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Error Enhancement in Geomagnetic Models Derived from Scalar Data

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Abstract

Models of the main geomagnetic field are generally represented by a scalar potential γ expanded in a finite number of spherical harmonics. In the last decade, such models have been derived mainly by a recursive iteration method from the field magnitude F observed by satellites in low-altitude polar orbits. Very accurate observations of F were used, but indications exist that the accuracy of models derived from them is considerably lower. One problem is that F does not always characterize γ uniquely: Backus has derived a class of counterexamples in which two different choices of γ correspond to the same F. It is not clear whether such ambiguity can be encountered in deriving γ from F in geomagnetic surveys, but there exists a connection, due to the fact that the counterexamples of Backus are related to the dipole field, while the geomagnetic field is dominated by its dipole component. If the models are recovered with a finite error (i.e. they cannot completely fit the data and consequently have a small spurious component) this connection allows the error in certain sequences of harmonic terms in γ to be enhanced without unduly large effects on the fit of F to the model. Computer simulations have demonstrated this effect, producing as a result models which fit the data of F quite closely but yield much poorer fits to the direction of the magnetic vector. Possible remedies are discussed. An appendix also discusses a particular class of fields - related to the counterexamples of Backus - for which it can happen that the recursive iteration deriving γ from F does not converge to the correct solution.

INTRODUCTION

In order to derive a general vector field in space, three independent scalars must be observed at each point - e.g. 3 orthogonal components, or the field's magnitude and two angles defining its direction. For the main geomagnetic field $\underline{B}$ - i.e. the part which originates deep inside the earth - such an observation is redundant, since that field is completely defined by a scalar potential γ

$$\underline{B} = -\nabla\gamma \qquad (1)$$

The information needed to derive $\underline{B}$ may thus be given by a single scalar function of position. In fact, there exists the additional constraint that γ must be harmonic ; this may be viewed as reducing the "information content" of γ to that of a two-dimensional scalar function, since only 2 indices appear in the spherical harmonic expansion of γ , traditionally written as

$$\gamma = a \sum_{n=1} \sum_{m=0}^{m=n} (a/r)^{n+1} P_n^m(\theta) \left\{ g_n^m \cos m\varphi + h_n^m \sin m\varphi \right\} \qquad (2)$$

(a is the earth's radius). In satellite surveys it is relatively easy to measure accurately one of the three scalars associated with the field, namely the magnitude $|\underline{B}|$, commonly denoted in geomagnetic literature by the capital letter F . Accurate observation of F requires neither information on the attitude of the observing spacecraft, not observation of the flexing of the boom carrying the magnetometer ; such information is required for accurate determination of the field's direction, which consequently is much harder to perform.

Furthermore, satellites conducting surveys of the field generally do not vary their altitude by great amounts, so that their observations of F tend to be distributed more or less over a two-dimensional surface - that of a sphere concentric with the earth. The preceding qualitative arguments do however suggest that the amount of information thus gathered may suffice to determine the field.

This has led to attempts to trade one scalar for the other, i.e. to derive γ from observations of F (of course, observed data must first be "cleaned" of fields of external origin, but this important point is not considered here). Zmuda [1958] appears to have been the first to propose this approach and its first major use occured in the analysis of Vanguard 3 data by Cain et al. [1962]. The derivation of models of the main field from satellite data has generally been based on F [Heppner, 1963 ; Cain et al., 1967 ; Cain, 1971], in particular the models derived from observations of OGO 2,4 and 6 [Cain and Sweeney, 1970]. The method used in such work is described further on. Because F is easily observed down to intensities of 1 γ (= 10^{-5} gauss) this has become the method of choice for mapping the geomagnetic field and models which fit observed F within an r.m.s. deviation of 4 - 7 γ have been derived by it.

