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A SOLUTION TO THE PROBLEM OF OPTIMIZING THE FUEL BIAS FOR A LIQUID PROPELLANT ROCKET BY AN APPLICATION OF THE CENTRAL LIMIT THEOREM

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16. ABSTRACT

This paper presents an accurate and efficient method of determining the fuel bias for a bipropellant liquid rocket that minimizes outage associated penalties on payload potential. A fuel bias so derived is normally called "the optimum fuel bias."

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DEFINITION OF SYMBOLS

EMR	Random variable, stage effective mixture ratio
emr	Outcome of EMR
FM	Nominal mainstage fuel
f	$f \epsilon R_F$
g	g:EMR $\rightarrow R_L$
h	h:EMR $\rightarrow R_F$
LM	Nominal mainstage LOX
ℓ	$\ell \epsilon R_F$
MB	Fuel bias
R	Random variable, outage $(R_L \cup R_F)$
R_F	Set of all possible fuel outage
R_L	Set of all possible LOX outage
r	Outcome of R
$u_{c/o}$	Nominal stage mixture ratio at cutoff
u_T	Nominal tanking ratio $\frac{LM}{FM}$
W(r)	Probability density function of R
$W_1(\ell)$	Frequency of occurrence of ℓ
$W_2(f)$	Frequency of occurrence of f
M	Mean value of payload potential, excluding the effects of outage
μ_R	Mean outage
η	Normal probability density function
Σ	Standard deviation of payload potential, excluding the effects of outage
σ_{EMR}	Standard deviation of EMR
σ_R	Standard deviation of outage

SECTION I. INTRODUCTION

Outage is defined as the amount of propellant (fuel or oxidizer, but never both) which cannot be burned because of an insufficient quantity of the other propellant. Outage obviously reduces the vehicles' capability potential in the form of unwanted inert weight at cutoff. An outage will occur when a variation exists between the targeted and actual tanked mixture ratio and/or between the nominal and the actual burned mixture ratio.

Since outage is clearly a random phenomenon, any payload study involving outage will be statistical in nature. The tanked mixture ratio and burned mixture ratio are assumed normal with known mean and standard deviation. The effective mixture ratio (EMR) is the statistical combination (root sum square) of these two ratios. In Section II the probability density function (pdf) of outage is developed analytically as a function of LM, FM, σ_{EMR}, $u_{c/o}$ and fuel bias. The equations show how the fuel bias helps shape the pdf of outage and hence, how it influences the mean and standard deviation of outage.

SECTION II. THE PROBABILITY DENSITY FUNCTION OF OUTAGE

Consider the maps $h: EMR \rightarrow R_F$ and $g: EMR \rightarrow R_L$ defined as follows (Figure 1):

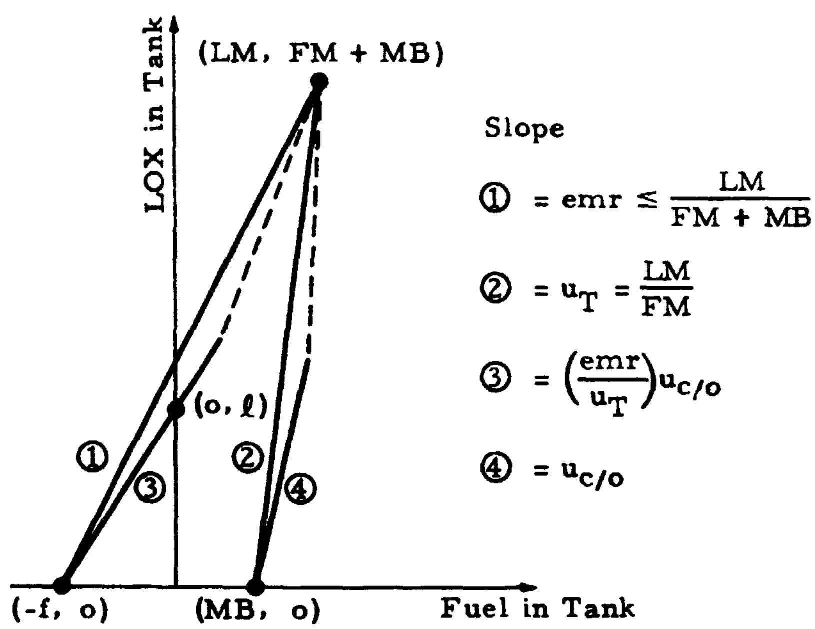

FIGURE 1. STAGE MIXTURE RATIO

$$f = h(emr)$$

$$= FM + MB - \frac{LM}{emr}, \qquad emr > \frac{LM}{FM + MB}$$

and

$$\ell = g(emr)$$

$$= \left(\frac{emr}{u_T}\right) u_{c/o} \left| f \right|$$

$$= \left(\frac{emr}{u_T}\right) u_{c/o} \left| FM + MB - \frac{LM}{emr} \right|, \qquad emr \leq \frac{LM}{FM + MB}$$

