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ANALYSIS OF THE DYNAMICS OF A NUTATING BODY

by William J. Anderson

Lewis Research Center
Cleveland, Ohio 44135

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16. Abstract

The equations for the displacement, velocity, and acceleration of a point in a nutating body are developed. These are used to derive equations for the inertial moment developed by a nutating body of arbitrary shape. Calculations made for a previously designed nutating plate transmission indicate that that device is severely speed limited because of the very high magnitude inertial moment.

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SUMMARY

The equations for the displacement, velocity, and acceleration of a point in a nutating body are developed. These are used to derive equations for the inertial moment developed in a nutating body of arbitrary shape. The moment equations were applied to calculate the inertial moment acting on the support bearing system in a nutating plate transmission. For calculation purposes a previously designed nutating plate transmission was used. The inertial moment was found to be 28 times as great as the moment resulting from reaction forces. This indicates that nutating plate drives are severely speed limited.

The equations developed are considered to be potentially useful for calculating squeeze film and distortion effects in thrust bearings and face seals which are subject to high-frequency, low-amplitude nutation.

INTRODUCTION

The development of mechanical power transmissions that utilize a nutating member (ref. 1) has led to a need for a thorough understanding of the dynamics of a nutating body. Figure 1 (from ref. 2) shows a sketch of the arrangement of the components in a split nutating drive main rotor gearbox. The nutating plates describe a nutation cycle for each revolution of the input shaft. The nature of the nutating motion considered here can be visualized by referring to figure 2(a). The X, Y, Z coordinate system is fixed in the nutating body. The X axis describes a cone of angle 2α in space. Points along the Y and Z axes describe lemniscate-like figures. There is no rotation of the body about the X axis. In this respect the nutation considered here differs from that of spinning bodies that rotate about the nutation axis. Pins on the nutating plates (fig. 1) engage teeth on the fixed reaction plates and also on the drive plates. The drive plates advance one tooth for each revolution of the input shaft.

The loads on the nutating plate support bearing, which result from both contact and inertial forces, must be known. In reference 2 a design analysis showed that contact forces on the nutating plate could result in severe nutator support bearing loads and in short bearing life. An approximate analysis of inertia loads assuming simple harmonic motion of the nutating body revealed that these, too, could be very significant, especially at high speeds.

A search of the literature revealed a number of papers dealing with the dynamics of nutating bodies such as references 3 to 6. All of these, however, are concerned with the force systems that create the nutation and damping techniques for attenuating the nutation. None of them treat the dynamic forces generated by the nutating body.

Therefore, the object of the work reported herein was to develop the displacement, velocity, and acceleration equations for a point in a nutating body and to apply these to the calculation of inertial moments. The resulting equations can also be used in the design of nutating transmissions since they describe the kinematics of nutating motion. They may also be useful in the design and performance prediction of seals since nutating motions may occur in face seals. High-frequency nutating motions, even of very small amplitude, could conceivably play an important role in the dynamics of seals.

SYMBOLS

M	inertial moment, $N \cdot sec^2/cm^4$ ($lb \cdot sec^2/in.^4$)
r	radius, cm (in.)
T	thickness, cm (in.)
t	time, sec
v	volume, cm^3 ($in.^3$)
X, Y, Z	coordinate system fixed in the nutating body
X_0, Y_0, Z_0	fixed inertial coordinate system
x_1, y_1, z_1	point coordinates in X, Y, Z coordinate system
x_0, y_0, z_0	point coordinates in X_0, Y_0, Z_0 coordinate system
α	nutation angle (between X and X_0, fig. 2), deg
β	projection of φ into the X_0Y_0 plane (fig. 2), $\arcsin(\sin \alpha \cos \omega t/\cos \psi)$, deg
η	harmonic oscillation angle, deg
θ	projection of ψ into the X_0Z_0 plane (fig. 2), $\arcsin(\sin \alpha \sin \omega t/\cos \varphi)$
ξ	angular coordinate for integration purposes

ρ density, $N \cdot sec^2/cm^4$ ($lb \cdot sec^2/in.^4$)

φ angle in XY_0Y plane between X and intersection of XY_0Y and X_0Z_0 planes (fig. 2), arcsin(sin α cos ωt)

ψ angle in XZZ_0 plane between X and intersection of XZZ_0 and X_0Y_0 planes (fig. 2), arcsin(sin α sin ωt)

ω nutation frequency, rad/sec

Subscripts:

h harmonic

I inertial

i inner

o outer

ANALYSIS

Consider a body of arbitrary shape subjected to a nutating motion at a constant frequency ω and nutation angle α. Displacements, velocities, and accelerations of a point $P(x_1, y_1, z_1)$ will be derived relative to a fixed set of inertial coordinates X_0, Y_0, Z_0. The nutation motion will be described by the body fixed coordinate system X, Y, Z, as shown in figure 2. The axis of nutation is the X_0 axis.

The nature of nutating motion is such that the Y_0 axis is always in the XY plane, and the Z_0 axis is always in the XZ plane.

