

THE SOLUTION OF TRANSCENDENTAL EQUATIONS

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by

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## ABSTRACT

This report deals with some of the existing methods to "globally" approximate the roots of transcendental equations namely, Graeffe's method, Summation of the reciprocated roots, Whittaker-Bernoulli method and the extension of Bernoulli's method via Koenig's theorem. The Aitken's  $\delta^2$  process is used to accelerate the convergence. Finally, the suitability of these methods is discussed in various cases.

## NOMENCLATURE

- C.P.U. = a measure of computer time
- M = a constant representing the upper bound for a set
- $\psi(z)$  = a function analytic at  $z = r$
- h = values derived recursively from Bernoulli's difference equation
- $\gamma$  = the maximum modulus of  $\psi(z)$
- $H_{\nu p}$  = a sequence of Hankel determinants
- $\rho$  = radius of convergence of a power series
- $k_{\nu p}$  = a normalizing polynomial
- $K_{\nu p}$  = a Hankel determinant of order p
- $R_{\nu+1}$  = the remainder for the Taylor series after  $\nu$  terms
- $P_n$  = an approximate to the root of least modulus using the method of summing the reciprocated roots.

## CHAPTER I

### INTRODUCTION AND SUMMARY

In the area of applied mathematics, there is an urgent need to effect more efficient means of approximating the roots of transcendental equations. Existing standard methods employed in finding the roots of algebraic equations are of little aid in obtaining satisfactory results in this situation. Incidentally, there exist no formulated method for finding the roots of algebraic equations above the fourth degree [14]. The same fact applies to a transcendental equation when it is represented by a power series that has been truncated above the fourth degree term.

There have been various attempts made in locating or approximating the roots of certain types of transcendental equations. One of the more basic means of obtaining information about the nature of the roots of an equation (algebraic or transcendental), such as,  $h(x) = g(x) - f(x) = 0$ , is through graphing the functions  $g(x)$  and  $f(x)$  on the same coordinate axes and noting the point (s) of intersection. After having made some suitable approximation to the desired root, then an iterative method may be used to refine the approximation.

On the other hand, there are methods which are capable of yielding, in a more consistent manner, information about the roots of a given transcendental equation. One such method is the Graeffe method [15]. Graeffe's method guarantees convergence to a root through repeated root squaring [4].

There are other methods, though not discussed in this paper,

that are 'self starting' or 'global' in the manner in which they approximate the roots to transcendental equations. These methods include Rutishauser's q-d algorithm and Bairstow's method. The above methods are fairly effective in determining the roots of polynomial equations, but possess certain inherent disadvantages such as their sensitivity to truncation error where power series are involved. Furthermore, they are impractical from the standpoint of their compatibility with the digital computer where C.P.U. time may be crucial.

There exist another method of great prominence and it is known as Bernoulli's method. However, its convergence is somewhat less rapid than Graeffe's method. In addition, it may be more difficult to deal with complex roots [4].

Later, Koenig proposed a theorem which extends Bernoulli's method to non-algebraic equations [5]. In this extension, provisions are made for treating equations having "equal" roots or in the case in which there are complex conjugates. Besides computing the root of least modulus within the region of analyticity, it is theoretically possible to compute all the roots of a transcendental equation that are within the radius of convergence of its power series representation[5]. However, in some cases the convergence of Bernoulli's method may be somewhat less than ideal and in this case one may hope to employ some method of accelerating this convergence. One method of achieving this aim is the use of Aitken's  $\delta^2$  process [5].

Another method, although less popular, can be derived from the Bernoulli method is the Whittaker expansion [3,11]. This method relates the coefficients of a polynomial or a truncated power series to the roots via determinants.

Among the miscellaneous methods, the root of least modulus for a transcendental equation may be evaluated by utilizing the standard relationship that exist between roots and coefficients of the truncated power series representing a particular transcendental equation.

Thus in chapter 3 these methods will be applied to certain transcendental equations in order to obtain information (on a comparative basis) regarding their suitability to the computer, speed of convergence, ease of computations, and accuracy of approximation.

We will study the nature of the non-algebraic equation by studying the coefficients of its power series representation. From a practical standpoint, we cannot consider all the terms in the series, hence, we must truncate it at some point where the truncation error [15] will be minimal. Afterwards, we consider the truncated power series as a special case of a polynomial. However since the original equation is likely to consist of meromorphic functions, we must have some previous knowledge about the nature of their convergence in the region of analyticity. The following well known theorems and definitions are relevant to our further discussion in later chapters.



Roots and coefficients. Given the polynomial equation

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0,$$

the following relationships exist between coefficients and roots [14]:

$$(a) \ a_1/a_0 = -(r_1 + r_2 + r_3 + \dots + r_n)$$

$$(b) \ a_2/a_0 = (r_1 r_2 + r_1 r_3 + \dots + r_1 r_n + \dots + r_{n-1} r_n)$$

$$(c) \ a_3/a_0 = -(r_1 r_2 r_3 + r_1 r_2 r_4 + \dots + r_{n-2} r_{n-1} r_n)$$

.....

$$(d) \ a_n/a_0 = (-1)^n (r_1 r_2 r_3 \dots r_n)$$

Theorem 1. Every power series  $\sum c_n (z-z_0)^n$  has a "radius of convergence"

R such that when  $0 < R < \infty$  the series converges absolutely for

$|z - z_0| < R$  and diverges for  $|z - z_0| > R$ . When  $R = 0$ , the series converges only for  $z = z_0$ . When  $R = \infty$ , the series converges for all  $z$ .

