

A MACROSCOPIC PLASMA LAGRANGIAN AND ITS APPLICATION TO WAVE
INTERACTIONS AND RESONANCES

by

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National Aeronautics and Space Administration
Grant NGL 05-020-176

National Science Foundation
Grant GK-32788X

SUIPR Report No. 575

June 1974

Institute for Plasma Research
Stanford University
Stanford, California

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ABSTRACT

This thesis is concerned with derivation of a macroscopic plasma Lagrangian, and its application to the description of nonlinear three-wave interaction in a homogeneous plasma and linear resonance oscillations in an inhomogeneous plasma.

One approach to obtain the Lagrangian is via the inverse problem of the calculus of variations for arbitrary first and second order quasilinear partial differential systems. Necessary and sufficient conditions for the given equations to be Euler-Lagrange equations of a Lagrangian are obtained. These conditions are then used to determine the transformations that convert some classes of non-Euler-Lagrange equations to Euler-Lagrange equation form. The Lagrangians for a linear resistive transmission line and a linear warm collisional plasma are derived as examples.

Using energy considerations, the correct macroscopic plasma Lagrangian is shown to differ from the velocity-integrated Low Lagrangian, by a macroscopic potential energy that equals twice the particle thermal kinetic energy plus the energy lost by heat conduction. The generalized variables are the macroscopic plasma cell position (Eulerian coordinates) defined in Lagrangian coordinates, and the vector and scalar potentials

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defined in Eulerian coordinates. With the continuity and heat flow equations treated as constraints, the Euler-Lagrange equations are shown to be the force law and Maxwell's equations. The effects of viscosity, heat conduction, and elastic collisions are included in this variational principle. The corresponding macroscopic Hamiltonian, and the microscopic Hamiltonian corresponding to the Low Lagrangian, are also derived.

Under the assumptions of scalar pressure and adiabatic processes, the macroscopic Lagrangian is approximated by expansions in weak perturbations of the generalized variables. The averaged Lagrangian method is then used to derive nonlinear three-wave coupling coefficients in a warm homogeneous two-component plasma. The effects of wave damping are included phenomenologically in the coupled mode equations. The general results are then specialized to make detailed quantitative comparisons between theory and available experimental results on parametrically excited ion-acoustic waves.

The approximate quadratic Lagrangian is also used to estimate the electrostatic (Tonks-Dattner) resonance properties of an inhomogeneous plasma. The Rayleigh-Ritz procedure is applied directly to the Lagrangian corresponding to a system of Euler-Lagrange equations. Use of an appropriate set of trial functions then leads to frequency and eigenfunction estimates in excellent agreement with the existing theoretical and experimental results for a low pressure positive column. Since this method mainly involves evaluating finite integrals, and solving algebraic eigenvalue equations, it is found to be more efficient than numerically solving differential equations, and more accurate than inner-outer expansions.

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LIST OF SYMBOLS

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
	(1) <u>Latin Alphabet</u>	
a	constant defined in (5.34)	5;159
\bar{a}	arbitrary constant	2;27
$a_{ij}^{\alpha\beta}$	coefficient defined in (2.36)	2;18
$a_{ij}^{\alpha\beta\gamma}$	coefficient defined in (2.36)	2;18
$a_{ijk}^{\alpha\beta\gamma}$	coefficient defined in (2.36)	2;18
$A_{ij}^{\alpha\beta}$	coefficient defined in (2.21)	2;14
\tilde{A}	vector potential	2,3,4;10
\tilde{A}'	perturbed vector potential	4;87
\tilde{A}_1	perturbation vector potential	4;87
\tilde{A}_M	macroscopic vector potential	3;41
\tilde{A}_R	random vector potential	3;41
$\underline{\underline{A}}$	normalized \tilde{A}_1	4;96
b	constant defined in (5.34)	5;159
\bar{b}	arbitrary constant	2;27
b^α	coefficient defined in (2.36)	2;18
$b^{\alpha\beta}$	coefficient defined in (2.36)	2;18
$b^{\alpha\beta\gamma}$	coefficient defined in (2.36)	2;18
$b_i^{\alpha\beta}$	coefficient defined in (2.36)	2;18

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used; page of definition</u>
$b_i^{\alpha\beta\gamma}$	coefficient defined in (2.36)	2;18
$b_{ij}^{\alpha\beta\gamma}$	coefficient defined in (2.36)	2;18
B^α	coefficient defined in (2.21)	2;14
\tilde{B}	magnetic field	1,2,3;1
\underline{B}	static magnetic field	4;96
c	speed of light in vacuum	4;96
\bar{c}	arbitrary constant	2;27
$c_i^{\alpha\beta}$	coefficient defined in (2.15)	2;12
C	normalized capacitance	2;23
C	collision term in Boltzmann equation	3;48
C	$ c_{\psi} $	4,5;113
C_{ij}	differential transformation operator	2;27
$C_i^{\alpha\beta}$	coefficient defined in (2.1)	2;7
C_{ψ}	nonlinear wave-wave coupling coefficient	4,5;104
$C_{\underline{K}\underline{K}'\underline{K}''}^{UVW}$	nonlinear wave-wave coupling coefficient	4,5;104
\bar{d}	arbitrary constant	2;27
$d\tilde{x}$	differential volume in \tilde{x}	3;36
$d^{\alpha\beta}$	coefficient defined in (2.15)	2;12

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used:</u> <u>page of definition</u>
D^α	coefficient defined in (2.1)	2;7
D_s	constant defined in (5.5)	5;139
D_{xx}, D_{xy}, D_{zz}	elements of \underline{D}	4;100
\underline{D}	wave dispersion tensor	4;100
$D_{ijk}^{\alpha\beta\gamma}$	function defined in (2.39)	2;19
e	electron charge	2,4;26
\bar{e}	arbitrary constant	2;27
e_0	$\exp(1) = 2.718$	4;117
e^α	coefficient defined in (2.15)	2;12
\underline{e}	unit wave polarization vector in \underline{E}	4,5;100
\underline{e}'	$\underline{e}'_{\underline{K}'} = \underline{U}'$	4,5;101
\underline{e}''	$\underline{e}''_{\underline{K}''} = \underline{U}''$	5;158
\underline{e}^U	\underline{e} of U wave	5;139
$\underline{e}_{\underline{K}}^U$	\underline{e} of U wave with wavenumber \underline{K}	4;101
E	perturbation electric field	2;26
\bar{E}	normalized E	2;26
\underline{E}	electric field vector	1,3;1
\underline{E}_1	perturbation in \underline{E}	4;99
\underline{e}'	$\underline{e}'_{\underline{K}'} = \underline{U}'$	4;101

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used; page of definition</u>
ϵ_T	constant defined in (4.83)	4;122
ϵ_{in}	thermal fluctuation field in ion-acoustic wave	5;147
ϵ_{th}	threshold of ϵ_p	4,5;112
ϵ_K^U	wave amplitude in \underline{g}	4;101
$\epsilon_{p,s,i}$	pump, signal, or idler wave amplitudes, respectively	4,5;105
$\epsilon_{p,s,i}^U$	$\epsilon_{p,s,i}$ of U wave	4;104
\underline{g}	normalized E_1	4;99
f	new dependent variable	2;25
f	Boltzmann distribution function	1,3;1
f_M	smoothed f	3;41
f_R	random portion of f	3;41
\underline{F}_M	macroscopic force on a particle	3;43
\underline{F}_R	random force on a particle	3;43
\mathcal{F}_{si}	i th component of $\underline{\mathcal{F}}_s$	3;67
\mathcal{F}_{EMi}	i th component of $\underline{\mathcal{F}}_{EM}$	3;67
$\underline{\mathcal{F}}$	energy flux defined in (3.82)	3;68
$\underline{\mathcal{F}}^E$	energy flux in macroscopic Eulerian coordinates	3;68

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
\tilde{f}_k	quadratic wave energy flux density	4;102
\tilde{f}_L	energy flux defined in (3.94)	3;71
\tilde{f}_s	particle functional (energy) flux	3;66
\tilde{f}_{EM}	electromagnetic field functional (energy) flux	3;66
g	new dependent variable	2;25
G	arbitrary physical quantity	3;65
g, g_s, g_{EM}	arbitrary physical quantities defined in (3.76) or (3.91)	3;66
h	new dependent variable	2;25
h	single particle Hamiltonian	3;70
\hbar	Planck's constant	3;63
H	macroscopic plasma Hamiltonian	3;62
H_L	microscopic plasma Hamiltonian	3;69
H_T	total microscopic plasma Hamiltonian	3;80
H_{QM}	quantum mechanical Hamiltonian of electron gas	3;63
\mathcal{H}	plasma Hamiltonian density in Eulerian coordinates	3;62
\tilde{H}_K	quadratic wave energy density	4;102
\tilde{H}_L	microscopic plasma Hamiltonian density	3;70

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used; page of definition</u>
H_s	plasma Hamiltonian density defined in (3.63)	3;62
H_{QM}	quantum mechanical Hamiltonian density of electron gas	3;63
\hat{i}_z	unit vector in the positive z direction	5;139
I	action integral	2,3;7
I	normalized electric current	2;23
I^E	action integral defined in (3.42)	3;54
I_i	action integral defined from \mathcal{L}_i	4;98
$\int_{\tilde{K}}^U$	wave action	4;103
$\mathcal{J}_{p,s,i}$	pump, signal, or idler wave actions, respectively	4,5;104
$\mathcal{J}_{p,s,i}^U$	$\mathcal{J}_{p,s,i}$ of U wave	4;104
J	Jacobian of \tilde{x} with respect to \tilde{x}_0	3;57
J	Jacobian of \tilde{x}' with respect to \tilde{x}	4;89
J_ν	ν th Bessel function of the first kind	4;116
\mathcal{J}	function defined in (2.25)	2;15
$\tilde{k}, \tilde{k}', \tilde{k}''$	wavenumbers	4;96
K_z	z-component of \tilde{K}	5;139
K_{ij}	inverse matrix of x_{ij}	3;57

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
$\underline{\underline{K}}, \underline{\underline{K}}', \underline{\underline{K}}''$	normalized $\underline{\underline{k}}, \underline{\underline{k}}',$ or $\underline{\underline{k}}''$, respectively	4,5;96
$\underline{\underline{K}}_{p,s,i}$	pump, signal, or idler wavenumber, respectively	4,5;104
ℓ	single particle Lagrangian	3;37
L	normalized inductance	2;23
L	macroscopic plasma Lagrangian	3,4;45
L'	perturbed L	4;87
L^E	Lagrangian defined in (3.27)	3;47
L_L	Low's Lagrangian (Low, 1958)	3;37
L'_L	velocity integrated L_L	3;41
L_s	Lagrangian defined in (3.52)	3;58
L_T	total microscopic plasma Lagrangian	3;40
L'_T	velocity integrated L_T	3;41
\mathcal{L}	Lagrangian density	2;7
\mathcal{L}	macroscopic Lagrangian density	3;45
\mathcal{L}'	perturbed \mathcal{L}	4;87
\mathcal{L}''	Lagrangian correction defined in (3.12)	3;41
\mathcal{L}^E	Lagrangian density defined in (3.27)	3;47

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
\mathcal{L}_i	i th order perturbation expansion of \mathcal{L}' , ($i = 1, 2, \dots$)	4;93
\mathcal{L}_L	Lagrangian density of L_L	3;37
\mathcal{L}'_L	velocity integrated \mathcal{L}_L	3;41
\mathcal{L}_N	Newcomb's Lagrangian density (Newcomb, 1962)	3;36
\mathcal{L}_s	Lagrangian density defined in (3.51)	3;58
m	electron mass	2;26
m	particle mass of a species	1,3;1
$m_{e,i}$	electron or ion mass, respectively	4;96
M	total number of dependent variables	2;7
M	function defined in (4.66)	4,5;114
M_0	constant defined in (4.66)	4,5;114
$M_i^{\alpha\beta}$	function defined in (2.17)	2;12
$M_{xx}^s, M_{xy}^s, \dots, M_{zz}^s$	elements of \tilde{M}^s	4;99
\tilde{M}^s	polarization constant of the s th particle species	4,5;99
m_i^α	function defined in (2.4)	2;8
n	arbitrary function of x_i	2;12
n	perturbation electron density	2;26

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
n	density of a particle species	1,3;1
n'	density at \tilde{x} after virtual displacement	3;47
n'	density at \tilde{x} after perturbation	4;87
\bar{n}	normalized n	2;26
n_0	static electron density	2;26
n_0	initial value of n	3;58
$n_{e,i}$	electron or ion density, respectively	4,5;97
n_i	i th order nonlocal expansion of n' , ($i = 1,2,\dots$)	4;91
n_{Li}	i th order local expansion of n' , ($i = 1,2,\dots$)	4;91
n_n	neutral density	5;135
N	total number of independent variables	2;7
N_ν	ν th order Bessel function of the second kind	4;116
η	function defined in (2.4)	2;8
η	K/Ω	4;100
$p_i^{\alpha\beta}$	function defined in (2.43)	2;19
$p_i^{\alpha\beta\gamma}$	function defined in (2.43)	2;19
P	perturbation electron pressure	2;26

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
P	plasma pressure	3;36
P	scalar pressure of a particle species	4;89
P	static background pressure	5;133
P'	perturbed P	4;89
P_0	static electron pressure	2;26
P_i	i th order nonlocal expansion of P'	4;92
P_{ij}	matrix element of \underline{P}	3;60
P_{Li}	i th order local expansion of P'	4;92
\underline{P}	pressure tensor of a particle species	3;41
\underline{P}'	perturbed \underline{P}	4;87
P_i^α	function defined in (2.25)	2;15
q	function defined in (2.46)	2;20
q	particle charge of a species	1,3;1
$q_{e,i}$	electron or ion charge, respectively	4;97
$q^{\alpha\beta}$	function defined in (2.46)	2;20
$q^{\alpha\beta\gamma}$	function defined in (2.46)	2;20
\underline{q}	heat flux of a particle species	3;42
Q_i	i th component of \underline{Q}	3;60

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
\tilde{Q}	energy transport by heat conduction	3;43
Q	function defined in (2.25)	2;15
R	normalized resistance	2;23
R^α	function defined in (2.49)	2;22
\tilde{R}	position of the center of ΔV	3;43
$R_i^{\alpha\beta}$	function defined in (2.33)	2;17
S	entropy	3;73
\bar{S}	entropy function defined in (3.96)	3;73
\bar{S}_0	entropy constant	3;73
$S^{\alpha\beta}$	function defined in (2.49)	2;22
t	displacement in time	1,2,3;1
t_1, t_2	limits of integration in time	3;54
T	normalized t according to (2.57)	2;26
T	normalized t according to (4.30)	4;96
T_0	quiescent plasma temperature	4;122
$T_{e,i}$	electron or ion temperature, respectively	5;133
T_{exp}	explosive instability period	4;110
$T_i^{\alpha\beta}$	function defined in (2.49)	2;22
\mathcal{I}_N	kinetic energy contribution to \mathcal{I}_N	3;36

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
u	constant defined in (5.34)	5;157
\underline{u}	hydromagnetic plasma velocity	3;36
U^α	α th dependent variable	2;7
U_i^α	derivative of U^α with respect to x_i	2;7
U_{ij}^α	second derivative of U^α with respect to x_i and x_j	2;14
u^U	group velocity of the U wave	5;155
$u_{ix,z}$	x- or z-component of \underline{u}_i , respectively	4,5;119
$u_{sx,z}$	x- or z-component of \underline{u}_s , respectively	4,5;119
\underline{u}_K^U	group velocity of the U wave with wavenumber \underline{K}	4;103
$\underline{u}_{p,s,i}$	pump, signal, or idler wave group velocity, respectively	4,5;104
$\underline{u}_{p,s,i}^U$	$\underline{u}_{p,s,i}$ of the U wave	4;104
v	perturbation electron drift velocity	2;26
\bar{v}	normalized v	2;26
v_i	i th component of \underline{v}	3;59
v_t	electron thermal speed	2;26
\underline{v}	particle velocity	1,3;1
\underline{v}_0	initial particle velocity	3;31

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
\tilde{v}_D	macroscopic drift velocity of a particle species	3;41
$\dot{\tilde{v}}_D$	$d\tilde{v}_D/dt$	3;65
\tilde{v}'_D	\tilde{v}_D at \tilde{x}' after virtual displacement	3;49
\tilde{v}''_D	perturbed \tilde{v}_D at \tilde{x}'	4;87
\tilde{v}_i	i th order nonlocal expansion of \tilde{v}'_D ($i = 1, 2, \dots$)	4;90
\tilde{v}_{Li}	i th order local expansion of \tilde{v}'_D ($i = 1, 2, \dots$)	4;90
\tilde{v}_R	particle random velocity with respect to \tilde{v}_D	3;45
V	normalized voltage	2;23
V	spatial volume	3;36
V'	perturbed V	4;87
V_0	initial plasma volume of a particle species	3;37
V^α	new α th dependent variable	2;22
V_i^α	derivative of V^α with respect to x_i	2;22
v_a	ion-acoustic speed	5;139
v_A	Alfvén speed	5;158
$v_{e,i}$	normalized electron or ion thermal speed, respectively	4,5;96

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
V_p	pump wave exciting voltage	5;135
V_{QM}	quantum mechanical potential	3;63
V_{th}	threshold of V_p	5;135
U	macroscopic potential energy density of a particle species	3;42
U_H	heat conduction contribution to U	3;42
U_N	potential energy contribution to \mathcal{L}_N	3;36
U_R	random collisional contribution to U	3;42
x	one-dimensional displacement in space	2;23
x_{0i}	ith component of \tilde{x}_0	3;57
x_i	ith independent variable	2;7
x_i	ith component of \tilde{x}	3;57
\dot{x}_i	ith component of $\dot{\tilde{x}}$	3;59
x_{ij}	$\partial x_i / \partial x_{0j}$	3;57
\dot{x}_{ij}	dx_{ij} / dt	3;59
\tilde{x}	particle position vector	3;31
\tilde{x}	macroscopic cell position of a particle species	3;42
\tilde{x}_0	initial particle position	3;31
\tilde{x}_0	initial cell position of a particle species	3;43

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
\tilde{x}'	macroscopic plasma cell position after virtual displacement	3;49
\tilde{x}''	perturbed macroscopic plasma cell position	4;87
$\dot{\tilde{x}}$	\tilde{v}_D	3;58
$\ddot{\tilde{x}}$	$d\tilde{v}_D/dt$	3;65
\tilde{x}_R	particle random position with respect to \tilde{R}	3;43
X	normalized x according to (2.57)	2;26
X	normalized x according to (4.30)	4;119
\tilde{X}	normalized \tilde{x} according to (4.30)	4;96
z	displacement in the z -direction	4;106
z	peak position of $e_i(z)$	5;135
Z	normalized z according to (4.30)	4,5;106
Z_0	normalized z_0 according to (4.30)	5;138
Z_{exp}	explosive instability length	4;110
	(2) <u>Greek Alphabet</u>	
α	constant defined in (4.89)	4;125
$\tilde{\alpha}$	canonical momentum conjugate to \tilde{A}	3;61
γ	adiabatic index for electrons	2;26
γ	adiabatic index of a particle species	3;89

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
$\gamma_{e,i}$	adiabatic index for electrons or ions, respectively	4,5;97
γ_{Li}	Landau damping rate of the ion-acoustic wave	5;141
Γ_0	spatial parametric amplification rate	4,5;106
Γ_{0i}	idler wave spatial growth constant	4;106
Γ^U	normalized temporal damping rate for the U wave	5;141
Γ_c	normalized collisional damping rate	5;155
Γ_L	normalized electron Landau damping rate	4,5;120
Γ_r	cyclotron resonance damping rate for whistlers	5;115
$\Gamma_{p,s,i}$	normalized temporal damping rate of \mathcal{E}_p , \mathcal{E}_s , or \mathcal{E}_i , respectively	4,5;112
δ	local variation	3;48
δ'	nonlocal variation	3;48
δ_{ij}	Kronecker delta	3,4;58
$\delta(\Omega-\Omega')$	delta function in Ω'	4;98
$\delta(\underline{k}-\underline{k}')$	delta function in \underline{k}'	4;98
Δ_s	function defined in (4.41)	4;100
ΔK	departure from synchronism in wavenumber	4;124

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
ΔL	distance from the point of perfect synchronism	4;124
ΔL_0	size of homogeneous plasma	4,5;124
Δt	small time period	3;49
ΔT	scale of slow variation in T	4;97
ΔV	small macroscopic volume	3;43
$\Delta \underline{x}$	small spatial displacement during Δt	3;49
$\Delta \underline{X}$	scale of slow variation in \underline{X}	4;97
ϵ	arbitrary parameter	3;48
ϵ_0	free space permittivity	2,3,4;26
ϵ_{kmn}	antisymmetric unit tensor	2;20
ξ	canonical momentum conjugate to microscopic particle position	3;69
$\xi_{e,i}$	normalized ξ_e or ξ_i , respectively	4;96
η_i	ith component of $\underline{\eta}$	3;65
$\dot{\eta}_i$	ith component of $\dot{\underline{\eta}}$	3;67
$\underline{\Pi}$	macroscopic canonical momentum conjugate to \underline{x}	3;61
$\dot{\underline{\Pi}}$	$d\underline{\Pi}/dt$	3;63
$\underline{\Pi}_E$	trial macroscopic momentum conjugate to \underline{x} in Eulerian coordinates	3;56

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
θ	angle between \underline{K} and the static magnetic field	5;139
θ'	angle between \underline{K}' and the static magnetic field	5;158
θ''	angle between \underline{K}'' and the static magnetic field	5;152
κ	temporal parametric amplification rate with wave damping included	4;112
κ_0	temporal parametric amplification rate	4;106
κ_{0i}	idler wave temporal growth constant	4;106
κ_{th}	constant defined in (4.63)	4;112
$\underline{\kappa}$	plasma equivalent permittivity tensor	4;110
Λ_{D_s}	normalized Debye length	4;96
Λ_I^s	field-particle interaction contribution to $\Lambda_{\underline{K}\underline{K}'}^{\underline{K}''}$	4;101
Λ_T^s	particle thermal contribution to $\Lambda_{\underline{K}\underline{K}'}^{\underline{K}''}$	4;101
$\Lambda_{\underline{K},\Omega}^{(2)}$	quadratic Lagrangian density in (\underline{K},Ω) space	4;98
$\Lambda_{\underline{K},\Omega}^{e,i(2)}$	electron, ion contribution to $\Lambda_{\underline{K},\Omega}^{(2)}$	4;98
$\Lambda_{\underline{K},\Omega}^{F(2)}$	electromagnetic field contribution to $\Lambda_{\underline{K},\Omega}^{(2)}$	4;98
$\Lambda_{\underline{K}\underline{K}'}^{\underline{K}''}$	cubic Lagrangian density defined in (4.46)	4;101

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
$\Lambda_{\underline{KK} \underline{K}''}^{UVW}$	$\Lambda_{\underline{KK} \underline{K}''}$	4,5;104
μ_0	free space permeability	3,4;36
μ^U	normalized spatial damping rate of the U wave	5;141
$\mu_{p,s,i}$	normalized spatial damping rate of ϵ_p , ϵ_s , or ϵ_i , respectively	4,5;112
$\mu_{ix,z}$	μ_i in the X or Z direction, respectively	4,5;119
$\mu_{sx,z}$	μ_s in the X or Z direction, respectively	4,5;119
ν	effective electron-ion and -neutral momentum transfer collision frequency	2;26
ν	constant defined in (4.69)	4;116
$\bar{\nu}$	normalized ν	2;26
ν_e	effective electron-ion and -neutral energy transfer collision frequency	4;123
ν_{en}	effective electron-neutral momentum transfer collision frequency	5;133
ν_{in}	effective ion-neutral momentum transfer collision frequency	5;133
ξ	polarization vector in particle position	3;32
ξ	virtual displacement in macroscopic plasma cell position	3;49

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
ξ	perturbation in macroscopic plasma cell position	4;87
$\dot{\xi}$	$d\xi/dt$	4;90
$\xi_{e,i}$	perturbation in macroscopic electron or ion cell position, respectively	4;96
ρ	plasma mass density	3;36
σ	ratio of ion-to-electron mass densities	4;96
$\tilde{\sigma}$	surface area	3;37
$\tilde{\sigma}$	surface area of V	3;66
$\tilde{\sigma}_0$	surface area of V_0	3;66
Σ	summation over all particle species	1,3;1
τ_e	-1	4;101
τ_i	$1/\sigma^2$	4;101
φ	scalar potential	3;37
φ'	perturbed φ	4;87
φ_1	perturbation in φ	4;87
φ_M	macroscopic φ	3;41
φ_R	random φ	3;41
$\tilde{\varphi}$	normalized φ	4;96

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
Φ_N	equivalent potential defined in (4.84)	4;123
χ	function defined in (4.69)	4;116
$\chi_{1i,s}$	functions defined in (4.71)	4;117
$\chi_{2i,s}$	functions defined in (4.73)	4;117
Ψ	Schrödinger wave function	3;63
$\Psi_{s,i}$	angle between the static magnetic field and \underline{u}_s or \underline{u}_i , respectively	4,5;119
$\omega, \omega', \omega''$	frequencies	4;96
ω_p	electron plasma frequency	2;26
ω_{pe}	electron plasma frequency	4,5;96
$\Omega, \Omega', \Omega''$	normalized $\omega, \omega',$ or ω'' , respectively	4,5;96
Ω_B	electron bounce frequency	4,5;120
Ω_c	$ \underline{\Omega}_c $	4,5;100
Ω_e	$-\Omega_c$	4;100
Ω_i	Ω_c/σ	4,5;100
$\Omega_{p,s,i}$	pump, signal, or idler wave frequency, respectively	4,5;104
$\underline{\Omega}_c$	normalized electron cyclotron frequency	4;96

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
	(3) <u>Subscripts</u>	
O	at $t = 0$	3;58
O	static	2;26
l	perturbation	4;87
a	acoustic	5;139
A	Alfvén	5;158
B	bounce	4;120
C	collisional	5;155
D	drift	3;49
e,i	of electron or ion, respectively	4;96
i	ith order in perturbation amplitude	4;93
H	of heat conduction	3;42
i,j	ith or jth component of a vector	3;57
i,j,k ...	i, j, or kth independent variable	2;10
L	Low's (collisionless, Vlasov) plasma model	3;37
L	Landau damping	4,5;120
M	macroscopic	3;41
n	neutral	5;135
N	Newcomb's hydromagnetic model	3;36

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
p,s,i	for pump, signal, or idler, respectively	4,5;104
r	cyclotron resonance	5;115
R	random	3;41
S	sth particle species	3;66
t	thermal	2;26
T	total	3;80
x	component in the x direction	4,5;99
y	component in the y direction	4;99
z	component in the z direction	4,5;99
EM	electromagnetic	3;66
en	electron-neutral	5;133
ij	elements of a tensor or matrix	3;57
in	noise in ion-acoustic waves	5;147
in	ion-neutral	5;133
Li	lth order in local expansion	4;90
QM	quantum mechanical	3;63
th	threshold	4,5;112
exp	explosive instability	4;110
$\underline{K}, \underline{K}', \underline{K}''$	of wave with wavenumber $\underline{K}, \underline{K}'$, or \underline{K}''	4,5;104

LIST OF SYMBOLS (cont.)

<u>Symbol</u>		<u>Section where used;</u> <u>page of definition</u>
	(4) <u>Superscripts</u>	
s	sth particle species	4;99
E	in Eulerian coordinates	3;68
F	of electromagnetic field	4;98
U,V,W	of the U,V, or W wave	4;104
$\alpha, \beta, \gamma, \dots$	$\alpha, \beta,$ or γ th dependent variable	2;10

ACKNOWLEDGMENTS

I am deeply indebted to my research advisor, Professor Frederick W. Crawford, for his patient guidance throughout this work and for his proficient help in organizing the thesis. His advices have profoundly influenced my view of plasma physics.

I also would like to express my sincere appreciation to Dr. Kenneth J. Harker for making available numerous hours of discussion during my study at Stanford and for making constructive criticism in preparing the thesis.

I am grateful to Professor Peter A. Sturrock for reading and commenting on the dissertation, and to Professor Oscar Buneman for his beneficial commentary on computer applications of Lagrangian formulations. Thanks are also due to Dr. Hongjin Kim for many stimulating discussion sessions and for verifying the derivations. Numerous fruitful conversations with Mr. David M. Sears, Dr. Dragan Ilic, and Dr. James J. Galloway are also acknowledged with pleasure.

I want to thank Evelyn Mitchell, who expertly and cheerfully typed and retyped this manuscript, and to thank Jane Johnston and Joanne Russell, who have offered efficient assistance during my study at the Institute for Plasma Research.

Finally, my deepest gratitude belongs to my wife San-San for her understanding and encouragement during the entire period of my graduate work, to my father for his instruction, and to God for His provisions.

1. INTRODUCTION

In theoretical descriptions of plasmas, three approximate models are commonly employed. These are the cold, the microscopic, and the macroscopic plasma models, all of which use Maxwell's equations. To complete the system of equations, the cold plasma model uses Newton's force law,

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad , \quad (1.1)$$

where \mathbf{E} and \mathbf{B} are the electric and magnetic fields, and \mathbf{v} , m , and q are the particle velocity, mass, and charge of a species. The plasma is then regarded as consisting of interpenetrating cold fluids with charge and current densities, $\sum qn$ and $\sum qn\mathbf{v}$, respectively, where \sum sums over all particle species, and n is the particle number density.

The microscopic plasma model uses Newton's force law in (1.1) for each particle, and the Boltzmann-Vlasov equation (Clemmow and Dougherty, 1969)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{d\mathbf{v}}{dt} \cdot \nabla_{\mathbf{v}} f = 0 \quad , \quad (1.2)$$

where $f(\mathbf{x}, \mathbf{v}, t)$ is the Boltzmann distribution function for each particle species. The expressions for charge and current densities now become $\sum q \int f d\mathbf{v}$ and $\sum q \int f \mathbf{v} d\mathbf{v}$, respectively. The plasma is regarded as a system of charged particles evolving under the influence of their own electromagnetic fields, and externally applied fields (if any). In principle, complete solutions to the particle force law for all particles will automatically generate the solution of $f(\mathbf{x}, \mathbf{v}, t)$ because (1.2) is a statement

of particle conservation along the particle trajectories in a six-dimensional phase space $(\underline{x}, \underline{v})$. When it is not necessary to obtain the particle trajectories, solutions of $f(\underline{x}, \underline{v}, t)$ are obtained by use of (1.2) together with Maxwell's equations.

The macroscopic plasma model uses velocity moments of the Boltzmann-Vlasov equation (1.2). Examples of these equations are expressed by (3.29) and (3.46). The charge and current densities are now written as Σqn and $\Sigma qn\underline{v}_D$, where \underline{v}_D is the drift velocity, i.e. the averaged local velocity of a particle species [see (3.13)]. Here, the plasma is approximated in terms of localized variables such as density, drift velocity, pressure, and heat flux. In terms of degree of approximation, the macroscopic model falls between the other two models.

The appropriate Lagrangians for the cold plasma model (Galloway and Crawford, 1970) and the microscopic plasma model (Low, 1958) are already well-known. However, the Lagrangian for the macroscopic plasma model, that corresponds to Maxwell's equations and the moments of the Boltzmann-Vlasov equation, has not been established. The gap will be filled in this thesis.

The interests in developing this variational principle stem from the fact that current theoretical investigations in nonlinear wave-wave and wave-particle interaction properties of homogeneous plasmas, and in linear properties of inhomogeneous plasmas, are at the limit of analytic tractability. While a suitable variational principle does not provide new fundamental laws, it leads to a relatively concise formulation and easy manipulation for these otherwise difficult problems.

To obtain the suitable macroscopic Lagrangian, the first problem that arises is the inverse problem of the calculus of variations, i.e. the

derivation of Lagrangians from arbitrary equations. This mathematical approach will be examined in Section 2 in contrast to the approach via energy considerations commonly used for physical problems. From this general point of view, it will be shown that energy dissipation effects can be included in variational (minimal) principles, in general, and the results will be demonstrated with examples.

The approach of the inverse problem treats all dependent variables as generalized variables. The plasma variational principle to be presented in Section 3 will treat only the macroscopic plasma cell position and the electromagnetic potentials as generalized variables. This type of formulation for a system of discrete charged particles, was described in a relativistically covariant form by Landau and Lifschitz (1965), and in the non-relativistic form by Goldstein (1950). Extensions of these Lagrangian densities to the microscopic plasma model, to include a velocity-distributed system of particles, have been proposed by Sturrock (1958a) and Low (1958). Based on Low's Lagrangian, and using energy considerations, we shall obtain the corresponding Lagrangian and Hamiltonian for the macroscopic plasma model, with the effects of viscosity, heat conduction, and elastic collisions taken into account.

In applications of Lagrangians to problems involving homogeneous plasmas, there has been progress in the areas of linear waves (Kim, 1972), nonlinear three-wave interactions (Galloway and Crawford, 1970), wave-background interactions (Dewar, 1970), wave kinetic equations (Suramlishvili, 1964 and 1965; Galloway, 1972), higher order nonlinear wave processes (Dewar, 1972; Dysthe, 1974), and statistical analysis of plasma turbulence (Kim and Wilhelm, 1972). For problems in inhomogeneous plasmas, results

have been presented in the area of energy principles (Newcomb, 1962), and in the use of the Rayleigh-Ritz procedure to obtain approximate solutions (Dorman, 1969). As compared with other branches of physics, these cases comprise a disproportionately small fraction of the theoretical effort in plasma physics.

In this work, the new macroscopic Lagrangian will be used in two ways. In Section 4 we shall be concerned with the general description of nonlinear three-wave interactions in a homogeneous plasma by use of the averaged Lagrangian technique (Whitham, 1965). These results will be specialized in Section 5 to parametric amplification of ion-acoustic waves to make quantitative comparisons with available experimental data. The second application of the Lagrangian will be presented in Appendix B, where the Rayleigh-Ritz procedure is applied to obtain approximate solutions for electrostatic resonances in a low pressure positive column. It will be shown that the results compare favorably with available experimental data and conventional numerical calculations by others (Parker, Nickel, and Gould, 1964).

Some conclusions are drawn in Section 6, where new contributions and future extensions of this research are briefly discussed.

2. INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

2.1 Introduction

In the calculus of variations Hamilton's principle is applied to the integral of a given Lagrangian density, often referred to as the action integral, to obtain the Euler-Lagrange equations extremizing the action integral (see for example, Courant and Hilbert, 1966). The inverse problem of the calculus of variations is to find the conditions that arbitrary differential equations must satisfy to be the Euler-Lagrange equations of a certain Lagrangian density, and to determine an appropriate Lagrangian density from the given differential equations. The term 'Lagrangian density' as used here will include cases with multiple independent variables. Recent developments of a method of studying weakly nonlinear wave propagation in distributed systems makes the inverse problem for a system of partial differential equations of particular interest (Whitham, 1965; Galloway and Crawford, 1970; see Chapter 4). A general approach is required to obtain appropriate Lagrangian densities for systems of equations including effects of energy loss due to heat flow, viscosity, and collisions.

The inverse problem of the calculus of variations attracted attention over a century ago, when Jacobi (1837) examined the characteristic properties of an ordinary Euler-Lagrange differential equation of second order (Kürschák, 1906; Akhiezer, 1962). More involved problems have since been studied. Such work includes that by LaPaz (1930), who treated the case of one dependent variable in many independent variables using the necessary property of self-adjointness of the Euler-Lagrange equation. Douglas (1941) has studied the case of many dependent variables

in one independent variable, by considering the characteristic forms of the coefficients in a system of Euler-Lagrange equations. Van der Vaart (1969) has applied the results of Douglas (1941) to a system of ordinary linear differential equations of second order. A survey of the literature on this approach to the inverse problem may be found in the book by Funk (1970).

The general case of many dependent and independent variables has been considered by Vainberg (1964). By treating the dependent variables as points in a coordinate system of functions, the invariance of an action integral under the variation of the functions is shown to be analogous to the invariance of a potential under the variation of the path of integration (Tonti, 1969a). This analogy has led to definition of potentiality conditions for the operators of the system of equations. For differential equations, these potentiality conditions are then the necessary and sufficient conditions for solutions of the inverse problem of the calculus of variations (Tonti, 1969b).

The treatment to be presented in Sections 2.2 and 2.3 will be confined to quasilinear differential systems of first and second order, respectively. We shall emphasize special forms of the Euler-Lagrange differential system, following the approach used by Douglas (1941), while generalizing to the case of partial differential equations. Since we are looking for some scheme that generates a Lagrangian density, the conditions for the given equations to be Euler-Lagrange equations will be established in such a way that, once satisfied by the differential equations, an explicit Lagrangian will be derivable. The cases of linear and weakly nonlinear differential equations are treated as examples in Sections 2.2.3 and 2.3.3, respectively.

In supplement to these conditions, the nonuniqueness in the form of the given differential system requires discussion: there are operations, such as multiplication of the system by some matrix expressions (Davis, 1929), and changes of the dependent variables, that can convert apparently non-Euler-Lagrange equations to equivalent Euler-Lagrange equations. In Section 2.4, we shall discuss one of these techniques, differential transformation of the dependent variables. As examples, we derive appropriate Lagrangian densities for a resistive transmission line, and for a warm collisional plasma in Sections 2.4.2 and 2.4.3, respectively.