Displacement Equations

$$x_0 = x_1 \cos \alpha - y_1 \sin \varphi \cos \theta + z_1 \cos \beta \sin \psi \tag{1a}$$

$$y_0 = y_1 \cos \varphi + x_1 \sin \alpha \cos \omega t + z_1 \sin \psi \sin \beta \tag{1b}$$

$$z_0 = z_1 \cos \psi - x_1 \sin \alpha \sin \omega t + y_1 \sin \varphi \sin \theta \tag{1c}$$

In terms of α and ωt equations (1) become

$$x_0 = x_1 \cos\alpha - \frac{y_1 \sin 2\alpha \cos\omega t}{2(1 - \sin^2\alpha \cos^2\omega t)^{1/2}} + \frac{z_1 \sin 2\alpha \sin\omega t}{2(1 - \sin^2\alpha \sin^2\omega t)^{1/2}} \tag{2a}$$

$$y_0 = y_1(1 - \sin^2\alpha \cos^2\omega t)^{1/2} + x_1 \sin\alpha \cos\omega t + \frac{z_1 \sin^2\alpha \sin 2\omega t}{2(1 - \sin^2\alpha \sin^2\omega t)^{1/2}} \tag{2b}$$

$$z_0 = z_1(1 - \sin^2\alpha \sin^2\omega t)^{1/2} - x_1 \sin\alpha \sin\omega t + \frac{y_1 \sin^2\alpha \sin 2\omega t}{2(1 - \sin^2\alpha \cos^2\omega t)^{1/2}} \tag{2c}$$

Velocity Equations

$$\dot{x}_0 = \frac{\omega \sin 2\alpha \sin\omega t}{2(1 - \sin^2\alpha \cos^2\omega t)^{3/2}} y_1 + \frac{\omega \sin 2\alpha \cos\omega t}{2(1 - \sin^2\alpha \sin^2\omega t)^{3/2}} z_1 \tag{3a}$$

$$\dot{y}_0 = -(\omega \sin\alpha \sin\omega t)x_1 + \frac{\omega \sin^2\alpha \sin 2\omega t}{2(1 - \sin^2\alpha \cos^2\omega t)^{1/2}} y_1 + \frac{\omega \sin^2\alpha(\cos 2\omega t + \sin^2\alpha \sin^4\omega t)}{(1 - \sin^2\alpha \sin^2\omega t)^{3/2}} z_1 \tag{3b}$$

$$\dot{z}_0 = -(\omega \sin\alpha \cos\omega t)x_1 + \frac{\omega \sin^2\alpha(\cos 2\omega t - \sin^2\alpha \cos^4\omega t)}{(1 - \sin^2\alpha \cos^2\omega t)^{3/2}} y_1 - \frac{\omega \sin^2\alpha \sin 2\omega t}{2(1 - \sin^2\alpha \sin^2\omega t)^{1/2}} z_1 \tag{3c}$$

Acceleration Equations

$$\ddot{x}_0 = \frac{\omega^2 \sin 2\alpha(\cos\omega t - 3\sin^2\alpha \cos\omega t + 2\sin^2\alpha \cos^3\omega t)}{2(1 - \sin^2\alpha \cos^2\omega t)^{5/2}} y_1 + \frac{\omega^2 \sin 2\alpha(-\sin\omega t + 3\sin^2\alpha \sin\omega t - 2\sin^2\alpha \sin^3\omega t)}{2(1 - \sin^2\alpha \sin^2\omega t)^{5/2}} z_1 \tag{4(a)}$$

$$\ddot{y}_0 = -(\omega^2 \sin\alpha \cos\omega t)x_1 + \frac{\omega^2 \sin^2\alpha(\cos 2\omega t - \sin^2\alpha \cos^4\omega t)}{(1 - \sin^2\alpha \cos^2\omega t)^{3/2}} y_1$$

$$+ \frac{\omega^2 \sin^2\alpha(-2 \sin 2\omega t + 2 \sin^2\alpha \sin^3\omega t \cos\omega t + 3 \sin^2\alpha \sin\omega t \cos\omega t - \sin^4\alpha \sin^5\omega t \cos\omega t)}{(1 - \sin^2\alpha \sin^2\omega t)^{5/2}} z_1$$

4(b)

$$\ddot{z}_0 = (\omega^2 \sin\alpha \sin\omega t)x_1 + \frac{\omega^2 \sin^2\alpha(-2 \sin 2\omega t + 2 \sin^2\alpha \cos^3\omega t \sin\omega t + 3 \sin^2\alpha \cos\omega t \sin\omega t - \sin^4\alpha \cos^5\omega t \sin\omega t)}{(1 - \sin^2\alpha \cos^2\omega t)^{5/2}} y_1$$

4(c)

$$- \frac{\omega^2 \sin^2\alpha(\cos 2\omega t + \sin^2\alpha \sin^4\omega t)}{(1 - \sin^2\alpha \sin^2\omega t)^{3/2}} z_1$$

Inertial Moments

The convention used to calculate inertial moments developed by an arbitrary nutating body is shown in figure 3. The moment equations then become

$$M_{x_0} = \int_v (\ddot{z}_0 y_0 - \ddot{y}_0 z_0)\rho \, dv \tag{5a}$$

$$M_{y_0} = \int_v (\ddot{x}_0 z_0 - \ddot{z}_0 x_0)\rho \, dv \tag{5b}$$

$$M_{z_0} = \int_v (\ddot{y}_0 x_0 - \ddot{x}_0 y_0)\rho \, dv \tag{5c}$$

RESULTS AND DISCUSSION

For purposes of calculation, conditions approximating those of the transmission design developed in reference 2 were used. This design (ref. 2) was for an input power 1.86 megawatts (2500 hp), and a 28:1 speed reduction. The nutating plate pitch diameter was 91.5 centimeters (36 in.). A point P_1 was chosen on the pitch diameter of the nutating plate (see fig. 4). Values of $\omega = 650$ radians per second and $\alpha = 9^\circ$ were used for calculating displacements, velocities, and accelerations.

