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16. Abstract

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SUMMARY

A general expression for the co-moving derivative of a tensor is derived. A variable describing the coordinate velocity field is introduced. Time dependency of the metric elements is expressed in terms of this velocity field. The resulting description of motion is one of which the Eulerian and Lagrangian viewpoints are special cases. This general description is useful in problems involving moving boundaries or discontinuities.

INTRODUCTION

The field equations (as opposed to the constitutive equations) of any continuum are simply statements of the conservation of mass, momentum, and energy. These equations can be written in a tensorial form (ref. 1) which is, by design, valid in any classical coordinate system. The most common practice is to adopt an inertially fixed coordinate system (Eulerian formulation) in which to describe the motion. Another classical, but seldom used, approach is to fix coordinate points in identified particles of the continuum (Lagrangian formulation, ref. 2). In most problems, the Eulerian approach is preferable and completely satisfactory. However, in problems which contain moving boundaries, free surfaces, or discontinuities, finite difference algorithms developed from an Eulerian description become extremely complex at these boundaries. On the other hand, the Lagrangian approach, which overcomes the problems associated with moving boundaries, can lead to extremely complex grid networks in the presence of vorticity (ref. 3). Both of these problems suggest the desirability of a mixed Eulerian-Lagrangian description which can be obtained by introducing a completely general coordinate system. Time dependence of the space metric introduces additional terms in the governing differential equations. The equations of motion for surface flow are derived in reference 1 by considering time dependent metric elements. By a similar approach, the methods can be extended to three-dimensional problems.

The purpose of the present paper is to derive an equation for the tensorial co-moving derivative of a tensor. In order to define the arbitrary coordinate system, a variable equal to the velocity of coordinate points relative to the primary inertial system and expressed in the arbitrary coordinate system, is introduced. Differential equations in time for the elements of the metric are expressed in terms of this variable. By using this approach, it is possible to develop the field equations in any arbitrary classical coordinate system.

SYMBOLS

A	tensor quantity
$A^{ij\ldots m}_{pq\ldots n}$	components of a general tensor A of any order
$A^i(y)$	denotes that tensor A is resolved in y^s
D	mean divergence of coordinate velocity field
g_{ij}	i,j element of the metric of y^s
I	identity matrix
$(\ )_I$	denotes evaluation in inertial coordinate system
n^i	surface normal
Q	scalar invariant
S^i	velocity of the nodal points of y^s in primary inertial coordinate system
$S^i{}_{\mid j}$	denotes covariant derivative of S^i with respect to y^j
T	matrix formed by elements T_{ij}
T_{ij}	coefficient of covariant transformation law (see eqs. (7) and (8))
t	time
u^i	velocity of continuum relative to x^s
V	control volume for defining mean divergence
v^i	velocity of continuum relative to y^s

x^s primary inertial rectangular Cartesian coordinate system

y^s arbitrary coordinate system

δ^i_j Kronecker delta

ϵ_{ijk} permutation symbol

λ inverse of matrix T, $\lambda = T^{-1}$

λ_{ij} i,j element of the matrix λ

Φ coordinate velocity potential

Ω^i angular velocity tensor

Repeated indices denote summation over the range of the index; Roman indices (for example, i,j) denote a range of three; and Greek indices (for example, α,β) denote a range of four.

ANALYSIS

In the analysis to follow, the co-moving derivative of a tensor quantity is defined, and physically described, in an inertial rectangular Cartesian reference frame. Based on this definition, an expression for the co-moving derivative in a completely arbitrary coordinate system is formulated.

Description of Co-Moving Differentiation

Consider the motion and deformation of a continuum in an inertial rectangular Cartesian coordinate system x^s. Let $A^{ij\ldots m}_{pq\ldots n}$ be the components of a tensor quantity A representing a characteristic of the continuum. Assume that the components $A^{ij\ldots m}_{pq\ldots n}$ have a functional form given by

$$A^{ij\ldots m}_{pq\ldots n} = A^{ij\ldots m}_{pq\ldots n}(x^s,t) \tag{1}$$

The co-moving derivative of a variable is defined (ref. 2) as the total time derivative of that variable evaluated for an identified particle in the continuum. That is, the co-moving derivative, as the name implies, is the time derivative seen by an observer moving with the particle and is evaluated in an inertial frame. Assuming that $A^{ij\ldots m}_{pq\ldots n}$ has continuous first partial derivatives, the total time derivative can be expanded by using the chain rule of differentiation and written

$$\frac{dA^{ij\ldots m}_{pq\ldots n}}{dt} = \frac{\partial A^{ij\ldots m}_{pq\ldots n}}{\partial t} + \frac{\partial A^{ij\ldots m}_{pq\ldots n}}{\partial x^s}\frac{dx^s}{dt} \tag{2}$$

which is equivalent to

$$\frac{dA^{ij\ldots m}_{pq\ldots n}}{dt} = \frac{\partial A^{ij\ldots m}_{pq\ldots n}}{\partial t} + u^s\frac{\partial A^{ij\ldots m}_{pq\ldots n}}{\partial x^s} \tag{3}$$

where u^s are the rectangular Cartesian components of the continuum velocity in the primary inertial frame. The first and second terms on the right-hand side of equation (3) are commonly called the local and convective derivatives, respectively. This form of the co-moving derivative describes the Eulerian viewpoint of change in A and is of fundamental importance in physical and mechanical laws.