2.2 First Order Differential Equations

A system of first order quasilinear differential equations has the general form,

$$C_i^{\alpha\beta} U_i^\beta + D^\alpha = 0 \quad (i=1, \dots, N; \alpha, \beta = 1, \dots, M), \quad (2.1)$$

where the coefficients $C_i^{\alpha\beta}$ and D^α are explicit expressions in independent variables x_i and dependent variables U^β , and U_i^α is written for the derivative $\partial U^\alpha / \partial x_i$. Repeated indices of i (or j, k, \dots etc.) are summed over the N independent variables; repeated indices of α (or β, γ, \dots etc.) are summed over the M dependent variables. We see that, in general, the numbers of the $C_i^{\alpha\beta}$ and D^α are $M^2 N$ and M , respectively.

The inverse problem of the calculus of variations aims at deriving a Lagrangian density, $\mathcal{L} [= \mathcal{L}(U_i^\alpha, U^\alpha, x_i)]$, that gives (2.1) as the set of Euler-Lagrange equations obtained by extremizing the action integral,

$$I = \int d^N x \mathcal{L}(U_i^\alpha, U^\alpha, x_i) \quad , \quad (2.2)$$

through variations of the dependent variables, U^α . For an arbitrary $\mathcal{L}(U_i^\alpha, U^\alpha, x_i)$, the Euler-Lagrange equations take the form,

$$\frac{d}{dx_i} \left(\frac{\partial \mathcal{L}}{\partial U_i^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial U^\alpha} = 0 \quad (i=1, \dots, N, \quad \alpha=1, \dots, M) \quad , \quad (2.3)$$

where d/dx_i operates on both the explicit and the implicit x_i -dependences through $U_i^\alpha(x_j)$ and $U^\alpha(x_i)$.

Equation (2.3) is a highly specialized form of (2.1): the coefficients $C_i^{\alpha\beta}$ and D^α are functions only of the derivatives of $\mathcal{L}(U_i^\alpha, U^\alpha, x_i)$. The necessary conditions for (2.1) to be a set of Euler-Lagrange equations will be obtained in Section 2.2.1 by elimination of $\mathcal{L}(U_i^\alpha, U^\alpha, x_i)$ from the expressions for $C_i^{\alpha\beta}$ and D^α . More important, however, are the sufficient conditions on $C_i^{\alpha\beta}$ and D^α so that a corresponding $\mathcal{L}(U_i^\alpha, U^\alpha, x_i)$ exists. In Section 2.2.2, we shall establish conditions which will enable us to solve the expressions of $C_i^{\alpha\beta}$ and D^α for $\mathcal{L}(U_i^\alpha, U^\alpha, x_i)$.

2.2.1 Necessary Conditions

In view of (2.3), \mathcal{L} must be linear in the U_i^α to give a set of Euler-Lagrange equations of the form (2.1). We write

$$\mathcal{L} = m_i^\alpha U_i^\alpha + n \quad , \quad (2.4)$$

where $m_i^\alpha [= m_i^\alpha(U^\beta, x_j)]$ and $n [= n(U^\alpha, x_i)]$ are functions of U^α and x_i . The Euler-Lagrange equations of this Lagrangian density with respect to U^α then become

$$\left(\frac{\partial m_i^\alpha}{\partial U^\beta} - \frac{\partial m_i^\beta}{\partial U^\alpha} \right) U_i^\beta + \frac{\partial m_i^\alpha}{\partial x_i} - \frac{\partial n}{\partial U^\alpha} = 0 \quad . \quad (2.5)$$

Comparing (2.1) and (2.5) gives,

$$c_i^{\alpha\beta} = \frac{\partial m_i^\alpha}{\partial u^\beta} - \frac{\partial m_i^\beta}{\partial u^\alpha} \quad , \quad D^\alpha = \frac{\partial m_i^\alpha}{\partial x_i} - \frac{\partial \eta}{\partial u^\alpha} \quad . \quad (2.6)$$

The $c_i^{\alpha\beta}$ and D^α must satisfy the following equations, obtained by eliminating the m_i^α and η from (2.6),

$$c_i^{\alpha\beta} + c_i^{\beta\alpha} = 0 \quad , \quad \frac{\partial c_i^{\alpha\beta}}{\partial x_i} = \frac{\partial D^\alpha}{\partial u^\beta} - \frac{\partial D^\beta}{\partial u^\alpha} \quad , \quad \frac{\partial c_i^{\alpha\beta}}{\partial u^\gamma} + \frac{\partial c_i^{\beta\gamma}}{\partial u^\alpha} + \frac{\partial c_i^{\gamma\alpha}}{\partial u^\beta} = 0 \quad . \quad (2.7)$$

2.2.2 Sufficient Conditions

Given $c_i^{\alpha\beta}(U^\gamma, x_j)$ and $D^\alpha(U^\beta, x_i)$, we require conditions sufficient to guarantee that (2.6) can be solved for the m_i^α and η in terms of the U^β and x_i . Note that the uniqueness of solutions is not required.

The first equation of (2.6) represents at most M^2N equations for the MN unknowns m_i^α , with U^β as independent variables. For m_i^α to exist, (2.7) must be satisfied. In particular, $c_i^{\alpha\beta} = -c_i^{\beta\alpha}$, so that the number of distinct equations covered by the first equation of (2.6) is reduced to $MN(M-1)/2$. These can be divided into N independent groups of $M(M-1)/2$ equations for each i . Consider in each of these groups the subset of equations relating m_i^α , m_i^β , and m_i^γ . If we assume a form for m_i^α , we may then use the first expression in (2.6), which yields

$$\frac{\partial m_i^\beta}{\partial u^\alpha} = \frac{\partial m_i^\alpha}{\partial u^\beta} - c_i^{\alpha\beta} \quad , \quad \frac{\partial m_i^\gamma}{\partial u^\alpha} = \frac{\partial m_i^\alpha}{\partial u^\gamma} - c_i^{\alpha\gamma} \quad , \quad (2.8)$$

to solve for m_i^β and m_i^γ , subject to the constraint that the particular solutions of m_i^β and m_i^γ must satisfy

$$C_i^{\beta\gamma} = \frac{\partial m_i^\beta}{\partial u^\gamma} - \frac{\partial m_i^\gamma}{\partial u^\beta} . \quad (2.9)$$

This self-consistency condition requires that $C_i^{\alpha\beta}$, $C_i^{\beta\gamma}$, and $C_i^{\gamma\alpha}$ satisfy the final relation of (2.7).

As a digression, it is of interest to note that this self-consistency condition is analogous to the well-known condition that magnetic fields are source-free. For the magnetic field, \underline{B} , and the corresponding vector potential, \underline{A} , we have

$$\nabla \cdot \underline{B} = 0 \quad , \quad \underline{B} = \nabla \times \underline{A} . \quad (2.10)$$

The analogy follows by taking $\alpha = 1$, $\beta = 2$, $\gamma = 3$, and

$$\underline{B} = \{C_i^{23}, C_i^{31}, C_i^{12}\} \quad , \quad \underline{A} = \{m_i^1, m_i^2, m_i^3\} \quad , \quad \nabla = \left\{ \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^3} \right\} . \quad (2.11)$$

The foregoing argument is valid for any triplet from the set $\{\alpha\}$. Therefore, with m_i^α arbitrarily chosen, and the final expression of (2.7) satisfied, we can always use the first relation of (2.6) to solve for m_i^β , m_i^γ , etc. Having obtained this set of particular solutions, they can be substituted in the second expression of (2.6), together with the given D^α , to obtain M equations for the single unknown function $\mathfrak{N}(U^\alpha)$. A solution for \mathfrak{N} exists if we have

$$\frac{\partial}{\partial u^\beta} \left(\frac{\partial \mathfrak{N}}{\partial u^\alpha} \right) = \frac{\partial}{\partial u^\alpha} \left(\frac{\partial \mathfrak{N}}{\partial u^\beta} \right) . \quad (2.12)$$

Use of (2.6) indicates that this condition implies that the D^α must be related according to (2.7).

The consistency relation of (2.7) for triplets such as $C_i^{\alpha\beta}$, $C_i^{\beta\gamma}$, $C_i^{\gamma\alpha}$ reduces the number of independent equations in (2.6) from $M(M-1)/2$ to $(M-1)$ for each i . This follows since use of the equations,

$$C_i^{12} = \frac{\partial m_i^1}{\partial U^2} - \frac{\partial m_i^2}{\partial U^1}, \quad C_i^{13} = \frac{\partial m_i^1}{\partial U^3} - \frac{\partial m_i^3}{\partial U^1}, \quad (2.13)$$

in the final expression of (2.7) will result in

$$C_i^{23} = \frac{\partial m_i^2}{\partial U^3} - \frac{\partial m_i^3}{\partial U^2} + \Delta(U^2, U^3), \quad (2.14)$$

where $\Delta(U^2, U^3)$ is the constant of integration. Since m_i^2 and m_i^3 also contain arbitrary functions of U^2 and U^3 , because of (2.13), it is always possible to adjust them to make $\Delta(U^2, U^3) = 0$. Symbolically, we can express the result that (2.14) follows from (2.13) as

$$12, 13 \rightarrow 23,$$

$$12, 13, 14 \rightarrow 23, 24, 34,$$

⋮
⋮
⋮

$$12, 13, 14, 1M \rightarrow 23, 24, \dots, (M-1)M.$$

Thus, by imposition of the third condition of (2.7), the first expression of (2.6) is reduced to only $M-1$ independent equations for each i . By choosing m_i^1 arbitrarily, m_i^β ($\beta > 1$) can then be obtained consistently by use of (2.6).

It is now clear that the relations expressed by (2.7) are the necessary and sufficient conditions for (2.1) to be a set of Euler-Lagrange equations of a Lagrangian density of the form given in (2.4). When expressions for $C_i^{\alpha\beta}$ and D^α satisfying these conditions are given, the procedure for deriving the corresponding Lagrangian density will involve only direct integrations. It should be noted that since \mathfrak{M}_i^1 is arbitrary, the set of \mathfrak{M}_i^α and the function \mathfrak{N} are not unique. In addition to the freedom in choosing \mathfrak{M}_i^α , the set of \mathfrak{M}_i^α and the function \mathfrak{N} are determined only to within arbitrary functions of the x_i .

2.2.3 Linear Differential Equations

As an exercise, we shall derive a Lagrangian density for a system of first order linear differential equations for which

$$C_i^{\alpha\beta} = c_i^{\alpha\beta}(x_i) \quad , \quad D^\alpha = d^{\alpha\beta}(x_i)U^\beta + e^\alpha(x_i) \quad . \quad (2.15)$$

The sufficient conditions of (2.7) then reduce to

$$c_i^{\alpha\beta} = -c_i^{\beta\alpha} \quad , \quad \frac{\partial c_i^{\alpha\beta}}{\partial x_i} = d^{\alpha\beta} - d^{\beta\alpha} \quad , \quad (2.16)$$

where the final relation of (2.7) has dropped out. We shall choose \mathfrak{M}_i^α to be the form, $M_i^{\alpha\beta}U^\beta$, where, by (2.6), the $M_i^{\alpha\beta}$ are related according to

$$c_i^{\alpha\beta} = M_i^{\alpha\beta} - M_i^{\beta\alpha} \quad . \quad (2.17)$$

For convenience, we shall impose the condition $M_i^{\alpha\beta} = -M_i^{\beta\alpha}$.

With the assumed form of \mathfrak{M}_i^α , we obtain by use of (2.6)

$$\frac{\partial \mathfrak{N}}{\partial U^\alpha} = \left(\frac{\partial M_i^{\alpha\beta}}{\partial x_i} - d^{\alpha\beta} \right) U^\beta - e^\alpha \quad . \quad (2.18)$$

Accordingly, \mathcal{L} assumes the form,

$$\mathcal{L} = N^{\alpha\beta} U^\alpha U^\beta - e U^\alpha U^\alpha + n(x_i) \quad , \quad N^{\alpha\beta} = \frac{1}{2} \left(\frac{\partial M^{\alpha\beta}}{\partial x_i} - d^{\alpha\beta} \right) = N^{\beta\alpha} \quad . \quad (2.19)$$

Combining these results into (2.4) gives the Lagrangian density

$$\mathcal{L} = \frac{1}{2} c_i^{\alpha\beta} U_i^\alpha U_i^\beta + \frac{1}{4} \left(\frac{\partial c_i^{\alpha\beta}}{\partial x_i} - 2d^{\alpha\beta} \right) U^\alpha U^\beta - e U^\alpha U^\alpha + n \quad , \quad (2.20)$$

where n is an arbitrary function of the x_i .

2.3 Second Order Differential Equations

A system of second order quasilinear differential equations can always be transformed into a first order differential system by treating the derivatives of the dependent variables as new dependent variables (Courant and Hilbert, 1966). However, this approach to the inverse problem is unsatisfactory for plasma problems because the introduction of new dependent variables is equivalent to introducing artificial degrees of freedom. Also, if we leave the number of degrees of freedom unchanged, the Lagrangian density for second order differential equations describing physical systems generally has a corresponding Hamiltonian density, \mathcal{K} . Conditions sufficient for the existence of a Lagrangian density, \mathcal{L} , will then also guarantee the existence of \mathcal{K} , and allow it to be used in such applications as evaluating nonlinear wave coupling coefficients (Sturrock, 1960a; Harker, 1970).

For purposes of example, the system of quasilinear second order differential equations will be assumed to have the form

$$A_{ij}^{\alpha\beta} U_{ij}^\beta + B^\alpha = 0 \quad (i = 1, \dots, N; \alpha, \beta = 1, \dots, M), \quad (2.21)$$

where U^α is the set of dependent variables; the coefficients $A_{ij}^{\alpha\beta}$ and B^α are functions of U_i^α , U^α , and x_i , and U_i^α and U_{ij}^α denote the first and second derivatives of U^α , respectively. Since $U_{ij}^\alpha = U_{ji}^\alpha$, the matrix $A_{ij}^{\alpha\beta}$ can be assumed symmetric in the indices i and j without loss of generality.

2.3.1 Necessary Conditions

Equation (2.3) may be written

$$\frac{\partial^2 \mathcal{L}}{\partial U_i^\alpha \partial U_j^\beta} U_{ij}^\beta + \frac{\partial^2 \mathcal{L}}{\partial U_i^\alpha U^\beta} U_i^\beta + \frac{\partial^2 \mathcal{L}}{\partial U_i^\alpha \partial x_i} - \frac{\partial \mathcal{L}}{\partial U^\alpha} = 0. \quad (2.22)$$

Comparing with (2.21) gives the necessary conditions for that system to be the set of Euler-Lagrange equations of \mathcal{L} as

$$2A_{ij}^{\alpha\beta} = \frac{\partial^2 \mathcal{L}}{\partial U_i^\alpha \partial U_j^\beta} + \frac{\partial^2 \mathcal{L}}{\partial U_j^\alpha \partial U_i^\beta}, \quad B^\alpha = U_i^\beta \frac{\partial^2 \mathcal{L}}{\partial U_i^\alpha \partial U^\beta} + \frac{\partial^2 \mathcal{L}}{\partial U_i^\alpha \partial x_i} - \frac{\partial \mathcal{L}}{\partial U^\alpha}. \quad (2.23)$$

Eliminating \mathcal{L} allows us to express these conditions in the form

$$\frac{\partial A_{ij}^{\alpha\beta}}{\partial U_k^\gamma} + \frac{\partial A_{jk}^{\alpha\beta}}{\partial U_i^\gamma} = \frac{\partial A_{ki}^{\beta\gamma}}{\partial U_j^\alpha} + \frac{\partial A_{ki}^{\gamma\alpha}}{\partial U_j^\beta},$$

$$A_{ij}^{\alpha\beta} = A_{ij}^{\beta\alpha}, \quad \frac{\partial B^\alpha}{\partial U_i^\beta} + \frac{\partial B^\beta}{\partial U_i^\alpha} = 2 \left(\frac{\partial}{\partial x_j} + U_j^\gamma \frac{\partial}{\partial U^\gamma} \right) A_{ij}^{\alpha\beta},$$

$$\left[\left(\frac{\partial}{\partial x_k} + U_k^\delta \frac{\partial}{\partial U^\delta} \right) \frac{\partial}{\partial U_k^\beta} - \frac{\partial}{\partial U^\beta} \right] A_{ij}^{\alpha\gamma} + \frac{\partial A_{ij}^{\alpha\beta}}{\partial U^\gamma} + \frac{\partial A_{ij}^{\beta\gamma}}{\partial U^\alpha} = \frac{1}{2} \left(\frac{\partial^2 B^\beta}{\partial U_i^\alpha \partial U_j^\gamma} + \frac{\partial^2 B^\beta}{\partial U_j^\alpha \partial U_i^\gamma} \right). \quad (2.24)$$

2.3.2 Sufficient Conditions

We can assume the general form of the Lagrangian density as

$$\mathcal{L} = \mathcal{J}(U_i^\alpha, U_j^\beta, x_j) + \mathcal{P}_i^\alpha(U_j^\beta, x_j) U_i^\alpha + Q(U_i^\alpha, x_i) \quad , \quad (2.25)$$

where \mathcal{J} is responsible for $A_{ij}^{\alpha\beta}$. Then it follows from the first relation of (2.23) that

$$2A_{ij}^{\alpha\beta} = \frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U_j^\beta} + \frac{\partial^2 \mathcal{J}}{\partial U_j^\alpha \partial U_i^\beta} \quad , \quad (2.26)$$

where only the set $\{U_i^\alpha\}$ are treated as independent variables. By interchanging α and β , we see that this differential equation can be solved only when $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, which is the second relation of (2.24).

Equation (2.26) constitutes $MN(M+1)(N+1)/4$ linear differential equations to be solved for \mathcal{J} . For \mathcal{J} to exist, consistency among the third order partial derivatives is required, i.e.

$$\frac{\partial}{\partial U_k^\gamma} \left(\frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U_j^\beta} \right) = \frac{\partial}{\partial U_j^\beta} \left(\frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U_k^\gamma} \right) \quad . \quad (2.27)$$

This implies that the first condition of (2.24) must be satisfied.

In the cases where either $\alpha = \beta$ or $i = j$, \mathcal{J} can be obtained by simple integration of (2.26). When both $\alpha \neq \beta$ and $i \neq j$, (2.26) becomes an ultrahyperbolic differential equation with constant coefficients (Koshlyakov, Smirnov, and Gliner, 1964). Its particular solutions can be obtained by Fourier transforms (Koshlyakov, Smirnov, and Gliner, 1964). The general solution of \mathcal{J} should include that of the homogeneous equation,

$$\frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U_j^\beta} + \frac{\partial^2 \mathcal{J}}{\partial U_j^\alpha \partial U_i^\beta} = 0 \quad , \quad (2.28)$$

which can be solved by separation of variables (see Appendix A). We shall show below that this nonuniqueness in \mathcal{J} will be restricted by other sufficient conditions.

From the second expression of (2.23) and (2.25), we obtain,

$$\left(\frac{\partial \mathcal{P}_i^\alpha}{\partial U^\beta} - \frac{\partial \mathcal{P}_i^\beta}{\partial U^\alpha} \right) U_i^\beta + \frac{\partial \mathcal{P}_i^\alpha}{\partial x_i} - \frac{\partial Q}{\partial U^\alpha} = -U_i^\beta \frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U^\beta} - \frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial x_i} + \frac{\partial \mathcal{J}}{\partial U^\alpha} + B^\alpha \quad . \quad (2.29)$$

Since (2.29) is an identity in U_i^α , U^α , and x_i , the right-hand side must be linear in U_i^α . Taking the derivative with respect to U_i^α , yields

$$\frac{\partial \mathcal{P}_i^\alpha}{\partial U^\beta} - \frac{\partial \mathcal{P}_i^\beta}{\partial U^\alpha} = \left(\frac{\partial}{\partial x_j} + U_j^\gamma \frac{\partial}{\partial U^\gamma} \right) \frac{\partial^2 \mathcal{J}}{\partial U_j^\alpha \partial U_i^\beta} + \frac{\partial^2 \mathcal{J}}{\partial U^\alpha \partial U_i^\beta} - \frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U^\beta} + \frac{\partial B^\alpha}{\partial U_i^\beta} \quad , \quad (2.30)$$

the left-hand side of which is antisymmetric in α and β . The third expression in (2.24) guarantees that the right-hand side is also antisymmetric in α and β . However, the fourth expression, which shows that (2.31) is symmetric in i and j , is not sufficient to establish that the right-hand side of (2.30) is independent of U_i^α . This requirement can be obtained by differentiating (2.30) with respect to U_j^γ ,

$$\left[\left(\frac{\partial}{\partial x_k} + U_k^\delta \frac{\partial}{\partial U^\delta} \right) \frac{\partial}{\partial U_k^\beta} - \frac{\partial}{\partial U^\beta} \right] \frac{\partial^2 \mathcal{J}}{\partial U_i^\alpha \partial U_j^\gamma} + \frac{\partial^3 \mathcal{J}}{\partial U^\gamma \partial U_i^\alpha \partial U_j^\beta} + \frac{\partial^3 \mathcal{J}}{\partial U^\alpha \partial U_i^\beta \partial U_j^\gamma} = \frac{\partial^2 B^\beta}{\partial U_i^\alpha \partial U_j^\gamma} \quad . \quad (2.31)$$

This is more restrictive than the third necessary condition, and represents limitations on \mathcal{J} in addition to the fourth.

Additional conditions from (2.30) for φ_i^α to exist are

$$\frac{\partial \mathcal{R}_i^{\alpha\beta}}{\partial u^\gamma} + \frac{\partial \mathcal{R}_i^{\beta\gamma}}{\partial u^\alpha} + \frac{\partial \mathcal{R}_i^{\gamma\alpha}}{\partial u^\beta} = 0, \quad (2.32)$$

analogous to the last expression of (2.8), where $\mathcal{R}_i^{\alpha\beta}$ is defined by

$$\mathcal{R}_i^{\alpha\beta} = \frac{\partial B^\alpha}{\partial u_i^\beta} - \left(\frac{\partial}{\partial x_j} + u_j^\gamma \frac{\partial}{\partial u^\gamma} \right) \frac{\partial^2 \mathcal{L}}{\partial u_j^\alpha \partial u_i^\beta}. \quad (2.33)$$

The solution of (2.30) enjoys one arbitrary choice of φ_i^α , just as the solutions of the first expression of (2.6) for \mathcal{M}_i^α , for each i . But if (2.29) is to give a consistent solution for Q , additional conditions restricting the set of φ_i^α then follow from (2.29), from the requirement that

$$\frac{\partial}{\partial u^\alpha} \left(\frac{\partial Q}{\partial u^\beta} \right) = \frac{\partial}{\partial u^\beta} \left(\frac{\partial Q}{\partial u^\alpha} \right), \quad (2.34)$$

analogous to (2.12). We have

$$\left(\frac{\partial}{\partial x_i} + u_i^\gamma \frac{\partial}{\partial u^\gamma} \right) \left(\frac{\partial^2 \mathcal{L}}{\partial u^\alpha \partial u_i^\beta} - \frac{\partial^2 \mathcal{L}}{\partial u^\beta \partial u_i^\alpha} \right) + \frac{\partial B^\alpha}{\partial u^\beta} - \frac{\partial B^\beta}{\partial u^\alpha} = \left(\frac{\partial}{\partial x_i} + u_i^\gamma \frac{\partial}{\partial u^\gamma} \right) \left(\frac{\partial \varphi_i^\alpha}{\partial u^\beta} - \frac{\partial \varphi_i^\beta}{\partial u^\alpha} \right). \quad (2.35)$$

In summary, sufficient conditions for (2.21) to represent a set of Euler-Lagrange equations of a Lagrangian density, \mathcal{L} , of the form (2.25) are as follows: for a solution of (2.26) for \mathcal{L} to exist, $A_{ij}^{\alpha\beta}$ must satisfy the first two conditions of (2.24); for a solution of (2.30) for φ_i^α to exist, \mathcal{L} must be further restricted by (2.31) and (2.32), while $A_{ij}^{\alpha\beta}$ and B^α must satisfy the third condition of (2.24); for a solution

of (2.29) for Q to exist, p_i^α , \mathcal{J} , and B^α must be restricted by (2.35). These conditions, as well as those of Section 2.2.2 for first order equations, agree with the general forms obtained by Tonti (1969b) by potentiality analysis of differential operators in function space (Vainberg, 1964; Tonti, 1969a).

2.3.3 Nonlinear Differential Equations of the Second Rank

As a demonstration of the foregoing results, sufficient conditions, and the corresponding Lagrangian density, will be obtained for the differential system

$$A_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} + a_{ij}^{\alpha\beta\gamma} U^\gamma + a_{ijk}^{\alpha\beta\gamma} U_k^\gamma \quad ,$$

$$B^\alpha = b^\alpha + b^{\alpha\beta} U^\beta + b^{\alpha\beta\gamma} U^\beta U^\gamma + b_i^{\alpha\beta} U_i^\beta + b_i^{\alpha\beta\gamma} U_i^\beta U^\gamma + b_{ij}^{\alpha\beta\gamma} U_i^\beta U_j^\gamma \quad . \quad (2.36)$$

The coefficients, $a_{ij}^{\alpha\beta}$, etc. and b^α , etc., are functions of x_i ; the $a_{ij}^{\alpha\beta}$ are symmetric in i and j , the $b^{\alpha\beta\gamma}$ in β and γ , and the $b_{ij}^{\alpha\beta\gamma}$ in (β, i) and (γ, j) . With the coefficients of (2.36), the equations of (2.21) are nonlinear and of the second rank because of the presence of $a_{ij}^{\alpha\beta\gamma}$, $a_{ijk}^{\alpha\beta\gamma}$, $b^{\alpha\beta\gamma}$, $b_i^{\alpha\beta\gamma}$ and $b_{ij}^{\alpha\beta\gamma}$.

Substituting (2.36) into (2.24) yields for the $a_{ij}^{\alpha\beta}$, etc.,

$$a_{ij}^{\alpha\beta} = a_{ij}^{\beta\alpha} \quad , \quad a_{ij}^{\alpha\beta\gamma} = a_{ij}^{\beta\alpha\gamma} \quad , \quad a_{ijk}^{\alpha\beta\gamma} = a_{ijk}^{\beta\alpha\gamma} \quad ,$$

$$a_{ijk}^{\alpha\beta\gamma} + a_{jki}^{\alpha\beta\gamma} = a_{kij}^{\beta\gamma\alpha} + a_{kij}^{\gamma\alpha\beta} \quad , \quad (2.37)$$

and for the b^α , etc.,

$$b_i^{\alpha\beta} + b_i^{\beta\alpha} = 2 \left(\frac{\partial a_{ij}^{\alpha\beta}}{\partial x_j} \right) , \quad b_i^{\alpha\beta\gamma} + b_i^{\beta\alpha\gamma} = 2 \left(\frac{\partial a_{ij}^{\alpha\beta\gamma}}{\partial x_j} \right) ,$$

$$b_{ij}^{\alpha\beta\gamma} + b_{ij}^{\beta\alpha\gamma} = a_{ij}^{\alpha\beta\gamma} + \frac{\partial a_{ijk}^{\alpha\beta\gamma}}{\partial x_k} . \quad (2.38)$$

We can now assume the general form for \mathcal{J} as

$$\mathcal{J} = \mathcal{D}_{ijk}^{\alpha\beta\gamma} U_i U_j U_k + \frac{1}{2} \left(a_{ij}^{\alpha\beta} + a_{ij}^{\alpha\beta\gamma} U^\gamma \right) U_i U_j , \quad (2.39)$$

and choose to impose

$$\mathcal{D}_{ijk}^{\alpha\beta\gamma} = \mathcal{D}_{jik}^{\beta\alpha\gamma} = \mathcal{D}_{ikj}^{\alpha\gamma\beta} . \quad (2.40)$$

The requirement of (2.31) reduces to

$$\frac{\partial \mathcal{D}_{ijk}^{\alpha\beta\gamma}}{\partial x_k} = \frac{1}{6} \left(2b_{ij}^{\gamma\alpha\beta} + a_{ij}^{\alpha\beta\gamma} - a_{ij}^{\beta\gamma\alpha} \right) , \quad (2.41)$$

while (2.32) becomes

$$b_i^{\alpha\beta\gamma} + b_i^{\beta\gamma\alpha} + b_i^{\gamma\alpha\beta} = \frac{\partial}{\partial x_j} \left(a_{ij}^{\alpha\beta\gamma} + a_{ij}^{\beta\gamma\alpha} + a_{ij}^{\gamma\alpha\beta} \right) . \quad (2.42)$$

Using (2.30) we find that p_i^α takes the form

$$p_i^\alpha = p_i^{\alpha\beta} U^\beta + p_i^{\alpha\beta\gamma} U^\beta U^\gamma , \quad (2.43)$$

where $p_i^{\alpha\beta}$ and $p_i^{\alpha\beta\gamma}$ are to be obtained from the differential relations,

$$p_i^{\alpha\beta} - p_i^{\beta\alpha} = b_i^{\alpha\beta} - \frac{\partial a_{ij}^{\alpha\beta}}{\partial x_j} , \quad p_i^{\alpha\beta\gamma} - p_i^{\beta\alpha\gamma} = \frac{1}{2} \left(b_i^{\alpha\beta\gamma} - \frac{\partial a_{ij}^{\alpha\beta\gamma}}{\partial x_j} \right) . \quad (2.44)$$

The conditions of (2.35) reduce to

$$b^{\alpha\beta} - b^{\beta\alpha} = \frac{\partial}{\partial x_i} \left(b_i^{\alpha\beta} - \frac{\partial^a \alpha\beta}{\partial x_j} \right), \quad b^{\alpha\beta\gamma} - b^{\beta\alpha\gamma} = \frac{1}{2} \frac{\partial}{\partial x_i} \left(b_i^{\alpha\beta\gamma} - \frac{\partial^a \alpha\beta\gamma}{\partial x_j} \right), \quad (2.45)$$

without further restricting the choices of $p_i^{\alpha\beta}$ and $p_i^{\alpha\beta\gamma}$.

The general form of Q is

$$Q = -b^{\alpha\alpha} U + q^{\alpha\beta} U U^\beta + q^{\alpha\beta\gamma} U U^\beta U^\gamma + q(x_i), \quad (2.46)$$

where, according to (2.29),

$$q^{\alpha\beta} = -\frac{1}{2} \left(b^{\alpha\beta} - \frac{\partial p_i^{\alpha\beta}}{\partial x_i} \right) = q^{\beta\alpha}, \quad q^{\alpha\beta\gamma} = -\frac{1}{3} \left(b^{\alpha\beta\gamma} - \frac{\partial p_i^{\alpha\beta\gamma}}{\partial x_i} \right) = q^{\beta\alpha\gamma}, \quad (2.47)$$

and $q(x_i)$ is arbitrary.

We find that the $p_{ijk}^{\alpha\beta\gamma}$ are determined from (2.41) only to within an arbitrary curl tensor, $\epsilon_{kmn} \partial d_{ijn}^{\alpha\beta\gamma} / \partial x_m$, where ϵ_{kmn} is the antisymmetric unit tensor. According to (2.44) the $p_i^{\alpha\beta}$ and $p_i^{\alpha\beta\gamma}$ are both determined only to within an arbitrary tensor symmetric in α and β .

In the special case that the set of differential equations expressed by (2.21) are linear, the relevant sufficient conditions reduce to

$$a_{ij}^{\alpha\beta} = a_{ij}^{\beta\alpha}, \quad b_i^{\alpha\beta} + b_i^{\beta\alpha} = 2 \left(\frac{\partial^a \alpha\beta}{\partial x_j} \right), \quad b^{\alpha\beta} - b^{\beta\alpha} = \frac{\partial}{\partial x_i} \left(b_i^{\alpha\beta} - \frac{\partial^a \alpha\beta}{\partial x_j} \right). \quad (2.48)$$

These are equivalent to those used by van der Vaart (1967), who considered the case $N = 1$ [Equation (16) by van der Vaart (1967)].

2.4 Differential Transformations

In Sections 2.2 and 2.3, we assumed that the α th equation of the differential systems represented by (2.1) and (2.21) is an Euler-Lagrange equation with respect to variation of U^α . Now these sets of equations are transformable, for example by change of variables, to equivalent sets which no longer satisfy the sufficiency conditions. Solution of the inverse problem of the calculus of variations is consequently less restrictive than is suggested by the sufficiency conditions: it may be possible to convert seemingly non-Euler-Lagrange equations to Euler-Lagrange equation form by use of appropriate transformations. The problem this poses is how to recognize when such transformation is possible. Here we shall comment briefly on the transformations likely to be involved. We cannot provide a general solution.

Among the range of possible transformations, those retaining the number of dependent variables unchanged include: (a) matrix transformation of the dependent variables; (b) matrix transformation of the differential equations, by use of integration factors, leaving the dependent variables unchanged (Davis, 1929); and (c) differential transformation that raises the order of the differential equations. For a self-consistent system of M differential equations, Method (a) involves M^2 functions of x_i as the elements of the transformation matrix; Method (b) involves M^2 functions of x_i as matrix elements, the dependent variables, and perhaps their first derivatives, while Method (c) involves only M functions of x_i , the new dependent variables, and their derivatives. In each of these cases, the number of functions at our disposal in general falls short of the number of sufficient conditions to be applied.

Consequently, we cannot expect that these methods will always be successful in transforming non-Euler-Lagrange equations to Euler-Lagrange equations.

Despite their inability to guarantee successful transformation, methods (a)-(c) are of interest in dealing with equations of forms, such as those occurring in some physical and engineering problems, where the number of sufficient conditions is reduced. Well-known examples can be readily found in the cases of replacing electromagnetic field variables by potential variables (Goldstein, 1950), and velocity variables by Clebsch variables (Lamb, 1930), before the Lagrangian densities can be obtained. To illustrate their use here, we shall apply a linear differential transformation to convert a first order linear non-Euler-Lagrange system to a second order Euler-Lagrange system in Section 2.4.1. Using the sufficiency conditions obtained in Section 2.3.3, we shall then determine the maximum number of dependent and independent variables for a successful transformation to be possible. As examples, a resistive transmission line, and a warm collisional plasma, will be studied in Sections 2.4.2 and 2.4.3, respectively.

2.4.1 Differential Transformation of Linear Differential Equations

Under the linear transformation,

$$U^\alpha = T_i^{\alpha\beta}(x_j)V_i^\beta + S^{\alpha\beta}(x_i)V^\beta + R^\alpha(x_i) \quad , \quad (2.49)$$

the first order linear differential equations of (2.1) and (2.15) become second order equations, with coefficients of the form (2.36) given by

$$\begin{aligned}
a_{ij}^{\alpha\beta} &= \frac{1}{2} \left(c_i^{\alpha\gamma} T_j^{\gamma\beta} + c_j^{\alpha\gamma} T_i^{\gamma\beta} \right) , & b_i^{\alpha\beta} &= c_j^{\alpha\gamma} \left(\frac{\partial T_i^{\gamma\beta}}{\partial x_j} \right) + d^{\alpha\gamma} T_i^{\gamma\beta} + c_i^{\alpha\gamma} S^{\gamma\beta} , \\
b^{\alpha\beta} &= c_i^{\alpha\gamma} \left(\frac{\partial S^{\gamma\beta}}{\partial x_i} \right) + d^{\alpha\gamma} S^{\gamma\beta} , & b^\alpha &= c_i^{\alpha\beta} \left(\frac{\partial R^\beta}{\partial x_i} \right) + d^{\alpha\beta} R^\beta + e^\alpha .
\end{aligned} \tag{2.50}$$

The sufficiency conditions imposed on these coefficients by (2.48) constitute $MN(M-1)(N+1)/4 + MN(M+1)/2 + M(M-1)/2$ conditions to be satisfied by the $M^2(N+1)$ unknown functions, $T_i^{\alpha\beta}$ and $S^{\alpha\beta}$. The number of unknowns will consequently be no less than that of the conditions only when

$$M(M-1)(N+1)(N-2) \leq 4M . \tag{2.51}$$

When (2.51) is satisfied, the equations obtained by substituting (2.50) into (2.48) are in general solvable for the $T_i^{\alpha\beta}$, $S^{\alpha\beta}$, and R^α , for the given coefficients $c_i^{\alpha\beta}$, $d^{\alpha\beta}$, and e^α . The resulting second order differential equations will automatically become Euler-Lagrange equations, with the corresponding Lagrangian density derivable using the results of Section 2.3.3.

2.4.2 Resistive Transmission Line

The first order equations for a linear resistive transmission line, as shown in figure 2.1, take the form,

$$\frac{\partial I}{\partial x} + C \frac{\partial V}{\partial t} = 0 , \quad L \frac{\partial I}{\partial t} + \frac{\partial V}{\partial x} + RI = 0 , \tag{2.52}$$

where I , V , C , L , and R denote the normalized current, voltage, distributed capacitance, inductance, and resistance, respectively. According to (2.16), (2.52) is not in Euler-Lagrange equation form.