However, two problems arose - one experimental, the other theoretical. Experimentally, the few observations in which the vector direction of the field could be compared to the model indicated considerable deviations [Cohen, 1971 ; Woodman, 1971]. Also, the results of airborne surveys indicate (J. Cain, private communication) that field components often deviate by 100-200 γ from the values prescribed by models.

The theoretical difficulty was pointed out by G. Backus [1968, 1970] who investigated the question of whether F observed on a sphere uniquely defines the scalar potential γ from which it can be derived. If the expansion (2) contains a finite number of terms it may be shown [Backus, 1968] that F and γ are uniquely related. However, if an infinite number of terms is allowed, that assertion turns out to be false and Backus [1970] furnished a class of counterexamples in which two quite distinct infinite expansions, similar to (2), yield identical forms of F on the unit sphere r = 1 .

This discovery has cast some doubt on the assertion that the geomagnetic scalar potential γ is uniquely derivable from its corresponding F. So far, this question has not been resolved.

What the present work shows is that the experimental discrepancies between vector field observations and the predictions of the model may have a mathematical origin, related to the work of Backus. Specifically, it will be shown that when perfect recovery of γ is not possible - due to finite observational errors of various origins - and each of the coefficients of (2) contains a certain error, then this error is enhanced for certain sequences of terms related to those derived by Backus. The basic reason for the connection is that the counterexamples constructed by Backus are related to the dipole field, while the geomagnetic field is known to be dominated by its dipole component. Computer simulations of the effect and possible remedies will also be described.

THE COUNTEREXAMPLE OF BACKUS

Backus [1970] sought to determine whether it was possible, in three dimensional space, to find two harmonic functions ϕ_1 and ϕ_2, vanishing at infinity and differing by more than their algebraic signs, such that on the unit sphere $r = 1$

$$(\nabla\phi_1)^2 = (\nabla\phi_2)^2 \tag{3}$$

He found it convenient to reformulate the problem in terms of the sum u and the difference v of these functions, which on $r = 1$ have to satisfy

$$\nabla u \cdot \nabla v = 0 \tag{4}$$

Choosing u to be the dipole potential

$$u = \gamma_d = \cos\theta / r^2 \tag{5}$$

Backus assumed v to be an infinite series of the form (2) and he then examined the coefficients to see whether they could be made to satisfy (4) . Not only did this turn out to be possible but a whole class of such solutions was found to exist. The general form of these solutions is best expressed if one interchanges the summations of equation (2) and writes

$$v = \sum_{m=1} \left\{ a_m \cos m\varphi \sum_{n=m}^{\infty} G_n^m (a/r)^{n+1} P_n^m(\theta) + \right.$$

$$\left. + \; b_m \sin m\varphi \sum_{n=m}^{\infty} H_n^m (a/r)^{n+1} P_n^m(\theta) \right\} \qquad (6)$$

where for sake of definiteness the leading term in each of the summations over n is set equal to unity

$$G_m^m = H_m^m = 1 \qquad (7)$$

Then, provided the series in the inner summations are suitably chosen (there exist two such summations for each value of m , and they are actually the same - i.e. $H_n^m = G_n^m$) , equation (4) is satisfied for any arbitrary choice of the coefficients a_m and b_m . The series over n are derived by recursion from their leading terms, and since these terms are fixed by (7), the series are unique. They are described in more detail in Appendix I and contain nonzero terms only for values of n and m which sum up to an even number.

Strictly speaking, (6) satisfies (4) only if an infinite number of terms is used. We shall denote this limit of v by a superscript ∞ and write

$$\nabla \gamma_d \cdot \nabla v^{\infty} = 0 \qquad (8)$$

If the number of terms in v is finite, (8) no longer holds; however, the scalar product contributed by such a truncated v (denoted by a superscript f for "finite") may be small if v^{∞} is approached sufficiently closely. This may be symbolically written as

$$\nabla \gamma_d \cdot \nabla v^f = O(\varepsilon) \qquad (9)$$

where $\varepsilon \ll 1$ is a small parameter. In what follows we indicate for brevity all small quantities by ε ; at least 3 distinct quantities of this sort appear, but devoting to each of them a separate notation merely encumbers the expressions and yields no new results. The magnitude of the right hand side of (9) depends on the exact form of v^f - in particular, on the number k of terms in the shortest summations over n which appear in it . If k is sufficiently large, however, the expression (9) may be made arbitrarily small.

