} \sigma^{-1} \exp\left(-\frac{x^2}{2\sigma^2}\right) \cdot (2\pi)^{-1/2} \sigma^{-1} \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad . \tag{4}$$

Since $w^2 = x^2 + y^2$, equation (4) becomes

$$f_{X,Y}(x, y) = (2\pi\sigma^2)^{-1} \exp\left(-\frac{w}{2\sigma^2}\right) = f_g(w) \quad , \tag{5}$$

where the functional designation $f_g(w)$ represents the fact that the probability density depends not on x and y individually, but only on the function of the two of them, which is given by

$$g(x, y) = w \quad , \tag{6}$$

*Captial letters indicate random variables, and lower-case letters designate the outcomes of the random experiment represented by the random variables.

where $g(x, y) = x^2 + y^2$. In this case, the probability density $f_W(w)$, dependent on w only, can be obtained from the joint distribution $f_g(w)$, dependent on both x and y, by use of the geometrical method (Parzen, 1960). First, we obtain the generalized volume V_g (which is an area in this two-dimensional example):

$$V_g = \iint_{g(x, y) \leq w} dx\, dy = \pi a^2 = \pi w \quad , \tag{7}$$

from which the derivative is $dV_g/dw = \pi$. Thus,

$$f_W(w) = f_g(w) \cdot \frac{dV_g(w)}{dw} = (2\sigma^2)^{-1} \exp\left(- \frac{w}{2\sigma^2}\right) \quad . \tag{8}$$

The mean value of this density function is

$$\overline{w} = (2\sigma^2)^{-1} \int_0^\infty w \exp\left(- \frac{w}{2\sigma^2}\right) dw = 2\sigma^2 \quad ,$$

and thus

$$f_W(w) = \overline{w}^{-1} \exp\left(- \frac{w}{\overline{w}}\right) \quad , \quad w \geq 0 \quad , \tag{9}$$

which is the negative-exponential energy-density distribution given by Goodman in his equation (5). If the received energy is expressed by a continuous variable equal to the mean number of received photoelectrons, equation (9) can be written

$$p(n) = \overline{n}^{-1} \exp\left(- \frac{n}{\overline{n}}\right) \quad , \quad n \geq 0 \quad , \tag{10}$$

where p(n) is the probability that the energy entering the receiver is equivalent to n photoelectrons.

The random variable N, which corresponds to n, represents the effects of target roughness. Another random effect is due to the cathode emission: If K is the random variable representing the electrons emitted from the cathode when the energy input to the receiver is N, we have the following Poisson conditional probability function,

$$p(K|N) = \left(\frac{n^k}{k!}\right) \exp(-n) \quad , \quad k \geq 0 \quad . \tag{11}$$

Our interest is in finding the probability p(k) of the emission of k electrons when there is a return whose mean value is $\bar{n}$, the quantity obtained from the range equation. We derive p(k) from equations (10) and (11) and from

$$p(k) = \int_0^\infty p(K|N)\, p(n)\, dn \quad . \tag{12}$$

Then from equations (10), (11), and (12), we have

$$p(k) = \frac{1}{\bar{n}k!} \int_0^\infty n^k \exp(-n) \exp\left(-\frac{n}{\bar{n}}\right) dn = \frac{1}{\bar{n}k!} \int_0^\infty n^k \exp\left[-n\left(1 + \frac{1}{\bar{n}}\right)\right] dn$$

$$= \frac{1}{\bar{n}k!\, b^{k+1}} \int_0^\infty u^k \exp(-u)\, du = \frac{\Gamma(k+1)}{\bar{n}k!\, b^{k+1}} = \frac{1}{\bar{n}b^{k+1}} \quad ,$$

where $b = 1 + 1/\bar{n}$ and $u = nb$. Hence,

$$p(k) = \left(\frac{\bar{n}}{1+\bar{n}}\right)^k \cdot \frac{1}{1+\bar{n}} \quad , \quad k \geq 0 \quad . \tag{13}$$

Equation (13) has the form of a geometric distribution starting with k = 0 rather than with the usual convention of k = 1. Goodman presents it as his equation (31) and calls it by its other designation, a Bose-Einstein distribution. Its mean value is $\bar{n}$ and its standard variation is $(\bar{n}^2 + \bar{n})^{1/2}$, which approaches $\bar{n}$ as $\bar{n}$ increases.