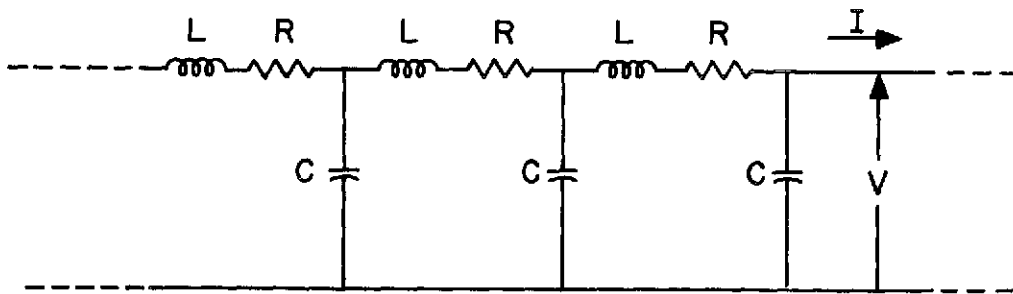


Figure 2.1 Resistive Transmission Line

With $\{I, V\} = \{U^1, U^2\}$ and $\{f, g\} = \{V^1, V^2\}$, the procedure outlined in Section 2.4.1 may be followed to give

$$\begin{bmatrix} I \\ V \end{bmatrix} \begin{bmatrix} C \frac{\partial}{\partial x} + C \frac{\partial}{\partial t} & \frac{\partial}{\partial x} + LC \frac{\partial}{\partial t} - RC \\ \frac{\partial}{\partial x} + LC \frac{\partial}{\partial t} & L \frac{\partial}{\partial x} + L \frac{\partial}{\partial t} - R \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, \quad (2.53)$$

as an appropriate transformation. The corresponding Lagrangian density is obtained, through the use of (2.25), (2.39), (2.43), (2.44), (2.46), and (2.48), in the form,

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \left[C \left(\frac{\partial f}{\partial x} \right)^2 + 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + L \left(\frac{\partial g}{\partial x} \right)^2 \right] + C \frac{\partial f}{\partial x} \left(\frac{\partial f}{\partial t} + L \frac{\partial g}{\partial t} \right) \\ & + L \frac{\partial g}{\partial x} \left(C \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} \right) + \frac{LC}{2} \left[C \left(\frac{\partial f}{\partial t} \right)^2 + 2 \frac{\partial f}{\partial t} \frac{\partial g}{\partial t} + L \left(\frac{\partial g}{\partial t} \right)^2 \right] \\ & - \frac{RC}{2} \left[g \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} \right) - f \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} \right) \right] - \frac{C}{2} R^2 g^2. \end{aligned} \quad (2.54)$$

The Euler-Lagrange equations of (2.54) can be shown to agree with the result of using (2.53) in (2.52). Because the number of sufficient conditions is less than the number of transformation coefficients, $T_i^{\alpha\beta}$ and $S^{\alpha\beta}$, (2.54) is but one example among an unlimited number of appropriate Lagrangian densities.

2.4.3 Warm Collisional Plasma

For small one-dimensional perturbations in a plasma with a homogeneous immobile neutralizing positive ion background, the macroscopic equations can be written as,

$$\frac{\partial E}{\partial x} + \frac{en}{\epsilon_0} = 0 \quad , \quad \frac{\partial n}{\partial t} + n_0 \frac{\partial v}{\partial x} = 0 \quad , \quad m \frac{\partial v}{\partial t} + m\nu v + \gamma P_0 \left(\frac{\partial n}{\partial x} \right) n_0 + eE = 0 \quad , \quad (2.55)$$

where m and e are the electron mass and charge; n_0 and P_0 are the quiescent electron density and pressure; ν is the effective electron-ion and electron-neutral momentum transfer collision frequency; γ is the adiabatic index, and ϵ_0 is the permittivity of free space. The pressure term, $\gamma P_0 \frac{\partial n}{\partial x} n_0$, results from assuming an adiabatic equation of state for the electrons,

$$(P_0 + P)(n_0 + n)^{-\gamma} = P_0 n_0^{-\gamma} \quad . \quad (2.56)$$

In (2.55), the dependent variables are the electric field, E , the electron density perturbation, n , and the electron drift velocity, v . Application of (2.16) indicates that these equations are not in Euler-Lagrange equation form. They satisfy (2.51), however, so that the differential transformation defined in Section 2.4.1 can be used. To reduce apparent complexity, it is helpful to rewrite (2.55) in terms of normalized variables, defined by

$$\begin{aligned} \bar{E} &= \frac{eE}{m\nu_t \omega_p} \quad , \quad \bar{n} = \frac{n}{n_0} \quad , \quad \bar{v} = \frac{v}{\nu_t} \quad , \\ X &= \frac{\omega_p x}{\nu_t} \quad , \quad T = \omega_p t \quad , \quad \bar{v} = \frac{v}{\omega_p} \quad , \end{aligned} \quad (2.57)$$

where $\nu_t [= (P_0/m)^{1/2}]$ and $\omega_p [= (n_0 e^2 / m \epsilon_0)^{1/2}]$ are the electron thermal velocity and plasma frequency, respectively. We then have, in place of (2.55),

$$\frac{\partial \bar{E}}{\partial X} + \bar{n} = 0 \quad , \quad \frac{\partial \bar{n}}{\partial T} + \frac{\partial \bar{v}}{\partial X} = 0 \quad , \quad \frac{\partial \bar{v}}{\partial T} + \bar{v} \bar{v} + \gamma \frac{\partial \bar{n}}{\partial X} + \bar{E} = 0 \quad . \quad (2.58)$$

From (2.48) - (2.50), and (2.58), it can be shown that are appropriate transformation,

$$\begin{bmatrix} \bar{E} \\ \bar{v} \\ \bar{n} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} f \\ g \\ h \end{bmatrix} \quad , \quad (2.59)$$

should have the following elements:

$$\begin{aligned} c_{11} &= \bar{b} \frac{\partial}{\partial X} + \bar{a} \quad , \quad c_{12} = \bar{c} \frac{\partial}{\partial X} - \bar{a} \frac{\partial}{\partial T} \quad , \\ c_{13} &= -\gamma \bar{a} \frac{\partial}{\partial X} + \bar{c} \frac{\partial}{\partial T} - \bar{b} - \bar{v} \bar{c} \quad , \quad c_{21} = \bar{c} \frac{\partial}{\partial X} \quad , \\ c_{22} &= \bar{d} \frac{\partial}{\partial X} \quad , \quad c_{23} = \bar{d} \frac{\partial}{\partial T} - \bar{c} - \bar{v} \bar{d} \quad , \quad c_{31} = -\bar{a} \frac{\partial}{\partial X} - \frac{\bar{e}}{\gamma} \quad , \\ c_{32} &= \frac{\bar{e}}{\gamma} \left(\frac{\partial}{\partial T} \right) \quad , \quad c_{33} = \bar{e} \frac{\partial}{\partial X} + \bar{a} \quad , \end{aligned} \quad (2.60)$$

where \bar{a} , \bar{b} , \bar{c} , \bar{d} , and \bar{e} are arbitrary constants, except that they should make (2.59) a reversible transformation.

The corresponding Lagrangian density, obtained through using (2.25), (2.39), (2.43), (2.46), and (2.48), may be expressed in terms of f , g , and h as,

$$\begin{aligned}
\mathcal{L} &= \frac{\bar{b}}{2} \left(\frac{\partial f}{\partial X} \right)^2 + \bar{c} \frac{\partial f}{\partial X} \frac{\partial g}{\partial X} + \frac{\bar{d}}{2} \left(\frac{\partial g}{\partial X} \right)^2 - \gamma \bar{a} \frac{\partial f}{\partial X} \frac{\partial h}{\partial X} + \frac{\gamma \bar{e}}{2} \left(\frac{\partial h}{\partial X} \right)^2 \\
&= \frac{\bar{e}}{2\gamma} \left(\frac{\partial g}{\partial T} \right)^2 + \frac{\bar{d}}{2} \left(\frac{\partial h}{\partial T} \right)^2 - \bar{a} \frac{\partial f}{\partial X} \frac{\partial g}{\partial T} + \bar{c} \frac{\partial f}{\partial X} \frac{\partial h}{\partial T} + (\bar{d} + \bar{e}) \frac{\partial g}{\partial X} \frac{\partial h}{\partial T} \\
&+ (\bar{b} + \bar{\nu} \bar{c} - \bar{e}) \frac{\partial h}{\partial X} f + (\bar{c} + \bar{\nu} \bar{d}) \frac{\partial h}{\partial X} g + \frac{\bar{e}}{\gamma} \frac{\partial f}{\partial T} g + \bar{a} \frac{\partial g}{\partial T} h \\
&+ \frac{\bar{e}}{2\gamma} f^2 - \bar{a} f h + \left(\frac{\bar{b}}{2} + \bar{\nu} \bar{c} + \frac{\bar{\nu}^2 \bar{d}}{2} \right) h^2 . \tag{2.61}
\end{aligned}$$

The Euler-Lagrange equations of (2.61) can be shown to agree with the result of using (2.59) and (2.60) in (2.58).

2.5 Discussion

In this section, we have considered the inverse problem of the calculus of variations for systems of first and second order quasilinear partial differential equations. The approach has been to compare the form of the Euler-Lagrange equations with that of an arbitrarily chosen set of equations. The resulting sufficiency conditions agree with general results obtained previously by the more abstract method of imposing conditions of potentiality of operators in a function space (Vainberg, 1964; Tonti, 1969). By restricting ourselves to equations of quasilinear form, we were able to determine the Lagrangian density explicitly, if the sufficient conditions are satisfied by the given set of differential equations. As examples, the results were applied to systems of first order linear equations, and second order nonlinear equations.

The explicit formulation described here has led to some success in using differential transformations to convert non-Euler-Lagrange equations to Euler-Lagrange equation form by changing the dependent variables. The

examples on the resistive transmission line and warm collisional plasma served to show their importance. This operation, together with the other transformation techniques discussed in Section 2.4, constitute powerful mathematical ways to define appropriate generalized (dependent) variables in a given inverse problem. This is complementary to the procedure of choosing suitable 'geometric' and 'force' variables in physical variational problems, described by Penfield and Haus (1967).

Our discussion of transformation techniques suggests some significant problems for further study in the inverse problem of the calculus of variations. For instance, recognition of equivalent systems of formally different differential equations becomes important if the area of application of the inverse problem is to be enlarged.

Although the mathematical approach is rigorous, the need to choose appropriate generalized variables, and to rewrite the equations in Euler-Lagrange equation form, reduces the practical value of our approach to the inverse problem in those physical situations in which intuitive energy considerations can be readily used to guess at, and establish appropriate Lagrangian densities. The latter approach relies on the experience that a physical system with well-defined energy can be described in terms of Hamilton's variational principle. The trial procedure then involves assigning generalized variables, according to the number of degrees of freedom, before guessing a Lagrangian density that includes various forms of kinetic and potential energies. The trial Lagrangian is then checked by the relatively simple process of applying Hamilton's principle to produce Euler-Lagrange equations that are the correct equations of the problem. A small number of trial-and-error exercises may then result in a suitable Lagrangian density.

In anticipation of the Lagrangian applications to plasma physics, to be discussed in Section 3, we may point out two other problems. The first concerns the explicit distinction between dependent and independent variables generally assumed in the inverse problem; the variation principle established in Section 3 relies on an unorthodox arrangement which we term the 'dual role' of the variables. This dual role can be understood to be some implicit dependence, in the form $x[y(z)]$, with both x and y assuming the role of generalized variables. The extension of the present results to incorporate this dual role is a problem that we have not had time to study in detail.

The second problem is related to the so-called 'Lagrange (or Bolza) problem' (Rund, 1966; Bliss, 1946). This deals with the variation principles whose solutions are to be determined under subsidiary constraints. This type of problem immediately introduces serious practical difficulties into the corresponding inverse problem, since there is no mathematical rule to determine which of the given differential equations are to be treated as constraint equations. In the corresponding physical problem, however, no such difficulty arises once the degrees of freedom are determined.

3. LAGRANGIAN AND HAMILTONIAN DENSITIES FOR PLASMAS

3.1 Introduction

In this section, appropriate Lagrangian and Hamiltonian densities will be established for the macroscopic model, in which the plasma is described by Maxwell's equations and moments of the Boltzmann equation. We shall first construct the Lagrangian density from energy considerations, rather than the formal mathematical approach of Section 2, and then verify its validity by applying Hamilton's principle to obtain the required Euler-Lagrange equations. Next, the corresponding Hamiltonian density will be derived.

Lagrangian densities have already been obtained for plasmas under various assumptions. In the microscopic model, in which the plasma is described by Maxwell's equations and the Vlasov equation, an appropriate Lagrangian has been obtained by Low (1958). The Lagrangian in a relativistically covariant formulation was discussed by Sturrock (1958a), who pointed out a difficulty associated with the choice of variables: the calculus of variations distinguishes between the generalized (dependent) and the integration (independent) variables; the description of plasmas, however, involves the charged particle trajectory, which is conventionally treated in Lagrangian coordinates, and the electromagnetic field, which is conventionally treated in Eulerian coordinates. The distinction is illustrated in figure 3.1, which shows a particle trajectory in phase space. The Eulerian coordinates, $(\underline{x}, \underline{v}, t)$, describe the particle with relation to a fixed set of axes, whereas in the Lagrangian description, the axes follow the trajectory and the particle location is denoted by $(\underline{x}_0, \underline{v}_0, 0)$.

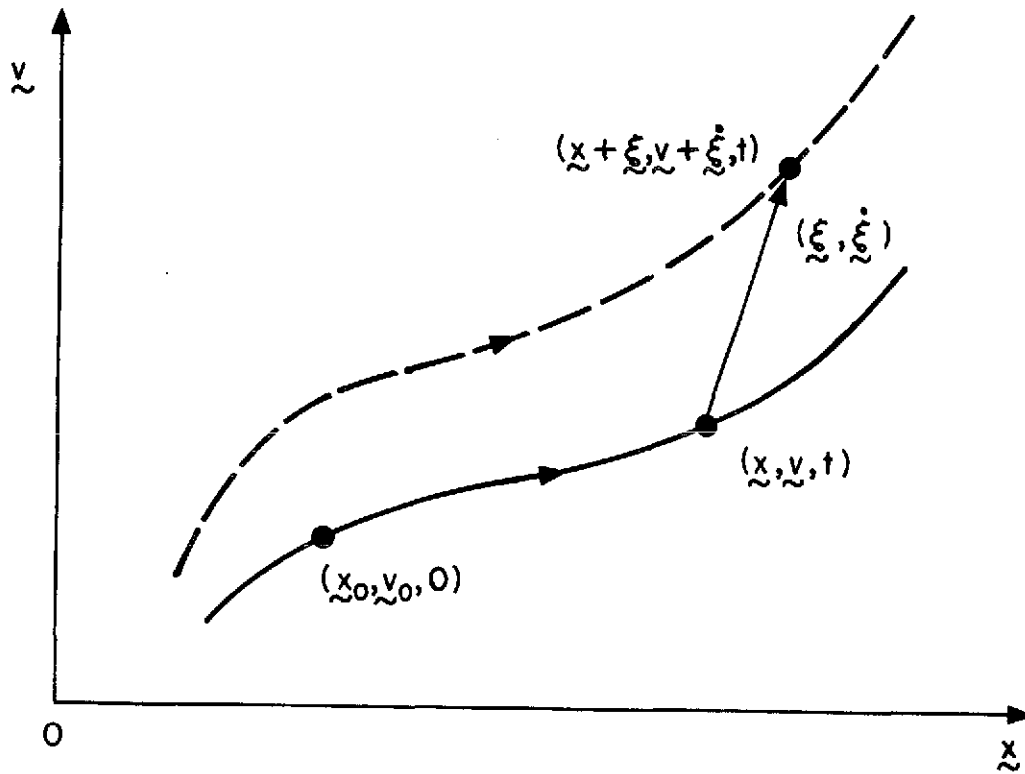


Figure 3.1 In phase space, a particle trajectory (—) can be specified by its Lagrangian (initial) coordinates, $(\underline{x}_0, \underline{v}_0, 0)$, or its Eulerian (present) coordinates, $(\underline{x}, \underline{v}, t)$. The polarization vector, $\underline{\xi}$, used by Sturrock (1958), connects the real particle trajectory (---) to a specified trajectory (—).

To resolve this difficulty, Sturrock introduced a "field-like" expansion of the total Lagrangian with the particle position vector written as $\underline{x} + \underline{\xi}$, where $\underline{\xi}$ is the polarization vector with respect to a specified particle position \underline{x} . The case when $\underline{\xi}$ is defined by comparing the two position vectors simultaneously is also shown in figure 3.1. Thus \underline{x} becomes the integration variable for both the particle displacement, $\underline{\xi}$, and the electromagnetic field. On the other hand, Low preferred to modify the conventional formulation of the calculus of variations in order to incorporate the two types of variables into his Lagrangian: it is composed of two parts, one for the motion of the particles in a given electromagnetic field, and the other for Maxwell's equations with given particle trajectories. In the application of Hamilton's principle, this requires that the electromagnetic potentials be treated as given functions of particle spatial position, \underline{x} , in the variations with respect to the particle trajectory, $\underline{x}(x_0, v_0, t)$, and that \underline{x} be treated as an integration variable in the variations with respect to the electromagnetic potentials. Since the particle spatial coordinates play the dual role of generalized and integration variables, while the electromagnetic potentials play the dual role of given functions and generalized variables, in what follows we shall describe this as the "dual role" approach.

A Lagrangian analogous to that of Low (1958) and Sturrock (1958a) has not yet been presented for the macroscopic plasma model. There have been successful attempts at derivation for simplified models, often using Lagrange multipliers that necessitate the use of generalized variables not corresponding to the physical degrees of freedom. For example,

Katz (1961) has given an appropriate Lagrangian for a one-fluid, compressible plasma, with scalar pressure and adiabatic motions assumed. He obtained Maxwell's equations and the equation of fluid motion by application of Hamilton's principle, with the use of two Lagrange multipliers to incorporate the fluid mass and charge continuity conditions. Su (1961) has obtained a suitable Lagrangian for an inviscid, adiabatic plasma with infinite electrical conductivity. The resulting Euler-Lagrange equations include Maxwell's equations, number density continuity equations, and three other equations from which the correct force law can be derived. Because he used some implicit form for the pressure term in his Lagrangian, the number density, n , was also treated as a generalized variable. Newcomb (1962) derived the Lagrangian for a hydromagnetic plasma, without using Lagrange multipliers, for the cases of scalar and axisymmetric pressure. The mass and magnetic flux continuity equations, and the plasma adiabatic pressure state equation, were treated as subsidiary constraints that define the variation of the Lagrangian and result in the correct force law.

The foregoing results (Katz, 1961; Su, 1961; Newcomb, 1962) are characterized by the absence of the dual role of the variables used in their formulations. In contrast, our own approach in Section 3.2 will follow that of Low closely: the plasma cell trajectories and the electromagnetic potentials will be used in dual roles. The Lagrangian thus established includes tensor pressure and elastic collisions. Following the technique of Newcomb, the first (continuity) and third (heat flow) moment equations are used as subsidiary constraints to show that the Euler-Lagrange equations obtained by application of Hamilton's principle

are the second (momentum) moment equation and Maxwell's equations. Our reasons for taking the "dual role" Low approach, rather than the polarization vector approach of Sturrock, will become clear in Section 4.2, where we shall show that the perturbation expansion necessary to treat nonlinear wave-wave interactions is thus greatly facilitated.

After the appropriate macroscopic Lagrangian is established through the dual role approach, an important question arises: does this form of Hamilton's principle, synthesized at the expense of the explicit distinction between the dependent and independent variables, form an acceptable basis for a plasma canonical formulation that agrees with the known plasma equations? If this can be answered affirmatively, the macroscopic, as well as the Low Lagrangian, will be acceptable in their total form. This constitutes our motivation to derive the appropriate plasma Hamiltonian and canonical mechanics through the dual role approach. Furthermore, the corresponding Hamiltonians will be readily subjected to perturbation approximations, analogous to those applied to the Lagrangians (Low, 1958; Sturrock, 1958a; Newcomb, 1962; Galloway and Kim, 1971; see also Section 4). The resulting approximate Hamiltonians, although not to be derived here, should have important applications to nonlinear plasma problems (Sturrock, 1960a; Harris, 1969; Harker, 1970).

Sturrock (1958a) and Newcomb (1962) have derived Hamiltonians for their quadratic plasma Lagrangians. In classical mechanics, if a given Lagrangian is at least quadratic in the derivatives of the generalized variables, a corresponding Hamiltonian can always be obtained (Goldstein, 1950). Obvious difficulties arise in the case of the macroscopic Lagrangian to be derived, in which the variables play

dual roles. Sections 3.3.2 to 3.3.4 will show how to modify the well-known Legendre transformation and Poisson brackets so that the dual role is embodied in a self-consistent Hamiltonian formulation. For completeness, a Hamiltonian is derived in Section 3.3.6 from the Low Lagrangian to facilitate a comparison between the microscopic and macroscopic models of plasma Hamiltonian mechanics.

3.2 Lagrangian Density

Newcomb (1962) has given the following Lagrangian density, \mathcal{L}_N ,

$$\mathcal{L}_N = \mathcal{J}_N - \mathcal{U}_N \quad , \quad \mathcal{J}_N = \frac{1}{2} \rho u^2 \quad , \quad \mathcal{U}_N = \frac{P}{\gamma-1} + \frac{B^2}{2\mu_0} \quad , \quad (3.1)$$

where \mathcal{J}_N and \mathcal{U}_N denote the kinetic and potential energy densities, respectively; ρ is the mass density; \underline{u} and P are the plasma velocity and scalar pressure; \underline{B} is the magnetic field; γ is the adiabatic index, and μ_0 is the permeability of free space. Gravity has been neglected, and the hydromagnetic assumption of infinite conductivity,

$$\underline{E} + \underline{u} \times \underline{B} = 0 \quad , \quad (3.2)$$

has been made.

Choosing \underline{x} as the generalized variable, he applies Hamilton's principle to the total Lagrangian,

$$L = \int_V dx \mathcal{L} \quad . \quad (3.3)$$

The variations in ρ , P , and \underline{B} , due to that in \underline{x} , are defined by the constraint equations of mass continuity, adiabatic state, and magnetic

flux conservation, respectively,

$$\rho d\tilde{x} = \text{constant} \quad , \quad P\rho^{-\gamma} = \text{constant} \quad , \quad \tilde{B} \cdot d\tilde{\sigma} = \text{constant} \quad , \quad (3.4)$$

where $d\tilde{\sigma}$ denotes an area element moving with the fluid. The resulting Euler-Lagrange equation is the appropriate force law in the hydromagnetic approximation,

$$\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (\rho u u) + \nabla P - (\nabla \times \tilde{B}) \times \tilde{B} = 0 \quad . \quad (3.5)$$

With the general form of (3.1) in mind, the main part of \mathcal{L} will be obtained in Section 3.2.1 by integrating the Low Lagrangian in velocity space. Use of Eulerian coordinates in Section 3.2.2 enables us to establish the necessary corrections to complete this approximate Lagrangian. Section 3.2.3 confirms its validity by demonstrating that the corresponding Euler-Lagrange equations are Maxwell's equations and the force law for macroscopic plasmas.

3.2.1 Macroscopic Approximation to the Low Lagrangian

The total Low Lagrangian, L_L , has the form

$$L_L = \sum_{\tilde{v}_0} \int_{\tilde{x}_0} d\tilde{x}_0 \int_{\tilde{v}_0} d\tilde{v}_0 f(\tilde{x}_0, \tilde{v}_0) \ell + \int_{\tilde{V}} d\tilde{x} \left(\frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \right) \quad ,$$

$$\ell = \frac{m}{2} v^2(\tilde{x}_0, \tilde{v}_0, t) - q[\varphi(\tilde{x}, t) - \tilde{v}(\tilde{x}_0, \tilde{v}_0, t) \cdot \tilde{A}(\tilde{x}, t)] \quad , \quad (3.6)$$

where the summation Σ implies more than one particle species; f is the velocity distribution function; \tilde{x} and \tilde{v} are the particle position

and velocity at time t ; \underline{x}_0 and \underline{v}_0 are their values at $t = 0$; V denotes the spatial volume occupied by all the particles under consideration at time t ; V_0 denotes the spatial volume occupied by the particles of a species at $t = 0$, as is schematically shown in figure 3.2; m and q denote the particle mass and charge; ϵ_0 is the permittivity of free space; \underline{E} , \underline{B} , ϕ , and \underline{A} represent the electric and magnetic fields, and the scalar and vector potentials, respectively, where

$$\underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t} \quad , \quad \underline{B} = \nabla \times \underline{A} \quad . \quad (3.7)$$

Note that we have chosen to use V and V_0 , as shown in figure 3.2, to confine ourselves to a plasma temporarily enclosed in volume V at time t , as opposed to letting $V \rightarrow \infty$, as in Low's and Newcomb's work. The double integral contains the part \mathcal{L} , the Lagrangian due to each particle, whose first term represents kinetic energy and second term represents the interaction energy of each particle with the electromagnetic field. The second integral describes the energy associated with the free space electromagnetic field. A careful discussion of the application of Hamilton's principle to L_L has been given by Galloway and Kim (1971). The variation of L_L due to the variation in $\underline{x}(\underline{x}_0, \underline{v}_0, t)$, constrained by the continuity of particles in phase space,

$$f(\underline{x}, \underline{v}, t) \, d\underline{x} \, d\underline{v} = \text{constant} \quad , \quad (3.8)$$

and with the electromagnetic field treated as a given function of \underline{x} , gives the particle force law. The variations in ϕ and \underline{A} , with \underline{x} treated as the integration variable, yield Maxwell's equations.

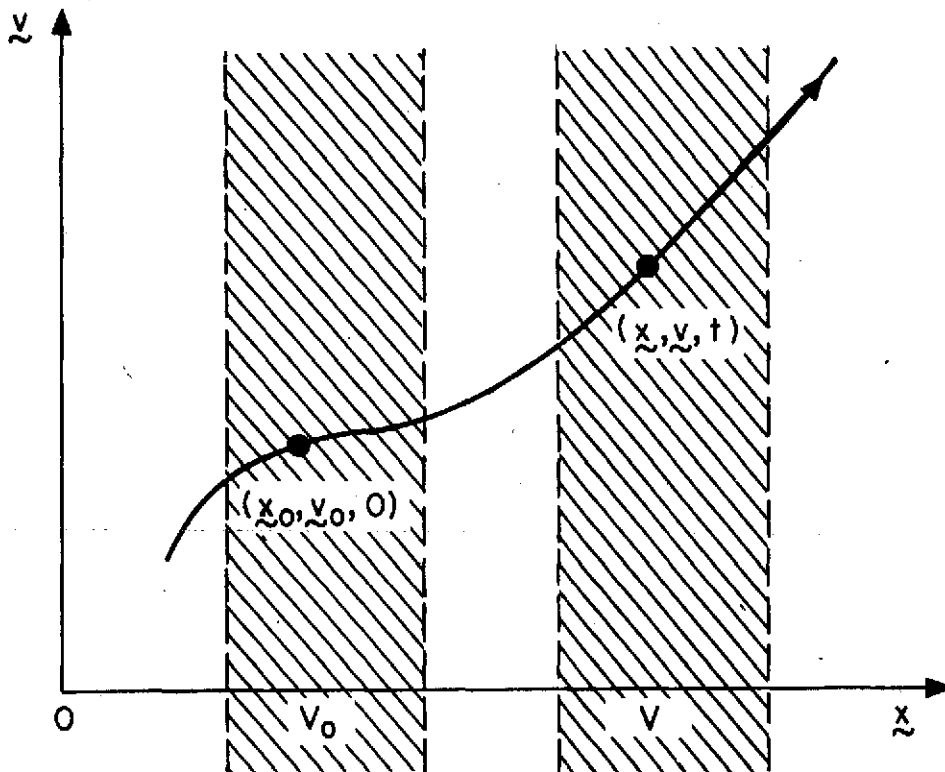


Figure 3.2 Particles of a species that occupy a spatial volume V at time t are assumed to occupy a volume V_0 at $t = 0$. This approximation is acceptable when t is sufficiently close to $t = 0$.

In anticipation of Section 3.2.2, the integration variables in L_L may be transformed from the Lagrangian (initial) phase space $(\underline{x}_0, \underline{v}_0)$ to the Eulerian phase space $(\underline{x}, \underline{v})$. Applying the conservation of particles in phase space

$$f(\underline{x}_0, \underline{v}_0) d\underline{x}_0 d\underline{v}_0 = f(\underline{x}, \underline{v}, t) d\underline{x} d\underline{v} \quad , \quad (3.9)$$

transforms (3.2) to

$$L_L = \int_V d\underline{x} \mathcal{L}_L \quad ,$$

$$\mathcal{L}_L = \sum \int d\underline{v} f(\underline{x}, \underline{v}, t) \left(\frac{m}{2} v^2 - q\phi + q\underline{v} \cdot \underline{A} \right) + \frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \quad . \quad (3.10)$$

Equation (3.10) applies to a collisionless (Vlasov) plasma, for which f is the smoothed velocity distribution function. The elementary volume, or cell, $d\underline{x}$, must be much larger than the mean particle spacing to justify the smoothing, and much smaller than the mean free path to justify neglect of collisions. The electromagnetic field variables are treated as constants within $d\underline{x}$, and represent the collective charge effects. We shall refer to this model, used by Low, as the "collisionless microscopic model".

The Lagrangian, L_T , for plasmas in the "collisional microscopic model", is identical to that of (3.10), but with different interpretations for f , $d\underline{x}$, \underline{E} , \underline{B} , ϕ , and \underline{A} : the distribution function, f , is now fine grained; the cell, $d\underline{x}$, roughly speaking, becomes smaller than the mean particle spacing and larger than the size of the particles; the electromagnetic field variables include the microscopic particle self-fields effective in collisions.

Despite their similarity in form, L_L and L_T are not equal in value. The difference can be determined by relating the quantities in the two models by

$$\varphi = \varphi_M + \varphi_R \quad , \quad \underline{\sim} A = \underline{\sim} A_M + \underline{\sim} A_R \quad , \quad \text{etc.},$$

$$f(\underline{\sim} x, \underline{\sim} v, t) = f_M(\underline{\sim} x, \underline{\sim} v, t) + f_R(\underline{\sim} x, \underline{\sim} v, t) \quad , \quad (3.11)$$

where subscript R denotes quantities which average to zero in a cell of the collisionless microscopic model. Substituting (3.11) in (3.10) yields the macroscopic approximations, L'_T and L'_L , of L_T and L_L , as

$$L'_T = L'_L + L'' = \int_V dx (\underline{\sim} \mathcal{L}'_L + \mathcal{L}'') \quad ,$$

$$\underline{\sim} \mathcal{L}'_L = \sum \left[\frac{m}{s} n v_D^2 + \frac{1}{2} \text{Tr} \underline{\sim} P - qn(\varphi_M - \underline{\sim} v_D \cdot \underline{\sim} A_M) \right] + \frac{\epsilon_0 E_M^2}{2} - \frac{B_M^2}{2\mu_0} \quad ,$$

$$\mathcal{L}'' = - \sum \left[\int \underline{\sim} dv f_R q(\varphi_R - \underline{\sim} v \cdot \underline{\sim} A_R) \right]_M + \left[\frac{\epsilon_0 E_R^2}{2} - \frac{B_R^2}{2\mu_0} \right]_M \quad , \quad (3.12)$$

where Tr represents the trace of a tensor; the macroscopic density, n , drift velocity, $\underline{\sim} v_D$, and pressure tensor, $\underline{\sim} P$, are defined by

$$n = \int \underline{\sim} dv f_M(\underline{\sim} x, \underline{\sim} v, t) \quad , \quad \underline{\sim} v_D = \frac{1}{n} \int \underline{\sim} dv \underline{\sim} v f_M(\underline{\sim} x, \underline{\sim} v, t) \quad ,$$

$$\underline{\sim} P = m \int \underline{\sim} dv (\underline{\sim} v - \underline{\sim} v_D)(\underline{\sim} v - \underline{\sim} v_D) f_M(\underline{\sim} x, \underline{\sim} v, t) \quad , \quad (3.13)$$

and the size of $\underline{\sim} dx$ corresponds to that of the collisionless microscopic model.

If Hamilton's principle is applied to L'_L , the correct macroscopic force law cannot be obtained as one of the Euler-Lagrange equations. The discrepancies arise in two ways. First, there is no term in L'_L that accounts for the energy associated with the heat flux, \tilde{q} , defined by,

$$\tilde{q} = \frac{m}{2} \int_{\tilde{D}} d\mathbf{v} (\mathbf{v} - \mathbf{v}_D) |\mathbf{v} - \mathbf{v}_D|^2 f_M(\mathbf{x}, \mathbf{v}, t) \quad . \quad (3.14)$$

To perform the variation of the term $\text{Tr } P/2$, one then has to assume \tilde{q} negligible. Second, the Euler-Lagrange equation resulting from variation in $\tilde{x}(\tilde{x}_0, t)$ is found to be correct only if the sign preceding the term $\text{Tr } P/2$ in L'_L is reversed. Clearly, it would be convenient if we could prove that $L'' \simeq -\Sigma \text{Tr } P$. This would involve a statistical theory of plasma fluctuations (Harker and Crawford, 1973), which may be considered as elastic multiparticle collisions, with the added complication of elastic two-particle collisions. We shall take a much more direct approach, however. Since we already know that the corrections should contain $-\Sigma \text{Tr } P$, and may result from including energy contributions from the plasma fluctuations and particle collisions, we shall obtain them from energy considerations.

3.2.2 Macroscopic Potential, \tilde{U}

We postulate that, for each particle species, the corrections can be considered as a macroscopic potential energy density, \tilde{U} , that has two parts: a random (collisional and fluctuational) potential, \tilde{U}_R , (when particle collisions are sufficiently frequent) and a heat transport potential, \tilde{U}_H , defined by,

$$U = U_R + U_H \quad , \quad U_R = \frac{-1}{\Delta V} \int dx \int dv f(\underline{x}, \underline{v}, t) \underline{F}_R \cdot \underline{x}_R \quad , \quad U_H = \nabla \cdot \underline{Q} \quad , \quad (3.15)$$

where, as shown in figure 3.3, ΔV is a macroscopic volume with size much larger than the particle mean free path and a boundary moving with a fixed number of particles of a species; the random collisional force, \underline{F}_R , and position of the particles, \underline{x}_R , are defined with respect to the macroscopic force, \underline{F}_M , and the position, \underline{R} , of ΔV , and \underline{Q} is the total density of heat energy transported across a macroscopic cell as it moves along its trajectory, $\underline{x}(\underline{x}_0, t)$,

$$\underline{Q} = \int_0^t dt \underline{q}[\underline{x}(\underline{x}_0, t), t] \quad . \quad (3.16)$$

To obtain an explicit macroscopic expression for U_R , consider the following equation which appears in proving the virial theorem in classical mechanics (Goldstein, 1950b),

$$\frac{d}{dt} \int_{\Delta V} dx \int dv f(\underline{x}, \underline{v}, t) m \underline{v} \cdot \underline{x} = \int_{\Delta V} dx \int dv f(\underline{x}, \underline{v}, t) [m v^2 + (\underline{F}_M + \underline{F}_R) \cdot \underline{x}] \quad , \quad (3.17)$$

where we have used the fact that the number of particles of a species in ΔV is constant. Rather than averaging over a long time period, as in classical mechanics, we shall consider the approximation over a large number of particles within ΔV . Expressing (3.17) in terms of the macroscopic and random quantities yields

$$\frac{d}{dt} \int_{\Delta V} dx \int dv f m (\underline{v}_D \cdot \underline{R} + \underline{v}_R \cdot \underline{x}_R) = \Delta V (m n v_D^2 + \text{Tr } \underline{P} + n \underline{F}_M \cdot \underline{R}) + \int_{\Delta V} dx \int dv f \underline{F}_R \cdot \underline{x}_R \quad , \quad (3.18)$$

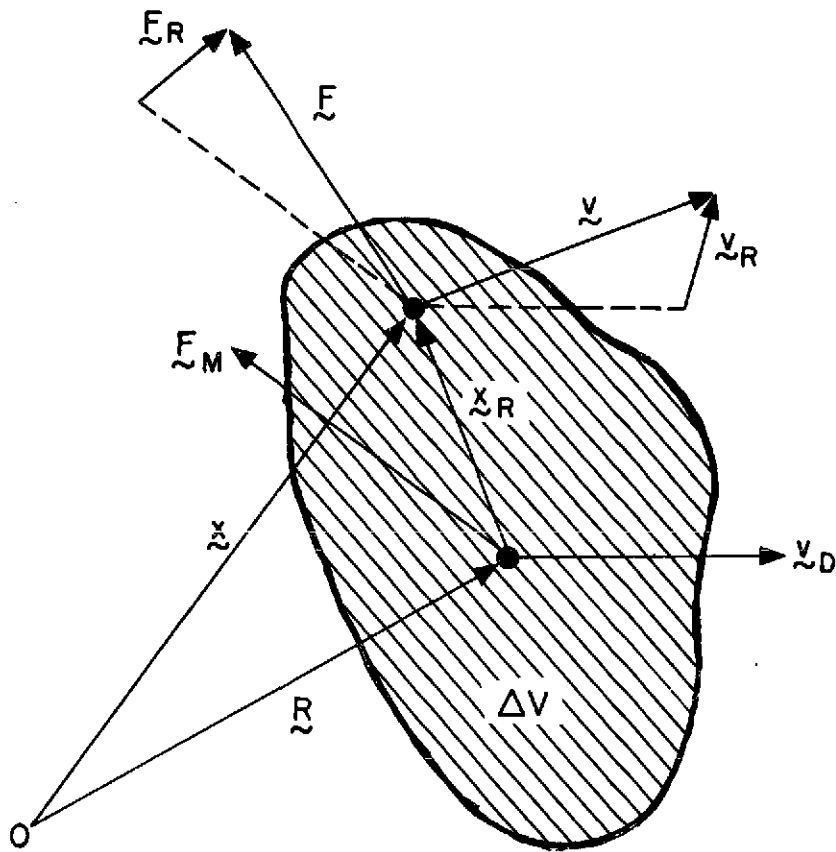


Figure 3.3 The size of ΔV is assumed to be much larger than the particle mean free path so that the same particles of a species are contained in ΔV for a time period much larger than the mean particle transit time across ΔV . This assumption of large ΔV size enables us to justify the necessary correction to the \underline{y} -integrated Low Lagrangian. It will be seen in Section 3.5 that the large size assumption is not necessary for the Lagrangian of (3.22).

where \underline{v}_R is the random particle velocity with respect to \underline{v}_D . Since $\underline{v}_D = dR/dt$ and $\underline{F}_M = m d\underline{v}_D/dt$, (3.18) simplifies to

$$\frac{d}{dt} \int_{\Delta V} dx \int_{\underline{v}} dv f m \underline{v}_R \cdot \underline{x}_R = \Delta V \text{Tr } \underline{P} + \int_{\Delta V} dx \int_{\underline{v}} dv f \underline{F}_R \cdot \underline{x}_R \quad (3.19)$$

The left-hand side of (3.19) involves $d(\Delta V)/dt$. In the case when this is zero, e.g., in a stationary plasma, combination of (3.16) and (3.13) then gives the expected expression,

$$\underline{U}_R = \text{Tr } \underline{P} \quad (3.20)$$

This agrees with a result obtained by Gartenhaus (1961), who showed that the collisional interaction potential of a stationary gas is proportional to the particle thermal energy.

