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GEOMETRICAL GEODESY TECHNIQUES IN GODDARD EARTH MODELS

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JANUARY 1974

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GREENBELT, MARYLAND

Presented at the International Association of Geodesy Symposium on Computational Methods in Geometrical Geodesy, Oxford, United Kingdom, September 2-8, 1973.

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ABSTRACT

The method for combining geometrical data with satellite dynamical and gravimetry data for the solution of geopotential and station location parameters is discussed. Geometrical tracking data (simultaneous events) from the global network of BC-4 stations are currently being processed in a solution that will greatly enhance the geodetic world system of stations. Previously the stations in Goddard Earth Models have been derived only from dynamical tracking data. In this paper a linear regression model is formulated for combining the data, based upon the statistical technique of weighted least squares. Reduced normal equations, independent of satellite and instrumental parameters, are derived for the solution of the geodetic parameters. Exterior standards for the evaluation of the solution and for the scale of the earth's figure are discussed.

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GEOMETRICAL GEODESY TECHNIQUES IN GODDARD EARTH MODELS

I. INTRODUCTION

A matrix model employing a reduced form of the general least squares adjustment process is developed. The model provides a method for combining geometrical BC-4 optical data with satellite dynamic and gravimetric data into a general geodetic solution. The geodetic solution consists of a geocentric system of station coordinates and spherical harmonic coefficients for the geopotential. A previous solution, Lerch et al (1972), for a Goddard Earth Model (GEM 4) contained 514 geodetic parameters but did not include any geometric data.

A map of station locations is presented in Figure 1, illustrating the distribution of 45 BC-4 stations associated with the geometric data and 61 stations associated with dynamic data from electronic, laser, and optical tracking systems used previously in GEM 4. Figure 2 illustrates local datum ties between the dynamic and geometric stations and BC-4 baselines obtained from the geodimeter and tellurometer systems that are to be employed in the new combination solution.

II. BASIC DATA SYSTEMS AND REPRESENTATION

1. Data Systems

Four basic geodetic data systems are employed in the combination solution and are listed with their associated solution parameters in Table 1. These systems are the BC-4 geometric ($\bar{B}$), gravimetry ($\bar{G}$), dynamic satellite ($\bar{D}$), and survey ($\bar{S}$).

In the survey data system $\bar{S}$, the baselines and ties were illustrated in Figure 2. The ties correspond to relative position coordinates for nearby geometrical and dynamical stations and are treated as observations with statistical errors based upon local survey accuracy. Similarly the baseline distances are treated as statistical observations.

The satellite related parameters associated with the BC-4 system $\bar{B}$ and the satellite dynamic system $\bar{D}$ require preliminary processing for initial estimates of the parameters. The initial processing will be discussed in a later section. There are some 20,000 satellite position parameters associated with 1100 BC-4 events of two, three, and four stations observing the satellite simultaneously at 6 to 7 reduced points of satellite position for each event. There are some 350 weekly arcs of satellite dynamic data on 27 satellites, where some 400,000 observations of electronic, laser, and optical tracking data have been processed.