The number R is given by

$$R = 1 / \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}$$

Theorem 2. Let  $f(z)$  be analytic in the interior of a circle C with center at  $z_0$  and radius R. Then at each point  $z$  interior to C

$$f(z) = \sum f^{(n)}(z_0) / n! (z-z_0)^n$$

Definition 1. A function  $f(z)$  which is analytic in a domain D, except at some points of D at which it has poles, is said to be meromorphic.

## CHAPTER II

### THEORETICAL DEVELOPMENTS OF GLOBAL METHODS

Among the existing methods \* of approximating the roots of polynomial equations, Bernoulli's method [1] is perhaps among the oldest known. In its original form, Bernoulli's method was proven valid for algebraic equations. By expressing the equation in terms of a power series, it would seem possible that a modified Bernoulli method could be used to determine information concerning non-algebraic equations.

Due mainly to the efforts of Koenig, the Bernoulli method was shown to be valid in the case of non-algebraic or transcendental equations. The theorems which follow were reproduced in order to (1) extend this method to the case of transcendental equations and (2) show that Aitken's  $\delta^2$  process can be applied jointly with the Bernoulli method.

**THEOREM A** (Koenig's theorem) Let  $h(z) = g(z)/f(z) = c_0 + c_1z + \dots$  be convergent for  $|z| < R$  where  $f(z)$  has a simple zero at  $z = r$  and  $g(z)$  be any function which is analytic throughout the circle but  $g(r) \neq 0$ . If  $|r| < \delta R < R$ , then  $c_v / c_{v+1} = r + O(\sigma^{v+1})$  [5].

From the hypothesis, since  $z = r$  is a pole then it is apparent that

$\psi(z) = (r-z)(h(z)) = b_0 + b_1z + b_2z^2 + \dots$  is analytic for  $|z| < R$ .

Let  $|r| < \delta R < R$  and  $\sigma|r| < \rho < R$ , then  $\psi(\rho)$  also converges having a term that is maximal in modulus which we call  $\gamma$ . Then for any  $v$ ,  $|b_v| \leq \gamma \rho^{-v}$  and this establishes the fact that each  $b_v$  is bounded. Expanding  $\psi(z)$  and equating similar coefficients, then

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\*

See [14] for a detailed development of Graeffe's method.

$$rc_0 = b_0$$

$$rc_1 - c_0 = b_1$$

.....

$$rc_v - c_{v-1} = b_v$$

Upon multiplying the first  $v+1$  equations by  $1, r, r^2, r^3, \dots, r^v$

and adding we have  $c_v r^{v+1} = \psi_v(r)$  where  $\psi_v(z) = b_0 + b_1 z + \dots + b_v z^v$ .

Letting  $R_v(z) \equiv \psi(z) - \psi_v(z)$ , then  $R_v(r) = r^{v+1} (b_{v+1} + b_{v+2} r + \dots)$

$$\begin{aligned} |R_v(r)| &\leq \gamma(1 + |r|/\rho + |r^2|/\rho^2 + \dots) |r^{v+1}|/\rho^{v+1} \\ &= \gamma |r/\rho|^{v+1} / (1 - |r/\rho|) \\ &= o(|r/\rho|^{v+1}). \end{aligned}$$

$$\begin{aligned} \text{But } c_v/c_{v+1} &= r\psi_v(r) / \psi_{v+1}(r) \\ &= r[1 - R_v(r) / \psi(r)] / [1 - R_{v+1}(r) / \psi(r)] \\ &= r[1 + o(|r/\rho|^{v+1})] \end{aligned}$$

and choosing an appropriate  $\rho$ , then  $|r/\rho| < \delta$ . In conclusion, since  $0 < \delta < 1$  we may conclude that  $\lim_{v \rightarrow \infty} c_v/c_{v+1} \rightarrow r$ . Also  $|r/\rho|$  is a common ratio of  $(c_v/c_{v+1} - r)$  thus making the sequence geometric. (This fact is very essential before the application of Aitken's  $\delta^2$  process; which we will discuss later in this chapter.) This theorem may be extended to include the case in which the power series  $h(z)$  has  $p$  simple poles in  $|z| < R$ , though in the examples that we have considered, this extension will be impracticable to apply.

THEOREM B [5] Let  $h(z)$  be a meromorphic function in  $|z| < R$  having exactly  $p$  poles  $r_1, r_2, \dots, r_p$  not necessarily distinct in the disk. Let

$|r_1| \leq |r_2| \leq |r_3| \leq \dots \leq |r_p| < \sigma R < R$  and let

$$\varphi(z) = 1 + a_1 z + \dots + a_p z^p \Xi (1 - r_1^{-1} z) \dots (1 - r_p^{-1} z)$$

setting  $K_\nu(z) = K_{\nu p}(z) \Xi 1 + \alpha(z) + \dots + \alpha z^p$  where

$k_{\nu p}(z)$  is defined as  $H_{\nu p} K_{\nu p}(z)$ . Then

$$= a_1 + o(\sigma^\nu), \text{ and } K_\nu(z) = \varphi(z) + o(\sigma^\nu)$$

In this section, we will extend the coverage of Theorem A by including the case when  $p = 2$ . Let  $r_1$  and  $r_2$  be 2 distinct roots. Then

$\psi(z) = \varphi(z)h(z) = b_0 + b_1 z + b_2 z^2 + \dots$  is analytic in  $|z| < R$  and we have  $(1 + a_1 z + a_2 z^2 + \dots) (c_0 + c_1 z + c_2 z^2 + \dots) =$

$\psi(z) = b_0 + b_1 z + b_2 z^2 + \dots$  and equating similar coefficients

$$c_0 = b_0$$

$$c_1 + a_1 c_0 = b_1$$

$$c_2 + a_1 c_1 + a_2 c_0 = b_2$$

.....