Combination of (3.16) and (3.20) gives a guess for \mathcal{L}'' , at least for the case when sufficiently frequent collisions can be assumed,

$$\mathcal{L}'' = -\Sigma \underline{U} \quad , \quad \underline{U} = \text{Tr } \underline{P} + \nabla \cdot \underline{Q} \quad (3.21)$$

Substitution of (3.21) in (3.12) then gives the total Lagrangian as

$$\begin{aligned} L &= \int_V dx \mathcal{L} \quad , \\ \mathcal{L} &= \sum \left[\frac{m}{2} n v_D^2 - \frac{1}{2} \text{Tr } \underline{P} - \nabla \cdot \underline{Q} - qn(\phi_M - \underline{v}_D \cdot \underline{A}_M) \right] + \frac{\epsilon_0 E_M^2}{2} - \frac{B_M^2}{2\mu_0} \quad , \quad (3.22) \end{aligned}$$

where the subscript M will be dropped from here on.

First, we should point out that the assumption of a stationary plasma in obtaining (3.20) can be relaxed for the purpose of obtaining

L in (3.22), if we let V tend to infinity. This can be seen by making $\Delta V \rightarrow dx$ in (3.19), and integrating dx over $V \rightarrow \infty$, making the left-hand side automatically zero because no particle now crosses the boundary of V. Second, the assumption of sufficiently frequent collisions requires that the size of ΔV be much larger than the particle mean free path. In the following, it will be seen that this requirement is overly restrictive since the macroscopic quantities in (3.22) are well-defined in a cell whose size need only be much larger than the mean particle spacing. Further discussion of this point will be included in Section 3.4.

3.2.3 Application of Hamilton's Principle

We shall now test the validity of (3.22) by application of Hamilton's principle. Note that L contains the generalized variables $\underline{x}(x_0, t)$, $\underline{A}(x, t)$, and $\varphi(x, t)$.

Variation with respect to \underline{A} and φ : The variation of L with respect to \underline{A} and φ is comparatively simple since \underline{x} is now treated as an integration variable. For variations $\delta \underline{A}$ and $\delta \varphi$, which vanish on the boundary of V and at times t_1 and t_2 , the Euler-Lagrange equations corresponding to

$$I = \int_{t_1}^{t_2} dt L \quad , \quad (3.23)$$

take the standard form (Schiff, 1968),

$$\frac{\partial}{\partial t} \left[\frac{\partial L}{\partial (\partial \underline{A} / \partial t)} \right] - \frac{\partial L}{\partial \underline{A}} = 0 \quad , \quad \frac{\partial}{\partial t} \left[\frac{\partial L}{\partial (\partial \varphi / \partial t)} \right] - \frac{\partial L}{\partial \varphi} = 0 \quad , \quad (3.24)$$

where the functional derivatives of L with respect to \underline{A} and φ , in Eulerian coordinates, are defined by

$$\frac{\delta L}{\delta \underline{A}} = \frac{\partial \mathcal{L}}{\partial \underline{A}} - \frac{\partial}{\partial x_i} \left[\frac{\partial \mathcal{L}}{\partial (\partial \underline{A} / \partial x_i)} \right] , \quad \frac{\delta L}{\delta \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x_i} \left[\frac{\partial \mathcal{L}}{\partial (\partial \varphi / \partial x_i)} \right] . \quad (3.25)$$

Using (3.7) in (3.22), and then applying (3.24), leads to the Euler-Lagrange equations,

$$\frac{1}{\mu_0} \nabla \times \underline{B} = \epsilon_0 \frac{\partial \underline{E}}{\partial t} + \Sigma q n \underline{v}_D , \quad \epsilon_0 \nabla \cdot \underline{E} = \Sigma q n . \quad (3.26)$$

These equations, together with (3.7), form the set of Maxwell equations.

Variation with respect to \underline{x} : For \underline{x} variation, the Lagrangian is simply,

$$L^E = \int_V dx \mathcal{L}^E ,$$

$$\mathcal{L}^E = \frac{m}{2} n v_D^2 - \frac{1}{2} \text{Tr} \underline{P} - \nabla \cdot \underline{Q} - qn(\varphi - \underline{v}_D \cdot \underline{A}) , \quad (3.27)$$

where the superscript E signifies Eulerian coordinates, and the integral of the electromagnetic field energy in (3.22) has been dropped because it has no effect on the resulting Euler-Lagrange equation (Hill, 1951; Galloway and Kim, 1971). The quantities \underline{v}_D , n , $\text{Tr} \underline{P}$, and $\nabla \cdot \underline{Q}$ are to be considered as implicit functions of \underline{x} via subsidiary constraints. [Otherwise, the resulting Euler-Lagrange equation will be a zero identity (Hill, 1951).] These constraints are the first and third moment equations derived from the Boltzmann equation (Braginskii, 1965),

$$\frac{\partial f_M}{\partial t} + \underline{v} \cdot \nabla f_M + \frac{d\underline{v}}{dt} \cdot \nabla_v f_M = C \quad , \quad (3.28)$$

where C represents the contribution of elastic collisions. These are the continuity and heat flow equations (Braginskii, 1965),

$$\frac{dn}{dt} + n \nabla \cdot \underline{v}_D = 0 \quad ,$$

$$\left[\frac{1}{2} \frac{d}{dt} (\text{Tr } \underline{P}) + \frac{1}{2} \text{Tr } \underline{P} \nabla \cdot \underline{v}_D + \underline{P} : \nabla \underline{v}_D + \nabla \cdot \underline{q} \right] = \int d\underline{v} \frac{m}{2} (v^2 - 2\underline{v} \cdot \underline{v}_D) C \quad . \quad (3.29)$$

The dyad notation will be used for dot products of adjacent vectors and tensors. The continuity equation is equivalent to the mass conservation law, the first expression of (3.4), used by Newcomb (1962), but the heat flow equation is not in the form of a conservation law such as the second expression of (3.4).

Virtual displacement method: A modified version of the virtual displacement method used by Serrin (1959) and Lundgren (1963) will now be employed to derive the variations of L from (3.29). This method consists of introducing an arbitrary parameter, ϵ , into $n(\underline{x}, t; \epsilon)$, etc. The corresponding nonlocal and local variations are then defined as, respectively,

$$\begin{aligned} \delta'(\) &= \frac{\partial(\)}{\partial \epsilon} \quad (\underline{x}_0, t \text{ fixed}, \epsilon = 0) \quad , \\ \delta(\) &= \frac{\partial(\)}{\partial \epsilon} \quad (\underline{x}, t \text{ fixed}, \epsilon = 0) \quad . \end{aligned} \quad (3.30)$$

For our problem, it is more convenient to use t in this role, without introducing ϵ .

The nonlocal and local variations are defined in figure 3.4, from which we can write

$$\begin{aligned}\delta' n(\underline{x}, t) &= n'(\underline{x}', t) - n(\underline{x}, t) \quad , \\ \delta n(\underline{x}, t) &= n'(\underline{x}, t) - n(\underline{x}, t) \quad ,\end{aligned}\tag{3.31}$$

with n' denoting the density after the virtual displacement of

$$\underline{\xi}(\underline{x}, t) = \underline{x}' - \underline{x} \quad .\tag{3.32}$$

Variation in \underline{v}_D : A schematic definition of $\delta' \underline{v}_D$ and $\delta \underline{v}_D$ is shown in figure 3.5, from which we obtain immediately

$$\begin{aligned}\delta' \underline{v}_D &= \underline{v}'_D(\underline{x}', t) - \underline{v}_D(\underline{x}, t) = \frac{d\underline{x}'}{dt} - \frac{d\underline{x}}{dt} = \frac{d\underline{\xi}}{dt} \quad , \\ \delta \underline{v}_D &= \delta' \underline{v}_D - \underline{\xi} \cdot \nabla \underline{v}_D = \frac{d\underline{\xi}}{dt} - \underline{\xi} \cdot \nabla \underline{v}_D \quad ,\end{aligned}\tag{3.33}$$

where (3.32) has been used to obtain the first expression, and Taylor series expansion of $\underline{v}'_D(\underline{x}', t)$ at \underline{x} has been used for the second expression.

Variation in n : We shall now use the first expression of (3.29) to obtain the virtual displacement in n . First, we perform a real displacement, $\Delta \underline{x}(\underline{x}, t)$, of a plasma cell in a short time period, Δt , with the particles of the species in the cell conserved. This operation is illustrated in figure 3.6, which shows both $\Delta \underline{x}$ and $\underline{\xi}$. Applying the first expression of (3.29) along the path, $\Delta \underline{x}$, we obtain,

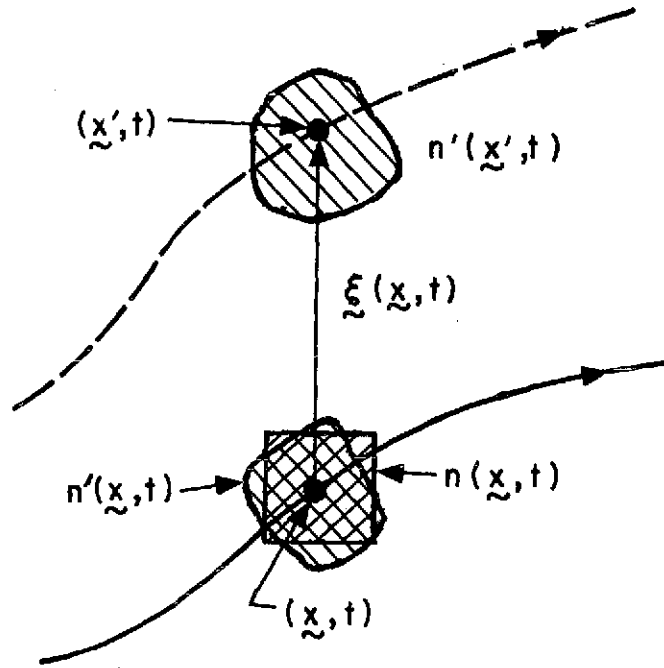


Figure 3.4 The definition of nonlocal and local variations in n due to the virtual displacement $\xi(\underline{x}, t)$. The nonlocal variation, $\delta' n$, is defined by comparing n' with n for the same cell, while the local variation, δn , is for the same coordinate \underline{x} , before and after the virtual displacement. The trajectories of the same cell before and after virtual displacement are denoted by — and ---, respectively.

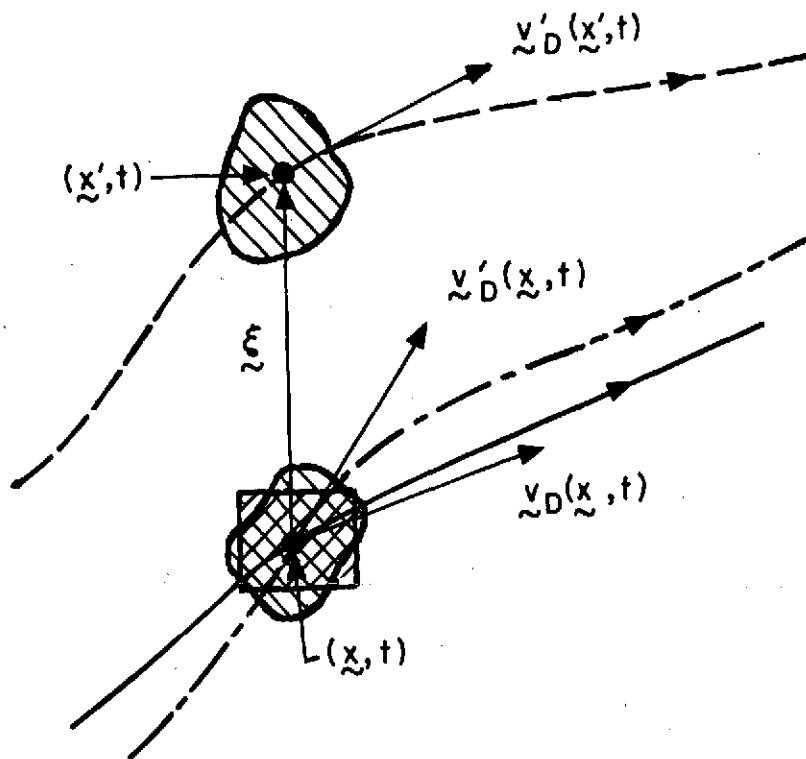


Figure 3.5 The definition of nonlocal and local variations in \underline{v}_D due to the virtual displacement $\xi(\underline{x}, t)$. The nonlocal variation, $\delta' \underline{v}_D$, is obtained by comparing \underline{v}'_D with \underline{v}_D for the same cell, while the local variation, $\delta \underline{v}_D$, is for the same coordinate \underline{x} . The trajectories of the same cell before and after virtual displacement are denoted by — and ---, respectively. The virtually displaced trajectory that passes through (\underline{x}, t) is denoted by - · - · -.

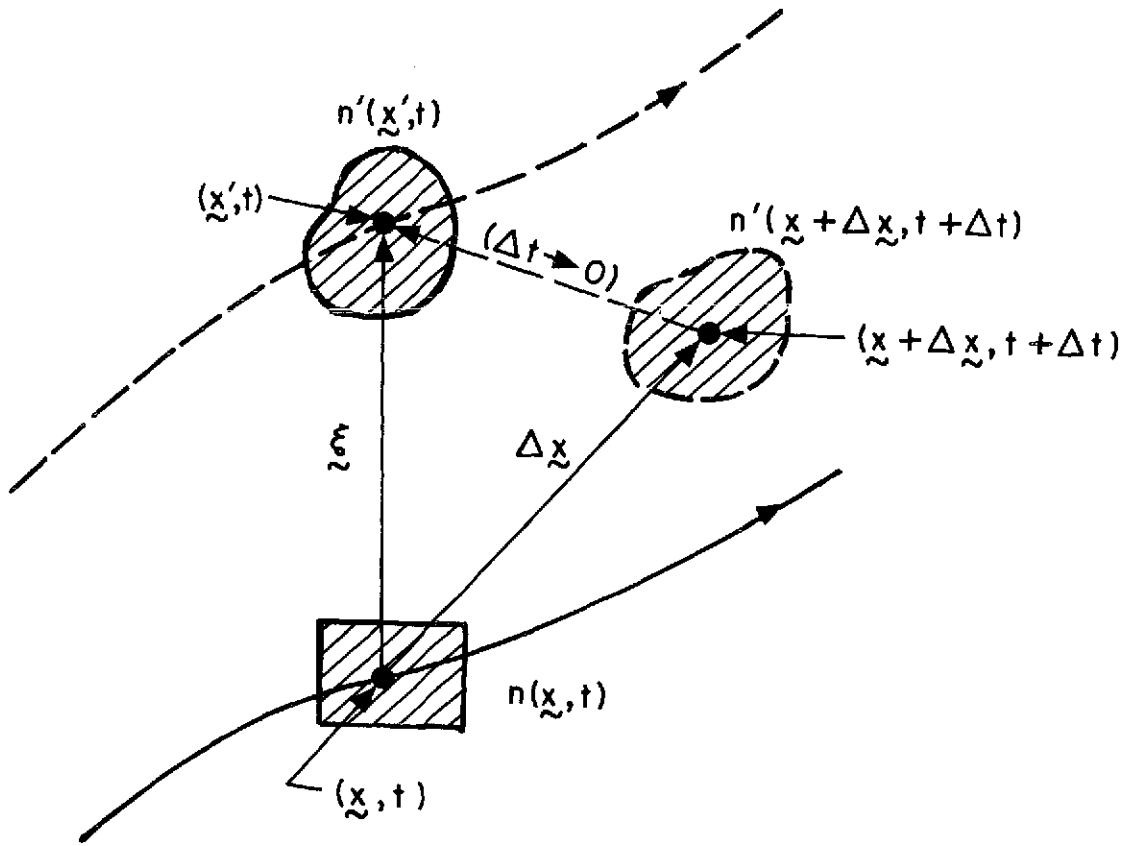


Figure 3.6 The approach of a real displacement, $\Delta \underline{x}$, performed in a short time period, Δt , to the virtual displacement, ξ , as Δt diminishes to zero. Since (3.29) describes the changes in n and $\text{Tr } \underline{P}/2$ along the path $\Delta \underline{x}$, the virtual displacements in n and $\text{Tr } \underline{P}/2$ are then described by (3.29) as a special case $\Delta t \rightarrow 0$ and $\Delta \underline{x} \rightarrow \xi$.

$$\frac{\Delta n}{\Delta t} + n \nabla \cdot \frac{\Delta \underline{x}}{\Delta t} = 0 \quad , \quad \Delta n = n'(\underline{x} + \Delta \underline{x}, t) - n(\underline{x}, t) \quad . \quad (3.34)$$

Since $\Delta \underline{x}$ and Δt are arbitrary, we can let $\Delta \underline{x} \rightarrow \underline{\xi}$ when $\Delta t \rightarrow 0$. Hence by definition, Δn , the change of n in the real displacement, approaches the nonlocal variation $\delta' n$. By cancelling Δt in (3.34) before approaching the limit $\Delta t = 0$, we obtain

$$\Delta n \rightarrow \delta' n = -n \nabla \cdot \underline{\xi} \quad . \quad (3.35)$$

The local variation in n is obtained by subtracting $\underline{\xi} \cdot \nabla n$ from $\delta' n$, giving

$$\delta n = -\nabla \cdot (\underline{\xi} n) \quad . \quad (3.36)$$

Variation in $(\text{Tr } \underline{P}/2 + \nabla \cdot \underline{Q})$: The foregoing procedure can be applied to any equation linear in d/dt and \underline{v}_D , such as the second expression in (3.29). Caution must be exercised, however, for those terms which do not contain \underline{v}_D explicitly. From (3.29), we have along the displacement $\Delta \underline{x}$,

$$\begin{aligned} \frac{1}{2} \Delta(\text{Tr } \underline{P}) + \frac{1}{2} \text{Tr } \underline{P} \nabla \cdot (\Delta \underline{x}) + \underline{P}_s : \nabla(\Delta \underline{x}) + (\nabla \cdot \underline{q}) \Delta t \\ = \left(\frac{m}{s} \int v^2 C \, d\underline{v} \right) \Delta t - m \int \underline{v} \cdot (\Delta \underline{x}) C \, d\underline{v} \quad . \end{aligned} \quad (3.37)$$

When $\Delta t \rightarrow 0$, the term $(m \int v^2 C \, d\underline{v}/2) \Delta t$ will vanish if the macroscopic displacement $\Delta \underline{x}$ does not significantly alter the microscopic random processes within the cell. The term $(\nabla \cdot \underline{q}) \Delta t$ does not vanish: the identity

$$\nabla \cdot \underline{q} = \nabla \cdot \left(\frac{d}{dt} \underline{Q} \right) = \frac{d}{dt} (\nabla \cdot \underline{Q}) + \nabla_{\underline{v}_D} : \nabla \underline{Q} \quad , \quad (3.38)$$

implies the relation,

$$(\nabla \cdot \underline{q}) \Delta t = \Delta (\nabla \cdot \underline{Q}) + \nabla (\Delta \underline{x}) : \nabla \underline{Q} \quad . \quad (3.39)$$

Substitution of (3.39) into (3.37) before taking the limit $\Delta t = 0$, and $\Delta = \delta'$, gives the following nonlocal variations:

$$\delta' \left(\frac{1}{2} \text{Tr} \underline{P} + \nabla \cdot \underline{Q} \right) = - \frac{1}{2} \text{Tr} \underline{P} \nabla \cdot \underline{\xi} - \underline{P} : \nabla \underline{\xi} - \nabla \underline{\xi} : \nabla \underline{Q} - \underline{\xi} \cdot \int_m \underline{v} \underline{C} \, d\underline{v} \quad . \quad (3.40)$$

The local variation corresponding to (3.40) is obtained by subtracting $\underline{\xi} \cdot \nabla (\text{Tr} \underline{P}/2 + \nabla \cdot \underline{Q})$,

$$\delta \left(\frac{1}{2} \text{Tr} \underline{P} + \nabla \cdot \underline{Q} \right) = - \nabla \cdot \left(\frac{1}{2} \underline{\xi} \text{Tr} \underline{P} + \underline{\xi} \cdot \nabla \underline{Q} \right) - \underline{P} : \nabla \underline{\xi} - \underline{\xi} \cdot \int_m \underline{v} \underline{C} \, d\underline{v} \quad . \quad (3.41)$$

The Euler-Lagrange equation with respect to \underline{x} : We are now in a position to derive the Euler-Lagrange equation from L^E by variation with respect to \underline{x} . The starting point is the integral of L^E from time t_1 to t_2 ,

$$I^E = \int_{t_1}^{t_2} dt L^E \quad . \quad (3.42)$$

Because the integration variables are \underline{x} and the time t (for fixed \underline{x}), local variations are to be used. The vector $\underline{\xi}(\underline{x}, t)$ is assumed to vanish on the boundary of V , and at times t_1 and t_2 . Using (3.27), the

variation of (3.42) becomes,

$$\delta I^E = \int_{t_1}^{t_2} dt \delta L^E, \quad (3.43)$$

$$\delta L^E = \int_V dx \left\{ \delta n \left[\frac{m}{2} v_D^2 - q(\varphi - v_D \cdot A) \right] + n \delta v_D \cdot (m v_D + qA) - \delta \left(\frac{1}{2} \text{Tr } P + \nabla \cdot Q \right) \right\}. \quad (3.43)$$

Substituting (3.33), (3.36), and (3.41) for δv_D , δn , and $\delta(\text{Tr } P/2 + \nabla \cdot Q)$, in (3.43), integrating by parts those terms involving $(\partial \xi / \partial t)$ and $\nabla \xi$, dropping any surface integral that involves ξ on the boundary of V , and dropping any initial (t_1) and final (t_2) terms that involve ξ , we obtain from (3.43) by straightforward manipulation the following result,

$$\delta L^E = \int_V dx \xi \cdot \left\{ -mn \left(\frac{\partial v_D}{\partial t} + v_D \cdot \nabla v_D \right) - \nabla \cdot P - qn \left(\nabla \varphi + \frac{\partial A}{\partial t} \right) \right. \\ \left. + qn (\nabla A \cdot v_D - v_D \cdot \nabla A) - (m v_D + qA) \left[\frac{\partial n}{\partial t} + \nabla \cdot (n v_D) \right] + \int m v C dv \right\}. \quad (3.44)$$

Use of the dyadic relation,

$$v_D \times (\nabla \times A) = \nabla A \cdot v_D - v_D \cdot \nabla A, \quad (3.45)$$

and substitution of (3.7) and the first expression of (3.29) in (3.44), finally gives the Euler-Lagrange equation as

$$mn \frac{dv_D}{dt} + \nabla \cdot P = qn (\underline{E} + v_D \times \underline{B}) + \int m v C dv, \quad (3.46)$$

which we recognize as the macroscopic force law, with general pressure tensor and elastic particle collisions (Braginskii, 1965).

In the above derivation, \underline{Q} is not restricted any further than by its definition in (3.16). The form $\nabla \cdot \underline{Q}$ in the integrand of L is not sufficient to make \underline{Q} disappear from the final form of δL^E , since \underline{Q} is in general nonzero on the boundary of V . In arriving at (3.44), however, it can be shown that \underline{Q} appears in the integrand of δL^E only in the form, $\nabla \cdot (\underline{\xi} \cdot \nabla \underline{Q})$, and hence drops out since $\underline{\xi}$ vanishes on the boundary of V .

3.3 Hamiltonian Density

When the independent (integration) variables of a Lagrangian, L , are distinctly different from the generalized variables, and L is quadratic in the time derivatives of the generalized variables, the Hamiltonian, H , can be obtained by straightforward application of the well-known Legendre transformation (Goldstein, 1950). The canonical Hamilton system of equations is then obtained by varying each of the generalized variables, and their conjugate momenta, separately.

For L of (3.22), however, the dual role of the variables calls for additional care in formulating the correct Legendre transformation. If the momentum conjugate to \underline{x} is defined as

$$\underline{\eta}_E = \frac{\partial L}{\partial \underline{v}_D} = n(m\underline{v}_D + q\underline{A}) \quad , \quad (3.47)$$

by the use of (3.25) in the Eulerian coordinates, $\underline{\eta}_E$ will be dependent on \underline{x} , because \underline{x} is the integration variable. This violates the requirement of the Legendre transformation that $\delta \underline{\eta}_E$ and $\delta \underline{x}$ ($= \underline{\xi}$) are to be independent of each other. A way to avoid this difficulty is to transform the integration variable from \underline{x} to \underline{x}_0 , for that part of L describing the particles, so that the integration is effectively performed

at $t = 0$. The dual roles played by \underline{A} and φ require that their dependence on \underline{x}_0 be only through their explicit dependence on $\underline{x}(\underline{x}_0, t)$. The momenta conjugate to the electromagnetic potentials must be defined only in the Eulerian coordinates (\underline{x}, t) (see Section 3.3.2).

The validity of the foregoing modifications to the Legendre transformation will be established in Section 3.3.3, where the Hamilton equations will be shown to agree with the Euler-Lagrange equations obtained in Section 3.2.3. After making similar changes in the definition of the Poisson brackets, the energy equation is obtained correctly, proving that the canonical formulation is self-consistent.

For completeness, a Hamiltonian corresponding to the Low Lagrangian will be obtained in Section 3.3.6. The resulting total microscopic and macroscopic energy conservation equations will be compared to help clarify the meaning of the macroscopic potential, ψ .

3.3.1 Lagrangian Coordinates

In deriving their variational principles, Katz (1961) and Newcomb (1962) used the transformations between \underline{x} and \underline{x}_0 extensively. These transformation relations include, for $\underline{x}(\underline{x}_0, t)$, the Jacobian, J , the transformation matrix, x_{ij} , and its inverse matrix, K_{ij} ,

$$J = \left| \frac{\partial \underline{x}}{\partial \underline{x}_0} \right|, \quad x_{ij} = \frac{\partial x_i}{\partial x_{0j}}, \quad K_{ij} = \frac{\partial x_{0j}}{\partial x_i}, \quad (3.48)$$

whence it is easily shown that

$$\frac{\partial J}{\partial x_{ij}} = JK_{ji}, \quad K_{ki} x_{ij} = \delta_{jk}, \quad \frac{\partial (JK_{ji})}{\partial x_{0j}} = 0, \quad d\underline{x} = J d\underline{x}_0, \quad (3.49)$$

where δ_{ij} is the Kronecker delta with respect to the Cartesian coordinate indices i and j , and the summation convention has been implied. The initial value of $n(\underline{x}, t)$ is $n_0(\underline{x}_0)$. By conservation of particles along the cell trajectory, we have

$$n d\underline{x} = n_0 d\underline{x}_0 \quad , \quad n_0 = Jn \quad . \quad (3.50)$$

Under these transformations, the Lagrangian, L , of (3.22) becomes

$$L = \sum \int_{V_0} d\underline{x}_0 \mathcal{L}_s + \int_V d\underline{x} \left(\frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \right) \quad ,$$

$$\mathcal{L}_s = \frac{m}{2} n_0 \dot{\underline{x}}^2 - qn_0 (\varphi - \dot{\underline{x}} \cdot \underline{A}) - J \left(\frac{1}{2} \text{Tr} \underline{P} + \nabla \cdot \underline{Q} \right) \quad . \quad (3.51)$$

The time derivative, $\dot{\underline{x}}$ ($= d\underline{x}/dt$), used above is taken with fixed \underline{x}_0 . The \underline{x} -dependence of \underline{A} and φ in the integrand should be retained, in keeping with the dual roles of these variables.

The Euler-Lagrange equations with respect to \underline{A} and φ need not be explicitly rederived in the Lagrangian coordinates (\underline{x}_0, t) . They should agree with those in the Eulerian coordinates since Hamilton's principle is invariant under transformation of integration variables (Courant and Hilbert, 1953). The variation of L with respect to \underline{x} , however, should be re-examined, since \underline{x} is no longer the integration variable. To do so, only the part L_s , corresponding to a particle species,

$$L_s = \int_{V_0} d\underline{x}_0 \mathcal{L}_s \quad , \quad (3.52)$$

has to be considered. The standard form of the Euler-Lagrange equation for $\underline{\tilde{x}}$ is then

$$\frac{d}{dt} \left(\frac{\partial L_s}{\partial \dot{\underline{\tilde{x}}}} \right) - \frac{\partial L_s}{\partial \underline{\tilde{x}}} = 0 \quad . \quad (3.53)$$

In Lagrangian coordinates, the functional derivatives of L_s with respect to $\underline{\tilde{x}}$ and $\dot{\underline{\tilde{x}}}$ have the forms

$$\frac{\partial L_s}{\partial x_i} = \frac{\partial f_s}{\partial x_i} - \frac{\partial}{\partial x_{0j}} \left(\frac{\partial f_s}{\partial x_{ij}} \right) \quad , \quad \frac{\partial L_s}{\partial \dot{x}_i} = \frac{\partial f_s}{\partial \dot{x}_i} - \frac{\partial}{\partial x_{0j}} \left(\frac{\partial f_s}{\partial \dot{x}_{ij}} \right) \quad , \quad (3.54)$$

which may be compared to those for $\partial L/\partial A$, etc. in Eulerian coordinates in (3.25).

The only part in f_s whose partial derivatives with respect to x_i and x_{ij} are unknown at this point is $\text{Tr } \underline{\tilde{P}}/2 + \nabla \cdot \underline{\tilde{Q}}$. To obtain them, we use the heat flow equation, but in a different fashion from that of Section 3.2.3. In Lagrangian coordinates, (3.29) becomes

$$\frac{1}{2} \frac{d}{dt} (\text{Tr } \underline{\tilde{P}}) + \frac{1}{2} \text{Tr } \underline{\tilde{P}} \nabla \cdot \dot{\underline{\tilde{x}}} + \underline{\tilde{P}} : \nabla \dot{\underline{\tilde{x}}} + \nabla \cdot \underline{\tilde{q}} = \int_s^m (v^2 - 2\underline{v} \cdot \dot{\underline{x}}) C \, d\underline{v} \quad , \quad (3.55)$$

where the gradient, ∇ , is to be read as $K_{ji} \partial/\partial x_{0j}$. Replacing $\nabla \cdot \underline{\tilde{q}}$ by the dyad identity (3.39) converts (3.55) to the form

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \text{Tr } \underline{\tilde{P}} + \nabla \cdot \underline{\tilde{Q}} \right) &= - \frac{1}{2} (\text{Tr } \underline{\tilde{P}}) K_{ji} \dot{x}_{ij} - P_{ij} K_{kj} \dot{x}_{ik} \\ &\quad - \left(\frac{\partial Q_j}{\partial x_i} \right) K_{kj} \dot{x}_{ik} - \int m v_i \dot{x}_i C \, d\underline{v} + \int \frac{m}{2} v^2 C \, d\underline{v} \\ &= \frac{\partial}{\partial x_{ij}} \left(\frac{1}{2} \text{Tr } \underline{\tilde{P}} + \nabla \cdot \underline{\tilde{Q}} \right) \dot{x}_{ij} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} \text{Tr } \underline{\tilde{P}} + \nabla \cdot \underline{\tilde{Q}} \right) \dot{x}_i + \frac{\partial}{\partial t} \left(\frac{1}{2} \text{Tr } \underline{\tilde{P}} + \nabla \cdot \underline{\tilde{Q}} \right) \quad , \quad (3.56) \end{aligned}$$

where the third formal expression is obtained by treating $\text{Tr } \underline{P}/2 + \nabla \cdot \underline{Q}$ as a uniquely determined function of $x_{ij}(t)$, $x_i(t)$, and t . Thus, by equating the coefficients of \dot{x}_{ij} and \dot{x}_i , we obtain

$$\frac{\partial}{\partial x_{ij}} \left(\frac{1}{2} \text{Tr } \underline{P} + \nabla \cdot \underline{Q} \right) = -\frac{1}{2} \text{Tr } \underline{P} K_{ji} - P_{ik} K_{jk} - \frac{\partial Q_k}{\partial x_i} K_{jk} \quad ,$$

$$\frac{\partial}{\partial x_i} \left(\frac{1}{2} \text{Tr } \underline{P} + \nabla \cdot \underline{Q} \right) = -\int_m v_i C dv \quad , \quad \frac{\partial}{\partial t} \left(\frac{1}{2} \text{Tr } \underline{P} + \nabla \cdot \underline{Q} \right) = \int_m \frac{1}{2} v^2 C dv \quad . \quad (3.57)$$

These relations enable us to find expressions for $\partial L_s / \partial x_i$ by deriving $\partial \mathcal{L}_s / \partial x_i$ and $\partial \mathcal{L}_s / \partial x_{ij}$ from (3.51). After some algebra, we obtain

$$\begin{aligned} \frac{\partial L_s}{\partial x_i} &= -q n_0 \left(\frac{\partial \varphi}{\partial x_i} - \dot{x}_j \frac{\partial A_j}{\partial x_i} \right) + J \int_m v_i C dv \\ &\quad - J \left[K_{jk} \frac{\partial}{\partial x_{0j}} P_{ik} + K_{ji} \frac{\partial}{\partial x_{0j}} (\nabla \cdot \underline{Q}) - K_{jk} \frac{\partial}{\partial x_{0j}} \left(\frac{\partial Q_k}{\partial x_i} \right) \right] \quad , \\ \frac{\partial L}{\partial \dot{x}} &= n_0 (m \dot{x} + qA) \quad . \end{aligned} \quad (3.58)$$

By replacing $K_{ji} \partial / \partial x_{0j}$ by $\partial / \partial x_i$, we find that the two terms involving \underline{Q} in the first expression in (3.58) cancel each other. It is now straightforward to use (3.49), (3.50), and (3.58) in (3.53) to show that the resulting Euler-Lagrange equation is identical to (3.46), except for a multiplicative factor, J .

3.3.2 Modified Legendre Transformation

Since \underline{x} is no longer formally used as an integration variable in (3.52), the momentum variable conjugate to \underline{x} for each particle species can be properly defined as (Schiff, 1968)

$$\frac{\partial L_s}{\partial \underline{x}} = \underline{\eta} = n_0 (\underline{m} \dot{\underline{x}} + q \underline{A}) \quad , \quad (3.59)$$

where the second expression of (3.58) has been used.

The momentum variable, $\underline{\alpha}$, conjugate to \underline{A} , must be defined differently. It is reasonable to let $\underline{\alpha}$ have exactly the same dual role property as \underline{A} has in L . The way to assure this is to make (\underline{x}, t) , rather than (\underline{x}_0, t) , the independent variable in the definition of $\underline{\alpha}$. Thus, we have

$$\underline{\alpha} = \frac{\partial L}{\partial (\partial \underline{A} / \partial t)} \quad , \quad (3.60)$$

which, by using (3.22), may be put in the familiar form (Schiff, 1968),

$$\underline{\alpha} = \epsilon_0 \left(\nabla \varphi + \frac{\partial \underline{A}}{\partial t} \right) = -\epsilon_0 \underline{E} \quad . \quad (3.61)$$

A momentum variable conjugate to φ does not exist, because L does not involve $\partial \varphi / \partial t$.

