DESIGN PROCEDURE FOR SATISFYING

TIME DOMAIN BOUNDS FOR

NONMINIMUM-PHASE SYSTEMS

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Abstract--This paper presents two design techniques applicable to nonminimum-phase systems. Both are designed to handle plants with one right-half-plane zero which may vary and any other variation of the plant parameters within known limits. The specifications that must be designed to are given as a set of step response bounds in the time domain. A completed design will yield responses that stay within the time domain bounds at all times and utilize the entire region of allowed variation.

The first method is an analytic one which finds an equivalent set of minimum-phase step response bounds. Once these are found, current design techniques may be used to complete the design. This method can be extended easily to design for multiple right-half-plane zeros, but the actual design will be very complicated for this problem. When there is variation of the right-half-plane zero, this method requires many designs before the best one is found. For these reasons, this method is not very useful.

The second method makes a direct transfer of the time domain bounds to the frequency domain. From these new boundaries and other resulting specifications, a worst case zero is found that will yield a design that is satisfactory as the zero varies. This method is thoroughly derived and explained for a plant with one right-half-plane zero and the plant does not depend on the position of the zero in any other way. These design techniques are the first contributions to the non-minimum-phase problem.

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INTRODUCTION

1.1   Problem Statement

The purpose of this paper is to present two methods
of design which are applicable to nonminimum-phase prob-
lems. Both methods are thoroughly derived for one right-
half-plane zero and the extensions are indicated in the
first method to handle multiple right-half-plane zeros.
One method of design is analytical whereas the other is
best suited to digital computer implementation. Both
methods do permit insight into the means of design. The
program utilized in the computer implementation of the
second example is in the appendix.

The problem specifications are given in general as
a set of time domain boundaries within which the final
step response must remain at all times. There may be
ignorance in the right-half-plane zero(s) and in any
other parameters the plant may contain. The only thing
known about the parameters are the limits that they are
permitted to vary within. The rate of parameter varia-
tion is assumed to be slower than the system response
time so time dependence of the parameters may be neg-
lected. A successful design will yield satisfactory step
responses for all possible sets of plant parameters.

The step response boundaries are unique because they restrict the initial undershoot that must occur in a nonminimum-phase system. The optimum design will permit step responses that utilize the entire allowed region of variation, both negative and positive. In this way the compensation is not more than it needs to be in complexity and bandwidth.

The feedback structure used in the design is a two degree of freedom structure shown in figure 1.1. A two degree of freedom structure is used because both sensitivity and minimum bandwidth must be designed for. The feedback structure is required to meet the sensitivity constraints, i.e., the plant variation cannot result in more variation in the step response then allowed by the step response boundaries. The prefilter F will adjust the bandwidth to the minimum allowed so as to make the system as immune to noise as possible.

R(s) ─────[F(s)]──────+ ⊙ ──────[G(s)]──[P(s)]──────── C(s)
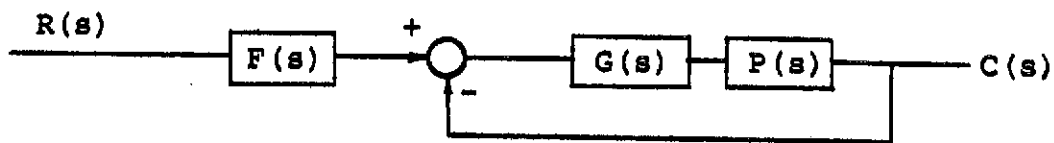
$$L(s) = G(s)P(s)$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{F(s)L(s)}{1 + L(s)}$$

Figure 1.1 Closed Loop Structure

1.2 History of the Problem

Right-half-plane zeros occur in many circumstances. Some examples are where there are multiple channels over which information can be passed, lattice networks, transmission lines, piping-flow delays, transportation lags, semiconductor diffusion [1], heat exchangers, mercury thermometers [2], and in aircraft control [3]. The presence of a right-half-plane zero greatly affects the stability conditions of a design. The loop transmission cannot be allowed to decrease too quickly or the additional phase lag will be so great as to cause conditional or complete instability. In a stable nonminimum-phase system, there is an upper limit to the crossover frequency [4]. A nonminimum-phase system is one that has a zero in the right half of the s-plane. Nonminimum-phase systems are slow in response because they initially go in the wrong direction [5]. A magnitude plot will completely determine the transfer function of a minimum-phase system, but both magnitude and phase plots are required for a nonminimum-phase system.

These considerations make the design for a nonminimum-phase plant a unique one, and from the areas where the problem arises it is an important one. Inspite of this, there is no design technique available for the solution of this problem.

1.3   Method of Approach

There is a design procedure presented in a paper written by Horowitz and Sidi which can formulate a design for a plant with parameter variations from a set of minimum-phase time domain bounds [6]. The first method presented here solves for the set of minimum-phase bounds equivalent to the given set of nonminimum-phase bounds. That is, a design meeting the minimum-phase bounds will also meet the nonminimum-phase ones. The minimum-phase bounds are found by solving the differential equation relating them to the nonminimum-phase bounds. Because the goal of this method is to yield an equivalent set of minimum-phase time domain specifications, this method is called the "time domain method." The remainder of the design is completed by the above procedure.

The second method approximates the transfer function by a third order function and finds a set of boundaries on the allowed variation of the magnitude of the transfer function in the frequency domain. This is done by taking the composite of the Bode plots of many transfer functions which have satisfactory step responses. It was found that a design for the smallest right-half-plane zero and the lowest set of specifications found above will be satisfactory as the right-half-plane zero varies. Since the design is carried out for a fixed position of the zero, a magnitude plot will be sufficient

to describe the transfer function. Thus, once the transformation to the frequency domain is carried out, the Horowitz and Sidi design technique can be used to complete the design.

CHAPTER II

TIME DOMAIN METHOD

2.1 Motivation of Method

Horowitz and Sidi have written a paper [6] which presents a design method utilizing a set of minimum-phase step response boundaries. One design approach for a set of nonminimum-phase time domain boundaries is to start by finding an equivalent set of minimum-phase tolerances and applying the Horowitz and Sidi method to these tolerances to complete the design. This transformation can be made by solving the differential equation relating the nonminimum-phase (nmp) transfer function and its minimum-phase (mp) component. This method will be known as "the time domain method" because its goal is to yield an equivalent set of mp time domain specifications and is not concerned with the techniques used to complete the design once the equivalent bounds are found.

2.2 Technique

If there is only one right-half-plane (RHP) zero, the transfer function may be written as

$$T(s) = T_m(s)(1 - \frac{s}{a})$$

where $T_m(s)$ is the minimum-phase portion of $T(s)$, and "a" is the location of the RHP zero. If the input is

$R(s) = \frac{1}{s}$, a unit step, then the output response $C(s)$, assuming zero initial conditions, can be written as

$$C(s) = R(s)T(s) = \frac{T(s)}{s} = \frac{T_m(s) - T_m(s)(\frac{s}{a})}{s}$$

$$C(t) = C_m(t) - \frac{1}{a}\dot{C}_m(t)$$

where the roles of $R(s)$, $C(s)$, and $T(s)$ are clarified in figure 2.1 and

$$C_m(t) = \mathcal{L}^{-1}\left[\frac{T_m(s)}{s}\right].$$

This means:

$$C_m(s) = \frac{C(s) - \frac{1}{a}C_m(0)}{1 - \frac{s}{a}} \qquad (2.1)$$

for the general case when there are initial conditions.

This equation points out that an initial condition is required to solve for the mp set of step response bounds. A point is required where both the mp and nmp responses are known or the same so the initial condition can be solved for. This is a point where the response has settled to a constant. Start at this point and work backwards solving for the initial conditions. An initial condition is required at every time when the nmp boundary is discontinuous.

2.3 Example of Design Method

To demonstrate the technique the equivalent mp boundaries for the nmp ones in figure 2.1 will be found. The results are shown dotted in figure 2.1, the mp bounds. From t_2 to t_4, the upper bound is constant at $1 + \alpha$. The equivalent mp boundary can be found by solving equation 2.1. The initial condition at t_2 can be solved for by knowing that at t_4 both the mp and nmp boundaries must equal one. From equation 2.1:

$$C_m(s) = \frac{\frac{1+\alpha}{s} - \frac{1}{a} C_m(0)}{1 - \frac{s}{a}}$$
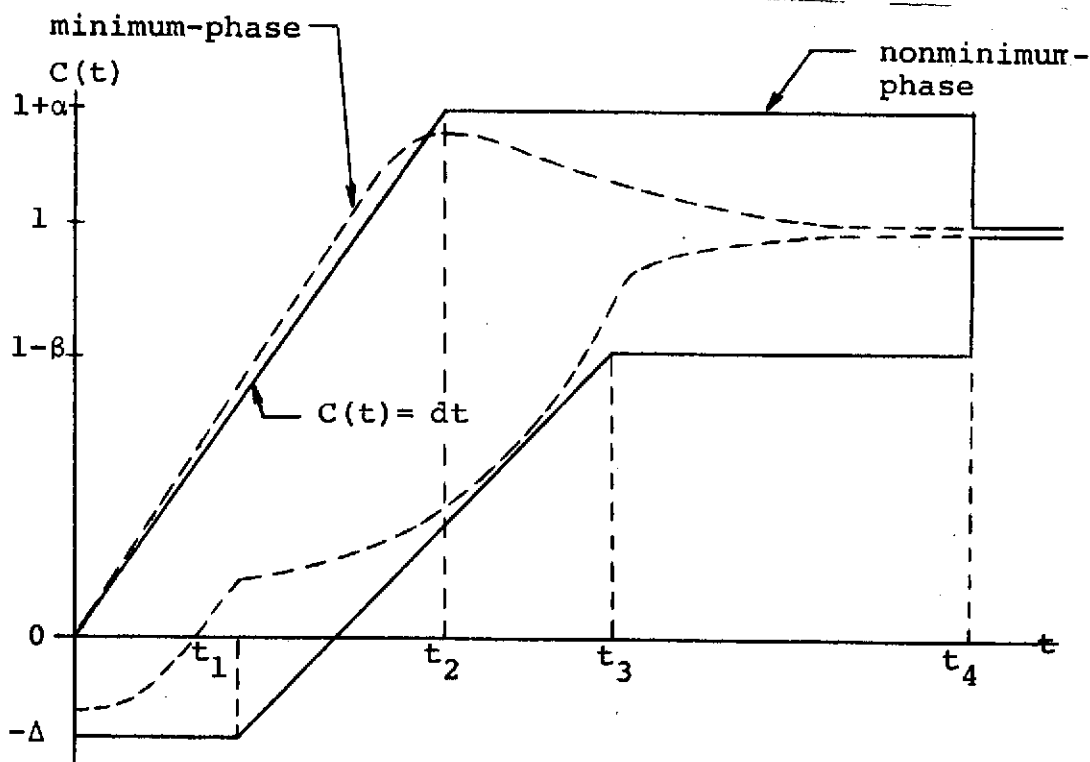
Figure 2.1 Hypothetical set of nonminimum-phase boundaries and equivalent set of minimum-phase bounds.

which yields

$$C_m(t) = 1 + \alpha - (1 + \alpha - C_m(t_2))e^{a(t-t_2)}$$

To solve for $C_m(t_2)$, the initial condition, one can solve the equation at a point where $C_m(t)$ is known, t_4.

$$C_m(t_4) = 1 = 1 + \alpha - (1 + \alpha - C_m(t_2))e^{(t_4 - t_2)a}$$

which yields

$$C_m(t_2) = 1 + \alpha - \alpha e^{-a(t_4 - t_2)}$$

Thus the value at t_2 is known and can be used to solve for the value at $t = 0$ and thus specify the entire upper bound. For this region:

$$C_m(s) = \frac{\dfrac{d}{s^2} - \dfrac{C_m(0)}{a}}{1 - \dfrac{s}{a}}$$

which yields

$$C_m(t) = \frac{d}{a} + dt - (\frac{d}{a} - C_m(0))e^{at}$$

$$C_m(t_2) = 1 + \alpha - \alpha e^{-a(t_4 - t_2)} = \frac{d}{a} + dt_2 - (\frac{d}{a} - C_m(0))e^{at_2}$$

$$C_m(0) = \left[1 + \alpha - \alpha e^{-a(t_4 - t_2)} - (\frac{d}{a} + dt_2) \right]e^{-at_2} + \frac{d}{a}$$

Now the entire equivalent mp upper boundary is known.

The same method is utilized to find the mp lower bound. Starting at t_4 as before:

$$C_m(s) = \frac{\frac{1-\beta}{s} - \frac{C_m(0)}{a}}{1 - \frac{s}{a}}$$

$$C_m(t) = 1 - \beta - (1 - \beta - C_m(0))e^{a(t-t_3)}$$

$$C_m(t_4) = 1 = 1 - \beta - (1 - \beta - C_m(t_3))e^{a(t_4-t_3)}$$

$$C_m(t_3) = 1 - \beta + \beta e^{-a(t_4-t_3)}$$

now for $t_1 < t < t_3$

$$C(t) = dt - \Delta$$

$$C_m(s) = \frac{\frac{d}{s^2} - \frac{\Delta}{s} - \frac{C_m(0)}{a}}{1 - \frac{s}{a}}$$

$$C_m(t) = \frac{d}{a} - \Delta + dt + [\Delta - \frac{d}{a} + C_m(t_1)]e^{a(t-t_1)}$$

$$C_m(t_3) = 1 - \beta + \beta e^{-a(t_4-t_3)}$$

$$= \frac{d}{a} - \Delta + dt_3 + [\Delta - \frac{d}{a} + C_m(t_1)]e^{a(t_3-t_1)}$$

$$C_m(t_1) = \frac{d}{a} - \Delta + \left[\Delta - \frac{d}{a} + \beta e^{-a(t_4-t_3)}\right]e^{-a(t_3-t_1)}$$

for the boundary from $0 < t < t_1$

$$C_m(s) = \frac{-\frac{\Delta}{s} - \frac{C_m(0)}{a}}{1 - \frac{s}{a}}$$

$$C_m(t) = -\Delta + (\Delta + C_m(0))e^{at}$$

$$C_m(t_1) = \frac{d}{a} - \Delta + \left[\Delta - \frac{d}{a} + \beta e^{-a(t_4-t_3)}\right]e^{-a(t_3-t_1)}$$

$$= -\Delta + (\Delta + C_m(0))e^{at_1}$$

$$C_m(0) = -\Delta + \left[\frac{d}{a} + \left[\Delta - \frac{d}{a} + \beta e^{-a(t_4-t_3)}\right]e^{-a(t_3-t_1)}\right]e^{-at_1}$$

Each of these equations, with the initial conditions substituted into their proper place, yield the complete set of mp boundaries shown in figure 2.1. The Horowitz-Sidi technique could now be applied to these boundaries to complete a design.

Because the problem is a nonminimum-phase one, designing to this set of bounds is not sufficient to insure a satisfactory response. A set of frequency based magnitude specifications alone is not enough to control the response. From the equation describing the response due to one RHP zero, $C(t) = C_m(t) - \frac{1}{a}\dot{C}_m(t)$, it is clear there must also be bounds on the derivative of the minimum-phase response to insure satisfactory nmp responses. If both $C_m(t)$ and $\dot{C}_m(t)$ fall within their bounds, $C(t)$ will also satisfy its specifications. This second set of boundaries is easily derived from the boundaries already found. It is found by solving this equation where all variables are known:

$$\dot{C}_m(t) = a[C_m(t) - C(t)]$$

Using the Horowitz and Sidi technique on these two sets of tolerances will yield a design for the original set of bounds. This is a fairly direct method for designing to a set of nonminimum-phase specifications as long as there is no ignorance in the position of the RHP zero. If there is a range of positions where the RHP zero could be, designs must be made over the entire range and the design that meets all conditions must be chosen. This would mean doing many designs before the needed one is found. The only time this would be unnecessary is when a worst-case set of specifications could be designed for and this design would be satisfactory for all other possible positions of the RHP zero. Since a worst-case would not be obvious, many designs would probably be required for each set of bounds. This is not a practical method of design when there is ignorance in the RHP zero.

2.4 Method's Worth

This method becomes very impractical when there are multiple RHP zeros that must be designed for. More and more sets of boundaries are required to specify the allowed variation of each additional derivative that is required as the number of RHP zeros increase. Not only do the number of boundaries increase, but the equations

that must be solved to find the boundaries become increasingly complex. Thus, for multiple zeros, the method becomes much too complicated and involved to be useful. This method holds little promise for a more general solution because of the increasing difficulty with more than one RHP zero.

CHAPTER III

FREQUENCY DOMAIN METHOD

3.1 Introduction

This method is to enable a priori design to meet the nonminimum-phase (nmp) tolerances specified. It was to be similar to the procedure used to design for minimum-phase systems (mp) except both magnitude and phase tolerances of the frequency response must be studied, but a simplifying discovery was made during the investigation. This will be clarified later in this section. In a mp system, the magnitude uniquely defines the transfer function, but in the nmp system this is only true if the position of the RHP zero is known. If a design procedure utilizes both the magnitude and phase tolerances of the frequency response, several designs will not be required as was true with the time domain method.

3.2 Method

The first step of this method is to transfer the nmp step response bounds into magnitude and phase bounds in the frequency domain. This is accomplished by approximating whatever the final transfer function may be by the following:

$$T(s) = \frac{-\frac{\omega_n^2 \, b}{a}(s-a)}{(s+b)(s^2 + 2z\omega_n s + \omega_n^2)} \qquad (3.1)$$

If "a" takes on the values of the RHP zero of the plant
and z and ω_n are allowed to vary over many values, many
representative step responses can be examined to see
which ones satisfy the required time domain boundaries.
A computer program called TEST2 has been written to per-
form this task. It varies all the parameters of the
transfer function and finds all the responses which fall
within the input time domain bounds. The bounds are de-
scribed by a series of "if statements." All parameter
sets yielding satisfactory responses and the parameters
whose responses "hit" the boundaries are printed out.
These "hits" are stored and when many are found, they
are used to approximate the time domain bounds by equi-
valent frequency domain tolerances, magnitude and phase.
The maxima and minima composite curves and their dif-
ference are printed. This allowed variation of $|\Delta T(j\omega)|$
is what is required to use the Horowitz and Sidi method
of design.

3.3 Examination of Method

To examine the method and to gain insight into
what is really necessary to complete a design, various
phases of the general method will be examined. For this
a useful set of step response boundaries must be found.
These bounds would be given as specifications in a real
problem. To accomplish the above, the far pole in

equation 3.1, (b) is set at -20, and ω_n and z were made
one for a critically damped system. The RHP zero (a) was
set at 10 so as to have little effect on the step res-
ponse. A set of boundaries were than centered about this
response. The boundary size was then adjusted to include
some responses and to exclude others. The boundaries had
to be narrow enough so the boundaries are not trivial
and cannot be exceeded, but wide enough so they are not
impossible to meet. The TEST2 program was run for the
set of bounds shown in figure 3.1 with a RHP zero at
.75, 1.0, 1.5, 3.0, or 8.0. Figures 3.2, 3.3, and 3.4
show the output plots of the allowed magnitude variation,
the allowed phase variation, and the acceptable regions
of LHP pole variation that yield satisfactory step res-
ponses.