$$c_\nu + a_1 c_{\nu-1} + a_2 c_{\nu-2} = b_\nu$$

Suppose  $r$  represents  $r_1$  or  $r_2$ . Multiplying the equations above

by  $1, r, r^2, \dots, r^\nu$  and adding, we obtain

$$(1) r^\nu c_\nu + (1 + a_1 r) r^{\nu-1} c_{\nu-1} = \psi_\nu(r). \text{ In a similar way we obtain}$$

$$(2) r^{\nu+1} c_{\nu+1} + (1 + a_1 r) r c_{\nu-1} = \psi_{\nu+1}(r) \quad \text{and}$$

$$(3) r^{\nu+2} c_{\nu+2} + (1 + a_1 r) r^{\nu+1} c_{\nu+2} = \psi_{\nu+2}(r).$$

and multiplying the first equation (1) by  $r^2$  and (2) by  $r$ , the equations

then show that the determinant

$$\begin{vmatrix} c_{v-1} & c_v & r^2 \psi_v(r) \\ c_v & c_{v+1} & r \psi_{v+1}(r) \\ c_{v+1} & c_{v+2} & \psi_{v+2}(r) \end{vmatrix} = 0$$

where they express the linear dependence of the three columns.

Since  $\psi_v(r) = \psi(r) - R_v(r)$ , it can be seen that in the limit

both  $r_1$  and  $r_2$  satisfy the equation  $K_{v2}(z) = 0$  or equivalently,

$k_{v2}(z) = 0$ . However, in case  $r_1 = r_2$  is a pole of  $h(z)$  of

multiplicity 2, a further argument is necessary to show that the

limiting equation has  $r_1$  as a double root. Accordingly, a

different argument has been developed by Householder [5] in which

confluence does not require special consideration.

At this point, it is appropriate to develop the difference equation enabling us to compute the coefficients of the power series which was derived from the quotient  $g(z)/f(z)$ ,  $f(z) \neq 0$ . Consider  $f(z) = a_0 + a_1 z + \dots$  which has a root  $r$  in its region of analyticity and  $g(z) = b_0 + b_1 z + \dots$  which is analytic within the same disk as  $f(z)$  but not vanishing at  $z = r$ , then

$$g(z)/f(z) = h_0 + h_1 z + \dots$$

Therefore,

$$\begin{aligned} g(z) &= (a_0 + a_1 z + a_2 z^2 + \dots)(h_0 + h_1 z^2 + \dots) \\ &= b_0 + b_1 z + b_2 z^2 + \dots \end{aligned}$$

On equating similar coefficients, we have

$$a_0 h_0 = b_0 \quad \text{where } a_0 \neq 0$$

$$a_0 h_1 + a_1 h_0 = b_1$$

$$a_0 h_2 + a_1 h_1 + a_2 h_0 = b_2$$

.....

$$a_0 h_\nu + a_1 h_{\nu-1} + a_2 h_{\nu-2} + \dots + a_n h_{\nu-n} = b_\nu$$

making it possible to compute  $h_\nu$  while making some arbitrary selection for  $g$  where  $h_\nu / h_{\nu+1}$  approaches  $r$  for some sufficiently large  $\nu$ .

With the knowledge gained from previous results we can compute a very useful expansion for approximating the root of least modulus for a particular transcendental equation. This is known as the WHITTAKER-BERNOULLI expansion. In reference to theorem A, the root of least modulus may be expressed as

$$\begin{aligned} r &= h_0/h_1 + (h_1/h_2 - h_0/h_1) + (h_2/h_3 - h_1/h_2) + \dots \\ &= h_0/h_1 + (h_1^2 - h_0 h_2)/h_1 h_2 + (h_2^2 - h_1 h_3)/h_2 h_3 + \dots \end{aligned}$$

and furthermore, suppose  $h(z) = 1/f(z)$  or  $h(z)f(z) = 1$  where  $h(z)$  is meromorphic for  $|z| < R$  having a simple pole at  $z = r$  within the disk. Let  $f(z)h(z) = h_0 + h_1 z + h_2 z^2 + \dots$

then multiplying and equating coefficients

$$a_0 h_0 = 1$$

$$a_0 h_1 + a_1 h_0 = 0$$

$$a_0 h_2 + a_1 h_1 + a_2 h_0 = 0$$

.....

$$a_0 h_v + a_1 h_{v-1} + \dots + a_v h_0 = 0$$

Solving the system of  $v+1$  equations in  $v+1$  unknowns we find that

$$h_0 = 1/a_0$$

$$h_1 = -a_1/a_0^2$$

$$h_2 = a_1^2/a_0^3 - a_2/a_0^2$$

.....

Thus we find  $r$  to be

$$r = \frac{-a_0}{a_1} - \frac{a_0^2 a_2}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3}{\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}} - \dots$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}$$

Occasionally, the sequence resulting from the application of the Bernoulli method to a particular transcendental equation may not provide satisfactory convergence to the root of least modulus, i.e. subsequent recursive computations of  $h_v/h_{v+1}$  determines a sequence of slow convergence. To accelerate convergence, there exist a method that is both highly effective and convenient, namely, Aitken's  $\delta^2$  method. A sufficient condition for the employment of this method is that the given sequence be geometric. Recall that this condition was established in Theorem A.