Displacements, Velocities, and Accelerations

Figure 4 shows the generalized motion of P_1 projected into both the X_0-Z_0 and X_0-Y_0 planes. The displacements projected into the X_0-Z_0 plane is a lemniscate-like curve (fig. 4(a)). In the X_0-Y_0 plane the displacement describes an arc (fig. 4(b)). Locations of P_1 at various values of ωt are shown in figures 4(a) and (b). In the case of the nutating plate transmission, these displacement curves are necessary for the proper design of the engagement between the nutator and the drive and reaction plates.

Figures 4(c) and (d) show the velocities of point P_1 projected into the X_0-Z_0 and X_0Y_0 planes, respectively. The velocities $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ can be added vectorially to get the absolute velocity of P_1 at any point in the nutation cycle. This is necessary for the calculation of the relative velocity of engagement between the nutator and the drive and reaction plates. This, in turn, is required for the calculation of elastohydrodynamic film thicknesses, traction forces, temperature rise, power loss, and other operating parameters.

Accelerations of point P_1 in the X_0-Z_0 and X_0-Y_0 planes are shown, respectively, in figures 4(e) and (f). These must be integrated over the volume of the nutator at a specific value of ωt using equations (5) to obtain the moments about the inertial axes. These are then added vectorially to get the total inertial moment that must be reacted by the bearing system that supports the nutator.

Equations (2) to (5) can also be used to analyze squeeze film and distortion effects in hydrodynamic thrust bearings and face seals subjected to high-frequency nutations. Small-amplitude, high-frequency nutations develop in face seal runners. The extremely small film thicknesses in face seals coupled with the typically fragile structure of face seal runners lead one to believe that the inertial forces that result from nutating motions may significantly affect the performance of the seal.

Inertial Moments

Expressions for the inertial moments M_{x_0}, M_{y_0}, and M_{z_0} acting about the inertial coordinate axes X_0, Y_0, and Z_0, respectively, are derived in appendix A for a nutating body of arbitrary shape. Equations (A10) to (A12) are applicable at $\omega t = 0$. Equations (A19) to (A21) are applicable at $\omega t = \pi/2$. Because the inertial moment must obviously be a rotating moment of constant magnitude for a constant frequency of nutation, its components can be calculated at any convenient value of ωt and summed vectorially.

The equations were derived for two values of ωt as a check on their correctness. A comparison of equations (A10) to (A12) with equations (A19) to (A21) shows their equivalence. In other words, for a circular disk of uniform thickness

$$M_{z_0}\Big|_{\omega t=0} = M_{y_0}\Big|_{\omega t=\pi/2}$$

$$M_{y_0}\Big|_{\omega t=0} = M_{z_0}\Big|_{\omega t=\pi/2}$$

$$M_{x_0}\Big|_{\omega t=0} = M_{x_0}\Big|_{\omega t=\pi/2}$$

The derivation and calculation of inertial moments for a thick nutating plate are given in appendix B. As expected, M_{x_0}, the moment about the rotational axis (in the X_0-Z_0 plane) acting at the nutation frequency, is identically zero for a constant ω. At $\omega t = 0$ the inertial moment acts about the Z_0 axis, and at $\omega t = \pi/2$ it acts about the Y_0 axis. Its rotational character is apparent from these relations.

The magnitude of the inertial moment acting on the nutating plate used in the transmission design in reference 2 (the plate dimensions are given in appendix B) was calculated to be 7.51×10^7 newton-centimeters (6.66×10^6 lb-in.). This exceeds by a factor of 28 the moment load imposed by the static axial reaction loads from the teeth of both the drive and reaction plates (ref. 2). It is quite obvious from these results that nutating plate drives are unusable at any operating conditions other than very low speeds. Even at very low speeds the magnitude of the reaction load and inertial moments on the nutating plate support bearing is likely to impose a severe bearing life problem.

It is interesting to note from equation (B7) that, unless the plate thickness T is substantial relative to its radius r, the contribution of its thickness (as given by the first term of eq. (B7)) to the inertial moment is insignificant. For the plate dimensions used in the calculations, the first term in equation (B7) increased the inertial moment by approximately 0.3 percent. Therefore, for relatively thin plates, the first term in equation (B7) can be neglected.

Comparison of Inertial Moments Due to Nutating and Simple Harmonic Motion

Appendix C contains a derivation of the inertial moment developed in a body subject to harmonic oscillation. Equation (C7) gives the maximum inertial moment due to harmonic motion, which is developed in a circular plate. It can be compared with equations (B4) and (B7). For a thin plate equation (C8) is compared with the second term in equations (B4) and (B7). Notice that the harmonic and nutating moment equations are identical except that equation (C8) contains the factor α (expressed in radians), while the nutating body equations contain the factor $\tan \alpha(\sin^2\alpha + 1)$.

A comparison of the magnitude of these terms as a function of α is given in table I, expressed as a percent error if equation (C7) is used, rather than the exact equation. The error is small at small values of α, but becomes appreciable as α increases. Thus it appears that the approximate inertial moment equation assuming simple harmonic oscillation can be used with good accuracy at small values of α. At larger values of α the exact nutation equations should be used.

An additional value of the displacement, velocity, and acceleration equations (eqs. (2) to (4)) may lie in their use for determining local distortions in nutating bodies. Different points in a nutating body experience varying accelerations at given times. This condition will result in varying stresses and local distortions. These may be significant in a machine element such as a face seal whose performance is a strong function of its geometry during operation.