Since the conjugate momenta of (3.59) and (3.61) are defined in Lagrangian and Eulerian coordinates, respectively, the Hamiltonian must be defined by adding contributions separately from the plasma variables $(\underline{x}, \underline{\eta})$, and the electromagnetic field variables $(\underline{A}, \underline{\alpha}, \varphi)$. It is given by the following modified form of Legendre transformation

$$H = \sum_{V_0} \int dx_0 \underline{\eta} \cdot \dot{\underline{x}} + \int_V dx \underline{\alpha} \cdot \frac{\partial \underline{A}}{\partial t} - L, \quad (3.62)$$

which is assembled in a similar form to the Low Lagrangian, L_L in (3.6), and the macroscopic Lagrangian, L in (3.51). Substitution of L from (3.51), and elimination of $\dot{\underline{x}}$ and $\partial \underline{A} / \partial t$ by use of (3.59) and (3.61), reduce H to the form,

$$H = \sum_{V_0} \int dx_0 \mathcal{H}_s + \int_V dx \left[\frac{\alpha^2}{2\epsilon_0} + \frac{|\nabla \times \underline{A}|^2}{2\mu_0} - \underline{\alpha} \cdot \nabla \varphi \right],$$

$$\mathcal{H}_s = \frac{1}{2mn_0} |\underline{\eta} - qn_0 \underline{A}|^2 + qn_0 \varphi + J \left(\frac{1}{2} \text{Tr } \underline{P} + \nabla \cdot \underline{Q} \right). \quad (3.63)$$

Alternatively, H can be transformed to Eulerian coordinates by use of the last expression of (3.49) and the first expression of (3.50),

$$H = \int_V dx \mathcal{H},$$

$$\mathcal{H} = \sum \left[\frac{1}{2mn} \left| \frac{\underline{\eta}}{J} - qn \underline{A} \right|^2 + \frac{1}{2} \text{Tr } \underline{P} + \nabla \cdot \underline{Q} + qn \varphi \right] + \frac{\alpha^2}{2\epsilon_0} + \frac{|\nabla \times \underline{A}|^2}{2\mu_0} - \underline{\alpha} \cdot \nabla \varphi. \quad (3.64)$$

By use of (3.59) and (3.61), we have from (3.64)

$$\mathcal{H} = \sum \left(\frac{mn}{2} v_D^2 + \frac{1}{2} \text{Tr } \underline{P} + \nabla \cdot \underline{Q} + qn \varphi \right) + \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} + \epsilon_0 \underline{E} \cdot \nabla \varphi, \quad (3.65)$$

which is the familiar form of energy density for a plasma, plus the term $\nabla \cdot \underline{Q}$. The latter represents all of the energy density lost through heat conduction prior to time t , by a cell of a particle species, as it moves along its trajectory.

It is of interest to compare H with that given by Schiff for a quantum mechanical system of charged particles in an electromagnetic field [Schiff, 1968, equation (57.4)]

$$H_{QM} = \int d\mathbf{x} \mathcal{H}_{QM} \quad ,$$

$$\mathcal{H}_{QM} = \frac{1}{2m} |(-i\hbar\nabla - q\mathbf{A})\Psi|^2 + v_{QM}|\Psi|^2 + q\varphi|\Psi|^2 + \frac{\alpha^2}{2\epsilon_0} + \frac{|\nabla \times \mathbf{A}|^2}{2\mu_0} - \mathbf{\alpha} \cdot \nabla\varphi \quad , \quad (3.66)$$

now written in MKS units. In (3.66), Ψ and v_{QM} are the Schrödinger wave function and potential, respectively, and $-i\hbar\nabla$ is the particle momentum operator. With the understanding that $|\Psi|^2$ is equivalent to particle number density, a term-by-term correspondence is observed between (3.66) and (3.64), with the potential energy density $v_{QM}|\Psi|^2$ corresponding to $\text{Tr } \mathbf{P}/2 + \nabla \cdot \mathbf{Q}$, in agreement with the interpretation of the negative sign that the latter has in the macroscopic Lagrangian.

3.3.3 Hamilton Equations

The approach of Section 3.3.2 has been plausible. To establish that the formulation is self-consistent, we should verify that the Hamilton equations derived from H agree with the Euler-Lagrange equations derived from L .

Using (3.59) and (3.60), the Euler-Lagrange equations, (3.53) and (3.24), become

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{j} \quad , \quad \frac{\partial L}{\partial \mathbf{A}_i} = \frac{\partial \alpha_i}{\partial t} \quad , \quad \frac{\partial L}{\partial \varphi} = 0 \quad . \quad (3.67)$$

The variation of L of (3.51),

$$\delta L = \sum \int_{V_0} dx_0 \left(\frac{\partial L}{\partial \underline{x}} \cdot \underline{\xi} + \frac{\partial L}{\partial \dot{\underline{x}}} \cdot \dot{\underline{\xi}} \right) + \int_V dx \left[\frac{\partial L}{\partial \underline{A}} \cdot \delta \underline{A} + \frac{\partial L}{\partial (\partial \underline{A} / \partial t)} \cdot \frac{\partial (\delta \underline{A})}{\partial t} + \frac{\partial L}{\partial \varphi} \delta \varphi \right], \quad (3.68)$$

can be put in the form

$$\delta L = \sum \int_{V_0} dx_0 \left[\delta' (\underline{\eta} \cdot \dot{\underline{x}}) + \dot{\underline{\eta}} \cdot \underline{\xi} - \dot{\underline{x}} \cdot \delta' \underline{\eta} \right] + \int_V dx \left[\delta \left(\underline{\alpha} \cdot \frac{\partial \underline{A}}{\partial t} \right) + \frac{\partial \underline{\alpha}}{\partial t} \cdot \delta \underline{A} - \frac{\partial \underline{A}}{\partial t} \cdot \delta \underline{\alpha} \right], \quad (3.69)$$

where (3.59), (3.60), and (3.67) have been used, and δ (δ') is the local (nonlocal) variation defined in Section 3.2.3. The variation of (3.62) consequently becomes

$$\delta H = - \sum \int_{V_0} dx_0 \left(\dot{\underline{\eta}} \cdot \underline{\xi} - \dot{\underline{x}}_0 \cdot \delta' \underline{\eta} \right) - \int_V dx \left(\frac{\partial \underline{\alpha}}{\partial t} \cdot \delta \underline{A} - \frac{\partial \underline{A}}{\partial t} \cdot \delta \underline{\alpha} \right). \quad (3.70)$$

Since the variables in H to be separately varied are \underline{x} , $\underline{\eta}$, \underline{A} , $\underline{\alpha}$, and φ , it is convenient to write δH as

$$\delta H = \sum \int_{V_0} dx_0 \left(\frac{\partial H}{\partial \underline{x}} \cdot \underline{\xi} + \frac{\partial H}{\partial \dot{\underline{x}}} \cdot \delta' \underline{\eta} \right) + \int_V dx \left(\frac{\partial H}{\partial \underline{A}} \cdot \delta \underline{A} + \frac{\partial H}{\partial \underline{\alpha}} \cdot \delta \underline{\alpha} + \frac{\partial H}{\partial \varphi} \delta \varphi \right), \quad (3.71)$$

where the functional derivatives of H are analogous to those of L given in (3.54) and (3.25).

Comparison of (3.70) and (3.71) shows that the correct forms of the Hamilton equations, for macroscopic plasmas described in terms of variables with dual role, are

$$\dot{\underline{x}} = \frac{\partial H}{\partial \underline{\eta}}, \quad \frac{\partial \underline{A}}{\partial t} = \frac{\partial H}{\partial \underline{\alpha}}, \quad (3.72)$$

which are identical to (3.59) and (3.61), and

$$\dot{\eta} = -\frac{\partial H}{\partial \tilde{x}} \quad , \quad \frac{\partial \alpha}{\partial t} = -\frac{\partial H}{\partial \tilde{A}} \quad , \quad \frac{\partial H}{\partial \varphi} = 0 \quad . \quad (3.73)$$

The last two expressions of (3.73) lead to (3.26) by straightforward differentiations of (3.64) before using (3.61). The right-hand side of the first expression of (3.73) can be written as $-\partial \mathcal{H}_s / \partial x_i + (\partial / \partial x_{0j})(\partial \mathcal{H}_s / \partial x_{ij})$. Its first term can be derived from (3.63) as

$$\frac{\partial \mathcal{H}_s}{\partial x_i} = -\frac{q}{m} (\eta_j - qn_0 A_j) \frac{\partial A_j}{\partial x_i} + qn_0 \frac{\partial \varphi}{\partial x_i} - J \int m v_i c \, d\tilde{v} \quad , \quad (3.74)$$

where the second expression of (3.57) has been used. Its second term can be shown to be

$$\frac{\partial}{\partial x_{0j}} \left(\frac{\partial \mathcal{H}_s}{\partial x_{ij}} \right) = JK_{ji} \frac{\partial}{\partial x_{0j}} \left(\frac{\partial Q_k}{\partial x_k} \right) - JK_{jk} \frac{\partial P_{ik}}{\partial x_{0j}} - JK_{jk} \frac{\partial}{\partial x_{0j}} \left(\frac{\partial Q_k}{\partial x_i} \right) = -J \frac{\partial P_{ik}}{\partial x_k} \quad , \quad (3.75)$$

where the first and third expressions of (3.49), and the first expression of (3.57), have been used to obtain the first result, and the relation, $\partial / \partial x_i = K_{ji} \partial / \partial x_{0j}$, has been used to obtain the second result. It is then straightforward to show that the first expression of (3.73) is equivalent to (3.46) by the use of (3.75), (3.74), (3.59), (3.45), and the relation, $\ddot{\tilde{x}} = \dot{\tilde{v}}_D$.

3.3.4 Poisson Brackets

Poisson brackets are convenient in writing down the time derivative of a physical quantity in a system that is formulated in terms of a Hamiltonian. They must also be modified on account of the dual role of the variables. Consider in general a physical quantity G , which is a functional of \tilde{x} , $\tilde{\eta}$, \tilde{A} , $\tilde{\alpha}$, $\tilde{\varphi}$, and their first derivatives. Without

loss of generality, we may write

$$G = \int_V d\tilde{x}^{\mathcal{G}} = \sum \int_{V_0} d\tilde{x}_0^{\mathcal{G}_s} + \int_V d\tilde{x}^{\mathcal{G}_{EM}} , \quad (3.76)$$

where \mathcal{G}_{EM} is a function of $\tilde{A}, \tilde{\alpha}, \varphi$, and their first derivatives, only, and \mathcal{G}_s contains terms that must involve $\tilde{x}, \tilde{\eta}$, their first derivatives, and $\text{Tr } \tilde{P}/2 + \nabla \cdot \tilde{Q}$. The time derivative of G can be obtained by accounting for the dual roles played by the variables, $\tilde{x}, \tilde{\eta}, \tilde{A}, \tilde{\alpha}$, and φ , in G . This requires that

$$\frac{dG}{dt} = \sum \int_{V_0} d\tilde{x}_0 \frac{d\mathcal{G}_s}{dt} + \frac{\partial}{\partial t} \int_V d\tilde{x}^{\mathcal{G}} , \quad (3.77)$$

where, in the right-hand side, the first time derivative is defined with $\tilde{A}, \tilde{\alpha}$, and φ treated as given functions of \tilde{x} , and the second time derivative is defined with \tilde{x} and $\tilde{\eta}$ treated as constants. Use of the Hamilton equations, (3.72) and (3.73), transforms (3.77) to

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \sum \left\{ \mathcal{G}_s, \mathcal{H}_s \right\}_{\tilde{x}, \tilde{\eta}} + \left\{ \mathcal{G}, \mathcal{H} \right\}_{\tilde{A}, \tilde{\alpha}, \varphi} - \sum_{S_0} \oint d\tilde{\sigma}_0 \cdot \tilde{\mathcal{F}}_s - \oint_S d\tilde{\sigma} \cdot \tilde{\mathcal{F}}_{EM} , \quad (3.78)$$

where the plasma and electromagnetic field Poisson brackets, and the functional fluxes, $\tilde{\mathcal{F}}_s$ and $\tilde{\mathcal{F}}_{EM}$, are defined by

$$\left\{ \mathcal{G}_s, \mathcal{H}_s \right\}_{\underline{x}, \underline{\eta}} = \int_{V_0} d\underline{x}_0 \left(\frac{\partial \mathcal{G}}{\partial \underline{x}} \cdot \frac{\partial \mathcal{H}}{\partial \underline{\eta}} - \frac{\partial \mathcal{G}}{\partial \underline{\eta}} \cdot \frac{\partial \mathcal{H}}{\partial \underline{x}} \right) ,$$

$$\left\{ \mathcal{G}, \mathcal{H} \right\}_{\underline{A}, \underline{\alpha}, \varphi} = \int_V d\underline{x} \left(\frac{\partial \mathcal{G}}{\partial \underline{A}} \cdot \frac{\partial \mathcal{H}}{\partial \underline{\alpha}} - \frac{\partial \mathcal{G}}{\partial \underline{\alpha}} \cdot \frac{\partial \mathcal{H}}{\partial \underline{A}} + \frac{\partial \mathcal{G}}{\partial \varphi} \frac{\partial \varphi}{\partial t} \right) ,$$

$$-\underline{f}_{sj} = \frac{\partial \mathcal{G}_s}{\partial x_{ij}} \dot{x}_i + \frac{\partial \mathcal{G}_s}{\partial (\partial \eta_1 / \partial x_{0j})} \dot{\eta}_1 ,$$

$$-\underline{f}_{EMj} = \frac{\partial \mathcal{G}}{\partial (\partial A_i / \partial x_j)} \frac{\partial A_i}{\partial t} + \frac{\partial \mathcal{G}}{\partial (\partial \alpha_i / \partial x_j)} \frac{\partial \alpha_i}{\partial t} + \frac{\partial \mathcal{G}}{\partial (\partial \varphi / \partial x_j)} \frac{\partial \varphi}{\partial t} . \quad (3.79)$$

In the integrals of (3.78), S_0 and S are the bounding surfaces of V_0 and V , and $d\underline{\sigma}_0$ and $d\underline{\sigma}$ are the elementary surface areas, with vectors taken along the outward normals. The partial time derivative, $\partial \mathcal{G} / \partial t$, takes care of any explicit time-dependence that \mathcal{G} may have in addition to its implicit time-dependence through \underline{x} , $\underline{\eta}$, \underline{A} , $\underline{\alpha}$, and φ .

3.3.5 Energy Conservation Equation

Equation (3.78) can be used to formally separate the time derivative of any physical quantity into the bulk and the surface contributions, and the plasma and electromagnetic field terms. As a demonstration that the foregoing formulations are self-consistent, the total energy conservation equation is derived by substituting \mathcal{H} for \mathcal{G} in (3.78).

We obtain

$$\frac{d\mathcal{H}}{dt} + \sum_{S_0} \int d\underline{\sigma}_0 \cdot \underline{f}_s + \int_S d\underline{\sigma} \cdot \underline{f}_{EM} = 0 , \quad (3.80)$$

where the plasma and electromagnetic energy flux densities, \underline{f}_s and \underline{f}_{EM} , respectively, are given by

$$\mathcal{F}_{sj} = J \dot{x}_i \left[K_{jk} \left(P_{ik} + \frac{\partial Q_k}{\partial x_i} \right) - K_{ji} \frac{\partial Q_k}{\partial x_k} \right] ,$$

$$\mathcal{F}_{EMj} = - \frac{1}{\mu_0} \frac{\partial A}{\partial t} \times \underline{\underline{B}} - \epsilon_0 \underline{\underline{E}} \frac{\partial \varphi}{\partial t} . \quad (3.81)$$

A more familiar form may be obtained from (3.80) by use of the third expression of (3.48) and (3.49), and the second expression of (3.50).

After some manipulation, we have

$$\frac{d}{dt} \left[\mathcal{H} d\underline{\underline{x}} - \Sigma (\nabla \cdot \underline{\underline{Q}} d\underline{\underline{x}}) \right] + d\underline{\underline{x}} (\nabla \cdot \underline{\underline{F}}) = 0 ,$$

$$\underline{\underline{F}} = \Sigma (\underline{\underline{P}} \cdot \underline{\underline{v}}_D + \underline{\underline{q}}) - \frac{1}{\mu_0} \frac{\partial A}{\partial t} \times \underline{\underline{B}} - \epsilon_0 \underline{\underline{E}} \frac{\partial \varphi}{\partial t} . \quad (3.82)$$

By use of the identity,

$$\frac{d}{dt} (d\underline{\underline{x}}) = d\underline{\underline{x}} (\nabla \cdot \underline{\underline{v}}_D) , \quad (3.83)$$

and the differentiation scheme of (3.77), we can further transform (3.82) to the following form in Eulerian coordinates,

$$\frac{\partial}{\partial t} (\mathcal{H} - \Sigma \nabla \cdot \underline{\underline{Q}}) + \nabla \cdot \underline{\underline{F}}^E = 0 ,$$

$$\underline{\underline{F}}^E = \sum \left[\underline{\underline{v}}_D \left(\frac{mn}{2} v_D^2 + \frac{1}{2} \text{Tr } \underline{\underline{P}} \right) + \underline{\underline{P}} \cdot \underline{\underline{v}}_D + \underline{\underline{q}} \right] + \frac{1}{\mu_0} \underline{\underline{E}} \times \underline{\underline{B}} - \frac{\partial}{\partial t} (\epsilon_0 \underline{\underline{E}} \varphi) . \quad (3.84)$$

Since $\mathcal{H} - \Sigma \nabla \cdot \underline{\underline{Q}}$ is the energy density in the plasma at time t , (3.81), to which (3.84) is equivalent, is the correct expression of energy conservation in the mixture of Eulerian and Lagrangian coordinate systems.

It is also possible to derive (3.84) by evaluating $\partial H/\partial t$ from (3.65), and then using (3.20), (3.46), and Maxwell's equations.

The derivation of (3.84) completes our development of a self-consistent Hamiltonian description of a plasma in the macroscopic approximation; the difficulties associated with the dual role of the variables in the description of the field-particle system, originally pointed out by Sturrock (1958a), have been resolved. Since the Low Lagrangian also assumes dual role, the microscopic Hamiltonian description of a Vlasov plasma can also be established in a similar fashion, as will be shown briefly in the next section.

3.3.6 Microscopic Hamiltonian Density

By the use of the Low Lagrangian in (3.6) and (3.10), the conjugate momenta can be defined in forms analogous to (3.59) and (3.60) as

$$\begin{aligned} \zeta(\underline{x}_0, \underline{v}_0, t) &= \frac{\partial L_L}{\partial \dot{\underline{x}}} = \frac{\partial \ell}{\partial \dot{\underline{x}}} = m\dot{\underline{x}} + qA(\underline{x}, t) \quad , \\ \alpha(\underline{x}, t) &= \frac{\partial L_L}{\partial (\partial A/\partial t)} = \frac{\partial \ell_L}{\partial (\partial A/\partial t)} - \frac{\partial}{\partial x_i} \left[\frac{\partial \ell_L}{\partial (\partial^2 A/\partial x_i \partial t)} \right] = \epsilon_0 \nabla \varphi + \frac{\partial A}{\partial t} \quad , \quad (3.85) \end{aligned}$$

where the second expression for ζ follows because ℓ does not involve $\partial/\partial x_0$ and $\partial/\partial v_0$ terms.

The corresponding Hamiltonian is defined analogously to H as in (3.63),

$$H_L = \sum \int_{V_0} dx_0 \int dv_0 f(\underline{x}_0, \underline{v}_0) \zeta \cdot \dot{\underline{x}} + \int_V dx \alpha \cdot \frac{\partial A}{\partial t} - L_L \quad . \quad (3.86)$$

After some manipulation, this takes the form

$$H_L = \sum_{V_0} \int dx_0 \int dv_0 f(\underline{x}_0, \underline{v}_0) h + \int_V dx \left[\frac{\alpha^2}{2\epsilon_0} + \frac{|\nabla \times \underline{A}|^2}{2\mu_0} - \underline{\alpha} \cdot \nabla \varphi \right] ,$$

$$h = \frac{1}{2m} |\underline{\zeta} - q\underline{A}|^2 + q\varphi . \quad (3.87)$$

The Eulerian form of H_L can be obtained from (3.87) by use of (3.9),

$$H_L = \int_V dx \underline{H}_L ,$$

$$\underline{H}_L = \sum \int dv f(\underline{x}, \underline{v}, t) \left(\frac{m}{2} v^2 + q\varphi \right) + \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} + \epsilon_0 \underline{E} \cdot \nabla \varphi . \quad (3.88)$$

With the definitions of (3.85), the Euler-Lagrange equations of L_L are

$$\frac{\partial L_L}{\partial \underline{x}} = \dot{\underline{\zeta}} , \quad \frac{\partial L_L}{\partial \underline{A}} = \frac{\partial \alpha}{\partial t} , \quad \frac{\partial L_L}{\partial \varphi} = 0 . \quad (3.89)$$

These can be shown to represent the microscopic particle force law and Maxwell's equations. Taking the variation of H_L , and using (3.89), the Hamilton equations are easily obtained as

$$\dot{\underline{x}} = \frac{\partial H_L}{\partial \underline{\zeta}} , \quad \frac{\partial \underline{A}}{\partial t} = \frac{\partial H_L}{\partial \alpha} , \quad \dot{\underline{\zeta}} = - \frac{\partial H_L}{\partial \underline{x}} , \quad \frac{\partial \alpha}{\partial t} = - \frac{\partial H_L}{\partial \underline{A}} , \quad \frac{\partial H_L}{\partial \varphi} = 0 .$$

$$(3.90)$$

Since H_L does not involve the derivatives of \underline{x} and $\underline{\zeta}$ with respect to \underline{x}_0 and \underline{v}_0 , it suffices to consider a physical quantity,

G, for plasma with the same simplification. We may write

$$\begin{aligned}
 G &= \int_V d\tilde{x} \mathcal{G}(\tilde{x}, \tilde{\zeta}; \tilde{\alpha}, \tilde{A}, \varphi) \\
 &= \sum_{V_0} \int d\tilde{x}_0 \int d\tilde{v}_0 f(\tilde{x}_0, \tilde{v}_0) \mathcal{G}_s(\tilde{x}, \tilde{\zeta}) + \int_V d\tilde{x} \mathcal{G}_{EM}(A, \alpha, \varphi) . \quad (3.91)
 \end{aligned}$$

The expression for dG/dt analogous to that of (3.78) is then obtained as

$$\frac{dG}{dt} = \left\{ G, H_L \right\}_P + \left\{ G, H_L \right\}_{EM} - \int_V d\tilde{\sigma} \cdot \tilde{F}_{EM} , \quad (3.92)$$

where the microscopic Poisson brackets, and \tilde{f}_{EMj} , are defined by

$$\begin{aligned}
 \left\{ G, H_L \right\}_P &= \sum_{V_0} \int d\tilde{x}_0 \int d\tilde{v}_0 f(\tilde{x}_0, \tilde{v}_0) \left(\frac{\partial G}{\partial \tilde{x}} \cdot \frac{\partial H_L}{\partial \tilde{\zeta}} - \frac{\partial G}{\partial \tilde{\zeta}} \cdot \frac{\partial H_L}{\partial \tilde{x}} \right) , \\
 \left\{ G, H_L \right\}_{EM} &= \int_V d\tilde{x} \left(\frac{\partial G}{\partial \tilde{A}} \cdot \frac{\partial H}{\partial \tilde{\alpha}} - \frac{\partial G}{\partial \tilde{\alpha}} \cdot \frac{\partial H}{\partial \tilde{A}} + \frac{\partial G}{\partial \varphi} \frac{\partial \varphi}{\partial t} \right) , \\
 -\tilde{f}_{EMj} &= \frac{\partial \tilde{A}}{\partial t} \cdot \frac{\partial \mathcal{G}}{\partial (\partial \tilde{A} / \partial x_j)} + \frac{\partial \tilde{\alpha}}{\partial t} \cdot \frac{\partial \mathcal{G}}{\partial (\partial \tilde{\alpha} / \partial x_j)} + \frac{\partial \varphi}{\partial t} \frac{\partial \mathcal{G}}{\partial (\partial \varphi / \partial x_j)} . \quad (3.93)
 \end{aligned}$$

The energy conservation equation in the microscopic plasma model is obtained by replacing G by H_L in (3.92). We obtain

$$\frac{dH_L}{dt} + \int_V d\tilde{\sigma} \cdot \tilde{f}_L = 0 , \quad \tilde{f}_L = -\frac{1}{\mu_0} \frac{\partial \tilde{A}}{\partial t} \times \tilde{B} - \epsilon_0 \tilde{E} \frac{\partial \varphi}{\partial t} . \quad (3.94)$$

By use of (3.88), the following is obtained from (3.94)

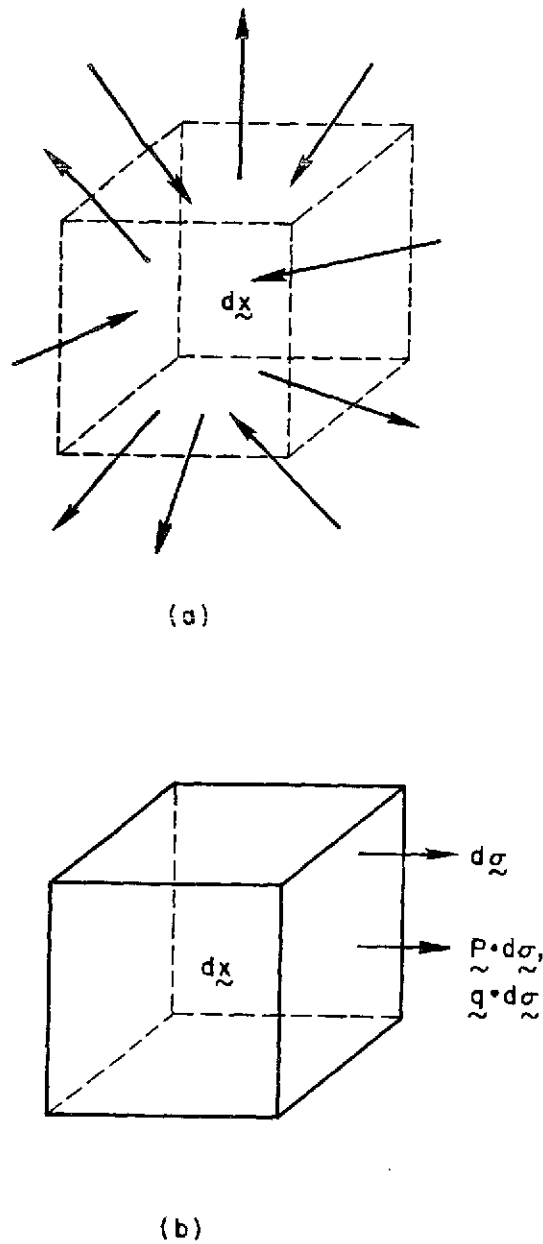


Figure 3.7 The conceptual difference between the cells of the (a) microscopic and (b) macroscopic models in a collisionless plasma. The arrows in (a) indicate the motions of the particles that carry the boundary surfaces of $\tilde{d}\underline{x}$ with them. In (b), $\tilde{d}\underline{\sigma}$ is the surface area of $\tilde{d}\underline{x}$ with an outward normal. The cell, $\tilde{d}\underline{x}$, in the macroscopic model, can only move with the macroscopic drift, \tilde{v}_D . The particle thermal motion then gives rise to momentum and energy transfer across $\tilde{d}\underline{\sigma}$.

$$\frac{d}{dt} (\underline{dx} \mathcal{H}_L) + \underline{dx} (\nabla \cdot \underline{f}_L) = 0 \quad . \quad (3.95)$$

A comparison of the energy conservation equations in the microscopic and macroscopic models, (3.95) and (3.82), respectively, can be found in Section 3.4.3.

3.3.7 Entropy for Plasmas

In this section the Lagrangian and Hamiltonian formulations have been developed, so far, for the purpose of obtaining self-consistency and consistency with the well-known equations describing plasmas in the macroscopic model, e.g. the force law, Maxwell's equations, and the energy conservation equation. Now, the dual role approach has resulted in a canonical formulation of plasma dynamics in which the plasma and electromagnetic field variables are all treated as canonical variables. It is proposed that these canonical variables may also be effectively used to express other plasma quantities of interest, such as the appropriate plasma entropy.

The question of whether the Gibbs or the Boltzmann H-function is the correct definition for a many-particle system with arbitrary inter-particle forces, has been discussed by Jaynes (1965). He concluded that an appropriate definition of entropy must include all of the degrees of freedom that we intend to use for the physical system. In this section, the number of degrees of freedom equals the number of generalized field-like variables. Thus an appropriate expression for entropy in the macroscopic plasma model can be obtained only in terms of all of the generalized variables and momenta. Since the Boltzmann H-function, in

its hydrodynamic form (Braginskii, 1965),

$$s = \int dx \bar{s} n \quad , \quad \bar{s} = \frac{3}{2} \ln (\text{Tr } \underline{P}) - \frac{5}{2} \ln n + \bar{s}_0 \quad , \quad (3.96)$$

is defined without including the degrees of freedom associated with the electromagnetic field variables, it clearly does not represent the total entropy of a plasma in the macroscopic model.

Consequently, the Boltzmann H-theorem is likely to be violated in plasmas whenever the process involves an appreciable change in the electromagnetic field energy. This conclusion agrees with recent results of Jaynes (1971), who has shown that the necessary condition for violating the Boltzmann H-theorem is that the initial kinetic energy associated with particle velocities, both drift and random, be greater than the equilibrium kinetic energy. The difference is then converted into potential energy, associated with particle interaction forces, both collective and random, during the evolution towards equilibrium.

An important question which we have not had time to pursue is the nature of an appropriate expression for plasma entropy based on the canonical models established in this section.

3.4 Discussion of the Macroscopic Potential, \mathcal{U}

In this section, we have been mainly concerned with a rigorous verification of the Lagrangian and Hamiltonian formulations for a plasma in the macroscopic model. However, in order to obtain the correct macroscopic Lagrangian, a heuristic argument that assumes sufficiently frequent particle collisions was used in Section 3.2.2.

This assumption was in turn found to be overly restrictive because the macroscopic Lagrangian and Hamiltonian, verified in Sections 3.2.3 and 3.3.3, respectively, are equally applicable to a collisionless macroscopic model when the moments of the Vlasov equation are appropriate. Therefore, it is of great interest to find a reasonable explanation for the necessity of introducing \tilde{U} of (3.21) also for the collisionless macroscopic plasma model. We shall proceed by considering the relation of \tilde{U} to viscosity and heat conduction first.

3.4.1 Relation of \tilde{U} to Viscosity and Heat Conduction

According to (3.21), \tilde{U} represents twice the thermal energy density of a particle species plus the energy density lost due to heat conduction by a macroscopic plasma cell, as it moves along its trajectory. The relation of \tilde{U} to viscosity and heat conduction can be further demonstrated by considering the particles of a species in a macroscopic cell, \underline{dx} , that has a size much larger than the mean particle spacing and follows some trajectory in the plasma. By the use of (3.83), (3.29), and (3.38), the total time derivative of \tilde{U} within \underline{dx} is given by

$$\frac{d}{dt} (\underline{dx} \tilde{U}) = \underline{dx} \left[-2\underline{P} : \underline{\nabla} \underline{v}_D + \int m(v^2 - 2\underline{v} \cdot \underline{v}_D) C \, d\underline{v} + \underline{\nabla} \cdot \left(\underline{v}_D \underline{\nabla} \cdot \underline{Q} + \frac{\partial \underline{Q}}{\partial t} - 2\underline{q} \right) \right]. \quad (3.97)$$

The first term in the right-hand side brackets represents twice the time rate of heat generation in \underline{dx} due to viscosity and mechanical compression (Braginskii, 1965). The second term is twice the time rate of heat generation in a particle species resulting from collisions with particles of other species. The third term contains twice the rate of energy gain due to heat conduction.

If (3.97) is integrated over V , before summing over all species of particles, the collisional term will drop out because of conservation of energy and momentum in elastic collisions. Equation (3.97) then becomes

$$\frac{d}{dt} \int_V dx \left(\sum \tilde{U} \right) = \int_V dx \sum \left[-2P: \tilde{\nabla}_D + \tilde{\nabla} \cdot \left(\tilde{\mathbf{v}}_D \tilde{\nabla} \cdot \tilde{\mathbf{Q}} + \frac{\partial \tilde{\mathbf{Q}}}{\partial t} - 2\tilde{\mathbf{q}} \right) \right] . \quad (3.98)$$

The phenomena of viscosity and heat conduction, which are usually regarded as dissipative, are being compensated for. The total energy of the plasma in the macroscopic model, as denoted by H in (3.63)-(3.65), is conserved through the inclusion of $-\tilde{U}$ in \mathcal{L}'_L of (3.12).

3.4.2 Loss of Particle Discreteness in Applying Macroscopic Approximation

Equations (3.97) and (3.98) are applicable also in the case when collisions are negligible, e.g. in the macroscopic approximation of a Vlasov plasma with the size of dx much larger than the mean particle spacing. In this case, the argument of Section 3.2.2 that leads to the expression for \tilde{U} is no longer appropriate because the assumption of frequent particle collisions within dx is no longer valid.

The following question consequently arises: why is it still necessary to include \tilde{U} in \mathcal{L}'_L , in the macroscopic approximation of a Vlasov plasma, to obtain the correct macroscopic Lagrangian, L , of (3.22), now that U_R in (3.15) can no longer be considered as the energy associated with particle collisional interactions? This question suggests the following argument; that it is actually the loss of particle discreteness in applying the macroscopic approximation of (3.12) which must be compensated by the correction expressed by (3.15).

In the collisionless microscopic model, the trajectory of each particle is, in theory, to be solved. When the Low Lagrangian is written, the cell \underline{dx} is used for the purpose of summing the contributions of particles within \underline{dx} . Owing to the assumed particle continuity in phase space, the bounding surface of \underline{dx} in this model is considered completely flexible; if \underline{dx} is cubic at time t , we can, in theory, always deform the shape of \underline{dx} to include the same particles at some later time. This definition of \underline{dx} is illustrated in figure 3.7(a), which implies that, in the microscopic model, as long as the particle trajectories are solved, there is no momentum and energy transfer across the flexible boundaries of \underline{dx} . In the macroscopic model, however, particles of the same species are indistinguishable. Only the spatial coordinates of the macroscopic cell, \underline{dx} , can be used to identify the plasma, as shown in figure 3.7(b). Although moving with the drift velocity, \underline{v}_D , the macroscopically smooth boundaries of \underline{dx} are penetrated by particles due to thermal motion. This then gives rise to the well-defined outward macroscopic momentum and energy transfers $\nabla \cdot \underline{P} \underline{dx}$ and $\nabla \cdot \underline{q} \underline{dx}$, respectively, from \underline{dx} . Furthermore, since elastic collisions conserve momentum and energy, they do not affect these macroscopic transport phenomena. In terms of these momentum and energy transfers, the dynamics of the cell, \underline{dx} , and that of a cell, ΔV , introduced in Section 3.2.2, become identical. The macroscopic potential energy of (3.15) must correspondingly be introduced for a collisionless macroscopic plasma.

3.4.3 Definitions of Plasma Cell Boundary

That the microscopic and macroscopic models define the boundary of a spatial volume differently can also be seen by comparing the microscopic and macroscopic energy conservation equations, (3.95) and (3.82), respectively. We see that the difference between the microscopic and macroscopic energy flux densities, \mathfrak{F}_L in (3.94) and \mathfrak{F} in (3.82), respectively, is the flux density, $\Sigma (\underline{P} \cdot \underline{v}_D + \underline{q})$, associated with particle thermal motion. Mathematically, this flux density is absent in \mathfrak{F}_L , because the microscopic Hamiltonian, H_L in (3.87), does not contain any derivatives of \underline{x} and $\underline{\zeta}$ with respect to \underline{x}_0 and \underline{v}_0 . Physically, this difference between (3.95) and (3.82) is attributable to a difference in defining the microscopic and macroscopic plasma cell boundaries.

Consider the time derivatives of H_L in (3.88) using two different definitions of the volume, V . First, define V as a stationary volume. We have,

$$\frac{\partial H_L}{\partial t} = \sum_V \int \underline{dx} \int \underline{dv} \left[\frac{m}{2} v^2 \frac{\partial f}{\partial t} + q \frac{\partial}{\partial t} (f\varphi) \right] + \int_V \underline{dx} \frac{\partial}{\partial t} \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} + \epsilon_0 \underline{E} \cdot \nabla \varphi \right), \quad (3.99)$$

which can be reduced to the macroscopic energy conservation equation, (3.82), by use of the Vlasov equation, Maxwell's equations, and integration over velocity space (Van Kampen and Felderhof, 1967). Second, define the boundary of V as moving with the enclosed particles, so that the Liouville theorem can be applied. We then have

$$\sum_V \int \underline{dx} \int \underline{dv} f(\underline{x}, \underline{v}, t) = \sum_{V_0} \int \underline{dx}_0 \int \underline{dv}_0 f(\underline{x}_0, \underline{v}_0) = \text{constant}, \quad (3.100)$$

because of (3.8). The total time derivative now becomes,

$$\frac{dH_L}{dt} = \sum_V \int_{\underline{v}} d\underline{v} \int_{\underline{x}} d\underline{x} \left[f m \underline{v} \cdot \frac{d\underline{v}}{dt} + q \frac{d}{dt} \varphi(\underline{x}, t) \right] + \frac{d}{dt} \int_V d\underline{x} \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} + \epsilon_0 \underline{E} \cdot \nabla \varphi \right), \quad (3.101)$$

which can easily be reduced to the microscopic energy conservation equation in (3.95) by use of the particle force law, Maxwell's equations, and integration over velocity space.

Since these energy conservation equations, (3.95) and (3.82), are also derivable from their corresponding Hamiltonians, (3.87) and (3.63), which are in turn derived from the corresponding Lagrangians, (3.6) and (3.51), it is seen that the difference between the Low Lagrangian, L_L , and the macroscopic Lagrangian, L , is to reflect the difference in defining the microscopic and macroscopic volumes (or cells) and their boundaries.