By construction, the functions g and h are 1-1 and they map the disjoint sets

$$emr \leq \frac{LM}{FM + MB}, \qquad emr > \frac{LM}{FM + MB}$$

onto the disjoint sets

$$R_L, \; R_F,$$

respectively.

Hence, the frequency of LOX outage occurrence may be given by

$$W_1(r) = \eta\left[g^{-1}(r)\right] \left|\frac{dg^{-1}}{dr}\right|_r, \qquad r \in R_L, \text{ zero elsewhere}$$

where η is the probability density function (pdf) of EMR and

$$\left|\frac{dg^{-1}}{dr}\right|_r$$

is the absolute value of the Jacobian of the inverse of the transformation g (References 1 and 2).

Similarly for fuel

$$W_2(r) = \eta\left[h^{-1}(r)\right] \left|\frac{dh^{-1}}{dr}\right|_r, \qquad r \in R_F, \text{ zero elsewhere}$$

Now

$$r \in R = R_L \cup R_F$$

and

$$R_L \cap R_F = \phi$$

implies that

$$P(r \in R) = P(r \in R_L) + P(r \in R_F)$$

which is to say that the pdf of R is

$$W(r) = W_1(r) + W_2(r)$$

$$= \eta[g^{-1}(r)] \left|\frac{dg^{-1}}{dr}\right|_r + \eta[h^{-1}(r)] \left|\frac{dh^{-1}}{dr}\right|_r, \qquad r \in R, \text{ zero elsewhere.}$$

Let $r \in R_L$, then there exists an

$$emr \le \frac{LM}{FM + MB}$$

such that

$$\begin{aligned} r &= g(emr) \\ &= \frac{emr}{u_T} u_{c/o} \left|(FM + MB) - \frac{1}{emr} LM\right| \\ &= \frac{u_{c/o}}{u_T} LM - \frac{u_{c/o}}{u_T} (FM + MB) emr \; . \end{aligned}$$

Solving for $g^{-1}(r)$

$$\begin{aligned} g^{-1}(r) &= emr \\ &= \frac{u_{c/o} LM - u_T r}{u_{c/o}(FM + MB)}, \qquad r \in R_L, \text{ zero elsewhere} \end{aligned}$$

and

$$\left|\frac{dg^{-1}}{dr}\right|_r = \left|\frac{-u_T}{u_{c/o}(FM + MB)}\right|, \qquad r \in R_L, \text{ zero elsewhere.}$$

Similarly

$$h^{-1}(r) = \frac{LM}{FM + MB - r}, \qquad r \in R_F, \text{ zero elsewhere}$$

and

$$\left|\frac{dh^{-1}}{dr}\right|_r = \left|\frac{LM}{(FM + MB - r)^2}\right|, \quad r \in R_F, \text{ zero elsewhere.}$$

Hence,

$$\begin{aligned} W(r) &= \eta[g^{-1}(r)]\left|\frac{-u_T}{u_{c/o}(FM + MB)}\right| + \eta[h^{-1}(r)] \left|\frac{LM}{(FM + MB - r)^2}\right| \\ &= \frac{1}{\sigma_{EMR}\sqrt{2\pi}} \exp\left[-\frac{\left(\frac{u_{c/o} LM - u_T r}{u_{c/o}(FM + MB)} - u_T\right)^2}{2\sigma_{EMR}^2}\right] \left|\frac{-u_T}{u_{c/o}(FM + MB)}\right| \\ &\quad + \frac{1}{\sigma_{EMR}\sqrt{2\pi}} \exp\left[-\frac{\left(\frac{LM}{FM + MB - r} - u_T\right)^2}{2\sigma_{EMR}^2}\right] \left|\frac{LM}{(FM + MB - r)^2}\right| \end{aligned}$$

which is the form of the pdf of outage [W(r)] used for digital computation of the mean outage

$$u_R = \int rW(r)$$

and standard deviation of outage

$$\sigma_R = \left\{\left[\int r^2 W(r)\right] - u_R^2\right\}^{\frac{1}{2}}$$

in the computer program that is discussed in the following sections.