MODEL DERIVATION FROM SCALAR DATA

In the derivations of models of the main geomagnetic field from a set of observations of $F = |\nabla \gamma|$, the value of γ is determined by iterated linearization, starting from an initial approximation γ_0 of the scalar potential. Let $\varepsilon \gamma_1$ be the correction to be added to γ_0 in order to obtain the true potential γ .

$$\gamma = \gamma_0 + \varepsilon \gamma_1 \qquad (10)$$

Then

$$\varepsilon \nabla \gamma_1 \cdot \nabla \gamma_0 = \frac{1}{2} \left\{ F^2 - (\nabla \gamma_0)^2 \right\} - \frac{1}{2} \varepsilon^2 (\nabla \gamma_1)^2 \qquad (11)$$

If the term of order ε^2 is neglected and γ_1 is expanded in a finite series of form (2) with N unknown coefficients, then (11) represents one linear equation with N unknowns for each point at which an observation of F was made.

Typically the number M of such observations greatly exceeds N . If, for instance, the series (2) for γ_1 is truncated past $n = n_{max}$, then

$$N = (n_{max} + 1)^2 - 1 \qquad (12)$$

Typically, $n_{max} = 10$, $N = 120$, while the number of equations (one for each observation of F) may be of the order of 10,000 . The best-fitting set of coefficients for γ_1 is then derived by least squares fitting, after which one replaces γ_0 by $(\gamma_0 + \varepsilon\gamma_1)$ and repeats the process, seeking a further correction term γ_1' . This procedure is continued until the r.m.s. difference between observed values of F and those obtained from the model no longer shows any improvement. Various refinements may be included, e.g. dividing each of the equations (11) by $|\nabla\gamma_0|$ in order to avoid giving greater weight to observations in polar regions, where F is larger.

Cases are known in which this procedure does not lead to the correct solution (Appendix II), but in analysis of simulated data from models representing the geomagnetic field this method tends to converge rapidly - to the "correct" solution, if the derived series is capable of expressing it, or to a small range of final parameters corresponding to a "best-fitting series" if the data contain unresolved higher harmonics or other sources of small error.

E R R O R   E N H A N C E M E N T

The basic equations used in the iteration are

$$\varepsilon\,\nabla\gamma_0 \cdot \nabla\gamma_1 = \frac{1}{2}\left\{F^2 - (\nabla\gamma_0)^2\right\} \qquad (13)$$

with one such equation being contributed by each observation point. These equations will lack a unique solution if the set of equations

$$\nabla\gamma_0 \cdot \nabla\gamma_1 = 0 \qquad (14)$$

evaluated for the same points, <u>does</u> have a solution. In that case, given one solution of (13) , new solutions may be obtained by adding to it arbitrary multiples of solutions of (14) .

If all observations were conducted at the same radial distance, if γ_0 were the dipole field and if γ_1 contained an infinite number of terms, then by (8) there would have existed a solution to (14) and (13) would not have defined γ_1 uniquely. None of these conditions holds exactly in the analysis of the geomagnetic field (in fact. γ_0 changes from one iteration to the next), but they are all approximately valid and this, in the presence of finite errc may cause problems.

In this section the finite spread in the radial distance r will be neglected (its effects are considered separately later on). Let γ_0 now be separated into a dipole part and a non-dipole part

$$\gamma_0 = \gamma_d + \varepsilon \gamma_{nd} \qquad (15)$$

where $\varepsilon \ll 1$ is yet another small quantity.

