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Models of the Earth's Electric Field

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Introduction

This is an attempt to construct simple models of the electric field of the magnetosphere, based on a combination of observation and plausible theory.

The origin of the electric field is assumed to be in interplanetary space, where a field $-\underline{v} \times \underline{B}$ is created by the motion of the solar wind with velocity $\underline{v}$ relative to the earth. An open magnetosphere is assumed, in which some interplanetary magnetic field lines reach the top of the atmosphere in the vicinity of the earth's magnetic poles (the name polar caps will be used for the regions to which such field lines are connected), conveying with them the interplanetary electric field. Observations indicate that this field is quite variable [Cauffman and Gurnett, 1972] but that on the average it points from dawn to dusk within a circular polar cap centered on the magnetic pole [e.g. Heppner 1972].

In this work, the radius ϱ_o of the polar cap will be taken as $a/4$, where a is the radius of the earth. Equipotential lines of the average field inside the polar caps - assuming that it is constant and in the dawn-dusk direction - are then aligned with the noon-midnight direction (Figure 1). Outside the polar cap the conducting ionosphere will extend the equipotentials to closed magnetic field lines, creating a two-celled pattern there with corresponding ionospheric currents.

Detailed models for this field will be derived in several stages, in all of which the conductivity along field lines will be assumed to be high enough to ensure the vanishing of $\underline{E}\cdot\underline{B}$ everywhere except in the ionosphere. At first the rotation of the earth is ignored completely and a simple model is constructed which fits the observed properties listed earlier. Next the rotation of the earth is taken into account, but the field is assumed to be that of a magnetic dipole rotating around its symmetry axis; this allows the concept of the electric potential to be retained, which permits the derivation of interesting properties, including the use of a conjugate potential u which paces the drift of charged particles in the field.

Finally, the general case, involving asymmetrical rotation, is briefly discussed.

Simple Model Without Rotation

If there exists no time dependence the electric field may be derived from a scalar potential ϕ and the assumption that in the magnetosphere $\underline{E} \cdot \underline{B} = 0$ then reduces to

$$\phi = \phi(\alpha, \beta) \tag{1}$$

where (α, β) are Euler potentials of the magnetic field [Stern, 1970 and references cited there]. Assuming the magnetic field to be that of a magnetic dipole and denoting by a the radius of the earth and by g_1^0 the dipole term in the expansion of the scalar potential ($g_1^0 \sim -3 \cdot 10^{-5}$ MKS)

$$\gamma = a\, g_1^0\, (a/r)^2 \cos\theta \tag{2}$$

one finds

$$\alpha = a\, g_1^0\, (a/r) \sin^2\theta$$
$$\beta = a\varphi \tag{3}$$

The solution for ϕ could now be derived as follows. Assuming an externally generated electric field above the polar cap ionosphere ($\alpha < \alpha_0$) which is constant and directed from dawn to dusk, one can solve for ϕ in the ionosphere (where it depends not only on α and β but on a third coordinate as well) both for the polar cap ionosphere and for the fringing values of ϕ which spill out over the edges of the polar caps into the sub-polar ionosphere. The values of ϕ in the latter case will then determine the potential on closed field lines: due to the electric field on such lines the magnetospheric plasma will experience a drift motion, leading to general convective motion in the magnetosphere.

This leads to a fairly complicated boundary value problem, which will not be treated here (the input field used is already very simplified, anyway). Instead a form of ϕ will be assumed which appears to be in agreement with observations. This form is

$$\phi = - \phi_o \, (\alpha / \alpha_o)^{1/2} \, \sin(\beta/a) \qquad \alpha \leq \alpha_o \qquad (4\text{-a})$$

$$\phi = - \phi_o \, (\alpha_o / \alpha)^{k/2} \, \sin(\beta/a) \qquad \alpha \geq \alpha_o \qquad (4\text{-b})$$

with k an appropriately chosen constant.