General Form of the Co-Moving Derivative

In many problems, it may be advantageous (as will be discussed later) to express physical laws (and hence, the co-moving derivative) in a completely arbitrary coordinate system. In such instances, the form of the co-moving derivative must be expressed in tensorial form. In the following development, a general form of the co-moving derivative is derived for a covariant first-order tensor. These results are then generalized to mixed tensors of any order.

Consider a completely arbitrary coordinate system y^s superimposed on the inertial space. Typical parametric line segments for such a coordinate system are shown in figure 1. Consider also an identified particle of the continuum moving about in this space. Assume that associated with this particle is a first-order covariant tensor A whose components A_s have continuous first partial derivatives. Let $A_s(x)$ and $A_s(y)$ denote covariant components of A in the x^s and y^s coordinate systems, respectively. Treating time as an invariant, the complete transformation (including time) between the components can be written

$$A_\alpha(y) = \frac{\partial x^\beta}{\partial y^\alpha} A_\beta(x) \tag{4}$$

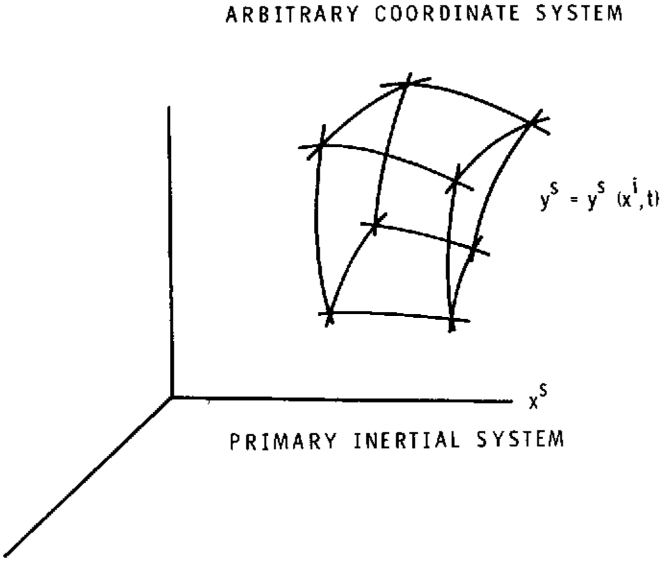


Figure 1.- Coordinate systems.

where

$$y^4 = x^4 = t \tag{5}$$

This transformation can also be written in the partitioned form

$$\begin{Bmatrix} A_i(y) \\ \hdashline A_4(y) \end{Bmatrix} = \left[\begin{array}{c:c} T_{ij} & 0 \\ \hdashline \dfrac{\partial x^j}{\partial t} & 1 \end{array}\right] \begin{Bmatrix} A_j(x) \\ \hdashline A_4(x) \end{Bmatrix} \tag{6}$$

where

$$T_{ij} = \frac{\partial x^j}{\partial y^i} \tag{7}$$

and Roman indices have a range of three and denote spatial components. Therefore,

$$A_i(y) = T_{ij}\, A_j(x) \tag{8}$$

That is, the spatial components (covariant) of A transform independently of the fourth component. Note, however, that the fourth component does not transform by invariance.

Now consider the contravariant components of A. These components transform according to

$$A^{\alpha}(y) = \frac{\partial y^{\alpha}}{\partial x^{\beta}} A^{\beta}(x) \tag{9}$$

which can be written

$$\left\{\begin{array}{c} A^i(y) \\ \hline A^4(y) \end{array}\right\} = \left[\begin{array}{c:c} \lambda_{ij} & \frac{\partial y^i}{\partial t} \\ \hdashline 0 & 1 \end{array}\right] \left\{\begin{array}{c} A^j(x) \\ \hline A^4(x) \end{array}\right\} \tag{10}$$

where

$$\lambda_{ij} = \frac{\partial y^i}{\partial x^j} \tag{11}$$

Assuming that the coordinate transformation has a nonvanishing Jacobian, it follows that

$$\frac{\partial y^{\beta}}{\partial x^{\gamma}} \frac{\partial x^{\alpha}}{\partial y^{\beta}} = \delta^{\alpha}_{\gamma} \tag{12}$$

and therefore

$$\left.\begin{array}{l} \lambda = T^{-1} \\ \\ \left.\dfrac{\partial x^i}{\partial t}\right|_{y^s} = - \left.\dfrac{\partial x^i}{\partial y^j} \dfrac{\partial y^j}{\partial t}\right|_{x^s} \end{array}\right\} \tag{13}$$

where $|_{y^s}$ indicates y_s was held constant during the partial differentation. Note also from equation (10) that the fourth covariant component transforms by invariance. However, the spatial components transform independently of the fourth component if, and only if, either the fourth component is identically zero or the coordinate transformation is time independent. Since time dependence of the transformation is an essential element in the present analysis and a restriction to spatial components is desirable, the analysis will be limited to those tensor quantities whose fourth component is identically zero. This restriction eliminates consideration of velocities relative to moving coordinate systems. However, velocity relative to the primary inertial frame and force vectors are admissable and of primary importance in formulation of the field equations.