If an exact solution for equations (13) does not exist - due to truncation, experimental error, unresolved contributions of external field sources etc. - then the "best fitting" result for γ_1 will consist of the sum of the "true solution" γ_t and an "error" γ_ϵ, each of which will be expressed by a finite series similar to the one in equation (2) . One can then write

$$\varepsilon(\nabla\gamma_d + \nabla\gamma_{nd})\cdot(\nabla\gamma_t + \nabla\gamma_e) \cong \frac{1}{2}\left\{F^2 - (\nabla\gamma_0)^2\right\} \qquad (16)$$

where $\cong$ will signify "best fitting by the least squares method" (for our particular set of observations).

Now a finite series γ_e of form (2) can in turn be resolved into two parts: a sum γ_{eb} of "Backus terms" similar to those in (6), plus a sum γ_{ei} of independent harmonics. To perform such a separation one first isolates all terms with n = m and includes them in γ_{eb}, while the remaining terms are assigned to γ_{ei} . Each of the terms in γ_{eb} is now viewed as the leading term of a Backus-type series of the type listed in equation (6) and in Appendix I (including the coefficient a_m or b_m multiplying such a series), and appropriate higher-order terms are added to it to form the rest of its series, up to the highest n and m allowed by the finite expansion of γ_{eb} ; to maintain the balance, the same terms that are added to γ_{eb} are subtracted from γ_{ei} . Finally, those expansions in γ_{eb} deemed to have (after truncation) too few terms for (9) to be valid (with the right hand side small enough) are transferred from γ_{eb} to γ_{ei} . This will certainly include all those Backus series which contain only one term - and perhaps also those containing two . Equation (16) now may be written

$$\varepsilon\,(\nabla\gamma_d + \nabla\gamma_{nd})\cdot(\nabla\gamma_t + \nabla\gamma_{eb} + \nabla\gamma_{ei}) \cong \frac{1}{2}\left\{F^2 - (\nabla\gamma_0)^2\right\} \qquad (17)$$

The contributions of the error terms to the left hand side are mainly due to scalar products with the dipole term, which overshadows the non-dipole term. If all coefficients involved in γ_{eb} and γ_{ei} were equal, then the contribution to the scalar product associated with a coefficient of γ_{eb} will on the average be one order in ε smaller than the contribution associated with one of the coefficients of γ_{ei} , since by equation (9) the former contribution gains an extra order in ε .

Alternatively, if the least squares optimization adjusts all sources of error so that they contribute approximately equally, then the coefficients of γ_{eb} will be larger by a factor of the order of ε^{-1} than those of γ_{ei} , leading to an enhanced error in these terms.

A different view of the same effect is gained by writing the basic relation prior to linearization

$$\left\{\nabla\left[\gamma_d + \varepsilon(\gamma_{nd} + \gamma_t + \gamma_{eb} + \gamma_{ei})\right]\right\}^2 \cong F^2 \qquad (18)$$

When the left hand side is expanded in orders of ε , the ε^0 term is simply $(\nabla \gamma_d)^2$ and all $O(\varepsilon)$ terms of the scalar potential contribute to the term of order ε , except for γ_{eb} , which due to (9) only contributes at the ε^2 level . Thus F^2 is inherently less sensitive to fluctuations of γ_{eb} , allowing its errors to be relatively large without excessive effects on the fit of F^2 . Intuitively, the Backus series may be viewed as modes at which a near-dipole field, derived in the manner described here, prefers to fluctuate - just as a bridge or a beam tend to oscillate in certain modes related to their geometrical properties.

S I M U L A T I O N

A test of the preceding argument was performed as follows. A 120-term expansion of γ (n_{max} = 10) was used to derive the corresponding values of F at 1440 points on the unit sphere. A program analyzing these simulated data by the previously described method was then applied, but the derived models contained only 63, 80 or 99 terms (n_{max} = 7, 8 or 9). Up to 10 iterations were performed, although the model generally showed little variation after the second iteration; the results cited here therefore refer to the fit after 2 iterations. As a check a different input for γ was used and similar effects were observed, although they are not included in the results listed here.