To derive some intuitive insight into this, let the polar cap be approximated by a plane surface, tangential to the magnetic pole. In this plane, let (ρ, Ψ) be plane polar coordinates and let $\rho = \rho_o$ be the boundary of the polar cap. Since $\rho = a \sin\theta$, one finds

$$\rho / \rho_o = (\alpha / \alpha_o)^{1/2} \qquad (5)$$

Thus for $\rho < \rho_o$

$$\phi \cong - \phi_o \, (\rho / \rho_o) \sin \Psi = - \phi_o \, y / \rho_o \qquad (6)$$

which gives a constant electric field $E_o = \phi_o / \rho_o$ in the y direction. For $\rho > \rho_o$, on the other hand

$$\phi = - \phi_o \, (\rho_o / \rho)^k \, \sin \Psi \qquad (7)$$

giving a field which drops off approximately as $\rho^{-(k+1)}$. All values of k give the familiar two-celled structure (Figure 1); for k = 1 this resembles the well known two dimensional solution satisfying Laplace's equation for $\rho > \rho_o$ and continuing a constant field which exists in $\rho \leq \rho_o$ [e.g. Panofsky and Phillips, 1955 , section 4-9] , while higher values of k progressively compress the equipotentials towards the circle $\rho = \rho_o$.

The choice $k = 2$ leads to a constant electric field in the equatorial plane (as assumed by Chen [1970]) , but observations from OGO 6 on the profile of $|\underline{E}|$ suggest $k = 4$. To fit observations, we shall also take

$$E_o = 0.02 \text{ volt/meter}$$

$$\rho_o = a/4$$

In a dipole field α is given by (3) and therefore (4-b) leads in the equatorial plane to

$$\phi = - \phi_o (\alpha_o/a\, g_1^o)^{1/2} (r/a)^{k/2} \sin \varphi$$

$$= - \phi_o (\alpha_o/a\, g_1^o)^{1/2}\, y\, (r/a)^{(k-2)/2} \qquad (8)$$

If $k = 2$ this field is uniform and equipotentials are all parallel to the noon-midnight direction. For higher values of k (as are assumed here) the equipotentials still tend to follow the noon-midnight direction, but they curve towards the noon-midnight line, their closest approach to that line occuring as they cross the dawn-dusk axis. A schematic description of these lines for $k = 2$ is given in Figure 2 , while Figure 3 shows them for $k = 4$.

Axisymmetric Rotation

If we assume that the earth and the material threaded by closed field lines are undergoing steady rotation with angular velocity ω around the symmetry axis, the electric field can still be represented by a scalar potential, since $\partial/\partial t = 0$. Assume then that equations (4) represent the electric field $\underline{E}^*$ in the rotating frame. Then the electric field observed in a non-rotating frame - e.g. by a charged particle entering the magnetosphere - is

$$\underline{E} = \underline{E}^* - \underline{v} \times \underline{B} \qquad (9)$$

where $\underline{v}$ in this case represents the velocity of co-rotation. One then finds, given $\alpha = \alpha(r, \theta)$, $\beta = a\varphi$

$$\underline{v} \times \underline{B} = a \omega r \sin\theta \; \hat{\underline{\varphi}} \times \left(\nabla \alpha(r,\theta) \times \nabla \varphi \right) = \nabla (a \omega \alpha) \qquad (10)$$

Hence on closed field lines

$$\phi = - \phi_o \left(\alpha_o / \alpha \right)^{k/2} \sin(\beta/a) + a \omega \alpha \qquad (11)$$

As far as the rotation of the polar caps is concerned, it will tend to twist open field lines into helical shape. The total amount of twisting experienced by such lines does not seem to be large, since the length of the geomagnetic tail is apparently only of the order of 500 a . An open polar field line sweeping past the earth with the velocity of the solar wind spends only about 3 hours in contact with the polar cap, from the time it merges with a closed field line to the time it again breaks connection with the earth's field, and during that time the earth will only rotate through 45^o .