The curves for allowed phase variation are explained
by their region of acceptable pole variations. Since the
phase variation is entirely due to the pole variation, a
plot was made for the phase due to the pole locations
possible in figure 3.4 and is shown in figure 3.5. The
frequencies where the poles may be on the Bode plot
account for the behavior of the phase curves. These phase
curves will not be required in the final design method
since only one value of the RHP zero will be designed
for. Since its position is known, magnitude alone is
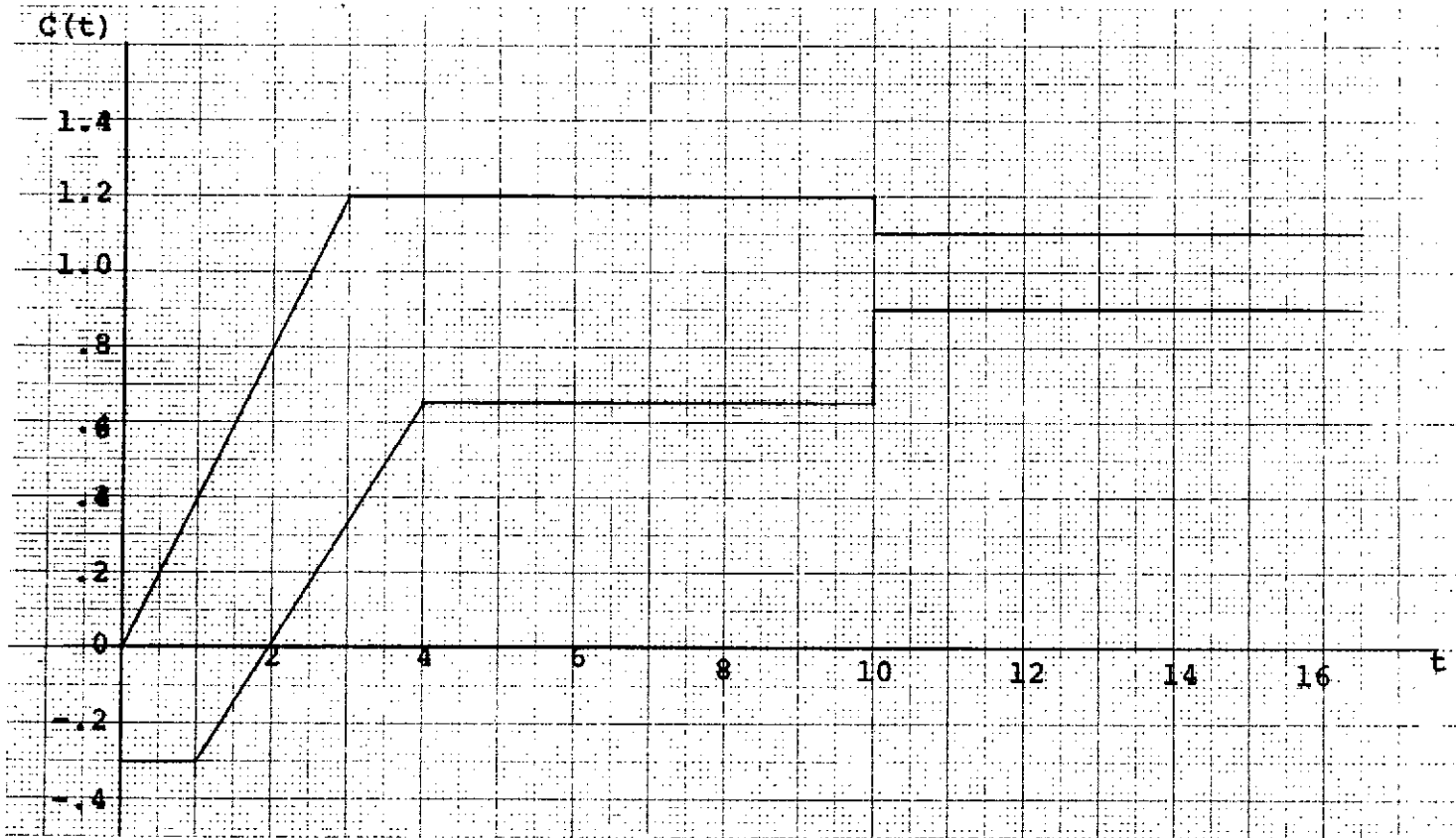sufficient to determine the transfer function. Since the

Figure 3.1   Test set of Nonminimum-Phase Boundaries
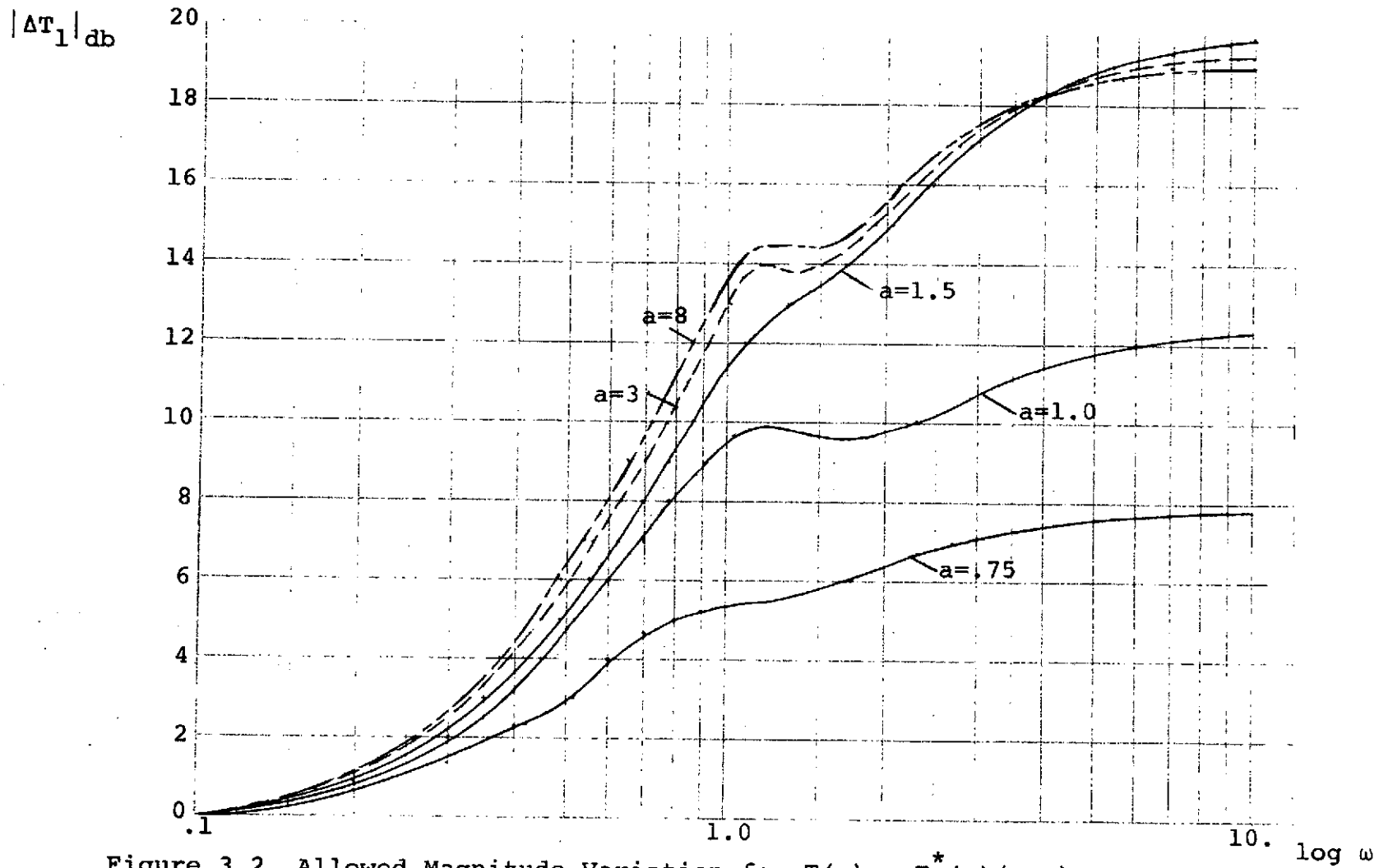
Figure 3.2   Allowed Magnitude Variation for $T(s) = T^*(s)(s-a)$
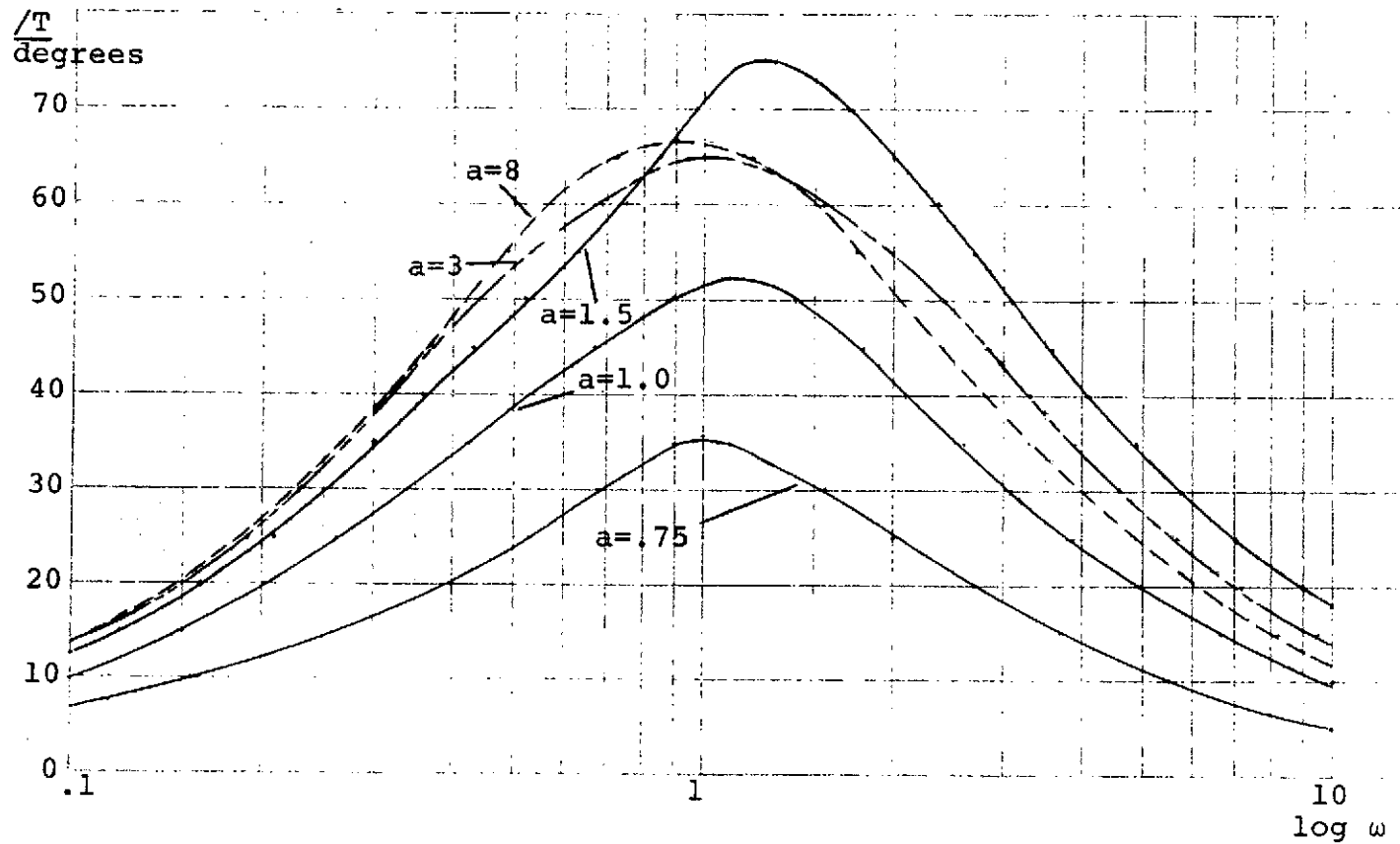
Figure 3.3   Phase Variation of $T(s) = T^*(s)(s-a)$
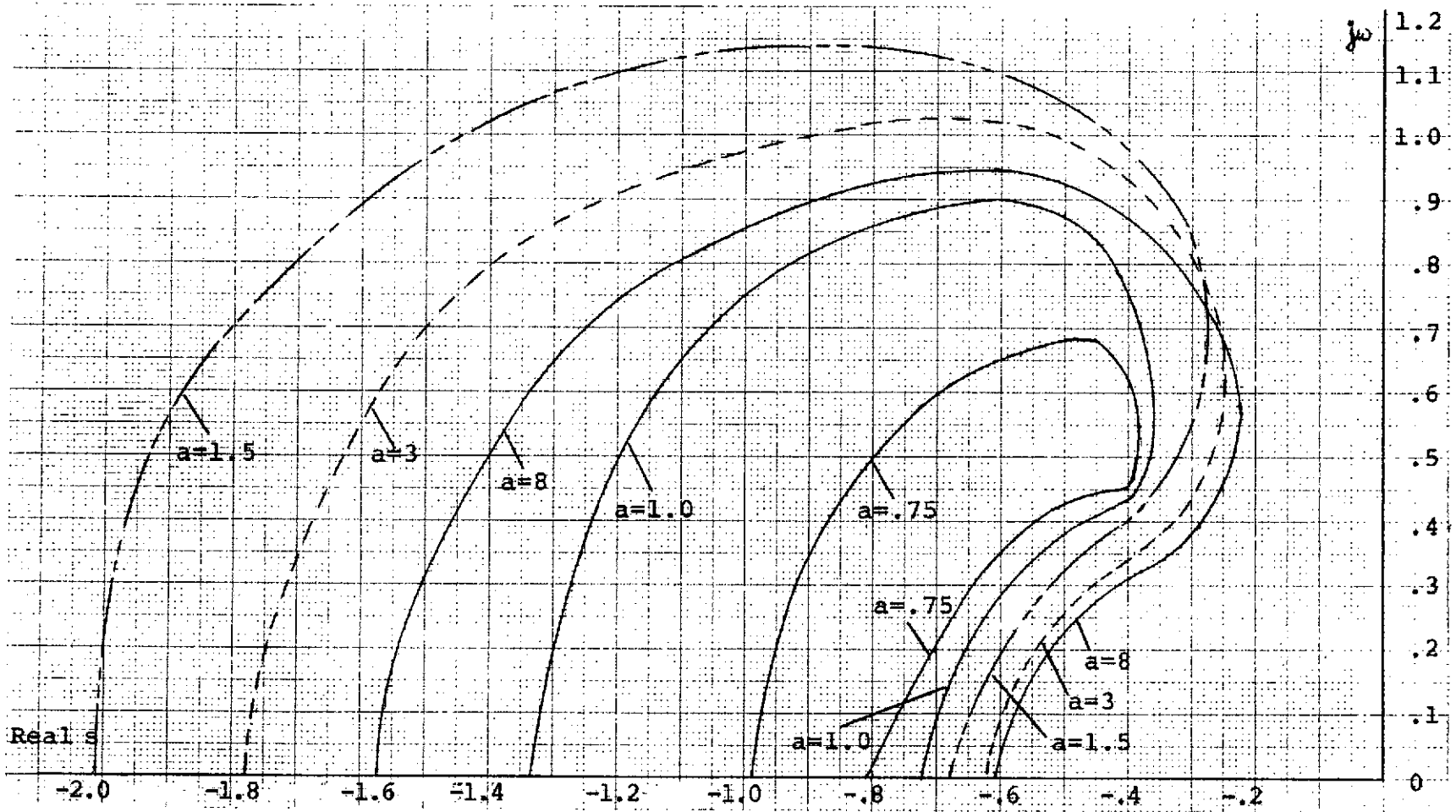
Figure 3.4 Acceptable LHP Pole Variation Regions for Various RHP Zeros
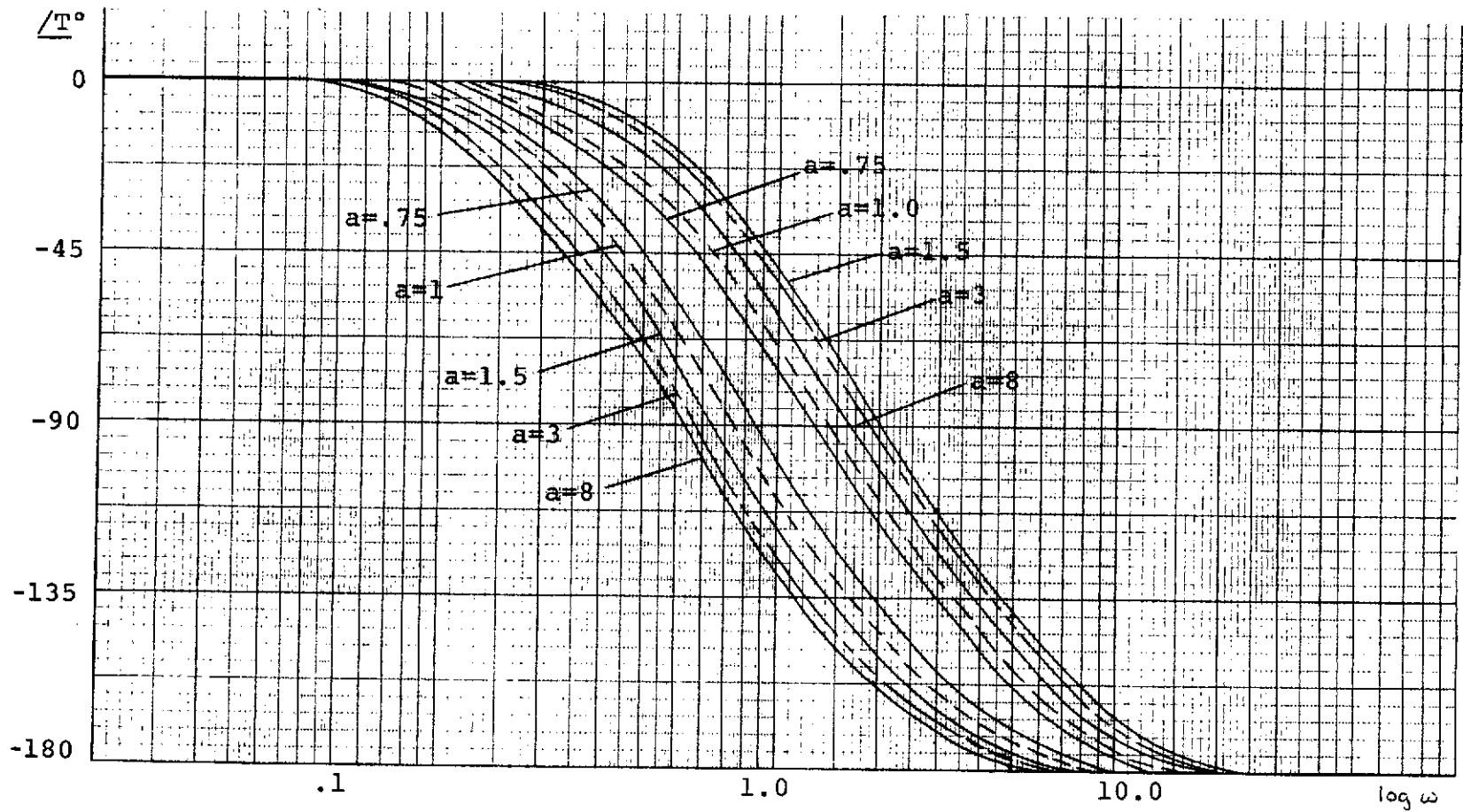
Figure 3.5   Explanation of $\Delta\angle T$ Behavior of $T(s) = T^*(s)(s-a)$

curves do agree with what is expected though, they do serve a purpose in that they verify the program's validity.

The regions of acceptable pole variation (figure 3.4) get larger as the RHP zero moves further out, only up to a certain point, then they start getting smaller. The behavior of these curves must be explained. First, the lower portions of the bounds will be examined. A transfer function with a RHP zero far from the origin will have little undershoot and will approach a minimum-phase system. One with a RHP zero very close to the origin will have a very large undershoot. The allowable regions for the complex poles relate to the speed of the system responses for various zeros. Clearly, a system with a RHP zero at one will require a faster response than one with the zero at ten in order not to violate the lower response boundary (figure 3.6). This means a slower system, smaller ω_n, will be acceptable with the RHP zero further out. This means that as the RHP zero moves further from the origin, pole variation is allowed at lower frequencies. This explains the lower portion of the region.