CONCLUSIONS

The equations developed for the displacement, velocity, and acceleration of a point in a nutating body can be used for the design of nutating plate transmissions. The equations developed for the calculation of the inertial moment of a nutating body of arbitrary shape, when applied to the plate of a previously designed nutating plate drive, indicate that this device is severely speed limited. The support bearing load due to the inertial moment was calculated to be 28 times as great as the load due to reaction forces.

At small values of the nutation angle, the approximate formula for the inertial moment, which is based on an assumption of simple harmonic oscillation, can be used with good accuracy. At nutation angles of 10° and greater the error becomes appreciable, and the exact formula for nutating motion should be used.

The equations for displacement, velocity, and acceleration should be useful in assessing squeeze film and local distortion effects in thrust bearings, and particularly face seals, in which nutating motions occur.

Lewis Research Center,
National Aeronautics and Space Administration,
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501-24.

APPENDIX A

DERIVATION OF INERTIAL MOMENTS FOR A BODY OF ARBITRARY SHAPE AT $\omega t = 0$ AND $\omega t = \pi/2$

$$M_{y_0} = \int_v (\ddot{x}_0 z_0 - \ddot{z}_0 x_0)\rho \, dv \tag{A1}$$

$$M_{z_0} = \int_v (\ddot{y}_0 x_0 - \ddot{x}_0 y_0)\rho \, dv \tag{A2}$$

$$M_{x_0} = \int_v (\ddot{z}_0 y_0 - \ddot{y}_0 z_0)\rho \, dv \tag{A3}$$

At $\omega t = 0$

$$x_0 = (\cos\,\alpha)x_1 - (\sin\,\alpha)y_1 \tag{A4}$$

$$y_0 = (\cos\,\alpha)y_1 + (\sin\,\alpha)x_1 \tag{A5}$$

$$z_0 = z_1 \tag{A6}$$

$$\ddot{x}_0 = \frac{\omega^2 \sin\,\alpha}{\cos^2\alpha} y_1 \tag{A7}$$

$$\ddot{y}_0 = -(\omega^2 \sin\,\alpha)x_1 + \left(\frac{\omega^2 \sin^2\alpha}{\cos\,\alpha}\right) y_1 \tag{A8}$$

$$\ddot{z}_0 = -(\omega^2 \sin^2\alpha)z_1 \tag{A9}$$

$$\ddot{x}_0 z_0 = \frac{\omega^2 \sin\,\alpha}{\cos^2\alpha} y_1 z_1$$

$$\ddot{z}_0 x_0 = -(\omega^2 \sin^2\alpha \cos\,\alpha)x_1 z_1 + (\omega^2 \sin^3\alpha)y_1 z_1$$

$$\ddot{x}_0 z_0 - \ddot{z}_0 x_0 = \omega^2 \sin\alpha\left[\left(\frac{1 - \sin^2\alpha\cos^2\alpha}{\cos^2\alpha}\right)y_1 z_1 + (\sin\alpha\cos\alpha)x_1 z_1\right]$$

$$M_{y_0} = \int_v \omega^2 \sin\alpha\left[\left(\frac{1 - \sin^2\alpha\cos^2\alpha}{\cos^2\alpha}\right)y_1 z_1 + (\sin\alpha\cos\alpha)x_1 z_1\right]\rho\, dv \tag{A10}$$

$$\ddot{y}_0 x_0 = -(\omega^2 \sin\alpha\cos\alpha)x_1^2 + 2(\omega^2 \sin^2\alpha)x_1 y_1 - \omega^2 \frac{\sin^3\alpha}{\cos\alpha} y_1^2$$

$$y_0 \ddot{x}_0 = \omega^2 \frac{\sin\alpha}{\cos\alpha} y_1^2 + \omega^2 \frac{\sin^2\alpha}{\cos^2\alpha} x_1 y_1$$

$$\ddot{y}_0 x_0 - y_0 \ddot{x}_0 = -\left[\omega^2 \tan\alpha(1 + \sin^2\alpha)y_1^2 + \omega^2 \tan^2\alpha(1 - 2\cos^2\alpha)x_1 y_1 + \omega^2(\sin\alpha\cos\alpha)x_1^2\right]$$

$$M_{z_0} = -\int_v \omega^2 \left[(\sin\alpha\cos\alpha)x_1^2 + \tan\alpha(\sin^2\alpha + 1)y_1^2 + \tan^2\alpha(1 - 2\cos^2\alpha)x_1 y_1\right]\rho\, dv \tag{A11}$$

$$\ddot{z}_0 y_0 = -(\omega^2 \sin^2\alpha\cos\alpha)y_1 z_1 - (\omega^2 \sin^3\alpha)x_1 z_1$$

$$\ddot{y}_0 z_0 = -(\omega^2 \sin\alpha)x_1 z_1 + \left(\frac{\omega^2 \sin^2\alpha}{\cos\alpha}\right) y_1 z_1$$

$$M_{x_0} = \int_v \omega^2 \sin\alpha\left\{(\cos^2\alpha)x_1 z_1 - \left[\tan\alpha(\cos^2\alpha + 1)\right] y_1 z_1\right\}\rho\, dv \tag{A12}$$

At $\omega t = \pi/2$

$$x_0 = x_1 \cos\alpha + z_1 \sin\alpha \tag{A13}$$

$$y_0 = y_1 \tag{A14}$$

$$z_0 = z_1 \cos\alpha - x_1 \sin\alpha \tag{A15}$$

$$\ddot{x}_0 = -\left(\frac{\omega^2 \sin\alpha}{\cos^2\alpha}\right) z_1 \tag{A16}$$

$$\ddot{y}_0 = -(\omega^2 \sin^2\alpha) y_1 \tag{A17}$$

$$z_0 = (\omega^2 \sin\alpha) x_1 + \left(\frac{\omega^2 \sin^2\alpha}{\cos\alpha}\right) z_1 \tag{A18}$$