Particle Motion

Consider a particle of very low energy, so that its low drift velocity in the magnetic field is essentially the electric drift velocity

$$\underline{v}_d = B^{-2} \left(\underline{E} \times \underline{B} \right) \qquad (12)$$

Then in the present case

$$\underline{v}_d \cdot \nabla \phi = 0 \qquad (13)$$

showing that such particles stay on surfaces of constant ϕ . In particular, in the fields of the present model the electric potential in the equatorial plane is

$$\phi = - \phi_o \left(\alpha_o / a\, g_1^o \right)^{\frac{k}{2}} (r/a)^{\frac{k}{2}} \sin \varphi + a^2 g_1^o \omega (a/r) \qquad (14)$$

A low energy particle moving in the equatorial plane will conserve the above quantity. For very small r , the second term dominates and equipotential lines in the equatorial plane approximate closed circles around

around the origin. For large values of r the second term is negligible and the first one determines the structure of the equipotentials; in particular, for $k \geq 2$ we obtain a bunch of equipotentials generally aligned with the noon-midnight direction, pinched together near the earth if $k > 2$.

These two regimes are separated by the "last closed equipotential" L , which also extends to infinity and has the form schematically shown in Figure 2 . At the point marked P in the figure, the equipotential surface crosses itself, and since the direction of $\nabla\phi$ at P is thus undefined, this gradient must vanish there. From (14) , the vanishing of $\partial\phi/\partial\varphi$ implies

$$\sin\varphi = \pm 1 \tag{15}$$

while the radial component gives

$$(k\,\phi_o/2)\;(\alpha_o/a\;g_1^o)^{\frac{1}{2}}(r/a)^{\frac{k}{2}}\sin\varphi = -\,a^2\,g_1^o\,\omega\;(a/r) \tag{16}$$

clearly requires the positive sign in equation (15), i.e. P is on the dusk side. Using numbers introduced earlier (including $k = 4$) gives for P

$$r = 6\,a$$

which is a little high for the distance to the plasmapause bulge - which is the usual interpretation of the point P [Brice, 1967] - but has the proper order of magnitude.

The Conjugate Potential

The Euler potentials (α,β) are not unique and other potentials $\alpha'(\alpha,\beta)$, $\beta'(\alpha,\beta)$ will also describe the field, provided

$$\partial(\alpha',\beta')/\partial(\alpha,\beta) = 1 \tag{17}$$

Furthermore, given a well-behaved function $\alpha'(\alpha,\beta)$, some $\beta'(\alpha,\beta)$ can be generally found for which (17) holds. In particular, if the electric

potential $\phi(\alpha, \beta)$ is regarded as an Euler potential, a conjugate potenti
$u(\alpha, \beta)$ will exist such that

$$\underline{B} = \nabla\phi \times \nabla u \tag{18}$$

It is easy to calculate the rate at which u changes for a particle moving with the electric drift velocity $\underline{v}_d$:

$$du/dt = \underline{v}_d \cdot \nabla u = - B^{-2} (\nabla\phi \times \underline{B}) \cdot \nabla u = 1 \tag{19}$$

Thus the rate is constant at all places and for all times, and a collection of particles which started with a given constant u at $t = 0$ will share a surface of constant u at all subsequent times. In other words, just as surfaces of constant ϕ guide the drift of particles along them, surfaces of constant u pace the rate at which such particles advance.

It may appear somewhat surprising that the same function $u(\alpha, \beta)$ paces the electric drift motion of all particles, regardless of their equatorial pitch angle λ_o. Consider for example two such particles starting at $t = 0$ from the same field line, one with $\lambda_o = \pi/2$ and the other one with a relatively small λ_o. It is not a-priori clear here why the two always stay on the same field line: the result is however true and the reason is that the electric drift velocity is a magnetic field line velocity [Newcomb, 1958 ; Stern, 1966 ; Stern, 1970] .

If gradient drifts and curvature drifts are included in the calculation of the motion, u is no longer conserved and particles with different equatorial pitch angles tend to drift onto different field lines. This type of motion, for the case of constant equatorial electric field ($k = 2$) has been thoroughly investigated by Chen [1970] .

