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A Reciprocal Theorem For A Mixture Theory

by

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1. Introduction. In the classical linear theory of elasticity there is a widely used integral theorem called the reciprocal theorem of Betti and Rayleigh. Sokolnikoff [1] and Love [2] have amply illustrated the versatility of this theorem for elastostatic problems while Payton [3], [4], and Beitin [5] have employed a dynamic version to study elastodynamic problems involving moving point and line loadings. Fung [6] has generalized the theorem to cover the case of a linear viscoelastic solid and his book contains references to versions of this theorem useful to thermoelasticity and shell theories.

In this paper we intend to establish a dynamic reciprocal theorem for a linearized theory of interacting media postulated in a paper by Steel [7]. The constituents of the mixture are a linear elastic solid and a linearly viscous fluid. In addition to Steel's field equations we use boundary conditions and inequalities on the material constants that have been shown by Atkin, Chadwick and Steel [8] to be sufficient to guarantee uniqueness of solution to initial-boundary value problems.

The elements of the theory are given in section 2 and two different boundary value problems are considered. The reciprocal theorem is derived in section 3 with the aid of the Laplace transform and the divergence theorem and this section is concluded with a discussion of the special cases which arise when one of the constituents of the mixture is absent.

As an illustration of the theorem we obtain the response of the mixture occupying an infinite region and subjected to an impulsively applied moving point load acting on the solid constituent. The displacement of the solid component and the velocity of the fluid constituent are found and discussed. This is the content of section 4.

2. Field equations for the mixture. We formulate the field equations appropriate to a mixture of linear elastic solid and linearly viscous fluid using the field equations and boundary conditions given in [7] and [8]. All equations are given referred to a cartesian coordinate system $x = (x_1, x_2, x_3)$, and time t. The mixture is assumed to occupy a regular region of three-dimensional Euclidean space, D, with bounding surface, S. The conventional indicial subscript notation is used to specify vector or tensor components with an index appearing twice indicating a sum over 1,2,3. Subscripts preceded by a comma indicate spatial differentiation with respect to that variable while time derivatives are indicated by a dot.

According to [7] and [8], the field equations consist of the following:

continuity equations

$$\rho_1 = \bar{\rho}_1(1 - e_{kk}), \quad \mathring{\eta} + \bar{\rho}_2 v_{k,k} = 0 \tag{2.1}$$

equations of motion

$$\begin{aligned} &\sigma_{ij,j} - \pi_i + \bar{\rho}_1 f_i = \bar{\rho}_1 \ddot{w}_i , \\ &\pi_{ij,j} + \pi_i + \bar{\rho}_2 g_i = \bar{\rho}_2 \dot{v}_i , \quad i = 1,2,3 \end{aligned} \tag{2.2}$$

strain-displacement equations

$$2e_{ij} = w_{i,j} + w_{j,i} , \quad i,j = 1,2,3 \tag{2.3}$$

rate of deformation-velocity relations

$$2f_{ij} = v_{i,j} + v_{j,i} , \quad i,j = 1,2,3 \tag{2.4}$$

constitutive relations

$$\begin{aligned} &\sigma_{ij} = \alpha_1 \delta_{ij} + 2\beta_3 e_{ij} + \beta_2 \delta_{ij} e_{kk} + \beta_1 \eta \delta_{ij} , \\ &\pi_{ij} = -\bar{\rho}_2 \alpha_2 \delta_{ij} - \gamma_2 \delta_{ij} e_{kk} + 2\mu f_{ij} + \lambda \delta_{ij} f_{kk} - \gamma_1 \eta \delta_{ij} , \\ &\pi_i = \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}} \eta_{,i} - \frac{\bar{\rho}_2 \alpha_1}{\bar{\rho}} e_{mm,i} + \alpha(\mathring{w}_i - v_i) , \quad i,j = 1,2,3. \end{aligned} \tag{2.5}$$

To complete the formulation we add to the above the initial and boundary conditions. Thus when $t=0$ we require

$$w_i(x,0) = w_i^{(1)}(x), \quad v_i(x,0) = v_i^{(1)}(x),$$
$$\dot{w}_i(x,0) = w_i^{(2)}(x), \quad \eta(x,0) = 0, \quad i=1,2,3 \tag{2.6}$$

for all points x in D, while on the boundary S we prescribe for $t\geq 0$

$$(\sigma_{ij} + \pi_{ij})n_j = t_i \, ,$$
$$\dot{w}_i - v_i = r_i \, , \; i = 1,2,3 \tag{2.7}$$

where n_i are the components of the unit outward normal to S.

Quantities appearing in (2.1) to (2.5) which are associated with the solid component of the mixture are ρ_1, $\bar{\rho}_1$, w_i, e_{ij}, σ_{ij} and f_i. Here ρ_1 is the density at time t and place x, $\bar{\rho}_1>0$ its initial value, w_i the displacement components, f_i the body force components, and e_{ij}, σ_{ij}, respectively, the strain and partial stress tensor component. In the fluid, η is the current density minus its initial value, $\bar{\rho}_2>0$, v_i the fluid velocity components, f_{ij} the rate-of-deformation tensor, π_{ij} the fluid partial stress tensor, and g_i the fluid body force components. The vector components π_i in (2.5) are those of the diffusive resistance vector. The material constants α_1, α_2, β_1, β_2, β_3, μ, λ, γ_1, γ_2 and α are assumed to obey the inequalities given in [8] as well as the equalities

$$\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2$$
$$\bar{\rho}_2\beta_1 = \gamma_2 + \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}} + \frac{\bar{\rho}_1\bar{\rho}_2\alpha_2}{\bar{\rho}} \tag{2.8}$$ *

Quantities $w_i^{(1)}$, $w_i^{(2)}$, $v_i^{(1)}$, t_i and r_i are assumed known throughout t the appropriate domains.

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* This follows from the definition of β_1, β_2, β_3, γ_1, γ_2 given by Steel [7].

Now consider a second problem for the same region D. Let the corresponding field equations for this problem be

$$\rho_1^{(2)} = \bar{\rho}_1 (1 - E_{kk}), \quad \dot{E} + \bar{\bar{\rho}}_2 v_{k,k} = 0 , \qquad (2.1)'$$

$$S_{ij,j} - P_i + \bar{\rho}_1 F_i = \bar{\rho}_1 \ddot{W}_i , \qquad (2.2)'$$

$$P_{ij,j} + P_i + \bar{\rho}_2 G_i = \bar{\bar{\rho}}_2 \dot{V}_i , \quad i = 1,2,3$$

$$2E_{ij} = W_{i,j} + W_{j,i} , \quad i,j = 1,2,3 \qquad (2.3)'$$

$$2F_{ij} = V_{i,j} + V_{j,i} , \quad i,j = 1,2,3 \qquad (2.4)'$$

$$S_{ij} = \alpha_1 \delta_{ij} + 2\beta_3 E_{ij} + \beta_2 \delta_{ij} E_{kk} + \beta_1 E \delta_{ij} ,$$

$$P_{ij} = - \bar{\rho}_2 \alpha_2 \delta_{ij} - \gamma_2 \delta_{ij} E_{kk} + 2\mu F_{ij} + \lambda \delta_{ij} F_{kk} - \gamma_1 E \delta_{ij} . \qquad (2.5)'$$

$$P_i = \frac{\bar{\bar{\rho}}_1 \alpha_2}{\bar{\rho}} E_{,i} - \frac{\bar{\bar{\rho}}_2 \alpha_1}{\bar{\rho}} E_{kk,i} + \alpha(\dot{W}_i - V_i) , \quad i,j = 1,2,3.$$

In place of (2.6) and (2.7) we require that when $t = 0$,

$$W_i(x,0) = W_i^{(1)}(x), \quad V_i(x,0) = V_i^{(1)}(x),$$
$$\dot{W}_i(x,0) = W_i^{(2)}(x), \quad E(x,0) = 0, \quad i = 1,2,3. \qquad (2.6)'$$

at all points x of D, and when x is on S,

$$(S_{ij} + P_{ij})n_i = T_i ,$$
$$\dot{W}_i - V_i = R_i , \quad i = 1,2,3. \qquad (2.7)'$$

Equations (2.1)' to (2.7)' differ from (2.1) to (2.7) only in allowing different body forces, initial conditions and surface conditions. The notational changes are obvious and are used for the sake of clarity in what follows.

In the next section we intend to show that there exists an integral relation between the solutions of problem 1 specified by (2.1) to (2.7) and the solutions of problem 2 given by (2.1)' to (2.7)'. This relation is established with the aid of the Laplace transform and the divergence theorem.