3.4.4 Relations Among the Variational Principles of Various Models

The arguments of Sections 3.2.2 and 3.4.1 - 3.4.3 are helpful for the purpose of understanding the macroscopic Lagrangian, L , and potential energy density, \mathcal{U} . The relation between these viewpoints can be understood from figure 3.8, which compares the plasma variational principles of the collisional and collisionless microscopic models defined in Section 3.2.1, and the macroscopic model. Procedure 1A involves dropping the random terms defined in (3.11) to relate L_T to L_L , while Procedure 1B involves only smoothing within the macroscopic cell, $d\underline{x}$, as described in (3.12), to obtain L'_T from L_T , by keeping the quadratic random terms. Procedure 2 reflects the difference in definitions of the plasma volume, V , or cell, $d\underline{x}$, in the microscopic and macroscopic

Models:

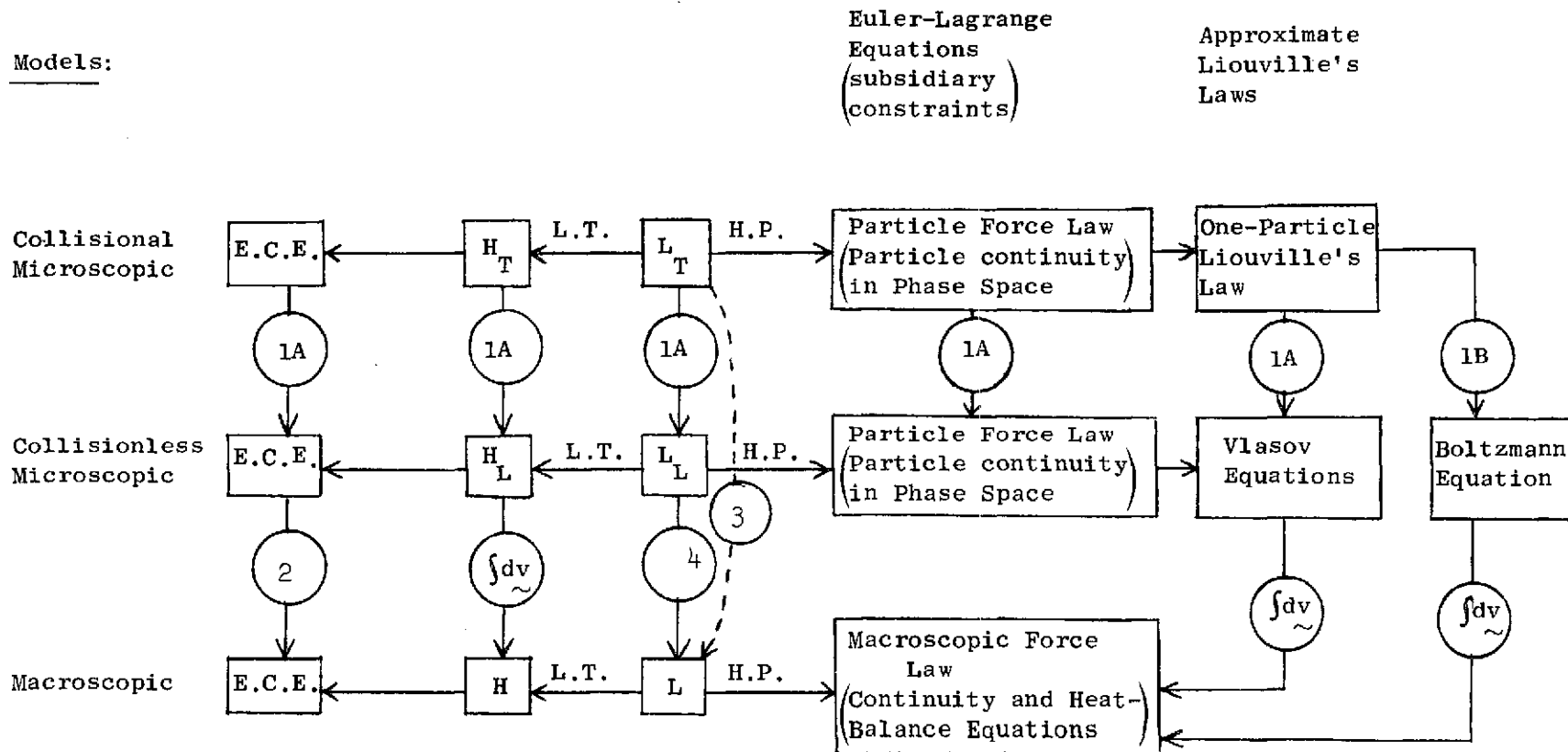


Figure 3.8 Relations among the variational principles of various plasma models. Here L_T and H_T represent the total Lagrangian and Hamiltonian, respectively, in the collisional microscopic model; L_L and H_L represent the (Low) Lagrangian and Hamiltonian, respectively, in the collisionless microscopic model; H.P. represents Hamilton's principle; L.T. represents the Legendre transformation, and E.C.E. represents the total energy conservation equation.

plasma models, as discussed in Section 3.4.3, with regard to the energy conservation equations in the two plasma models. In the case of a collisional plasma, Procedure 3 represents the heuristic approach of Section 3.2.2 that led to the appropriate expression of \mathcal{U} in (3.21). For a collisionless plasma, Procedure 4 represents the argument of Section 3.4.2 that discusses the loss of particle discreteness when we go from the microscopic model to the macroscopic model, and the dynamic equivalence between the macroscopic cells of the collisionless and collisional plasmas, so that the introduction of \mathcal{U} is seen to be necessary also for the collisionless macroscopic model.

All of these arguments should be considered less rigorous, for the purpose of proving the validity of L of (3.22), than the rigorous verification by application of Hamilton's principle in Section 3.2.3. These different viewpoints do, however, converge to the conclusion that, in general, the macroscopic potential energy density, \mathcal{U} , is a result of the combined contributions of plasma fluctuations (which is equivalent to elastic multiparticle collisions), elastic two-particle collisions, the loss of particle discreteness, and a redefinition of the plasma cell, \underline{dx} , in the process of making the macroscopic approximation.

3.5 Discussion

The principal contributions of this section verification of an appropriate Lagrangian density in Section 3.2.3 and Hamiltonian density in Section 3.3.2 for the macroscopic plasma model, including a pressure tensor, heat conduction, and elastic particle collisions. In Section 3.3.2, it was shown to be necessary to introduce a macroscopic potential energy density $\mathcal{U} (= \text{Tr } \underline{P} + \nabla \cdot \underline{Q})$. In order to obtain the Hamiltonian density,

it was necessary to modify the conventional Legendre transformation to conform to the dual role of the generalized variables. Low's Lagrangian (Low, 1958) shares the same property; the modified Legendre transformation was used successfully to derive a corresponding Hamiltonian in Section 3.3.6. The modified form of the Hamilton equations and Poisson brackets presented in Sections 3.3.3 and 3.3.4 may have further interesting implications in the canonical formulation of plasma dynamics.

The plasma model used in this section is, of course, a macroscopic approximation to the microscopic model described by the Boltzmann equation, or the Vlasov equation when collisions are negligible. The Euler-Lagrange equations, (3.26) and (3.46), consequently have a one-to-one correspondence with those of the Low Lagrangian. The continuity of particles in phase space, $(\underline{x}, \underline{v})$, is used by Low to constrain the variation with respect to the particle trajectory, $\underline{x}(\underline{x}_0, \underline{v}_0, t)$, while here the first (particle continuity) and third (heat balance) moment equations are used to constrain the variations with respect to the plasma macroscopic cell trajectory, $\underline{x}(\underline{x}_0, t)$. That only two moment equations are sufficient here does not imply that the mathematical system is closed at the second order moment equation. What we have done is to apply Hamilton's principle to a mathematically open system of equations: the Maxwell equations, and the moment equations. The latter are not truncated, since \underline{q} appears in the heat flow equation, and can only be obtained by use of higher order moment equations. In practice, truncation will be accomplished by making some arbitrary assumption about \underline{q} , typically $\underline{q} = 0$.

Whenever the scale length of variation of the macroscopic quantities becomes comparable with the mean particle spacing, the results presented

in this section for the macroscopic model becomes questionable. In addition, the assumption of elastic collisions neglects excitation, ionization, and recombination phenomena, and electromagnetic radiation associated with charged particle collisions (bremsstrahlung). Nevertheless, the variational principle developed here is appropriate to a multicomponent plasma, including neutral components.

In later sections, we shall apply our results to a number of plasma problems involving perturbation expansions. While these are straightforward for \tilde{v}_D and n (Newcomb, 1962), difficulty arises in connection with the term $\text{Tr } \tilde{P}/2 + \nabla \cdot \tilde{Q}$ in \mathcal{L} . Inspection of the heat flow equation in (3.29), reveals that it is not self-consistent: even with an explicit form given for C , we have to solve for all of the elements of \tilde{P} as well as the components of \tilde{q} ; these depend on higher order moment equations. Even when truncation at a specified moment of the Boltzmann equation is assumed, the heat flow equation is not generally equivalent to some constant of motion for a particle species. For the case in which \tilde{q} and C are neglected, approximate expressions for \tilde{P} in terms of the perturbation in \tilde{x} are generally impossible to derive mathematically (see Section 4.2.2). It is not surprising that only problems in which collisions between different particle species are neglected, and adiabatic processes are assumed to occur with either a scalar or axisymmetric pressure, have been successfully studied in terms of perturbation expansions. Such problems form the subject of Section 4.

4. THREE-WAVE INTERACTIONS: THEORY

4.1 Introduction

In this section, we shall show how the macroscopic Lagrangian obtained in Section 3 can be applied to the description of nonlinear wave-wave interactions in homogeneous plasmas. After developing a perturbation expansion of the Lagrangian, the method of averaged Lagrangians will be employed to derive general coupled mode equations. In Section 5, these equations will be specialized to a number of cases involving parametric amplification of ion-acoustic waves for which experimental data are available.

The use of Lagrangians in describing nonlinear wave phenomena was considered by Sturrock in 1961. He showed that in a conservative distributed system the time-averaged Poincaré invariants, which he later extended to the case of field variables (Sturrock, 1962), are equivalent to power-balance relations of the type well-known to electrical engineers as "Manley-Rowe relations" (Penfield, 1960). These results were extended by Whitham (1965) who averaged the Lagrangian in such a way as to remove rapidly varying terms, but conserve the slow variations in amplitude, frequency, and wave vector characteristic of a wave train in a weakly nonlinear medium. Vedenov and Rudakov (1965) used this approach to describe the interaction between ion-acoustic and Langmuir waves in a plasma. Using Low's Lagrangian (Low, 1958) and its perturbation approximation, several cases were examined by Suramlishvili, who derived the wave coupling coefficients for interactions between Langmuir and ion-acoustic waves (Suramlishvili, 1964), between one transverse and two longitudinal waves (Suramlishvili, 1965), between Alfvén and

whistler waves (Suramlishvili, 1967), between Alfvén and ion-acoustic waves (Suramlishvili, 1970), and among magnetoacoustic waves (Suramlishvili, 1971). A variety of general results have been established by Galloway (1972) and Dysthe (1974), using the averaged Lagrangian method. In particular, the former demonstrated that the energy conservation equation may be used to derive the Manley-Rowe relations, and the coupled mode equations describing wave-wave interactions. He has also shown that the wave energy and energy flux follow directly from the energy and energy flux terms quadratic in the perturbations, while the nonlinear wave coupling coefficient follows directly from the term in the Lagrangian cubic in the perturbations. Similar general results were employed by Boyd and Turner (1972b), who used Low's Lagrangian to examine the generation of longitudinal plasma waves by two high frequency electromagnetic waves in a warm field-free plasma, and the interaction of three electromagnetic waves in a cold magnetized plasma. This approach will be followed in Section 4.3.

Generalizations of the averaged Lagrangian method to nonlinear plasma phenomena other than wave-wave interactions have been made. Using a hydromagnetic Lagrangian derived by Newcomb (1962), Dewar has shown that the interactions between the wave and the slowly varying background plasma may be derived by use of Hamilton's principle (Dewar, 1970). Dougherty (1970) has obtained ray tracing and coupled mode equations, and demonstrated the conservation of wave action in a relativistically covariant formulation. Four-wave interactions, self-action effects, and sideband decay phenomena have been treated in the paper by Dysthe (1974) mentioned above. Derivations of the wave kinetic equation have been

supplied by Suramlshvili (1964; 1965) and Galloway (1972); the latter has also provided a description of quasilinear wave-particle interaction. A Lagrangian theory for nonlinear wave-packets propagating in a collisionless plasma has been developed by Dewar (1972) which describes the nonlinear frequency shift, Landau damping, and modulational sideband instabilities.

It will be clear from the foregoing remarks that the averaged Lagrangian method has been utilized to discuss many of the significant nonlinear phenomena occurring in plasmas. It has the merits of conciseness and efficiency in the analysis, and gives considerable insight into the physical mechanisms involved.

4.2 Perturbation Approximations to L

Prior to its use in Section 4.3, we shall expand the Lagrangian, (3.22), in terms of perturbations in the generalized variables, \underline{x} , φ and \underline{A} , up to the third order. Similar to the well-known approach for the macroscopic Lagrangian (Newcomb, 1962), the perturbations in n , \underline{v}_D , and $\text{Tr } \underline{P}/2 + \nabla \cdot \underline{Q}$ will be considered as due only to those in the generalized variables. Denoting the perturbed and unperturbed Lagrangians by L' and L , respectively, we write

$$L' = L + L_1 + L_2 + L_3 + \dots, \quad L_i = \int_V d\underline{x} \mathcal{L}_i, \quad (4.1)$$

where L_i denotes the Lagrangian i th order in perturbations. The quadratic Lagrangian, L_2 , will provide the first order equations that determine the linear wave properties, and the cubic Lagrangian, L_3 , will lead directly to the nonlinear coupling coefficients among the waves.

4.2.1 Definition of Perturbations

The perturbations in the generalized variables are defined as,

$$\underline{x}' = \underline{x} + \underline{\xi}(\underline{x}, t) \quad , \quad \varphi' = \varphi + \varphi_1 \quad , \quad \underline{A}' = \underline{A} + \underline{A}_1 \quad , \quad (4.2)$$

where $(\underline{x}, \varphi, \underline{A})$ and $(\underline{\xi}, \varphi_1, \underline{A}_1)$ are the unperturbed and perturbation variables, respectively. Figure 4.1 illustrates the definition of $\underline{\xi}$, which is analogous to the particle displacement vector defined by Sturrock (1958a) and Low (1958), and the macroscopic cell displacement vector defined by Newcomb (1962). The quantities (φ, \underline{A}) and $(\varphi_1, \underline{A}_1)$ are considered as functions of the Eulerian coordinates (\underline{x}, t) , consistent with their dual roles discussed in Sections 3.1 and 3.2.1.

The perturbed Lagrangian differs from the unperturbed Lagrangian (3.22) due to two factors: first, the presence of $\underline{\xi}$, φ_1 , and \underline{A}_1 at a given \underline{x} , and second, the use of the integration variable \underline{x}' rather than \underline{x} . Thus the perturbed Lagrangian may be written as

$$L' = \int_{V'} d\underline{x}' \mathcal{L}' \quad ,$$

$$\mathcal{L}' = \sum \left[\frac{m}{2} n' v_D'^2 - \frac{1}{2} \text{Tr} \underline{P}' - \nabla \cdot \underline{Q}' - qn' (\varphi' - \underline{v}_D' \cdot \underline{A}') \right] + \frac{\epsilon_0 E'^2}{2} - \frac{B'^2}{2\mu_0} \quad , \quad (4.3)$$

where the summation applies to the particle species. In (4.3), and what follows, primed and unprimed quantities are functions of \underline{x}' and \underline{x} , respectively, unless otherwise noted.

Imposing the condition that the particles are conserved in the process of applying perturbations, implies that,

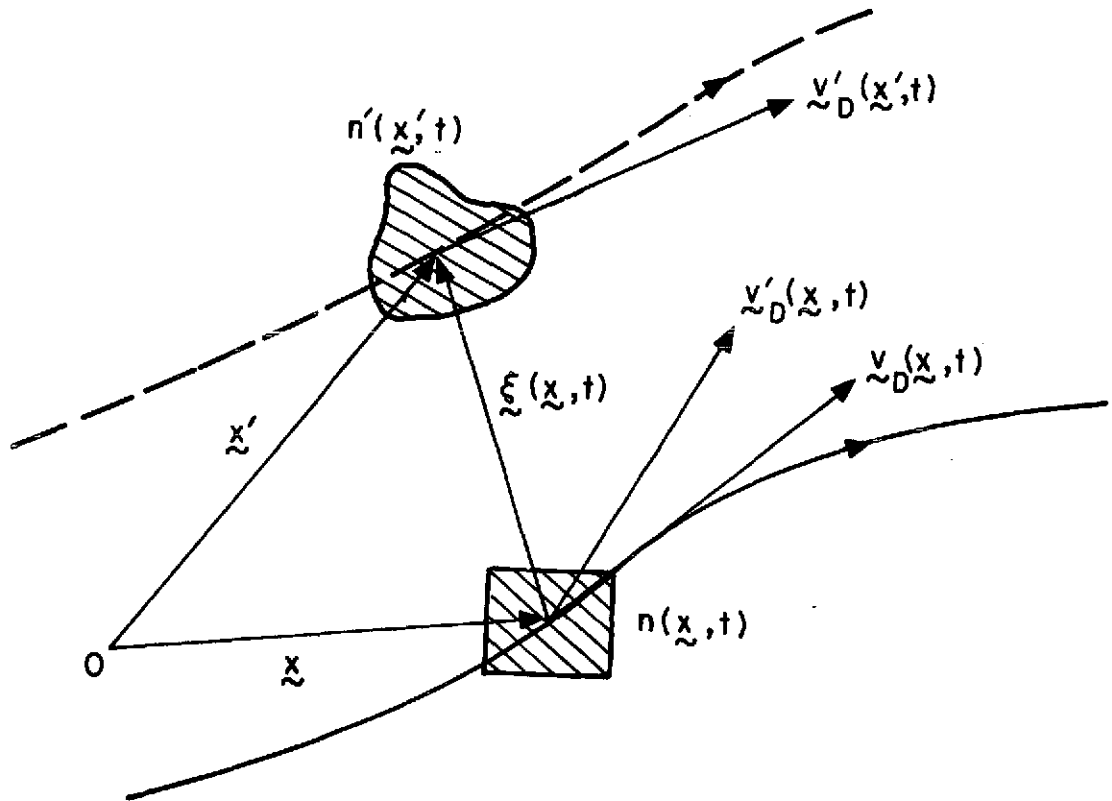


Figure 4.1 The definition of perturbation ξ in the macroscopic plasma cell position from \underline{x} to \underline{x}' . The perturbation is performed in such a way that the number of particles of a species in the cell is fixed. The perturbed and unperturbed cell trajectories are denoted as — and -----, respectively. The cell velocities before and after perturbation are denoted by $\underline{v}_D(\underline{x}, t)$ and $\underline{v}'_D(\underline{x}', t)$, respectively. The velocity of the cell that is at (\underline{x}, t) after perturbation is denoted by $\underline{v}'_D(\underline{x}, t)$.

$$n d\tilde{x} = n' d\tilde{x}' \quad , \quad (4.4)$$

while the Jacobian, J , of the transformation from \tilde{x} to \tilde{x}' is

$$J = \left| \frac{d\tilde{x}'}{d\tilde{x}} \right| \quad . \quad (4.5)$$

The integral in (4.3) can consequently be written in terms of an \tilde{x} -integration as

$$L' = \int_V d\tilde{x} f'' \quad ,$$

$$f'' = \sum \left[\frac{m}{2} n v_D'^2 - J \left(\frac{1}{2} \text{Tr} \tilde{P}' + \nabla \cdot \tilde{Q}' \right) - qn \left(\varphi' - \tilde{v}_D' \cdot \tilde{A}' \right) \right] + \left(\frac{\epsilon_0 E'^2}{2} - \frac{B'^2}{2\mu_0} \right)_{\tilde{x}} \quad . \quad (4.6)$$

4.2.2 Expansions of \tilde{v}_D' , n' , and \tilde{P}'

The terms that involve \tilde{v}_D' , J , φ' , \tilde{A}' , \tilde{E}' , and \tilde{B}' are easily expanded by the use of (4.2), (4.4), and (4.5), and the Taylor series expansions at \tilde{x} . However, expansion of the terms, $\text{Tr} \tilde{P}'/2 + \nabla \cdot \tilde{Q}'$, in (4.6) introduces considerable difficulty. We shall confine ourselves to the case of scalar pressure and adiabatic processes satisfying the equation of state

$$\frac{P'}{P} = \left(\frac{n'}{n} \right)^\gamma \quad , \quad \tilde{P} = \delta_{ij} P \quad , \quad (4.7)$$

for each particle species, with γ being the adiabatic index.

Expansion of \underline{v}'_D : Taking the time derivative of the first expression of (4.2), we obtain

$$\underline{v}'_D = \underline{v}_D + \dot{\underline{\xi}} \quad , \quad (4.8)$$

where $\dot{\underline{\xi}}$ denotes $d\underline{\xi}[\underline{x}(\underline{x}_0, t), t]/dt$. Then the nonlocal expansion of \underline{v}'_D is seen to be,

$$\begin{aligned} \underline{v}'_D &= \underline{v}_D + \underline{v}_1 + \underline{v}_2 + \underline{v}_3 + \dots \quad , \\ \underline{v}_1 &= \dot{\underline{\xi}} \quad , \quad \underline{v}_2 = \underline{v}_3 = \dots = 0 \quad . \end{aligned} \quad (4.9)$$

If we define the local expansion of $\underline{v}'_D(\underline{x}, t)$ as

$$\underline{v}'_D(\underline{x}, t) = \underline{v}_D + \underline{v}_{L1} + \underline{v}_{L2} + \underline{v}_{L3} + \dots \quad , \quad (4.10)$$

it can be shown by Taylor series expansion of $\underline{v}'_D(\underline{x}', t)$ at \underline{x} that

$$\underline{v}_i = \underline{v}_{Li} + \sum_{j=1}^i \frac{1}{j!} (\underline{\xi} \cdot \nabla)^j \underline{v}_{L, i-1} \quad . \quad (4.11)$$

As a result, we have

$$\underline{v}_{L1} = \dot{\underline{\xi}} - \underline{\xi} \cdot \nabla \underline{v}_D \quad , \quad \underline{v}_{L2} = -\underline{\xi} \cdot \nabla \dot{\underline{\xi}} + \frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla \underline{v}_D + \underline{\xi} \cdot \nabla \underline{\xi} \cdot \nabla \underline{v}_D \quad . \quad (4.12)$$

Expansion of \underline{n}' : Combination of (4.4) and (4.5) gives the particle conservation law along $\underline{\xi}$ as (Newcomb, 1962)

$$\underline{n}' \cdot \underline{j} = \underline{n} \quad . \quad (4.13)$$

With the Jacobian, J , expressed in its exact expansion,

$$J = 1 + \nabla \cdot \underline{\xi} + \frac{1}{2} \left[(\nabla \cdot \underline{\xi})^2 - \nabla \underline{\xi} : \nabla \underline{\xi} \right] + \left| \frac{\partial \underline{\xi}}{\partial \underline{x}} \right| ,$$

$$\left| \frac{\partial \underline{\xi}}{\partial \underline{x}} \right| = \frac{1}{6} (\nabla \cdot \underline{\xi})^3 - \frac{1}{2} (\nabla \cdot \underline{\xi}) (\nabla \underline{\xi} : \nabla \underline{\xi}) + \frac{1}{3} \nabla \underline{\xi} : (\nabla \underline{\xi} \cdot \nabla \underline{\xi}) , \quad (4.14)$$

the nonlocal expansion of n' becomes

$$n' = n + n_1 + n_2 + n_3 + \dots ,$$

$$n_1 = -\nabla \cdot \underline{\xi} \quad , \quad n_2 = \frac{n}{2} \left[(\nabla \cdot \underline{\xi})^2 + \nabla \underline{\xi} : \nabla \underline{\xi} \right] ,$$

$$n_3 = -n \left[(\nabla \cdot \underline{\xi}) (\nabla \underline{\xi} : \nabla \underline{\xi}) + \left| \frac{\partial \underline{\xi}}{\partial \underline{x}} \right| \right] . \quad (4.15)$$

By use of a relation between n_i and n_{Li} analogous to (4.11), the local expansion of n' then takes the form

$$n'(\underline{x}, t) = n + n_{L1} + n_{L2} + n_{L3} + \dots ,$$

$$n_{L1} = -\nabla \cdot (\underline{\xi} n) \quad , \quad n_{L2} = \frac{1}{2!} \nabla \nabla : (\underline{\xi} \underline{\xi} n) . \quad (4.16)$$

The results of (4.16) agree with Sturrock's generalization of the Lagrange expansion in the form (Sturrock, 1960b)

$$n_{Li} = \frac{(-1)^i}{i!} \nabla_k \dots \nabla_j (\xi_k \dots \xi_j n) \quad (k, j = 1, 2, 3; i = 1, 2, \dots) , \quad (4.17)$$

while n_{L1} and n_{L2} have been derived by Newcomb (1962).

Expansion of P' : The nonlocal expansion of P' can be obtained by substituting (4.15) and the second expression of (4.14) in the first expression of (4.7),

$$P' = P + P_1 + P_2 + P_3 + \dots ,$$

$$P_1 = -\gamma P(\nabla \cdot \underline{\xi}) \quad , \quad P_2 = \frac{\gamma}{2} P \left[\gamma(\nabla \cdot \underline{\xi})^2 + \nabla \underline{\xi} : \nabla \underline{\xi} \right] \quad ,$$

$$P_3 = -\gamma P \left[\frac{\gamma^2}{6} (\nabla \cdot \underline{\xi})^3 + \frac{\gamma}{2} (\nabla \cdot \underline{\xi})(\nabla \underline{\xi} : \nabla \underline{\xi}) + \frac{1}{3} \nabla \underline{\xi} : (\nabla \underline{\xi} \cdot \nabla \underline{\xi}) \right] \quad . \quad (4.18)$$

By use of a relation between P_i and P_{Li} analogous to (4.11), we obtain from (4.18),

$$P'(\underline{x}, t) = P + P_{L1} + P_{L2} + \dots ,$$

$$P_{L1} = -\gamma P \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla P \quad ,$$

$$P_{L2} = P \left[\frac{\gamma^2}{2} (\nabla \cdot \underline{\xi})^2 + \frac{\gamma}{2} \nabla \underline{\xi} : \nabla \underline{\xi} + \gamma \underline{\xi} \cdot \nabla(\nabla \cdot \underline{\xi}) \right]$$

$$+ \left[\underline{\xi} \cdot \nabla \underline{\xi} + \gamma(\nabla \cdot \underline{\xi}) \underline{\xi} \right] \cdot \nabla P + \frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla P \quad . \quad (4.19)$$

Because \underline{v}_{L3} , n_{L3} , and P_{L3} will not be needed to derive the second order expansions of the perturbed force law (3.46) and Maxwell's equations (3.26), these local expansions of \underline{v}' , n' , and P' were not given in (4.12), (4.16), and (4.19). The nonlocal expansions, \underline{v}_3 , n_3 , and P_3 , however, are included in (4.9), (4.15), and (4.18) because they will be used in deriving L_3 .

4.2.3 Expansion of the Lagrangian

The expressions for the \mathcal{L}_i of (4.1) can be obtained by substitution of (4.8), (4.14), (4.18), and the second expression of (4.7) in (4.6).

Then, for the unperturbed Lagrangian density, we have

$$\mathcal{L} = \sum \left[\frac{m}{2} n v_D^2 - \frac{P}{\gamma-1} - qn(\varphi - \underline{v}_D \cdot \underline{A}) \right] + \frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0}, \quad (4.20)$$

where the term $P/(\gamma-1)$ is a result of generalization from $P/(\frac{5}{3}-1)$ by replacing $5/3$ by γ (Newcomb, 1962). For the perturbation Lagrangian densities, we have after using integration by parts,

$$\mathcal{L}_1 = \sum \left[m n \underline{v}_D \cdot \dot{\underline{\xi}} + P \nabla \cdot \underline{\xi} - qn(\varphi_1 + \underline{\xi} \cdot \nabla \varphi - \dot{\underline{\xi}} \cdot \underline{A}) + qn \underline{v}_D \cdot (\underline{A}_1 + \underline{\xi} \cdot \nabla \underline{A}) \right] + \epsilon_0 \underline{E} \cdot \underline{E}_1 - \frac{1}{\mu_0} \underline{B} \cdot \underline{B}_1,$$

$$\mathcal{L}_2 = \sum \left\{ \frac{m}{2} n \dot{\underline{\xi}}^2 - \frac{P}{2} \left[(\gamma-1)(\nabla \cdot \underline{\xi})^2 + \nabla \underline{\xi} : \nabla \underline{\xi} \right] - qn \left(\underline{\xi} \cdot \nabla \varphi_1 + \frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla \varphi \right) \right. \\ \left. + qn \left[\underline{v}_D \cdot \left(\underline{\xi} \cdot \nabla \underline{A}_1 + \frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla \underline{A} \right) + \dot{\underline{\xi}} \cdot (\underline{A}_1 + \underline{\xi} \cdot \nabla \underline{A}) \right] + \frac{\epsilon_0 E_1^2}{2} - \frac{B_1^2}{2\mu_0} \right\},$$

$$\mathcal{L}_3 = \sum \left\{ P \left[\frac{(\gamma-1)^2}{6} (\nabla \cdot \underline{\xi})^3 + \frac{\gamma-1}{2} (\nabla \cdot \underline{\xi})(\nabla \underline{\xi} : \nabla \underline{\xi}) + \frac{1}{3} \nabla \underline{\xi} : (\nabla \underline{\xi} \cdot \nabla \underline{\xi}) \right] \right. \\ \left. - qn \left(\frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla \varphi_1 + \frac{1}{6} \underline{\xi} \underline{\xi} \underline{\xi} : \nabla \nabla \nabla \varphi \right) \right. \\ \left. + qn \left[\underline{v}_D \cdot \left(\frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla \underline{A}_1 + \frac{1}{6} \underline{\xi} \underline{\xi} \underline{\xi} : \nabla \nabla \nabla \underline{A} \right) + \dot{\underline{\xi}} \cdot \left(\underline{\xi} \cdot \nabla \underline{A}_1 + \frac{1}{2} \underline{\xi} \underline{\xi} : \nabla \nabla \underline{A} \right) \right] \right\}. \quad (4.21)$$

The Euler-Lagrange equations of \mathcal{L} , from \mathcal{L} of (4.20), with respect to \underline{x} , \underline{A} , and φ , are the unperturbed force law and Maxwell's equations,

$$m\dot{v}_{\sim D} + \nabla P - qn(\underline{E} + \underline{v}_D \times \underline{B}) = 0 \quad ,$$

$$\frac{1}{\mu_0} \nabla \times \underline{B} - \epsilon_0 \frac{\partial \underline{E}}{\partial t} - \sum qn\dot{v}_{\sim D} = 0 \quad , \quad \epsilon_0 \nabla \cdot \underline{E} - \sum qn = 0 \quad . \quad (4.22)$$

Use of these shows that L_1 , from \mathcal{L}_1 of (4.21), is identically zero. The Euler-Lagrange equations of L_2 , from \mathcal{L}_2 of (4.21), form the system of linear equations in $\underline{\xi}$, \underline{A}_1 , and φ_1 , describing wave propagation. We shall not derive them at this point, but consider the Euler-Lagrange equations resulting from variation of $L_2 + L_3$, describing nonlinear wave-wave interactions.

Variation in φ_1 : Taking the variation of $L_2 + L_3$ with respect to φ_1 yields

$$\epsilon_0 \nabla \cdot \underline{E}_1 + \sum q \left[\nabla \cdot (n \underline{\xi}) - \frac{1}{2} \nabla \nabla : (n \underline{\xi} \underline{\xi}) \right] = 0 \quad . \quad (4.23)$$

Local expansion of the second expression of (3.26), and use of the last expression of (4.22), gives

$$\epsilon_0 \nabla \cdot \underline{E}_1 - \sum q (n_{L1} + n_{L2}) = 0 \quad . \quad (4.24)$$

By use of (4.16), it is easy to demonstrate that (4.23) and (4.24) are identical.

Variation in \underline{A}_1 : Taking the variation of $L_2 + L_3$ with respect to \underline{A}_1 yields

$$\frac{1}{\mu_0} \nabla \times \underline{B}_1 - \epsilon_0 \frac{\partial \underline{E}_1}{\partial t} - \sum q \left[n \dot{\underline{\xi}} - \nabla \cdot (n \underline{\xi} \underline{v}_D) - \nabla \cdot (n \dot{\underline{\xi}} \underline{\xi}) + \frac{1}{2} \nabla \nabla : (n \underline{\xi} \underline{\xi} \underline{v}_D) \right] = 0 \quad . \quad (4.25)$$

Local expansion of the first expression of (3.26), and use of the second expression of (4.24), gives

$$\frac{1}{\mu_0} \nabla \times \underline{\underline{B}}_1 - \epsilon_0 \frac{\partial \underline{\underline{E}}_1}{\partial t} - \sum q \left(n_{\underline{\underline{L}}_1} \underline{\underline{v}} + n_{\underline{\underline{L}}_1 \underline{\underline{D}}} \underline{\underline{v}} + n_{\underline{\underline{L}}_2} \underline{\underline{v}} + n_{\underline{\underline{L}}_1 \underline{\underline{L}}_1} \underline{\underline{v}} + n_{\underline{\underline{L}}_2 \underline{\underline{D}}} \underline{\underline{v}} \right) = 0. \quad (4.26)$$

Use of (4.12) and (4.16) shows that (4.25) and (4.26) are identical.

Variation in $\underline{\underline{\xi}}$: Taking the variation of $L_2 + L_3$ with respect to $\underline{\underline{\xi}}$ yields

$$\begin{aligned} mn \ddot{\underline{\underline{\xi}}} - (\gamma-1) \nabla P \nabla \cdot \underline{\underline{\xi}} - \nabla \underline{\underline{\xi}} \cdot \nabla P + \gamma P \left[(\gamma-1) (\nabla \cdot \underline{\underline{\xi}}) \nabla (\nabla \cdot \underline{\underline{\xi}}) + \nabla (\nabla \underline{\underline{\xi}} : \nabla \underline{\underline{\xi}}) + \nabla \underline{\underline{\xi}} \cdot \nabla (\nabla \cdot \underline{\underline{\xi}}) \right] \\ + \frac{\gamma-1}{2} \nabla P \left[(\gamma-1) (\nabla \cdot \underline{\underline{\xi}})^2 + \nabla \underline{\underline{\xi}} : \nabla \underline{\underline{\xi}} \right] + (\gamma-1) (\nabla \cdot \underline{\underline{\xi}}) \nabla \underline{\underline{\xi}} \cdot \nabla P + \nabla \underline{\underline{\xi}} \cdot \nabla \underline{\underline{\xi}} \cdot \nabla P \\ - qn \left(\underline{\underline{E}}_1 + \underline{\underline{\xi}} \cdot \nabla \underline{\underline{E}} + \underline{\underline{v}}_D \times \underline{\underline{B}}_1 - \underline{\underline{\xi}} \cdot \nabla \underline{\underline{B}} \times \underline{\underline{v}}_D + \dot{\underline{\underline{\xi}}} \times \underline{\underline{B}} \right) \\ - qn \left[\underline{\underline{\xi}} \cdot \nabla \underline{\underline{E}}_1 + \frac{1}{2} \underline{\underline{\xi}} \underline{\underline{\xi}} : \nabla \nabla \underline{\underline{E}} - \left(\underline{\underline{\xi}} \cdot \nabla \underline{\underline{B}}_1 + \frac{1}{2} \underline{\underline{\xi}} \underline{\underline{\xi}} : \nabla \nabla \underline{\underline{B}} \right) \times \underline{\underline{v}}_D - \left(\underline{\underline{B}}_1 + \underline{\underline{\xi}} \cdot \nabla \underline{\underline{B}} \right) \times \dot{\underline{\underline{\xi}}} \right] = 0. \quad (4.27) \end{aligned}$$

The local expansion of (3.46) at $\underline{\underline{x}}$, with $\nabla \cdot \underline{\underline{P}}$ replaced by ∇P , does not agree with (4.27). The discrepancy occurs because the former is inconsistent with the nonlocal expansion of L' of (4.3). The appropriate expansion of the force law must be constrained by the particle conservation law, i.e. by comparing the first expression of (4.22) with the force law

$$J \left[mn' \frac{d\underline{\underline{v}}_D'}{dt} + \nabla' P' - qn' (\underline{\underline{E}}' + \underline{\underline{v}}_D' \times \underline{\underline{B}}') \right] = 0, \quad (4.28)$$

which describes the same cell of particles in its perturbed motion.