The equation relating the mp and nmp responses must be examined to explain the behavior of the upper boundary. As long as the lower response bound is the one that will be violated, $\dot{C}_m(t)$ must be kept small for a zero close to the origin in the equation

$$C(t) = C_m(t) - \frac{1}{a} \dot{C}_m(t) \ .$$

If the zero is close to the origin, $\frac{1}{a}$ is large, there-fore to make sure the undershoot does not violate the lower bound, the response must be slow to keep $\dot{C}_m(t)$ small. If the zero were further out, "a" is larger, then $\dot{C}_m(t)$ can be larger without violating the lower bound. Therefore, while the lower bound is still the one being violated, the allowable region increases in size as the RHP zero moves further out, a faster response is allowed.
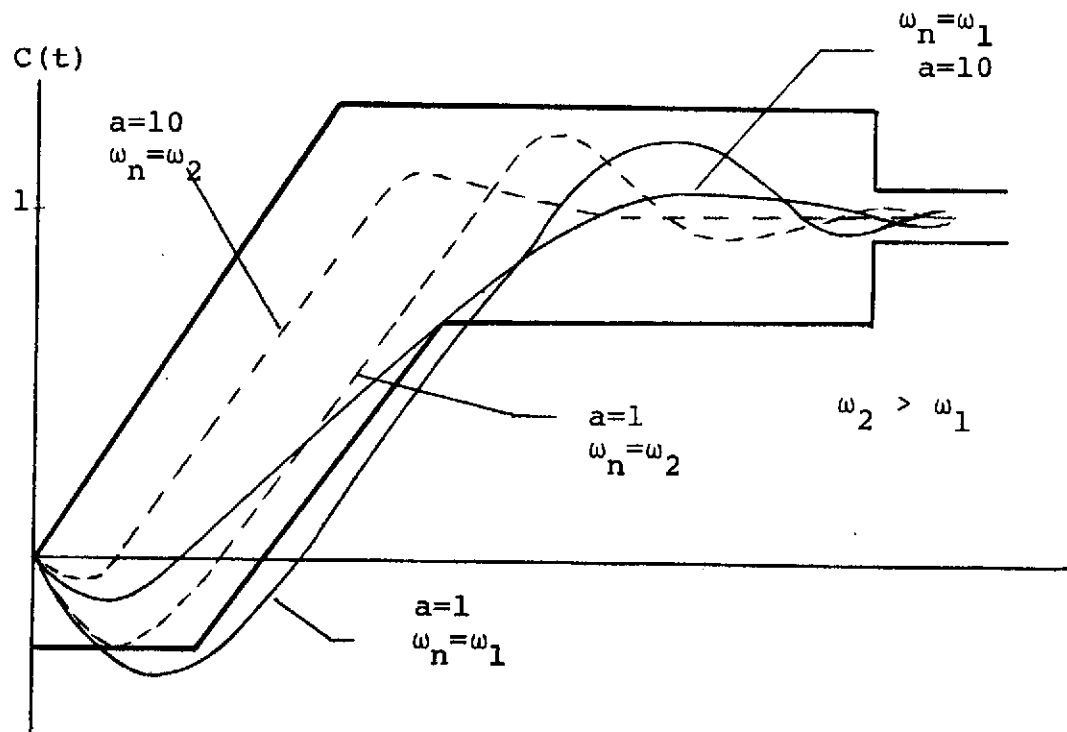
Figure 3.6 Lower Bound Violation by Nonminimum-Phase Responses

At some point the response will be so fast, it will violate the upper bound. This occurs as the zero goes far enough out to allow a response that rises faster than the upper boundary does. This has been verified by examining various responses to see which portions of the boundaries are violated. When this starts to occur, the allowed pole variation regions begin to close in. The regions begin getting smaller when the upper boundary becomes the critical one. Figure 3.7 shows that the response with a RHP zero at one will be satisfactory, whereas the response with the zero at ten will be unsatisfactory. As "a" increases now, the system must be slower to prevent violation of the upper boundary.
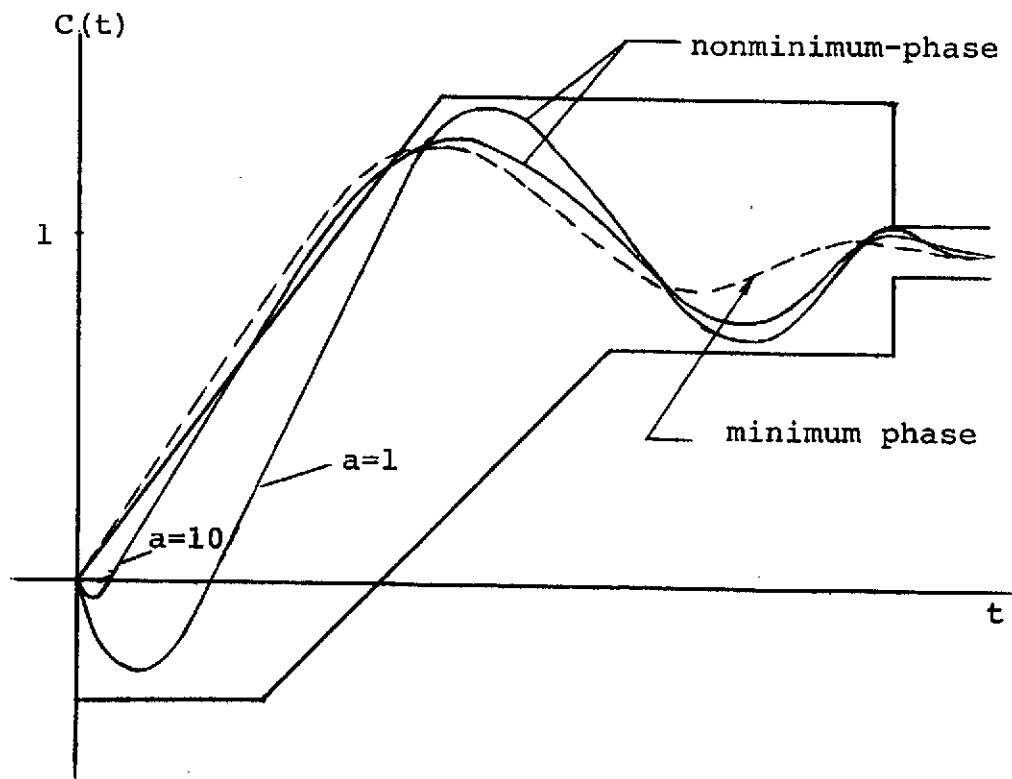
Figure 3.7 Upper Bound Violation

3.4 Design for Worst Case

Given a set of $|\Delta T|_{db}$ variation curves as shown in figure 3.2, can a design for a certain set of specifications and one location of the RHP zero be satisfactory as the position of the RHP zero changes? Consider a plant P(s) which can be written as P(s) = P^*(s)(s-a), where P^*(s) is independent of "a", the RHP zero. In using the Horowitz and Sidi design technique, it is necessary to have a plant template which can be used in connection with the Nichols Chart to design a compensation G. To gain insight into the effect of the RHP zero, use a plant template for each value of "a". For increasing values of "a"

$$P(s) = P^*(s)(s-a)$$

$$P(j\omega) = P^*(j\omega)\left[\sqrt{\omega^2+a^2} \ \angle 180° - \tan^{-1}\frac{\omega}{a}\right]$$

this equation clearly shows that both the magnitude and phase of the term (s-a) increases. This means that if a plant template is made for each value of "a" the templates will be identical but will have greater magnitude and phase positions with increasing "a" (figure 3.8).

The worst case that can be designed for is when the specifications are the same over the range of RHP variation. In this case the same variation must be met whereas other times, as the zero increases, more variation is allowed (figure 3.2) and the control can be decreased for

this case. Certainly the most difficult problem is when the strictest set of specifications must be met over the entire range of parameter variation.

If designs for the compensation G(s) were made using the plant templates of figure 3.8, the same $L(j\omega)$ boundaries would result for both zeros if the same set of worst-case $|\Delta T(j\omega)|$ specifications were used. Figure 1.1 explains the meaning of G(s) and L(s). This happens because the size and shape of the plant templates are independent of the location of the RHP zero, only the plant template values vary ($|P_i|, \angle P_i$). The Nichols Chart
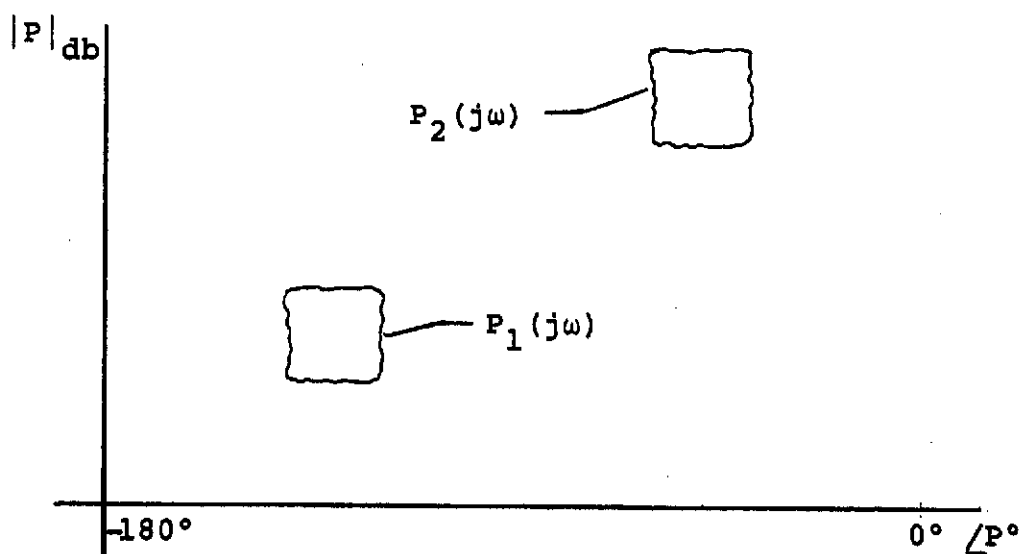
$$P_1(s) = P^*(s)(s-a_1)$$
$$P_2(s) = P^*(s)(s-a_2)$$
$$\text{where } a_1 < a_2$$

Figure 3.8   Template Variation

boundary on $L(j\omega)$ is found by sliding the plant template along a constant phase line until it fits between the $\frac{L(j\omega)}{1+L(j\omega)}$ curves that correspond to the allowed variation in $T(j\omega)$. Since the same template is used for various "a's", and the same variation in $\frac{L(j\omega)}{1+L(j\omega)}$ is allowed, the same $L(j\omega)$ boundaries will result. The compensation $G(j\omega)$ boundaries are found by:

$$|G(j\omega)|_{db} = |L(j\omega)|_{db} - |P(j\omega)|_{db}$$

$$|G_1(j\omega)|_{db} = |L(j\omega)|_{db} - |P_1(j\omega)|_{db}$$

$$|G_2(j\omega)|_{db} = |L(j\omega)|_{db} - |P_2(j\omega)|_{db}$$

from before $|P_1(j\omega)|_{db} < |P_2(j\omega)|_{db}$

which implies $|P_1(j\omega)|_{db} + \Delta = |P_2(j\omega)|_{db}$ where $\Delta > 0$ db.

$$|G_1(j\omega)|_{db} + |P_1(j\omega)|_{db} = |G_2(j\omega)|_{db} + |P_2(j\omega)|_{db}$$

$$|G_1(j\omega)|_{db} - \Delta = |G_2(j\omega)|_{db}$$

which implies $|G_1(j\omega)|_{db} > |G_2(j\omega)|_{db}$

The same argument also will yield $\angle G_1(j\omega) > \angle G_2(j\omega)$. On a plot of $|G(j\omega)|$ vs $\angle G(j\omega)$, this means each corresponding point on $G_1(j\omega)$ will be higher and to the right of $G_2(j\omega)$ (figure 3.9). If $G_1(j\omega)$ is above $G_2(j\omega)$ at every point, then a design for the smallest value of the RHP zero will be satisfactory for all positions of the zero. If $G_1(j\omega)$

is lower than $G_2(j\omega)$ at some phase angle, even though each corresponding point is higher (figure 3.10), then the design must be more complicated because more than one location of the RHP zero must be designed for.

When the design is completed, one set of magnitude and phase of $L(j\omega)$ on the $L(j\omega)$ boundary is realized. The question whether the $G(j\omega)$ boundaries cross as in figure 3.10, is equivalent to asking another question. Utilizing the compensation $G(j\omega)$ that yielded the point on the $L(j\omega)$ boundary with one position of the RHP zero, will the $L(j\omega)$ found as the zero moves also be on or above the $L(j\omega)$ boundary? If the $G_1(j\omega)$ used to get the point L_1 on the boundary is below the $G_2(j\omega)$ boundary, then the point L_2 found with a_2 and $G_1(j\omega)$ is not necessarily above the $L(j\omega)$ boundary.
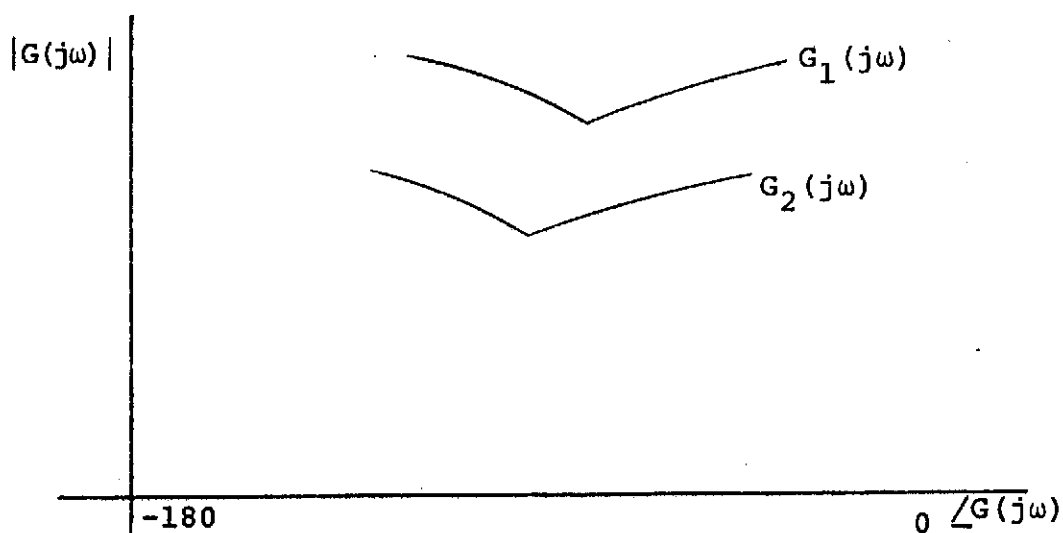
Figure 3.9   Desirable G Boundaries
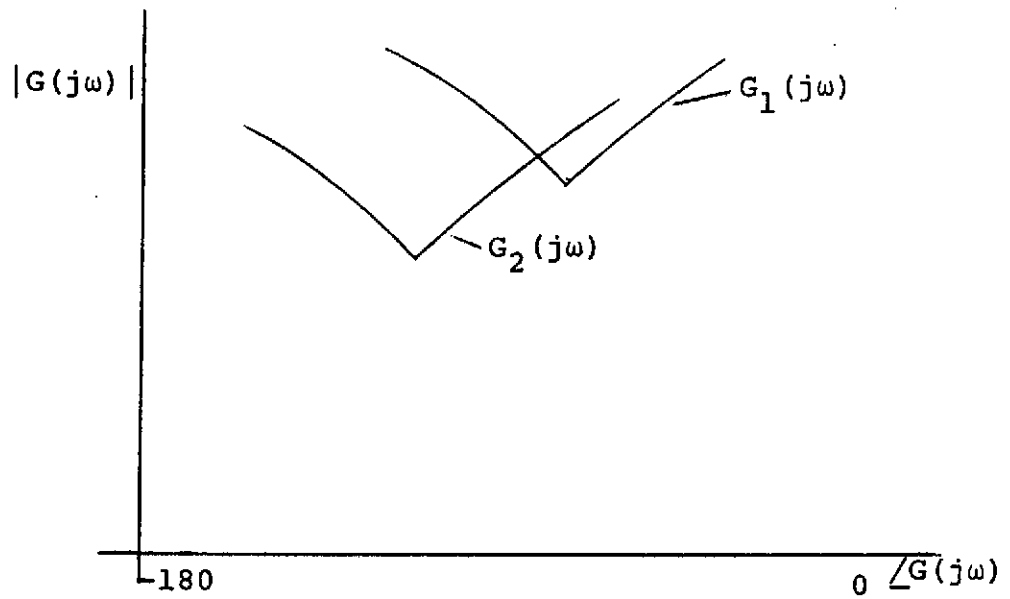
Figure 3.10   Undesirable G Boundaries
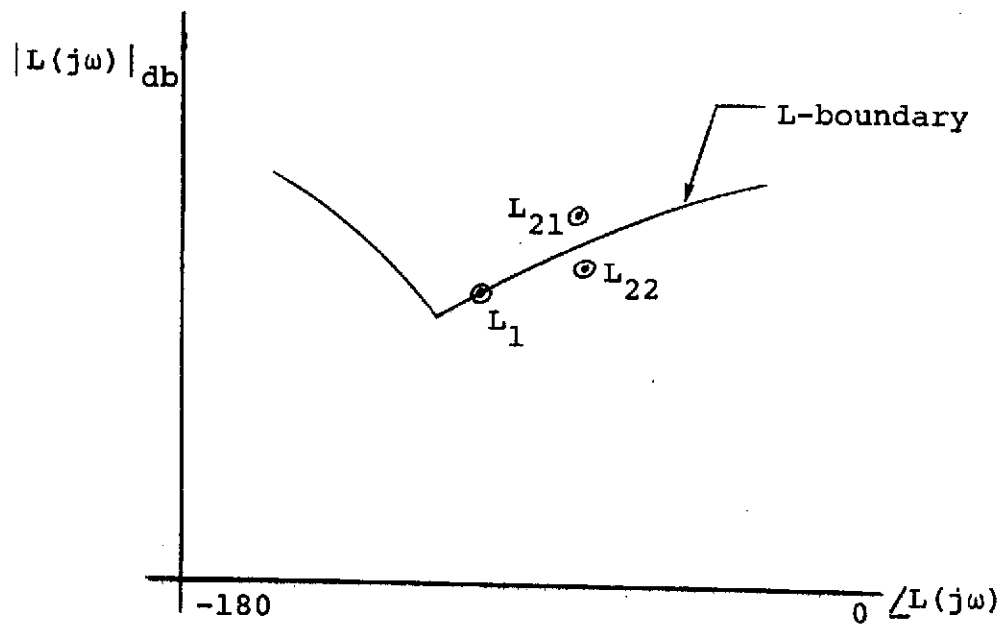
Figure 3.11   Possible Loop Transmissions

It is known that if L_1 is the $L(j\omega)$ designed for with $G_1(j\omega)$ and a_1, that the resulting point L_2 as the zero moves to a_2, where $a_2 > a_1$, using the designed compensation G_1 will be above and to the right of L_1 on a $|L(j\omega)|$ vs $\underline{/}L(j\omega)$ plot. Whether L_2 is above or below the L boundary, L_{21} or L_{22} in figure 3.11, depends on the lead added by (s-a) at that frequency by the increase in "a" and the slope of the L boundary. For these reasons it cannot be proven when it will or will not happen, but an argument can be given that it will not happen in most problems.

In the majority of problems specifications will not be met at frequencies higher than $\omega = \dfrac{a_1}{2}$, where "a_1" is the minimum RHP zero location. At frequencies much less than "a_1", almost pure gain is added as the zero moves out, eq. 3.2, $\dfrac{j\omega}{a} \simeq 0$. This means that at frequencies

$$P(s) = P^*(s)(s-a) \rightarrow P_1(j\omega)(1 - \dfrac{j\omega}{a}) \qquad (3.2)$$

where $\omega \ll a$, the design for a_1 will certainly be satisfactory for a_2. As ω approaches a_1, more and more lead is added as the RHP zero increases to a_2. For most plants, and an exception has yet to be found, the L boundaries appear as in 3.12.

It is clear that if the phase added by the increase in the RHP zero is to yield an $L(j\omega)$ that falls below the bounds, it would happen at low frequencies where the

boundaries are most peaked. It is at these frequencies
though where the change is almost pure gain and there is
no problem. As the frequencies increase, it becomes
harder to be above some L_1 and fall below the boundary
also since they flatten out and actually start going ne-
gative. Many examples have been tried and in all cases
a design for the smallest zero will be satisfactory as
the zero varies. This argument at least makes this
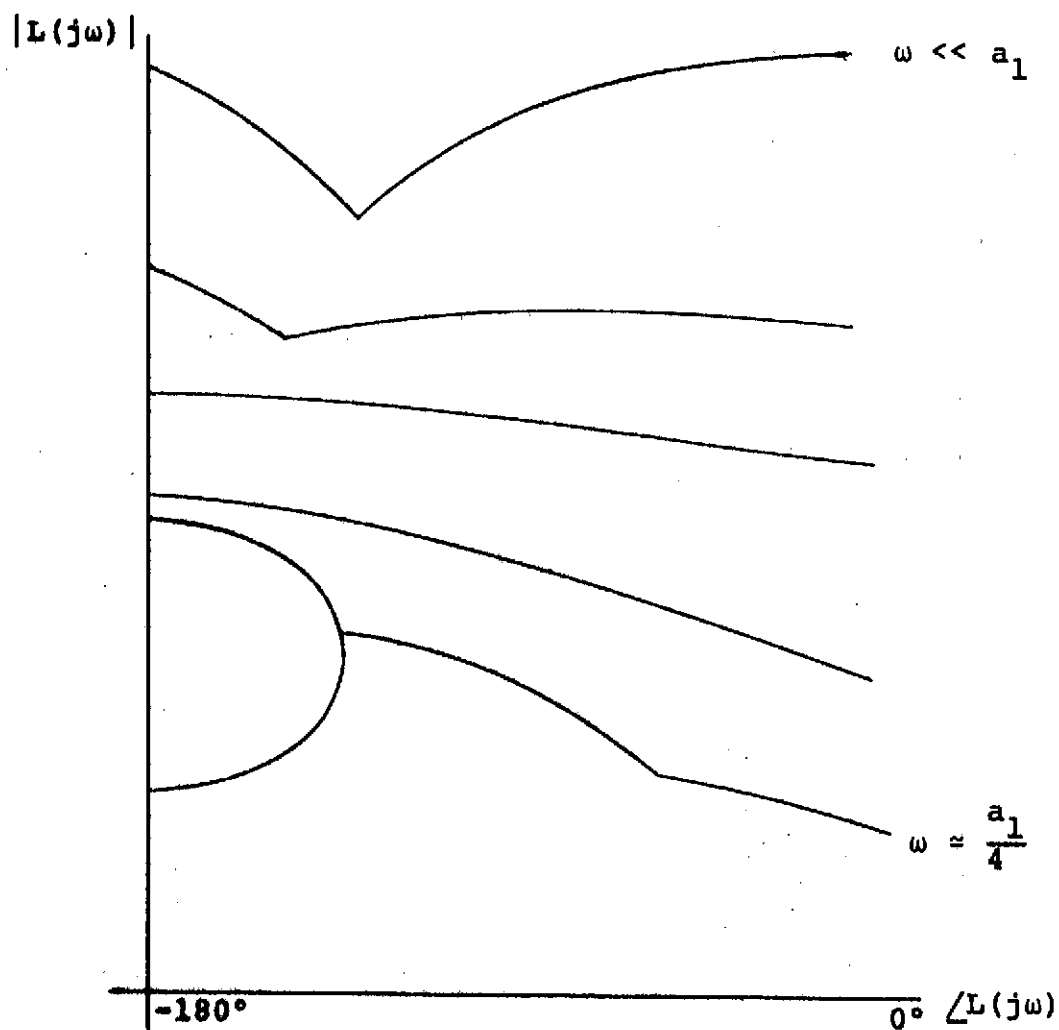
Figure 3.12   Typical L(jω) Boundaries

theory more plausible and the examples fail to expose
any possible problem. A rigorous proof is not possible
since it depends on so many circumstances. Certainly if
the range of RHP zero variation is small or if the fre-
quency range requiring feedback is much smaller than the
smallest RHP zero, a design for the smallest RHP zero
will be satisfactory for all allowed positions of the
RHP zero. In the majority of problems this will be true.
A quick test would be to find all the G(jω) compensation
curves for possible RHP zero locations to see if the
G(jω) boundaries ever cross. No plant has been found yet
where this has happened.