3. The reciprocal theorem. We begin by defining the Laplace transform with respect to time of a function $f(t)$ to be

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt \tag{3.1}$$

and by recalling that the inverse of the product of $\hat{f}_1(s)\hat{f}_2(s)$ is given by

$$L^{-1}\{\hat{f}_1(s)\hat{f}_2(s)\} = \int_0^t f_1(t-\tau)f_2(\tau)d\tau \tag{3.2}$$

Applying (3.1) to (2.1), (2.2) and (2.5), and, using the initial conditions (2.6) we obtain

$$s\hat{\eta}(x,s) + \bar{\rho}_2\,\hat{v}_{m,m}(x,s) = 0\ , \tag{3.3}$$

$$\begin{aligned} &\hat{\sigma}_{ij,j}(x,s) - \hat{\pi}_i(x,s) + \bar{\rho}_1\,\hat{f}_i(x,s) = \\ &- w_i^{(1)}(x) - s w_i^{(2)}(x) + s^2\,\hat{w}_i(x,s)\ , \\ &\hat{\pi}_{ij,j}(x,s) + \hat{\pi}_i(x,s) + \bar{\rho}_2\,\hat{g}_i(x,s) = \\ &\bar{\rho}_2[-v_i^{(1)}(x) + s\hat{v}_i(x,s)]\ , \quad i = 1,2,3 \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\hat{\sigma}_{ij}(x,s) = \frac{\alpha_1}{s}\delta_{ij} + \beta_2\delta_{ij}\hat{e}_{kk}(x,s) + 2\beta_3\hat{e}_{ij}(x,s) + \beta_1\delta_{ij}\hat{\eta}(x,s), \\ &\hat{\pi}_{ij}(x,s) = \frac{-\bar{\rho}_2\alpha_2}{s}\delta_{ij} - \gamma_2\delta_{ij}\hat{e}_{kk}(x,s) + 2\mu\hat{f}_{ij}(x,s) + \\ &\qquad \lambda\delta_{ij}\hat{f}_{mm}(x,s) - \gamma_1\delta_{ij}\hat{\eta}(x,s), \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\hat{\pi}_i(x,s) = \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}}\hat{\eta}_{,i}(x,s) - \frac{\bar{\rho}_2\alpha_1}{\bar{\rho}}\hat{e}_{kk,i}(x,s) + \\ &\qquad \alpha[-w_i^{(1)} + s\hat{w}_i(x,s) - \hat{v}_i(x,s)]\ , \quad i,j = 1,2,3. \end{aligned}$$

The boundary conditions (2.7) when transformed by (3.1) become

$$[\hat{\sigma}_{ij}(x,s) + \hat{\pi}_{ij}(x,s)]n_j = \hat{t}_i(x,s) \;,$$
$$- v_i^{(1)}(x) + s\hat{w}_i(x,s) - \hat{v}_i(x,s) = \hat{r}_i(x,s) \;, \quad i,j = 1,2,3 \tag{3.6}$$

for x on the surface S.

Now consider the solution to (2.1)' to (2.7)' to be given by $W_i(x,t)$ and $V_i(x,t)$. Apply (3.1) to this solution, multiply $(3.4)_1$ by $s\hat{W}_i(x,s)$ and $(3.4)_2$ by $\hat{V}_i(x,s)$, then sum on i. Integrate both equations over D and add. We then have

$$\iiint_B \{ s\hat{W}_i(x,s)[\hat{\sigma}_{ij,j}(x,s) - \hat{\pi}_i(x,s)] + \hat{V}_i(x,s)[\hat{\pi}_{ij,j}(x,s) + \hat{\pi}_i(x,s)] \} \, d\tau =$$

$$\iiint_B \{ s\hat{W}_i(x,s)[-\bar{\rho}_1\hat{f}_i(x,s) + \bar{\rho}_1'(-w_i^{(2)}(x) - sw_i^{(1)}(x) + s^2\hat{w}_i(x,s))] + \hat{V}_i(x,s)[\bar{\rho}_2(-v_i^{(1)}(x) + s\hat{v}_i(x,s)) - \bar{\rho}_2\hat{g}_i(x,s)] \} \, d\tau \;. \tag{3.7}$$

Define

$$I_1 = \iiint_B s\hat{W}_i\hat{\sigma}_{ij,j} d\tau \;,$$
$$I_2 = \iiint_B s\hat{W}_i\hat{\pi}_i \, d\tau \;,$$
$$I_3 = \iiint_B \hat{V}_i\hat{\pi}_{ij,j} d\tau \;,$$
$$I_4 = \iiint_B \hat{V}_i\hat{\pi}_i \, d\tau \;. \tag{3.8}$$

Rewrite I_1 as

$$I_1 = \iiint_B [(s\hat{W}_i\hat{\sigma}_{ij})_{,j} - s\hat{W}_{i,j}\hat{\sigma}_{ij}] d\tau \ldots$$

and this form in turn, upon using the divergence theorem, (2.3), (2.4), (2.3)', (3.1), (3.3) and $(3.5)_1$, becomes

$$I_1 = \iint_S s\hat{W}_i\hat{\sigma}_{ij}n_j\, d\sigma$$

$$= \iiint_B sW_{i,j}\Big[\frac{\alpha_1}{s}\delta_{ij} + \beta_2\hat{w}_{k,k}\delta_{ij} + \beta_3(\hat{w}_{i,j} + \hat{w}_{j,i})$$

$$- \frac{\bar{\varphi}_2\beta_1}{s}\hat{v}_{m,m}\delta_{ij}\Big]d\tau\ ;$$

An application of the divergence theorem to the new volume integral yields

$$I_1 = \iint_S \big[s\hat{W}_i\hat{\sigma}_{ij}n_j - \alpha_1\hat{W}_jn_j - s\beta_2\hat{E}_{mm}\hat{w}_jn_j - 2s\beta_3\hat{E}_{ij}\hat{w}_in_j$$

$$+ \bar{\varphi}_2\beta_1\hat{E}_{mm}\hat{v}_jn_j\big]\, d\sigma$$

$$+ \iiint_B \big[s\hat{w}_i(2\beta_3\hat{E}_{ij,j} + \beta_2\hat{E}_{jj,i}) - \bar{\varphi}_2\beta_1\hat{E}_{jj,i}\hat{v}_i\big]\, d\tau\ .$$

If we now use $(2.1)_2'$, $(2.5)_1'$ and (3.1) to eliminate $\hat{E}_{ij}$, then

$$I_1 = \iint_S \big[s\hat{W}_i\hat{\sigma}_{ij}n_j - \alpha_1\hat{W}_jn_j - s\hat{w}_in_j(\hat{S}_{ij} - \alpha_1\delta_{ij} + \bar{\varphi}_2\beta_1\hat{F}_{kk}\delta_{ij})$$
$$+ \bar{\varphi}_2\beta_1\hat{E}_{kk}\hat{v}_in_i\big]\, d\sigma$$

$$+ \iiint_B \big[s\hat{w}_i(\hat{S}_{ij,j} + \frac{\bar{\varphi}_2\beta_1}{s}\hat{F}_{kk,i}) - \bar{\varphi}_2\beta_1\hat{E}_{jj,i}\hat{v}_i\big]\, d\tau\ .$$

Finally, use (3.1), $(2.1)_2'$, $(2.2)_1'$, $(2.5)_3'$ and (2.6)' to obtain

$$I_1 = \iint_S \big[s\hat{W}_i\hat{\sigma}_{ij}n_j - \alpha_1\hat{W}_jn_j - s\hat{w}_in_j(\hat{S}_{ij} - \alpha_1\delta_{ij} + \bar{\varphi}_2\beta_1\hat{F}_{kk}\delta_{ij})$$

$$+ \bar{\varphi}_2\beta_1\hat{E}_{kk}\hat{v}_in_i\big]\, d\sigma$$

$$+ \iiint_B \big[s\hat{w}_i\big[\frac{\bar{\varphi}_2\beta_1}{s}\hat{F}_{kk,i} - \bar{\varphi}_1\hat{F}_i + \bar{\varphi}_1(s^2\hat{W}_i - sW_i^{(1)} - W_i^{(2)})$$

$$+ \alpha(s\hat{W}_i - W_i^{(1)} - \hat{V}_i) - \frac{\bar{\varphi}_2\alpha_1}{\bar{\varphi}}\hat{E}_{kk,i} - \frac{\bar{\varphi}_1\bar{\varphi}_2\alpha_2}{\bar{\varphi}s}\hat{F}_{kk,i}\big]$$

$$- \bar{\varphi}_2 \beta_1 \hat{E}_{kk,i} \hat{v}_i \Big] \, d\tau \, , \tag{3.9}$$

its final form.