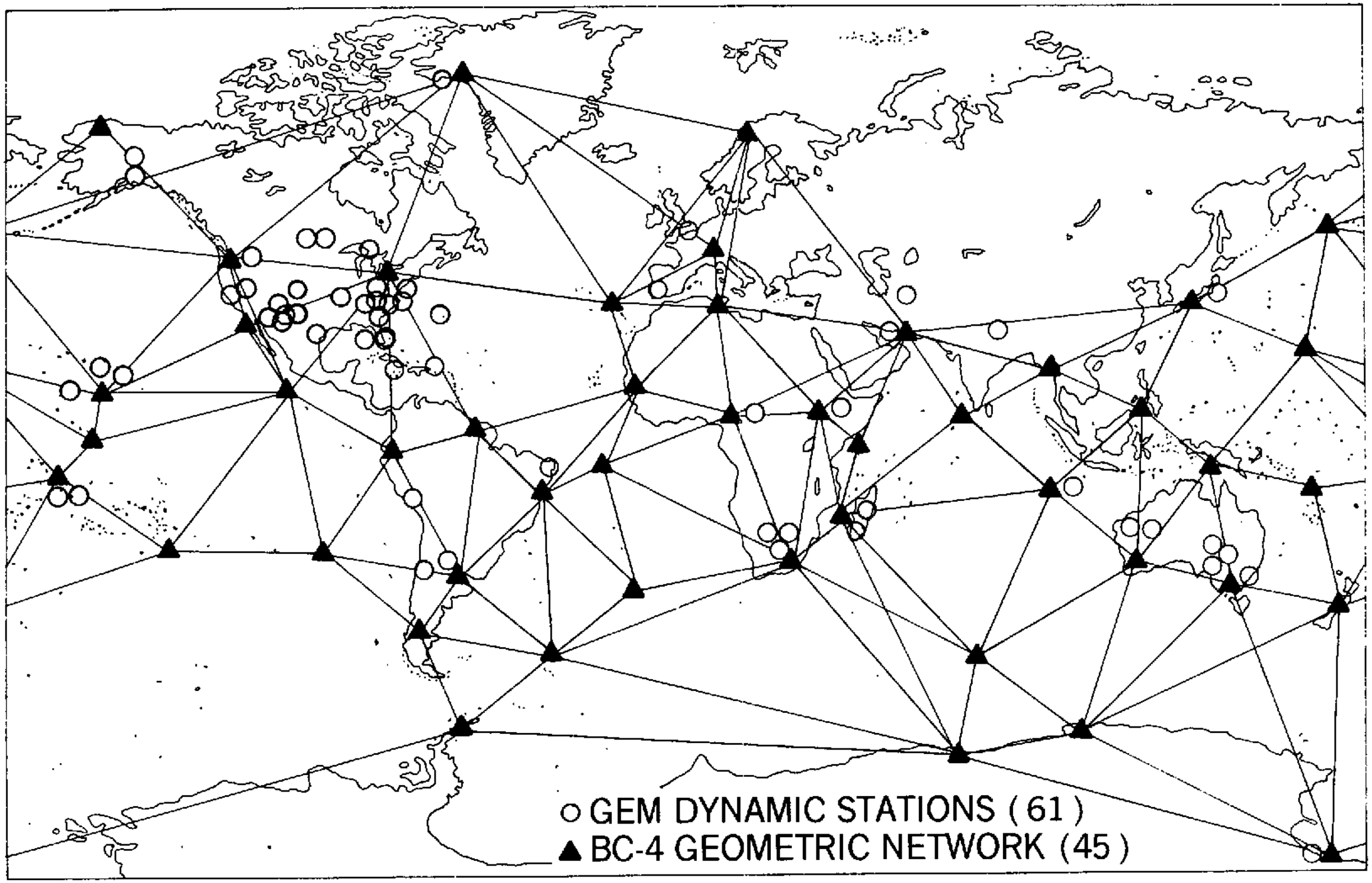

Figure 1. Station Locations

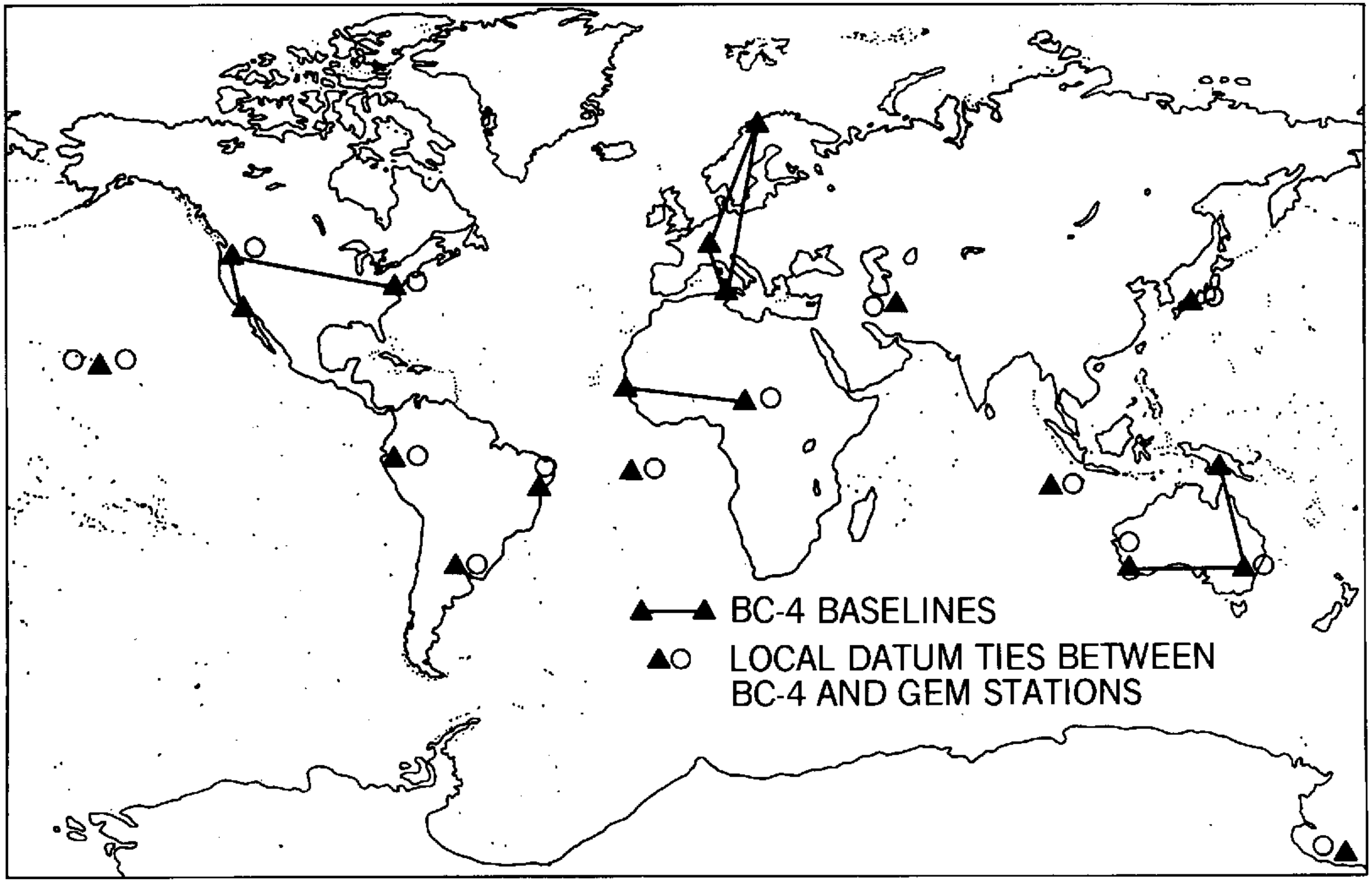

Figure 2. Baselines and Station Ties (Survey Data)

Table 1
Basic Data Systems and Solution Parameters

Symbol	Data System	Solution Parameters (unknowns)	Unknown Symbol
$\bar{B}$	BC-4 geometric	station location coordinates satellite position components for each geometrical point j	x q_i
$\bar{G}$	Gravimetry	potential coefficients (spherical harmonic)	c
$\bar{D}$	Dynamic satellite	potential coefficients station coordinates satellite orbital elements and tracking system parameters associated with each satellite arc i	c z p_i
$\bar{S}$	Survey (Baselines and ties)	station coordinates connecting BC-4 baselines x_b in x station coordinates connecting ties x_t in x, z_t in z	 x x, z

The satellite dynamic parameters consist of six orbital elements and modeled force parameters at a given epoch on each weekly arc. Tracking system parameters for certain electronic systems have been included in the modeling and are associated with the initial processing on a weekly arc. The gravimetry data system $\bar{G}$ consists of a global distribution of 5° equal area blocks of mean gravity anomalies, Rapp, (1972).

Because of the large number of satellite parameters in the systems $\bar{B}$ and $\bar{D}$, a least squares matrix model is developed that will reduce the matrix to a form containing just the geodetic parameters. The reduced form will then be suitable for computational solution.

2. Representation of Data Systems

A total data system $\bar{C}$, which will be used to encompass the four basic data systems, is represented as

$$\bar{C} : (O, C, v; y) \tag{1}$$

where each symbol denotes a column vector as follows:

O — observations

C — computed quantities corresponding to the observations

v — observation residuals ($v = O - C$)

y — the solution parameters or unknowns (see Table 1)

The column vectors are further defined in Section III where the method for the solution is developed. The four basic data systems, subsystems of $\bar{C}$, are similarly represented and defined as in the above form (1), namely

$$\bar{B} : (O_B, B, b; x, q) \tag{2}$$

$$\bar{G} : (O_G, G, g; c) \tag{3}$$

$$\bar{D} : (O_D, D, d; c, z, p) \tag{4}$$

$$\bar{S} : (O_S, S, s; z, x) \tag{5}$$

where the solution parameters c, z, x, $p = [p_i]$, $q = [q_j]$ are defined in Table 1 for each of the data subsystems. Using the form above, data subsystems of $\bar{D}$ and $\bar{B}$ are represented for each dynamic satellite arc i and geometric satellite position point j, respectively as follows:

$$\bar{D}_i : (O_{Di}, D_i, d_i; c, z, p_i) \tag{6}$$

$$\bar{B}_j : (O_{Bj}, B_j, b_j; x, q_j) \tag{7}$$

III. DEVELOPMENT OF THE METHOD

1. General Matrix Model for the Least Squares Solution

A general matrix model of the least squares adjustment process is developed for the complete data system $\bar{C}$ given in (1). The model will be subsequently employed to develop a reduced form of solution for the geodetic data subsystems.

In the data system $\bar{C}$ the vector symbols, O, C, v, and the unknown solution y were defined under (1). The vector C = C(y) and the residual vector is

$$v = O - C(y) \tag{8}$$

which in component form is

$$[v_n] = [O_n - C_n(y)] \quad \text{for } n = 1 \text{ to } N \tag{9}$$

where $C_n(y)$ is the computed quantity corresponding to the observation O_n and is a function of the parameters in y. The linear condition equation for v in the least squares adjustment process is given by use of Taylor's expansion as

$$v = v_o - C_y \Delta y , \tag{10}$$

where $v_o = O - C(y_o)$, $y = \Delta y + y_o$, y_o is a column vector of initial estimates for each component y_k in y for k = 1 to K, and the matrix of partial (derivative) coefficients

$$C_y = \left[\frac{\partial C_n(y)}{\partial y_k}\right]_{NXK} \tag{11}$$

in which the partial coefficient element lies in row n and column k and $C_n(y)$ is given under (9) above. C_y is evaluated at $y = y_o$.