From the models, the predicted intensity F_m was derived for the 1440 input point and at each point the deviation

$$\delta F = |F - F_m|$$

was computed. The quantity $\delta F/F$ was then derived, its average was also found and a representative table of its values at every 3rd point was printed out, from which the worst (i.e. largest) value was picked visually. In addition, the mean angle $\delta\lambda$ (in radians) between the model field and the true field was derived: it is of the same order as the relative discrepancy $\delta B_i / B_i$ in one of the components of $\underline{B}$. As before, the worst case was also picked out visually from a representative table.

The results are given in Table 1 . The relative accuracy of the direction is more than 10 times worse than that of the magnitude, and the worst fit in direction has an error about 10 times larger still. This is an effect slightly larger than what is found with models of the geomagnetic field.

Inspection of the coefficients revealed that the error in the derived models indeed contained relatively large contributions from the Backus sequences. For this purpose the differences Δg_n^m and Δh_n^m between the harmonic coefficients of the model potential and those of the potential used as input were tabulated. The coefficients sequences

$$g_m^m \;,\; g_{m+2}^m \;,\; g_{m+4}^m \;\; \dots$$

and

$$h_m^m \;,\; h_{m+2}^m \;,\; h_{m+4}^m \;\; \dots$$

corresponding to terms which participate in a single Backus series almost invariably showed the characteristic alternation of signs between consecutive terms which is typical of any such series. Furthermore, when the ratio R of any such term to the leading term of its series is compared to the corresponding ration R_b for the Backus series (Table 4) , the two ratios are roughly similar.

Results from the test for some of these sequences, as well as for some terms not belonging to any sequence, are shown in Table 2 . As can be seen, some of the sequences grow to relatively large amplitudes (e.g. the g_2^2 series for 99 term recovery, the g_3^3 series for 80 term recovery) , others (e.g. the one headed by h_2^2 in the 99 term case) fail to develop.

POSSIBLE REMEDIES

The counterexamples discovered by Backus satisfy equation (4) only on a single spherical surface. If observations are distributed over a finite region in three dimensions, the series (6) no longer satisfy (4) over the entire region of observation and it is thus possible, in principle, to overcome the problem described earlier. In fact Backus proved [Backus, 1974] that equation (14) cannot be satisfied over a spherical shell of any finite thickness.

In practical observations such a three dimensionality always exists due to the finite eccentricity of the orbit of the observing satellite. For those satellites of the OGO series which were used for geomagnetic surveys this eccentricity was rather small: the altitude range (above sea level) was about 400 to 1500 km for OGO 2 , 400 to 900 km for OGO 4 and 400 to 1100 km for OGO 6 . Unfortunately, it appears that in the presence of finite error such a variation in altitude is only able to reduce somewhat the effects described in this work and will not eliminate them altogether.

Suppose observations are confined to a spherical shell with

$$r_0 - \delta r \leq r \leq r_0 + \delta r$$

If $\delta r/r = O(\varepsilon)$, then equation (9) - which is the one relevant here - may still be used, for by Taylor expansion

$$\begin{aligned} u(r,\theta,\varphi) &= u(r_0,\theta,\varphi) + O(\varepsilon) \\ v(r,\theta,\varphi) &= v(r_0,\theta,\varphi) + O(\varepsilon) \end{aligned} \tag{19}$$

If u and v satisfy (9) on the sphere $r = r_0$, then the above equations show that they satisfy a similar equation over the entire spherical shell, although the "smallness" of the right hand side is somewhat impaired. Thus equations (17) and (18) still hold and error enhancement still exists, although not as strongly as in the case of observations in an infinitely thin spherical shell.