Eliminating the details, which are similar to I_1, we give the forms forms which the remaining integrals in (3.8) take. The integrals I_2, I_3, and I_4 become

$$I_2 = \iint_S \Big[\frac{\bar{\varphi}_1 \bar{\varphi}_2 \alpha_2}{\bar{\varphi}} (\hat{E}_{kk} \hat{v}_i n_i - \hat{W}_i \hat{F}_{kk} n_i) + \frac{\bar{\varphi}_2 \alpha_1}{\bar{\varphi}} (s \hat{W}_i \hat{E}_{kk} n_i - s \hat{W}_i \hat{e}_{kk} n_i) \Big] \, d\sigma$$

$$+ \iiint_B \Big[\alpha s \hat{W}_i (s \hat{w}_i - w_i^{(1)} - v_i^{(1)}) - \frac{\bar{\varphi}_1 \bar{\varphi}_2 \alpha_2}{\bar{\varphi}} \hat{E}_{kk,i} \hat{v}_i - \frac{s \bar{\varphi}_2 \alpha_1}{\bar{\varphi}} \hat{w}_i \hat{E}_{mm,i} \Big] \, d\tau \tag{3.10}$$

$$I_3 = \iint_S \Big[\hat{v}_i \hat{\pi}_{ij} n_j + \frac{\bar{\varphi}_2 \alpha_2}{s} \hat{v}_i n_i - \hat{v}_i n_j \Big(\hat{P}_{ij} + \frac{\bar{\varphi}_2 \alpha_2}{s} \delta_{ij} + \gamma_2 \delta_{ij} \hat{E}_{kk} \Big) + \gamma_2 \hat{F}_{kk} \hat{w}_i n_i \Big] \, d\sigma$$

$$+ \iiint_B \Big[-\frac{\gamma_1 \bar{\varphi}_2}{s} \hat{F}_{mm,i} \hat{v}_i - \gamma_2 \hat{F}_{mm,i} \hat{w}_i + \hat{v}_i \Big\{ \gamma_2 \hat{E}_{kk,i} - \frac{\gamma_1 \bar{\varphi}_2}{s} \hat{F}_{mm,i} + \bar{\varphi}_2 (- V_i^{(1)} + s \hat{V}_i) - \bar{\varphi}_2 \hat{G}_i + \frac{\bar{\varphi}_1 \bar{\varphi}_2 \alpha_2}{s \bar{\varphi}} \hat{F}_{kk,i} + \frac{\bar{\varphi}_2 \alpha_1}{\bar{\varphi}} \hat{E}_{kk,i} - s \alpha (- W_i^{(1)} + s \hat{W}_i - \hat{V}_i) \Big\} \Big] \, d\tau \, , \tag{3.11}$$

$$I_4 = \iint_S \Big[\frac{\bar{\varrho}_1\bar{\varrho}_2\alpha_2}{\bar{\varrho}\, s} (\hat{F}_{kk}\hat{v}_i n_i - \hat{V}_i \hat{f}_{kk} n_i)$$

$$+ \frac{\bar{\varrho}_2\alpha_1}{\bar{\varrho}} (\hat{F}_{kk}\hat{w}_i n_i - \hat{V}_i \hat{e}_{kk} n_i)\Big]\, d\sigma$$

$$+ \iiint_B \Big[\alpha \hat{V}_i (s\hat{w}_i - w_i^{(1)} - \hat{w}_i) + \frac{\bar{\varrho}_1\alpha_2\bar{\varrho}_2}{s\,\bar{\varrho}} \hat{v}_i F_{kk,i}$$

$$- \frac{\bar{\varrho}_2\alpha_1}{\bar{\varrho}} \hat{w}_i \hat{F}_{kk,i}\Big]\, d\tau \, . \qquad (3.12)$$

Expressions (3.9) to (3.12) are now used in (3.7) and volume and surface surface integrals are collected. If new we recall the material identity (2.8) and if we apply (3.1) to the boundary conditions (2.7), (2.7)' and use these results in the surface integral we achieve

$$\hat{L}_1 + \hat{L}_2 = 0 \qquad (3.13)$$

where

$$\hat{L}_1 = \iiint_B \Big[s\hat{W}_i (\bar{\varrho}_1 \hat{f}_i + \bar{\varrho}_1 w_i^{(2)} + \alpha w_i^{(1)} + s\bar{\varrho}_1 w_i^{(1)})$$

$$+ \hat{V}_i (\bar{\varrho}_2 \hat{g}_i + \bar{\varrho}_2 v_i^{(1)} - \alpha w_i^{(1)})$$

$$- s\hat{w}_i (\bar{\varrho}_1 \hat{F}_i + \bar{\varrho}_1 W_i^{(2)} + \alpha W_i^{(1)} + s\bar{\varrho}_1 W_i^{(1)})$$

$$- \hat{v}_i (\bar{\varrho}_2 \hat{G}_i + \bar{\varrho}_2 V_i^{(1)} - \alpha W_i^{(1)})\Big]\, d\tau \, , \qquad (3.14)$$

$$\hat{L}_2 = \iint_S \Big[\tfrac{1}{2}(s\hat{W}_i + \hat{V}_i)(\hat{t}_i + \frac{\bar{\varrho}_2\alpha_2 - \alpha_1}{s} \delta_{ij} n_j)$$

$$+ \tfrac{1}{2} \hat{R}_i (\hat{\sigma}_{ij} - \hat{\pi}_{ij} - \frac{\alpha_1 + \bar{\varrho}_2\alpha_2}{s} \delta_{ij} + \frac{2\,\bar{\varrho}_1\bar{\varrho}_2\alpha_2}{\bar{\varrho}\, s} \hat{f}_{kk}\delta_{ij}$$

$$+ \frac{2\,\bar{\varrho}_2\alpha_1}{\bar{\varrho}} \hat{e}_{kk}\delta_{ij}) n_j$$

$$- \frac{1}{2} (s\hat{w}_i + \hat{v}_i)(\hat{T}_i + \frac{\bar{\varrho}_2\alpha_2 - \alpha_1}{s} \delta_{ij}n_j)$$

$$- \frac{\hat{r}_i}{2} (S_{ij} - P_{ij} - \frac{\alpha_1 + \bar{\varrho}_2\alpha_2}{s} \delta_{ij} + \frac{2\,\bar{\varrho}_1\bar{\varrho}_2\alpha_2}{\bar{\varrho}\, s} \hat{F}_{kk}\delta_{ij})$$

$$+ \frac{2\,\bar{\varrho}_2\alpha_1}{\bar{\varrho}} \hat{E}_{kk}\delta_{ij})n_j\Big]\, d\sigma \,; \qquad (3.15)$$

Equation (3.13) is the statement of the reciprocal in the transformed variables. A direct inversion to real time yields the final form of the theorem and this can be accomplished by means of the convolution (3.2). We give several versions that are useful.

A. <u>Zero initial data</u>. If the initial conditions (2.6), (2.6)' are homogeneous then by (3.2) and (3.13) to (3.15) we obtain

$$\int_0^t \iiint_B \Big[\bar{\varrho}_1 F_i(x,t-\xi) \frac{\partial w_i(x,\xi)}{\partial\xi} + \bar{\varrho}_2 G_i(x,t-\xi) v_i(x,\xi)\Big] d\tau\, d\xi$$

$$+ \int_0^t \iint_S \Big[\frac{1}{2} R_i(x,t-\xi) n_j(x)\Big\{\sigma_{ij}(x,\xi) - \pi_{ij}(x,\xi)$$

$$- (\alpha_1 + \bar{\varrho}_2\alpha_2)\delta_{ij} - \frac{2\,\bar{\varrho}_1\alpha_2}{\bar{\varrho}} \eta(x,\xi)\delta_{ij} + \frac{2\,\bar{\varrho}_2\alpha_1}{\bar{\varrho}} e_{kk}(x,\xi)\delta_{ij}\Big\}$$

$$+ \frac{1}{2} (\frac{\partial w_i(x,\xi)}{\partial\xi} + v_i(x,\xi))\Big\{T_i(x,t-\xi) \;[illegible]$$

$$+ (\bar{\varrho}_2\alpha_2 - \alpha_1)\delta_{ij}n_j(x)H(t-\xi)\Big\}\Big]\, d\sigma\, d\xi$$

$$=$$

$$\int_0^t \iiint \Big[\bar{\varrho}_1 f_i(x,t-\xi) \frac{\partial W_i(x,\xi)}{\partial\xi} + \bar{\varrho}_2 g_i(x,t-\xi) V_i(x,\xi)\Big] d\tau\, d\xi$$

$$+ \int_0^t \iint_S \Big[\frac{1}{2} r_i(x,t-\xi) n_j(x)\Big\{S_{ij}(x,\xi) - P_{ij}(x,\xi)$$

$$- (\alpha_1 + \bar{\varrho}_2\alpha_2)\delta_{ij} - \frac{2\,\bar{\varrho}_1\alpha_2}{\bar{\varrho}} E(x,\xi)\delta_{ij} + \frac{2\,\bar{\varrho}_2\alpha_1}{\bar{\varrho}} E_{kk}(x,\xi)\delta_{ij}\Big\}$$

$$+ \frac{1}{2} (\frac{\partial W_i(x,\xi)}{\partial\xi} + V_i(x,\xi)) \Big\{ t_i(x,t-\xi)$$

$$+ (\bar{\varrho}_2\alpha_2 - \alpha_1)\delta_{ij}n_j(x)H(t-\xi)\Big\}\Big]\, d\sigma\, d\xi \;. \qquad (3.16)$$

B. Infinite region. If in place of (2.7) and (2.7)' we use the condition that velocities and stresses vanish as distance increases from the origin then from (3.16) there remains

$$\int_0^t \iiint_{-\infty}^{+\infty} \left[\bar{\rho}_1 F_i(x,t-\xi) \frac{\partial w_i(x,\xi)}{\partial \xi} + \bar{\rho}_2 G_i(x,t-\xi) v_i(x,\xi) \right] d\tau \, d\xi$$

$$=$$

$$\int_0^t \iiint_{-\infty}^{+\infty} \left[\bar{\rho}_1 f_i(x,t-\xi) \frac{\partial W_i(x,\xi)}{\partial \xi} + \bar{\rho}_2 g_i(x,t-\xi) V_i(x,\xi) \right] d\tau \, d\xi \quad . \qquad (3.17)$$

It is this version that we will use in section 4.