Whatever the design, it must of course meet the
tightest set of specifications. Thus a design for the
smallest RHP zero and the lowest set of allowed $|\Delta T(j\omega)|$
specifications will meet the time response specification
as the RHP zero takes on other values. As shown in
figure 3.2, the smallest RHP zero also has the lowest
set of specifications. This is always true from low fre-
quencies up to some higher frequency where the specifi-
cations may cross, i.e., a design for $a\varepsilon[3,8]$, a=3 is
lowest until ω ≈ 3.5.

3.5 What are the Lowest Specifications

From figure 3.2, it is clear that the smallest RHP
zero has the lowest specifications at least at fre-
quencies where $\omega < \frac{a}{2}$. From studying figure 3.13, a
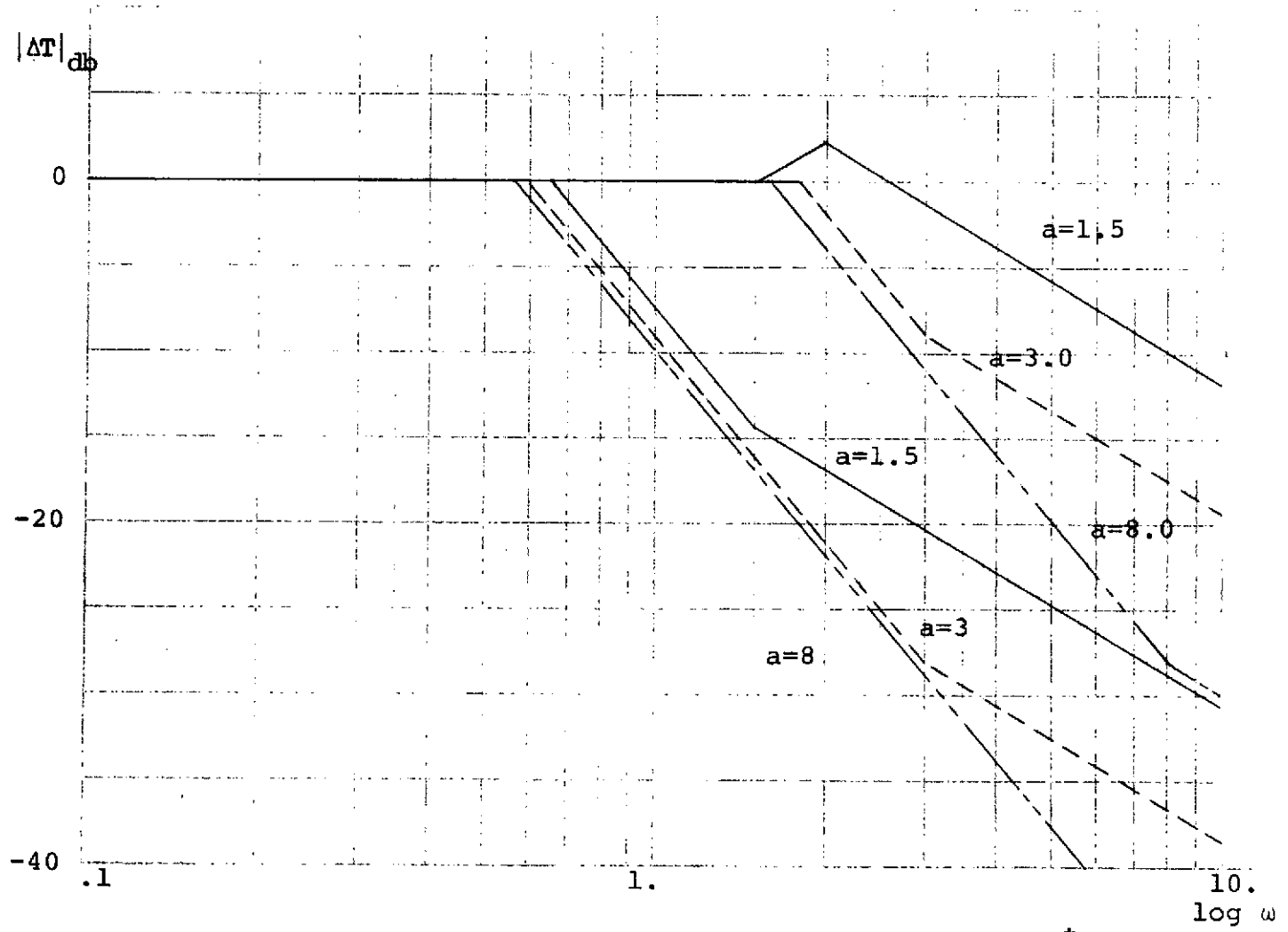
Figure 3.13 Straightline Approximation of $|\Delta T|$ where $T(s) = T^*(s)(s-a)$

straight-line approximation of the $|\Delta T(j\omega)|$ curves for the regions of acceptable pole variation, figure 3.4, some general rules may be found. First, the RHP zero, a_1, yields the lowest bounds at least until the frequency of the maximum position of a varying pole if it is less than the value of the RHP zero. If this frequency is greater than the RHP zero, the smallest zero, a_1, will yield the smallest $|\Delta T(j\omega)|$ at least till $\omega = a_1$. The smallest value of "a" will yield the lowest specifications at all frequencies if the allowed region of parameter variation of a_2 encloses that of a_1, where $a_1 < a_2$. This is shown by $a\epsilon[.75,1.0]$ in figures 3.2 and 3.4. The specs for a = .75 are lower than those of a = 1 at all frequencies, and the poles of a = 1 could vary at lower and higher frequencies than those of a = .75.

These guidelines may help in deciding which RHP zero yields the lowest specs over the frequency range of interest. The surest way is to run program TEST2 on several positions of the RHP zero and see what specifications are lowest. These specifications are required in the design anyway.

For any set of specifications a design for the smallest RHP zero will be satisfactory for all locations of the RHP zero. A design for the smallest set of allowed variation specs on $|\Delta T(j\omega)|$ will certainly be satisfactory when more variation is allowed. Therefore,

a design for the smallest RHP zero and the lowest set of
specs will yield satisfactory step responses as the zero
is allowed to vary.

3.6 When Cannot the Design Be Completed

One major drawback of a system with a RHP zero is
that the bandwidth is limited. This means that an ar-
bitrary set of sensitivity or disturbance specifications
are not realizable. Some plant and specification combi-
nations may require a large bandwidth, but this in turn
results in an unstable system. Therefore, it would be
nice to have some guidelines as to when the design can
or cannot be completed so a lot of time is not wasted
in trying to find a stable design when it is not possible
for there to be one.

One of the steps in the design is to ascertain the
frequency range over which feedback is required to meet
the specifications. This is done by seeing where the
curves of $|\Delta T(j\omega)| = |\Delta P(j\omega)|$. Up until ω_1, $|\Delta P(j\omega)| >$
$|\Delta T(j\omega)|$. This means feedback is required up to ω_1 to
constrain the variation of the transfer function due to
the plant variation to the specifications required in
$|\Delta T(j\omega)|$. The curve $|\Delta T(j\omega)|$ is obtained as before by
the program TEST2 which takes a composite Bode plot of
the transfer functions which yield satisfactory step
responses. The curve $|\Delta P|$ is the maximum difference be-
tween all the Bode plots of the plant as all parameters

but the RHP zero vary.

The frequency where these curves cross roughly cor-
responds to the bandwidth of the system. If this fre-
quency is less than half the value of the smallest RHP
zero, the design can be completed satisfactorily. If
this frequency range is $\frac{a_1}{2} < \omega < 2a_1$, where "$a_1$" is the
smallest RHP zero, the design may be completed, but it
will be very difficult to do if it can indeed be done.
If the crossover is at a frequency greater than twice
the smallest RHP zero, the design is impossible. This
should give a rough idea of how difficult a design is,
if it is possible.

Another means of judging if the design can be com-
pleted concerns the Nichols Chart. The loop transmission
can be written as:

$$L(s) = L_1(s)(s-a) = L_1(s)(s-a)\frac{(s+a)}{(s+a)} = L_1(s)(s+a)[\frac{s-a}{s+a}]$$

$$L(s) = L_m(s)A(s)$$

where A(s) is an all-pass function.
If figure 3.12 was the Nichols Chart boundaries on L(s),
a plot of those for $L_m(s)$ would be shifted to the right
by the phase contributed by the all-pass function at each
frequency. If there is a disturbance in the system, it
must be damped out. The damping can be approximated by
a dominant pair of complex poles whose damping factor z
can be related to the disturbance overshoot allowed.

This can then be related to a magnitude peaking curve which yields a forbidden region on the Nichols Chart. The region specifying the damping of a disturbance, the ellipse-like shape in figure 3.12, will move also, but it will be distorted because each point on it corresponds to a different frequency. This is clear if the L found wraps right around this region. Each point will move a different amount because the all-pass contributes a different amount of phase at each frequency.

If the last boundary corresponds to a frequency $\omega > 2a_1$, the damping region will probably cross the zero degree line, figure 3.14. When this happens, a design
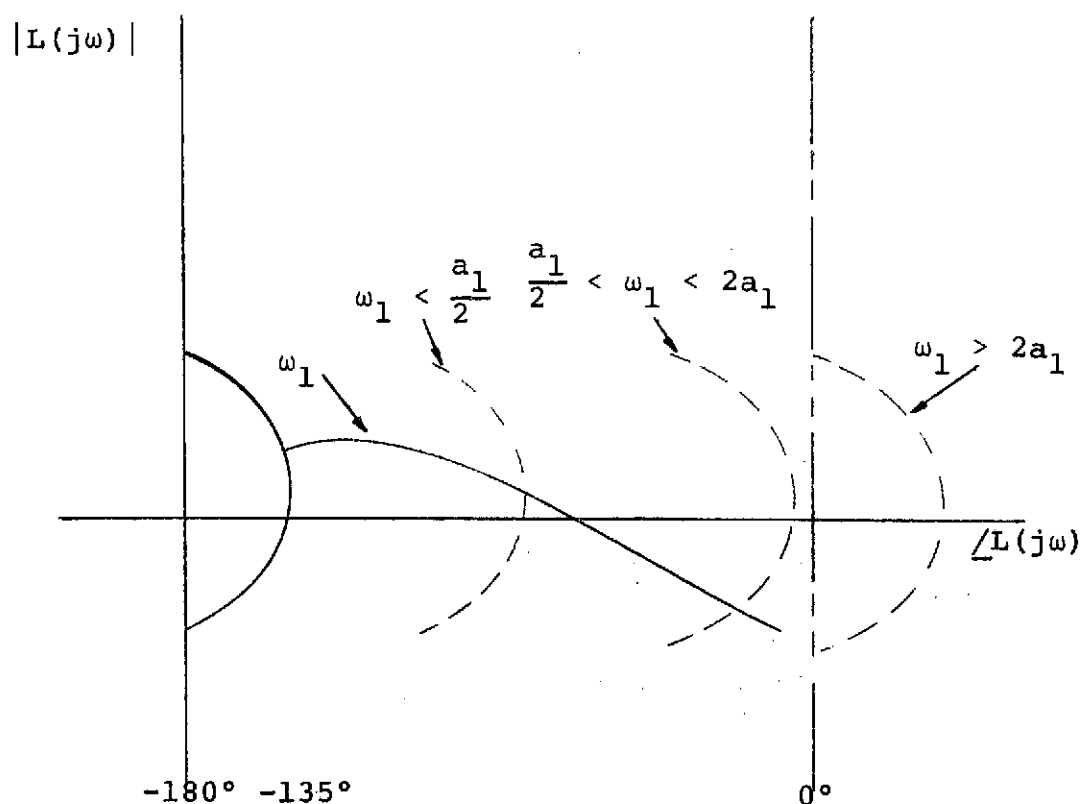
Figure 3.14   Minimum-Phase Disturbance Regions

cannot be made because it is not possible to come down in magnitude any distance in decibels with phase lead. If $\omega_1 < \frac{a_1}{2}$, it is clear a design can be found. If $2a_1 > \omega_1 > \frac{a_1}{2}$, it will be more difficult to stay outside the disturbance region. This method of looking at the problem yields the same results as before but by being concerned with the damping of a disturbance signal.

If a lot of gain variation is present in the plant, the last specification curve must be at $\omega_1 << a_1$, because it will take a wide frequency range to clear the disturbance region. This means that if ω_1 was near a_1, the disturbance region would almost certainly cross the zero degree line, and probably over a large magnitude range. This would definitely make the design impossible to complete (figure 3.15).
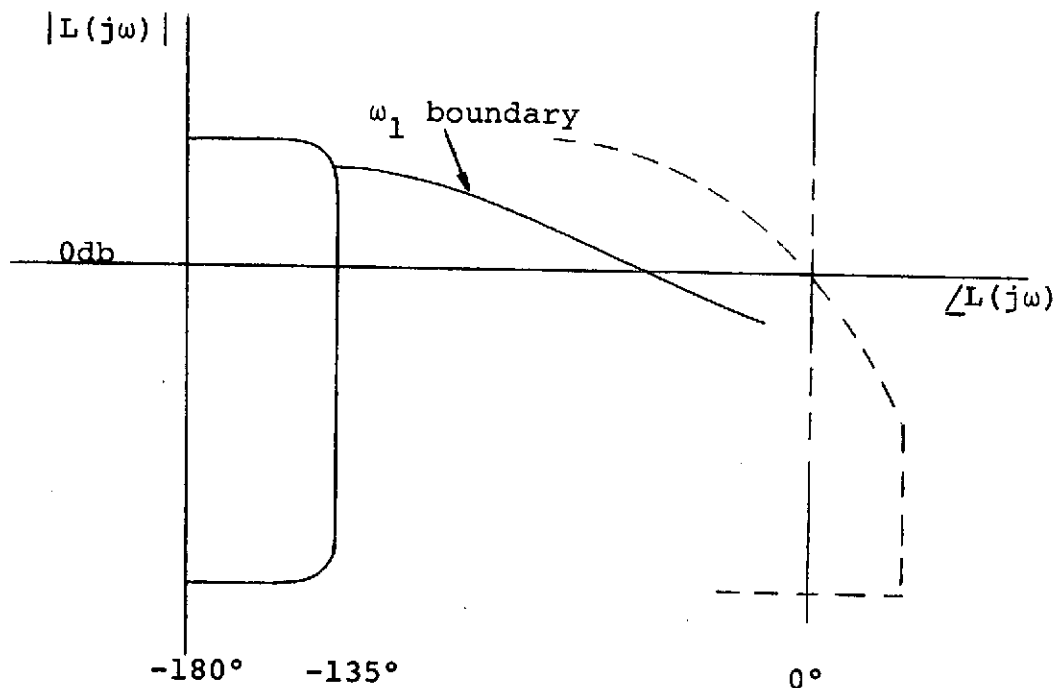
Figure 3.15   Impossible Design Case

3.7   Summary of Design Procedure

This section will present the design steps of the proposed procedure. First, find the set of frequency domain specifications on $|\Delta T(j\omega)|$ by running a program like TEST2 which approximates T(s) by a third order function and searches for valid transfer functions. Next, ascertain the set of specifications that will be designed to, the zero designed for, and the frequency where feedback is no longer required. This is usually the smallest set of tolerances, the smallest possible RHP zero, and the frequency where $|\Delta P(j\omega_1)| = |\Delta T(j\omega_1)|$. With this information use the Horowitz and Sidi design technique to find the boundaries on the loop transmission up to ω_1.

Now by some technique, possibly trial and error, find an L(jω) that meets these requirements and yields the desired amount of disturbance damping to the accuracy desired. Knowing the plant and the loop transmission, the compensation G(s) can be solved for. With this G(s) and the varying plant a set of actual $|\Delta T(j\omega)|$ specs can be found. The prefilter F(s) can be solved for which shapes the transfer functions to the same shape and bandwidth of the specifications. This completes the design and should yield satisfactory step responses.

CHAPTER   IV

FIRST EXAMPLE

Now that a design technique has been postulated, it is necessary to work a meaningful example that will test this postulate. The first step in postulating an example is to have a valid set of time domain specifications. The specifications must be tight enough to make the example meaningful, non-trivial, and yet not too tight so as to make it impossible to complete the design. This was accomplished by approximating the transfer function by:

$$T(s) = \frac{-\dfrac{b\omega_n^2}{a}(s-a)}{(s+b)(s^2 + 2z\omega_n s + \omega_n^2)}$$

as was done in the earlier theory section. This approximates the transfer function essentially by the RHP zero and a dominant pair of complex poles. As before, the time domain bounds are centered around a critically damped response of $T(s)$ and then situating the bounds so some responses are confined to the interior and other responses exceed the bounds. The resulting bounds are shown in figure 4.1.

To use the Horowitz and Sidi design technique a set of $|\Delta T|_{db}$ specifications in the frequency domain are required. A program, TEST2, was written to vary z
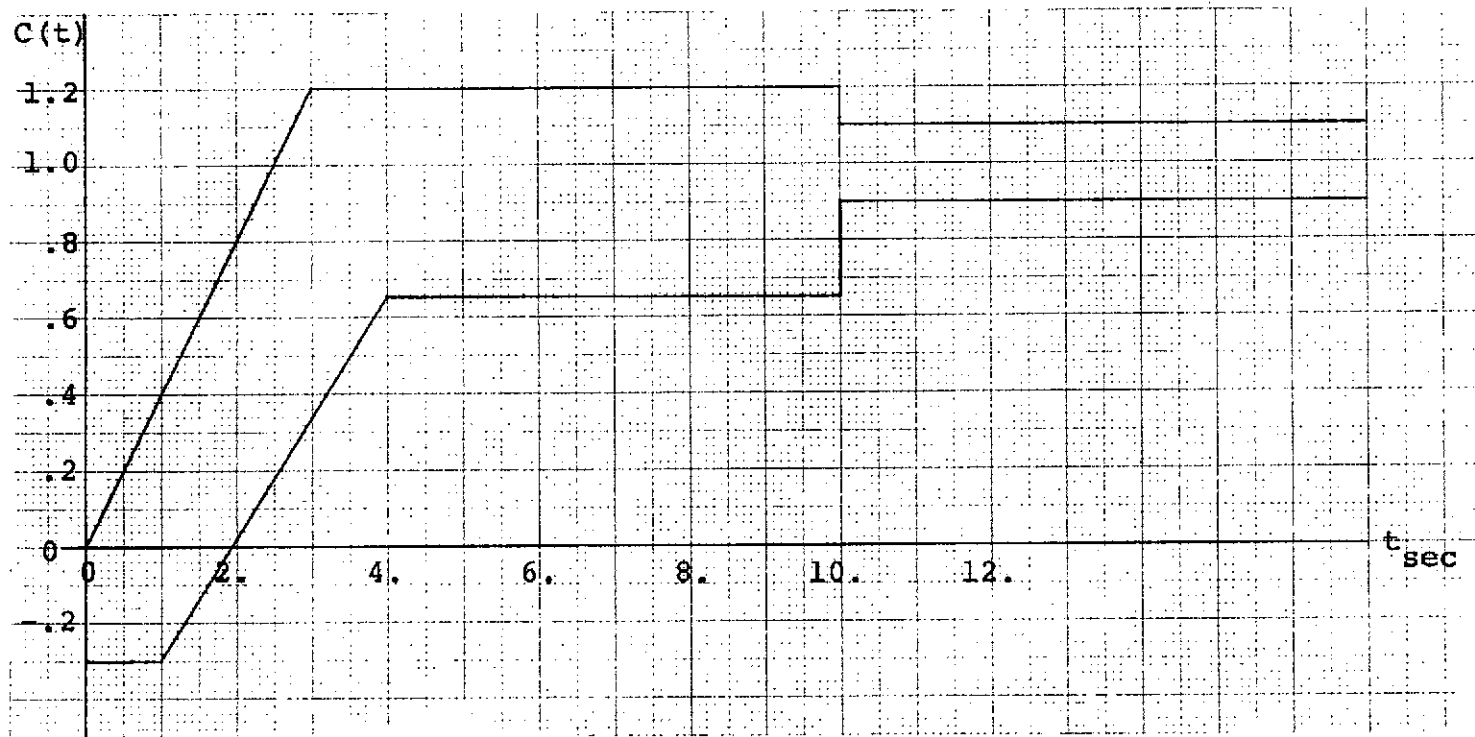
Figure 4.1   Time Domain Bounds

and ω_n in T(s) and find many T(s)'s whose step responses satisfy the time domain specification. Frequency domain plots of those that "hit" the boundaries yield the equivalent frequency domain bounds of the original time domain specifications. The required $|\Delta T_1(j\omega)|$ specifications are the differences between the minimum and maximum plots at each frequency.