Consider a sequence  $a_1, a_2, \dots, a_v, \dots$  converging geometrically to a limit, say,  $a$ , for some  $v$ , where  $a_v = h_v/h_{v+1}$ . Now the Aitken  $\delta^2$  algorithm may be derived in the following manner. Under the assumption that the sequence above converges geometrically to a value  $a$  or  $a - a_v = bh^v + \epsilon_v(h^v)$  where  $\epsilon \rightarrow 0$  whenever  $h \rightarrow 0$  which may be expressed as  $a - a_v = bh + o(h)$ . From this result, we obtain  $(a - a_{v+1})/(a - a_v) = h + o(h)$  and  $(a - a_v)/(a - a_{v-1}) = h + o(h)$  thus upon subtracting we have

$$(a - a_{v+1})/(a - a_v) - (a - a_v)/(a - a_{v-1}) = o(h) \text{ where}$$

$$a_{v-1}a_{v+1} - a_v^2 - a(a_{v-1} - 2a_v + a_{v+1}) = o(h^{2v})$$

Since

$$a_{v-1} - 2a_v + a_{v+1} = -bh^{v-1}(1-h)^2 = o(h^{v-1}) \text{ thus}$$

$$a = (a_{v-1}a_{v+1} - a_v^2)/(a_{v-1} - 2a_v + a_{v+1}) + o(h^{v+1})$$



In the following discussion, we will investigate a method of solving transcendental equations whose theoretical developments are unrelated to the previously discussed materials. An algorithm will be developed which will enable us to locate the real root of least modulus through summing the reciprocated roots of a power series. The motivating reason underlying such an undertaking is that situations do exist where a transcendental equation may have a power series representation.

Using basic principles from the theory of equations one can state explicitly the sum of the roots of a regular polynomial equation. We focus our attention to the general case where  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$ . Upon dividing through by  $a_0 \neq 0$  we have  $(a_n/a_0)x^n + (a_{n-1}/a_0)x^{n-1} + \dots + 1 = 0$  then  $1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n = 0$ , In addition from the theory of equations, we know that the sum of the product of the reciprocals of  $\lambda$  taken  $k$  at a time is  $(-1)^k \alpha_k$ . It also follows that if given the equation  $1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n = 0$  then the sum of the squares of the products of the reciprocals of its positive roots taken  $k$  at a time is  $(-1)^k (\alpha_k)$ . Suppose if given the real roots of a particular equation, say  $\lambda_1, \lambda_2, \lambda_3, \dots$  where we are find the sum of the reciprocals of various powers of the roots. First, we may represent the sum of the reciprocals of the given roots as

$$(1) \quad 1/\lambda_1 + 1/\lambda_2 + \dots + 1/\lambda_n = \alpha_1$$

$$(2) \quad 1/\lambda_1 \lambda_2 + 1/\lambda_2 \lambda_3 + \dots + 1/\lambda_{n-1} \lambda_n = \alpha_2$$

Now assuming that  $\lambda_1, \lambda_2, \lambda_3, \dots$  are roots to the equation

$1 + \alpha_1 x + \alpha_2 x^2 + \dots = 0$  and combining equations (1) and (2) we have

$$(3) \quad 1/\lambda_1^2 + 1/\lambda_2^2 + 1/\lambda_3^2 + \dots + 1/\lambda_n^2 = \alpha_1^2 - 2\alpha_2$$

In a similar way we can find

$$(4) \quad 1/\lambda_1^3 + 1/\lambda_2^3 + 1/\lambda_3^3 + \dots + 1/\lambda_n^3 = 3/2(1/\lambda_1^2 + \dots + 1/\lambda_n^2) \times \\ (1/\lambda_1 + 1/\lambda_2 + \dots + 1/\lambda_n) \\ + 3(1/\lambda_1 \lambda_2 \lambda_3 + 1/\lambda_1 \lambda_3 \lambda_4 + \dots) \\ - 1/2 (1/\lambda_1 + 1/\lambda_2 + \dots + 1/\lambda_n)^3$$

Thus from equations (1), (2) and (3) we have

$$(5) \quad 1/\lambda_1^3 + 1/\lambda_2^3 + \dots + 1/\lambda_n^3 = 3\alpha_1\alpha_2 - 3\alpha_3 - \alpha_1^3 \quad \text{and similarly}$$

$$(6) \quad \sum_1^n 1/\lambda_k^4 = \alpha_1^4 - 4\alpha_1^2\alpha_2 + 2\alpha_2^2 + 4\alpha_1\alpha_3 - 4\alpha_4$$

At this point it should be noted that the reason for seeking higher powers of  $1/\lambda_k$  ( $k=1,2,3,\dots$ ) is to obtain a better approximation to the root of least modulus. Also in the manner in which (5) and (6) were derived, one may extend this method to include higher powers of  $1/\lambda_k$  if further "refinement" is desired. Note that the rapidity of convergence of the expressions on the left of (5) and (6) is such that all but a finite number of terms will cluster about the "dominating" term thus allowing us to approximate the root of least modulus.

## CHAPTER III

### EXAMPLES OF SOLVING TRANSCENDENTAL EQUATIONS

The methods of solving transcendental equations that were indicated in Chapter II will now be applied to particular equations. The motivating reasons for the employment of these methods are consistent with those outlined in the abstract.

Among the facilities used for aiding computations was an APL 360 computer terminal. Because of the limited capacity of our compiler, some difficulties were encountered in cases where the coefficients of series under consideration converge absolutely at a fast rate. However, inspite of these limitations, considerable progress was made in making the objectives of this paper a reality.

## SUMMATION OF THE RECIPROCAL OF ROOTS

Consider some transcendental equations and their solutions:

(1)  $\sin x / x = 0$ , using the Maclaurin series expansion for

$$\sin x, \text{ we find } \sin x / x = 1 - x^2/6 + x^4/120 - x^6/5040 + \dots$$

From previous considerations, we see that  $\alpha_1 = -1/6$ .