C. Single constituent. If one of the constituents is absent then from (3.13) to (3.15) we obtain a reciprocity relation valid for a linear elastic solid or a linearly viscous fluid alone.

Suppose first that the solid is absent. Then $\rho_1 = 0$, $\rho_2 = \rho$ and the fluid equations are obtained from (2.1) to (2.7) by equating to zero the constants

α_1 , β_1 , β_2 , β_3 , γ_1 , γ_2 and α

and by identifying λ , μ as the viscosities and $\bar{\rho}\alpha_2 = \bar{p}$ with the fluid pressure in the rest state.

Making these adjustment and replacing the boundary conditions (2.7) by either

$$\pi_{ij} n_j = t_i \quad \text{on } S \qquad (3.18a)$$

or

$$v_i = r_i \text{ on } S \qquad (3.18b)$$

then, using (3.18a) and zero initial data, (3.14) and (3.15) yield

$$\int_0^t \iiint_B \bar{\rho} g_i(x,t-\xi) V_i(x,\xi)\, d\tau\, d\xi$$

$$+ \int_0^t \iint_S V_i(x,\xi) \left[t_i(x,t-\xi) + \bar{p} n_i(x) H(t-\xi) \right] d\sigma\, d\xi$$

$$=$$

$$\int_0^t \iiint_B \bar{\rho} G_i(x,t-\xi) v_i(x,\xi)\, d\tau\, d\xi$$

$$+ \int_0^t \iint_S v_i(x,\xi) \left[T_i(x,t-\xi) + \bar{p} n_i(x) H(t-\xi) \right] d\sigma\, d\xi\,. \qquad (3.19)$$

Similarly, if the fluid component is absent, then $\rho_1 = \rho$, $\rho_2 = 0$, and we set equal to zero

α_2, λ, μ, γ_1, γ_2, α, α_1 and β_1.

We identify β_2, β_3 as the Lame constants and replace the boundary conditions (2.7) by either $\sigma_{ij} n_j = t_i$ or $w_i = r_i$ on S.

Hence, using zero initial data, and prescribing traction boundary conditions on S yields from (3.13) to (3.15)

$$\int_0^t \iiint_B \bar{\rho} W_i(x,\xi) f_i(x,t-\xi)\, d\tau\, d\xi$$

$$+ \int_0^t \iint_S W_i(x,\xi) t_i(x,t-\xi)\, d\sigma\, d\xi$$

$$=$$

$$\int_0^t \iiint_B \bar{\rho} w_i(x,\xi) F_i(x,t-\xi)\, d\tau\, d\xi$$

$$+ \int_0^t \iint_S w_i(x,\xi) T_i(x,t-\xi)\, d\sigma\, d\xi \qquad (3.20)$$

which is the standard form of the Betti-Rayleigh theorem [6].

4. <u>Motion of a mixture of a linear elastic solid and viscous fluid due to a moving point load.</u>

As a preliminary problem we seek $w_i(x,t)$, $i = 1,2,3$ satisfying (2.1) to (2.5) and homogeneous initial conditions,

$$w_i(x,0) = \dot{w}_i(x,0) = v_i(x,0) = 0, \quad i = 1,2,3$$
$$\eta(x,0) = 0 \tag{4.1}$$

for the infinite region defined by $-\infty < x_1, x_2, x_3 < +\infty$ and $t \geq 0$. In place of (2.7) we require $w_i(x,t)$, $v_i(x,t)$, σ_{ij} and π_{ij} to vanish as $(x_i x_i)^{\frac{1}{2}}$ increases without bound.

In particular we consider body forces to be given by

$$\vec{f}(x,x_0,t) = \vec{a}_1\, \delta(x_1 - x_{10})\, \delta(x_2 - x_{20})\, \delta(x_3 - x_{30})\, \delta(t)$$
$$\vec{g}(x,x_0,t) = \vec{a}_1\, \delta(x_1 - x_{10})\, \delta(x_2 - x_{20})\, \delta(x_3 - x_{30})\, \delta(t) \tag{4.2}$$

where $\vec{a}_1$ is the unit vector in the x_1 direction and δ is the Dirac delta function. The symbol $\vec{f}$ stands for the usual vector statement $\vec{f} = f_i \vec{a}_i$.

The physical problem described above corresponds to that of finding $\vec{w}(x,x_0,t)$ at place x and at time t due to a unit force applied at x_0 in the direction parallel to the x_1 axis at $t = 0$. The vectors sought clearly play the role of Greens functions for the theory used here.

The problem has been examined in [9] and [10] for the body force loading (4.2) with $\vec{g} = \vec{0}$ and with several restrictions on the material constants. In [9], the solution was given for α, the diffusive resistance parameter, zero for the cases when the fluid is inviscid or viscous. In [10], the same problem was examined by using (1) an early time approximation and (2) a perturbation expansion for α small. In addition to small α we used the restrictions, which can be removed

$$\bar{\rho}_2\beta_1 = \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}}, \quad \bar{\rho}_2\gamma_1 = \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}}, \quad \bar{\rho}_2\alpha_2 = \alpha_1. \tag{4.3}$$

This last case is used here.

From [10] we take the solution $w_i(x,x_0,t)$, valid for α small and subject to (2.1) - (2.5), (4.1), (4.3) and (4.2) with $\vec{g} = \vec{0}$, to be terms up to order α

$$w_1(x,x_0,t) = \frac{t}{4\pi R_0^2} F_1(x,x_0,t), \tag{4.4}$$

$$w_\nu(x,x_0,t) = \frac{t(x_1 - x_{10})(x_\nu - x_{\nu 0})}{4\pi R_0^4} F_2(x,x_0,t), \quad \nu = 2,3 \tag{4.5}$$

where

$$F_1(x,x_0,t) = \frac{(x_1 - x_{10})^2}{c_1 R_0^2}\delta(t - \frac{R_0}{c_1}) + \left[1 - \frac{(x_1 - x_{10})^2}{R_0^2}\right]\frac{1}{v_s}\delta(t - \frac{R_0}{v_s})$$

$$+ \frac{1}{R_0}\left[\frac{3(x_1 - x_{10})^2}{R_0^2} - 1\right]\left[1 - \frac{\alpha}{2\bar{\rho}_1}(t - \frac{R_0}{c_1})\right] H(t - \frac{R_0}{c_1})$$

$$- \frac{1}{R_0}\left[\frac{3(x_1 - x_{10})^2}{R_0^2} - 1\right]\left[1 - \frac{\alpha}{2\bar{\rho}_1}(t - \frac{R_0}{v_s})\right] H(t - \frac{R_0}{v_s}), \tag{4.6}$$

$$F_2(x,x_0,t) = \frac{1}{c_1}\delta(t - \frac{R_0}{c_1}) - \frac{1}{v_s}\delta(t - \frac{R_0}{v_s})$$

$$+ \frac{3}{R_0}\left[1 - \frac{\alpha}{2\bar{\rho}_1}(t - \frac{R_0}{c_1})\right] H(t - \frac{R_0}{c_1}) - \frac{3}{R_0}$$

$$- \frac{3}{R_0}\left[1 - \frac{\alpha}{2\bar{\rho}_1}(t - \frac{R_0}{v_s})\right] H(t - \frac{R_0}{v_s}). \tag{4.7}$$

The wave speeds $c_1 > v_s > 0$ are associated with the elastic component and are defined by $c_1^2 = k_1/\bar{\varrho}_1$, $v_s^2 = \beta_3/\bar{\varrho}_1$. R_0 is the spherical distance measured from the point x_0.

Expressions for the fluid velocity components were also found in [10] but since we do not intend to use them here they will not be reproduced.