From equations (A1), (A13), (A15), (A16), and (A18)

$$M_{y_0} = -\int_v \omega^2 \left[(\sin\alpha \cos\alpha) x_1^2 + \tan\alpha(\sin^2\alpha + 1) z_1^2 - \tan^2\alpha(1 - 2\cos^2\alpha) x_1 z_1\right] \rho \, dv \tag{A19}$$

From equations (A2), (A13), (A14), (A16), and (A17)

$$M_{z_0} = \int_v \omega^2 \sin\alpha \left[(\sin\alpha \cos\alpha) x_1 y_1 - \left(\frac{1 - \sin^2\alpha \cos^2\alpha}{\cos^2\alpha}\right) y_1 z_1\right] \rho \, dv \tag{A20}$$

From equations (A3), (A14), (A15), (A17), and (A18)

$$M_{x_0} = \int_v \omega^2 \sin\alpha \left\{(\cos^2\alpha) x_1 z_1 + \left[\tan\alpha(\cos^2\alpha + 1)\right] y_1 z_1\right\} \rho \, dv \tag{A21}$$

APPENDIX B

DERIVATION AND CALCULATION OF INERTIAL MOMENTS FOR A THICK NUTATING PLATE

From figure 5

$$x_1 = x_1$$

$$y_1 = r \sin \xi$$

$$z_1 = r \cos \xi$$

$$dv = r\, dr\, d\xi\, dx_1$$

From equation (A10) and at $\omega t = 0$,

$$M_{y_0} = \omega^2 \rho \sin \alpha \left[\frac{1 - \sin^2\alpha \cos^2\alpha}{\cos^2\alpha}\right] \int_0^{2\pi} \int_{r_i}^{r_o} \int_{-T/2}^{T/2} r^3 \sin \xi \cos \xi\, dx_1\, dr\, d\xi$$

$$+ \omega^2 \rho \sin^2\alpha \cos \alpha \int_0^{2\pi} \int_{r_i}^{r_o} \int_{-T/2}^{T/2} x_1 r^2 \cos \xi\, dx_1\, dr\, d\xi \qquad \text{(B1)}$$

Both integrals vanish so that

$$M_{y_0} = 0 \qquad \text{(B2)}$$

From equation (A11) and at $\omega t = 0$

$$M_{z_0} = -\omega^2 \rho \sin \alpha \cos \alpha \int_0^{2\pi} \int_{r_i}^{r_o} \int_{-T/2}^{T/2} x_1^2 r\, dx_1\, dr\, d\xi$$

$$-\omega^2 \rho \tan \alpha (\sin^2\alpha + 1) \int_0^{2\pi} \int_{r_i}^{r_o} \int_{-T/2}^{T/2} r^3 \sin^2\xi\, dx_1\, dr\, d\xi$$

$$-\omega^2 \rho \tan^2\alpha (1 - 2 \cos^2\alpha) \int_0^{2\pi} \int_{r_i}^{r_o} \int_{-T/2}^{T/2} x_1 r^2 \sin \xi\, dx_1\, dr\, d\xi \qquad \text{(B3)}$$

$$M_{z_0} = -\pi\omega^2\rho T \sin\alpha \cos\alpha \left[\frac{T^2\left(r_o^2 - r_i^2\right)}{12}\right] - \pi\omega^2\rho T \tan\alpha(\sin^2\alpha + 1)\left[\frac{r_o^4 - r_i^4}{4}\right] \quad \text{(B4)}$$

From equation (A12) at $\omega t = 0$

$$M_{x_0} = \omega^2\rho \sin\alpha \cos^2\alpha \int_0^{2\pi}\int_{r_i}^{r_o}\int_{-T/2}^{T/2} x_1 r^2 \cos\xi \, dx_1 \, dr \, d\xi$$

$$-\omega^2\rho \tan\alpha(\cos^2\alpha + 1)\int_0^{2\pi}\int_{r_i}^{r_o}\int_{-T/2}^{T/2} r^3 \sin\xi \cos\xi \, dx_1 \, dr \, d\xi \quad \text{(B5)}$$

Both integrals vanish, so that

$$M_{x_0} = 0 \quad \text{(B6)}$$

as expected if ω is constant. From equations (A19) to (A21), it can be shown that, at $\omega t = \pi/2$,

$$M_{y_0} = -\pi\omega^2\rho T \sin\alpha \cos\alpha \left[\frac{T^2\left(r_o^2 - r_i^2\right)}{12}\right] - \pi\omega^2\rho T \tan\alpha(\sin^2\alpha + 1)\left[\frac{r_o^4 - r_i^4}{4}\right] \quad \text{(B7)}$$

$$M_{z_0} = 0 \quad \text{(B8)}$$

$$M_{x_0} = 0 \quad \text{(B9)}$$

As expected

$$M_{z_0}\Big|_{\omega t=0} = M_{y_0}\Big|_{\omega t=\pi/2}$$

$$M_{y_0}\Big|_{\omega t=0} = M_{z_0}\Big|_{\omega t=\pi/2}$$

$$M_{x_0}\Big|_{\omega t=0} = M_{x_0}\Big|_{\omega t=\pi/2} \equiv 0$$

At any value of ωt the inertial moment M_I will be

$$M_I = \sqrt{M_{y_0}^2 + M_{z_0}^2} \qquad \text{(B10)}$$

From these results it is clear that the inertial moment generated by the nutating plate acts about the axis that is in the Y_0Z_0 plane and rotates with the nutation frequency ω.