As a test, the 99-term recovery of the 120-term model, used in tables 1 and 2, was repeated with data points oscillating over a finite range in radial distance, comparable to that of the OGO orbits. A distinct improvement in the angular resolution was obtained (Table 3) but there still exists a pronounced discrepancy between $\delta F/F$ and $\delta\lambda$ (the values of $\delta F/F$ deteriorate somewhat with increasing δr, probably because the "error" is an unresolved set of $n = 10$ terms, the contribution of which increases rapidly for those points at which r is diminished). If an analysis similar to that of table 2 is performed, it is found that the Backus series are still enhanced, although the ratios R depart further from those in table 4.

More generally, the addition of some vector data to the data set can in general help resolve the problem. In such cases it might be useful to separate the model potential γ_m derived from F into two parts γ_{mb} and γ_{mi}, in a similar way to the resolution of γ_e in (16) into its two parts γ_{eb} and γ_{ei} in (17). One may then assume γ_{mi} to be accurately derived from F and use the added data for deriving γ_{mb} alone.

Suppose for example that one is given as added vector data an accurate location of the dip equator on the surface $r = a$. On the dip equator $\partial\gamma/\partial r = 0$ and therefore, by (6) (noting the equality of the two summations when (7) is assumed)

$$\partial\gamma_{mi}/\partial r\Big|_{r=a} = \sum_{m=1} (a_m \cos m\varphi + b_m \sin m\varphi) \sum_{n=m} (n+1)\, G_n^m P_n^m(\theta) \tag{20}$$

Here the left hand side is known, the coefficients G_n^m are given in table 4 and the unknowns to be derived are the factors a_m and b_m. If the variation of θ is neglected (assuming it to be near $\pi/2$) the solution of (20) reduces to the expansion of a given function of φ in a Fourier series. In practice, with θ kept as variable, the simplest way of deriving a_m and b_m would be by least squares fitting of (20).

APPENDIX I : THE BACKUS COUNTEREXAMPLE .

The published form of the counterexample of Backus [Backus, 1970] normalizes the Legendre functions P_n^m in the manner preferred by mathematicians and expresses the dependence on the longitude angle φ in terms of complex exponentials. For computer programs it is much more convenient to adopt Gaussian normalization, which avoids the need to calculate square roots when deriving either spherical harmonics or recursion coefficients. It is also more conventional in geomagnetic usage to express the dependence on φ by means of $\cos m\varphi$ and $\sin m\varphi$ (as in equation 2) instead of $\exp(\pm i m \varphi)$. The purpose of this appendix is to list some properties of the Backus counterexamples when these are expressed in Gaussian normalization with trigonometric coefficients in φ .

The general form of the counterexample is given in equation (6), and if condition (7) holds, then the two series there are identical - i.e. $G_n^m = H_n^m$. The basic recursion relation for either of these series is, with Gaussian normalization,

$$G^m_{m+2k} = - \frac{(m+2k)(2m+4k+1)(2m+4k-1)}{12k\ (m+2k+1)(m+k)} \quad G^m_{m+2k-2} \qquad \text{(A-I-1)}$$

A similar recursion holds for H_n^m ; if (7) is assumed, the results of this recursion are shown in Table 4 . Note that in all terms involved here the sum of indices is even; terms for which this sum is odd vanish identically.

Table 4 shows that for any given m , the magnitude of the coefficients increases with increasing lower index n . The resulting series nevertheless do converge (as was shown by Backus), because the normalization factors entering P_n^m also contribute to the convergence. For Gaussian normalization the P_n^m for any given m are most conveniently generated by a recursion relation starting from

$$P_m^m(\theta) = \sin^m\theta \qquad \text{(A-I-2)}$$

$$P_{m+1}^m(\theta) = \cos\theta \, \sin^m\theta \qquad \text{(A-I-3)}$$

. . .