$\alpha_2 = 1/120$ ,  $\alpha_3 = -1/5040$ . Now if we let  $P_n$  be a

root of  $\sin x/x = 0$ , then the sum of the reciprocated

fourth powers of the roots would be represented by

$$\sum 1/P_n^4 = 1/90, \text{ thus } P_n \approx \sqrt[4]{90} \approx 3.14. \text{ The actual value is } 3.14+.$$

Of course, an even more accurate approximation may be

made by calculating the sum of the reciprocated eighth

powers of the roots.

(2) Consider  $J_0(x) = 0$  where  $J_0(x)$  is a particular solution to the

$$\text{Bessel equation. } J_0(x) = 1 - x^2/2^2 + x^4/2^2 4^2 - x^6/2^2 4^2 6^2 + \dots = 0.$$

$$\alpha_1 = -1/4, \quad \alpha_2 = 1/64, \quad \alpha_3 = -1/2304$$

Letting  $P_n$  be the  $n^{\text{th}}$  positive root of  $J_0(x)$ , thus we have

$$\sum 1/P_n^2 = 1/4 \text{ where } \sum 1/P_n^4 = 1/32 \text{ and } \sum 1/P_n^6 = 1/192$$

thus  $P_n = \sqrt[6]{192} \approx 2.40$ . The actual value of this root is 2.4048.

(3) In our investigation of  $\tan x = x$ , and  $\tan x$ , and  $\tan x$  being an

non-integral function, we may write an equivalent expression

for the given. Namely,  $\sin x - x \cos x = 0$ ,  $\cos x \neq 0$ . Using

the power series for  $\cos x$  and  $\sin x$

we have the following series:

$$x^3/3 - x^5/30 + x^7/840 - x^9/45360 + \dots = 0. \text{ Multiplying}$$

both sides of the expression by  $3/x^3$ , we have

$$1 - x^2/10 + x^4/280 - x^6/15120 + \dots = 0; \quad x \neq 0.$$

We see that  $\alpha_1 = -1/10$ ,  $\alpha_2 = 1/280$ ,  $\alpha_3 = -1/15120$

assuming that  $P_n$  is the  $n^{\text{th}}$  positive root of

$\sin x - x \cos x = 0$  and from previous results, we have

$$\sum 1/P_n^2 = -1/10; \quad \sum 1/P_n^4 = 1/360, \quad \sum 1/P_n^6 = 1/7875$$

thus  $\sqrt[6]{7875} \approx 4.46$ . The actual value of this root is 4.4934.

(4) Now consider another equation which is associated with the trans-vibration of beams, namely  $\cosh x \cos x = 1$ . Again utilizing the Maclaurin series expansion for  $\cos x$  and  $\cosh x$  the expression

$$\text{becomes } 1 - x^4/12 + x^8/5040 - \dots = 0$$

$$\sum 1/P_n^4 = 1/12 \quad \text{and } P_n = \sqrt[4]{12} = 1.86$$

also  $\sum 1/P_n^8 = 11/1680$  and  $P_n = \sqrt[8]{1680/11} = 1.87 +$ . The actual value is 1.8751.

#### THE METHOD OF WHITTAKER - BERNOULLI

We now focus our attention on another method which may be alternately used to approximate the roots of a transcendental equation.

This method is known mainly by the Whittaker-Bernoulli method which was derived in chapter II. For some equation say

$a_0 + a_1 x + a_2 x^2 + \dots = 0$  the smallest root can be approximated in absolute value by the given expression:

$$\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}$$

$$x = a_0/a_1 - a_0 a_2/a_1 \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} - a_0^3 \frac{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}$$

Now we employ this method to treat the equations introduced previously.

$$J_0(x) = 1 - x^2/2^2 + x^4/2^2 4^2 - x^6/2^2 4^2 6^2 + \dots \text{ where we let } z = x^2$$

$$\text{then } J_0(x) = 1 - z/2^2 + z^2/2^2 4^2 - z^3/2^2 4^2 6^2 + \dots = 0$$

and

$$z = 1/1/4 + \frac{1^2 \cdot 1/64}{\begin{vmatrix} -1/4 & 1/64 \\ 1 & 1/4 \end{vmatrix}} + \frac{1^3 \begin{vmatrix} 1/64 & 1/2304 \\ -1/4 & 1/64 \end{vmatrix}}{\begin{vmatrix} -1/4 & 1/64 \\ 1 & -1/4 \end{vmatrix} \begin{vmatrix} -1/4 & 1/64 & 1/2304 \\ 1 & -1/4 & 1/64 \\ 0 & 1 & -1/4 \end{vmatrix}}$$

$$z = 4 + .3333 + .16447$$

$$\text{thus } x = 2.40$$

Evaluating the equation  $\cosh x \cos x = -1$

$$\text{where } x^4 = z \text{ and } \cos z \cosh z = 1 - z/12 + z^2/5040 - \dots = 0$$

$$\text{and } a_0 = 1, a_1 = -1/2, a_2 = 1/5040$$

then the root

$$z = \frac{1 \cdot (1/5040)}{\begin{vmatrix} -1/12 & 1/5040 \\ 1 & 1/12 \end{vmatrix}} = 1.8744$$