For the specific plate used in the transmission design in reference 2: $r_o = 45.7$ centimeter (18 in.), $r_i = 20.3$ centimeter (8 in.), $T = 4.65$ centimeter (1.83 in.), $\omega = 587$ radians per second, $\alpha = 10^{\circ}$, and $\rho = 7.9\times10^{-5}$ $N\cdot sec^2/cm^4$ (7.34×10^{-4} $lb\cdot sec^2/in.^4$).

With these values the inertial moment M_I becomes (from eqs. (B2), (B4), and (B10)) 7.51×10^7 newton centimeters (6.66×10^6 lb·in.).

APPENDIX C

INERTIAL MOMENT FOR A HARMONICALLY OSCILLATING PLATE

From figure 5, if the plate executes an oscillation of amplitude 2α at frequency ω about the Z_0 or Z axis

$$\eta = \alpha \cos \omega t \tag{C1}$$

$$\dot{\eta} = -\alpha\omega \sin \omega t \tag{C2}$$

$$\ddot{\eta} = -\alpha\omega^2 \cos \omega t \tag{C3}$$

The inertial moment due to harmonic motion will then be

$$M_h = I_{ZZ}\ddot{\eta} \tag{C4}$$

The maximum moment occurs at $\omega t = 0$ and π where

$$M_{h_{max}} = \mp I_{ZZ}\alpha\omega^2 \tag{C5}$$

$$I_{ZZ} = \frac{\pi\rho T}{4}\left(r_o^4 - r_i^4\right) + \frac{\pi\rho T}{12}\left[T^2\left(r_o^2 - r_i^2\right)\right] \tag{C6}$$

$$M_{h_{max}} = \mp \frac{\pi\rho T\alpha\omega^2}{4}\left[\left(r_o^4 - r_i^4\right) + \frac{T^2}{3}\left(r_o^2 - r_i^2\right)\right] \tag{C7}$$

where α is given in radians.

For a thin plate

$$M_{h_{max}} = \mp \frac{\pi\rho T\alpha\omega^2}{4}\left(r_o^4 - r_i^4\right) \tag{C8}$$

Again, for the specific plate used in the transmission design in reference 2 (dimensions and operating conditions given in appendix B), $M_{h_{max}} = 7.21\times10^7$ newton-centimeters (6.39×10^6 lb·in.).

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TABLE I. - COMPARISON OF INERTIAL MOMENTS IN THIN NUTATING PLATE USING EXACT NUTATING MOTION ANALYSIS AND SIMPLE HARMONIC OSCILLATION

Nutation angle, α, deg	Percent error using harmonic oscillation, $\frac{M_I - M_{h_{max}}}{M_I} 100$
1	0.1
5	1
10	3.9
15	8.4
20	14
30	27