Figures 1 to 4 show plots of equation (13) for $\bar{n} = 1$, 10, 100, and 1000 photoelectrons. Although the curves are drawn as continuous lines, they are valid only for discrete values of the ordinate k. These figures also plot the Poisson distribution, which accounts only for the randomness of photoelectron emission from the cathode. The Poisson curve is equation (11), with $n = \bar{n}$; the statistical variation in the energy entering the receiving telescope has been removed. For $\bar{n} = 1$, the Poisson and Bose-Einstein curves are similar. For this reason, we see that scintillation from the reflector has little effect on weak returns such as those obtained in lunar ranging. As $\bar{n}$ increases, the two distributions begin to differ significantly. Figure 4 shows that for $\bar{n} = 1000$ photoelectrons, the Bose-Einstein curve is so flat and extended that the received signal k may be very different from $\bar{n}$.

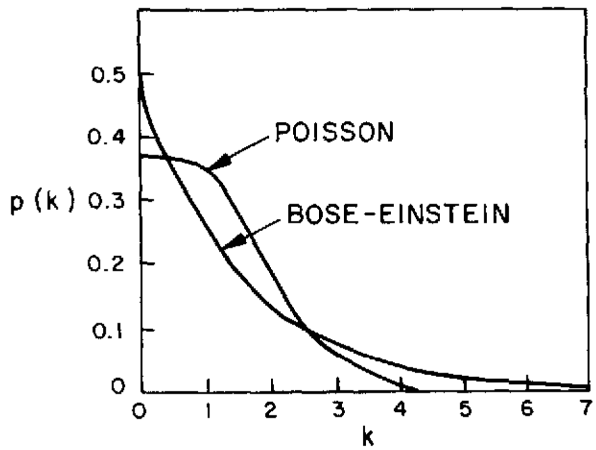

Figure 1. Photoelectron distribution for $\bar{n} = 1$.

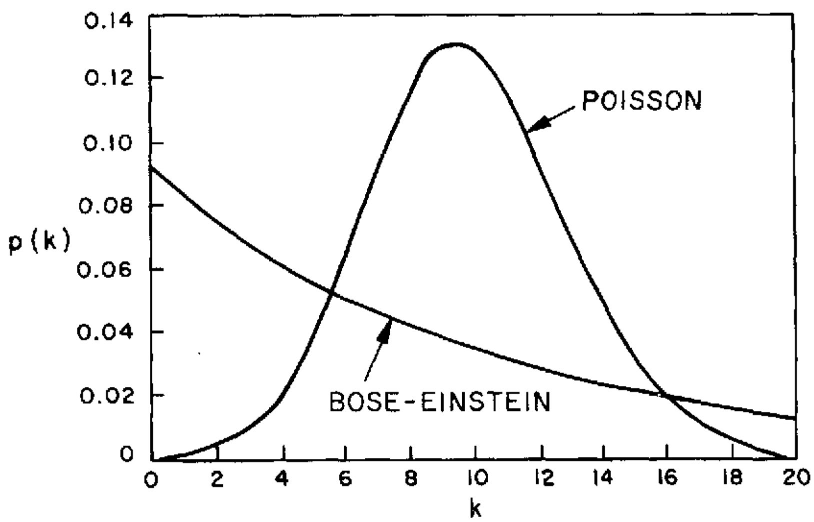

Figure 2. Photoelectron distribution for $\bar{n} = 10$.