Based on the above discussion, the transformation of the spatial covariant components of A can be written

$$A_i(x) = \lambda_{ij}\, A_j(y) \tag{14}$$

Since (for a time invariant coordinate transformation) time differentiation does not alter tensor character, the co-moving derivative of $A_i(x)$ is a tensor and can be expanded in the form

$$\frac{dA_i(x)}{dt} = \lambda_{ij}\left[\frac{\partial A_j(y)}{\partial t} + v^k \frac{\partial A_j(y)}{\partial y^k}\right] + A_j(y)\left[\frac{\partial \lambda_{ij}}{\partial t} + v^k \frac{\partial \lambda_{ij}}{\partial y^k}\right] \tag{15}$$

where the v^k are defined by $v^k = \dfrac{dy^k}{dt}$ and are the contravariant components of the continuum velocity relative to the arbitrary coordinate system. The components of this derivative in the arbitrary coordinate system are given by

$$\left(\frac{dA_i(y)}{dt}\right)_I = T_{ij}\,\frac{dA_j(x)}{dt} \tag{16}$$

where $(\ )_I$ indicates that the time differentiation was performed in an inertial coordinate system. Substituting equation (15) into equation (16) gives

$$\left(\frac{dA_i(y)}{dt}\right)_I = \frac{\partial A_i(y)}{\partial t} + v^s \frac{\partial A_i(y)}{\partial y^s} + T_{ij}\left[\frac{\partial \lambda_{jk}}{\partial t} + v^s \frac{\partial \lambda_{jk}}{\partial y^s}\right] A_k(y) \tag{17}$$

Equation (17) can be rearranged and written as

$$\left(\frac{dA_i(y)}{dt}\right)_I = \frac{\partial A_i(y)}{\partial t} + v^s\left[\frac{\partial A_i(y)}{\partial y^s} + T_{ij}\frac{\partial \lambda_{jk}}{\partial y^s} A_k(y)\right] + T_{ij}\frac{\partial \lambda_{jk}}{\partial t} A_k(y) \tag{18}$$

As is shown in appendix A, the bracketed term in equation (18) is the covariant derivative of the quantity $A_i(y)$. Therefore,

$$\left(\frac{dA_i(y)}{dt}\right)_I = \frac{\partial A_i(y)}{\partial t} + v^s A_i(y)_{|s} + T_{ij}\frac{\partial \lambda_{jk}}{\partial t} A_k(y) \tag{19}$$

The last term in equation (19) accounts for time dependence in the metric of the arbitrary coordinate system. Since, by definition

$$T_{ij}\lambda_{jk} = I \tag{20}$$

then,

$$T_{ij}\frac{\partial \lambda_{jk}}{\partial t} = -\frac{\partial T_{ij}}{\partial t}\lambda_{jk} \tag{21}$$

Also, since

$$T_{ij} = \frac{\partial x^j}{\partial y^i} \tag{22}$$

then,

$$\left.\frac{\partial T_{ij}}{\partial t}\right|_{y^s} = \frac{\partial}{\partial y^i}\left(\left.\frac{\partial x^j}{\partial t}\right|_{y^s}\right) \tag{23}$$

Therefore,

$$T_{ij} \left. \frac{\partial \lambda_{jk}}{\partial t} \right|_{y^s} = - \frac{\partial}{\partial y^i} \left(\left. \frac{\partial x^j}{\partial t} \right|_{y^s} \right) \lambda_{jk} \tag{24}$$

Since the arbitrary coordinate system must be specified in some manner, a new variable S is introduced and is defined as the velocity of coordinate points of the arbitrary coordinate system relative to the primary inertial coordinate system. In the arbitrary coordinate system, the contravariant components of S are

$$S^i = - \left. \frac{\partial y^i}{\partial t} \right|_{x^s} \tag{25}$$

Therefore,

$$\left. \frac{\partial x^j}{\partial t} \right|_{y^s} = - \frac{\partial x^j}{\partial y^s} \left. \frac{\partial y^s}{\partial t} \right|_{x^s} = T_{sj} \, S^s \tag{26}$$

and substituting equation (26) into equation (24) results in:

$$T_{ij} \left. \frac{\partial \lambda_{jk}}{\partial t} \right|_{y^s} = - \left[\frac{\partial S^k}{\partial y^i} + \frac{\partial T_{sj}}{\partial y^i} \, S^s \, \lambda_{jk} \right] \tag{27}$$

Again, as is shown in appendix A, the term on the right-hand side of equation (27) is the covariant derivative of the coordinate velocity; that is,

$$T_{ij} \left. \frac{\partial \lambda_{jk}}{\partial t} \right|_{y^s} = -S^k{}_{|i} \tag{28}$$