The least squares minimum condition is

$$Q = v^T W_v v = \text{minimum} \tag{12}$$

where W_v is a diagonal weight matrix with each element

$$w_n = \frac{1}{\sigma_n^2} \qquad \text{for } n = 1 \text{ to } N \tag{13}$$

in which σ_n^2 is the error variance for the observation O_n. The minimum condition for (12) is obtained from

$$\frac{\partial Q}{\partial y} = \left[\frac{\partial Q}{\partial y_k}\right] = 0 \qquad \text{for } k = 1 \text{ to } K \tag{14}$$

which with use of (10) and (12) will give the normal equation

$$C_y^T W_v v = 0 \tag{15}$$

or

$$(C_y^T W_v C_y) \Delta y - C_y^T W_v v_o = 0. \tag{16}$$

Solving (16) for Δy will give

$$\Delta y = (C_y^T W_v C_y)^{-1} C_y^T W_v v_o \tag{17}$$

and the solution y is

$$y = y_o + \Delta y.$$

Under ideal conditions, where the modeling of $C_n(y)$ for O_n is complete except for a random observation error which has normal distribution and variance σ_n^2, the least squares solution for y may be shown to satisfy the maximum likelihood principle. See Anderson (1958).

2. Reduced Form of the Normal Equations for the Geodetic Subsystems

Using the column matrices defined for the total data system $\bar{C}$ under (1) and those for the geodetic data subsystems in (2) through (5), the following matrix partitioning is given for $\bar{C}$ in terms of its data subsystems:

$$O = \begin{bmatrix} O_D \\ O_B \\ O_G \\ O_S \end{bmatrix} \qquad C = \begin{bmatrix} D \\ B \\ G \\ S \end{bmatrix} \qquad v = O - C = \begin{bmatrix} d \\ b \\ g \\ s \end{bmatrix} \tag{18}$$

Also partition

$$W_v = \begin{bmatrix} W_d & & & \\ & W_b & & \\ & & W_g & \\ & & & W_s \end{bmatrix} \qquad y = \begin{bmatrix} c \\ z \\ x \\ p \\ q \end{bmatrix} \qquad \Delta y = \begin{bmatrix} \Delta c \\ \Delta z \\ \Delta x \\ \Delta p \\ \Delta q \end{bmatrix} \tag{19}$$

where the solution parameters $p = [p_i]$, $q = [q_j]$, x z, and c are defined in Table 1, the weight matrix W_v was defined under (13) for $\bar{C}$ and W_d, W_b, W_g, W_s are defined as diagonal weight matrices respectively for the subsystem observations as ordered in O above. When v and W_v above are substituted into (12), Q then becomes

$$Q = d^T W_d d + b^T W_b b + g^T W_g g + s^T W_s s \tag{20}$$

In terms of the variables associated with the data subsystem as given in Table 1, the linearized residual equations may be expressed by Taylor's series for each subsystem as

$$
\begin{aligned}
d &= d_o - D_c \Delta c - D_z \Delta z - D_p \Delta p \\
b &= b_o - B_x \Delta x - B_q \Delta q \\
g &= g_o - G_c \Delta c \\
s &= s_o - S_x \Delta x - S_z \Delta z
\end{aligned}
\tag{21}
$$

where the subscript (o) corresponds to the initial estimate of the residual vector which is analogous to v_o under (10), and the matrices of partial coefficients are defined analogous to C_y under (11), for example, in D_z above D would correspond to C and z to y. Differentiating Q with respect to each of the variables as ordered in (19) for y and setting the result equal to zero in order to obtain the minimum, the following normal equations will result with use of (21) for each of the variables:

$$
\begin{aligned}
D_c^T W_d\, d + G_c^T W_g\, g &= 0 \qquad \text{for } c \\
D_z^T W_d\, d + S_z^T W_s\, s &= 0 \qquad \text{for } z \\
B_x^T W_b\, b + S_x^T W_s\, s &= 0 \qquad \text{for } x \\
D_p^T W_d\, d &= 0 \qquad \text{for } p \\
B_x^T W_b\, b &= 0 \qquad \text{for } q
\end{aligned}
\tag{22}
$$

Substituting the relations for d, b, g, and s in (21) into (22) will produce the normal matrix equations in terms of the unknown parameters and the initial residuals. This result between (21) and (22) is equivalent to the result between (15) and (16) for the total data system $\bar{C}$. If C_y is partitioned in terms of the partial coefficient matrices in (21) then by substituting its transpose, v from (18), and W_v from (19) into (15) the normal equations (22) will result directly.

The last two equations in (22) are used with d and b in (21) to express the satellite related parameters (Δp, Δq) in terms of the geodetic parameters (Δc, Δx, Δz).

When these results for Δp and Δq are substituted into d and b for the first three equations of (22), the reduced normal equations will then be obtained for the geodetic parameters. This process just described is carried out below.

Proceeding in this manner then from (22) and d in (21)

$$D_p^T W_d d = D_p^T W_d (d_o - D_c \Delta c - D_z \Delta z - D_p \Delta p) = 0. \tag{23}$$

Solving (23) for Δp and substituting the result back into d will give

$$d = \bar{P} (d_o - D_c \Delta c - D_z \Delta z) \tag{24}$$

where

$$\bar{P} = I - D_p \bar{\bar{P}} \tag{25}$$

$$\bar{\bar{P}} = (D_p^T W_d D_p)^{-1} D_p^T W_d \tag{26}$$

$$\Delta p = \bar{\bar{P}} (d_o - D_c \Delta c - D_z \Delta z) \tag{27}$$

Proceeding similarly as in (23) with

$$B_q^T W_b b = 0$$

then

$$b = \bar{Q} (b_o - B_x \Delta x) \tag{28}$$

where $\bar{Q}$ and Δq may be derived as in (25) through (27).

By use of d in (24), b in (28), and s and g in (21) the system (22) is expressed in matrix form for the geodetic parameters as follows:

$$\begin{bmatrix} Dcc + Gcc & Dcz & 0 \\ Dcz^T & Dzz + Szz & Szx \\ 0 & Szx^T & Bxx + Sxx \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta z \\ \Delta x \end{bmatrix} = \begin{bmatrix} Dco + Gco \\ Dzo + Szo \\ Bxo + Sxo \end{bmatrix} \tag{29}$$

where the above symbols are defined as follows:

<u>dynamic satellite system $\bar{D}$</u>

$$\begin{aligned} Dcc &= D_c^T \bar{W} D_c \qquad & Dcz &= D_c^T \bar{W} D_z \qquad & Dzz &= D_z^T \bar{W} D_z \\ Dco &= D_c^T \bar{W} d_o \qquad & Dzo &= D_z^T \bar{W} d_o \qquad & \bar{W} &= W_d \bar{P} \end{aligned} \tag{30}$$

<u>BC-4 geometric system $\bar{B}$</u>

$$Bxx = B_x^T W_b \bar{Q} B_x \qquad Bxo = B_x^T W_b \bar{Q} b_o \tag{31}$$

<u>gravimetric system $\bar{G}$</u>

$$Gcc = G_c^T W_g G_c \qquad Gco = G_c^T W_g g_o \tag{32}$$

<u>survey system $\bar{S}$</u>

$$\begin{aligned} Szz &= S_z^T W_s S_z \qquad & Szx &= S_z^T W_s S_x \qquad & Sxx &= S_x^T W_s S_x \\ Szo &= S_z^T W_s s_o \qquad & Sxo &= S_x^T W_s s_o & & \end{aligned} \tag{33}$$

In the computations the results for $\bar{D}$ in (30) and $\bar{B}$ in (31) are expressed in terms of reduced results obtained from the observations on each satellite arc i with satellite parameters p_i and on each satellite geometric point j with coordinates q_j. This situation is represented by the data subsystems $\bar{D}_i$ in (6) and $\bar{B}_j$ in (7). The data sets and associated partial coefficient matrices of $\bar{D}$ and $\bar{B}$ may be partitioned in terms of these quantities for their respective subsystems to obtain the detailed results.