SECTION III. COMPUTER PROGRAM LISTING

```
* * * INPUT * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*SIGE=ONE STANDARD DEVIATION OF EFFECTIVE MIXTURE RATIO             *
*UCO=STAGE CUTOFF MIXTURE RATIO                                     *
*FM=MAINSTAGE FUEL LOAD.                                            *
*LM=MAINSTAGE LOX LOAD.                                             *
*MB=FUEL BIAS                                                       *
* * * OUTPUT  * * * * * * * * * * * * * * * * * * * * * * * * * * * *
*R=RESIDULES (LBM)                                                  *
*PR=PROBABILITY OF R  OR LESS OCCURRING.
*FR=FREQUENCY OF OCCURRANCE OF R.
*UR=MEAN RESIDUAL.                                                  *
*SIGR=STANDARD DEVIATION OF RESIDUAL.                               *
*PRLIM= CUTOFF LIMIT FOR PR                                         *
*IPRT= OPTION FOR DETAIL PRINT OF R,FR,PR   IF(IPRT.GE.1)           *
*NCASES= 1 THRU 15                                                  *
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      REAL LM,MB
      COMMON /COM1/LPI,NNY
      DIMENSION A(2500,4)
      DIMENSION IDX(12,3),IDY(12,3),LABEL(7,3),LHEAD(6),ISYM(3),L(3)
      DIMENSION XL(3),XR(3),YB(3),YT(3),IDZ(12),LP(3)
      NAMELIST / NAME1/ SIGE,UCO,FM,LM,MB,UT,C1,C3,PRLIM,IPRT
      NAMELIST/NAME2/IDX,IDY,LABEL,LHEAD,IDZ,ISYM,XL,XR,YB,YT,L,LP
     *,IPLOT,NCASES
      READ(5,NAME2)
      WRITE(6,NAME2)
      IF(IPLOT .LT. 1) GO TO  5
      CALL CAMRAV(359)
      NNY=2
      *** SYSTEM SUBROUTINE ****
      CALL SCOUTV
      CALL RITE2V(80,550,900,90,1,36,1,LHEAD,NERR)
    5 CONTINUE
      WRITE(6,510)
      DO 200 IJ=1,NCASES
      READ(5,NAME1)
      UT=LM/FM
      PR=0.
      UR=0.
      RRFR=0.
      R2FR=0.
      PR2FR=0.
      SIGR2=0.
      DR=5.
      R=0.
      PRREV=0.
      A(1,1)=0.
      A(1,2)=0.
      A(1,3)=0.
```

```
      C1=1. /(SIGE*(SQRT(2.*3.1415927)))
      C2=UT/(UCO*(FM+MB))
      WRITE(6,NAME1)
      DO 100 I=1,2500
      II=I
      M=I-1
      R=M*5
      DR=R - RPREV