Specifications were obtained for several RHP zeros, figure 3.2. Since the allowed $|\Delta T(j\omega)|$ variation is almost the same for $a \in [1.5,8]$, it was decided to let the RHP zero vary from one to eight so the theory would be tested to the fullest by designing over a wide range of allowed variation. That is, if the theory were wrong, it should certainly fail when there is a large difference in specifications over the allowed range of RHP variation.

There is a rule of thumb that the maximum obtainable bandwidth in a nonminimum-phase system is $\omega = \frac{a}{2}$, where "a" is the position of the RHP zero. If the RHP zero varied from .75 to 8. instead of 1 to 8, this would mean feedback would be required over the range $\omega \in [0,.375]$ and the tolerances for $a = .75$ allowed very little plant variation. The remaining parameters of the plant could not have any meaningful variation without requiring a larger bandwidth. For these reasons $a = 1$ was chosen as the minimum RHP zero for this problem.

By examining the allowed $|\Delta T|$ variation a plant
was chosen which would not require feedback over $\omega = .25$
or $\frac{a}{4}$. The plant chosen was:

$$P(s) = \frac{s-a}{s(s+b)} , \text{ where } \begin{array}{l} a\epsilon[1,8] \\ \\ b\epsilon[.015,.15] \end{array}$$

Feedback is not required once $|\Delta P|_{a=1}$ is below $|\Delta T|_{a=1}$
where $|\Delta P|_{a=1} = \frac{j\omega + .15}{j\omega + .015}$, see figure 4.2. Since the
variation in the overall transfer function is due to the
variation in the plant, once the plant variation is be-
low the allowed variation of the transfer function no
further control over the plant is required to hold it
there. Now the example has been completely defined con-
sisting of a set of time domain boundaries and a plant
with RHP zero and LHP pole variations. The remainder of
this example will show how the design technique is ap-
plied to this type of problem and how well it solves
the problem.
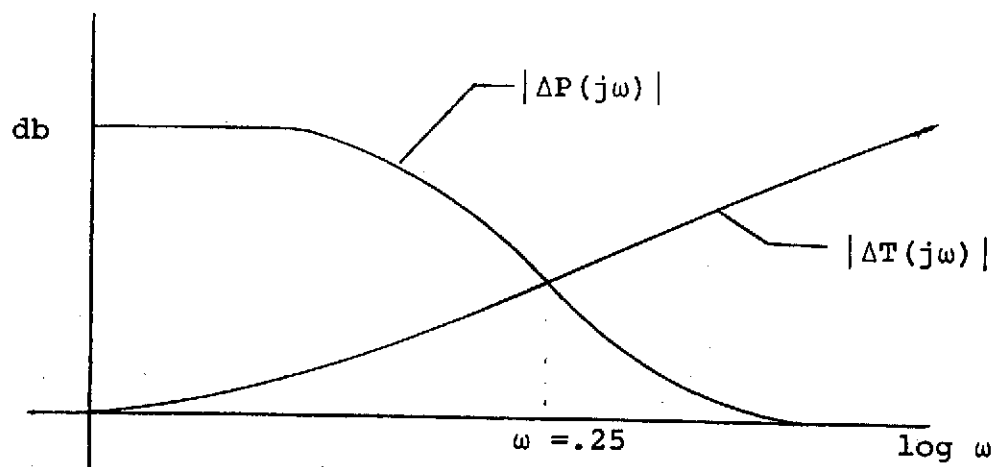
Figure 4.2   Range of ω Requiring Feedback

The first step in using the Horowitz-Sidi design technique is to arrive at a set of boundaries on the loop transmission $L_1(j\omega)$ where $P_1(j\omega)$ is the plant at a set of parameters yielding maximum phase lag at all frequencies. If the compensation G that is used with this P_1 yields an L_1 which does not violate the disturbance damping region and is stable, then the plots of the other L's resulting from all other sets of plants parameters will be to the right of L_1 on the Nichols Chart. This means that for all plant conditions, the resulting L is stable and has at least the required damping of a disturbance. This is why this P_1 is used in the design. In this problem $\underline{/}P(j\omega)$ can be expressed as:

$$\underline{/}P(j\omega) = 90° - \tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{b} .$$

It is clear that the most phase lag is available when both a and b are at their minimum conditions, i.e., a = 1, b = .015. The design technique indicates that a design for the smallest value of the RHP zero will work as the zero varies. Therefore the smallest value of "a" will be the only value of concern throughout the design.

A program implementing the Horowitz and Sidi technique was used to yield a set of L_1 boundaries that meet the input $|\Delta T(j\omega)|$ specifications by using a plant template that describes the allowed plant variation. (Figure 4.3) Any compensation which in connection with
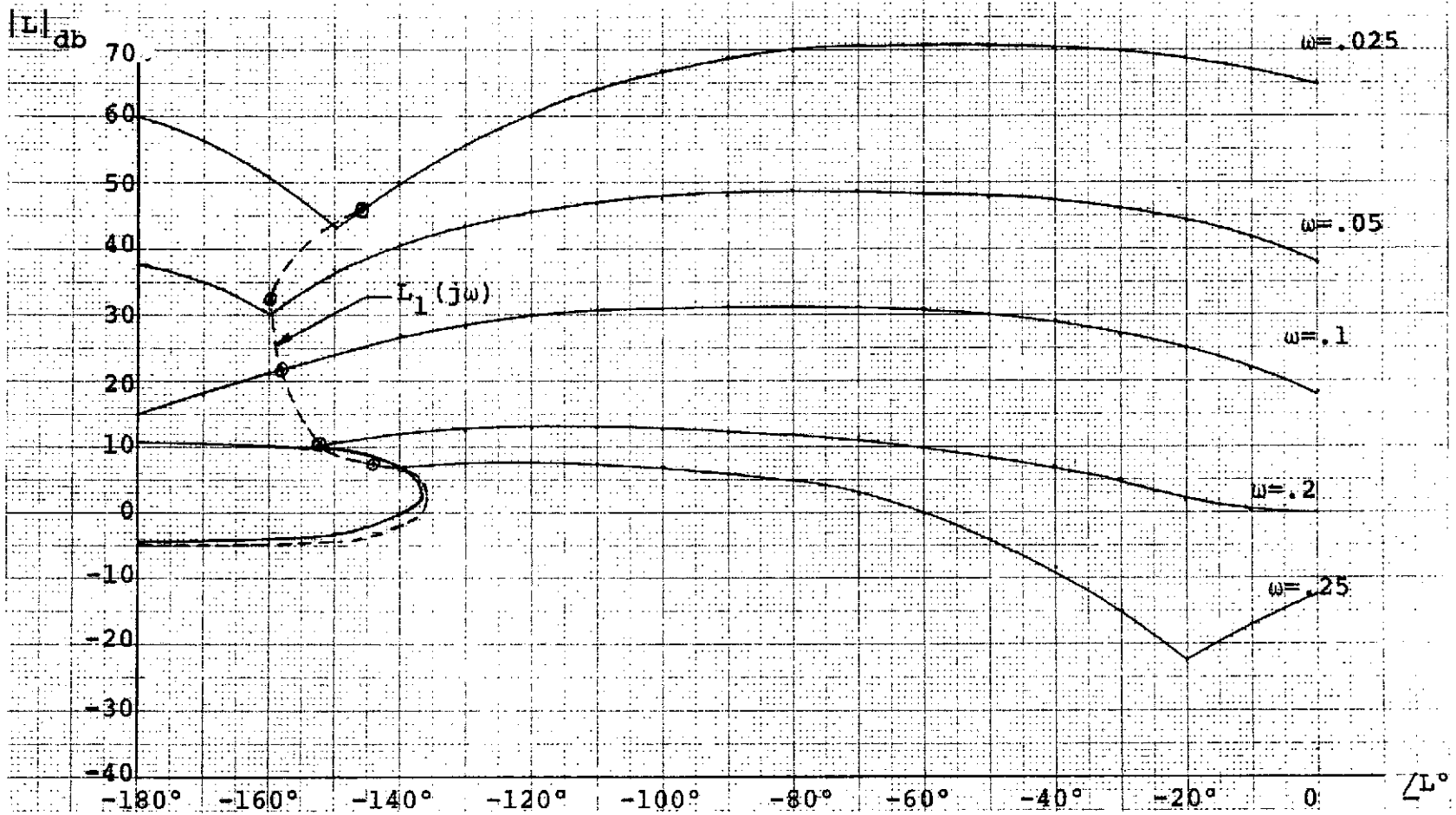
Figure 4.3 L_1-Boundaries

$P_1(j\omega)$ yield a set of magnitude and phase values of $L(j\omega)$ that fall above these boundaries at all frequencies will more than meet the allowed $|\Delta T(j\omega)|$ variations. The $L(j\omega)$ meeting all the boundaries is found by trying various pole zero combinations with the RHP zero at its lowest value until an acceptable solution is found.

Also shown in figure 4.3 is a small elliptical shaped region near the -180°, 0db point. If there is a disturbance in the system, it must be damped out. If the damping can be approximated by that of a pair of dominant complex poles, the damping factor z can be related to the disturbance overshoot that can be allowed. This z can then be related to a magnitude peaking curve which shows up as a region on the Nichols Chart. The region used here is approximately a 3db curve and corresponds to an allowable 25% of disturbance overshoot.

The optimum $L(j\omega)$ would fall on each L-boundary at each frequency and follow the 3db boundary without entering it. Using the trial and error method of design, this is the most time consuming portion of the design. As the frequency approaches that of the RHP zero, phase lag is introduced with gain. This makes it extremely difficult to stay below the 3db region. If feedback is required at half or three-fourths the value of the RHP zero, it may be impossible to find an $L(j\omega)$ that acts in the desired manner.

The $L_1(s)$ for this example is

$$L_{11}(s) = \frac{-640000(s-1)(s+.15)(s+.015)(s+.4)}{s(s+.015)(s+100)^3(s+.02)(s+.24)}$$

$$L_{12}(s) = \frac{(s^2+.22s+.0625)(s^2+1.4s+1)(s^2+.105s+.005625)}{(s^2+.35s+.065)(s^2+1.85+1)(s^2+.135s+.005625)}$$

$$L_1(s) = L_{11}(s)L_{12}(s)$$

Once a suitable $L_1(s)$ is found, it is necessary to solve for the required compensation $G(s)$. This is done as follows:

$$L_1(s) = G(s)P_1(s) \implies G(s) = \frac{L_1(s)}{P_1(s)}$$

For this problem, the following $G(s)$ resulted:

$$G_{11}(s) = \frac{-640000.(s+.15)(s+.015)(s+.4)}{(s+100)^3(s+.02)(s+.24)}$$

$$G_{22}(s) = \frac{(s^2+.22s+.0625)(s^2+1.4s+1)(s^2+.105s+.005625)}{(s^2+.35s+.065)(s^2+1.8s+1)(s^2+.135s+.005625)}$$

$$G(s) = G_{11}(s)G_{22}(s)$$

If this were the optimum $G(s)$, yielding an $L(j\omega)$ on the boundaries at all frequencies, it would yield a $|\Delta T(j\omega)|$, as the parameters of P varied, which would be identical to the $|\Delta T(j\omega)|$ specified. Since all points of $L(j\omega)$ are on or above the boundaries, the resulting $|\Delta T(j\omega)|$ for the example, call it $|\Delta T_2|$, should be the same or less than that specified, called $|\Delta T_1|$. Figure 4.4 compares these two curves. It is clear that the
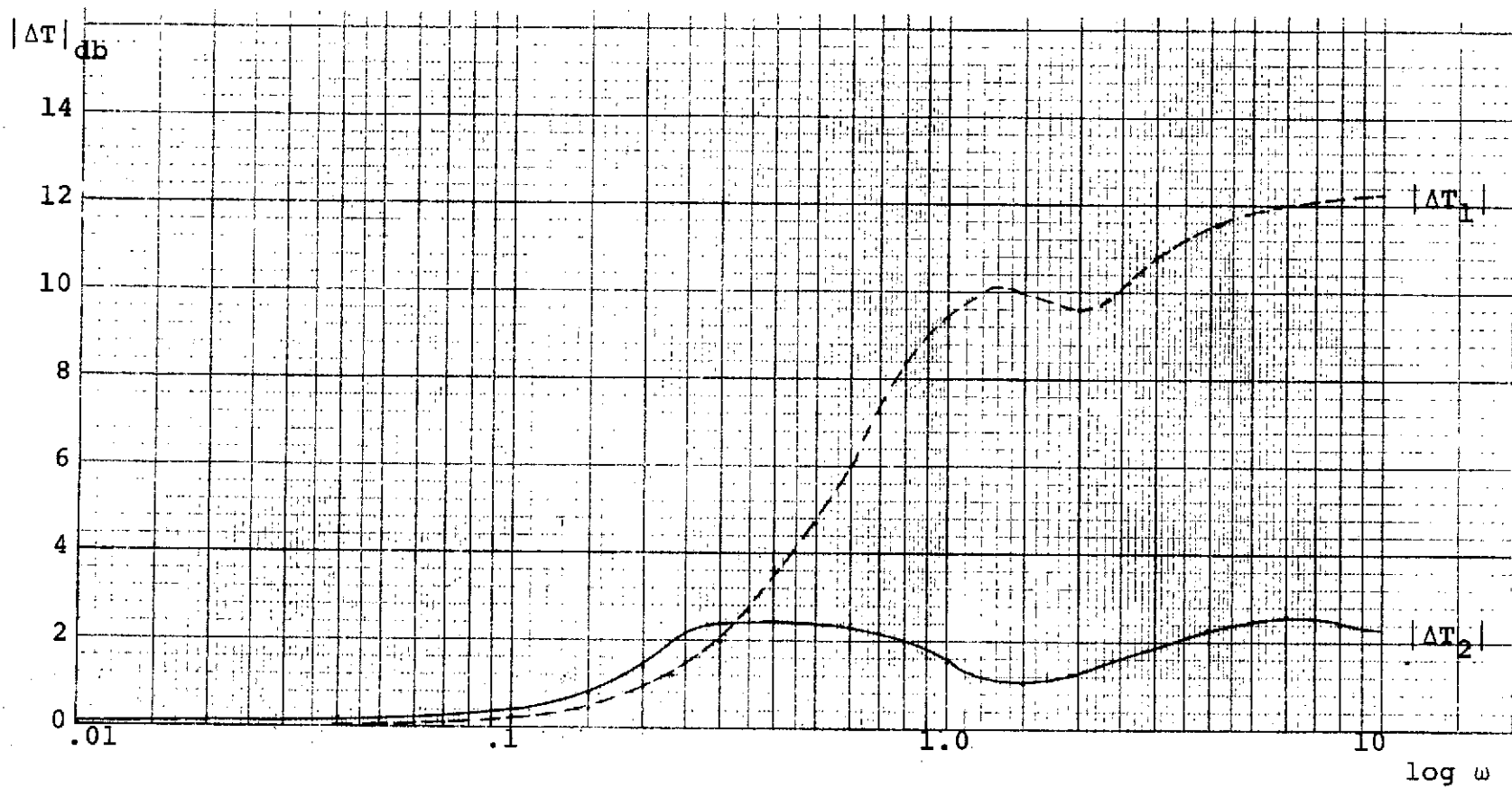
Figure 4.4 Allowed variation $|\Delta T_1|$ vs Actual Variation $|\Delta T_2|$

curves are not as expected. The error is not in the method of designing for the smallest zero since the zero has not been allowed to take on any other values. After examination of the program which finds the roots of T(s) from L(s), an error is clear. This program takes the roots of L(s) and outputs the poles of T(s). For one run there is a pole and a zero of L at .015 and therefore there should be a pole and a zero of T(s) at the same point. The program says the pole is at .0155 instead of .015. This error alone leads to a $|\Delta T_2|$ that is in error by easily the amount encountered. The method used by the subroutines of this program lead to a significant error when poles are used at such low frequencies. Since there is an error, but an unavoidable one for this example, the example will continue. This error does not appear to be a significant one.

The final step in the design is to find the prefilter that will put the variation in the shape and bandwidth of the allowed region indicated by the components of $|\Delta T_1|$. The compensation G holds the plant variation to an acceptable value and F shapes it to match the T_1 specifications. From the equation below, a way of calculating F(s) is to use the maximum curve making up $|\Delta T_1|$ and subtracting the maximum curve of $|\Delta T_2|$ in decibels. This difference can be realized by some F(s) that will make the curves coincide. If the optimum G(s)

had been obtained,

$$T_1(s) = \frac{F(s)L(s)}{1 + L(s)} = F(s)T_2(s)$$

$$|F(j\omega)| = \frac{T_1(j\omega)}{T_2(j\omega)}$$

$$|F(j\omega)|_{db} = |T_1(j\omega)|_{db} - |T_2(j\omega)|_{db}$$

this F(s) would force the minimum curves of T_1 and T_2 to be the same also since $|\Delta T_1| = |\Delta T_2|$ in this ideal case. Since the G(s) used is only close to ideal, two prefilters would result, but they are so close to being the same that either should yield a satisfactory response. Figure 4.5 shows these curves and the F(s) realized which was:

$$F(s) = \frac{53.352 \ (s^2 + .26s + .0625)}{(s+2)(s+3)^2(s^2 + .35s + .0625)}$$

Now the design for the closed loop structure of figure 1.1 is complete. Figure 4.6 shows the step responses of the system under various plant conditions. It is clear the design worked very well. The allowed region variation is utilized to the fullest and all responses are acceptable.