The relation is

$$P_{n+1}^m(\theta) = \cos\theta \, P_n^m(\theta) + \lambda_n^m \, P_{n-1}^m(\theta) \qquad \text{(A-I-4)}$$

where

$$\lambda_m^n = (n+m)(n-m)/\left[(2n+1)(2n-1)\right] \qquad \text{(A-I-5)}$$

The derivatives are obtained by recursions found from the above ones by differentiation, starting with

$$dP_m^m/d\theta = m \cos\theta \, \sin^{m-1}\theta \qquad \text{(A-I-6)}$$

APPENDIX II : CASE IN WHICH ITERATION FAILS.

It was noted that in practice the iteration (13) tends to converge rapidly to an "optimal model" of the field : the purpose of this appendix is to point out a case in which this is not true. The example given is closely related to the work of Backus [1970] and is probably rather atypical, but it does show that (13) is not universally useful.

Let ϕ_1^∞ and ϕ_2^∞ be two potentials of the Backus type, satisfying on r = 1 the condition

$$(\nabla \phi_1^\infty)^2 = (\nabla \phi_2^\infty)^2 = F^2 \qquad \text{(A-II-1)}$$

Let ϕ_1^n and ϕ_2^n be truncated versions of these series, with all terms having upper indices exceeding n omitted, and let

$$|\nabla \phi_1^n| = F_1^n$$
$$\text{(A-II-2)}$$
$$|\nabla \phi_2^n| = F_2^n$$

By the theorem of Backus proving the uniqueness of the relation between γ and F for potentials with a finite number of harmonic terms [Backus, 1968] , F_1^n uniquely characterizes ϕ_1^n and F_2^n uniquely characterizes ϕ_2^n . However, as $n \to \infty$, both F_1^n and F_2^n tend to F and therefore, for large values of n , the difference between the two becomes rather small.

This suggests that if the input to the recursion (13) consisted of F_1^n or F_2^n, with n large enough, the recursive iteration might experience difficulty in telling the two apart - e.g., given F_1^n, it might start converging to <u>either</u> ϕ_1^n <u>or</u> ϕ_2^n, depending on the initial choice of γ_0.

This is confirmed by computer simulations with F generated by 48-term expansions and the least-squares iteration capable of resolving an equal number of coefficients. If the program deriving the model of γ is presented by such an F_1^n, one of two things is found to occur. If the initial approximation resembles ϕ_1^n more than it resembles ϕ_2^n, the recursive iteration will rapidly converge to the correct coefficients. If, however, the initial approximation is nearer to ϕ_2^n, the coefficients will tend to evolve towards those of ϕ_2^n. The iteration then never converges - the search being conducted in the wrong range of parameters - and instead the values of the output coefficients wander within a finite range around those of ϕ_2^n.

For a recursion deriving 48 term expansions, each resulting γ may be viewed as defining a point in a 48-dimensional space σ_{48} in which each coefficient corresponds to one coordinate, and (13) may be viewed as prescribing a mapping of each point in σ_{48} into another point there. If the field magnitude used as input is F_1^n ($n \leq 6$), the mapping will have a fixed point ϕ_1^n which maps onto itself, and in some region surrounding this point the mapping will converge to ϕ_1^n. However, this convergence evidently does not hold for all of

σ_{48} , since for some initial points the process leads to the neighborhood of the "false solution" ϕ_2^n . The region from which the iteration converges to ϕ_1^n thus has a boundary, which will represent a limit cycle - i.e. if the initial point is located on that boundary, the mapping causes the resulting set of coefficients to wander along the boundary without either converging to ϕ_1^n or approaching ϕ_2^n .