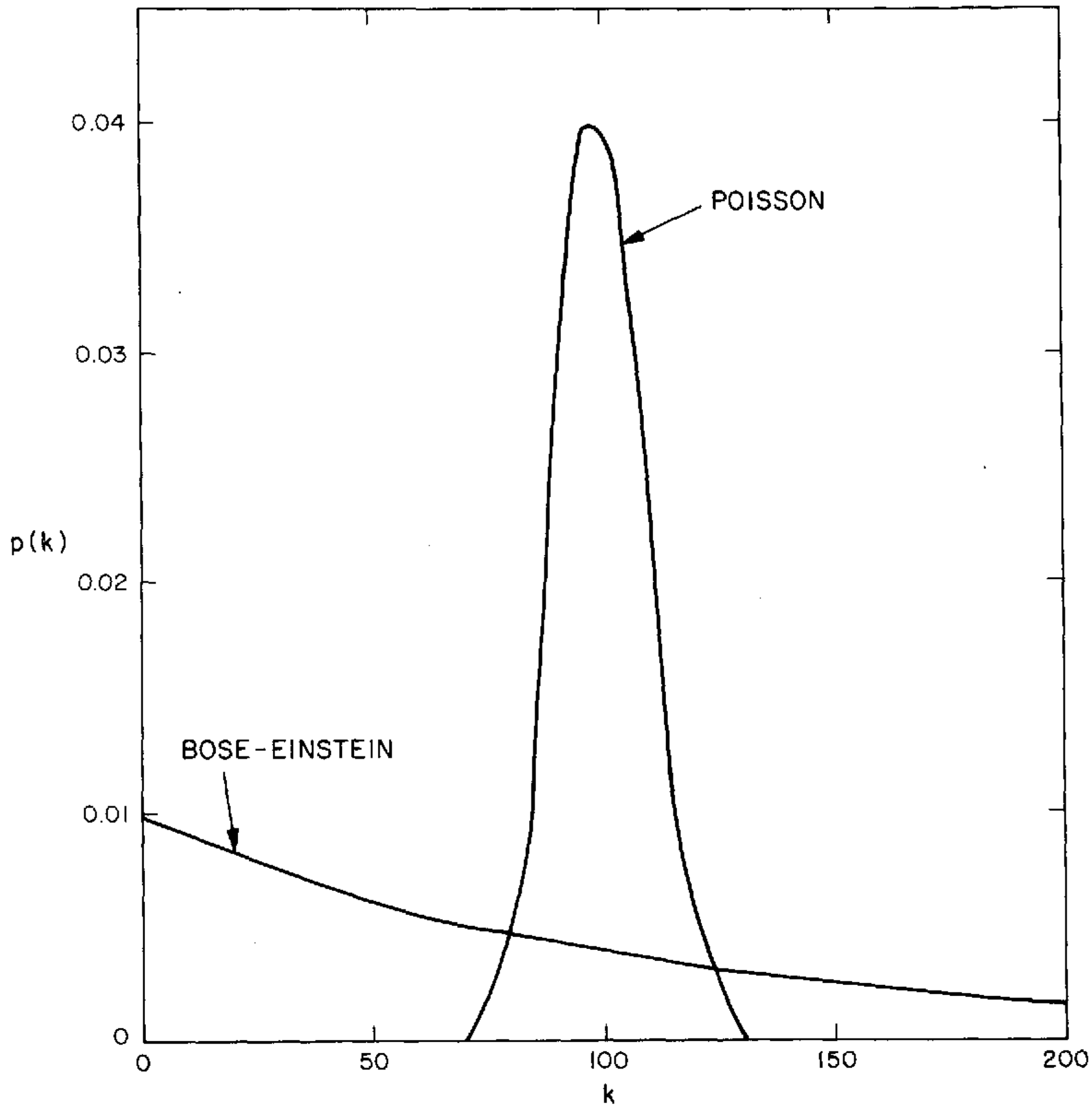

Figure 3. Photoelectron distribution for $\bar{n} = 100$.