Some things should be pointed out about the example. Usually the simplest problem is gain variation, but this was not possible here. The specifications on $|\Delta T_1|$ were so low in this frequency range that only an extremely
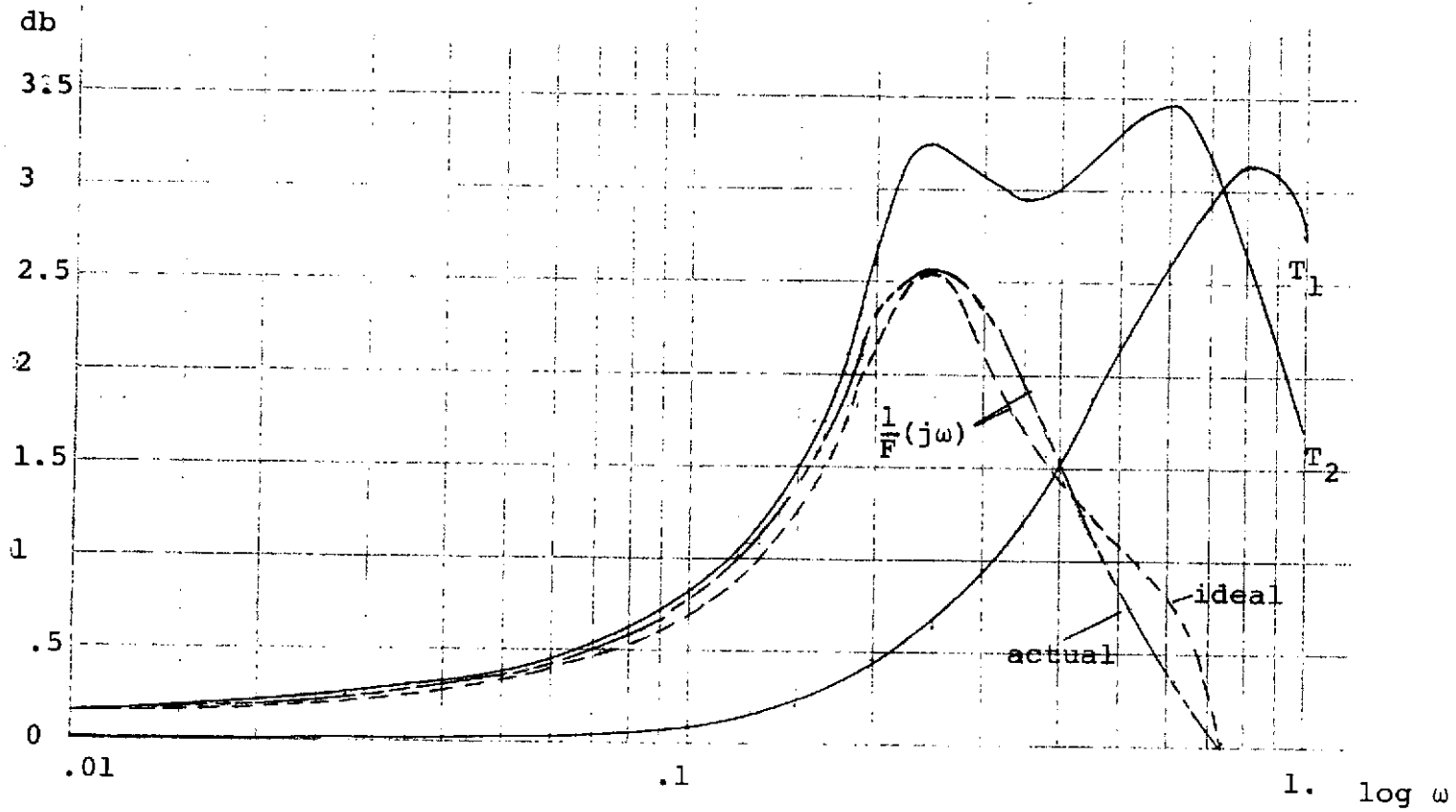
Figure 4.5   Obtaining F(jω)
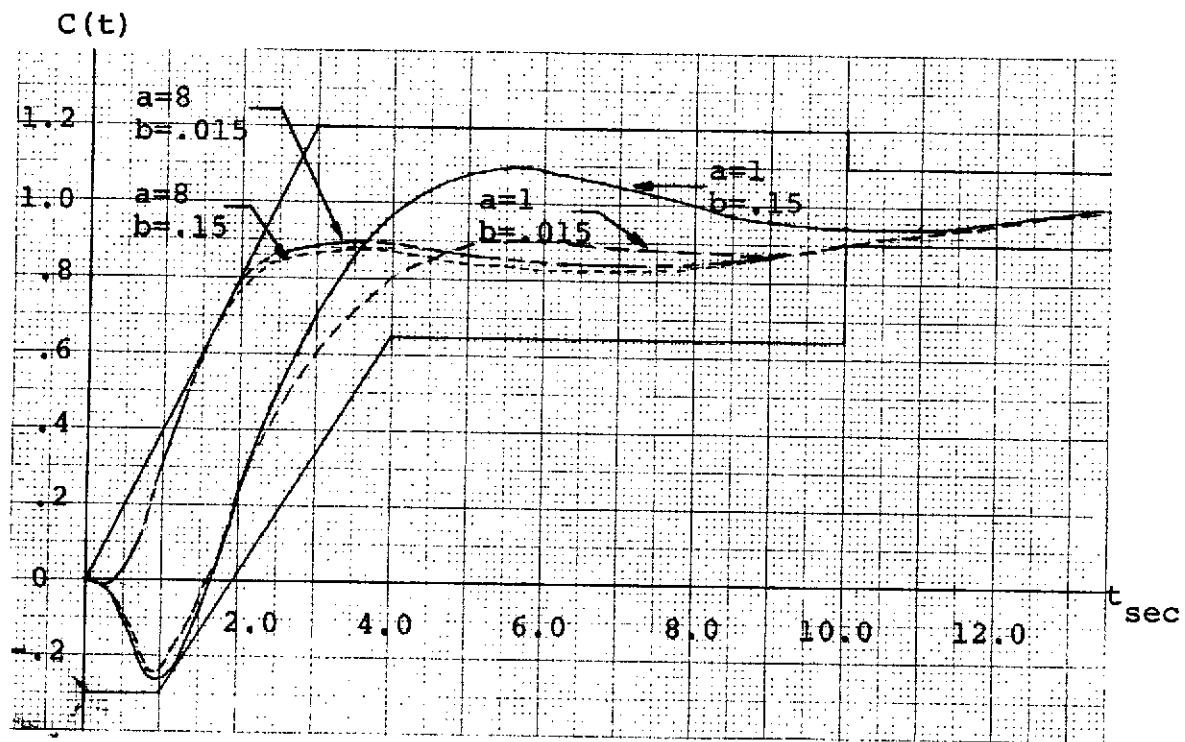
Figure 4.6 Step Responses

low gain variation could be used. Since it would be so impractically small, it was left out of this example. Another example with larger tolerances must be worked in order to demonstrate the technique's validity in problems with gain variation.

This example was to design over a range of RHP zeros where the $|\Delta T(j\omega)|$ specifications vary greatly. Variation from one to eight demonstrates how correct the premise is of designing for the smallest zero. If the specifications were RHP zeros from three to eight where the tolerances are almost constant, there could still be doubt in the usefulness of the theory here extended. A range of specifications will not be of main concern in the next example since this one showed how effective the design technique is.

Another example will therefore be worked in this area to investigate gain variation. A higher frequency range will be worked with to diminish the effect of errors encountered in the subroutines of the program that finds the roots of $T(s)$.

This should help clarify whether the subroutines are the cause of the error or not. Also another example will stress the effectiveness of the method. Modeling time domain boundaries by a third order transfer function gave a very adequate set of frequency specifications on $|\Delta T(j\omega)|$.

CHAPTER   V

SECOND EXAMPLE

Whereas the first example was concerned purely with verifying the design theory be designing over a range of specification variation, this example was to answer other questions. The first example, although verifying the design technique, had raised some questions that needed more than a possible explanation. First, was the excessive variation of the designed transfer function ($|\Delta T_2|$) really caused by the inaccuracy of the subroutines of the root finding program? Second, gain variation was impossible due to the low specs that had to be met in the first example. Does the design method work for gain variation, or are further problems encountered? To answer these questions, this example has gain variation and deals with a higher frequency range to help decrease the error in the subroutines.

This example uses the same time domain specifications as the first example, but the plant is changed to:

$$P(s) = \frac{K(s-a)}{s(s+b)} , \quad \text{where} \quad \begin{array}{l} a\varepsilon[3,8] \\ K\varepsilon[1,2.2] \\ b\varepsilon[.075,.75] \end{array}$$

This plant needs feedback to meet the design specs up to a frequency of .72 RPS. This was found by seeing where the $|\Delta P|_{db}$ specs become lower than allowed $|\Delta T_1|_{db}$

specs (Figure 5.1).

The next step is to find the L-boundaries for the plant over this frequency range. These are found by the program implementing the Horowitz and Sidi method and are shown in figure 5.2. Also shown in the figure is the region that must be avoided in order to permit only 25% overshoot of a disturbance signal.

Finding a loop transmission L(s) that meets these boundaries is the most difficult part of the design. The optimal solution is to meet each boundary at each frequency and to come as close as possible to the forbidden region. Shown in figure 4.2 is a plot of the
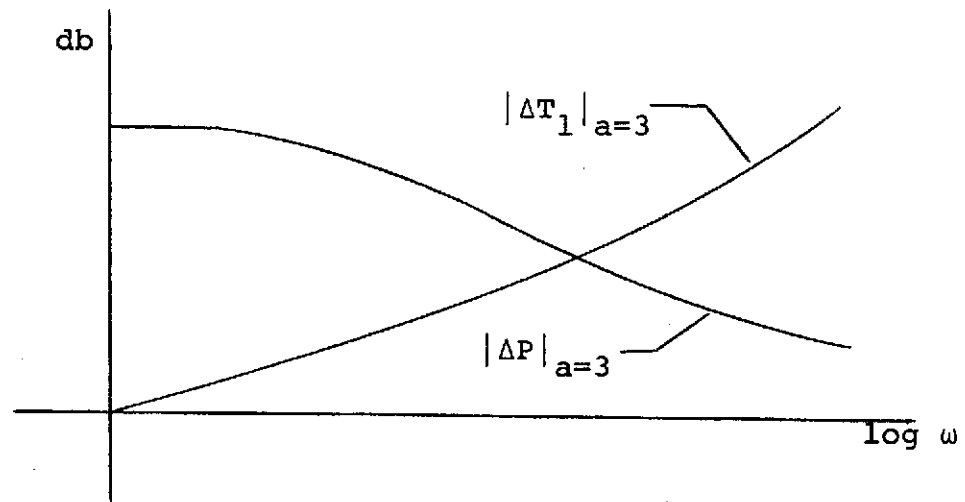
$$|P(j\omega)|_{a=3} = \left| \frac{2.2(j\omega+.75)}{1(j\omega+.75)} \right|$$

Figure 5.1   Frequency Range Requiring Feedback
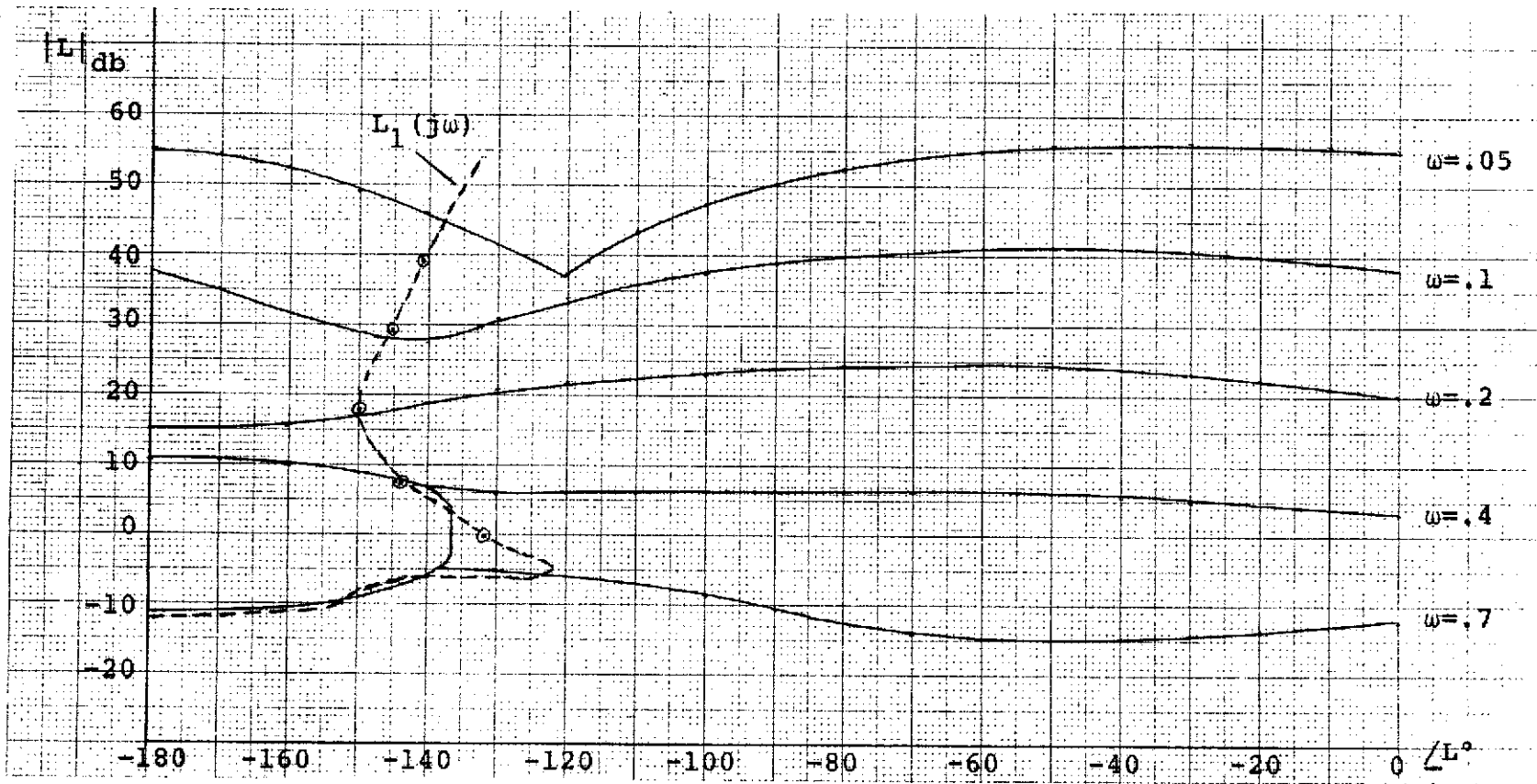
Figure 5.2   L-Boundaries and Resulting $L_1(j\omega)$

L(jω) found for this problem. A much better L(jω) can be found but requires a great deal more effort by the trial and error method employed. The loop transmission found suitable for $P_1(s) = \frac{s-3}{s(s+.075)}$ is:

$$L_1(s) = \frac{-140000.(s-3)(s+.75)(s^2+2.66s+3.61)(s^2+9.8s+49)}{s(s+.05)(s+100)^3(s^2+1.52s+3.61)(s^2+5.6s+49)}$$

Now the required compensation G may be found as follows:

$$L_1(s) = G(s)P_1(s) \implies G(s) = \frac{L_1(s)}{P_1(s)}$$

For this problem:

$$G(s) = \frac{-140000.(s+.075)(s+.75)(s^2+2.66s+3.61)(s^2+9.8s+49)}{(s+.05)(s+100)^3(s^2+1.52s+3.61)(s^2+5.6s+4.9)}$$

Using this compensation and allowing the plant parameters to vary throughout their entire range, holding a = 3, will yield the actual variaton of the transfer function $|\Delta T_2|_{db}$. As in example one, the maximum component of $|\Delta T_2|$ will be required to complete the design by solving for the required prefilter F(s). Figure 5.3 compares $|\Delta T_2|$ and the specs of $|\Delta T_1|$. If the L_1 found in figure 5.2 had points on all the L-boundaries instead of some above and some below, the plots of $|\Delta T_1|$ and $|\Delta T_2|$ would be identical. When a point is above the L-boundary, the $|\Delta T_2|$ at that frequency will be below

that for $|\Delta T_1|$ and the opposite is also true. This was the problem encountered in example one and thought to be caused by errors due to the techniques used in the subroutines of root finding program. Note that the problem is not encountered here as was hoped. Poles do not occur at the low frequencies encountered in the first example, and therefore if the subroutines were the cause of the error they should not and do not lead to erroneous results here. This answers or at least supports the suspected answer to the question concerning the origin of the error encountered in the first example.

Figure 5.4 shows four things, the maximum component of $|\Delta T_1|_{db}$ (T_{1m}), the maximum component of $|\Delta T_2|_{db}$ (T_{2m}), the prefilter required F_R, and the prefilter realized F_A. As before, F_R is simply the point by point subtraction of T_{2m} from T_{1m} as explained by:

$$T_{2m} = \frac{L_m}{1+L_m}$$

$$|T_{1m}| = \frac{F\,L_m}{1+L_m} \implies |F| = \frac{|T_{1m}|}{\dfrac{L_m}{1+L_m}} = \frac{|T_{1m}|}{|T_{2m}|}$$

$$|F|_{db} = |T_{1m}|_{db} - |T_{2m}|_{db}.$$
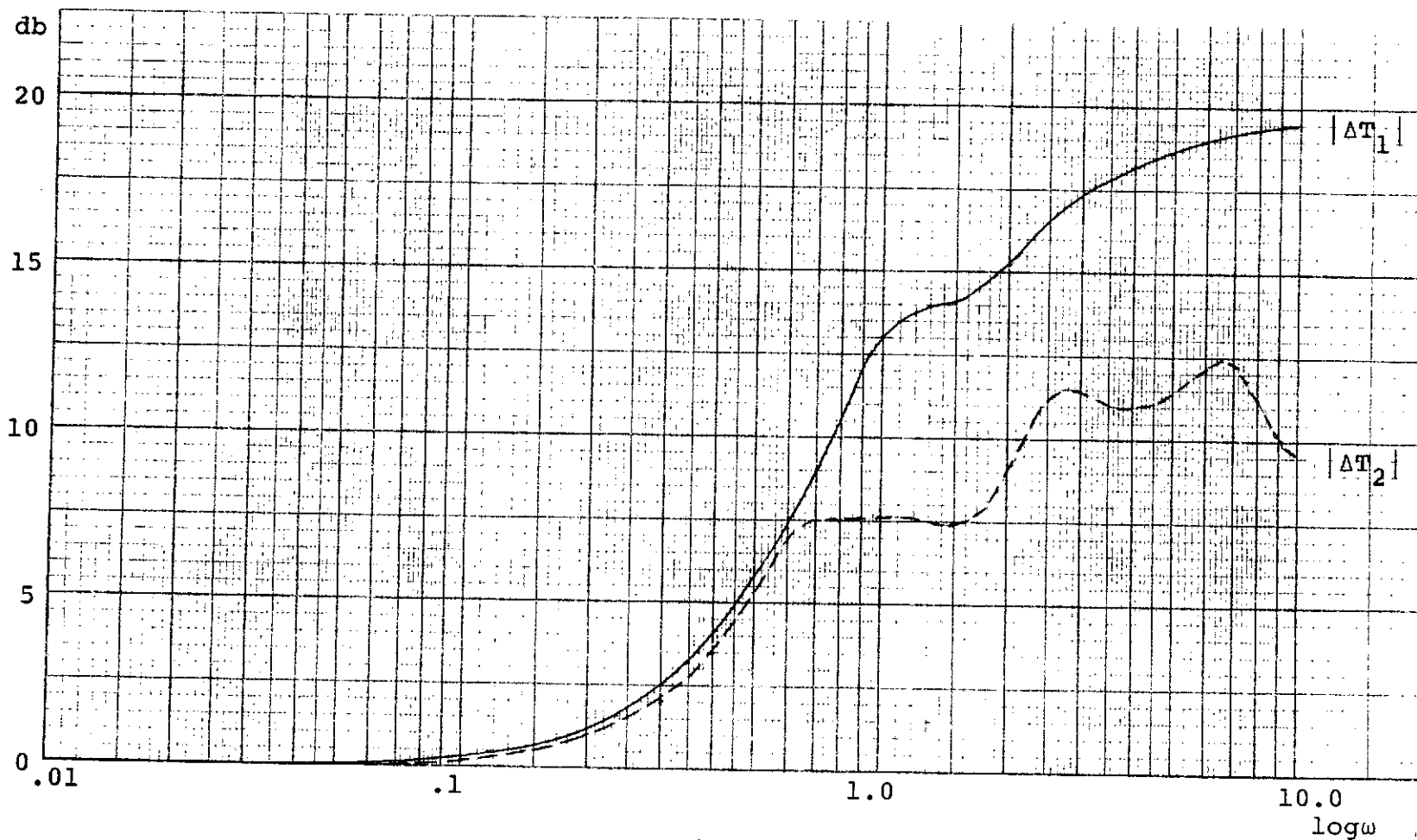
Figure 5.3 $|\Delta T_1|$ vs $|\Delta T_2|$
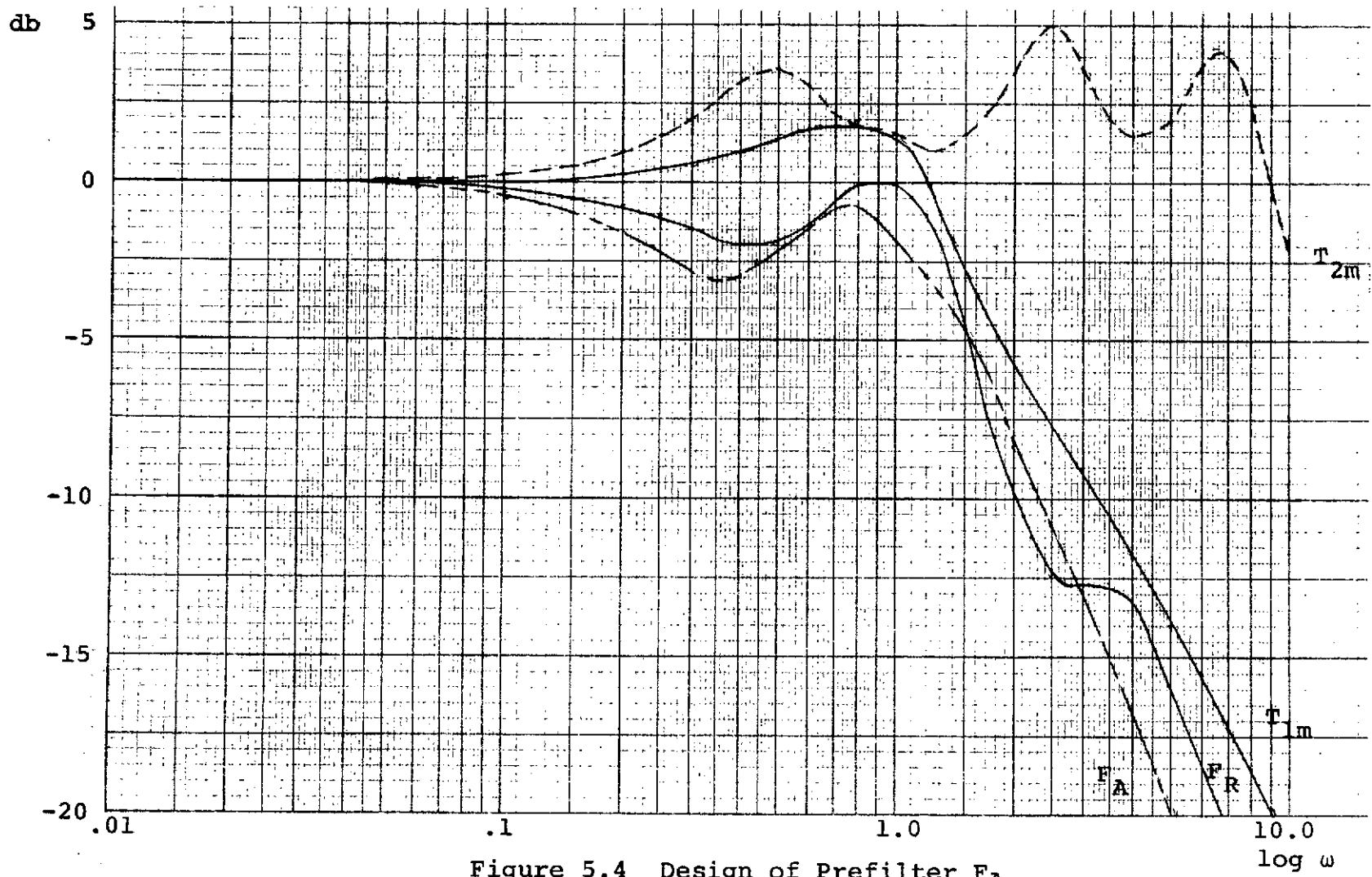
Figure 5.4 Design of Prefilter F_A