Consider the equation  $\sin x/x = 0$  whereby applying the Maclaurin series we have  $\sin x/x = 1 - x^2/6 + x^4/120 - x^6/5040 + \dots = 0$

where  $a_0 = 1$ ,  $a_1 = -1/6$ ,  $a_2 = 1/120$ ,  $a_3 = -1/5040$  and letting  $z = x^2$

$$\text{so } z = -1/1/6 + \frac{1 \cdot 1/20}{1/6} - \frac{\begin{vmatrix} 1/120 & -1/5040 \\ -1/6 & 1/120 \end{vmatrix}}{\begin{vmatrix} -1/6 & 1/120 \\ 1 & -1/6 \end{vmatrix}} + \frac{\begin{vmatrix} -1/6 & 1/120 & -1/5040 \\ 1 & -1/6 & 1/120 \\ 0 & 1 & 1/6 \end{vmatrix}}{\begin{vmatrix} -1/6 & 1/120 \\ 1 & -1/6 \end{vmatrix}}$$

thus  $z = 6 + 2.57201 + .9207885$  or  $x = \sqrt{9.49279} = 3.081$

Now let us consider the equation

$\tan x = x$  or in power series form

$1 - x^2/10 + x^4/280 - x^6/15120 + \dots = 0$  and letting  $z = x^2$

we have  $1 - z/10 + z^2/280 - z^3/15120 + \dots = 0$

where  $a_0 = 1$ ,  $a_1 = -1/10$ ,  $a_2 = 1/280$ ,  $a_3 = -1/15120$

and  $x = 4.49$

THE EXTENSION OF  
BERNOULLI'S METHOD

Consider the Bessel function  $J_0(x)$ . Following closely the theorem outlined in chapter 2, we shall obtain an approximation to the root of :

$$J_0(x) = 1 - x^2/2^2 + x^4/2^2 4^2 - x^6/2^2 4^2 6^2 + \dots = 0, \text{ and letting } z = x^2$$

$$J_0(z) = 1 - z/2^2 + z^2/2^2 4^2 - z^3/2^2 4^2 6^2 + \dots = 0$$

we find that  $h_0/h_1 = 1$ ,  $h_1/h_2 = 5.333$ ,  $h_2/h_3 = 5.684$ ,  $h_3/h_4 = 5.763$   
 $h_4/h_5 = 5.779$ ,  $h_5/h_6 = 5.782$ ,  $h_6/h_7 = 5.783$

Noting the relative magnitudes of the last 2 terms in the sequence, it becomes apparent that the sequence has begun to converge to a root.

Thus we find that  $z = 5.783$  or  $x = 2.405$  .

Now we consider a function which converges somewhat less slowly, namely,

$\log(1 + z) = 1/2$ . Expanding the function by the Maclaurin series, we

$$\text{have } -1 + 2z - z^2 + 2/3 z^3 - 1/2 z^4 + \dots = 0$$

Again computing recursively,  $h_v$ 's from the difference equation

$$h_v = 2(h_{v-1} + 1/2 h_{v-2} + 1/3 h_{v-3} + \dots) \text{ and } h_0=1, h_1=2, h_2=3, h_3=14/3$$

$$h_4 = 43/6, h_5 = 166/15, h_6 = 767/45 \text{ and } h_0/h_1 = .5, h_1/h_2 = .667$$

$$h_2/h_3 = .643, h_3/h_4 = .651, h_4/h_5 = .648, h_5/h_6 = .6493. \text{ According}$$

to this method, the indicated root is .6493. The actual root to the

above equation is  $e^{1/2} - 1 = .64872$ . At this point it would be inter-

esting to see whether the Aitken  $\delta^2$  process [5] would have given us

better accuracy while using fewer terms. This process is stated

$$\text{as } x \approx x_{n+2} - (x_{n+2} - x_{n+1})^2 / (x_{n+2} - 2x_{n+1} + x_n), \quad \text{thus}$$

$$\text{we find that } x \approx .6493 - (.6493 - .648)^2 / (.6493 - 1.298 + .651) = .6486.$$

The result compares very favorably with the actual least root of

the above equation, namely,  $\log(1 + z) = 1/2$ .



Let us now focus our attention on the equation  $-\tan\sqrt{z}/\sqrt{z} \approx$   
 $-1 - 1/3 z - 2/15 z^2 - 17/315 z^3 - 62/2835 z^4 - 1382/155925 z^5 -$   
 $21844/6081075 z^6 = 0$  and computing  $h_0, h_1, h_2, \dots, h_6$ , which are  
 respectively 1,  $-1/3$ ,  $-1/45$ ,  $-2/945$ ,  $-1/4725$ ,  $-2/93555$ ,  $-1382/648750375$ .  
 The ratios  $h_0/h_1, h_1/h_2, \dots, h_5/h_6$  becomes respectively  
 $-3, 15, 10, 5, 10, 9.9, 9.870$ . On the other hand, from the  
 Aitkin  $\delta^2$  process we find that the indicated root is 9.938. It  
 should be noted here that we used only four (4) terms from the sequence  
 to make this evaluation!

## GRAEFFE'S METHOD

In computing the root\* of least modulus to the equation  $\log(1 + x) = 1/2$  whose power series representation is  $-1 + 2x - x^2 + 2/3x^3 - 1/2x^4 + 2/5x^5 - 1/3x^6 + 2/7x^7 \dots$ , we find that the root of least modulus is .6448 where the actual root to this equation is .64872. Eight (8) terms and iterations were used.

For equation (2), we have  $\sin x/x = 0$  where  $x \neq 0$  and  $\sin x/x = 1 - 1/6x^2 + 1/120x^4 - 1/5040x^6 \dots$ . Employing the Graeffe Method to these coefficients, it was determined that root of least modulus  $x = 3.078 + \dots$ . On the other hand, the actual value of such root should have been  $3.14 + \dots$ . Four (4) terms and iterations were used.

In applying the Graeffe Method to the Bessel equation of order zero, namely  $J_0(x) = 0$ , to the coefficients of the Macluarin series (which was stated early) we find that the root of least modulus is  $x = 2.567 +$  where the actual root is  $x = 2.40 +$ . Six (6) terms and 6 iterations were used.