      RPREV=R
      C3=((((UCO*LM)-(UT*R))/(UCO*(FM+MB)))-UT)**2 /(2.*SIGE*SIGE)
      C4N=(LM/(FM+MB-R))-UT
      C4= C4N*C4N/(2.*SIGE*SIGE)
      C5=ABS(LM/((FM+MB-R)**2.))
      FR=C1*(EXP(-C2)*C3 + EXP(-C4)*C5)
      IF(I.EQ.1) GO TO 10
      PR=PR+DR*((FR+A(M,2))/2.)
      UR=UR + DR*(R*(FR+A(M,2))/2.)
      R2FR=R*R*FR
      RRFR=RRFR + DR*((R2FR+PR2FR)/2.)
      PR2FR=R2FR
      SIGR2=RRFR - (UR*UR)
   10 CONTINUE
      A(I,1)=R
      A(I,2)=FR
      A(I,3)=PR
      IF(PR .GT. PRLIM ) GO TO  101
  100 CONTINUE
  101 CONTINUE
      SIGR=SQRT(SIGR2)
      WRITE(6,501)UR,SIGR,PR
      IF(IPRT .LT. 1) GO TO 149
      DO 102 I=1,II,5
  102 WRITE(6,504)I,A(I,1),A(I,2),A(I,3)
  149 IF(IPLOT.LT. 1) GO TO 150
      DO 20 J=1,2
      JJ=J+1
      LPI=LP(J)
C     *** SYSTEM SUBROUTINE ****
      CALL QUIK3V(L(J),ISYM(J),IDX(1,J),IDY(1,J),II,A(1,1),A(1,JJ))
      CALL RITE2V(250,990,900,90,1,36,1,LHEAD,NERR)
      CALL RITE2V(230,960,980,90,1,42,1,LABEL(1,J),NERR)
   20 CONTINUE
  150 CONTINUE
      WRITE(6,520)
      WRITE(6,521)SIGE
      WRITE(6,522)UCO
      WRITE(6,523)FM
      WRITE(6,524)LM
      WRITE(6,525)MB
```

```
      WRITE(6,526)
      WRITE(6,527)SIGR
      WRITE(6,528)UP
      WRITE(6,502)
  200 CONTINUE
      IF(IPLOT .LT. 1) STOP
      WRITE(16,510)
      CALL RITE2V(80,550,900,90,1,36,1,LHEAD,NERR)
C     *** SYSTEM SUBROUTINE ****
      CALL CLEAN
      STOP
  501 FORMAT(4H UR=,E16.8,15H    SQRT(SIGR2)=,E16.8,6H    PR=,F16.8)
  502 FORMAT(////12H END OF CASE)
  503 FORMAT(105X,5H C4N=,F14.7)
  504 FORMAT (1H ,I4,6F14.7)
  510 FORMAT(1H1)
  520 FORMAT(1H1,////,27X,5HINPUT,/)
  521 FORMAT(48H STANDARD DEVIATION OF EFFECTIVE MIXTURE RATIO =,F9.4)
  522 FORMAT(28H STAGE CUTOFF MIXTURE RATIO=,20X,F9.4)
  523 FORMAT(22H MAINSTAGE FUEL LOAD=,21X,F14.4)
  524 FORMAT(22H MAINSTAGE  LOX LOAD =,21X,F14.4)
  525 FORMAT(12H FUEL BIAS =,31X,F14.4)
  526  FORMAT(////,1H ,27X,6HOUTPUT,/)
  527 FORMAT(32H STANDARD DEVIATION OF RESIDUAL=,9X,F14.7)
  528 FORMAT(35H EXPECTED VALUE OF RESIDUAL (MEAN)=,6X,F14.7)
      END
```