In the equatorial plane the two types of surface degenerate into two families of lines (generally not orthogonal) - one of them marking the trajectories of particles, the other one the successive positions of a "front" of advancing particles.

In order to derive u it is best to consider it as a function not of (α, β) but of $\phi(\alpha, \beta)$ and of α. Substituting this dependence in (18) gives, in a few steps

$$\left.\frac{\partial \phi}{\partial \beta}\right|_{\alpha} \left.\frac{\partial u}{\partial \alpha}\right|_{\phi} = -1 \qquad (20)$$

From this u is readily derived by integration, provided $\partial\phi/\partial\beta$ can be expressed in terms of (ϕ, α).

Because ϕ is given by different expressions inside and outside the polar cap, the same distinction must be made when u is derived. Suppose that inside the polar cap rotation can be neglected and ϕ is given by

$$\phi = -\phi_0 (\alpha/\alpha_0)^{\frac{1}{2}} \sin\Psi - a\omega\alpha_0 \qquad (21)$$

This reduces to a constant field in the y direction if plane polar coordinates can be used with the approximation $\alpha/\alpha_0 = (\rho/\rho_0)^2$. Then $(\beta = a\Psi)$

$$\begin{aligned} \partial\phi/\partial\beta &= -(\phi_0/a)(\alpha/\alpha_0)^{\frac{1}{2}} \cos\Psi \\ &= -\left[(\alpha\phi_0^2/\alpha_0 a^2) - (\phi + a\omega\alpha_0)^2 \right]^{\frac{1}{2}} \\ &= -(p\alpha - q^2)^{\frac{1}{2}} \end{aligned} \qquad (22)$$

From (20) then

$$u = \int (p\alpha - q)^{-\frac{1}{2}} d\alpha = -(2a/\phi_0)(\alpha_0 \alpha)^{\frac{1}{2}} \cos\varphi \qquad (23)$$

In the plane approximation which is used here, u is a multiple of x, which agrees with the advance of a front of particles given $\underline{E} = E_0 \hat{\underline{y}}$, $\underline{B} = -B_0 \hat{\underline{z}}$. Rotation can be taken into account by replacing α_0 in the second term of (21) with α, but the result then is no longer intuitively simple.

Outside the polar cap

$$\phi = -\phi_0 (\alpha_0/\alpha)^{\frac{k}{2}} \sin\varphi - a\omega\alpha \qquad (24)$$

giving

$$u = \int \left[\phi_0^2 (\alpha_0/\alpha)^k - (\phi + a\omega\alpha)^2 \right]^{-\frac{1}{2}} d\alpha \qquad (25)$$

This must be integrated numerically along a line of constant potential ϕ. The only difficulty here occurs near the intersection between the equipotential and the y axis, where $\cos\varphi = 0$, causing the denominator $-\partial\phi/\partial\beta$ to vanish. However, a transformation to a new variable $\psi = \varphi - \pi/2$ shows that the integrand behaves as $\psi^{-1/2}$ in this vicinity and the resulting integral therefore converges.

The function u thus obtained can be conveniently represented in the equatorial plane of the dipole, as was done in Figure 3. In that figure curves of constant ϕ are drawn solid, while broken lines represent lines of constant u. Distances are given in normalized units of the order of the dimensions of the plasmapause [Stern, 1974, equations 13 and 24]. Unit distance here is approximately $0.412\ E_0^{1/3}$ earth radii, where $E_0 = \phi_0/\rho_0$ and where $\rho_0 = a/4$ is assumed, while the line $u = 0$ is at a distance $R = 2$, parallel to the dawn-dusk line.

One can view the line $u = 0$ as the starting position of a group of very low-energy particles drifting earthward from the geomagnetic tail with the $\underline{E} \times \underline{B}$ drift. Other lines of constnt u then mark the position of the field lines to which these particles are attached at selected later times.

The values of u on these times are proportional to the times required to reach these positions, by virtue of equation (19). It will be noted that in the figure the elapsed time between consecutive curves almost doubles at each step, because the magnitude E/B of the drift velocity increases rapidly as the earth is approached, due to the increase in B. The line $u = 1$, in the normalized units used here, corresponds to a time of $1/2\pi$ days or about 4 hours and it will be noted that this is the order of the time required here by the particles to reach the vicinity of the plasmapause.