Therefore, by substituting equation (28) into equation (19), the co-moving derivative becomes

$$\boxed{\left(\frac{dA_i(y)}{dt}\right)_I = \frac{\partial A_i(y)}{\partial t} + v^s A_i(y)\big|_s - S^k\big|_i A_k(y)} \tag{29}$$

In order to carry out the covariant differentiation in equation (29), elements of the metric must be determined. Elements of the metric can be related to the coordinate velocities through a differential equation as follows. The required metric elements of the y coordinate system can be written

$$g_{ij} = \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\alpha}{\partial y^j} = T_{ik} T_{jk} \tag{30}$$

Differentiating gives

$$\frac{\partial g_{ij}}{\partial t} = T_{ik} \frac{\partial T_{jk}}{\partial t} + \frac{\partial T_{ik}}{\partial t} T_{jk} \tag{31}$$

From equations (21) and (28),

$$\lambda_{jk} \frac{\partial T_{ij}}{\partial t} = S^k\big|_i \tag{32}$$

which can be written

$$\frac{\partial T_{ij}}{\partial t} = T_{kj}\, S^k\big|_i \tag{33}$$

Substituting equation (33) into equation (31) and reducing gives

$$\frac{\partial g_{ij}}{\partial t} = g_{ip}\, S^p\big|_j + g_{jq}\, S^q\big|_i \tag{34}$$

which is equivalent to

$$\frac{\partial g_{ij}}{\partial t} = S_{i|j} + S_{j|i} \tag{35}$$

The equivalent form of the co-moving derivative (eq. (29)) for a first-order contravariant tensor can be formed as follows. Let Q be a scalar invariant. For a scalar, which transforms by invariance, the co-moving derivative (eq. (3)) reduces to

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + v^s \, Q|_s \tag{36}$$

Consider a particular scalar generated by the inner product

$$Q = A^i(y) \, A_i(y) \tag{37}$$

By generating the co-moving derivative of Q by product differentiation and substituting from equation (29) for the time derivative of the covariant components of A, the co-moving derivative for the contravariant form is found to be

$$\boxed{\left(\frac{dA^i(y)}{dt}\right)_I = \frac{\partial A^i(y)}{\partial t} + v^s \, A^i(y)|_s + S^i|_k \, A^k(y)} \tag{38}$$

The expressions for the co-moving derivatives (eqs. (29) and (38)) can be generalized to any order tensor by a process of inner product formation and product differentiation. For instance, let

$$B_i = A_{ij} \, A^j \tag{39}$$

Then,

$$\frac{dB_i}{dt} = A_{ij} \frac{dA^j}{dt} + A^j \frac{dA_{ij}}{dt} \tag{40}$$

The time derivatives of A^j and B_i can be expanded by using equations (29) and (38). From this result, the co-moving derivative of a second-order covariant tensor is found to be

$$\left(\frac{dA_{ij}(y)}{dt}\right)_I = \frac{\partial A_{ij}(y)}{\partial t} + v^s \, A_{ij}(y)|_s - S^k|_i \, A_{kj}(y) - S^k|_j \, A_{ik}(y) \tag{41}$$

By using a similar procedure, the results can be completely generalized to a mixed tensor of any order. This result can be written

$$\boxed{\begin{aligned}\left(\frac{dA^{ij\ldots m}_{pq\ldots n}(y)}{dt}\right)_I = \frac{\partial A^{ij\ldots m}_{pq\ldots n}(y)}{\partial t} + v^\alpha A^{ij\ldots m}_{pq\ldots n}(y)\Big|_\alpha + S^i\big|_\beta A^{\beta j\ldots m}_{pq\ldots n}(y) \\ + S^j\big|_\beta A^{i\beta\ldots m}_{pq\ldots n}(y) + \ldots + S^m\big|_\beta A^{ij\ldots\beta}_{pq\ldots n}(y) \\ - S^\beta\big|_p A^{ij\ldots m}_{\beta q\ldots n}(y) - S^\beta\big|_q A^{ij\ldots m}_{p\beta\ldots n}(y) - \ldots \\ - S^\beta\big|_n A^{ij\ldots m}_{pq\ldots\beta}(y)\end{aligned}} \tag{42}$$

RESULTS AND DISCUSSION

In the discussion to follow, the utility of the generalized co-moving derivative is demonstrated. Eulerian and Lagrangian descriptions are shown to be specialized cases of the generalized form. The significance of coordinate velocity derivatives for a rotating Cartesian coordinate system is discussed. Finally, application of the generalized form is described.