In one simulation the nonzero coefficients of ϕ_1^n were

$$g_1^0 = 10 \qquad g_2^2 = 2$$

$$g_4^2 = -2.8 \qquad g_6^2 = 3.375$$

(for the corresponding ϕ_2^n , the signs of the $m = 2$ terms are reversed). The coefficients of the initial field were chosen as

$$g_1^0 = 10 \qquad g_2^1 = 2$$

$$g_2^2 = -2\lambda \qquad g_4^2 = 2.8\lambda$$

For $\lambda = 1$ the last two coefficients of the initial field are equal to coefficients of the "false solution" ϕ_2^n and the recursion in that case heads for that solution (the spurious term g_2^1 is whittled down in the process). On the other hand, if $\lambda = 0$ the recursion heads for the "true" solution ϕ_1^n . The transition, corresponding to a limit cycle, occurs near $\lambda = 0.102$ and when that value was used, as many as 6 iterations were completed before a clear trend became evident.

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CAPTIONS TO TABLES

Table 1 Deviations of a model geomagnetic field, derived by means of F, from the "true" field it is supposed to represent, in a computer simulation. The input field is given by a scalar potential with 120 coefficients ($n_{max} = 10$), the number of points is 1440 and 2 iterations are used, except for one case where the results of 3 iterations are also shown for comparison. All results are in units of 10^{-4}.

Table 2 Some of the differences $\Delta g = g(\text{model}) - g(\text{true})$, where g stands for one of the harmonic coefficients g_n^m or h_n^m in a simulated recovery of γ from F as described in Table 1. Here R_b denotes the ratio between a term in a Backus sequence having the type and indices indicated on the table and the leading term of its sequence (these ratios are listed in Table 4), while R is the ratio between corresponding values of Δg.

Table 3 Results similar to those of Table 1, with a similar input model and 99-term recovery in 3 iterations, for cases in which data points are spread out over a spherical shell bounded by $r_0 \pm \delta r$.

Table 4 Ratios G^m_{m+2k}/G^m_m derived from the recursion relation A-I-1.

No. of terms in recovery	63	80	99	
			2 iter.	3 iter.
Mean $\delta F/F$	26	20	7.1	6.4
Worst $\delta F/F$	249	207	35.0	31.4
Mean $\delta\lambda$ (radians)	300	222	102.8	106.4
Worst $\delta\lambda$ (radians)	2497	2139	1298	1307

Table 1

type	n	m	63 terms (n_{max} = 7)	80 terms (n_{max} = 8)	99 terms (n_{max} = 9)		
					Δg	R	R_b
g	2	1	59	38	9		
g	2	2	186	199	- 240		
h	2	2	247	- 30	16		
g	3	0	- 130	- 122	46		
h	3	1	- 35	- 14	33		
h	3	2	- 244	- 198	94		
g	3	3	195	242	92		
g	4	0	116	98	13		
g	4	2	- 157	- 197	345	-1.44	-1.4
h	4	2	- 285	78	- 2		-1.4
g	5	3	- 276	- 303	- 130	-1.41	-1.72
g	6	2	187	197	- 409	1.71	1.79
h	6	2	337	- 98	3		1.79
g	7	3	466	468	170	1.85	2.44
g	8	2	-	- 163	412	-1.72	-2.25
h	8	2	-	62	- 3		-2.25

Table 2

$\delta r / r_0$	0	0.05	0.10
Mean $\delta F/F$	6.37	7.21	9.17
Worst $\delta F/F$	31.4	79.9	135.2
Mean $\delta\lambda$	106.4	85.6	79.7
Worst $\delta\lambda$	1307	1303	1050

Table 3

m \ k	0	1	2	3	4	5
1	1.000	-1.0938	1.2533	-1.4580	1.7987	-2.2121
2	1.000	-1.4	1.7875	-2.2509	2.8350	-3.5826
3	1.000	-1.7187	2.4438	-3.2890	4.3339	-5.6593
4	1.000	-2.0423	3.2156	-4.6285	6.3976	-8.6580
5	1.000	-2.3698	4.1006	-6.3039	9.1463	-12.8477
6	1.000	-2.6984	5.0978	-8.3512	12.7146	-18.5484

Table 4