Let us now consider the same problem with $\vec{f} = 0$ in (4.2). Following the methods presented in [10] we first translate the origin to the point x_0. Then using the Fourier exponential transform on each of the space variables and the Laplace transform on the time variable the equations (2.1) - (2.5), (4.1) and (4.3) yield

$$\begin{aligned} &\Delta_4 \hat{w}_m + (k_1 - \beta_3)\lambda_m \lambda_j \hat{w}_j - \alpha \hat{v}_m = 0, \\ &\Delta_1 \hat{v}_m + (k_2 - \mu)\lambda_m \lambda_k \hat{v}_k - \alpha p \hat{w}_m = \frac{\bar{\varrho}_2 \delta_{m1}}{(2\pi)^{3/2}} \end{aligned} \qquad (4.8)$$

$$\begin{aligned} &\Delta_1 \equiv \Delta_1(\lambda_j \lambda_j) = \mu \lambda_j \lambda_j + \alpha + \bar{\varrho}_2 p \\ &\Delta_4 \equiv \Delta_4(\lambda_j \lambda_j) = \beta_3 \lambda_j \lambda_j + \alpha p + \bar{\varrho}_1 p^2 \end{aligned} \qquad (4.9)$$

The notation $\hat{w}_m$ represents

$$\hat{w}_m(\lambda_1, \lambda_2, \lambda_3, p) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{+\infty} \int_0^{\infty} e^{-pt - i\lambda_j x_j} w_m(x_1, x_2, x_3, t)\, dt\, dx_1\, dx_2\, dx_3$$

If we multiply (4.8) by λ_m and sum, the resulting expressions can be used to eliminate the terms $\lambda_m \hat{w}_m$, $\lambda_m \hat{v}_m$. Doing this and solving, $\hat{w}_M$ is found to be

$$[\Delta_1\Delta_4 - \alpha^2 p]\hat{w}_m = \frac{\alpha\bar{\varphi}_2}{(2\pi)^{3/2}[\Delta_2\Delta_3 - \alpha^2 p]}[\delta_{m1}\{\Delta_1\Delta_4 - \alpha^2 p +$$

$$(\lambda_2^2 + \lambda_3^2)[(k_2 - \mu)\Delta_3 + (k_1 - \beta_3)\Delta_1]\}$$

$$- \lambda_1\lambda_\nu\delta_{m\nu}\{(k_2 - \mu)\Delta_3 + (k_1 - \beta_3)\Delta_1\}], \quad \nu = 2, 3, \qquad (4.10)$$

with no sum on ν, and

$$\Delta_2(\lambda_j\lambda_j) = k_2\lambda_j\lambda_j + \alpha + \bar{\varphi}_2\, p$$
$$\Delta_3(\lambda_j\lambda_j) = k_1\lambda_j\lambda_j + \alpha p + \bar{\varphi}_1\, p^2 \qquad (4.11)$$

Fourier inversion of (4.10) is accomplished in two steps. The denominator of (4.10) is factored into quadratics in λ_1^2 and (4.10) is expanded by partial fractions involving these factors. Inversion with respect to λ_1 is then easily performed. The inversion with respect to λ_2, λ_3 is next and is made easier if one exploits rotational symmetry of the expressions.

The net result of those two operations leaves the function $\hat{w}_m(x_1, x_2, x_3, p)$ with the inversion of the Laplace transform remaining. At this stage we introduce the perturbation in small α and retain only the leading term in the expansion. We have then for $\hat{w}_m(x, p)$, to terms of order α,

$$\hat{w}_m(x,p) = \frac{\alpha\bar{\varphi}_2\delta_{m1}}{4\pi\bar{\varphi}_1}\Bigg[\frac{1}{k_2 p\left(p - \frac{\bar{\varphi}_2 c_1^2}{k_2}\right)}\Big\{\left[\frac{1}{R} - \frac{1}{p_4} f_1(p_4, x_1, R)\right] e^{-p_4 R}$$

$$- \left[\frac{1}{R} + \frac{1}{p_3} f_1(p_3, x_1, R)\right] e^{-p_3 R}\Big\} +$$

$$+ \frac{1}{\mu p(p - \frac{\bar{\varphi}_2 v_s^2}{\mu})} \left\{ \frac{1}{p_1} f_1(p_1, x_1, R)\, e^{-p_1 R} + \frac{1}{p_2} f_1(p_2, x_1, R)\, e^{-p_2 R} \right\}\Bigg]$$

$$- \frac{\alpha \bar{\varphi}_2 x_1 (\delta_{m2} x_2 + \delta_{m3} x_3)}{4\pi \bar{\varphi}_1 R^3} \Bigg[\frac{1}{k_2 p(p - \frac{\bar{\varphi}_2 c_1^2}{k_2})} \Big\{ f_2(p_3, R)\, e^{-p_3 R}$$

$$+ f_2(p_4, R)\, e^{-p_4 R} \Big\} - \frac{1}{\mu p(p - \frac{\bar{\varphi}_2 v_s^2}{\mu})} \Big\{ f_2(p_1, R)\, e^{-p_1 R}$$

$$+ f_2(p_2, R)\, e^{-p_2 R} \Big\}\Bigg]. \qquad (4.12)$$

In (4.12) the p_k, $1 \leq k \leq 4$, represent the factors involved in the λ_1 inversion and to the first order in α are defined by

$$p_1 = \frac{p}{v_s}, \quad p_2 = \left(\frac{\bar{\varphi}_2\, p}{\mu}\right)^{\frac{1}{2}}, \quad p_3 = \frac{p}{c_1}, \quad p_4 = \left(\frac{\bar{\varphi}_2\, p}{k_2}\right)^{\frac{1}{2}}. \qquad (4.13)$$

In addition we have also used $R = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$,

$$f_1(p_k, x_1, R) = \frac{-p_k}{R}\left(1 - \frac{x_1^2}{R^2}\right) + \frac{1}{R^2}\left(\frac{3x_1^2}{R^2} - 1\right)\left(1 + \frac{1}{Rp_k}\right),$$

$$(4.14)$$

$$f_2(p_k, R) = 1 + \frac{3}{Rp_k} + \frac{3}{R^2 p_k^2}, \quad k = 1,2,3,4.$$

A direct term by term for the leading terms of (4.12))gives the displacement for early time;

$$\hat{w}_m(x,t) = \frac{\alpha \bar{\varphi}_2 \delta_{m1}}{4\pi \bar{\varphi}_1} G_1(x,t) + \frac{\alpha \bar{\varphi}_2 (\delta_{m2} x_2 + \delta_{m3} x_3)}{4\pi \bar{\varphi}_1} G_2(x,t), \qquad (4.15)$$

where

$$\theta_1(x,t) = -\frac{x_1^2(t-\frac{R}{c_1})}{k_2R^3}\left\{1+\frac{1}{2}\left(\frac{3c_1}{R}+\frac{\bar{\vartheta}_2 c_1^2}{k_2}-\frac{c_1R}{x_1^2}\right)(t-\frac{R}{c_1})\right\}H(t-\frac{R}{c_1})$$

$$+\frac{(x_1^2-R^2)(t-\frac{R}{\nu_s})}{\mu R^3}\left\{1+\frac{1}{2}\left(\frac{\nu_s(R^2-3x_1^2)}{R^3(R^2-x_1^2)}+\frac{\bar{\vartheta}_2\nu_s^2}{\mu}\right)(t-\frac{R}{\nu_s})\right\}H(t-\frac{R}{\nu_s})$$

$$+\frac{4t(2R^2-x_1^2)}{k_2R^3}\left\{i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{k_2t}\right)^{\frac{1}{2}}\right]-\frac{2(R^2-3x_1^2)}{R(2R^2-x_1^2)}\left[\frac{k_2t}{\bar{\vartheta}_2}\right]^{\frac{1}{2}} i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{k_2t}\right)^{\frac{1}{2}}\right]\right\}$$

$$+\frac{4t(x_1^2-R^2)}{\mu R^3}\left\{i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{\mu t}\right)^{\frac{1}{2}}\right]+\frac{2(3x_1^2-R^2)}{R(x_1^2-R^2)}\left[\frac{\mu t}{\bar{\vartheta}_2}\right]^{\frac{1}{2}} i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{\mu t}\right)^{\frac{1}{2}}\right]\right\},$$

$$\theta_2(x,t) = -\frac{(t-\frac{R}{c_1})}{k_2R^3}\left\{1+\frac{1}{2}\left(\frac{3c_1}{R}+\frac{\bar{\vartheta}_2c_1^2}{k_2}\right)(t-\frac{R}{c_1})\right\}H(t-\frac{R}{c_1})$$

$$+\frac{(t-\frac{R}{\nu_s})}{\mu R^3}\left\{1+\frac{1}{2}\left(\frac{\bar{\vartheta}_2\nu_s^2}{\mu}+\frac{3\nu_s}{R}\right)(t-\frac{R}{\nu_s})\right\}H(t-\frac{R}{\nu_s})$$

$$-\frac{4t}{k_2R^3}\left\{i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{k_2t}\right)^{\frac{1}{2}}\right]+\frac{6}{R}\left[\frac{k_2t}{\bar{\vartheta}_2}\right]^{\frac{1}{2}} i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{k_2t}\right)^{\frac{1}{2}}\right]\right\}$$

$$+\frac{4t}{\mu R^3}\left\{i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{\mu t}\right)^{\frac{1}{2}}\right]+\frac{6}{R}\left[\frac{\mu t}{\bar{\vartheta}_2}\right]^{\frac{1}{2}} i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\vartheta}_2}{\mu t}\right)^{\frac{1}{2}}\right]\right\}. \qquad (4.16)$$

The new notation introduced into (4.16) is the repeated integrals of the error function $i^k\mathrm{Erfc}(A)$ defined by

$$i^k\mathrm{Erfc}(A) = \int_A^{\infty} i^{k-1}\mathrm{Erfc}\ t\ dt\ , \quad k = 0,1,2,\ldots$$

when $k = 0$ we have $i^0\mathrm{Erfc}(A) = \mathrm{Erfc}(A)$.