SECTION IV. COMPUTER PROGRAM SAMPLE RESULTS

Table I shows the results of nine computer runs that were made to support a design phase, Space Shuttle sizing study. Total run time for the nine cases was 2.3 min on the Univac 1108 computer.

TABLE I. PROGRAM RESULTS

Case	Input Data					Output Data	
	σ_{EMR}	$u_{c/o}$	LM	FM	MB	μ_R	σ_R
1	0.021	6.0	1,399,175	233,196	300	1413	1543
2	0.021	6.0	1,399,175	233,196	500	1224	1264
3	0.021	6.0	1,399,175	233,196	800	1243	1046
4	0.021	6.0	1,399,175	233,196	900	1255	974
5	0.021	6.0	1,399,175	233,196	1000	1281	915
6	0.021	6.0	1,399,175	233,196	1100	1320	869
7	0.021	6.0	1,399,175	233,196	1200	1377	836
8	0.021	6.0	1,399,175	233,196	1400	1498	799
9	0.021	6.0	1,399,175	233,196	1600	1654	7[illegible]8

The computer program has the option of generating plots of the probability density function and distribution function of outage. Computer plots corresponding to case two (Figures 2 and 3) and to case six (Figures 4 and 5) are shown.

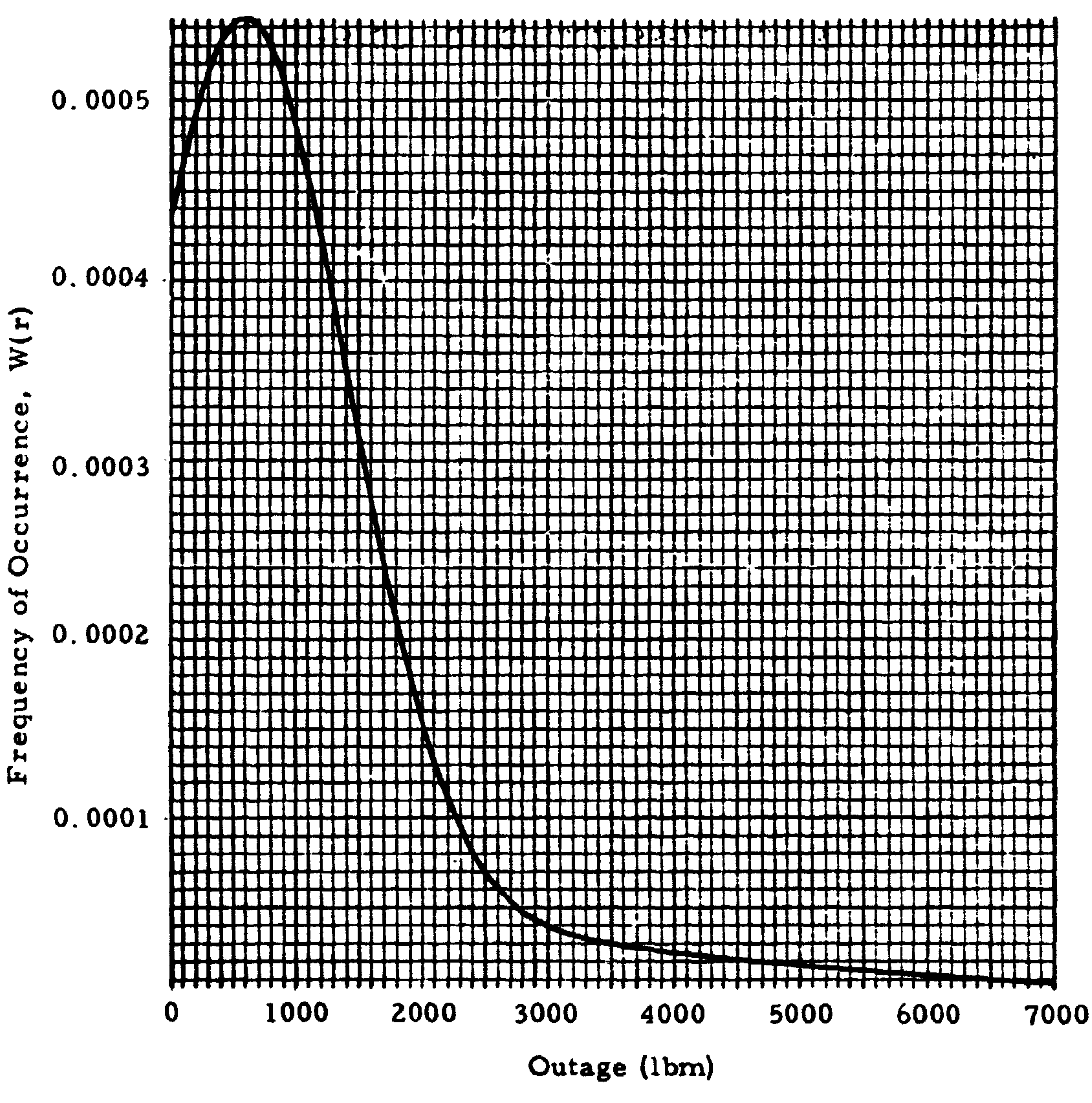

FIGURE 2. PROBABILITY DENSITY FUNCTION, W(r)--CASE TWO

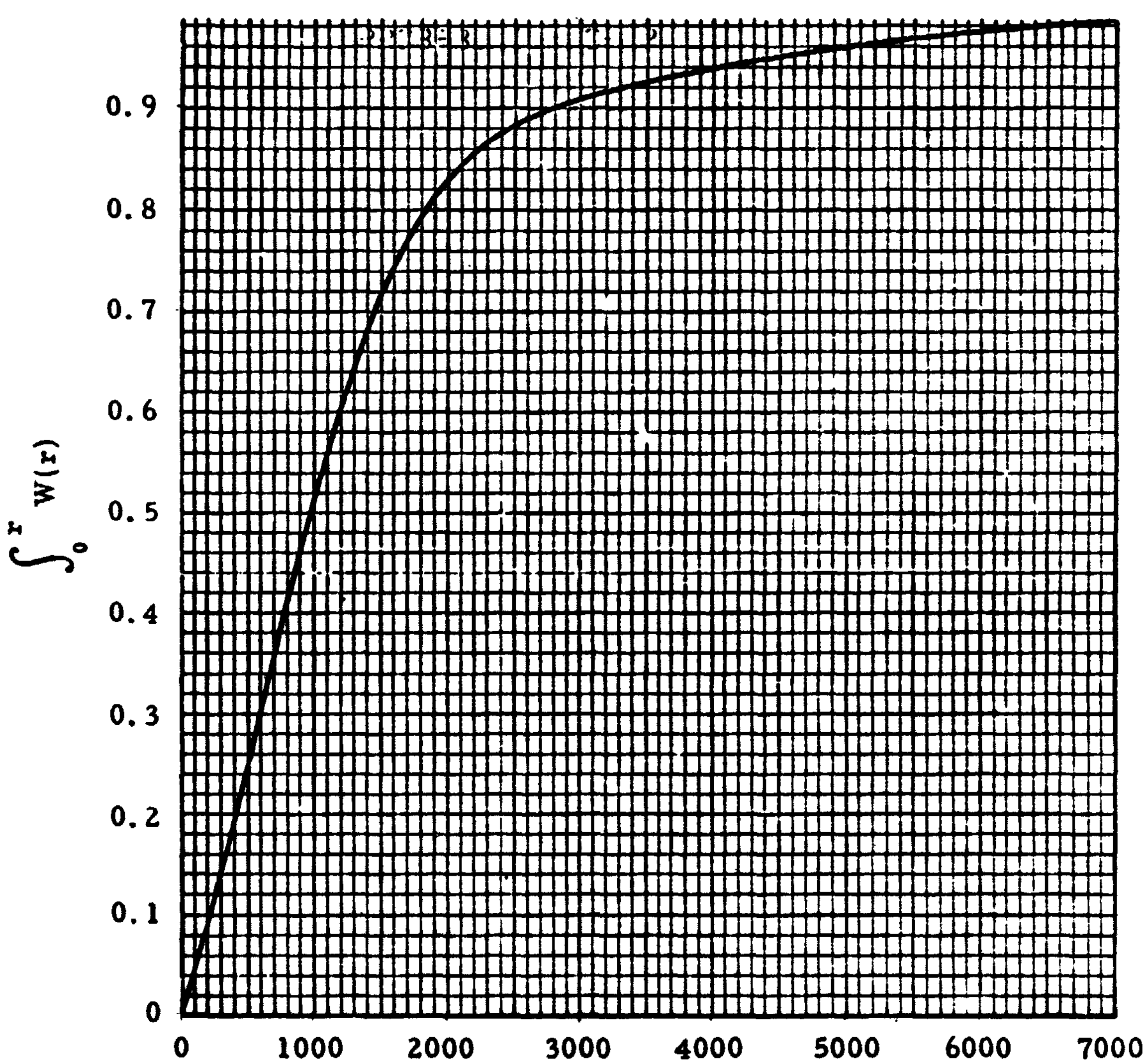

FIGURE 3. DISTRIBUTION FUNCTION--CASE TWO

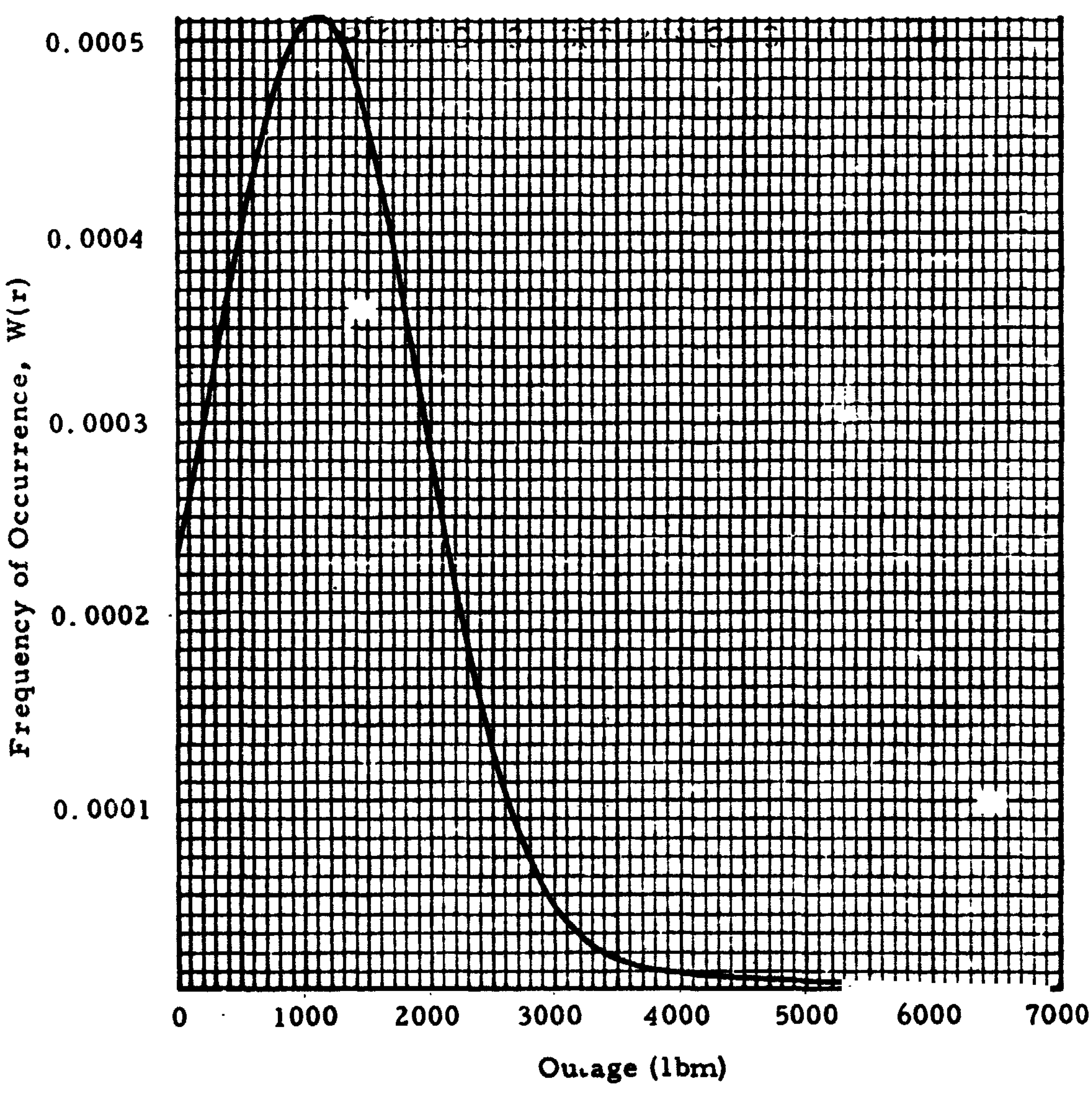

FIGURE 4. PROBABILITY DENSITY FUNCTION, W(r)--CASE SIX

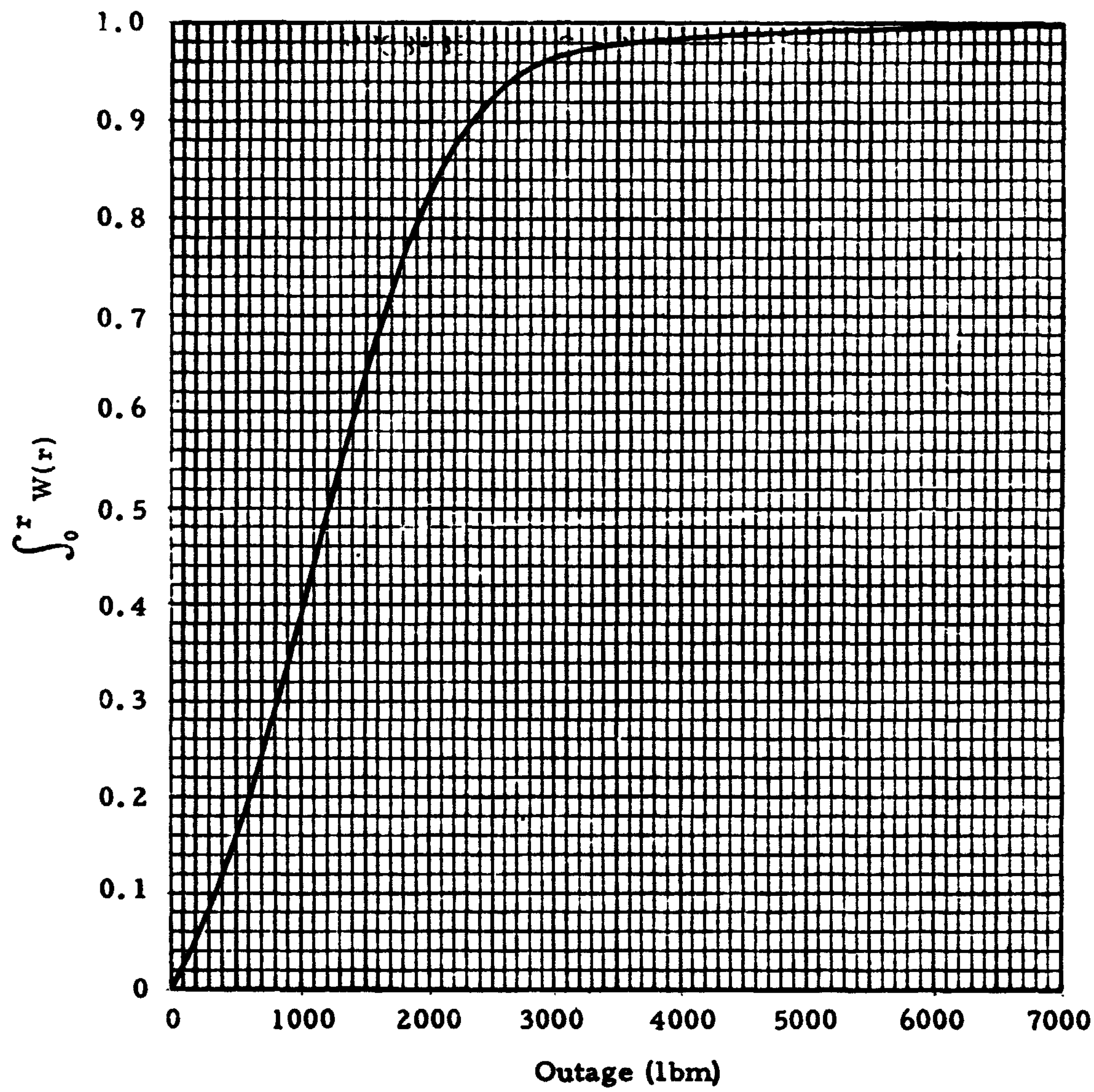

FIGURE 5. DISTRIBUTION FUNCTION--CASE SIX

SECTION V. CHOOSING THE OPTIMUM FUEL BIAS

Let M and Σ be the mean and the standard deviation of payload potential, excluding the effects of outage. M and Σ are the statistical combinations (usually root sum square) of payload potential variations caused by the randomness of thrust, ISP, propellant loads, and aerodynamic forces.

By an application of the Central Limit Theorem (Reference 3), the total standard deviation, including outage effects on payload potential, is closely approximated by

$$\sqrt{\sigma_R^2 + \Sigma^2}$$