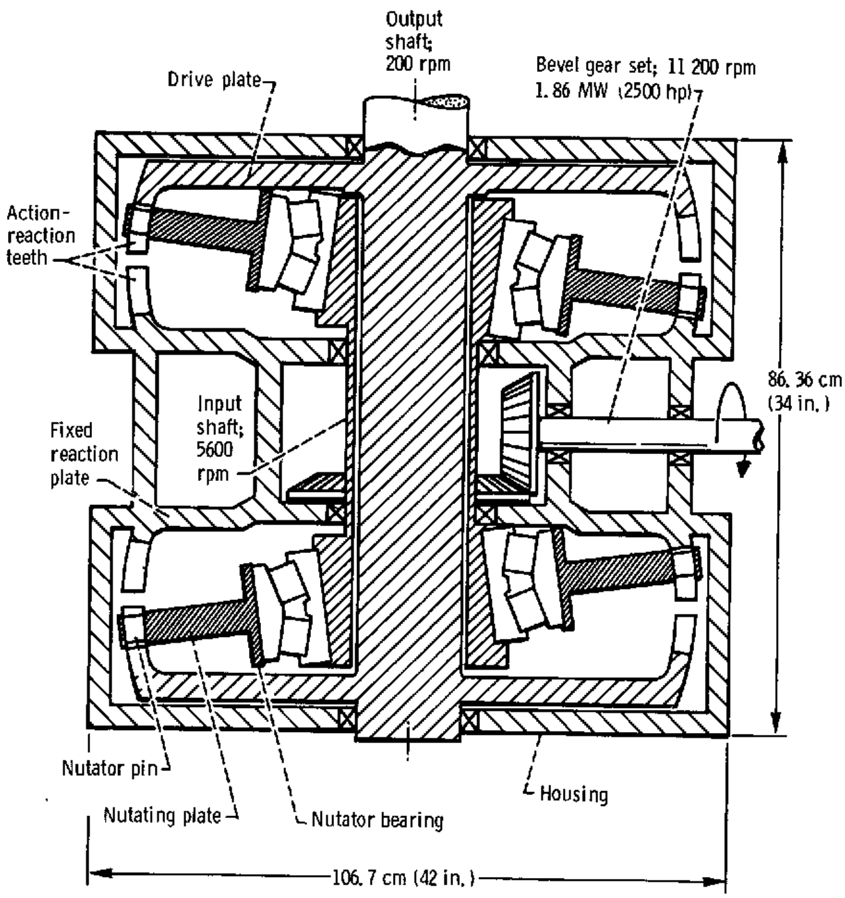
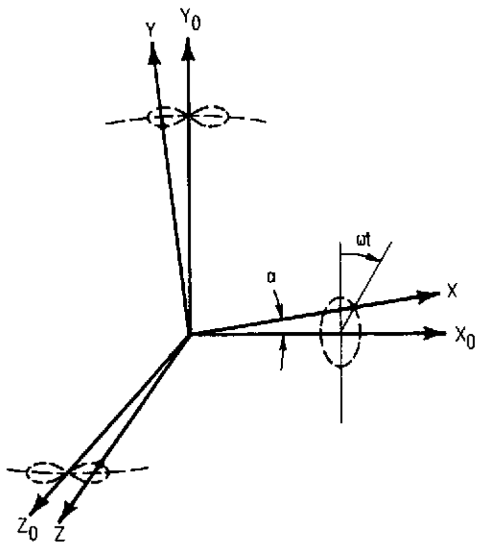
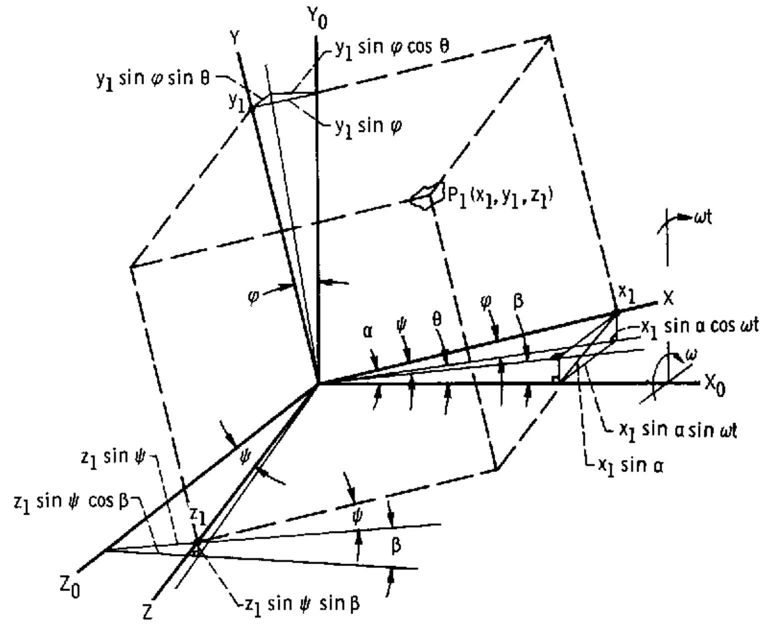


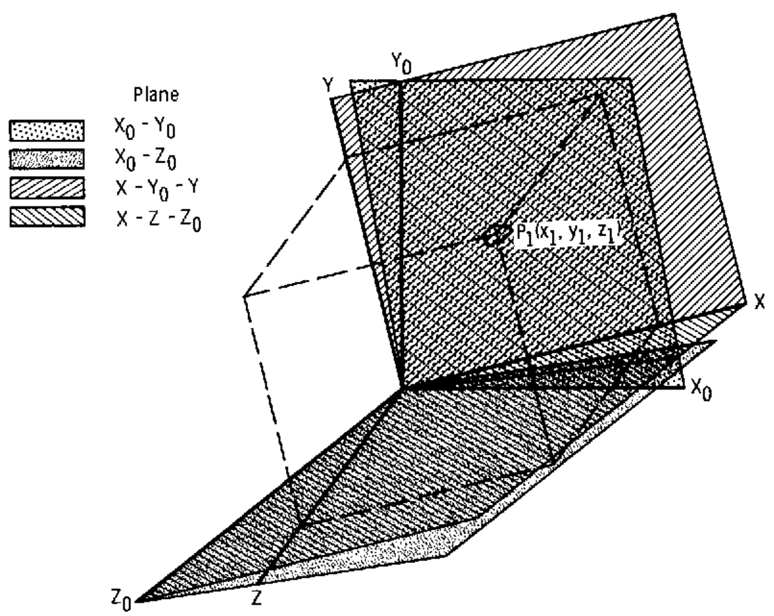
Figure 1. - Sketch showing physical size and arrangement of split nutating drive main rotor gearbox (ref. 2).



(a) Nutating motion described by points in the body.



(b) Angles and distances required for transferring from XYZ to $X_0Y_0Z_0$ coordinate system.



(c) Planar relationships among axes in XYZ and $X_0Y_0Z_0$ coordinate systems.

Figure 2. - Coordinate systems.