We shall now consider the solid displacement and fluid velocity field produced by a moving-point force. Let a point force be suddenly applied on the solid constituent at the origin at time $t = 0$ and maintained at a constant velocity v along positive x_3-axis, that is, we let in (3.14)

$$\vec{F}(x,t) = \vec{a}_3\, \delta(x_1)\delta(x_2)\delta(x_3 - vt)$$
$$\vec{G}(x,t) = 0\ . \tag{4.17}$$

The displacement field of the solid component is to be found first. Since we want to have the displacement field and not the velocity field of the solid component, we go back to the reciprocal statement in the transformed variables (3.13) - (3.15) instead of utilising (3.17). With the aid of (4.2) with $\vec{g} = 0$ and (4.17), we get a direct inversion of (3.13) - (3.15) to real time in the final form

$$\int_0^t \iiint_{-\infty}^{+\infty} \bar{\varphi}_1 F_1(x,t-\xi) w_1 \, dr d\xi = \int_0^t \iiint_{-\infty}^{+\infty} \bar{\varphi}_1 f_1(x,t-\xi) W_1 \, dr d\xi\ . \tag{4.18}$$

Eventhough we are dealing with the mixture, the relation, (4.18), between the displacement fields subjected to (4.2) with $\vec{g} = 0$ and (4.17) appears to be that of the single constituent (3.20). To determine the solid displacement W_1 subject to (4.17), we substitute (4.2) with $\vec{g} = 0$ and (4.4) - (4.7) into (4.18). Then, performing the integration gives

$$W_1 = \frac{1}{4\pi}\int_0^t \frac{(t-\xi)x_1(x_3 - v\xi)F_2(x_1,x_2,x_3;0,0,v\xi,t-\xi)}{R^4(\xi)}\, d\xi\ . \tag{4.19$_1$}$$

Similarly, employing $\vec{f}(x,x_0,t) = \vec{a}_2\delta(x_1-x_{10})\delta(x_2-x_{20})\delta(x_3-x_{30})\delta(t)$, and then $\vec{f}(x,x_0,t) = \vec{a}_3\delta(x_1-x_{10})\delta(x_2-x_{20})\delta(x_3-x_{30})\delta(t)$, then, performing the integration (4.18) give

$$W_2 = \frac{1}{4\pi}\int_0^t \frac{(t-\xi)x_2(x_3-v\xi)F_2(x_1,x_2,x_3,0,0,v\xi,t-\xi)}{R^4(\xi)}\,d\xi \qquad (4.19)_2$$

$$W_3 = \frac{1}{4\pi}\int_0^t \frac{(t-\xi)F_1(x_3,x_2,x_1,v\xi,0,0,t-\xi)}{R^2(\xi)}\,d\xi \qquad (4.19)_3$$

where we used the notation $R(\xi) = \left\{x_1^2+x_2^2+(x_3-v\xi)^2\right\}^{\frac{1}{2}}$, $F_1(x_1,x_2,x_3,x_{10},x_{20},x_{30},t) = F_1(\vec{x},\vec{x}_0,t)$ with vector notation being understood in (4.6) and (4.7).

The velocity field of the fluid component may be easily found by the reciprocal relation (3.17). Considering the initial and regular conditions, (4.2) with $\vec{f} = 0$, and (4.17), we get from (3.17)

$$\int_0^t\iiint_{-\infty}^{+\infty} \bar{\rho}_1F_1(x,t-\xi)\frac{\partial w_1(x,\xi)}{\partial\xi}\,d\tau d\xi = \int_0^t\iiint_{-\infty}^{+\infty} \bar{\rho}_2g_1(x,t-\xi)V_1(x,\xi)\,d\tau d\xi \qquad (4.20)$$

where $\frac{\partial w_1(x,\xi)}{\partial\xi}$ is the derivative with respect to time variable ξ from the displacement (4.15).

To determine V_1 subject to (4.17), we substitute (4.2) with $\vec{f} = 0$ and (4.15) into (4.20), then we get by performing the integration

$$V_1 = \frac{\alpha}{4\pi}\int_0^t x_1x_3\,H_2[R(\xi),t-\xi]\,d\xi \,. \qquad (4.21)_1$$

Similarly, by employing $\vec{g}(x,x_0,t) = \vec{a}_2\delta(x_1-x_{10})\delta(x_2-x_{20})\delta(x_3-x_{30})\delta(t)$, $\vec{g}(x,x_0,t) = \vec{a}_3\delta(x_1-x_{10})\delta(x_2-x_{20})\delta(x_3-x_{30})\delta(t)$, and performing the integration (4.20) we get

$$V_2 = \frac{\alpha}{4\pi}\int_0^t x_2 x_3\, H_2[R(\xi), t-\xi]\, d\xi \qquad (4.21)_2$$

$$V_3 = \frac{\alpha}{4\pi}\int_0^t H_1(x_3, x_2, x_1, v\xi, 0, 0, t-\xi)\, d\xi \qquad (4.21)_3$$

where $H_i(\vec{x},\vec{x}_0,t) = H_i(\vec{x}-\vec{x}_0,t)$ are defined to be from (4.15) as following

$$H_1(x,t) = \frac{\partial G_1(x,t)}{\partial t}$$

$$H_2(x,t) = \frac{\partial G_2(x,t)}{\partial t} \qquad (4.22)$$

For the integration of (4.19) and (4.21), a careful consideration must be given to the behavior of the function

$$g_1(\xi) = t - \xi - \frac{R(\xi)}{c_1}, \qquad g_2(\xi) = t - \xi - \frac{R(\xi)}{v_s},$$

because the integral of (4.19) and (4.21) depend upon the zeroes of $g_1(\xi)$, $g_2(\xi)$ and upon the interval where $g_1(\xi)$, $g_2(\xi)$ take positive values. Since the behavior of a similar function for the elastic solid was illustrated in [3], we omit the duplication.

The final solid displacement and fluid velocity fields are found to be in polar cylindrical coordinates (r, θ, z):

$$W_r = \frac{1}{4\pi v^2}\Big\{\Big[-\frac{(x_3-vt)}{rR_{c_1}} + \frac{r^2(x_3-vt)+x_3^3}{rR^3} + \frac{3\pi r v^2}{2\bar{\rho}_1}T_1(c_1,x,t)\Big\} H(t-\frac{R}{c_1})$$

$$-\Big[\frac{2(x_3-vt)}{rR_{c_1}} + \frac{3\pi r v^2}{2\bar{\rho}_1}T_2(c_1,x,t)\Big] S(c_1)$$

$$-\left[\left\{\frac{(x_3-\upsilon t)}{rR_{\nu_s}}+\frac{r^2(x_3-\upsilon t)+x_3^3}{rR^3}+\frac{3\pi r\upsilon^2}{2\,\bar{\varphi}_1}T_1(\nu_s,x,t)\right\}H(t-\frac{R}{\nu_s})\right.$$

$$\left.+\left\{\frac{2(x_3-\upsilon t)}{rR_{\nu_s}}+\frac{3\pi r\upsilon^2}{2\,\bar{\varphi}_1}T_2(\nu_s,x,t)\right\}\delta(\nu_s)\right] , \qquad (4.23)_1$$

where

$$T_1(c,x,t)=\frac{1}{3\upsilon^3}\left\{[r^2-(x_3-\upsilon t)^2]\left[\frac{(c^2-\upsilon^2)^3}{c^3M_1^3(c,x,t)}-\frac{1}{R^3}\right]+\frac{3}{R}\right.$$

$$+\frac{3}{2}\left(\frac{\upsilon^2-2c^2}{c^3}\right)\frac{(c^2-\upsilon^2)}{M_1(c,x,t)}-\frac{3\upsilon}{2cR}\tan^{-1}M_5(c,x,t)$$

$$\left.+2(x_3-\upsilon t)\left[\frac{(c^2-\upsilon^2)^2M_2(c,x,t)}{c^2M_1^3(c,x,t)}-\frac{x_3}{R^3}-\frac{M_6(c,x,t)}{r^2}\right]\right\}$$

$$=\frac{t}{\pi c\upsilon R^2} ,$$

$$T_2(c,x,t)=-\frac{1}{\upsilon^3}\left\{\frac{2(c^2-\upsilon^2)^3R_c}{3c^2M_1^3(c,x,t)M_3^3(c,x,t)}\left[(x_3-\upsilon t)^2\{\upsilon^2(x_3-\upsilon t)^2\right.\right.$$

$$\left.+r^2(\upsilon^2-2c^2)\}-2(c^2-\upsilon^2)r^4\right]+\frac{\upsilon}{2cr}\tan^{-1}M_{10}(c,x,t)$$

$$\left.+\frac{R_c}{\{r^2+(x_3-\upsilon t)^2\}}\left[\frac{\upsilon^2-2c^2}{c^2}-\frac{2(x_3-\upsilon t)^2}{3r^2}\right]\right\} ,$$

$$W_\theta=0 \qquad (4.23)_2$$

$$W_{x_3}=\frac{1}{4\pi\upsilon^2}\left[\left\{\frac{1}{R_{c_1}}-\frac{(R^2+\upsilon x_3 t)}{R^3}+\frac{\alpha\upsilon^2T_3(c_1,x,t)}{2\,\bar{\varphi}_1}\right\}H(t-\frac{R}{c_1})\right.$$