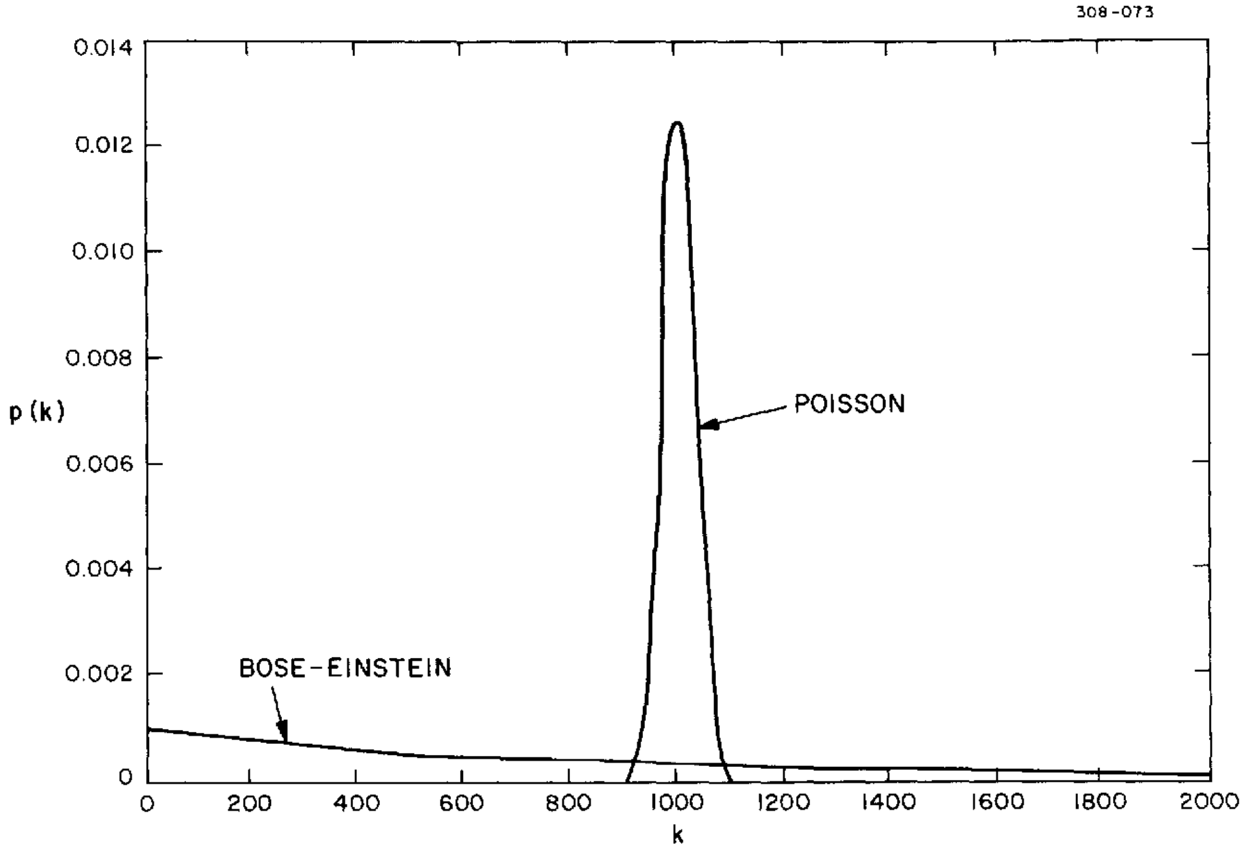

Figure 4. Photoelectron distribution for $\bar{n}$ = 1000.

3. EXAMPLE

Another way to look at the effect of equation (13) on laser returns is to generate random numbers on a computer and use them to compute a set of simulated returns. The first step is to calculate F(x), the distribution function corresponding to p(k), defined as

$$F(x) = P[k \leq x] \quad ; \tag{14}$$

this is the probability that the number of received photoelectrons will be less than or equal to some number x. The probability F(x) goes from 0 to 1, with all values between these limits equally probable. Hence, the returns can be simulated by generating random numbers between 0 and 1 for values of F(x). If $\bar{n} = \eta N_c$, where N_c is the mean number of received photons and η is the conversion efficiency (photoelectrons/photon) of the receiver, then

$$F(x) = \frac{1}{1+\bar{n}} \sum_{j=0}^{x} \left(\frac{\bar{n}}{1+\bar{n}}\right)^j = 1 - \left(\frac{\bar{n}}{1+\bar{n}}\right)^{x+1} = 1 - \left(\frac{\eta N_c}{1+\eta N_c}\right)^{x+1} \quad . \tag{15}$$

Solving equation (15) for x results in

$$x = \frac{\log\,[1 - F(x)]}{\log\,[\eta N_c/(1 + \eta N_c)]} - 1 \quad . \tag{16}$$

Figure 5 was obtained from equation (16) for a typical receiver efficiency of $\eta = 0.018$ electrons/photon and for values of N_c between 10^3 and 10^7. We randomly generated F(x) for each N_c, and then obtained x from equation (16). In this case, x represents the random variable K, the number of photoelectrons in a received pulse. The straight line in the figure represents $k = 0.018\ N_c$, which relates k and N_c through the receiver efficiency without consideration of statistical effects.