Within a given sigma probability, k; the total vehicle payload potential is

$$M - [\mu_R \pm k\sqrt{\sigma_R^2 + \Sigma^2}]$$

Now, given that LM, FM, σ_{EMR}, $u_{c/o}$ are known, μ_R and σ_R are uniquely determined for a given fuel bias (Section II). Thus optimizing the fuel bias amounts to choosing the fuel bias that minimizes the payload potential loss

$$\mu_R + k\sqrt{\sigma_R^2 + \Sigma^2}$$

within a given sigma probability, k.

Table II shows the payload potential loss associated with each of the output data of the sample computer run (Table I). The 3Σ used was 4151 lbm. The optimum fuel bias for this study was determined to be 1100 lbm (Table II).

TABLE II. PAYLOAD POTENTIAL LOSS (lbm) VERSUS FUEL BIAS

Case	MB	$\mu_R + \sqrt{(3\sigma_R)^2 + 4151^2}$
1	300	7630
2	600	6846
3	800	6441
4	900	6332
5	1000	6258
6	1100	6222
7	1200	6227
8	1400	6291
9	1600	6431

It is worth noting that the optimum fuel bias is not highly sensitive to reasonable values of Σ. Several fuel bias optimization studies were made for this Space Shuttle configuration with 3Σ ranging from 1000 to 6000 lbm. The resulting optimum fuel biases were between 1000 and 1300 lbm. However, the optimum fuel bias is sensitive to the total propellants tanked (LM + FM) and to EMR.

SECTION VI. SUMMARY

This paper presents an accurate and efficient method of determining the optimum fuel bias for a bipropellant liquid rocket. Basically, the paper shows that the mean and standard deviation of outage are uniquely determined for a given fuel bias (Section II), and how, because of this, probable loss in payload resulting from outage can be minimized by the proper choice of a fuel bias.

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APPROVAL

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By William Viera, Jr.

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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