The Eulerian viewpoint is generated by requiring the arbitrary coordinate system to be time invariant. That is, let

$$S^i \equiv 0 \tag{43}$$

Under these assumptions the time derivative is evaluated at an inertially fixed point and the co-moving derivative reduces to

$$\left(\frac{dA^{ij\ldots m}_{pq\ldots n}(y)}{dt}\right)_I = \frac{\partial A^{ij\ldots m}_{pq\ldots n}(y)}{\partial t} + v^s A^{ij\ldots m}_{pq\ldots n}(y)\Big|_s \tag{44}$$

In the Lagrangian formulation, coordinate points in the arbitrary coordinate system follow particles of the continuum. This viewpoint can be generated by assuming that the continuum is stationary in the arbitrary coordinate system

$$v^s \equiv 0 \tag{45}$$

and that the coordinate points move at the continuum velocity u^i; that is,

$$S^i = u^i \tag{46}$$

Note that the continuum velocity (and therefore, the coordinate velocity) is resolved in the arbitrary coordinate system. Under these assumptions, the co-moving derivative becomes

$$\left(\frac{dA^{ij\ldots m}_{pq\ldots n}(y)}{dt}\right)_I = \frac{\partial A^{ij\ldots m}_{pq\ldots n}(y)}{\partial t} + u^i{}_{|r}\, A^{rj\ldots m}_{pq\ldots n}(y) + u^j{}_{|r}\, A^{ir\ldots m}_{pq\ldots n}(y) + \ldots + u^m{}_{|r}\, A^{ij\ldots r}_{pq\ldots n}(y)$$

$$- u^s{}_{|p}\, A^{ij\ldots m}_{sq\ldots n}(y) - u^s{}_{|q}\, A^{ij\ldots m}_{ps\ldots n}(y) - \ldots - u^s{}_{|n}\, A^{ij\ldots m}_{pq\ldots s}(y) \tag{47}$$

As an additional example, let the arbitrary coordinate system be a rotating rectangular Cartesian coordinate system. In order to examine the significance of the coordinate velocity, assume a rigid continuum with body fixed axes such that

$$v^s \equiv 0 \tag{48}$$

Consider the co-moving derivative of the velocity du^i/dt at a point $\left(y^1,\ y^2,\ y^3\right)$ in the body axes system. For this situation the co-moving derivative becomes

$$\frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + S^i{}_{|\beta}\, u^\beta \tag{49}$$

where

$$S^i{}_{|j} = \frac{\partial S^i}{\partial y^j} + \left\{ {i \atop jk} \right\} S^k \tag{50}$$

Since the arbitrary coordinate system is Cartesian,

$$\left\{ {i \atop jk} \right\} \equiv 0 \tag{51}$$

Therefore, the co-moving derivative becomes

$$\frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + \frac{\partial S^i}{\partial y^s} u^s \tag{52}$$

For a right-hand coordinate system and constant angular velocity Ω_i

$$S^i = \epsilon^{ijk} \Omega^j y^k \tag{53}$$

Therefore,

$$\frac{du^i}{dt} = \frac{\partial u^i}{\partial t} + \epsilon^{ijk} \Omega^j u^k \tag{54}$$

which is equivalent to the classical result obtained using vector analysis.

Formulation of the equations of change in a completely arbitrary coordinate system is particularly useful in problems which involve moving boundaries, discontinuities, or free surfaces. By locating a parametric line along the moving boundary, boundary conditions become trivial and the computational algorithm is simplified. As an example, consider the flow of blood through a single chamber of the heart. This system is typified

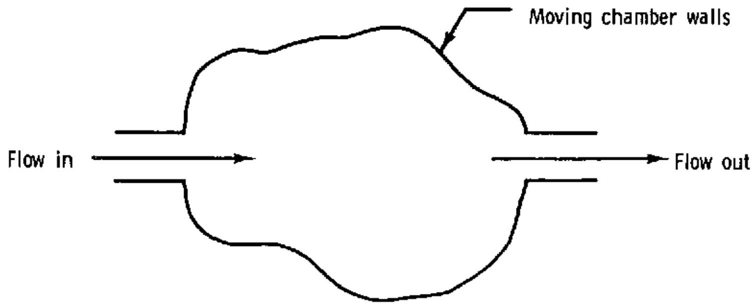


Figure 2.- Typical open system with moving boundaries.

in figure 2. This flow problem could be formulated in an Eulerian sense by covering the region of interest by an appropriate time invariant coordinate system. Figure 3 shows a typical approach. However, for fixed grid spacing, numerous problems are associated with boundary conditions. These problems are particularly complex when the boundary is moving. The problem of boundary conditions could be eliminated by adapting a Lagrangian formulation. However, numerous computation problems arise in the presence of vorticity which tends to twist the parametric lines into complex overlapping geometries.

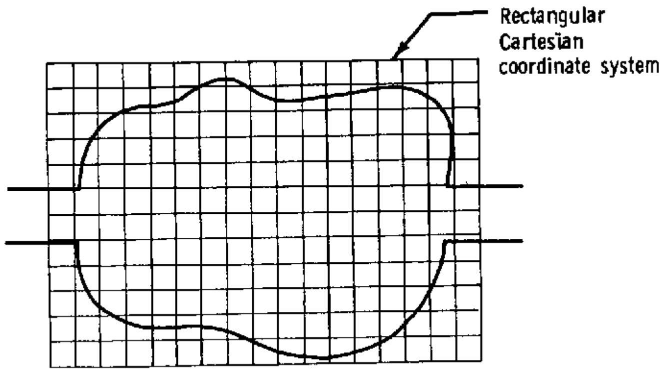


Figure 3.- Typical Cartesian coordinate frame.