The procedure for these results will be briefly derived. From $\bar{D}_i$ in (6) and $\bar{D}$ in (4)

$$d_i = O_{Di} - D_i\,(c, z, p_i) \qquad \text{for each } i = 1 \text{ to } I, \tag{34}$$

$$d = [d_i] \qquad D = [D_i] \qquad \Delta p = [\Delta p_i] \qquad p = [p_i] \tag{35}$$

Using Taylor's expansion for d_i in (34)

$$d_i = d_{io} - H_i\,\Delta c - Z_i\,\Delta z - P_i\,\Delta p_i \tag{36}$$

then, from d in (21) and since $d = [d_i]$,

$$d_o = [d_{io}] \qquad D_c = [H_i] \qquad D_z = [Z_i]$$

$$D_p = \begin{bmatrix} P_1 & & & o \\ & P_2 & & \\ & & \ddots & \\ o & & & P_I \end{bmatrix} \tag{37}$$

Defining the weight matrix W_{di} for O_{Di} and ordering it on the diagonal of W_d for $i = 1$ to I and using the above results for (34) through (37), then (23) through (27) become for each arc i

$$P_i^T W_{di} d_i = P_i^T W_{di} (d_{io} - H_i \Delta c - Z_i \Delta z - P_i \Delta p_i) = 0$$

$$d_i = \bar{P}_i (d_{io} - H_i \Delta c - Z_i \Delta z)$$

$$\bar{P}_i = I - P_i \bar{\bar{P}}_i \tag{38}$$

$$\bar{\bar{P}}_i = (P_i^T W_{di} P_i)^{-1} P_i^T W_{di}$$

$$\Delta p_i = \bar{\bar{P}}_i (d_{io} - H_i \Delta c - Z_i \Delta z)$$

and the results in (30) become

$$\begin{aligned} Dcc &= \sum_i (H_i^T \bar{W}_i H_i) \qquad Dcz = \sum_i (H_i^T \bar{W}_i Z_i) \qquad Dzz = \sum_i (Z_i^T \bar{W}_i Z_i) \\ Dco &= \sum_i (H_i^T \bar{W}_i d_{io}) \qquad Dzo = \sum_i (H_i^T \bar{W}_i d_{io}) \qquad \bar{W}_i = W_{di} \bar{P}_i \end{aligned} \tag{39}$$

where the sum ranges from $i = 1$ to I.

Treating the system $\bar{B}_j$ in (7) in a like manner to $\bar{D}_i$ above, the result in (31) would become

$$Bxx = \sum_{j=1}^{J} (X_j^T \bar{W}_j X_j) \qquad Bxo = \sum_{j=1}^{J} (X_j^T \bar{W}_j b_{jo}), \tag{40}$$

where X_j corresponds to Z_i, $\bar{W}_j$ to $\bar{W}_i$, and b_{jo} to d_{io}.

The solution for the geodetic parameters is obtained through the inverse of the reduced normal matrix in (29). The inverse matrix also represents the variance-covariance matrix for the geodetic parameters, as the weighting for the observations given in (13) is inversely equal to the variance of the observation errors.

3. Computational Procedure

In practice the contributions to the reduced normal matrix for each of the geodetic subsystems are computed separately and then combined as in (29). The terms for the reduced normal matrix for each of these four systems are given separately in equations (30) through (33). The above practice is similarly true in the computations for the subsystems $\bar{D}_i$ of $\bar{D}$ and $\bar{B}_j$ of $\bar{B}$ where their respective matrix terms are identified and combined as in (39) and (40).

The initial starting values for the geodetic parameters are obtained from a previous geodetic solution, survey data, or through a separate process of data analysis.

3.1 Satellite Dynamic System

The initial observation residuals and dynamic satellite parameters for the system $\bar{D}_i$ are obtained from a preliminary reduction of the observation data on a weekly satellite arc. In this process a bias parameter is modeled for certain electronic systems for the observation data on each satellite tracking pass. Since a large number of these parameters may occur within a weekly arc span of data, each bias parameter is eliminated through the back subsitution process at the end of the associated tracking pass. Numerical integration is employed for the satellite orbit and the preliminary reduction of the data on the weekly arc is carried out through the least square process of successive iterations. The normal matrix for the system $\bar{D}_i$ is formed immediately after the preliminary reduction.

3.2 Geometric Geodesy Technique

In the computations for the BC-4 data system the reduced normal equations (31) may be obtained through the formulation of condition equations which are independent of satellite parameters. This technique is described in the appendix of the report, and it includes the constraint equations for the station coordinate ties and baseline distances from datum survey. In addition it provides for use of simultaneous MOTS and laser data, which has recently been analyzed by Reece and Marsh (1973) for stations in the area of the United States.

IV. ANALYSIS OF SOLUTION AND GEODETIC RESULTS

The geodetic solution may be tested and analyzed with the use of survey data in several areas of investigation.

The mean sea level height (MSL) from station survey may be compared with the height of the station above the geoid as computed from the solution of the potential coefficients and the geocentric station coordinates including the reference ellipsoid, Lerch et al (1972). In view of the distribution of stations in Figure 1 the dispersion of the differences between survey and computed values about a mean line may be analyzed. The MSL heights from survey are generally reported to be accurate to about a meter for these stations. Any significant offset in the mean line from zero differences may be associated with the scale for the reference ellipsoid, and thus the equatorial radius (a_e) of the reference ellipsoid may be adjusted. Any significant dispersion in the differences (including systematic differences) may be analyzed in terms of geographical areas and in terms of various tracking systems such as the electronic, laser, and optical systems. The dispersion may be analyzed in terms of the geopotential model particularly in areas where the gravimetric measurements are not available. These analyses may be supported with separate tests for geoid heights and station coordinates as indicated below.

Geoid heights computed from the potential model may be tested and analyzed in certain major survey areas that contain astrogeodetic deflections of the vertical and detailed gravimetric data, Vincent et al (1972).

Through adjustment for scale, orientation, and datum shift the station coordinates between datum survey and the solution may be analyzed. In such an analysis for a geocentric station solution by Marsh et al (1971), an rms agreement of 3.5 meters on station coordinate differences has been obtained for 20 stations on the North American Datum.

With a solution derived from the geodetic data systems and use of the above analysis the following geodetic results may be obtained:

1. A global geoid represented in terms of spherical harmonic coefficients.

2. A world datum of station coordinates including an adjustment for scale, orientation, and datum shift for local datums.

3. Mean equatorial radius (a_e) for the Earth.

4. Mean equatorial value of normal gravity (g_e) may be derived from the gravimetric data, Rapp (1972).

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APPENDIX

Reduced Normal Equations for Geometric Satellite Geodesy Including Constraints from Datum Survey

CONTENTS

APPENDIX

Reduced Normal Equations for Geometric Satellite Geodesy Including Constraints from Datum Survey

The method developed here for the least squares normal equations is based upon the technique of formulating reduced condition equations where the satellite parameters have been eliminated. The data considered consists of simultaneous events from MOTS (Minitrack Optical Tracking System) and laser systems on GEOS-I and II, the BC-4 worldwide camera network on PAGEOS, and local datum survey ties and baselines. The reduced condition equations are developed in this appendix and a case is considered for the treatment of correlated observations.

1. Technique for the Normal Equations

The mathematical analysis leading to the formation of normal equations for the geometric adjustment of coordinates of tracking stations is based on the following type of events:

1. Two cameras observe the satellite simultaneously
2. Three cameras observe the satellite simultaneously
3. Four cameras observe the satellite simultaneously
4. Two cameras and one laser observe the satellite simultaneously.

Condition equations resulting from a given set of simultaneous observations are of two types:

- Coplanarity equation, which requires that two observing stations and their directions to the satellite lie in the same plane.
- Length equation, which requires that the satellite position satisfying the two-station coplanarity relationship also agrees with the range from a third station.