The prefilter found is:

$$F(s) = \frac{3.49(s+2.5)^2(s^2+.42s+.176)}{(s+.7)^2(s+4)^2(s^2+.98s+.49)}$$

Thus the design is now complete. Figure 5.5 shows exactly what is meant by a prefilter F and compensation G as to their part in the closed loop structure. To test the design, all parameters were varied over their allowed range and plots of their step responses were made. Figure 5.6 shows that again the design technique works perfectly. The allowed time response region is utilized to its fullest as the economical design should.

This example accomplished two things. First, it again verified how well the design technique functions when applied to a problem. Second, it helped answer the questions that prompted a second example. The inaccuracy
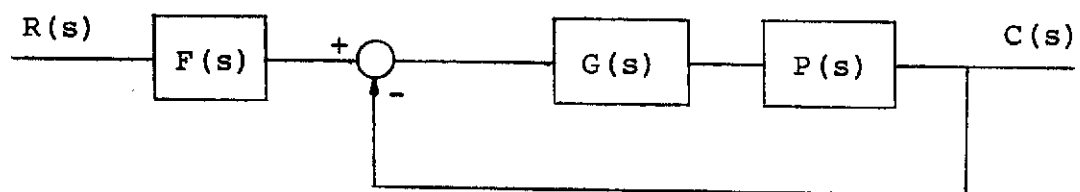
$$T(s) = \frac{C(s)}{R(s)}$$

$$L(s) = G(s)P(s)$$

Figure 5.5   Closed Loop Structure
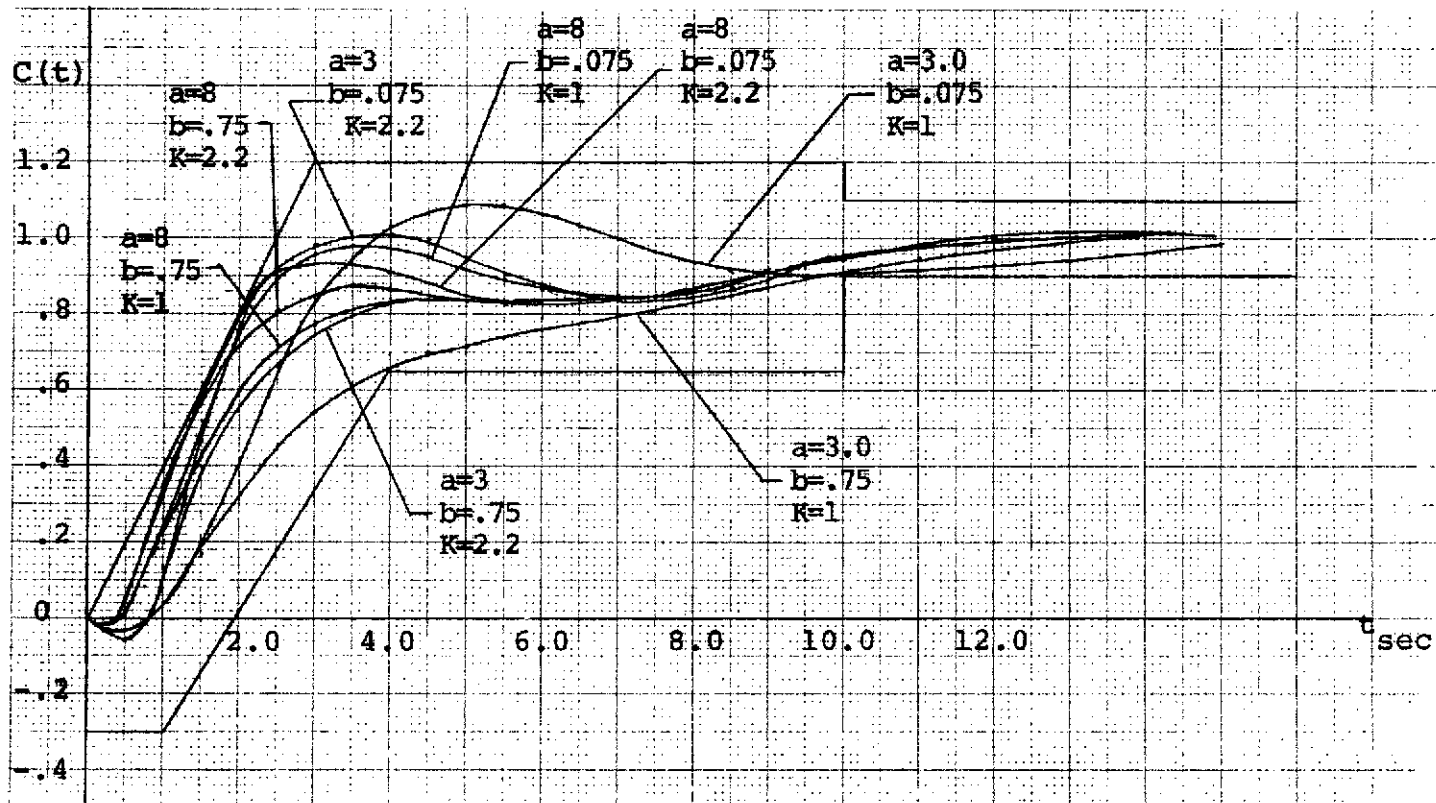
Figure 5.6 Step Responses of Final Design for $P(s) = \dfrac{K(s-a)}{s(s+b)}$

of the subroutines of the root finding program at very low frequencies certainly seems to be the cause of excessively large variation of $|\Delta T_2|$ at certain frequencies in the first example. The design technique works for gain variation without introducing any complications.

The example again strenghtened the feeling that a design for a system requiring feedback up to $\omega = \dfrac{a_{min}}{4}$ is not an easy problem, but is a workable one. A problem requiring feedback up to $\dfrac{a_{min}}{2}$ will be much more difficult if the trial and error method is used to find the L meeting the required L-boundaries. This is the only difficult part of the design and requires time in exponential proportion to the accuracy desired.

CHAPTER VI
CONCLUSIONS

The first design method presented in this paper is not very useful in the majority of real world problems because of its inability to handle variation of the right-half-plane zero. Its technique may yield a more general method or at least give some insight in the problems encountered in a nonminimum-phase problem.

The second method is very effective in presenting an a priori design technique for a plant with one right-half-plane zero and the plant is not a function of the right-half-plane zero in any other way. Necessary extensions of this method are to handle multiple right-half-plane zeros and plants that depend on the right-half-plane zeros in more complicated ways than just having a zero there, i.e., gain is a function of the position of the zeros.

Some of the simplicity of this method may be lost when these extensions are made, but this method allows an insight into phases of the design which point out short cuts. For example, it made it clear that under certain circumstances a design for one position of the zero will work for all variations. It also facilitated the presentation of criteria explaining when a design is not possible by this method. It seems very likely that a pursuit of the nonminimum-phase problem along these lines may yield a general design technique.

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Appendix A

DIGITAL COMPUTER IMPLEMENTATION OF TIME DOMAIN TO

FREQUENCY DOMAIN TRANSFORMATION

A.1 General Description

The program which yields the $|\Delta T|$ specs necessary

to make the design and the maxima and minima curves of

T to calculate a prefilter F is presented here. It is

coded in FORTRAN IV source language and has been exe-

cuted on a Control Data Corporation 6400 computer. A

series of IF STATEMENTS in the subroutine INVLAP de-

scribe the step response boundaries. A "hit" is a res-

ponse that at some time came to within ±.01 of the

boundary without ever exceeding this tolerance. This

tolerance and the amount the complex pair of poles are

incremented each time can be adjusted to suit the ac-

curacy desired.

The program varies the complex poles of

$$T(s) = \frac{(s-a)\left(\dfrac{\omega_n^2 \, 20}{-a}\right)}{(s^2 + 2z\omega_n s + \omega_n^2)(s+20)}$$

and finds the region of pole variation that yields step

responses within the boundaries. It will print out the

first and last set of parameters giving satisfactory

responses for a given real value of the poles, varying

the imaginary part. The sets of parameters that yield

"hits" are also printed out. Finally the program prints

out the maximum, minimum, and difference of the magnitude and phase bode plots of all the hits. All this information provides the insight necessary to design for one set of parameters and yield satisfactory responses as the parameters vary.

The following is a logic flow chart of the program and a complete listing of the program with a sample set of output.

A.2 Flow Chart of Time to Frequency Domain Transformation
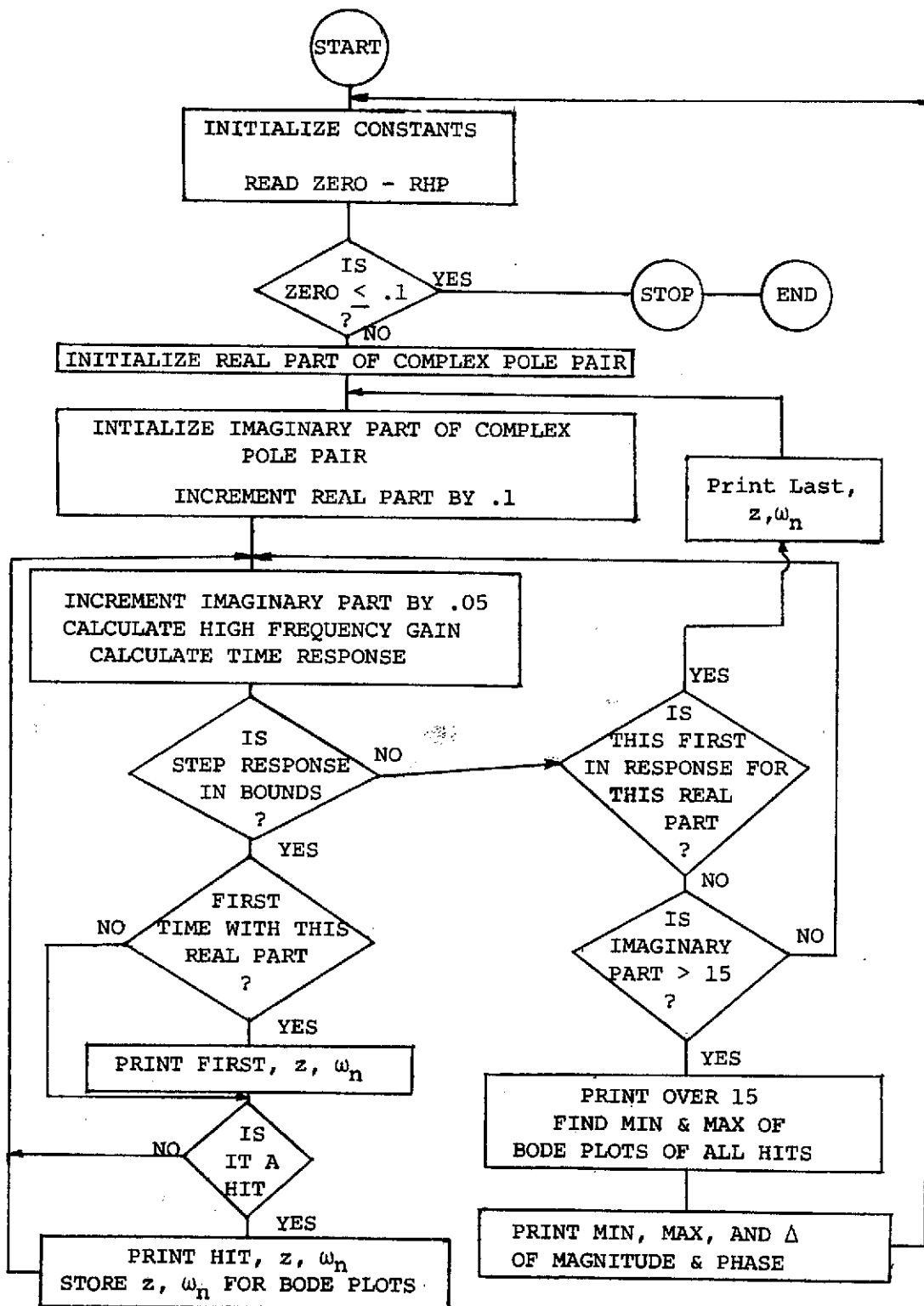
A.3 Program Listing

```
      PROGRAM TEST2(INPUT,OUTPUT)
      DIMENSION RD(20),RN(20),A(101,9),CN(20),P(5),Z(5),PC(2,10),
     2B(101,9),DAT(200,4),ZC(2,10)
      COMPLEX RN,RD
   10 ITEM=2
      CN(1)=0.
      IFINI=0
      A(1,1)=.1
      RD(1)=CMPLX(0.0,0.0)
      RD(2)=CMPLX(-20.0,0.0)
   20 READ 900,ZERO
  900 FORMAT(F10.6)
      PRINT 901,ZERO
  901 FORMAT(1X,F10.6)
      IHIT=1
      IF(ZERO .LE. .1) GO TO 500
      IFINI=IFINI+1
      RN(1)=CMPLX(ZERO,0.0)
      POLER=.3
   30 POLEI=0.
      POLER=POLER+.1
      FRST=0.
   35 POLEI=POLEI+.05
      IN=0
      RD(3)=CMPLX(-POLER,-POLEI)
      RD(4)=CMPLX(-POLER,POLEI)
      SF= -(20./ZERO)*(POLER**2+POLEI**2)
      CALL INVLAP(SF,RD,4,RN,1,CN,100,.1,1,POLER,POLEI,DAT,RET,ZERO,A,
     2ITEM,IHIT,IN,FRST)
      IF(RET-3.)35,30,40
   40 ITEM1=IHIT-1
      DO 39 I=1,100
      A(I,2)=-1000.
      A(I,3)=1000.
      B(I,2)=-1000.
   39 B(I,3)=1000.
      DO 100 I=1,ITEM1
      P(1)=DAT(I,1)
      Z(1)=DAT(I,2)
      PC(1,1)=DAT(I,3)
      PC(2,1)=DAT(I,4)
      G=1.
      NP=1
      NPC=1
      NZ=1
      DEL=.1
      NZC=0
      S=0.
      CALL BODE(S,G,NP,P,NPC,PC,NZ,Z,NZC,ZC,DB,Q,DEL,A,B)
  100 CONTINUE
      INOW=IFINI+3
      DO 200 I=1,100
      A(I,INOW)=A(I,2)-A(I,3)
      B(I,INOW)=B(I,2)-B(I,3)
  200 CONTINUE
  600 PRINT 910
      DO 300 I=1,100
      PRINT 905,(A(I,J),J=1,4),(B(I,J),J=2,4)
  300 CONTINUE
  905 FORMAT(1X,7F13.4)
  910 FORMAT(1H1)
      PRINT 910
      GO TO 10
  500 CONTINUE
  990 END
```

```
      SUBROUTINE INVLAP(SF,RD,ND,RN,NN,CN,NT,DT,K,POLER,POLEI,DAT,RET,
     2ZERO,A,ITEM,IHIT,IN,FRST)
COMMENT   ROUTINE EVALUATES THE INVERSE LAPLACE TRANSFORM OF A RATIONAL
C         FUNCTION AT NT POINTS AT INTERVALS DT.
C         INPUT FORMS
C         K=1, SF*((S-RN(1))*...*(S-RN(NN)))/((S-RD(1))*...*(S-RD(ND)))
C         K=2, SF*(S**NN+C(2)*S**(NN-1)+...+C(NN)*S+C(NN+1))/
C              ((S-RD(1))*...*(S-RD(ND))).
      DIMENSION CN(20),RD(20),RN(20),RES(20),DAT(200,4),A(101,9)
      COMPLEX CT1, RD, RN, RES
      INTEGER HIT
      HIT=0
    1 FORMAT(1H ,10X,*IMAGINARY PART OF TIME RESPONSE OUT OF LIMITS AT
     2T=*,F10.4,* CT=*,2E12.5)
      DO 7 I=1,ND
      RES(I)=SF
      IF(NN.EQ.0) GO TO 6
      GO TO(2,4),K
    2 DO 3 J=1,NN
    3 RES(I)=RES(I)*(RD(I)-RN(J))
      GO TO 6
    4 RES(I)=(RD(I)+CN(2))
      DO 5 J=2,NN
    5 RES(I)=RES(I)*RD(I)+CN(J+1)
      RES(I)=SF*RES(I)
    6 DO 7 J=1,ND
      IF(I.EQ.J) GO TO 7
      RES(I)=RES(I)/(RD(I)-RD(J))
    7 CONTINUE
      T1=A(1,1)
      DO 9 I=1,NT
      CT1=0.
      DO 8 J=1,ND
    8 CT1=CT1+RES(J)*EXP(T1*REAL(RD(J)))*CMPLX(COS(T1*AIMAG(RD(J))),
     2SIN(T1*AIMAG(RD(J))))
      IF(ABS(AIMAG(CT1)).GT..005*ABS(REAL(CT1)))   PRINT 1,T1,CT1
      A(I,1)=T1
      A(I,ITEM)=REAL(CT1)
   95 IF(T1 .GE. 9.87) GO TO 750
  100 IF(T1 .GE. (7.) .AND. A(I,ITEM) .LE.(1.11).AND. A(I,ITEM).GE.
     2(.89))GO TO 700
  102 IF(T1 .GE. 7.) GO TO 125
  105 IF(T1 .GE.(4.).AND.A(I,ITEM).LE.(1.21).AND.A(I,ITEM).GE.(.64))
     2GO TO 600
  107 IF(T1 .GE. 4.) GO TO 125
  110 IF(T1.GE.(3.).AND.A(I,ITEM).LE.(1.21) .AND. A(I,ITEM) .GE.
     2(.31667*T1-.63)) GO TO 500
  112 IF(T1 .GE. 3.) GO TO 125
  115 IF(T1 .GE. (1.) .AND. A(I,ITEM) .LE. (.4*T1+.01) .AND. A(I,ITEM)
     2.GE. (.31667*T1-.63)) GO TO 400
  117 IF(T1 .GE. 1.) GO TO 125
  120 IF (A(I,ITEM) .GE. (-.31) .AND. A(I,ITEM) .LE. (.4*T1+.01))
     2GO TO 300
  125 HIT=0
      IF (FRST .LE. .5) GO TO 130
  127 OMEGA=SQRT(POLER**2+POLEI**2)
      ZETA=POLER/OMEGA
      PRINT 971,POLER,POLEI,ZETA,OMEGA
  971 FORMAT(1X,5HREAL=,F10.6,3X,5HIMAG=,F10.6,3X,5HZETA=,F10.6,3X,
     26HOMEGA=,F10.6,4X,4HLAST,/)
      FRST=0.
      RET=3.
      GO TO 15
  130 IF (POLEI .GE. 15.) GO TO 150
  135 IF (IN .GE. 2) GO TO 370
  145 GO TO 10
  150 RET=4.
      PRINT 973
  973 FORMAT(1X,13HPOLEI OVER 15,//)
  195 GO TO 15
  300 IF (A(I,ITEM) .LE. (-.29) .OR. A(I,ITEM) .GE. (.4*T1+.01)) HIT=4
  310 IF (IN .LE. 2) GO TO 350
```

```
325 GO TO 9
350 IN=4
    GO TO 9
370 IN=0
    GO TO 10
400 IF (A(I,ITEM) .LE. (.31667*T1-.60667) .OR. A(I,ITEM) .GE.
   2(.4*T1-.01)) HIT=4
410 GO TO 310
500 IF (A(I,ITEM) .LE. (.31667*T1-.60667) .OR. A(I,ITEM) .GE. (1.19))
   2HIT=4
505 GO TO 310
600 IF (A(I,ITEM) .LE. (.66) .OR. A(I,ITEM) .GE. (1.19)) HIT=4
605 GO TO 310
700 IF(A(I,ITEM) .LE. (.91) .OR. A(I,ITEM) .GE. (1.09)) HIT=4
705 GO TO 310
750 IF (FRST .GE. (.5)) GO TO 775
    OMEGA=SQRT(POLER**2+POLEI**2)
    ZETA=POLER/OMEGA
    PRINT 970,POLER,POLEI,ZETA,OMEGA
970 FORMAT(1X,5HREAL=,F10.6,3X,5HIMAG=,F10.6,3X,5HZETA=,F10.6,3X,
   26HOMEGA=,F10.6,4X,5HFIRST)
    FRST=1.
775 IF (HIT .LE. 2) GO TO 10
780 OMEGA=SQRT(POLER**2+POLEI**2)
    ZETA=POLER/OMEGA
    PRINT 972,POLER,POLEI,ZETA,OMEGA
972 FORMAT(1X,5HREAL=,F10.6,3X,5HIMAG=,F10.6,3X,5HZETA=,F10.6,3X,
   26HOMEGA=,F10.6,4X,3HHIT,/)
    HIT=0
    DAT(IHIT,1)=20.
    DAT(IHIT,2)=-(ZERO)
    DAT(IHIT,3)=ZETA
    DAT(IHIT,4)=OMEGA
    IHIT=IHIT+1
  9 T1=T1+DT
 10 RET=2.
 15 RETURN
    END
_____
    SUBROUTINE BODE(S,G,NP,P,NPC,PC,NZ,Z,NZC,ZC,DB,Q,DEL,A,B)
COMMENT SUBROUTINE EVALUATES RATIONAL FUNCTION AT S.
COMMENT POLES=P(I), ZEROS=Z(I), COMPLEX POLES=ZETA,OMEGA=PC(1,I),PC(2,I)
COMMENT COMPLEX ZEROS=ZETA,OMEGA=ZC(1,I),ZC(2,I), ERROR CONSTANT=G.
    DIMENSION P(1),Z(1),PC(2,10),ZC(2,10),A(101,3),B(101,3)
    PI=3.1415927
    DO 100 IBID=1,100
    S=S+DEL
    A(IBID,1)=S
    B(IBID,1)=S
    DB=G**2
    Q=0.
    IF(NP.LE.0) GO TO 3
    DO 2 I=1,NP
    IF(P(I).EQ.0.) GO TO 1
    DB=DB*P(I)**2/(S**2+P(I)**2)
    Q=Q-ATAN(S/P(I))
    GO TO 2
  1 DB=DB/S**2
    Q=Q-PI/2.
  2 CONTINUE
  3 IF(NZ.LE.0) GO TO 6
    DO 5 I=1,NZ
    IF(Z(I).EQ.0.) GO TO 4
    DB=DB*(S**2+Z(I)**2)/Z(I)**2
    Q=Q+ATAN(S/Z(I))
    GO TO 5
  4 DB=DB*S**2
    Q=Q+PI/2.
  5 CONTINUE
  6 IF(NPC.LE.0) GO TO 9
```

```
      DO 8 I=1,NPC
      DB=DB*PC(2,I)**4/((PC(2,I)**2-S**2)**2+4.*(PC(2,I)*PC(1,I)*S)**2)
      Q=Q-ATAN(2.*PC(1,I)*PC(2,I)*S/(PC(2,I)**2-S**2))
      IF(S.LE.PC(2,I)) GO TO 8
      IF(PC(1,I).LT.0.) GO TO 7
      Q=Q-PI
      GO TO 8
    7 Q=Q+PI
    8 CONTINUE
    9 IF(NZC.LE.0) GO TO 12
      DO 11 I=1,NZC
      DB=DB*((ZC(2,I)**2-S**2)**2+4.*(ZC(1,I)*ZC(2,I)*S)**2)/ZC(2,I)**4
      Q=Q+ATAN(2.*ZC(1,I)*ZC(2,I)*S/(ZC(2,I)**2-S**2))
      IF(S.LE.ZC(2,I)) GO TO 11
      IF(ZC(1,I).LT.0.) GO TO 10
      Q=Q+PI
      GO TO 11
   10 Q=Q-PI
   11 CONTINUE
   12 DB=SQRT(ABS(DB))
      IF (DB .GT. 0.) GO TO 13
      PRINT 937,IBID
  937 FORMAT(10X,I10,/)
      PRINT 901,P(1),Z(1),PC(1,1),PC(2,1)
  901 FORMAT(1X,4F10.6)
   13 R=DB*COS(Q)
      AM=DB*SIN(Q)
      DB=20.*ALOG10(DB)
      Q=Q*180./PI
      IF(A(IBID,2) .GE. DB) GO TO 50
   45 A(IBID,2)=DB
   50 IF(A(IBID,3) .LE. DB) GO TO 60
   55 A(IBID,3)=DB
   60 IF(B(IBID,2) .GE. Q) GO TO 70
   65 B(IBID,2)=Q
   70 IF (B(IBID,3) .LE. Q) GO TO 80
   75 B(IBID,3)=Q
   80 CONTINUE
  100 CONTINUE
      RETURN
      END
```

```
  3.000000
REAL=    .400000   IMAG=    .350000   ZETA=    .752577   OMEGA=    .531507   FIRST
REAL=    .400000   IMAG=    .750000   ZETA=    .470588   OMEGA=    .850000   HIT

REAL=    .400000   IMAG=    .800000   ZETA=    .447214   OMEGA=    .894427   LAST

REAL=    .500000   IMAG=    .250000   ZETA=    .894427   OMEGA=    .559017   FIRST
REAL=    .500000   IMAG=    .250000   ZETA=    .894427   OMEGA=    .559017   HIT

REAL=    .500000   IMAG=    .950000   ZETA=    .465746   OMEGA=   1.073546   HIT

REAL=    .500000   IMAG=   1.000000   ZETA=    .447214   OMEGA=   1.118034   LAST

REAL=    .600000   IMAG=    .050000   ZETA=    .996546   OMEGA=    .602080   FIRST
REAL=    .600000   IMAG=    .050000   ZETA=    .996546   OMEGA=    .602080   HIT

REAL=    .600000   IMAG=    .100000   ZETA=    .986394   OMEGA=    .608276   HIT

REAL=    .600000   IMAG=   1.000000   ZETA=    .514496   OMEGA=   1.166190   LAST

REAL=    .700000   IMAG=    .050000   ZETA=    .997489   OMEGA=    .701783   FIRST
REAL=    .700000   IMAG=   1.000000   ZETA=    .573462   OMEGA=   1.220656   LAST
```