In reference to the equation  $\cos x \cosh x = -1$ , it was determined from the application of the method of Graeffe, that the root of least modulus is  $x = 1.875$  where the actual value of this root is given as  $1.87 +$ . Three (3) terms and 6 iterations were used.

Finally, in computing the roots to  $\tan x/x = 0$  and with 6 applications of the Graeffe Process, we find that the root as indicated by computation to be  $x = 4.147$  where the actual root is  $x = 4.49 +$ . Four (4) terms and 6 iterations were used.

---

\*

See Appendix II for the actual computation of the roots.

## CHAPTER IV

### REMARKS AND CONCLUSIONS

In Chapter III, we mentioned various existing methods of globally approximating the roots of transcendental equations. The various approaches mentioned are advantageous from a particular standpoint depending upon the type of solution desired, size of the root sought, computational aids available and more importantly, the effect of "roundoff" error on the approximation. At this point, we discuss the suitability of the various methods explored in this paper.

The method which employs the summing of the reciprocated powers of the roots of the transcendental equation offers fast convergence. Implicit in this method of approach is that one may obtain a fairly high degree of accuracy with the least number of terms from the power series representing the equation. In economizing the terms of the power series we reduce the effect of roundoff error on the root approximation. In addition, this method allows one to achieve good accuracy, especially, if the computational aids are limited only to the slide rule or desk calculator.

On the other hand, the Whittaker-Bernoulli method offers a somewhat different approach as an alternate to approximating the roots of transcendental equations. This method allows one to obtain a relationship between the root of the transcendental equation and the coefficients of the truncated power series. One advantage here is that we can reduce C.P.U. time by as much as 20% over the Graeffe method, and at the same time, this method yields acceptable accuracy

(for most cases) with the least amount of effort. Again we reduce the effect of "roundoff" error by employing fewer number of terms of the series.

The method of Bernoulli [Koenig's extension] does guarantee convergence to a root but not nearly as fast as other methods as indicated in the application to  $J_0(x) = 0$ . One of the advantages of this slower convergence is that it may provide suitable separation when the roots are nearly of the same magnitude. However, in the case of  $J_0(x) = 0$ , convergence was accelerated by employing the Aitken  $\delta^2$  process. This was possible because  $h_n/h_{n+1}$  ( $n = 0, 1, \dots$ ) converges geometrically as was shown in Chapter II. In using the Bernoulli method alone, it sometimes requires a large  $n$  in order to obtain accuracy comparable to the other methods mentioned in this paper.

Among the methods that have been studied in this paper, the Graeffe method is the most "global" in that it is capable of yielding all the roots to a transcendental equation from a theoretical standpoint. On the other hand, there are some inherent drawbacks associated with the process when applied to transcendental equations. First, when the power series is truncated and the Graeffe method applied, some sign alternation may result. When this situation is observed in the case of polynomials, then one is immediately alerted of the presence of complex roots. On the contrary, truncating the power series representing a particular transcendental equation tends to obscure the properties or behavior of roots subsequent to the first. Throughout the examples that were

included (see Appendix II), such erratic behavior was not evident when the root of least modulus was computed from the first coefficient of the derived equation(s), but instead with the roots subsequent to the first when computed from the remaining coefficients. Secondly, situations were encountered where the coefficients would "absolutely" tend to zero for increasing subscripts. In attempting to compute more terms to refine the accuracy in determining other roots and iterating Graeffe's process, one is rewarded with an inordinate number of zero coefficients making it nearly impossible to determine information about the roots. This is especially evident in the example displayed on page 33 of the appendix. In a positive sense, in observing the examples of Chapter III, it can be seen that the method of Graeffe yields acceptable accuracy and fast convergence in computing the root of least modulus. Also this method seemed very compatible with the capability of the APL programming language. Computer programs for the methods mentioned in this paper are available in Appendix I.

In the case involving polynomials, Bernoulli's method may be applied prior to Graeffe's method in order to increase the separation of roots [1]. Since the domain of our investigation did not include the approximation of real roots of larger modulus, it may be interesting to know the degree of separation of roots that can be expected from the employment of the above procedure to a truncated power series. As a contribution to this effort, to what extent should we incorporate the method of asymptotic expansions to the power series representing a particular transcendental equation?

## BIBLIOGRAPHY

1. Carnahan, B., Luther, H. A. and Wilkes, J. O., "Applied Numerical Methods" 1st edition. New York: John Wiley and Sons, Inc., 1969.
2. Gantmacher, F. R., "The Theory of Matrices." Volume 1. New York: Chelsea Publishing Company, 1960.
3. Higgins, T. J. "Note on Whittaker's Method for the Roots of a Power Series." Communication of the ACM Volume 9, 1960.
4. Hamming, Richard W. and Feigenbaum, E. A. "Introduction to Applied Numerical Analysis", 1st edition. New York: McGraw-Hill Book Company, 1971.
5. Householder, A. S. "The Numerical Treatment of a Simple Non-linear Equation", 1st edition. New York: McGraw-Hill, Inc., 1970.
6. Kowel, Stephen T. "Absolute Iteration and the Solution of Transcendental Equations", Journal of Franklin Institute, Volume 278, 1964.
7. Kreyszig, Erwin. "Advanced Engineering Mathematics", 3rd edition. New York: John Wiley and Sons, Inc., 1972.
8. Lehmer, D. H. "The Graeffe Process as Applied to Power Series," Mathematics Tables Aids Comp. Volume 1, 1945.
9. Maehly, Hans J. "Numerical Solution of a Certain Transcendental Equation Involving Exponentials," Journal of the Society of Industrial Applied Mathematics, Volume 10, 1962.
10. Pennisi, Louis L. "Elements of Complex Variables", 2nd edition. New York: Holt, Rinehart and Winston, Inc., 1963.
11. Retkory, K. "Applicable Mathematics", 1st edition. Massachusetts: The M.I.T. Press, 1968.
12. Spiegel, M. R. "The Summation of Series Involving Roots of Transcendental Equations and Related Applications", Journal of Applied Physics, Volume 24, 1953.
13. Thacher, Henry C. Jr. "Solutions of Transcendental Equations by Series Reversion", Communication of the ACM, Volume 9, 1966.
14. Uspensky, J. V. "Theory of Equations", 1st edition. New York: McGraw-Hill Book Company, Inc., 1948.
15. Wolford, J. C., Smith, G. M., James, M. L. "Applied Numerical Methods for Digital Computation with FORTRAN." 1st edition. Pennsylvania: International Textbook Company, 1967.