Corresponding to each event condition equations of the following form are used:

$$\sum_{i}^{m} a_i v_i + \sum_{j}^{n} b_j x_j + c = 0 \tag{1}$$

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where a_i, b_j, and c are known constants derived in the subsequent sections, c is the discrepancy in the condition equation

v_i are the unknown residuals (adjusted minus observed values)

x_j are the unknown corrections to stations' Cartesian coordinates (adjusted minus initial values)

m is the number of observed quantities

n is the number of unknown coordinates.

The number and types of condition equations for events 1 through 4 above are as follows:

- For a two-camera event, one coplanarity equation is used.
- For a three-camera event, three coplanarity equations are used.
- For a four-camera event, five coplanarity equations are used.
- For a two-camera, single-laser event, one coplanarity equation and one length equation are used.

The number of condition equations for each event corresponds to the number of observations less three, since the observation equations are reduced to a form where the three satellite position coordinates are eliminated. Each observing camera contributes two observations in an event and an observing laser contributes one observation. Each of the coplanarity equations for an event involves distinct pairs of observing stations.

Additional condition (constraint) equations specify coordinate differences and also take the form of equation (1). Constraints are treated statistically, similar to observation equations, where values and accuracies are obtained from a priori information based on datum survey. Two types of constraint equations are applied:

- Distance equations (baselines), which require the distance between two stations to remain near a given value
- Coordinate-shift equations, which require the differences between coordinates of two nearby stations to remain near a given value. This constraint is used to connect the stations in the geometric geodesy with those in the dynamic satellite geodesy.

For a geometric only solution a third type of statistical constraint may be applied on individual station coordinates in order to fix the origin of the system. A priori values for these constraints should be taken from a different source than datum survey such as from a previously determined geocentric solution in a center of mass reference frame. This constraint is not used in the combination solution with dynamic satellite geodesy.

For each event (or constraint) k, denote the associated condition equations of the form (1) in matrix notation as

$$A_k V_k + B_k X + C_k = 0, \tag{2}$$

for which an example of the dimensions and elements of the matrices are given below. Minimizing Q below w.r.t. the unknown station coordinates in X and residuals in V_k will lead to the formation of the normal matrix equation. The form Q is

$$Q = \sum_{k=1}^{K} (V_k^T W_k V_k - 2\lambda_k^T (A_k V_k + B_k X + C_k)) \tag{3}$$

where W_k is the diagonal weight matrix for the observations in V_k and each λ_k is a column vector of Lagrangian multipliers corresponding to the number of condition equations in event k. The resulting normal matrix equation to be combined with the gravimetric and dynamic satellite geodesy systems is

$$JX + \sum_{k=1}^{K} B_k^T M_k^{-1} C_k = N, \tag{4}$$

where

$$J = \sum_{k=1}^{K} (B_k^T M_k^{-1} B_k), \tag{5}$$

$$M_k = (A_k W_k^{-1} A_k^T) \tag{6}$$

The largest dimension of M_k is 5×5 corresponding to event of type 3 where there are 5 coplanarity equations of condition. This case has 8 observations and the dimensions of A_k, V_k, B_k, and C_k are respectively 5×8, 8×1, $5 \times N_x$, and 5×1. N_x is the total number of all station coordinates and B_k $(5 \times N_x)$ would contain for each of the 5 rows only 6 non-zero elements, corresponding to the b_j coefficients in (1) for each distinct coplanarity equation involving two observing stations. Each row of A_k has 4 non-zero elements corresponding to the a_i coefficients in (1), associated with the two observing stations in each coplanarity equation.

By employing a suitable set of constraints including those that fix the origin, N may be set equal to zero and a geometric only solution for X can be derived from (4).

Condition equations for coplanarity, length, and constraints are developed in sections 2 through 5 and section 6 treats a case for correlated observations.

2. Coordinate System

Camera observations in α and δ are transformed from right ascension α and declination δ to earth-fixed angles β and γ. The conversion of α and δ, as corrected for precession, nutation, and polar motion, to the angles β and γ is straightforward. The topocentric angle γ is measured with respect to the equatorial plane and is equivalent to δ, i.e., $\gamma = \delta$. The angle β is measured from the Greenwich meridian in a plane parallel to the equator and is

$$\beta = \alpha - \mathrm{GHA} \tag{7}$$

where GHA is the Greenwich Hour Angle at the epoch of the observation.

3. Coplanarity Equation

The coplanarity equation requires that the volume of the parallelepiped defined by the two station-to-satellite vectors and the station-to-station vector and their respective errors be zero. The two station-to-satellite vectors are defined in the local terrestrial coordinates as

$$\vec{p}_i = u_i \hat{i} + v_i \hat{j} + w_i \hat{K} \tag{8}$$

where

$$u_i = \cos \gamma_i \cos \beta_i$$

$$v_i = \cos \gamma_i \sin \beta_i \qquad (i = 1, 2)$$

$$w_i = \sin \gamma_i$$

The station-to-station direction vector $\vec{p}_3$ is similarly defined in spherical coordinates by use of

$$\beta_3 = \tan^{-1} \left(\frac{y_2 - y_1}{x_2 - x_1} \right) \qquad 0 \le \beta_3 \le 2\pi \tag{9}$$

$$\gamma_3 = \tan^{-1}\left[\frac{z_2 - z_1}{((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}}\right] - \frac{\pi}{2} \le \gamma_3 < \frac{\pi}{2}, \tag{10}$$

where x, y, z are the Cartesian coordinates and the range between the station is

$$r_3 = ((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2)^{1/2} \tag{11}$$

The volume of the parallelepiped defined by these vectors ($\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$) is given by their triple scalar product, which is the determinant

$$F_0 = \begin{vmatrix} \cos\gamma_1 \cos\beta_1 & \cos\gamma_2 \cos\beta_2 & \cos\gamma_3 \cos\beta_3 \\ \cos\gamma_1 \sin\beta_1 & \cos\gamma_2 \sin\beta_2 & \cos\gamma_3 \sin\beta_3 \\ \sin\gamma_1 & \sin\gamma_2 & \sin\gamma_3 \end{vmatrix} \tag{12}$$

and the adjusted volume through linear expansion is

$$F = F_0 + \Delta F = 0$$

The coefficients of the expansion are then given by

$$a_1 \equiv \frac{\partial F_0}{\partial \beta_1} = \cos\gamma_1 \sin\gamma_2 \cos\gamma_3 \cos(\beta_3 - \beta_1) - \cos\gamma_1 \cos\gamma_2 \sin\gamma_3 \cos(\beta_2 - \beta_1) \tag{13}$$

$$\begin{aligned} a_2 \equiv \frac{\partial F_0}{\partial \gamma_1} = {} & \cos\gamma_1 \cos\gamma_2 \cos\gamma_3 \sin(\beta_3 - \beta_2) - \sin\gamma_1 \cos\gamma_2 \sin\gamma_3 \sin(\beta_2 - \beta_1) \\ & + \sin\gamma_1 \sin\gamma_2 \cos\gamma_3 \sin(\beta_3 - \beta_1) \end{aligned} \tag{14}$$

$$a_3 \equiv \frac{\partial F_0}{\partial \beta_2} = \cos\gamma_2 [\cos\gamma_1 \sin\gamma_3 \cos(\beta_2 - \beta_1) - \sin\gamma_1 \cos\gamma_3 \cos(\beta_3 - \beta_2)] \tag{15}$$

$$a_4 \equiv \frac{\partial F_0}{\partial \gamma_2} = -\cos\gamma_1 \cos\gamma_2 \cos\gamma_3 \sin(\beta_3 - \beta_1) - \sin\gamma_1 \sin\gamma_2 \cos\gamma_3 \sin(\beta_3 - \beta_2)$$