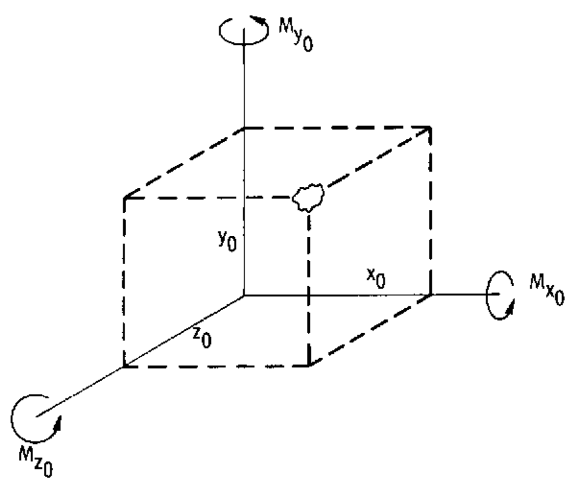
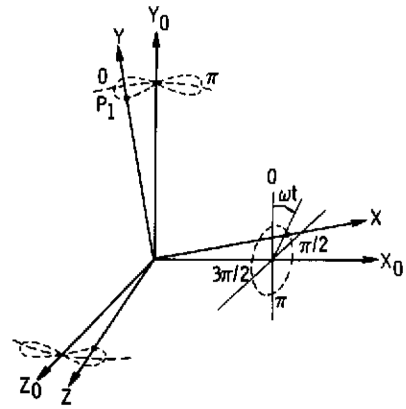
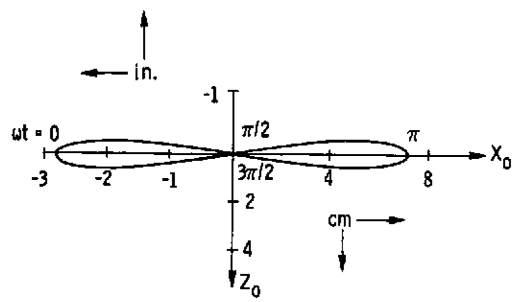
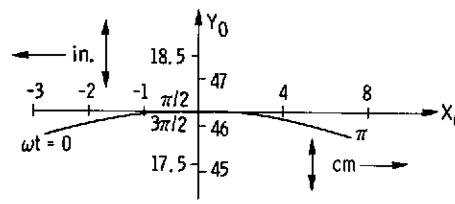


Figure 3. - Convention used to calculate inertial moments.

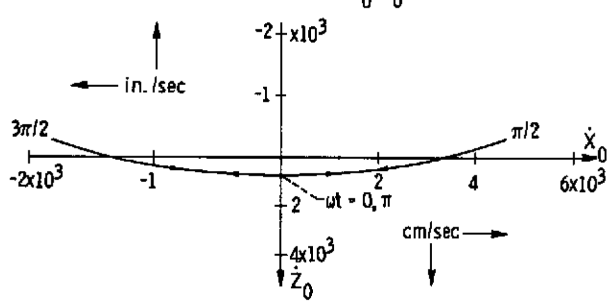




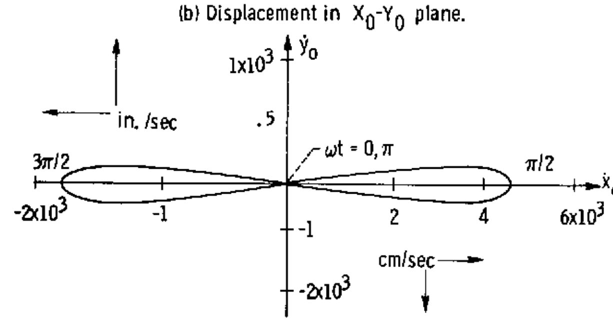
(a) Displacement in X_0-Z_0 plane.



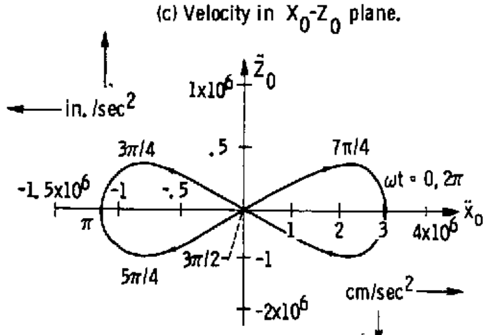
(b) Displacement in X_0-Y_0 plane.



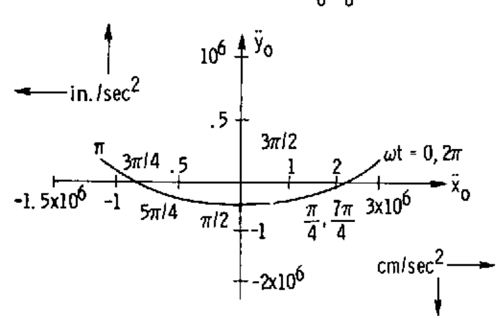
(c) Velocity in X_0-Z_0 plane.



(d) Velocity in X_0-Y_0 plane.



(e) Acceleration in X_0-Z_0 plane.



(f) Acceleration in X_0-Y_0 plane.

Figure 4. - Generalized motion of a point P_1 on nutating body. P_1 coordinates, (0, 45.7, 0) cm (or (0, 18, 0) in.); ω = 650 radians per second; $\alpha = 9^0$.

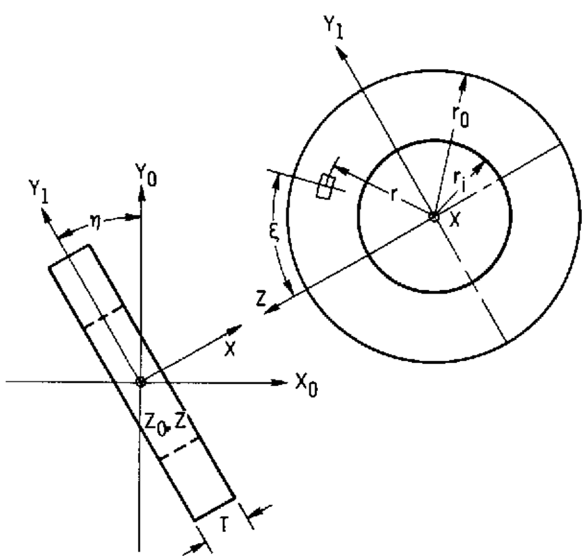


Figure 5. - Coordinate systems and symbols used to calculate nutating plate inertial moments. Plate position shown is at $\omega t = 0$.