The nonlocal expansion of (4.28) is

$$\begin{aligned}
mn\ddot{\xi} + \nabla P \left[\nabla \cdot \xi + \frac{1}{2} (\nabla \cdot \xi)^2 - \frac{1}{2} (\nabla \xi : \nabla \xi) \right] \\
+ \left[1 + \nabla \cdot \xi + \frac{1}{2} (\nabla \cdot \xi)^2 - \frac{1}{2} (\nabla \xi : \nabla \xi) \right] \left(\nabla P_{L1} + \xi \cdot \nabla \nabla P + \nabla P_{L2} + \xi \cdot \nabla \nabla P_{L1} + \frac{1}{2} \xi \xi : \nabla \nabla \nabla P \right) \\
- qn \left(\underline{E}_1 + \xi \cdot \nabla \underline{E} + \xi \cdot \nabla \underline{E}_1 + \frac{1}{2} \xi \xi : \nabla \nabla \underline{E} \right) - qn \left(\dot{\xi} \times \underline{B} + \underline{v}_D \times \underline{B}_1 - \xi \cdot \nabla \underline{B} \times \underline{v}_D \right. \\
\left. + \dot{\xi} \times \underline{B}_1 - \xi \cdot \nabla \underline{B} \times \dot{\xi} - \xi \cdot \nabla \underline{B}_1 \times \underline{v}_D - \frac{1}{2} \xi \xi : \nabla \nabla \underline{B} \times \underline{v}_D \right) = 0 \quad , \quad (4.29)
\end{aligned}$$

where (4.8), (4.14), (4.15), and the first expression of (4.22) have been used. Introduction of (4.19) establishes that (4.28) and (4.29) are identical. A similar force law expansion is examined in Appendix B.

4.3 Nonlinear Wave Coupling Coefficients

We shall now use $L_2 + L_3$ to obtain the coupling coefficients for waves in a homogeneous, stationary, two-component magnetoplasma. The general expression will be derived by the averaged-Lagrangian technique.

4.3.1 Averaged Lagrangian

It is convenient to introduce the normalized quantities

$$\begin{aligned}
\sigma = \frac{n_i m_i}{n_e m_e} \quad , \quad v_s = \left(\frac{P_s}{n_s m_s c^2} \right)^{1/2} = \Lambda_{Ds} \quad , \quad \tilde{\Omega} = \frac{eB}{\omega_{pe} m_e} \quad , \\
\tilde{\zeta}_s = \frac{\omega_{pe}}{c} \zeta_s \quad , \quad \tilde{\phi} = \frac{e}{m_e c^2} \phi_1 \quad , \quad \tilde{a} = \frac{e}{m_e c} a_1 \quad , \\
\tilde{\Omega} = \frac{\omega}{\omega_{pe}} \quad , \quad \tilde{K} = \frac{c}{\omega_{pe}} k \quad , \quad \tilde{X} = \frac{\omega_{pe}}{c} x \quad , \quad T = \omega_{pe} t \quad , \quad (4.30)
\end{aligned}$$

where the subscript s denotes electrons or ions; ω_{pe} is the electron plasma frequency, c is the free space speed of light, and Λ_{Ds} is the Debye length normalized against c/ω_{pe} . Then \mathcal{L}_2 and \mathcal{L}_3 from (4.21) may be written as

$$\begin{aligned}
\mathcal{L}_2 = & \sigma \left[\frac{1}{2} \dot{\xi}_i^2 - \frac{\gamma_i}{2} v_i^2 (\nabla \cdot \xi_i)^2 \right] + \dot{\xi}_i \cdot \underline{a} - \frac{1}{2} \dot{\xi}_i \cdot (\xi_i \times \underline{\Omega}_c) + \Phi \nabla \cdot \xi_i \\
& + \frac{1}{2} \dot{\xi}_e^2 - \frac{\gamma_e}{2} v_e^2 (\nabla \cdot \xi_e)^2 - \dot{\xi}_e \cdot \underline{a} + \frac{1}{2} \dot{\xi}_e \cdot (\xi_e \times \underline{\Omega}_c) - \Phi \nabla \cdot \xi_e \\
& + \frac{1}{2} |\nabla \Phi + \dot{\underline{a}}|^2 - \frac{1}{2} |\nabla \times \underline{a}|^2, \\
\mathcal{L}_3 = & \sigma v_i^2 \left[\frac{\gamma_i}{2} (\nabla \cdot \xi_i) (\nabla \xi_i : \nabla \xi_i) - \frac{\gamma_i}{6} (2 - \gamma_i) (\nabla \cdot \xi_i)^3 \right] + \xi_i \cdot \nabla \underline{a} \cdot \dot{\xi}_i - \frac{1}{2} \xi_i \xi_i : \nabla \nabla \Phi \\
& + v_e^2 \left[\frac{\gamma_e}{2} (\nabla \cdot \xi_e) (\nabla \xi_e : \nabla \xi_e) - \frac{\gamma_e}{6} (2 - \gamma_e) (\nabla \cdot \xi_e)^3 \right] - \xi_e \cdot \nabla \underline{a} \cdot \dot{\xi}_e + \frac{1}{2} \xi_e \xi_e : \nabla \nabla \Phi, \quad (4.31)
\end{aligned}$$

where the values of \mathcal{L}_2 and \mathcal{L}_3 have been normalized against $n_e m_e c^2$; the pressure terms have been reduced from their counterparts in (4.21) by integration by parts, and we have assumed the background charge to be neutral ($q_i n_i = e n_e$, $q_e = -e$) and the dc $\nabla \nabla A$ and $\nabla \phi$ to be zero.

The averaged forms of \mathcal{L}_2 and \mathcal{L}_3 may be obtained using the Fourier transformation

$$\xi_e(\underline{X}, T) = \frac{1}{(2\pi)^2} \int d\underline{K} \int d\underline{\Omega} \xi_e(\underline{K}, \underline{\Omega}) \exp i(\underline{\Omega} T - \underline{K} \cdot \underline{X}), \quad (4.32)$$

where $\xi_e(\underline{K}, \underline{\Omega})$, \underline{K} , and $\underline{\Omega}$ can vary slowly with the normalized space and time coordinates, \underline{X} and T , due to weak nonlinear wave-wave interaction. If the scales of these slow variations, $\Delta \underline{X}$ and ΔT , satisfy

$$\Delta T |\Omega - \Omega'| \gg \pi \quad , \quad |\Delta \underline{X} \cdot (\underline{K} - \underline{K}')| \gg \pi \quad , \quad (4.33)$$

then we have (Schiff, 1970)

$$\int_{\Delta \underline{X}} d\underline{X} \int_{\Delta T} dT \exp i [(\Omega - \Omega')T - (\underline{K} - \underline{K}') \cdot \underline{X}] \approx (2\pi)^4 \delta(\Omega - \Omega') \delta(\underline{K} - \underline{K}') \quad . \quad (4.34)$$

The action integral of (4.31),

$$I_i = \int d\underline{X} \int dT \mathcal{L}_i \quad (i = 2, 3) \quad , \quad (4.35)$$

can be rewritten by use of (4.32) and (4.34). For I_2 , we have

$$I_2 = \int d\underline{\Omega} \int d\underline{K} \Lambda_{\underline{K}, \underline{\Omega}}^{(2)} \quad , \quad \Lambda_{\underline{K}, \underline{\Omega}}^{(2)} = \Lambda_{\underline{K}, \underline{\Omega}}^{i(2)} + \Lambda_{\underline{K}, \underline{\Omega}}^{e(2)} + \Lambda_{\underline{K}, \underline{\Omega}}^{F(2)} \quad ,$$

$$\Lambda_{\underline{K}, \underline{\Omega}}^{i(2)} = \sigma \left[\frac{\Omega^2}{2} |\underline{\zeta}_i|^2 - \frac{\gamma_i}{2} v_i^2 |\underline{K} \cdot \underline{\zeta}_i|^2 \right] + i \Omega \underline{\zeta}_i \cdot \underline{\underline{a}}^* - \frac{1}{2} \Omega \underline{\zeta}_i \cdot (\underline{\zeta}_i^* \times \underline{\underline{\Omega}}) + i \Phi(\underline{K} \cdot \underline{\zeta}_i^*) \quad ,$$

$$\Lambda_{\underline{K}, \underline{\Omega}}^{e(2)} = \frac{\Omega^2}{2} |\underline{\zeta}_e|^2 - \frac{\gamma_e}{2} v_e^2 |\underline{K} \cdot \underline{\zeta}_e|^2 - i \Omega \underline{\zeta}_e \cdot \underline{\underline{a}}^* + \frac{1}{2} \Omega \underline{\zeta}_e \cdot (\underline{\zeta}_e^* \times \underline{\underline{\Omega}}) - i \Phi(\underline{K} \cdot \underline{\zeta}_e^*) \quad ,$$

$$\Lambda_{\underline{K}, \underline{\Omega}}^{F(2)} = \frac{1}{2} |\underline{K} \Phi - \Omega \underline{\underline{a}}|^2 - \frac{1}{2} K^2 |\underline{\underline{a}}|^2 \quad , \quad (4.36)$$

where the Coulomb gauge ($\nabla \cdot \underline{\underline{a}} = 0$), and the symmetry, $\underline{\zeta}_e(-\underline{K}, -\Omega) = \underline{\zeta}_e^*(\underline{K}, \Omega)$, have been assumed. The latter reflects the requirement that $\underline{\zeta}_e(\underline{X}, T)$ be real, and has also been applied to $\underline{\zeta}_i$, Φ , and $\underline{\underline{a}}$.

To reduce I_3 to its simplest form, it will be necessary to use the properties of linear waves, which can be obtained from I_2 . We shall therefore establish them before proceeding further.

4.3.2 Linear Waves

By taking the variations of I_2 with respect to $\underline{\zeta}_s^*(\underline{k}, \Omega)$ and $\underline{a}^*(\underline{k}, \Omega)$, we obtain the following linear equations

$$\begin{aligned} \sigma \left[\Omega^2 \underline{\zeta}_i - \gamma_i v_i^2 \underline{k}(\underline{k} \cdot \underline{\zeta}_i) \right] - i\Omega \underline{a} + i\underline{k}\underline{\phi} + i\Omega \underline{\zeta}_i \times \underline{\Omega}_c &= 0, \\ \Omega^2 \underline{\zeta}_e - \gamma_e v_e^2 \underline{k}(\underline{k} \cdot \underline{\zeta}_e) + i\Omega \underline{a} - i\underline{k}\underline{\phi} - i\Omega \underline{\zeta}_e \times \underline{\Omega}_c &= 0, \\ \underline{k}^2 \underline{a} + \Omega(\underline{k}\underline{\phi} - \Omega \underline{a}) - i\Omega(\underline{\zeta}_i - \underline{\zeta}_e) &= 0. \end{aligned} \quad (4.37)$$

By introducing the normalized electric field,

$$\underline{\mathcal{E}} = i(\underline{k}\underline{\phi} - \Omega \underline{a}) = \frac{e\mathbf{E}}{\omega \frac{m_e}{pe} c}, \quad (4.38)$$

we can obtain from (4.37)

$$\underline{\phi} = -\frac{i\underline{k} \cdot \underline{\mathcal{E}}}{K^2}, \quad \underline{a} = \frac{i}{\Omega} \left(\underline{\mathcal{E}} - \frac{\underline{k}\underline{k} \cdot \underline{\mathcal{E}}}{K^2} \right), \quad \underline{\zeta}_e = \underline{M}^e \cdot \underline{\mathcal{E}}, \quad \underline{\zeta}_i = -\frac{1}{\sigma} \underline{M}^i \cdot \underline{\mathcal{E}}. \quad (4.39)$$

Here, the polarization tensor, \underline{M}^s , is defined in a Cartesian coordinate system with $\underline{\Omega}_c$ in the positive z-direction. Its elements are given by

$$\begin{aligned} M_{xx}^s &= \frac{1}{\Delta_s} \left[\Omega^2 - \gamma_s v_s^2 (K_y^2 + K_z^2) \right], & M_{yy}^s &= \frac{1}{\Delta_s} \left[\Omega^2 - \gamma_s v_s^2 (K_x^2 + K_z^2) \right], \\ M_{zz}^s &= \frac{1}{\Delta_s} \left[\Omega^2 - \gamma_s v_s^2 (K_x^2 + K_y^2) - \Omega_s^2 \right], \\ M_{xy}^s &= M_{yx}^{s*} = \frac{1}{\Delta_s} \left[\gamma_s v_s^2 \left(K_x K_y + \frac{iK_z^2 \Omega_s}{\Omega} \right) - i\Omega \Omega_s \right], \\ M_{xz}^s &= M_{zy}^{s*} = \frac{\gamma_s v_s^2 K_z}{\Delta_s} \left(K_x - \frac{iK_y \Omega_s}{\Omega} \right), & M_{yz}^s &= M_{zy}^{s*} = \frac{\gamma_s v_s^2 K_z}{\Delta_s} \left(K_y + \frac{iK_x \Omega_s}{\Omega} \right), \end{aligned} \quad (4.40)$$

where * denotes the complex conjugate, and Ω_e , Ω_i , and Δ_s are defined as

$$\Omega_e = -\Omega_c \quad , \quad \Omega_i = \frac{\Omega_c}{\sigma} \quad , \quad \Delta_s = \Omega^2 \left(\Omega^2 - \gamma_s v_s^2 k^2 \right) - \Omega_s^2 \left(\Omega^2 - \gamma_s v_s^2 k_z^2 \right) \quad . \quad (4.41)$$

Using (4.39) in (4.37) yields the familiar results (Allis, Buchsbaum, and Bers, 1963)

$$\begin{aligned} \underline{\underline{D}} \cdot \underline{\underline{e}} &= 0 \quad , \quad \underline{\underline{D}} = \underline{\underline{N}} \underline{\underline{N}} - \underline{\underline{I}} + \underline{\underline{K}} \quad , \\ \underline{\underline{K}} &= \underline{\underline{I}} - \underline{\underline{M}} - \frac{\underline{\underline{M}}_i}{\sigma} \quad , \quad \underline{\underline{N}} = \frac{\underline{\underline{K}}}{\Omega} \quad , \end{aligned} \quad (4.42)$$

where $\underline{\underline{K}}$ is the plasma equivalent permittivity tensor. Nontrivial solutions for (4.42) exist when $\underline{\underline{K}}$ and Ω satisfy the dispersion relation,

$$\det \underline{\underline{D}} = 0 \quad . \quad (4.43)$$

For these values of $\Omega(\underline{\underline{K}})$, the electric field polarization vector is

$$\underline{\underline{e}} \propto \left\{ \left(\underline{\underline{D}}_{yy} \underline{\underline{D}}_{xz} - \underline{\underline{D}}_{xy} \underline{\underline{D}}_{yz} \right) \quad , \quad \left(\underline{\underline{D}}_{xx} \underline{\underline{D}}_{yz} - \underline{\underline{D}}_{yx} \underline{\underline{D}}_{xz} \right) \quad , \quad \left(\underline{\underline{D}}_{xy} \underline{\underline{D}}_{yx} - \underline{\underline{D}}_{xx} \underline{\underline{D}}_{yy} \right) \right\} \quad . \quad (4.44)$$

We shall postpone discussion of the various linear waves described by (4.43) until Section 5, where Langmuir, ion-acoustic, and whistler waves will be treated in applications of the results to be derived in this section.

4.3.3 Wave Coupling Coefficients

We now return to the derivation of wave coupling coefficients from $\underline{\underline{J}}_3$. We first separate $\underline{\underline{e}}(\underline{\underline{K}}, \Omega)$ into its component wave amplitudes and unit polarizations,

$$\underline{\underline{e}}(\underline{\underline{k}}, \Omega) = \sum_U \underline{\underline{e}}_{\underline{\underline{k}}}^U(\underline{\underline{x}}, T) \underline{\underline{e}}_{\underline{\underline{k}}}^U, \quad (4.45)$$

where $\underline{\underline{e}}_{\underline{\underline{k}}}^U(\underline{\underline{x}}, T)$ has slow $\underline{\underline{x}}$ - and T -dependences because of nonlinear wave interaction. Use of (4.45), (4.32), (4.34), (4.35), and (4.39), in (4.31) then gives

$$\begin{aligned} I_3 &= i \int_S d\underline{\underline{k}} d\underline{\underline{k}}' d\underline{\underline{k}}'' \underline{\underline{e}} \underline{\underline{e}}' \underline{\underline{e}}'' \Lambda_{\underline{\underline{k}}\underline{\underline{k}}'\underline{\underline{k}}''}^s, \\ \Lambda_{\underline{\underline{k}}\underline{\underline{k}}'\underline{\underline{k}}''}^s &= \sum_{\rho, s} \tau_s \left[\Lambda_T^s(\underline{\underline{k}}, \underline{\underline{k}}', \underline{\underline{k}}'') + \Lambda_I^s(\underline{\underline{k}}, \underline{\underline{k}}', \underline{\underline{k}}'') \right], \\ \Lambda_T^s(\underline{\underline{k}}, \underline{\underline{k}}', \underline{\underline{k}}'') &= v_s^2 (\underline{\underline{k}} \cdot \underline{\underline{M}}^s \cdot \underline{\underline{e}}) \left[\frac{\gamma_s}{2} (\underline{\underline{k}}' \cdot \underline{\underline{M}}^{s''} \cdot \underline{\underline{e}}'') (\underline{\underline{k}}'' \cdot \underline{\underline{M}}^{s'} \cdot \underline{\underline{e}}') \right. \\ &\quad \left. - \frac{\gamma_s}{6} (2 - \gamma_s) (\underline{\underline{k}}' \cdot \underline{\underline{M}}^{s'} \cdot \underline{\underline{e}}') (\underline{\underline{k}}'' \cdot \underline{\underline{M}}^{s''} \cdot \underline{\underline{e}}'') \right], \\ \Lambda_I^s(\underline{\underline{k}}, \underline{\underline{k}}', \underline{\underline{k}}'') &= \frac{\Omega''}{\Omega} (\underline{\underline{e}} \cdot \underline{\underline{M}}^{s''} \cdot \underline{\underline{e}}'') (\underline{\underline{k}} \cdot \underline{\underline{M}}^{s'} \cdot \underline{\underline{e}}') \\ &\quad - \frac{1}{K^2} \left(\frac{1}{2} + \frac{\Omega''}{\Omega} \right) (\underline{\underline{k}} \cdot \underline{\underline{e}}) (\underline{\underline{k}} \cdot \underline{\underline{M}}^{s'} \cdot \underline{\underline{e}}') (\underline{\underline{k}} \cdot \underline{\underline{M}}^{s''} \cdot \underline{\underline{e}}''), \end{aligned} \quad (4.46)$$

where τ_i , τ_e , $\underline{\underline{e}}'$, and $\underline{\underline{e}}''$ are given by

$$\tau_i = 1/\sigma^2, \quad \tau_e = -1, \quad \underline{\underline{e}}' = \underline{\underline{e}}_{\underline{\underline{k}}'}^U, \quad \underline{\underline{e}}'' = \underline{\underline{e}}_{\underline{\underline{k}}''}^U. \quad (4.47)$$

In I_3 , the integration, \int_S , over $\underline{\underline{k}}$, $\underline{\underline{k}}'$, and $\underline{\underline{k}}''$ is restricted by the synchronism conditions,

$$\underline{\underline{k}} + \underline{\underline{k}}' + \underline{\underline{k}}'' = 0, \quad \Omega(\underline{\underline{k}}) + \Omega'(\underline{\underline{k}}') + \Omega''(\underline{\underline{k}}'') = 0. \quad (4.48)$$

The summation in $\Lambda_{\underline{\underline{K}}\underline{\underline{K}}'\underline{\underline{K}}''}$ is assumed to cover all permutations of its following terms with respect to $\underline{\underline{K}}$, $\underline{\underline{K}}'$, and $\underline{\underline{K}}''$. By use of the frequency synchronism condition of (4.48), we find that the second term of $\Lambda_{\underline{\underline{I}}}^{\underline{\underline{S}}}(\underline{\underline{K}}, \underline{\underline{K}}', \underline{\underline{K}}'')$, when summed over all permutations, will make zero contribution to $\Lambda_{\underline{\underline{K}}\underline{\underline{K}}'\underline{\underline{K}}''}$.

It is noted that $\Lambda_{\underline{\underline{K}}\underline{\underline{K}}'\underline{\underline{K}}''}$ is composed of two types of nonlinear interaction energies. One is associated with the thermal motion of the charged particles, represented by $\Lambda_{\underline{\underline{T}}}^{\underline{\underline{S}}}(\underline{\underline{K}}, \underline{\underline{K}}', \underline{\underline{K}}'')$. The other, $\Lambda_{\underline{\underline{I}}}^{\underline{\underline{S}}}(\underline{\underline{K}}, \underline{\underline{K}}', \underline{\underline{K}}'')$ $\left[= \Omega'' \left(\underline{\underline{e}} \cdot \underline{\underline{M}}^{\underline{\underline{S}}} \cdot \underline{\underline{e}}'' \right) \left(\underline{\underline{K}} \cdot \underline{\underline{M}}^{\underline{\underline{S}}'} \cdot \underline{\underline{e}}' \right) / \Omega \right]$, is the field-particle interaction contribution associated with the current, charge density, and electric and magnetic fields. It consists of those terms in \mathcal{L}_3 of (4.31) that are multiplied by the unperturbed charge density, $q_s n_s$.

The quantity $(\Lambda_{\underline{\underline{K}}\underline{\underline{K}}'\underline{\underline{K}}''} + \text{c.c.})$, with c.c. representing the complex conjugate, can be interpreted as the rate of energy transfer to the $(-K)$ wave due to nonlinear interaction between the $\underline{\underline{K}}'$ and $\underline{\underline{K}}''$ waves of unit amplitude in the normalized electric field (Galloway, 1972). Because of this energy transfer, the energy of a single wave is no longer conserved, as in the linear case. Instead, we have (Galloway and Kim, 1971)

$$\frac{1}{\Omega} \left(\frac{\partial}{\partial T} \mathcal{H}_{\underline{\underline{K}}} + \frac{\partial}{\partial X} \cdot \mathcal{F}_{\underline{\underline{K}}} \right) = \int_S d\underline{\underline{K}}' d\underline{\underline{K}}'' \underline{\underline{e}} \underline{\underline{e}}' \underline{\underline{e}}'' \Lambda_{\underline{\underline{K}}\underline{\underline{K}}'\underline{\underline{K}}''}, \quad (4.49)$$

where S signifies that the synchronism conditions of (4.48) are satisfied. The wave energy and energy flux densities, $\mathcal{H}_{\underline{\underline{K}}}$ and $\mathcal{F}_{\underline{\underline{K}}}$, can be obtained by applying the Legendre transformation to \mathcal{L}_2 , but retaining the original dependent variables $\underline{\underline{\xi}}_s$, $\underline{\underline{\Phi}}$, and $\underline{\underline{Q}}$, before applying the averaging process. They are given by,

$$H_2 = \sum_U \int d\mathbf{K} \tilde{H}_{\mathbf{K}}^U, \quad F_2 = \sum_U \int d\mathbf{K} \tilde{F}_{\mathbf{K}}^U,$$

$$\tilde{H}_{\mathbf{K}}^U = |e_{\mathbf{K}}^U|^2 \Omega_{\mathbf{K}}^U, \quad \tilde{F}_{\mathbf{K}}^U = \underline{u}_{\mathbf{K}}^U \tilde{H}_{\mathbf{K}}^{(2)U},$$

$$\Omega_{\mathbf{K}}^U = \frac{1}{\sigma} \left(\frac{\Omega^2}{2} |\tilde{M}^i \cdot \tilde{e}|^2 + \frac{\gamma_i v_i^2}{2} |\tilde{K} \cdot \tilde{M}^i \cdot \tilde{e}|^2 \right) + \frac{\Omega^2}{2} |\tilde{M}^e \cdot \tilde{e}|^2 + \frac{\gamma_e v_e^2}{2} |\tilde{K} \cdot \tilde{M}^e \cdot \tilde{e}|^2 + \frac{(1+\eta)^2}{2} - \frac{|\mathbf{K} \cdot \tilde{e}|^2}{2\Omega^2},$$

$$\mathcal{J}_{\mathbf{K}-\mathbf{K}}^U = \frac{\gamma_i}{2\sigma} v_i^2 \left(\tilde{K} \cdot \tilde{M}^i \cdot \tilde{e} \right) \left(\tilde{M}^{i*} \cdot \tilde{e}^* \right) + \frac{\gamma_e}{2} v_e^2 \left(\tilde{K} \cdot \tilde{M}^e \cdot \tilde{e} \right) \left(\tilde{M}^{e*} \cdot \tilde{e}^* \right) + \frac{\mathbf{K} \cdot \mathbf{K} \cdot \tilde{e} \cdot \tilde{e}^*}{2\Omega^2}, \quad (4.50)$$

with * denoting the complex conjugate. When only three discrete waves are involved, the right-hand side of (4.49) reduces to a summation symmetric in \mathbf{K} , \mathbf{K}' , and \mathbf{K}'' . We have, by substitution of (4.50),

$$\begin{aligned} \mathcal{J}_{\mathbf{K}} \left(\frac{\partial}{\partial T} + \underline{u}_{\mathbf{K}} \cdot \frac{\partial}{\partial \mathbf{X}} \right) |e|^2 &= \mathcal{J}_{\mathbf{K}'} \left(\frac{\partial}{\partial T} + \underline{u}_{\mathbf{K}'} \cdot \frac{\partial}{\partial \mathbf{X}} \right) |e'|^2 \\ &= \mathcal{J}_{\mathbf{K}''} \left(\frac{\partial}{\partial T} + \underline{u}_{\mathbf{K}''} \cdot \frac{\partial}{\partial \mathbf{X}} \right) |e''|^2 = \mathcal{J}_{\mathbf{K}\mathbf{K}\mathbf{K}'} |e''|^2. \end{aligned} \quad (4.51)$$

The first two equations are the Manley-Rowe relations expressing the conservation of wave action, $\mathcal{J}_{\mathbf{K}} |e|^2$, in three-wave interaction (Louisell, 1960).

The coupled mode equation can be obtained from (4.51) by cancelling e^* , e'^* , and e''^* in (4.51),

$$\begin{aligned}
j_p^U \left(\frac{\partial}{\partial T} + \underline{u}_p^U \cdot \frac{\partial}{\partial \underline{X}} \right) e_p^U &= -c_{psi}^* e_s^V e_i^W , \\
j_s^V \left(\frac{\partial}{\partial T} + \underline{u}_s^V \cdot \frac{\partial}{\partial \underline{X}} \right) e_s^V &= c_{psi} e_i^{W*} e_p^U , \\
j_i^W \left(\frac{\partial}{\partial T} + \underline{u}_i^W \cdot \frac{\partial}{\partial \underline{X}} \right) e_i^W &= c_{psi} e_s^{V*} e_p^U , \tag{4.52}
\end{aligned}$$

where U , V , and W denote the waves, and the following substitutions have been assumed,

$$c_{psi} = \underbrace{C_{KK'K''}^{UVW}} = \underbrace{\Lambda_{-K, K' K''}^{U' VW}} , \quad \underline{e}_{\underline{K}}^U = \underline{e}_{\underline{K}}^{U'} , \quad \Omega_p = -\Omega ,$$

$$\underline{K}_p = -\underline{K} , \quad \underline{K}_s = \underline{K}' , \quad \underline{K}_i = \underline{K}'' , \quad j_p^U = -j_{-\underline{K}}^{U'} , \quad \underline{u}_p^U = \underline{u}_{-\underline{K}}^{U'} , \tag{4.53}$$

with the last two relations following from (4.50). The subscripts p , s , and i denote the pump, signal, and idler waves, respectively. This nomenclature is commonly used in describing parametric amplifiers, where a large amplitude pump wave, e_p^U , is applied to amplify the signal and idler waves (Louisell, 1960).

4.4 Parametric Wave Amplification

In Section 4.3 we have derived the general coupled mode equations for nonlinear three-wave interaction under three important assumptions: first, that the waves are undamped; second, that the background plasma is homogeneous, and third, that there are no additional nonlinear wave processes, such as wave-particle interaction. We shall now consider spatial solutions to the coupled mode equations, and assess the effects of relaxing some of

these assumptions. In particular, some results with damping taken into account will be considered. They will be of use in Section 5 for comparing theoretical predictions with experiments on parametric amplification.

4.4.1 Parametric Amplification Without Wave Damping

The solutions of the coupled mode equations in (4.52) are well-known, and will be briefly discussed in this subsection. We proceed with the assumption of a large pump wave amplitude.

$|e_p| \gg |e_s|, |e_i|$: In this case, e_p can be considered as a constant because the right-hand side of the first expression of (4.52) is now negligible. The remaining two equations become linear in e_s and e_i . Their solutions are then subject to the instability criteria well-known in the linear wave theory concerning absolute (temporal) and convective (spatial) growth (Sturrock, 1958b; Briggs, 1964; Derfler, 1967). This type of analysis has been extended by Harker and Crawford (1969a) and Van Hoven (1971). In particular, they have shown that, (a) if $j_s j_i > 0$ and $u_s u_i > 0$, e_s and e_i are convectively unstable with both temporal and spatial growth possible, (b) if $j_s j_i > 0$ and $u_s u_i < 0$, they become absolutely unstable, and evanescent in space but amplified in time, (c) if $j_s j_i < 0$ and $u_s u_i > 0$, they are evanescent in time but amplified in space, and (d) if $j_s j_i < 0$ and $u_s u_i < 0$, they vary like beat waves and no persistent amplification in space or time is possible. Cases (a) and (c), in which the signal and idler wave group velocities are in the same direction ($u_s u_i > 0$), are usually termed co-flow or forward scatter, while Cases (b) and (d), in which the signal

and idler wave group velocities have opposite directions ($u_s u_i < 0$), are usually termed contra-flow or back scatter. The distinction between the two types of parametric amplification is illustrated in figures 4.2(a) and 4.2(b).

The temporal (assuming space-independence) and spatial (assuming time-independence) parametric amplification rates, Γ_0 and κ_0 , respectively, are given by (Louisell, 1960),

$$\Gamma_0 = \kappa_0 (u_s u_i)^{1/2} = \frac{|c_{\text{psi}}| |\mathcal{E}_p|}{(g_s g_i)^{1/2}} \quad (4.54)$$

These expressions can be obtained from the last two relations of (4.52) by assuming that \mathcal{E}_s and \mathcal{E}_i vary as $\exp(\Gamma_0 T)$ or $\exp(\kappa_0 Z)$ in a one-dimensional approximation in the z-direction. Since imaginary Γ_0 or κ_0 amounts to a small nonlinear frequency or wavenumber shift, respectively, for the signal and idler, the results of (4.54) are seen to be consistent with those by Harker (1969) and Van Hoven (1971).

If we assume that the initial (or boundary) value of \mathcal{E}_i is zero, (4.54) breaks down because \mathcal{E}_i can no longer be an exponential function in T (or Z). We then find from the right-hand side of the second expression of (4.52) that the initial slope of \mathcal{E}_s in T (or Z) is zero. Also, from the last expression of (4.52), the starting behavior of \mathcal{E}_i is seen to be $\Gamma_{0i} Z$ (or $\kappa_{0i} T$), with

$$\Gamma_{0i} = u_i \kappa_{0i} = \frac{c_{\text{psi}} \mathcal{E}_s^*(0) \mathcal{E}_p(0)}{g_i} \quad (4.55)$$

When $|\mathcal{E}_i|$ becomes comparable to $|\mathcal{E}_s|$ due to this linear growth, the exponential growth of (4.54) is then applicable. The spatial behavior

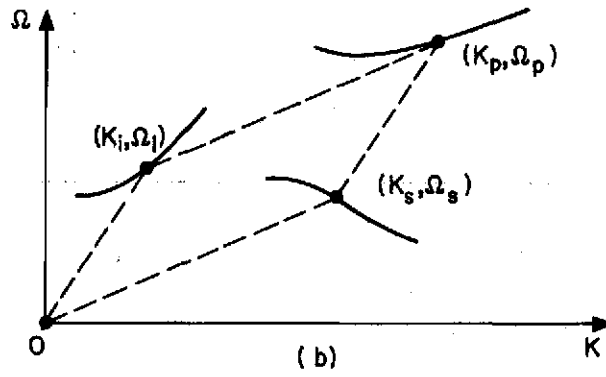
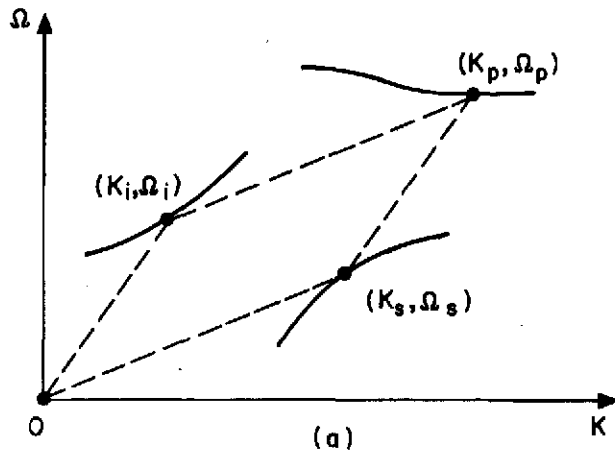


Figure 4.2 The distinction between (a) the co-flow (forward scatter) and (b) the contra-flow (back scatter) cases of parametric amplification. The solid lines indicate the dispersion curves near the (K, Ω) of the pump, signal, and idler waves. In the co-flow case, the group velocities ($U = d\Omega/dK$) of the signal and idler have the same sign, while in the contra-flow case, their group velocities have opposite signs.

of \mathcal{E}_s and \mathcal{E}_i in each of the above mentioned four cases were presented in a summary paper by Barnes (1964).

$|\mathcal{E}_p| \sim |\mathcal{E}_s| \sim |\mathcal{E}_i|$: In the case of parametric growth in \mathcal{E}_s and \mathcal{E}_i , the results of (4.54) and (4.55) can be considered accurate only near the boundary where, or soon after the initial time when, the pump wave is being injected. These results break down as soon as $|\mathcal{E}_s|$ and $|\mathcal{E}_i|$ become comparable to $|\mathcal{E}_p|$. In this case, the temporal solutions to (4.52) must be put in terms of elliptic integrals involving the wave actions $\mathcal{J}_p |\mathcal{E}_p|^2$, $\mathcal{J}_s |\mathcal{E}_s|^2$, and $\mathcal{J}_i |\mathcal{E}_i|^2$. The solutions of these wave actions, when all the \mathcal{J} 's have the same sign, are illustrated in figure 4.3 (see for example, Sagdeev and Galeev, 1969). Several features of this figure should be noted: first, that $|\mathcal{E}_p|$ can no longer be considered as a constant; second, that the interaction process is reversible and has a nonlinear period T_n , which can be shown to be roughly proportional to the inverse of the maximum value of $|\mathcal{E}_p|$, and third, that the Manley-Rowe type of wave action conservation laws, derivable from (4.51),

$$\mathcal{J}_p |\mathcal{E}_p|^2 - \mathcal{J}_s |\mathcal{E}_s|^2 = \text{constant} \quad , \quad \mathcal{J}_s |\mathcal{E}_s|^2 + \mathcal{J}_i |\mathcal{E}_i|^2 = \text{constant} \quad , \quad (4.56)$$

are satisfied. Finally, it should be noted that if all the \mathcal{U} 's have the same sign, the spatial solutions in this case are similar to the temporal solutions shown in figure 4.3. To complete the analogy, it is only necessary to replace T and the $\mathcal{J}|\mathcal{E}|^2$'s in figure 4.3 by Z and the corresponding $\mathcal{J}\mathcal{U}|\mathcal{E}|^2$'s, respectively. Also, (4.56) becomes,

$$\mathcal{J}_p \mathcal{U}_p |\mathcal{E}_p|^2 - \mathcal{J}_s \mathcal{U}_s |\mathcal{E}_s|^2 = \text{constant} \quad , \quad \mathcal{J}_s \mathcal{U}_s |\mathcal{E}_s|^2 + \mathcal{J}_i \mathcal{U}_i |\mathcal{E}_i|^2 = \text{constant} \quad , \quad (4.57)$$

which is the Manley-Rowe type of power-balance formula (Penfield, 1960).

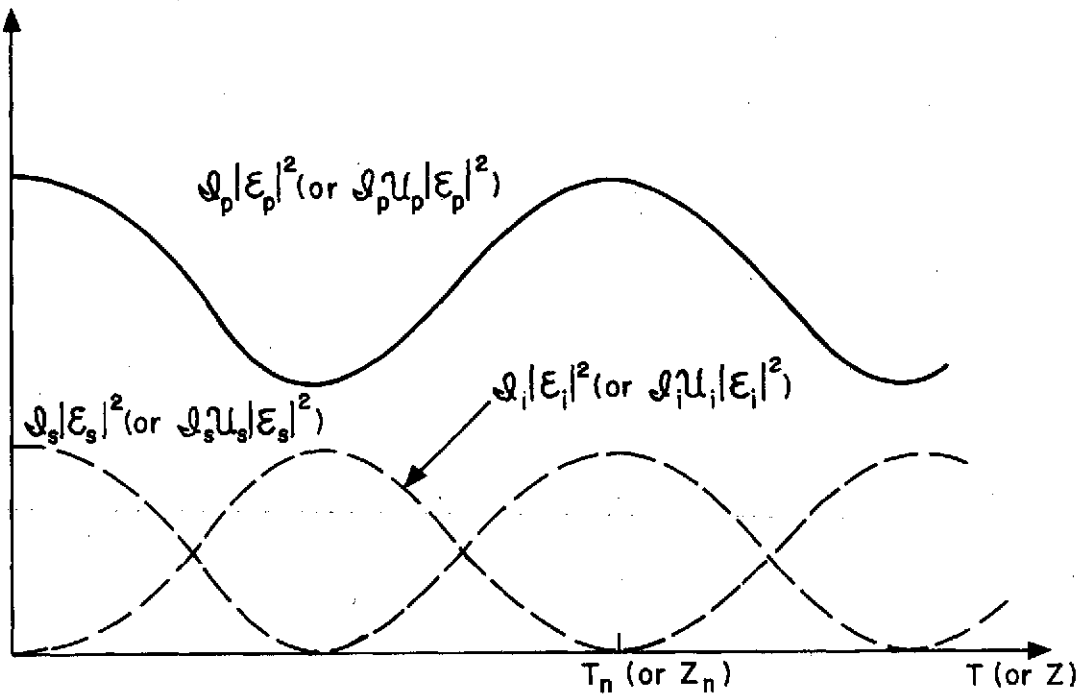


Figure 4.3 The solutions to (4.52) when $|\mathcal{e}_p|$, $|\mathcal{e}_s|$, and $|\mathcal{e}_i|$ are comparable in magnitude are in terms of the wave action, $\mathcal{J}|\mathcal{e}|^2$, for temporal behavior, and wave power transfer, $\mathcal{J}\mathcal{U}|\mathcal{e}|^2$, for spatial behavior. T_n (or Z_n) is the interaction periodicity time (or spatial length) of wave interaction.