As an alternate approach, consider a time varying coordinate system for which the boundary is along a parametric line. This condition is generated by requiring that on the boundary

$$n^i S_i = n^i u_i \tag{55}$$

where n^i is the outward boundary normal and u_i is the continuum velocity at the boundary.

In addition to locating a parametric line along the boundary, the coordinate velocity field interior to the boundary must also be constructed. One simple method of defining this geometry is through scaling of the boundary surface. Under this scheme, a typical representation of the arbitrary coordinate system is shown in figure 4. In this example,

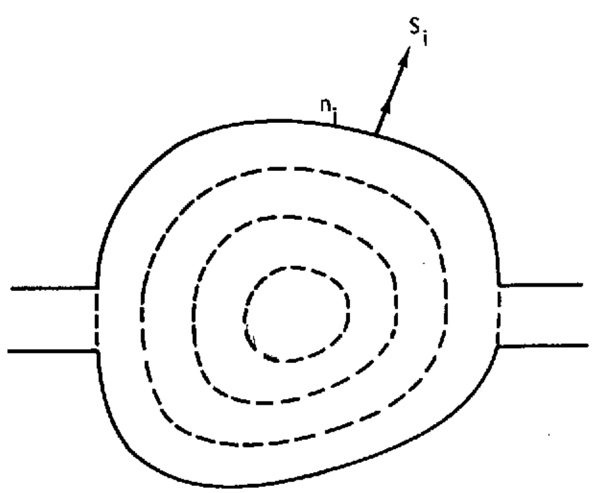


Figure 4.- Representation of arbitrary coordinate system.

orthogonality between the chamber normal and scaled chamber tangential lines is assumed.

An alternate approach for specifying the internal geometry is to generate an irrotational coordinate velocity field. Let the coordinate velocity be defined as the gradient of a scalar

$$S^i = g^{ij} \Phi_{|j}^{\bullet} \tag{56}$$

where Φ is the coordinate velocity potential. The divergence of this vector field is given by

$$S^i{}_{|i} = g^{ij} \Phi_{|ij} \tag{57}$$

The mean divergence of the vector field over the region is given by

$$D = \frac{1}{V} \oint\!\!\!\oint n_i S^i \, dA \tag{58}$$

Equation (58) is evaluated in units consistent with the moving coordinate system. Assume that the divergence is constant throughout the region; therefore

$$S^i{}_{|i} = D \tag{59}$$

or

$$g^{ij} \Phi_{|ij} = D \tag{60}$$

Equation (60) can be solved for the coordinate velocity potential and hence (using eq. (56)) for the coordinate velocity field. The methods described above for generating the coordinate velocity field are illustrated in appendix B.

As an additional example, consider the problem of atmospheric flow over complex terrain. In order to simplify the boundary conditions, a coordinate system can be defined for which the terrain is a parametric surface. For problems that are strongly conditioned by the time-dependent height of the inversion layer, it may be convenient to define this layer as an additional parametric surface of the coordinate system. As a result, the coordinate system has time-dependent metrical properties. Scaling between the parametric surfaces is arbitrary. A convenient choice is exponential scaling in order that the coordinate stretching will approximately parallel the atmospheric density variation. This coordinate system is illustrated in figure 5.

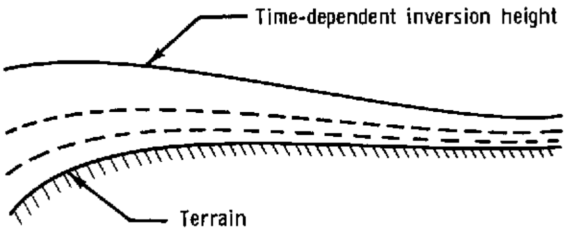


Figure 5.- A representation of scaled parametric surfaces.

CONCLUDING REMARKS

An expression for the co-moving derivative of a tensor has been derived. The expression derived is valid for coordinate systems having time-dependent metric elements. Time dependency of the metric elements was related to a variable describing the coordinate velocity. The resulting theory represents a generalized viewpoint (of which the Eulerian and Lagrangian viewpoints are limiting cases) of the field equations of continuum mechanics and is useful in problems containing moving boundaries or discontinuities.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., January 31, 1974.

APPENDIX A

A DESCRIPTION OF COVARIANT DIFFERENTIATION IN TERMS OF TRANSFORMATION LAWS

The purpose of this appendix is to show how the covariant derivative can be expressed in terms of the elements of the tensor transformation law and their derivatives. This development is similar to a related discussion in reference 4.