$$-\cos\gamma_1 \sin\gamma_2 \sin\gamma_3 \sin(\beta_2 - \beta_1) \tag{16}$$

$$b_1 \equiv \frac{\partial F_0}{\partial \beta_3} = \cos\gamma_3 [\sin\gamma_1 \cos\gamma_2 \cos(\beta_3 - \beta_2) - \cos\gamma_1 \sin\gamma_2 \cos(\beta_3 - \beta_1)] \tag{17}$$

$$b_2 \equiv \frac{\partial F_0}{\partial \gamma_3} = \cos\gamma_1 \cos\gamma_2 \cos\gamma_3 \sin(\beta_2 - \beta_1) - \sin\gamma_1 \cos\gamma_2 \sin\gamma_3 \sin(\beta_3 - \beta_2)$$

$$+\cos\gamma_1 \sin\gamma_2 \sin\gamma_3 \sin(\beta_3 - \beta_1) \tag{18}$$

Since β_1, γ_1, β_2, and γ_2 are observations, $\Delta\beta_1$, $\Delta\gamma_1$, $\Delta\beta_2$, and $\Delta\gamma_2$ are residuals and are designated v_1, v_2, v_3, and v_4, respectively. $\Delta\beta_3$ and $\Delta\gamma_3$ are the inter-station direction adjustments. The variables to be solved for are corrections to the stations Cartesian coordinates. The transformation of unknowns from inter-station direction to Cartesian coordinate corrections are given by equation 26. Then there results an equation of the form of (1).

4. Length Equation

The length equation is developed for two cameras and a laser DME observing the satellite simultaneously. Assume the existence of two cameras (A and B), the laser DME (L), and the satellite (S), where directions from the cameras to the satellite are observed simultaneously (A to S and B to S) and a range is observed at the same time from L to S. These quantities and auxiliary vectors and angles are shown in Figure A-1. Assumed values of coordinates of the cameras and the laser system are used to calculate initial estimates of the directions and distances between the cameras and the laser. By taking scalar products of the station-to-station and station-to-satellite vectors the cosines of the angles, ξ, η, and ζ are obtained as follows:

$$\cos\xi = \vec{\rho}_1 \cdot \vec{\rho}_2 = \sin\gamma_1 \sin\gamma_2 + \cos\gamma_1 \cos\gamma_2 \cos(\beta_2 - \beta_1) \tag{19}$$

$$\cos\eta = \vec{\rho}_1 \cdot \vec{\rho}_3 = \sin\gamma_1 \sin\gamma_3 + \cos\gamma_1 \cos\gamma_3 \cos(\beta_3 - \beta_1) \tag{20}$$

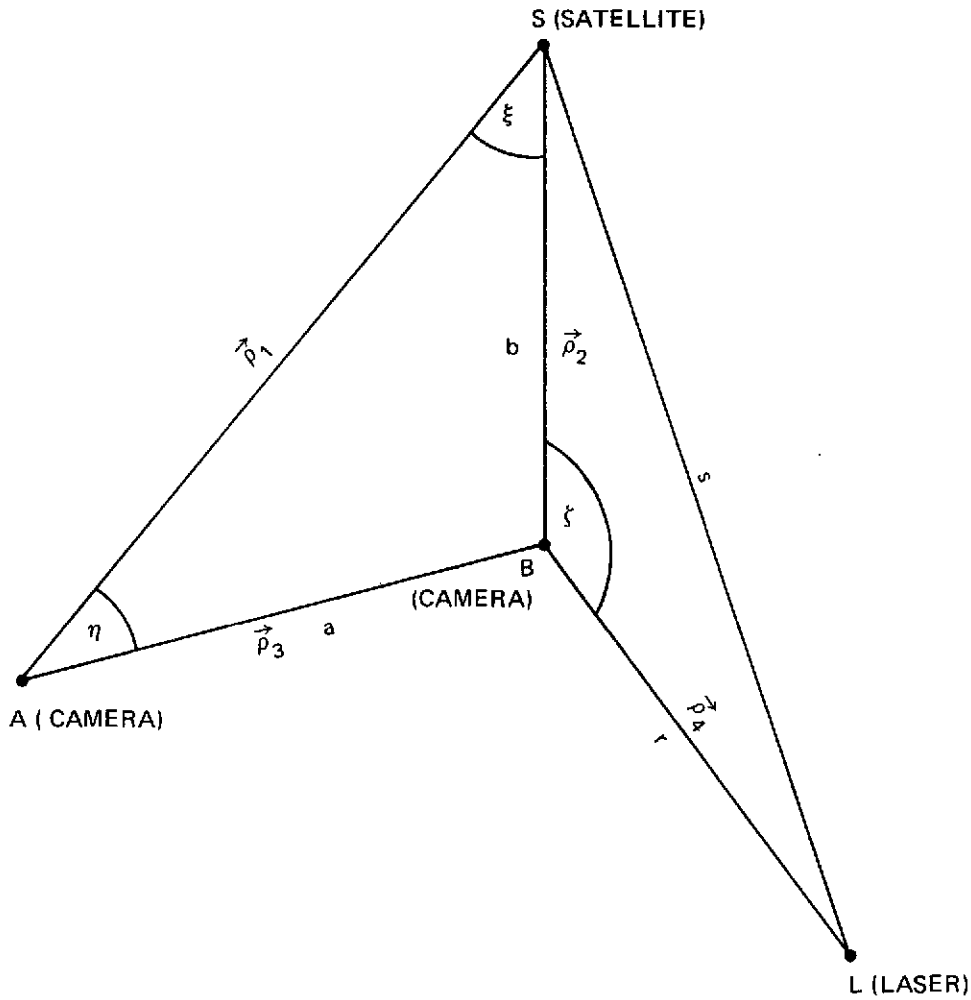

Figure A-1. Geometry for Two-Camera and One Laser DME Observing Simultaneously

$$\cos\zeta = \vec{\rho}_2 \cdot \vec{\rho}_4 = \sin\gamma_2 \sin\gamma_4 + \cos\gamma_2 \cos\gamma_4 \cos(\beta_4 - \beta_2) \tag{21}$$

where β_1, γ_1, β_2, and γ_2 are directions to the satellite,

β_3 , γ_3 are the inter-station angles for the two cameras, and

β_4, γ_4 are the inter-station angles for one camera and the laser system.

From Figure A-1 the law of cosines will give, corresponding to the laser length s

$$F = r^2 + b^2 - 2br \cos\zeta - s^2 = 0, \tag{22}$$

and the law of sines will give for b above

$$b = a\frac{\sin\eta}{\sin\xi}.$$

Through the use of (19) through (21) we expand F linearly about the values of a, r, β_3, γ_3, β_4, γ_4, obtained from the initial station coordinates, and the values of β_1 , γ_1, β_2, γ_2, and s_0 from the observations. Then we have using differentials as adjustments ($d \equiv \Delta$)

$$F = F_0 + \frac{\partial F}{\partial a} da + \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \beta_3} d\beta_3 + \ldots + \frac{\partial F}{\partial s} ds = 0$$