The method illustrated here can be adapted to the motion of particles of arbitrary energy and pitch angle [Chen and Stern, 1974]. The details and qualitative properties of this, however, are beyond the scope of this work and will be described in another paper.

Asymmetrical Rotation

If $\partial \underline{B}/\partial t$ does not vanish $\underline{E}$ can no longer be described by ϕ alone. Assume as before that the electric field at the top of the polar ionosphere is given by $-\nabla\phi$, with ϕ the same as in equations (4). The electric field in the magnetosphere is then still given by (9), with $\underline{v}$ the velocity of the local medium, but the added term $-(\underline{v} \times \underline{B})$ is no longer curl-free.

Suppose that on all closed field lines $\underline{v}$ corresponds to rigid co-rotation, satisfying

$$\underline{v} = \underline{\omega} \times \underline{r} \tag{26}$$

with $\underline{\omega}$ a constant vector. If

$$\underline{B} = \nabla \times \underline{A} \qquad \nabla \cdot \underline{A} = 0 \tag{27}$$

then it may be shown that

$$\underline{v} \times \underline{B} = \nabla(\underline{A} \cdot \underline{v}) + \nabla \times (\underline{v} \times \underline{A}) \tag{28}$$

The proof starts from the identities

$$\underline{v} \times \underline{B} = \nabla(\underline{v} \cdot \underline{A}) - \underline{A} \times (\nabla \times \underline{v}) - \underline{A} \cdot \nabla \underline{v} - \underline{v} \cdot \nabla \underline{A} \tag{29}$$

$$0 = \nabla \times (\underline{v} \times \underline{A}) - \underline{v}(\nabla \cdot \underline{A}) + \underline{A}(\nabla \cdot \underline{v}) - \underline{A} \cdot \nabla \underline{v} + \underline{v} \cdot \nabla \underline{A} \tag{30}$$

When these are added up, the last terms cancel. Also, from (26)

$$\underline{v} = -\frac{1}{2} \nabla \times r^2 \underline{\omega} \tag{31}$$

so that $\nabla \cdot \underline{v}$ vanishes. Furthermore, by explicit calculation in cartesian coordinates with one axis aligned with $\underline{\omega}$

$$\nabla \times \underline{v} = 2 \underline{\omega} \tag{32}$$

$$\underline{A} \cdot \nabla \underline{v} = \underline{\omega} \times \underline{A} \tag{33}$$

This shows that all terms except those in equation (28) cancel. In axisymmetrical poloidal fields one may choose Euler potentials $\alpha(r, \theta)$ and $\beta = a\varphi$ and take

$$\underline{A} = \alpha \nabla\beta$$

Equation (10) is then recovered as a special case. If the earth's field is approximated by a dipole tilted at a small angle relative to the rotation axis, this dipole may be resolved into components parallel and perpendicular to that axis, with corresponding contributions $\underline{A}_1$ and $\underline{A}_2$ to the vector potential. Then $\underline{A}_1$ contributes only to the second term in (28); if $\underline{A}_2$ is small, it may be neglected, leading again to the electric field of a dipole rotating around its symmetry axis. A scalar potential which approximately gives the electric field can then be derived and it resembles the one given in (11), except that in the first term there α refers to the entire dipole field while in the second term only the value of α associated with the aligned dipole component is used.

As a final note, it may be pointed out that a general asymmetric magnetosphere may be expected not to undergo purely rigid rotation, since its boundary is fixed in the frame of the solar wind. The external field of the earth then undergoes a time-dependent change even in the co-rotating frame and the actual electric effects are more difficult to analyze.

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CAPTIONS TO FIGURES

Figure 1 - The two-celled structure of equipotential lines in the polar cap, viewed from above.

Figure 2 - Schematic view of equipotentials in the dipole equatorial plane, for $k = 2$.