Consider a first-order covariant tensor A_i. Let $A_i(x)$ and $A_i(y)$ be components of the tensor A_i in an inertial rectangular Cartesian coordinate system x_i and an arbitrary coordinate system y_i, respectively. Assuming that $A_i(y)$ has continuous first partial derivatives, the covariant derivative is defined as

$$A_i(y)_{|\alpha} \equiv \frac{\partial A_i(y)}{\partial y^s} - \begin{Bmatrix} p \\ i\ s \end{Bmatrix} A_p(y) \tag{A1}$$

The equivalent definition of the covariant derivative of the associated contravariant first-order tensor is

$$A^i(y)_{|s} \equiv \frac{\partial A^i(y)}{\partial y^s} + \begin{Bmatrix} i \\ s\ p \end{Bmatrix} A^p(y) \tag{A2}$$

The Christoffel symbol of the second kind can be written as

$$\begin{Bmatrix} p \\ i\ s \end{Bmatrix} = g^{pr}\left[i\ s,\ r\right] \tag{A3}$$

where the Christoffel symbol of the first kind is given by

$$\left[i\ s,\ r\right] = \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial y^s} + \frac{\partial g_{sr}}{\partial y^i} - \frac{\partial g_{is}}{\partial y^r}\right) \tag{A4}$$

An element of the metric can be expressed in terms of the elements of the covariant transformation law and written as

$$g_{ij} = T_{ik} T_{jk} \tag{A5}$$

where

$$T_{ik} = \frac{\partial x^k}{\partial y^i} \tag{A6}$$

Assuming continuous first partial derivatives

$$\frac{\partial}{\partial y^i}\left(T_{qk}\right) = \frac{\partial}{\partial y^q}\left(T_{ik}\right) \tag{A7}$$

the Christoffel symbol of the first kind becomes

$$\left[i\ s,\ r\right] = T_{rq} \frac{\partial T_{iq}}{\partial y^s} \tag{A8}$$

Consider the matrix T formed by the elements T_{ij}. Let λ_{ij} be an element of the inverse of this matrix. Then,

$$\lambda_{kr} T_{rq} = I \tag{A9}$$

Note also that, by definition,

$$g^{pr} g_{sr} = \delta^p_s \tag{A10}$$

By substituting equation (A8) into equation (A3), the Christoffel symbol of the second kind can be written as

$$\begin{Bmatrix} p \\ i\ s \end{Bmatrix} = g^{pr} T_{rq} \frac{\partial T_{iq}}{\partial y^s} \tag{A11}$$

which, by substituting equations (A9), (A5), and (A10), reduces to

$$\begin{Bmatrix} p \\ i\ s \end{Bmatrix} = \lambda_{qp} \frac{\partial T_{iq}}{\partial y^s} \tag{A12}$$

As a consequence of equation (A9), the Christoffel symbol of the second kind can also be written as

$$\left\{ \begin{matrix} p \\ i\ s \end{matrix} \right\} = -T_{iq} \frac{\partial \lambda_{qp}}{\partial y^s} \tag{A13}$$

Therefore, for a covariant first-order tensor, the covariant derivative becomes

$$A_i(y)\big|_s = \frac{\partial A_i(y)}{\partial y^s} + T_{iq} \frac{\partial \lambda_{qp}}{\partial y^s} A_p(y) \tag{A14}$$

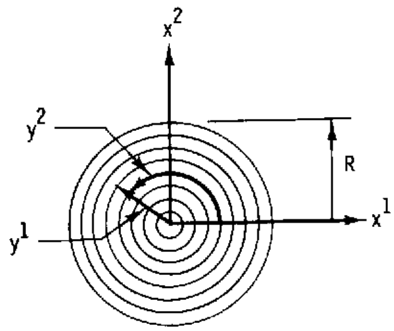
An equivalent expression for a contravariant tensor can be deduced from equation (A2). The result can be written

$$A^i(y)\big|_s = \frac{\partial A^i(y)}{\partial y^s} + \lambda_{qi} \frac{\partial T_{pq}}{\partial y^s} A^p(y) \tag{A15}$$

APPENDIX B

THE CO-MOVING DERIVATIVE OF VELOCITY IN A HARMONICALLY PULSATING TWO-DIMENSIONAL POLAR COORDINATE SYSTEM

Consider the two-dimensional region bounded by a harmonically pulsating circle. Over this region, define an orthogonal coordinate system in which the bounding circle is a parametric line and the family of similar parametric lines are generated by linear scaling. This geometry is shown in the following sketch:



Assume that the bounding circle has the following harmonic behavior:

$$R = \overline{R}\left[1 + \ell \sin(\omega t - \alpha)\right] \tag{B1}$$

where the mean radius $\overline{R}$, the amplitude of the oscillation ℓ, the frequency ω, and the phase angle α are constants.