Divide F through by $q = 2(b - r\cos\zeta)$ and denote the result by

$$a_1 d\beta_1 + a_2 d\gamma_1 + a_3 d\beta_2 + a_4 d\gamma_2 + a_5 ds + b_1 d\beta_3 + b_2 d\gamma_3 + b_3 da + b_4 d\beta_4 + b_5 d\gamma_4$$

$$+ b_6 dr + C = 0 \tag{23}$$

where $C = F_0/q$, and the differentials on the a_i coefficients are the observation residuals v_i for i = 1 to 5. This represents the laser length equation. The coefficients and C are evaluated from the initial values, where F_0 is obtained from the misclosure of (22) and the coefficients in (23) are as follows:

$$
\begin{array}{ll}
a_1 = P_1/\sin\xi - P_5/\sin\eta & b_1 = P_5/\sin\eta \\
a_2 = P_2/\sin\xi - P_6/\sin\eta & b_2 = -P_8/\sin\eta \\
 & b_3 = \sin\eta/\sin\xi \\
a_3 = -P_1/\sin\xi - P_9/\sin\zeta & b_4 = P_9/\sin\zeta \\
a_4 = P_4/\sin\xi - P_{10}/\sin\zeta & b_5 = -P_{12}/\sin\zeta \\
a_5 = -s/(b - r\cos\zeta) & b_6 = (r - b\cos\zeta)/(b - r\cos\zeta)
\end{array}
\tag{24}
$$

and where the P's are given as

$$P_1 = b\cot\xi[\cos\gamma_2\sin(\beta_2 - \beta_1)]\ \cos\gamma_1$$

$$P_2 = b\cot\xi[\cos\gamma_1\sin\gamma_2 - \sin\gamma_1\cos\gamma_2\cos(\beta_2 - \beta_1)]$$

$$P_3 = -P_1$$

$$P_4 = b\cot\xi[\sin\gamma_1\cos\gamma_2 - \cos\gamma_1\sin\gamma_2\cos(\beta_2 - \beta_1)]$$

$$P_5 = \frac{a\cos\eta}{\sin\xi}[\cos\gamma_1\cos\gamma_3\sin(\beta_3 - \beta_1)]$$

$$P_6 = \frac{a\cos\eta}{\sin\xi}[\cos\gamma_1\sin\gamma_3 - \sin\gamma_1\cos\gamma_3\cos(\beta_3 - \beta_1)]$$

$$P_7 = -P_5$$

$$P_8 = \frac{a\cos\eta}{\sin\xi}[\sin\gamma_1\cos\gamma_3 - \cos\gamma_1\sin\gamma_3\cos(\beta_3 - \beta_1)]$$

$$P_9 = \left(\frac{br\sin\zeta}{b - r\cos\zeta}\right)[\cos\gamma_2\cos\gamma_4\sin(\beta_4 - \beta_2)]$$

$$P_{10} = \left(\frac{br \sin \zeta}{b - r \cos \zeta}\right) [\cos \gamma_2 \sin \gamma_4 - \sin \gamma_2 \cos \gamma_4 \cos(\beta_4 - \beta_2)]$$

$$P_{11} = -P_9$$

$$P_{12} = \left(\frac{br \sin \zeta}{b - r \cos \zeta}\right) [\sin \gamma_2 \cos \gamma_4 - \cos \gamma_2 \sin \gamma_4 \cos(\beta_4 - \beta_2)]$$

In order to obtain the desired form (1) for Cartesian station coordinates, the coplanarity and length equations are transformed from γ, β, r variables to x, y, z variables by using the relationships

$$x_2 - x_1 = r \cos \gamma \cos \beta$$

$$y_2 - y_1 = r \cos \gamma \sin \beta$$

$$z_2 - z_1 = r \sin \gamma \tag{25}$$

Differentiating these expressions yields

$$\begin{bmatrix} dx_2 - dx_1 \\ dy_2 - dy_1 \\ dz_2 - dz_1 \end{bmatrix} = \begin{bmatrix} -r \cos \gamma \sin \beta & -r \sin \gamma \cos \beta & \cos \gamma \cos \beta \\ r \cos \gamma \cos \beta & -r \sin \gamma \sin \beta & \cos \gamma \sin \beta \\ 0 & r \cos \gamma & \sin \gamma \end{bmatrix} \begin{bmatrix} d\beta \\ d\gamma \\ dr \end{bmatrix} \tag{26}$$

Inverting Equation 26 produces the transformation

$$\begin{bmatrix} d\beta \\ d\gamma \\ dr \end{bmatrix} = \begin{bmatrix} \dfrac{-\sin \beta}{r \cos \gamma} & \dfrac{\cos \beta}{r \cos \gamma} & 0 \\ \dfrac{-\sin \gamma \cos \beta}{r} & \dfrac{-\sin \gamma \sin \beta}{r} & \dfrac{\cos \gamma}{r} \\ \cos \gamma \cos \beta & \cos \gamma \sin \beta & \sin \gamma \end{bmatrix} \begin{bmatrix} dx_2 - dx_1 \\ dy_2 - dy_1 \\ dz_2 - dz_1 \end{bmatrix} \tag{27}$$

5. Condition Equations for Constraints

The three types of condition equations are: (1) coordinate equations, (2) distance equations, and (3) coordinate-shift equations. As indicated previously constraints for station coordinates, distances (baselines), and coordinate shifts (local datum ties) are based upon a priori information. Such information is treated statistically as in the case of satellite observations, and hence weights are applied corresponding to a priori errors.

5.1 Coordinate Equation

Assume the input coordinates of the ith station are coordinates for which a priori information is available. Let the input values of X_i, Y_i, Z_i be X_{io}, Y_{io}, Z_{io}, and denote the adjusted coordinates as

$$\begin{aligned} dx_i &= X_1 - X_{io} \equiv x_{i-2} \\ dy_i &= Y_i - Y_{io} \equiv x_{i-1} \\ dz_i &= Z_i - Z_{io} \equiv x_i \end{aligned} \tag{28}$$

then the constraint equations in the form of equation (1) are simply

$$\begin{aligned} v_{i-2} - x_{i-2} &= 0 \\ v_{i-2} - x_{i-1} &= 0 \\ v_i - x_i &= 0 \end{aligned} \tag{29}$$

5.2 Distance Equations (Baselines)

The condition equation for the baseline distance q between the rth and sth ground stations is

$$-dq + \cos\gamma\cos\beta(dx_s - dx_r) + \cos\gamma\sin\beta(dy_s - dy_r) + \sin\gamma(dz_s - dz_r) = 0 \tag{30}$$

where γ and β are used as in (9) and (10) and the differentials are the unknown station adjustments. The adjustment

$$\begin{aligned} dq &= q - q_c = (q - q_0) - (q_c - q_0) \\ &= v - C \end{aligned} \tag{31}$$

where q is the solution value, q_c is the computed value based upon the initial station coordinates, and q_0 is the a priori value for the constraint.

In conformity with the previous notation the terms dx_s, dy_s, dz_s, dx_r, dy_r, and dz_r are replaced by x_{s-2}, x_{s-1}, x_s, x_{r-2}, x_{r-1}, s_r, and dq by (31) to obtain

$$-v + \cos\gamma\cos\beta(x_{s-2} - x_{r-2}) + \cos\gamma\sin\beta(x_{s-1} - x_{r-1}) + \sin\gamma(x_s - x_r) + C = 0 \quad (32)$$

5.3 Coordinate-Shift Equations

For two nearby stations denote the difference in coordinates as

$$\begin{aligned} D_x &= x_2 - x_1 \\ D_y &= y_2 - y_1 \\ D_z &= z_2 - z_1 \end{aligned} \quad (33)$$

Since the results are similar for the three equations we will treat just one equation of condition. For the x component the differential is

$$-dD_x + dx_2 - dx_1 = 0, \quad (34)$$

and as in the case of the distance equation (31)

$$dD_x = v_x - C_x, \qquad v_x = \bar{D}_x - D_{x_0}, \qquad C_x = D_{x_c} - D_{x_0} \quad (35)$$

where D_x is the unknown difference, D_{x_0} is obtained from local survey station coordinates, and D_{x_c} is computed from the initial input of the station coordinates. For the ith and jth stations, using the notation of the form (1) where differentials are replaced by corrections x_q and x_p, the condition equations for (33) are

$$\begin{aligned} -v_x + x_{j-2} - x_{i-2} + C_x &= 0 \\ -v_y + x_{j-1} - x_{i-1} + C_y &= 0 \\ -v_z + x_j - x_i + C_z &= 0 \end{aligned} \quad (36)$$

6. Correlated Observations

The model described above was developed for camera systems that observed simultaneously the flashing lamps on GEOS-I and II, and then the model was employed to include the BC-4 camera network that observed the PAGEOS satellite. A BC-4 photograph taken on PAGEOS by an observing station, s, was reduced to 7 time points (k = 1 to 7) of satellite observation angles (γ_k^s, β_k^s). The reduced observations γ_k^s and β_k^s are correlated separately in each type among the points k = 1 to 7. The modeling for the correlated observations is presented.