Figure 3 - Equipotential lines (solid) and lines of constant u (broken) in the equatorial plane of the dipole. The value of k is taken to be equal to 4 .

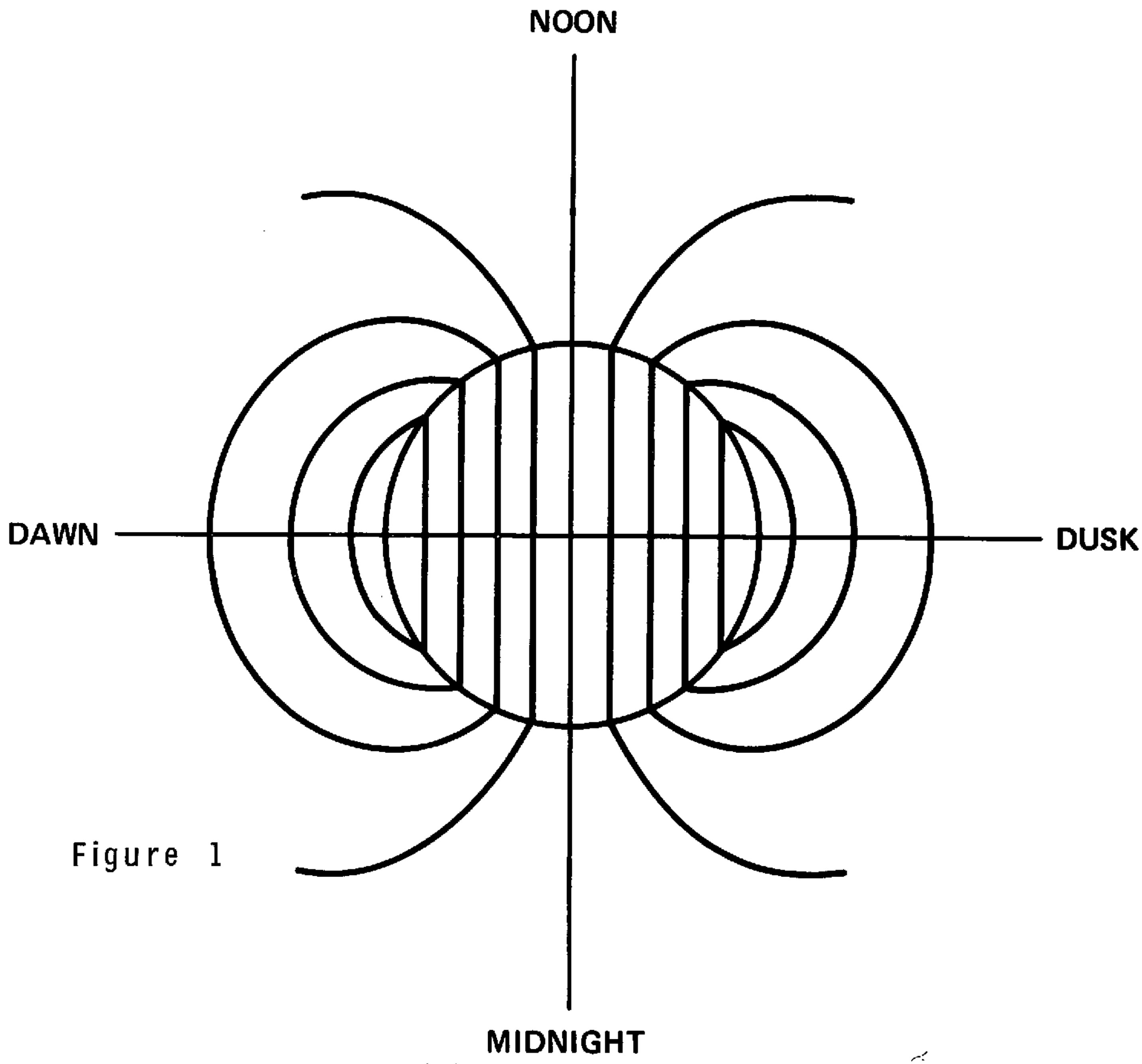


Figure 1

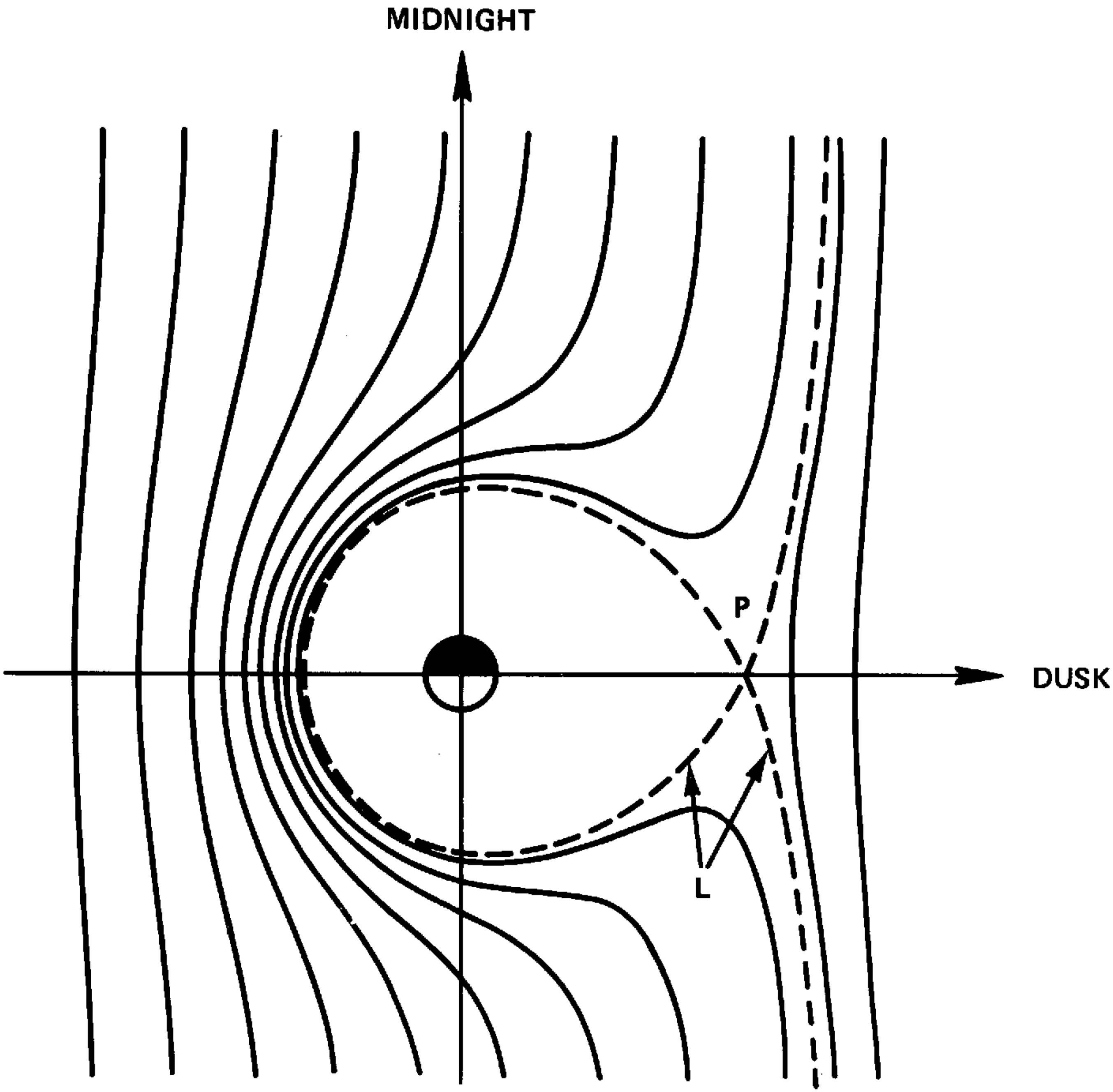
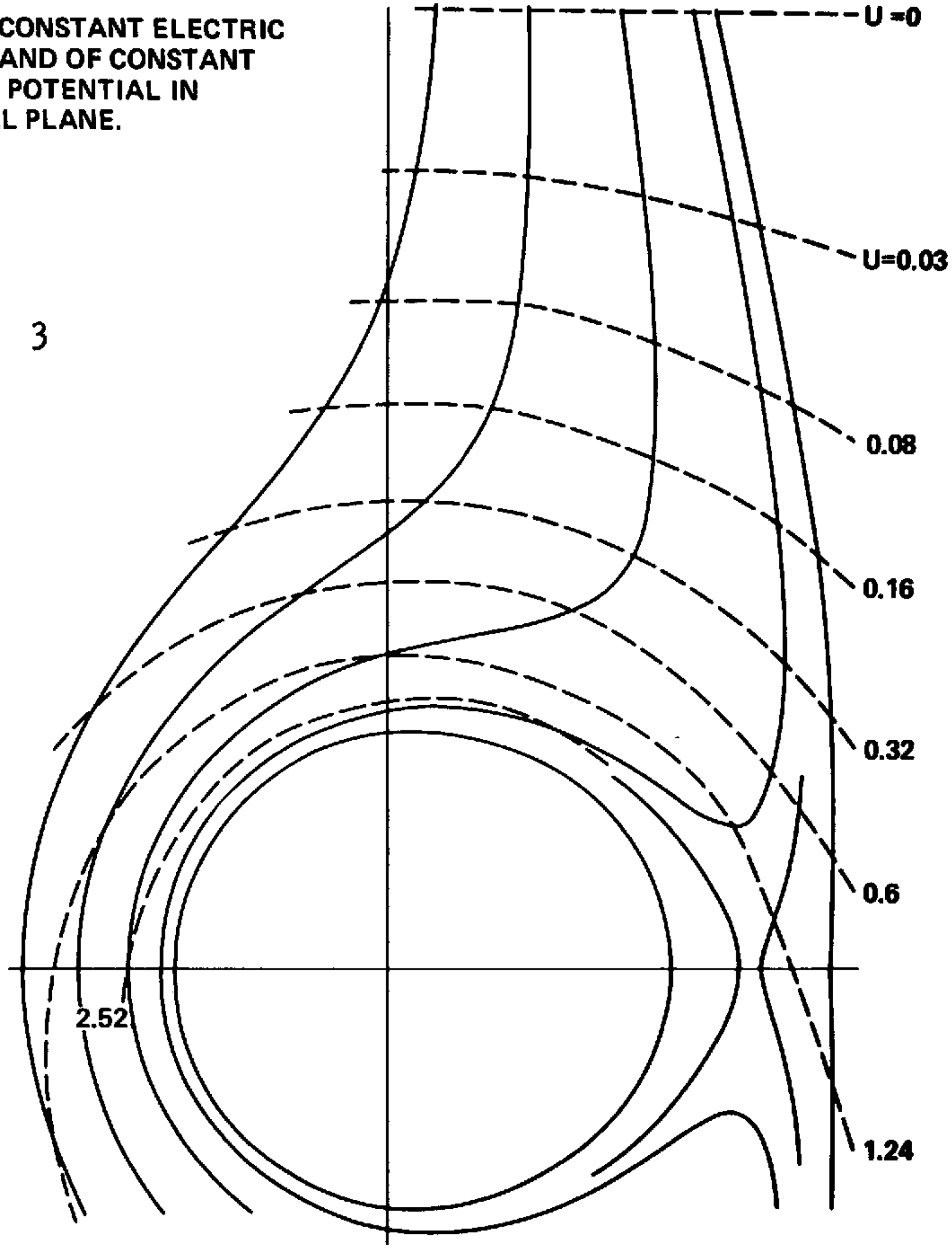


Figure 2

CURVES OF CONSTANT ELECTRIC POTENTIAL AND OF CONSTANT CONJUGATE POTENTIAL IN EQUATORIAL PLANE.

Figure 3



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