In this simple example, it is possible to write down a coordinate transformation and determine the metric elements directly. The coordinate transformations can be written

$$\left.\begin{aligned} x^1 &= \left(\frac{y^1}{y^1{}_b}\right) R \cos y^2 \\ x^2 &= \left(\frac{y^1}{y^1{}_b}\right) R \sin y^2 \end{aligned}\right\} \tag{B2}$$

where R is a function of time and where $y^1{}_b$ is the y^1 coordinate at the boundary (a constant). By definition, the metric elements are

$$\left.\begin{aligned} g_{12} &= g_{21} = 0 \\ g_{11} &= \left(\frac{R}{y^1{}_b}\right)^2 \\ g_{22} &= \left[\left(\frac{R}{y^1{}_b}\right) y^1\right]^2 \end{aligned}\right\} \tag{B3}$$

Note that these elements are time dependent; that is,

$$\left.\begin{aligned} \frac{\partial g_{11}}{\partial t} &= 2\left(\frac{1}{y^1{}_b}\right)^2 R \frac{\partial R}{\partial t} \\ \frac{\partial g_{22}}{\partial t} &= 2\left(\frac{y^1}{y^1{}_b}\right)^2 R \frac{\partial R}{\partial t} \end{aligned}\right\} \tag{B4}$$

For a two-dimensional time-dependent polar coordinate system, components of the coordinate velocity are

$$\left.\begin{aligned} S^1 &= \left(\frac{y^1}{R}\right)\frac{\partial R}{\partial t} \\ S^2 &\equiv 0 \end{aligned}\right\} \tag{B5}$$

In general, the co-moving derivative of the velocity u^i is

$$\left(\frac{du^i}{dt}\right)_I = \frac{\partial u^i}{\partial t} + v^\alpha \, u^i{}_{|\alpha} + S^i{}_{|k} \, u^k \tag{B6}$$

For a two-dimensional polar coordinate system, all Christoffel symbols vanish except the following:

$$\left.\begin{aligned} \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} &= \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{y^1} \\ \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= -y^1 \end{aligned}\right\} \tag{B7}$$

Therefore, the components of the co-moving derivative become

$$\left(\frac{du^1}{dt}\right)_I = \frac{\partial u^1}{\partial t} + v^1 \frac{\partial u^1}{\partial y^1} + v^2\left(\frac{\partial v^1}{\partial y^2} - y^1 u^2\right) + \frac{u^1}{R}\frac{\partial R}{\partial t} \tag{B8}$$

and

$$\left(\frac{du^2}{dt}\right)_I = \frac{\partial u^2}{\partial t} + v^1\left(\frac{\partial u^2}{\partial y^1} + \frac{u^2}{y^1}\right) + v^2 \frac{\partial u^2}{\partial y^2} + \frac{u^1 v^2}{y^2} \tag{B9}$$

It is interesting to note that the harmonically pulsating polar system is also generated from the assumptions of an irrotational coordinate velocity field and constant divergence over the region. The velocity of the boundary is equal to the time derivative of the radius transformed to the y^s coordinate system. The components are

$$\left.\begin{aligned} S^1 &= \frac{y^1 b}{R}\frac{\partial R}{\partial t} \\ S^2 &= 0 \end{aligned}\right\} \tag{B10}$$

Therefore, the mean divergence for the bounded region is given by

$$D = 2\,\frac{1}{R}\frac{\partial R}{\partial t} \tag{B11}$$

As a result,

$$g^{ij} \Phi_{|ij} = 2 \frac{1}{R} \frac{\partial R}{\partial t} \tag{B12}$$

The initial values of the metric elements are defined by the initial coordinate geometry. Assume that, initially, the parametric lines are concentric circles. Therefore, at $t = 0$,

$$\left.\begin{aligned} g_{12} &= g_{21} = 0 \\ g_{11} &= \left(\frac{R}{y^1{}_b}\right)^2 \\ g_{22} &= \left[\left(\frac{R}{y^1{}_b}\right) y^1\right]^2 \end{aligned}\right\} \tag{B13}$$

Expanding equation (B12) gives

$$\Phi_{|11} + \left(y^1\right)^2 \Phi_{|22} = 2\left(\frac{y^1{}_b}{R}\right)^2 \frac{1}{R} \frac{\partial R}{\partial t} \tag{B14}$$

which has the solution

$$\Phi = \frac{1}{2}\left(\frac{y^1{}_b}{R}\right)^2 \frac{1}{R} \frac{\partial R}{\partial t}\left(y^1\right)^2 + \text{Constant} \tag{B15}$$

The corresponding velocity field is

$$S^1 = g^{11} \Phi_{|1} \tag{B16}$$

which becomes

$$S^1 = y^1 \frac{1}{R} \frac{\partial R}{\partial t} \tag{B17}$$

Also,

$$S^2 = g^{22} \Phi_{|2} \tag{B18}$$

which becomes

$$S^2 = 0 \tag{B19}$$

Note that this velocity field is equivalent to the one originally specified.

Time variation of the metric elements is given by

$$\frac{\partial g_{ij}}{\partial t} = S_{i|j} + S_{j|i} \tag{B20}$$

which gives

$$\left.\begin{aligned}
\frac{\partial g_{11}}{\partial t} &= 2 \frac{R}{\left(y^1{}_b\right)^2} \frac{\partial R}{\partial t} \\
\frac{\partial g_{22}}{\partial t} &= 2 R \left(\frac{y^1}{y^1{}_b}\right)^2 \frac{\partial R}{\partial t} \\
\frac{\partial g_{12}}{\partial t} &= \frac{\partial g_{12}}{\partial t} = 0
\end{aligned}\right\} \tag{B21}$$

Since these derivatives are equivalent to the derivatives obtained in equations (B4), the coordinate velocity field is identical to the one specified initially.

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