Consider 7 events of the type (1), (2), or (3) described in section 1, where respectively 2, 3, or 4 stations (S = 2, 3, or 4) observe the satellite simultaneously at each of the 7 reduced photographic points k = 1 to 7. Thus for each event there are 2S simultaneous observations, namely (γ_k^s, β_k^s) for s = 1 to S. Let p denote this configuration of S stations and 7 events, then for each p there are 7 sets of matrix condition equations of the form (2). Denote these as

$$A_p V_p + B_p X + C_p = 0 \tag{37}$$

where by row partitioning for k = 1 to 7

$$\begin{aligned} V_p &= [V_k^p] \\ C_p &= [C_k^p] \\ B_p &= [B_k^p] \end{aligned} \tag{38}$$

and A_k^p lies along the diagonal submatrix path of A_p (with zero submatrices for off diagonal blocks)

$$A_p = [\mathrm{DIAG}\, A_k^p] \tag{39}$$

The submatrices V_k^p, B_k^p, and A_k^p are given as before in (2) for a particular event, but here the event type for S = 2, 3, or 4 stations is fixed for the 7 events for a given configuration p.

Denote the variance-covariance matrix of the observation errors as

$$P_p = E(\overline{V}_p \overline{V}_p^T) \tag{40}$$

where $\overline{V}_p$ corresponds to the observation errors (noise) in V_p.

The normal equations for the BC-4 observations is obtained by minimizing

$$Q = \sum_{p} Q_p \tag{41}$$

for the unknowns V_p and X, where

$$Q_p = V_p^T P_p^{-1} V_p - 2(A_p V_p + V_p X + C_p)^T \lambda_p \tag{42}$$

and

$$\lambda_p = [\lambda_k^p]$$

for which λ_k^p is a vector of Lagrangian multipliers defined as in (3) for a given event. Hence the normal equations will have the form given in (4) through (6). Thus for each p the normal equations are

$$J_p X + B_p^T C_p = N_p \tag{43}$$

where

$$\begin{aligned} J_p &= B_p^T M_p^{-1} B_p \\ M_p &= (A_p P_p A_p^T), \end{aligned} \tag{44}$$

and the total set of normal equations for all p are then

$$N' = \sum_{p} N_p.$$

It is of interest to compare the matrix M_p derived from 7 events to that of M_k given in (6) for a single event k. Take the case of $S = 4$ for which the dimensions A_k, V_k, and W_k were given under (6) respectively as 5×8, 8×1, and 8×8, and for which there were 8 observations and 5 coplanarity equations of condition from the 4 observing stations in a given event. Consider 7 events of the same type as in the configuration p for $S = 4$, denote M_k as M_k^p for $k = 1$ to 7, and assume correlations are absent as in the previous modeling. Then

$$M_p = \begin{bmatrix} M_1^p & & & & 0 \\ & M_2^p & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & M_7^p \end{bmatrix}_{35\times 35} \equiv [\mathrm{DIAG}\, M_k^p] \tag{45}$$

where

$$M_k^p = A_k^p (W_k^p)^{-1} (A_k^p)^T]_{5\times 5} \tag{46}$$

and since correlations are assumed absent here

$$P_p = [\mathrm{DIAG}(W_k^p)^{-1}].$$

With correlations the same diagonal blocks in (45) arise for M_p in (44) since in each event k all observations are uncorrelated, but similar off diagonal blocks also exist which is now shown. Using the submatrices V_k^p for V_p in (38) and dropping the superscript p on the submatrices, then the variance-covariance matrix in (40) becomes

$$P_p = E(\overline{V}_p \overline{V}_p^T) = [E(\overline{V}_k \overline{V}_\ell^T)] \quad \text{for } k = 1 \text{ to } 7 \text{ and } \ell = 1 \text{ to } 7, \tag{47}$$

which corresponds to 49 sublocks or submatrices in P_p. For a given event k and ℓ the only covariances occur when the station s and the angle γ_k^s or β_k^s are the same. Denote the observation errors for a given k as (where T ≡ true, o ≡ observed)

$$\overline{V}_k = \begin{bmatrix} \gamma_k^1(T) - \gamma_k^1(0) \\ \beta_k^1(T) - \gamma_k^1(0) \\ \cdot \\ \cdot \\ \cdot \\ \gamma_k^s(T) - \gamma_k^s(0) \\ \beta_k^s(T) - \beta_k^s(0) \end{bmatrix}_{2S\times 1} \equiv [\overline{v}_{ik}]; \quad i = 1 \text{ to } 2S, \tag{48}$$

then for a given k and ℓ

$$E(\bar{V}_k \bar{V}_\ell^T) = [E(\bar{v}_{ik}\bar{v}_{j\ell})]_{2S\times 2S} \quad i = 1 \text{ to } 2S, \quad j = 1 \text{ to } 2S. \tag{49}$$

and from the definition of the correlations

$$\begin{aligned} E(\bar{v}_{ik}\bar{v}_{j\ell}) &= \sigma_{ij}^2(k, \ell) = 0 && \text{for } i \neq j \text{ for all } k \text{ and } \ell \\ &= \sigma_{ii}^2(k, \ell) && \text{for covariances } k \neq \ell, \\ &= \sigma_{ii}^2(k, k) && \text{for variances } k = \ell. \end{aligned} \tag{50}$$

Thus, in each sublock of a given k and ℓ, the off diagonal elements are zero and

$$\begin{aligned} E(\bar{V}_k \bar{V}_\ell^T) &= [\text{DIAG } \sigma_{ii}^2(k, \ell)]_{2S\times 2S} \\ &\equiv D_{k\ell} \end{aligned} \tag{51}$$

and

$$P_p = [D_{k\ell}] \quad (k, \ell = 1 \text{ to } 7), \tag{52}$$

Denote

$$M_p = [M_{k\ell}] \quad (k, \ell = 1 \text{ to } 7), \tag{53}$$

and with use of (38), (39), and (52) in (44) then by (53)

$$M_{k\ell} = [A_k D_{k\ell} A_\ell^T]_{(2S-3)\times(2S-3)} \tag{54}$$

Now

$$\begin{aligned} M_{kk} &= A_k D_{kk} A_k^T \\ &= A_k W_k^{-1} A_k^T = M_k, \end{aligned} \tag{55}$$

which is the same as in (46) for the uncorrelated case.

The block form $[M_{k\ell}]$ for M_p, where $M_{k\ell}$ is given in (54), provides a convenient method for the computations of M_p. However in the present case of correlated observations there are 49 such blocks, whereas only the 7 diagonal blocks M_{kk} are computed for uncorrelated observations as in (45) or (55). The largest inverse matrix M_p to be inverted occurs for the case of $S = 4$ and which has dimension 35×35, whereas previously for the case of uncorrelated observations the largest matrix was 5×5. Correlations are generally large among the reduced observations of a photograph. Thus the geometric normal equations should be analyzed further to investigate the overall effect of the correlation on the final combination solution.