APPENDIX I

APL COMPUTER PROGRAMS

## APL PROGRAMS RELATING TO TRANSCENDENTAL EQUATIONS

## VDET[ ] V (WHITTAKER -BERNOULLI METHOD)

```

V Z←A DET B;I;J;K
[1] B←Q(1 0 +A)ρ(.Q(A,A)ρB),~J+1<I←A,K+1 0
[2] →Z+(V/O=K+B[1;1],K)ρ 0
[3] →(0≠I←I-ρρρB+1φ(J,1)φ[1] B-(J×B[;1])ρ.×B[1;]←B[1;]÷K[1])/2
[4] Z←×/K

```

∇

## VREC[ ] V (METHOD OF BERNOULLI VIA KOENIG'S EXTENSION)

```

V Z←REC B
[1] Z←ρρB+,φB
[2] →2×(ρB)ρZ←Z,R+,×(-ρB)↑Z

```

∇

## VCOE[ ] V (GRAEFFE'S METHOD)

```

VU1←COE V
[1] COUNT←0
[2] U1←(1,(R-1)ρ-2 2)+.×(((0,-2|ρV)+U),φU+Iρ.≤)×(V*2),[1]×/(1-1ρ.×-1↑I+1R)
φ(2,(-11+R+[0,5×ρV],ρV)ρV,V
[3] COUNT←(COUNT + 1)
[4] ρ+2*COUNT
[5] →(COUNT =K)/0
[6] ρ+V←U←U1
[7] →2
[8] N+1 ∇

```



APPENDIX II  
EQUATIONS EVALUATED BY GRAEFFE'S METHOD

COE SH (COS X COSH X = 0)

2  
1 0.006547619048 3.936759889E-8  
4  
1 4.279257999E-5 1.549807842E-15  
8  
1 1.831201803E-9 2.401904349E-30  
16  
1 3.353300043E-18 5.7691445E-60  
32  
1 1.124462118E-35 0  
64  
1 1.264415055E-70 0  
128  
1 0 0  
256  
1 0 0

R+1.264415055E-70\*(+256)

R

1.875165023

COE XS (TAN X / X = 0)

2  
1 0.002857142857  $-4.724111867E^{-7}$   $4.374177655E^{-9}$   
4  
1 9.10808768E<sup>-6</sup>  $-2.477212855E^{-11}$   $1.913343015E^{-17}$   
8  
1 1.325015183E<sup>-10</sup>  $2.651204342E^{-22}$   $3.660881495E^{-34}$   
16  
1 1.702641148E<sup>-20</sup>  $-2.672562663E^{-44}$   $1.340205332E^{-67}$   
32  
1 2.899521391E<sup>-40</sup> 0 0  
64  
1 0 0 0  
128  
1 0 0 0  
256  
1 0 0 0

R+1.702641148E<sup>-20</sup>\*(+32)

R

4.147414124

COE JX (J<sub>0</sub>(X) = 0)

2  
1 0.03125 0.0001899401347 1.094597219E<sup>-8</sup> 6.768187595E<sup>-16</sup> 1.321996248E<sup>-26</sup>  
4  
1 0.0005966822306 3.539313286E<sup>-8</sup> 1.195571972E<sup>-16</sup> 4.577942226E<sup>-31</sup> 1.747674078E<sup>-52</sup>  
8  
1 2.852434186E<sup>-7</sup> 1.252531178E<sup>-15</sup> 1.429389099E<sup>-32</sup> 2.095755085E<sup>-61</sup> 0  
16  
1 7.885874552E<sup>-14</sup> 1.568834344E<sup>-30</sup> 2.043153196E<sup>-64</sup> 0 0  
32  
1 6.215564077E<sup>-27</sup> 2.461241198E<sup>-60</sup> 0 0 0  
64  
1 3.863323187E<sup>-53</sup> 0 0 0 0  
128  
1 0 0 0 0 0  
256  
1 0 0 0 0 0

R+\*(6.215564077E<sup>-27</sup>)\*(+64)

R

2.567301452

COE SX (SIN X /X = 0)

2  
1 0.011111111111 3.306878307E-6 3.936759889E-8  
4  
1 0.0001168430335 -8.639000868E-10 1.549807842E-15  
8  
1 1.538009465E-8 3.841548606E-19 2.401904349E-30  
16  
1 2.357790018E-16 7.369192446E-38 5.7691445E-60  
32  
1 5.559159032E-32 2.710013468E-75 0  
64  
1 3.090424914E-63 0 0  
128  
1 0 0 0  
256  
1 0 0 0

p

R++3.090424914E-63\*(+128)

R

3.078642304