$$+\left\{\frac{2}{R_{c_1}}+\frac{\alpha v^2}{2\,\bar{\rho}_1}\,T_4(c_1,x,t)\right\}S(c_1)$$

$$-\left\{\frac{\nu_s^2-v^2}{\nu_s^2 R_{\nu_s}}-\frac{(R^2+vx_3t)}{R^3}+\frac{\alpha v^2 T_3(\nu_s,x,t)}{2\,\bar{\rho}_1}\right\}H\!\left(t-\frac{R}{\nu_s}\right)$$

$$-\left\{\frac{2(\nu_s^2-v^2)}{\nu_s^2 R_{\nu_s}}+\frac{\alpha v^2 T_4(\nu_s,x,t)}{2\,\bar{\rho}_1}\right\}S(\nu_s)\Big] \quad , \qquad (4.23)_3$$

where

$$T_3(c,x,t)=\frac{1}{v^3}\Big\{4(x_3-vt)\Big[\frac{c^2-v^2}{cM_1(c,x,t)}\Big\{1-\frac{r^2(c^2-v^2)^2}{2c^2M_1^2(c,x,t)}\Big\}-\frac{3v}{8c}M_8(c,x,t)$$

$$-\frac{1}{R}\Big\{1-\frac{r^2}{2R^2}\Big\}\Big]-3M_6(c,x,t)+\frac{v(x_3-vt)}{2cr}\tan^{-1}M_5(c,x,t)-\frac{2v}{c}\log\left|\frac{cM_1(c,x,t)}{R(c^2-v^2)}\right|$$

$$-\frac{3vr^2}{c}M_9(c,x,t)+\{r^2-(x_3-vt)^2\}\Big\{\frac{(c^2-v^2)^2M_2(c,x,t)}{c^2M_1^3(c,x,t)}-\frac{x_3}{R^3}\Big\}+2\log\left|\frac{c(x_3-vt+R_c)}{(c-v)(R+x_3)}\right|\Big\}$$

$$T_4(c,x,t)=\frac{1}{v^3}\Big\{M_{11}(c,x,t)+2\log\left|\frac{x_3-vt+R_c}{x_3-vt-R_c}\right|-\frac{v(x_3-vt)}{2cr}\tan^{-1}M_{10}(c,x,t)$$

$$+\frac{\{r^2+(x_3-vt)^2\}M_{12}(c,x,t)}{c^2}\Big[3v^2+\frac{2(c^4-v^4)\ r^2+(x_3-vt)^2}{M_1(c,x,t)M_3(c,x,t)}\Big]-\frac{2v}{c}\log\left|\frac{M_1(c,x,t)}{M_3(c,x,t)}\right|\Big\}.$$

Above we used the symbols $r=(x_1^2+x_2^2)^{\frac{1}{2}}$, $R=(r^2+x_3^2)^{\frac{1}{2}}$, $R_c=\{(x_3-vt)^2+(1-\frac{v^2}{c^2})r^2\}^{\frac{1}{2}}$, $H(t-\frac{R}{c})$ is the Heavyside unit step function, $S(c)$ is a function which has the value 1 inside the region $R>ct$, $t-\frac{x_3}{v}-\frac{r}{v}(\frac{v^2}{c^2}-1)^{\frac{1}{2}}>0$ and the value zero outside the region with c playing the role of c_1 or ν_s. We note that $R_c>0$ in each of the region where $H(t-\frac{R}{c})=1$ and $S(c)=1$.

The expressions $M_i(c,x,t)$, $i=1$, to $i=13$, are given on page 27.

The fluid velocity fields are found to be:

$$V_r = -\frac{\alpha r}{4\pi}\left\{J_1(c_1,k_2,x,t)H(t-\frac{R}{c_1}) + J_2(c_1,k_2,x,t)S(c_1)\right\}$$

$$+\frac{\alpha r}{4\pi}\left\{J_1(v_s,\mu,x,t)H(t-\frac{R}{v_s}) + J_2(v_s,\mu,x,t)S(v_s)\right\}$$

$$-\frac{\alpha r}{4\pi} J_3(c_1,k_2,x,t) + \frac{\alpha r}{4\pi} J_3(v_s,\mu,x,t) \;, \qquad (4.24)$$

where

$$J_1(c,k,x,t) = \frac{1}{vk}\left\{\frac{3c}{2v}\left[\frac{(c^2-v^2)^3 M_2(c,x,t)}{c^3 M_1^4(c,x,t)} - \frac{x_3}{R} - \frac{1}{r}\tan^{-1}M_5(c,x,t)\right]\right.$$

$$+\frac{\bar{\vartheta}_2 c^2}{k}\log\left|\frac{cM_1(c,x,t)}{R(c^2-v^2)}\right|$$

$$+\frac{\bar{\vartheta}_2 c^2}{vk}\left[\frac{M_2(c,x,t)}{M_1(c,x,t)} - \frac{x_3}{R} - \log\left|\frac{c(x_3-vt+R_c)}{(c-v)(R+x_3)}\right|\right]$$

$$+\left\{2+\frac{\bar{\vartheta}_2 c^2(x_3-vt)}{vk}\right\}\left[\frac{1}{R} - \frac{(c^2-v^2)}{c\,M_1(c,x,t)}\right]$$

$$\left.+\frac{3c}{2v}(x_3-vt)\left[\frac{1}{R^2} - \frac{(c^2-v^2)^2}{c^2 M_1^2(c,x,t)}\right]\right\},$$

$$J_2(c,k,x,t) = \frac{1}{vk}\left\{\frac{3c}{2v}\left[\frac{(c^2-v^2)^3}{c^3}\left(\frac{M_2(c,x,t)}{M_1^4(c,x,t)} - \frac{M_4(c,x,t)}{M_3^4(c,x,t)}\right) - \frac{1}{r}\tan^{-1}M_{10}(c,x,t)\right]\right.$$

$$+\frac{\bar{\vartheta}_2 c}{k}\left[\log\left|\frac{M_1(c,x,t)}{M_3(c,x,t)}\right| - \frac{c}{v}\log\left|\frac{x_3-vt+R_c}{x_3-vt-R_c}\right|\right]$$

$$2\,M_{11}(c,x,t)$$

$$J_3(c,k,x,t) = \frac{4tx_3}{kR\, M_{13}(x,t)}\, i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\varphi}_2}{kt}\right)^{\frac{1}{2}}\right]$$

$$+ \frac{16t^{\frac{3}{2}}}{(k\bar{\varphi}_2)^{\frac{1}{2}} M_{13}(x,t)} \left\{ \frac{7x_3}{4R^2} + \frac{\upsilon t x_3^2 - R^2(x_3+\upsilon t)}{R^2 M_{13}(x,t)} - \frac{2\upsilon^2 t^2 x_3}{M_{13}^2(x,t)} \right\} i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\varphi}_2}{kt}\right)^{\frac{1}{2}}\right]$$

$V_\theta = 0$, $(4.24)_2$

$$V_{x_3} = \left\{ \frac{\bar{\varphi}_2 c_1((\upsilon x_3 - c_1^2 t) + c_1 R_{c_1}}{k_2^2(\upsilon^2 - c_1^2)} + \frac{1}{\upsilon^2 k_2} \left\{ 2c_1 \log\left| \frac{c_1 M_1(c_1,x,t)}{R(c_1^2-\upsilon^2)} \right| \right.\right.$$

$$- \{\upsilon + \bar{\varphi}_2 c_1^2(x_3-\upsilon t)\} \log\left| \frac{c_1(x_3-\upsilon t+R_{c_1})}{(R+x_3)(c_1-\upsilon)} \right| - \frac{c_1}{2r}\{(x_3-\upsilon t) - 2\upsilon r^2\bar{\varphi}_2\} \tan^{-1} M_5(c_1,x,t)$$

$$+ \{2\upsilon + \bar{\varphi}_2 c_1^2(x_3-\upsilon t)\} M_6(c_1,x,t) + r^2\bar{\varphi}_2 M_7(c_1,x,t) + \frac{3c_1}{2}(x_3-\upsilon t) M_8(c_1,x,t)$$

$$\left.\left. + \frac{3c_1 r^2}{2} M_9(c_1,x,t) + \bar{\varphi}_2 c_1^2 \left\{ \frac{c_1 M_1(c_1,x,t)}{c_1^2-\upsilon^2} - R \right\} \right\} \right\} H\left(t - \frac{R}{c_1}\right)$$

$$+ \left\{ \frac{2\bar{\varphi}_2 c_1^2 R_{c_1}}{k_2^2(\upsilon^2-c_1^2)} + \frac{1}{\upsilon^2 k_2} \left\{ 2c_1 \log\left| \frac{M_1(c_1,x,t)}{M_3(c_1,x,t)} \right| - \{\upsilon + \bar{\varphi}_2 c_1^2(x_3-\upsilon t)\} \log\left| \frac{x_3-\upsilon t+R_{c_1}}{x_3-\upsilon t-R_{c_1}} \right| \right.\right.$$