REAL=	.800000	IMAG=	.050000	ZETA=	.998053	OMEGA=	.801561	FIRST
REAL=	.800000	IMAG=	1.000000	ZETA=	.624695	OMEGA=	1.280625	LAST
REAL=	.900000	IMAG=	.050000	ZETA=	.998460	OMEGA=	.901388	FIRST
REAL=	.900000	IMAG=	.950000	ZETA=	.687745	OMEGA=	1.308625	HIT
REAL=	.900000	IMAG=	1.000000	ZETA=	.668965	OMEGA=	1.345362	LAST
REAL=	1.000000	IMAG=	.050000	ZETA=	.998752	OMEGA=	1.001249	FIRST
REAL=	1.000000	IMAG=	.950000	ZETA=	.724999	OMEGA=	1.379311	HIT
REAL=	1.000000	IMAG=	1.000000	ZETA=	.707107	OMEGA=	1.414214	LAST
REAL=	1.100000	IMAG=	.050000	ZETA=	.998969	OMEGA=	1.101136	FIRST
REAL=	1.100000	IMAG=	.900000	ZETA=	.773957	OMEGA=	1.421267	HIT
REAL=	1.100000	IMAG=	.950000	ZETA=	.756823	OMEGA=	1.453444	LAST
REAL=	1.200000	IMAG=	.050000	ZETA=	.999133	OMEGA=	1.201041	FIRST
REAL=	1.200000	IMAG=	.850000	ZETA=	.816024	OMEGA=	1.470544	HIT
REAL=	1.200000	IMAG=	.900000	ZETA=	.800000	OMEGA=	1.500000	LAST
REAL=	1.300000	IMAG=	.050000	ZETA=	.999261	OMEGA=	1.300961	FIRST
REAL=	1.300000	IMAG=	.800000	ZETA=	.851658	OMEGA=	1.526434	HIT
REAL=	1.300000	IMAG=	.850000	ZETA=	.836970	OMEGA=	1.553222	LAST
REAL=	1.400000	IMAG=	.050000	ZETA=	.999363	OMEGA=	1.400893	FIRST
REAL=	1.400000	IMAG=	.700000	ZETA=	.894427	OMEGA=	1.565248	HIT
REAL=	1.400000	IMAG=	.750000	ZETA=	.881480	OMEGA=	1.588238	HIT
REAL=	1.400000	IMAG=	.800000	ZETA=	.868243	OMEGA=	1.612452	LAST
REAL=	1.500000	IMAG=	.050000	ZETA=	.999445	OMEGA=	1.500833	FIRST
REAL=	1.500000	IMAG=	.600000	ZETA=	.928477	OMEGA=	1.615549	HIT
REAL=	1.500000	IMAG=	.650000	ZETA=	.917556	OMEGA=	1.634778	HIT
REAL=	1.500000	IMAG=	.700000	ZETA=	.906183	OMEGA=	1.655295	LAST
REAL=	1.600000	IMAG=	.050000	ZETA=	.999512	OMEGA=	1.600781	FIRST
REAL=	1.600000	IMAG=	.450000	ZETA=	.962651	OMEGA=	1.662077	HIT
REAL=	1.600000	IMAG=	.500000	ZETA=	.954480	OMEGA=	1.676305	HIT
REAL=	1.600000	IMAG=	.550000	ZETA=	.945687	OMEGA=	1.691892	LAST
REAL=	1.700000	IMAG=	.050000	ZETA=	.999568	OMEGA=	1.700735	FIRST
REAL=	1.700000	IMAG=	.050000	ZETA=	.999568	OMEGA=	1.700735	HIT
REAL=	1.700000	IMAG=	.100000	ZETA=	.998274	OMEGA=	1.702939	HIT
REAL=	1.700000	IMAG=	.150000	ZETA=	.996130	OMEGA=	1.706605	HIT
REAL=	1.700000	IMAG=	.200000	ZETA=	.993151	OMEGA=	1.711724	HIT
REAL=	1.700000	IMAG=	.250000	ZETA=	.989359	OMEGA=	1.718284	HIT
REAL=	1.700000	IMAG=	.300000	ZETA=	.984784	OMEGA=	1.726268	HIT
REAL=	1.700000	IMAG=	.350000	ZETA=	.979457	OMEGA=	1.735655	LAST

POLEI OVER 15

ω	T_{max}	T_{min}	$|\Delta T|_{db}$	$\angle T_{max}°$	$\angle T_{min}°$	$\Delta \angle T°$
.1000	.0714	-.2285	.2999	-7.1978	-20.9955	13.7977
.2000	.2812	-.8796	1.1608	-14.5785	-41.0431	26.4646
.3000	.6136	-1.8648	2.4784	-22.3371	-60.0069	37.6697
.4000	1.0314	-3.0752	4.1067	-30.6910	-77.8711	47.1801
.5000	1.4581	-4.4106	5.8688	-39.8816	-93.7694	53.8878
.6000	1.7584	-5.7941	7.5525	-50.1576	-107.5548	57.3972
.7000	1.7780	-7.1734	8.9514	-58.8446	-119.3672	60.5226
.8000	1.9690	-8.5463	10.5153	-66.5168	-129.4850	62.9682
.9000	1.9261	-9.9803	11.9064	-73.9334	-138.2087	64.2752
1.0000	1.5797	-11.3373	12.9170	-81.0828	-145.8059	64.7230
1.1000	.9278	-12.6121	13.5400	-87.9580	-152.4957	64.5377
1.2000	.0358	-13.8057	13.8415	-94.5560	-158.4509	63.8949
1.3000	-1.0024	-14.9219	13.9195	-100.8775	-163.8056	62.9281
1.4000	-2.1026	-15.9657	13.8631	-106.9263	-168.6641	61.7378
1.5000	-3.0691	-16.9426	13.8735	-112.7084	-173.1074	60.3990
1.6000	-3.6366	-17.8581	14.2215	-118.2316	-177.7854	59.5538
1.7000	-4.2167	-18.7174	14.5007	-123.5052	-182.2910	58.7858
1.8000	-4.7988	-19.5252	14.7265	-128.5392	-186.3371	57.7979
1.9000	-5.2337	-20.2860	15.0523	-133.3443	-190.0116	56.6673
2.0000	-5.6656	-21.0038	15.3382	-137.9314	-193.3801	55.4487
2.1000	-6.0762	-21.6820	15.6058	-142.3115	-196.4924	54.1809
2.2000	-6.4376	-22.3241	15.8865	-146.4953	-199.3871	52.8919
2.3000	-6.7923	-22.9328	16.1405	-150.4933	-202.0945	51.6012
2.4000	-7.1400	-23.5110	16.3710	-154.3159	-204.6387	50.3228
2.5000	-7.4805	-24.0609	16.5804	-157.9726	-207.0391	49.0665
2.6000	-7.8136	-24.5848	16.7712	-161.4729	-209.3118	47.8388
2.7000	-8.1393	-25.0846	16.9453	-164.8257	-211.4700	46.6444
2.8000	-8.4575	-25.5621	17.1046	-168.0392	-213.5250	45.4858
2.9000	-8.7684	-26.0189	17.2505	-171.1214	-215.4861	44.3648
3.0000	-9.0720	-26.4565	17.3845	-174.0797	-217.3616	43.2819
3.1000	-9.3684	-26.8762	17.5078	-176.9211	-219.1585	42.2374
3.2000	-9.6579	-27.2793	17.6214	-179.6523	-220.8829	41.2306
3.3000	-9.9405	-27.6668	17.7263	-182.2792	-222.5402	40.2610
3.4000	-10.2165	-28.0398	17.8233	-184.8077	-224.1353	39.3276
3.5000	-10.4862	-28.3993	17.9131	-187.2432	-225.6724	38.4291
3.6000	-10.7495	-28.7460	17.9965	-189.5908	-227.1553	37.5645
3.7000	-11.0069	-29.0809	18.0740	-191.8552	-228.5875	36.7323
3.8000	-11.2585	-29.4045	18.1461	-194.0407	-229.9722	35.9314
3.9000	-11.5044	-29.7177	18.2133	-196.1517	-231.3121	35.1605
4.0000	-11.7450	-30.0210	18.2760	-198.1919	-232.6100	34.4181
4.1000	-11.9803	-30.3150	18.3347	-200.1650	-233.8681	33.7031
4.2000	-12.2106	-30.6002	18.3896	-202.0744	-235.0887	33.0143
4.3000	-12.4361	-30.8771	18.4410	-203.9233	-236.2738	32.3505
4.4000	-12.6569	-31.1462	18.4893	-205.7148	-237.4253	31.7105
4.5000	-12.8732	-31.4078	18.5346	-207.4516	-238.5448	31.0932

4.6000	-13.0853	-31.6625	18.5773	-209.1365	-239.6341	30.4976
4.7000	-13.2931	-31.9105	18.6174	-210.7718	-240.6945	29.9227
4.8000	-13.4970	-32.1523	18.6553	-212.3600	-241.7275	29.3676
4.9000	-13.6970	-32.3880	18.6910	-213.9031	-242.7344	28.8313
5.0000	-13.8933	-32.6181	18.7247	-215.4034	-243.7164	28.3130
5.1000	-14.0861	-32.8427	18.7566	-216.8628	-244.6746	27.8118
5.2000	-14.2754	-33.0622	18.7868	-218.2830	-245.6102	27.3271
5.3000	-14.4614	-33.2768	18.8154	-219.6659	-246.5240	26.8581
5.4000	-14.6442	-33.4867	18.8425	-221.0130	-247.4171	26.4040
5.5000	-14.8239	-33.6921	18.8682	-222.3260	-248.2903	25.9643
5.6000	-15.0006	-33.8933	18.8927	-223.6062	-249.1445	25.5383
5.7000	-15.1745	-34.0904	18.9159	-224.8550	-249.9804	25.1254
5.8000	-15.3456	-34.2836	18.9380	-226.0738	-250.7988	24.7250
5.9000	-15.5140	-34.4730	18.9590	-227.2638	-251.6004	24.3366
6.0000	-15.6799	-34.6589	18.9791	-228.4261	-252.3859	23.9598
6.1000	-15.8432	-34.8414	18.9982	-229.5619	-253.1558	23.5939
6.2000	-16.0042	-35.0206	19.0164	-230.6721	-253.9108	23.2387
6.3000	-16.1628	-35.1966	19.0338	-231.7579	-254.6514	22.8936
6.4000	-16.3192	-35.3696	19.0504	-232.8200	-255.3782	22.5582
6.5000	-16.4734	-35.5397	19.0663	-233.8595	-256.0917	22.2322
6.6000	-16.6254	-35.7070	19.0816	-234.8771	-256.7923	21.9152
6.7000	-16.7755	-35.8716	19.0961	-235.8738	-257.4805	21.6067
6.8000	-16.9235	-36.0336	19.1101	-236.8501	-258.1567	21.3066
6.9000	-17.0697	-36.1931	19.1234	-237.8070	-258.8214	21.0145
7.0000	-17.2140	-36.3502	19.1363	-238.7450	-259.4749	20.7300
7.1000	-17.3564	-36.5050	19.1486	-239.6648	-260.1177	20.4529
7.2000	-17.4972	-36.6575	19.1604	-240.5671	-260.7500	20.1829
7.3000	-17.6362	-36.8079	19.1717	-241.4524	-261.3721	19.9197
7.4000	-17.7736	-36.9562	19.1826	-242.3214	-261.9845	19.6632
7.5000	-17.9094	-37.1025	19.1931	-243.1745	-262.5875	19.4130
7.6000	-18.0437	-37.2468	19.2032	-244.0123	-263.1813	19.1689
7.7000	-18.1764	-37.3893	19.2129	-244.8353	-263.7661	18.9308
7.8000	-18.3077	-37.5299	19.2222	-245.6440	-264.3424	18.6984
7.9000	-18.4375	-37.6687	19.2312	-246.4388	-264.9103	18.4714
8.0000	-18.5660	-37.8059	19.2399	-247.2202	-265.4700	18.2498
8.1000	-18.6931	-37.9413	19.2482	-247.9885	-266.0219	18.0334
8.2000	-18.8189	-38.0752	19.2563	-248.7442	-266.5661	17.8219
8.3000	-18.9434	-38.2075	19.2641	-249.4876	-267.1028	17.6152
8.4000	-19.0667	-38.3383	19.2716	-250.2192	-267.6323	17.4131
8.5000	-19.1888	-38.4676	19.2788	-250.9392	-268.1548	17.2156
8.6000	-19.3097	-38.5955	19.2858	-251.6480	-268.6704	17.0224
8.7000	-19.4294	-38.7221	19.2926	-252.3459	-269.1793	16.8334
8.8000	-19.5481	-38.8472	19.2992	-253.0332	-269.6817	16.6485
8.9000	-19.6656	-38.9711	19.3055	-253.7103	-270.1778	16.4675