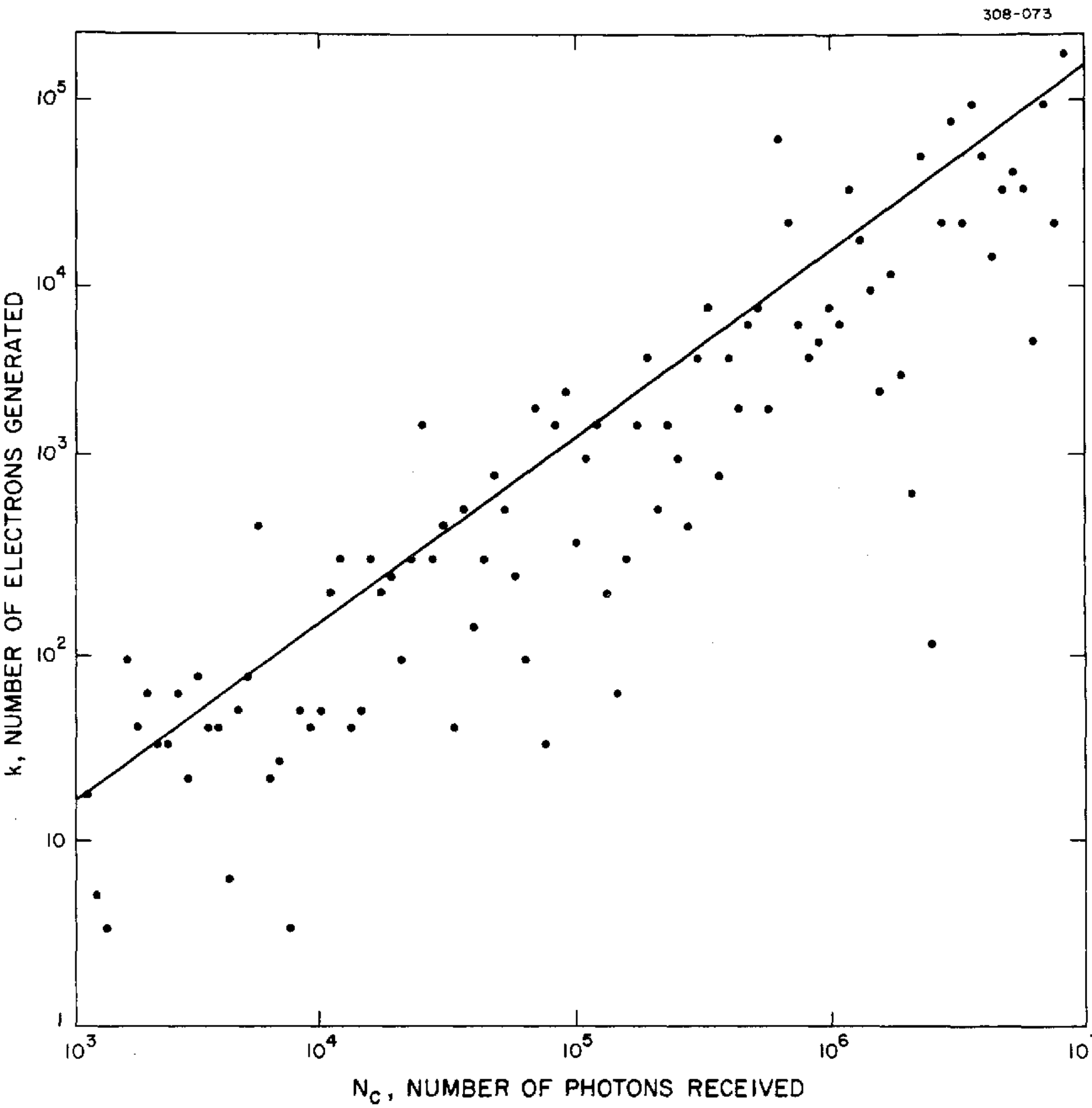

Figure 5. Statistical simulation of laser returns for a receiver efficiency of 0.018 photoelectrons/photon. (The straight line represents the curve $k = 0.018\ N_c$.)

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