$$- \frac{c_1}{2r}(x_3-\upsilon t-2r^2\bar{\varphi}_2\upsilon) \tan^{-1} M_{10}(c_1,x,t) + \left[2\upsilon + \bar{\varphi}_2 c_1^2 \left\{ (x_3-\upsilon t) \right.\right.$$

$$\left.\left. + \frac{r^2}{(x_3-\upsilon t)} \right\}\right] M_{11}(c_1,x,t) - 3\upsilon(x_3-\upsilon t)^2 M_{12}(c_1,x,t)$$

$$\left.\left. + \frac{2c_1^4\bar{\varphi}_2 R_{c_1}}{c_1^2-\upsilon^2} \right\} \right\} S(c_1)$$

$$+\left\{\frac{3\nu_s}{4\mu v^2}\left\{\frac{1}{R^3}\left(\frac{r^2}{R}+\frac{x_3(x_3-vt)}{R}+\frac{4vx_3}{3\nu_s}\right)_{r=}\frac{(\nu_s^2-v^2)^2}{\nu_s^4 M_1^3(\nu_s,x,t)}\left(\frac{r^2(\nu_s^2-v^2)^2}{M_1(\nu_s,x,t)}\right.\right.\right.$$

$$\left.\left.+M_2(\nu_s,x,t)\left(\frac{4\nu_s v}{3}+\frac{\nu_s(\nu_s^2-v^2)(x_3-vt)}{M_1(\nu_s,x,t)}\right)\right)+\frac{4}{3}M_9(\nu_s,x,t)-\frac{(x_3-vt)}{6r^2}M_8(\nu_s,x,t)\right\}$$

$$+\frac{1}{\mu v}\left\{\left(1-\frac{\bar{\varphi}_2\nu_s^2(x_3-vt)}{\mu v}\right)M_6(\nu_s,x,t)-\frac{r^2\bar{\varphi}_2}{\mu v}M_7(\nu_s,x,t)\right.$$

$$\left.\left.-\nu_s\left(\frac{r\bar{\varphi}_2}{\mu}+\frac{(x_3-vt)}{8r^3v}\right)\tan^{-1}M_5(\nu_s,x,t)\right\}\right\}H\left(t-\frac{R}{\nu_s}\right)$$

$$+\frac{1}{\mu v}\left\{\frac{(\nu_s^2-v^2)^2M_4(\nu_s,x,t)}{\nu_s^2M_3^4(\nu_s,x,t)}\left(M_3(\nu_s,x,t)+\frac{3(x_3-vt)(\nu_s^2-v^2)}{4v}+\frac{3r^2(\nu_s^2-v^2)^2}{4\nu_s vM_4(\nu_s,x,t)}\right)\right.$$

$$-\frac{(\nu_s^2-v^2)^2M_2(\nu_s,x,t)}{\nu_s^2M_1^4(\nu_s,x,t)}\left(M_1(\nu_s,x,t)+\frac{3(x_3-vt)(\nu_s^2-v^2)}{4v}+\frac{3r^2(\nu_s^2-v^2)^2}{4\nu_s M_2(\nu_s,x,t)}\right)$$

$$+\left(1-\frac{\bar{\varphi}_2\nu_s^2}{\mu v}\left(x_3-vt+\frac{r^2}{(x_3-vt)}\right)\right)M_{11}(\nu_s,x,t)$$

$$\left.-\left(\frac{9r^2-(x_3-vt)^2}{4r^2}\right)M_{12}(\nu_s,x,t)-\left(\frac{\nu_s r\bar{\varphi}_2}{\mu}+\frac{\nu_s(x_3-vt)}{8r^3v}\right)\tan^{-1}M_{10}(\nu_s,x,t)\right\}S(\nu_s)$$

$$+\frac{4t(2R^2-x_3^2)}{k_2R\,M_{13}(x,t)}\,i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\varphi}_2}{k_2t}\right)^{\frac{1}{2}}\right]+\frac{16\,t^{\frac{3}{2}}}{(k_2\bar{\varphi}_2)^{\frac{1}{2}}M_{13}(x,t)}\left(\frac{R^2(x_3-2R^2)-vtx_3^3}{R^2\,M_{13}(x,t)}\right.$$

$$\left.+\frac{2v^2t^2(x_3^2-2R^2)}{M_{13}^2(x,t)}+\frac{3x_3^2}{4R^2}\right)i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\bar{\varphi}_2}{k_2t}\right)^{\frac{1}{2}}\right]$$

$$-\frac{4r^2t}{\mu R\, M_{13}(x,t)}\, i^2\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\tilde{\varphi}_2}{\mu t}\right)^{\frac{1}{2}}\right] + \frac{4t^{\frac{3}{2}}}{(\mu\tilde{\varphi}_2)^{\frac{1}{2}}\, R^2\, M_{13}(x,t)}\left\{\frac{4r^2\,(R^2-\nu t x_3)}{M_{13}(x,t)}\right.$$

$$\left. + \frac{8r^2R^2\nu^2t^2}{M_{13}^2(x,t)} + 4x_3^2 - 3r^2\right\}\, i^3\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\tilde{\varphi}_2}{\mu t}\right)^{\frac{1}{2}}\right], \qquad (4.24)_3$$

where the symbols used are defined by

$$M_1(c,x,t) = \nu(x_3-\nu t) + cR_c\,, \qquad M_2(c,x,t) = c(x_3-\nu t) + \nu R_c$$

$$M_3(c,x,t) = \nu(x_3-\nu t) - cR_c\,, \qquad M_4(c,x,t) = c(x_3-\nu t) - \nu R_c$$

$$M_5(c,x,t) = \frac{\nu r\{(\nu x_3-c^2t) + cR_c\}}{r^2(c^2-\nu^2) + cx_3M_2(c,x,t)}$$

$$M_6(c,x,t) = \frac{M_2(c,x,t)}{M_1(c,x,t)} - \frac{x_3}{R}\,, \qquad M_7(c,x,t) = c\,\frac{c^2-\nu^2}{M_1(c,x,t)} - \frac{c}{R}$$

$$M_8(c,x,t) = \frac{(c^2-\nu^2)M_2(c,x,t)}{cM_1^2(c,x,t)} - \frac{x_3}{R^2}\,, \quad M_9(c,x,t) = \frac{(c^2-\nu^2)^2}{c^2M_1^2(c,x,t)} - \frac{1}{R^2}$$

$$M_{10}(c,x,t) = \frac{2\nu rcR_c(c^2-\nu^2)}{r^2(c^2-\nu^2)^2 + c^4(x_3-\nu t)^2 - \nu^2c^2R_c^2}$$

$$M_{11}(c,x,t) = \frac{2R_c(x_3-\nu t)}{r^2 + (x_3-\nu t)^2}\,, \quad M_{12}(c,x,t) = \frac{R_c(x_3-\nu t)(c^2-\nu^2)^2}{M_1^2(c,x,t)M_3^2(c,x,t)}$$

$$M_{13}(x,t) = R^2 - 2\nu x_3 t\,.$$

In (4.24) the terms with factor $i^4\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\tilde{\varphi}_2}{\mu t}\right)^{\frac{1}{2}}\right]$, $i^4\mathrm{Erfc}\left[\frac{R}{2}\left(\frac{\tilde{\varphi}_2}{k_2 t}\right)^{\frac{1}{2}}\right]$ are omitted.

Simple observation shows that the solid displacement and fluid velocity fields for the mixture exhibit components that depend upon the solid wave velocities c_1, ν_s and a diffusive component depending upon the fluid viscosities μ and k_2.

Moreover, if the velocity of the moving force is greater than the wave velocity ν_s but less than c_1, then there is a region whose points satisfy

$$R > \nu_s t \text{ and } t - \frac{x_3}{v} - \frac{r}{v}\left(\frac{v^2}{\nu_s^2} - 1\right)^{\frac{1}{2}} > 0 .$$

Inside this region we have $S(\nu_s) = 1$ and out side this region $S(\nu_s) = 0$. Therefore, the solid displacement and fluid velocity fields have a common propagating conical wave fronts

$t - \frac{x_3}{v} - \frac{r}{v}\left(\frac{v^2}{\nu_s^2} - 1\right) = 0$, besides the spherical one, $R = \nu_s t$. Similarly if the velocity of the moving force is greater than the wave velocity c_1, then there are two conical regions in which $S(c_1) = 1$ or $S(\nu_s) = 1$ but outside the regions $S(c_1) = 0$ and $S(\nu_s) = 0$, therefore, the solid displacement and fluid velocity fields have two common propagating conical wave fronts besides the spherical one.

The solid displacement and the fluid velocity fields have singularities and become unbounded when $R_c = 0$, so the singularities occur at $x_3 = vt$, $r = 0$ if $v < \nu_s$ and at the conical surfaces

$t - \frac{x_3}{v} - \frac{r}{v}\left\{\left(\frac{v}{c}\right)^2 - 1\right\}^{\frac{1}{2}} = 0$ if $v > c$ where c play the role of $c = c_1$ or $c = \nu_s$.

Finally we observe that the fluid velocity field are of order α for both wave and diffusive components. If α were zero, the fluid response would be identically zero. On the other hand, the solid displacement in the case of $\alpha = 0$ reduces to that of the elastic the elastic solid case [3].

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