A situation in which the wave amplitudes increase to infinity in a finite time period occurs when $g_p < 0$ and $g_s, g_i > 0$. We then have from (4.52),

$$|g_p| \frac{d\mathcal{E}_p}{dT} = C \mathcal{E}_s \mathcal{E}_i, \quad g_s \frac{d\mathcal{E}_s}{dT} = C \mathcal{E}_i \mathcal{E}_p, \quad g_i \frac{d\mathcal{E}_i}{dT} = C \mathcal{E}_s \mathcal{E}_p, \quad (4.58)$$

where $C = |c_{\text{psi}}|$. Now \mathcal{E}_p , \mathcal{E}_s , and \mathcal{E}_i can be made real in value by choosing $\pi/2$ as the phase angle difference between the pump and the signal and idler waves in (4.52) (Davidson, 1972). From (4.58), we see that, whenever \mathcal{E}_p , \mathcal{E}_s , and \mathcal{E}_i simultaneously have the same sign, the wave amplitudes grow without bound. As illustrated in figure 4.4, we have for the special case $\mathcal{E}_p = \mathcal{E}_s = \mathcal{E}_i$ (Davidson, 1972),

$$\mathcal{E}_p(T) = \mathcal{E}_p(0) \left/ \left[1 - \frac{C \mathcal{E}_p(0)}{|g_p|} T \right] \right., \quad (4.59)$$

which becomes infinite in a time period $T_{\text{exp}} = |g_p|/C \mathcal{E}_p(0)$. This is the simplest example of the so-called explosive instability (see for example, Coppi, Rosenbluth, and Sudan, 1969).

This infinitely large amplitude cannot occur in practice, however, because of the limiting effects of linear wave damping, and other nonlinear wave processes, which are omitted in the foregoing model of a wave-triplet. On the other hand, the spatial solutions to (4.52), when $g_p u_p < 0$, and $g_s u_s, g_i u_i > 0$, have a similar form to the temporal solutions shown in figure 4.4, with the explosive instability length being $Z_{\text{exp}} = |g_p u_p|/C \mathcal{E}_p(0)$. In the next subsection, it will be shown that the inclusion of weak wave damping will drastically change the behavior of \mathcal{E}_p , \mathcal{E}_s , and \mathcal{E}_i from that briefly indicated above.

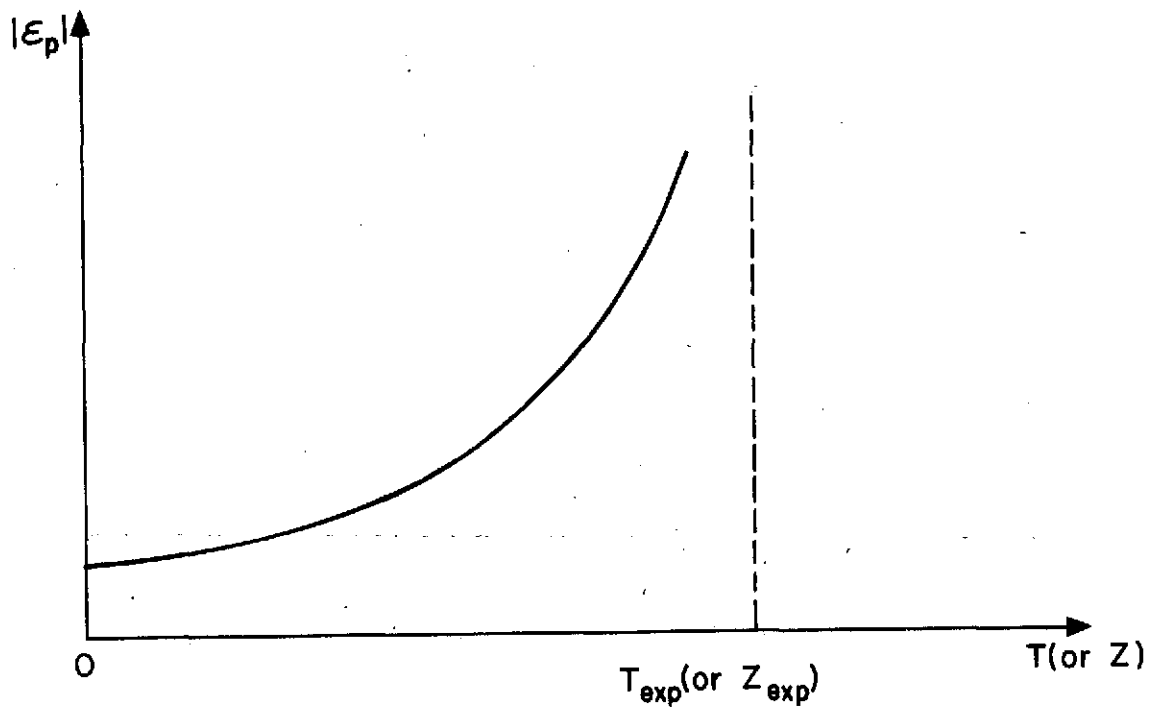


Figure 4.4 The behavior of $|E_p|$ in the explosive instability of three-wave interaction, occurring when g_p is different in sign from g_s and g_i for their temporal solutions, or $g_p u_p$ is different in sign from $g_s u_s$ and $g_i u_i$ for their spatial solutions.

4.4.2 Wave Damping

The effects of weak linear wave damping can be included phenomenologically in the coupled mode equations (4.52) (Sjölund and Stenflo, 1967),

$$\begin{aligned} \mathcal{J}_p \left(\frac{\partial}{\partial T} + \underline{u}_p \cdot \frac{\partial}{\partial \underline{X}} + \Gamma_p \right) \mathcal{E}_p &= -c_{psi}^* \mathcal{E}_s \mathcal{E}_i \quad , \\ \mathcal{J}_s \left(\frac{\partial}{\partial T} + \underline{u}_s \cdot \frac{\partial}{\partial \underline{X}} + \Gamma_s \right) \mathcal{E}_s &= c_{psi} \mathcal{E}_i^* \mathcal{E}_p \quad , \\ \mathcal{J}_i \left(\frac{\partial}{\partial T} + \underline{u}_i \cdot \frac{\partial}{\partial \underline{X}} + \Gamma_i \right) \mathcal{E}_i &= c_{psi} \mathcal{E}_s^* \mathcal{E}_p \quad , \end{aligned} \quad (4.60)$$

where the $\Gamma_{p,s,i}$ represent the normalized damping rates of the pump, signal, and idler. Due to the presence of $\Gamma_{p,s,i}$, $|\mathcal{E}_p|$ must be larger than some threshold before \mathcal{E}_s and \mathcal{E}_i can be amplified. This threshold field, \mathcal{E}_{th} , for amplification can be obtained by setting $\partial/\partial T = 0 = \partial/\partial \underline{X}$ in the last two expressions of (4.60). We then obtain

$$\mathcal{E}_{th} = \frac{(\mathcal{J}_s \mathcal{J}_i \Gamma_s \Gamma_i)^{\frac{1}{2}}}{|c_{psi}|} \quad , \quad (4.61)$$

after cancelling \mathcal{E}_s and \mathcal{E}_i . When the pump field is not far above the threshold, we may obtain the initial parametric amplification rate for waves propagating parallel to Z , in a fashion similar to that used in obtaining (4.54) (Louisell, 1960), as

$$\kappa^2 = \frac{\kappa_0^2 - \kappa_{th}^2}{\mu_s + \mu_i} \quad , \quad (4.62)$$

where κ_0 is given by (4.54); μ_s , μ_i , and κ_{th} are defined by

$$\mu_s = \frac{\Gamma_s}{U_s}, \quad \mu_i = \frac{\Gamma_i}{U_i}, \quad \kappa_{th} = \kappa_0 \left| \frac{e_p}{e_{th}} \right|, \quad (4.63)$$

and we have assumed $(\mu_s + \mu_i) \gg 4(\kappa_0^2 - \kappa_{th}^2)$.

For the case of a large amplitude undamped pump wave, the spatial solutions for linearly damped signal and idler waves have already been presented in the context of nonlinear optics (Bloembergen, 1965; Bobroff, 1965). The stability analysis, in line with the approach by Harker and Crawford (1969a) and Van Hoven (1971), has been extended to this case by Bers, Chambers, and Hawryluk (1973). Here, we wish to extend the analysis to account for a damped pump wave. A similar problem has been studied by Ohnuma and Hatta (1970), and by Porkolab and Chang (1970).

By making the phase of e_p different from those of e_s and e_i by $\pi/2$ in (4.60), we have for the one-dimensional case (Davidson, 1972),

$$U_s g_s \left(\frac{d}{dz} + \mu_s \right) e_s = C e_i e_p, \quad U_i g_i \left(\frac{d}{dz} + \mu_i \right) e_i = C e_s e_p, \quad (4.64)$$

where $C = |C_{psi}|$; e_p , e_s , and e_i are real in value, and we assume an exponentially decaying, large amplitude pump wave

$$e_p = e_p(0) \exp(-\mu_p z) \quad (|e_p| \gg |e_s|, |e_i|) \quad (4.65)$$

Case 1. $\mu_p \neq \mu_s = \mu_i = \mu$: The simplified case where $\mu_p = \mu_s = \mu_i = \mu$ has been treated by Ohnuma and Hatta (1970), and a further simplification to second harmonic generation by a large amplitude ion-acoustic wave has been studied by them and by Litzenberger, *et al.* (1972). For purposes of comparison between theory and experiment in Section 5.2.3, we shall

consider solutions for parametric amplification when $\mu_p \neq \mu_s = \mu_p = \mu$.

We then have as solutions of (4.64), for the case of co-flow (figure 4.2a)

$$\mathcal{E}_s = \left[\mathcal{E}_s(0) \cosh M + \mathcal{E}_i(0) \left(\frac{u_i j_i}{u_s j_s} \right)^{1/2} \sinh M \right] \exp(-\mu Z) \quad ,$$

$$\mathcal{E}_i = \left[\mathcal{E}_i(0) \cosh M + \mathcal{E}_s(0) \left(\frac{u_s j_s}{u_i j_i} \right)^{1/2} \sinh M \right] \exp(-\mu Z) \quad ,$$

$$M(Z) = M_0 \left[1 - \exp(-\mu Z) \right] \quad , \quad M_0 = \frac{c \mathcal{E}_p(0)}{\mu_p (u_s j_s u_i j_i)^{1/2}} = \frac{\mathcal{E}_p(0)}{\mathcal{E}_{th}^{\mu_p}} (u_s u_i)^{1/2} \quad , \quad (4.66)$$

where $\mathcal{E}_s(0)$ and $\mathcal{E}_i(0)$ are the values of \mathcal{E}_s and \mathcal{E}_i at $Z = 0$, and the second expression for M_0 is obtained by using (4.61).

By evaluating $d\mathcal{E}_s/dZ$ and $d\mathcal{E}_i/dZ$ at $Z = 0$, we find that \mathcal{E}_s and \mathcal{E}_i grow in the vicinity of $Z = 0$ according to (4.61) and (4.62). For Z such that $\mu_p Z \gg 1$, \mathcal{E}_s and \mathcal{E}_i both decay by linear damping. This behavior is illustrated in figure 4.5 and is drastically different from that shown in figure 4.3 where damping was not included. In the simpler theoretical model used by Ohnuma and Hatta (1970), in which $\mu_p = \mu$, the following assumptions were made,

$$M(Z) \ll 1 \quad , \quad \mathcal{E}_s(0) > \frac{\mathcal{E}_i(0)}{M_0} \left(\frac{j_i}{j_s} \right)^{1/2} \quad , \quad (4.67)$$

where the second condition is obtained by making $d\mathcal{E}_i/dZ > 0$ at $Z = 0$ according to the second expression of (4.66). It then follows that \mathcal{E}_i has its maximum value at $Z_0 = \ln 2 / \mu_p \approx 0.7 / \mu_p$, a result obtained without assuming $\mu_p = \mu$.

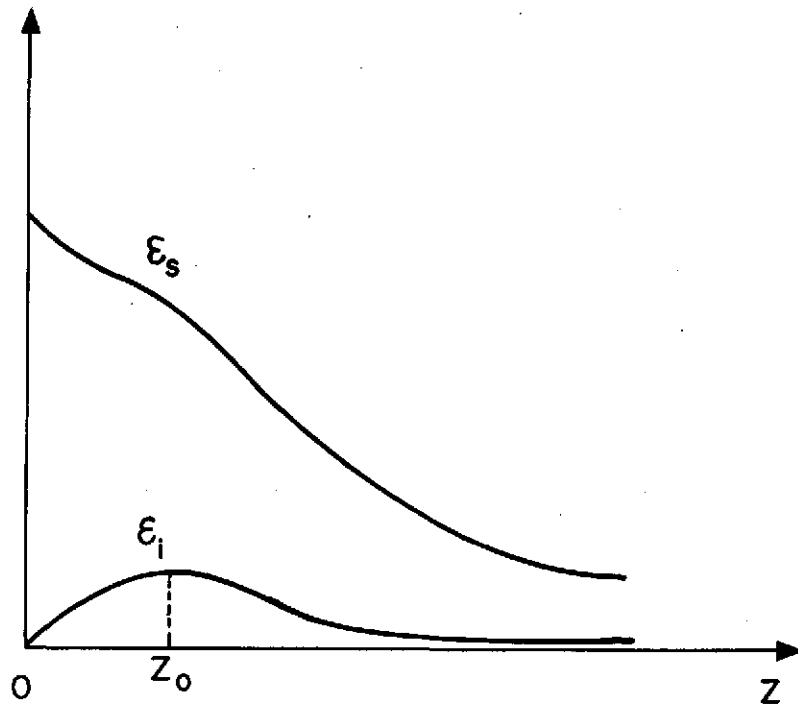


Figure 4.5 The schematic behavior of ϵ_s and ϵ_i according to (4.66), where wave damping of ϵ_p , ϵ_s , and ϵ_i has been included. When $\mu_p = \mu$, and under the condition (4.67), the peak location z_0 of ϵ_i becomes independent of $\epsilon_p(0)$.

Case 2. $\underline{\mu_p \neq \mu_s \neq \mu_i}$: The solutions for \mathcal{E}_s and \mathcal{E}_i must now be expressed in terms of Bessel functions (Porkolab and Chang, 1970). We shall consider only the case where $\mu_p > 0$ ($u_p > 0$), $\mu_s < 0$ ($u_s < 0$), and $\mu_i > 0$ ($u_i > 0$), i.e. the contra-flow or back scatter case (Barnes, 1964; Bloembergen, 1965), as illustrated in figure 4.2b, relevant to examples in Sections 5.3.2, 5.4.3, and 5.5.4. Under these conditions the solutions of (4.64) for \mathcal{E}_s and \mathcal{E}_i can be written as

$$\begin{aligned} \mathcal{E}_i &= \left[c_{i1} J_{\nu-1}(\chi) + c_{i2} N_{\nu-1}(\chi) \right] \exp\left[-\left(\mu_p + \mu_s + \mu_i\right) \frac{Z}{2}\right] , \\ \mathcal{E}_s &= \left[c_{s1} J_{\nu}(\chi) + c_{s2} N_{\nu}(\chi) \right] \exp\left[-\left(\mu_p + \mu_s + \mu_i\right) \frac{Z}{2}\right] , \end{aligned} \quad (4.68)$$

where the quantities χ , ν , $|M_0|$, $c_{i1,2}$, and $c_{s1,2}$ are defined by

$$\begin{aligned} \chi &= |M_0| \exp\left(-\mu_p Z\right) , \quad \nu = \frac{1}{2\mu_p} \left(\mu_p - \mu_s + \mu_i\right) , \\ |M_0| &= \frac{c_p \mathcal{E}_p(0)}{\mu_p \left(|u_s| g_s u_i g_i\right)^{1/2}} = \frac{\mathcal{E}_p(0)}{\mathcal{E}_{th}^{\mu_p}} \left(|\mu_s| |\mu_i|\right)^{1/2} , \\ c_{i1} &= \frac{\pi |M_0|}{2} \left[\left(\frac{|u_s| g_s}{u_i g_i}\right)^{1/2} N_{\nu-1}(|M_0|) \mathcal{E}_s(0) - N_{\nu}(|M_0|) \mathcal{E}_i(0) \right] , \\ c_{i2} &= \frac{\pi |M_0|}{2} \left[J_{\nu}(|M_0|) \mathcal{E}_i(0) - \left(\frac{|u_s| g_s}{u_i g_i}\right)^{1/2} J_{\nu-1}(|M_0|) \mathcal{E}_s(0) \right] , \\ c_{s1} &= \frac{\pi |M_0|}{2} \left[N_{\nu-1}(|M_0|) \mathcal{E}_s(0) - \left(\frac{|u_s| g_s}{u_i g_i}\right)^{1/2} N_{\nu}(|M_0|) \mathcal{E}_i(0) \right] , \\ c_{s2} &= \frac{\pi |M_0|}{2} \left[\left(\frac{|u_s| g_s}{u_i g_i}\right)^{1/2} J_{\nu}(|M_0|) \mathcal{E}_i(0) - J_{\nu-1}(|M_0|) \mathcal{E}_s(0) \right] , \end{aligned} \quad (4.69)$$

where $J_\nu(\chi)$ and $N_\nu(\chi)$ are Bessel functions of the first and second kinds (Abramovitz and Stegun, 1965).

Simplifications of (4.68) are possible under various conditions of interest for Section 5:

$\chi \ll 1$: In this case, \mathcal{E}_i and \mathcal{E}_s reduce to

$$\begin{aligned} \mathcal{E}_i &\approx \left\{ \mathcal{E}_i(0) \left[\frac{1}{\chi_{1i}} - \frac{|M_0|^2 \chi_{1i}}{4\nu(\nu-1)} \right] + \mathcal{E}_s(0) \frac{|M_0|}{(\nu-1)} \left(\frac{|u_s| g_s}{u_i \beta_i} \right)^{1/2} \sinh[(\nu-1)\mu_p Z] \right\} \\ &\quad \times \exp \left[- \left(\mu_p + \mu_s + \mu_i \right) \frac{Z}{2} \right] , \\ \mathcal{E}_s &\approx \left\{ \mathcal{E}_s(0) \left[\chi_{1s} - \frac{|M_0|^2}{4\nu(\nu-1)\chi_{1s}} \right] + \mathcal{E}_i(0) \frac{|M_0|}{\nu} \left(\frac{|u_s| g_s}{u_i \beta_i} \right)^{1/2} \sinh(\nu\mu_p Z) \right\} \\ &\quad \times \exp \left[- \left(\mu_p + \mu_s + \mu_i \right) \frac{Z}{2} \right] , \end{aligned} \quad (4.70)$$

where χ_{1i} and χ_{1s} are defined by

$$\chi_{1i} = \exp[(\nu-1)\mu_p Z] \quad , \quad \chi_{1s} = \exp(\nu\mu_p Z) \quad . \quad (4.71)$$

$\nu \gg \chi$: We now have for \mathcal{E}_i and \mathcal{E}_s the expressions

$$\begin{aligned} \mathcal{E}_i &\approx \left[\frac{\mathcal{E}_i(0)}{e_0} \left(\frac{\nu}{\nu-1} \right)^{\nu-1/2} - \mathcal{E}_s(0) \chi_{2i} \right] \exp(-\mu_i Z) \\ &\quad + \left[\mathcal{E}_s(0) \chi_{2i} - \mathcal{E}_i(0) \frac{e_0 |M_0|^2}{4\nu^2} \left(\frac{\nu-1}{\nu} \right)^{\nu-3/2} \right] \exp \left[- \left(\mu_p + \mu_s \right) Z \right] , \\ \mathcal{E}_s &\approx \left[\frac{\mathcal{E}_s(0)}{e_0} \left(\frac{\nu}{\nu-1} \right)^{\nu-1/2} - \mathcal{E}_i(0) \chi_{2s} \right] \exp(-\mu_s Z) \\ &\quad + \left[\frac{\mathcal{E}_i(0)}{\nu} \chi_{2s} - \mathcal{E}_s(0) \frac{e_0 |M_0|^2}{4\nu} \left(\frac{\nu-1}{\nu} \right)^{\nu-1/2} \right] \exp \left[- \left(\mu_p + \mu_i \right) Z \right] , \end{aligned} \quad (4.72)$$

where χ_{2i} and χ_{2s} are defined by

$$\chi_{2i} = \frac{|M_0|}{2(\nu-1)} \left(\frac{|u_s| j_s}{u_i j_i} \right)^{1/2}, \quad \chi_{2s} = \frac{|M_0|}{2\nu} \left(\frac{|u_s| j_s}{u_i j_i} \right)^{1/2}, \quad (4.73)$$

and e_0 has been written for $\exp(1) = 2.718\dots$.

$\chi > \nu \gg 1$: In this approximation, \mathcal{E}_i and \mathcal{E}_s reduce to

$$\begin{aligned} \mathcal{E}_i &\approx \left[\mathcal{E}_i(0) \cos M + \mathcal{E}_s(0) \left(\frac{|u_s| j_s}{u_i j_i} \right)^{1/2} \sin M \right] \exp \left[-(\mu_s + \mu_i) \frac{Z}{2} \right], \\ \mathcal{E}_s &\approx \left[\mathcal{E}_s(0) \cos M + \mathcal{E}_i(0) \left(\frac{|u_s| j_s}{u_i j_i} \right)^{1/2} \sin M \right] \exp \left[-(\mu_s + \mu_i) \frac{Z}{2} \right]. \end{aligned} \quad (4.74)$$

We note that,

$$M(Z) = |M_0| \left[1 - \exp(-\mu_p Z) \right] \approx |M_0| \mu_p Z, \quad (4.75)$$

where the second expression for $M(Z)$ is appropriate when $\mu_p Z \ll 1$. In (4.70), (4.72), and (4.75), the quantity $|M_0|$ has been defined in (4.69). It will be seen in Section 5 that the behavior of \mathcal{E}_s and \mathcal{E}_i according to (4.70)-(4.75) is drastically different from that indicated in Section 4.4.1 where linear damping is neglected.

Case 3. Noncollinear Propagation: So far we have assumed that the group velocities are collinear. In preparation for Section 5.5.4, we now consider the case when \underline{u}_p , \underline{u}_s , and \underline{u}_i are no longer collinear. The coupled mode equations (4.64) become

$$\begin{aligned}
g_s \left(u_{sx} \frac{\partial}{\partial X} + u_{sz} \frac{\partial}{\partial Z} + \Gamma_s \right) e_s &= c e_i e_p , \\
g_i \left(u_{ix} \frac{\partial}{\partial X} + u_{iz} \frac{\partial}{\partial Z} + \Gamma_i \right) e_i &= c e_s e_p , \quad (4.76)
\end{aligned}$$

where u_{sx} , u_{sz} , u_{ix} , and u_{iz} are defined by

$$u_{sx} = u_s \sin \psi_s , \quad u_{sz} = u_s \cos \psi_s , \quad u_{ix} = u_i \sin \psi_i , \quad u_{iz} = u_i \cos \psi_i , \quad (4.77)$$

with ψ_s and ψ_i being the angles of \underline{u}_s and \underline{u}_i with respect to the positive Z direction. Substituting

$$\begin{aligned}
e_s(X, Z) &= \bar{e}_s(Z) \exp(-\mu_{sx} X) , \quad e_i(X, Z) = \bar{e}_i(Z) \exp(-\mu_{ix} X) , \\
\mu_{sx} &= \frac{\Gamma_s}{u_s} \sin \psi_s , \quad \mu_{sz} = \frac{\Gamma_s}{u_s} \cos \psi_s , \quad \mu_{ix} = \frac{\Gamma_i}{u_i} \sin \psi_i , \quad \mu_{iz} = \frac{\Gamma_i}{u_i} \cos \psi_i , \quad (4.78)
\end{aligned}$$

(4.76) can be reduced to

$$\begin{aligned}
g_s u_{sz} \left(\frac{d}{dZ} + \mu_{sz} \right) \bar{e}_s(Z) &= c \bar{e}_i(Z) e_p(Z) \exp\left[(\mu_{sx} - \mu_{ix}) X \right] , \\
g_i u_{iz} \left(\frac{d}{dZ} + \mu_{iz} \right) \bar{e}_i(Z) &= c \bar{e}_s(Z) e_p(Z) \exp\left[(\mu_{ix} - \mu_{sx}) X \right] . \quad (4.79)
\end{aligned}$$

It will be seen that, along a path with X constant, (4.79) is equivalent to (4.64), but with different wave coupling coefficients. Along an appropriately chosen Z-axis, where $X = 0$, e.g. the central plane of a plasma slab, the form of (4.79) becomes identical to that of (4.64). The solutions, (4.68)-(4.75), become the solutions of (4.79) when u_s , u_i , μ_s , and μ_i are replaced by u_{sz} , u_{iz} , μ_{sz} , μ_{iz} , respectively.

4.4.3 Effects of Other Nonlinear Processes and Plasma Inhomogeneity

In addition to linear wave damping, other nonlinear wave processes and plasma background inhomogeneity may also significantly alter the behavior of the interacting wave-triplet. Here we shall only briefly discuss the conditions under which the solutions of \mathcal{E}_s and \mathcal{E}_i given in Section 4.4.2 are still acceptable in practice.

Other Nonlinear Processes: The competing nonlinear wave processes likely to occur in the presence of a large amplitude pump wave are primarily in the form of modifications of the background plasma. While causing them, the pump wave may have a significantly different behavior from the exponential decay assumed in (4.65), thus rendering the solutions for \mathcal{E}_s and \mathcal{E}_i given in Section 4.4.2 inaccurate.

(i) Particles trapped by an electrostatic pump wave: In a collisionless plasma, and assuming $K^2 V_e^2 \ll 1$, this process is important under the condition (see for example, Davidson, 1972),

$$\Omega_p \gg \Omega_B \gg \Gamma_L, \quad \Omega_B = \left(\frac{ek|E_p|}{\omega_p^2 m_e} \right)^{1/2} = \left(K|\mathcal{E}_p| \right)^{1/2}, \quad (4.80)$$

where Ω_B is the electron bounce frequency when trapped in a trough of the pump wave, and Γ_L is the normalized linear Landau damping rate,

$$\Gamma_L = \left(\frac{\pi}{8} \right)^{1/2} \frac{1}{(KV_e)^3} \exp\left(-\frac{1}{2KV_e^2} - \frac{3}{2} \right). \quad (4.81)$$

It is well known that, under the condition (4.80), the electrostatic wave decays like $\exp(-\Gamma_L T)$ only for $T \leq 1/\Omega_B$, but executes modulational oscillation when $T \geq 1/\Omega_B$. This behavior is illustrated in figure 4.6.

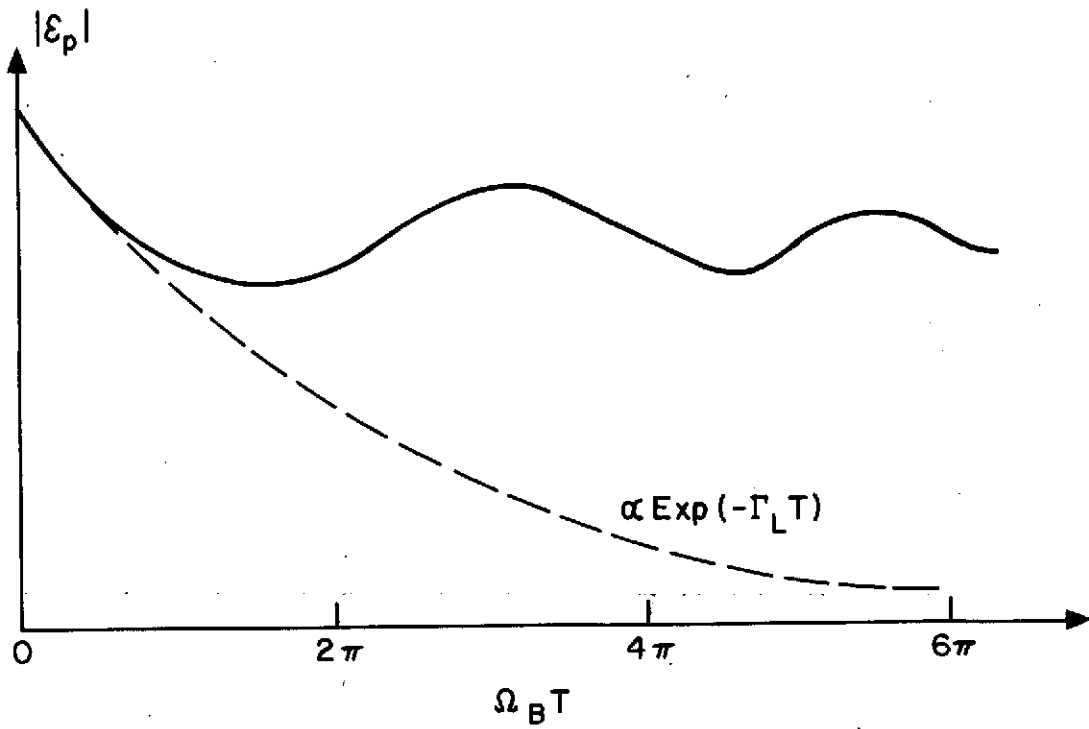


Figure 4.6 The behavior of an electrostatic pump wave amplitude under the influence of trapped electrons (—) in a collisionless plasma under condition (4.80). In the absence of electron-trapping, the wave decays according to linear Landau damping as $\exp(-\Gamma_L T)$ (----).

The results of Section 4.4.2 are therefore appropriate only for $T \ll 1/\Omega_B$, or for $Z \ll 1/\Omega_B u_p$ when considering spatial evolution.

The first condition in (4.80) is necessary to restrict the size of $|\mathcal{E}_p|$ so that the electron density perturbation is much smaller than the quiescent density (Davidson, 1972), and is consistent with the assumption of weak perturbation used in Section 4.2 in approximating the Lagrangian. The second condition in (4.80) is required to allow the trapped electrons to make many bounces in a wave trough before they become untrapped due to wave damping. Therefore, when the collisional damping Γ_c is much larger than Γ_L , Γ_c should replace Γ_L in (4.80). Following the same reasoning, Γ_c should be replaced by the total measured pump wave damping rate Γ_p , which may be a sum of Γ_c and the nonlinear damping rate Γ_{NL} that accounts for all the nonlinear effects on the pump wave. We then have from (4.80),

$$\Omega_B = \left(k |\mathcal{E}_p| \right)^{1/2} \ll \Gamma_p, \quad (4.82)$$

as the condition under which the trapped particle effects on an electrostatic pump wave are negligible.

(ii) Plasma heating: Substantial background modification in the form of plasma heating by an incident wave can occur in a collisional plasma due to collisional randomization of the ordered perturbation in electron motion. It has been shown that the electron temperature, T_e , increases from its quiescent value, T_0 , roughly according to (Ginzburg, 1970),

$$\frac{T_e}{T_0} = 1 + \left(\frac{|\mathcal{E}_p|}{\mathcal{E}_T} \right)^2, \quad \mathcal{E}_T = \left(\frac{6}{\sigma} \right)^{1/2} v_e \Omega, \quad (4.83)$$

when $\Omega^2 \gg v_e^2 \omega_{pe}^2$, with v_e being the electron-ion and electron-neutral energy transfer collisional frequency. When $|\mathcal{E}_p|^2 \gg \mathcal{E}_T^2$, it is seen that the corresponding temperature increase can be substantial, thus changing the wave coupling coefficients and linear wave properties. Thus, in the presence of a large amplitude pump wave, the electron temperature should be determined from (4.83) before using the results of Section 4.4.2.

(iii) Density Modifications: Particle trapping and plasma heating deal with the modification of the plasma background in velocity space. Modifications in configuration space can occur when the large amplitude wave is inhomogeneous in amplitude, as is the case in (4.65). This results in an equivalent potential energy, Φ_N (normalized to $m_e c^2$), for each charged particle (Ginzburg, 1970),

$$\Phi_N = \frac{|\mathcal{E}_p|^2}{\Omega^2} = \frac{|\mathcal{E}_p(0)|^2}{\Omega^2} \exp(-2\mu_p Z) \quad (4.84)$$

This nonlinear effect becomes important when Φ_N is comparable with the average electron thermal energy, $v_e^2/2$. So the condition under which the background plasma density is not significantly altered by an inhomogeneous large amplitude pump wave is then,

$$|\mathcal{E}_p|^2 \ll v_e^2 \Omega^2 \quad , \quad (4.85)$$

over the plasma region where $\mu_p Z \sim 1$. The effects of background inhomogeneity on the three-wave interaction process will be briefly discussed below.

Plasma Inhomogeneity: The size of the plasma region, ΔL_0 , in which nonlinear wave interaction occurs, may be limited either by boundaries in the case of a uniform plasma, or by satisfying the synchronism conditions (4.48) in the case of a slightly inhomogeneous plasma.

(i) Bounded homogeneous plasma: In this case, we require

$$k\Delta L_0 \gg 1 \quad , \quad k\Delta L_0 > 1 \quad , \quad (4.86)$$

to see significant nonlinear wave amplification within ΔL_0 . If there is a departure $\Delta K (= K - K' - K'')$ from synchronism, then the parametric growth will be substantially unaffected only if

$$\Delta K \cdot \Delta L_0 < \pi \quad . \quad (4.87)$$

This conclusion may be reached by considering the exact expression for the integral in (4.34) (Schiff, 1970; Phelps, Van Hoven, and Rynn, 1973)

$$\int_{\Delta L_0} dZ \exp(i\Delta K \cdot Z) = \frac{2}{\Delta K} \sin\left(\frac{\Delta K \cdot \Delta L_0}{2\Delta L_0}\right) \quad , \quad (4.88)$$

which has the significant value of roughly unity only when (4.87) is satisfied. Note here that we have neglected the reflection of waves from the boundaries.

(ii) Weakly inhomogeneous background: In this case, the quantity ΔK is a function of ΔL , where ΔL is the distance from the point of perfect synchronism. Then (4.87) can be used to determine the width, ΔL_0 , of the region over which synchronism is sufficiently well satisfied for parametric amplification to occur. Assuming linear dependence of ΔK on ΔL , we then have from (4.87)

$$\Delta L_0 < \left(\frac{\pi}{\alpha}\right)^{1/2}, \quad \Delta K = \alpha \Delta L, \quad (4.89)$$

which agrees with a result by Harker and Crawford (1970).

4.5 Discussion

The purpose of this section has been to apply the macroscopic Lagrangian obtained in Section 3 to the description of parametric wave amplification phenomena in a homogeneous plasma. In Section 4.2, the Lagrangian of (3.22) was expanded in terms of the perturbations, ξ , A , and φ_1 , in plasma cell position \underline{x} and the vector and scalar potentials \underline{A} and φ . In Section 4.3, the averaged Lagrangian technique was applied to obtain the energy densities and energy flux densities of the linear waves, and the nonlinear wave coupling coefficients. We specialized in Section 4.4 the analysis of the three-wave coupled mode equations to parametric amplification, and demonstrated how to solve them in the presence of linear wave damping. The results will be used in Section 5 to make comparisons between theory and experiment.

5. THREE-WAVE INTERACTIONS: APPLICATIONS

5.1 Introduction

Nonlinear wave-wave interaction represents the lowest order correction to linear plasma wave theory. It must be described and understood, together with the phenomena of nonlinear wave-particle interactions, which have been ignored in this thesis, before effective progress can be made towards understanding such significant plasma problems as turbulence and anomalous transport (Kadomtsev, 1965). In view of its importance, one would expect to find a variety of theoretical predictions for interactions between various types of plasma waves, and a corresponding series of experimental investigations designed to test the validity of the theory.

This desirable stage has not been reached, however. Although there have been many analyses of wave-wave interactions, there have been few experiments carried out to test them rigorously. A representative sample of the literature is given in Table 5.1. The list is not intended to be exhaustive, but illustrative. Most of the nonlinear wave coupling coefficients have been derived by the conventional iterative method. This approach is conceptually simple, but generally involves greater algebraic complexity than the Lagrangian method.

Although all of those results listed could be rederived by the averaged Lagrangian method, by use of an appropriate Lagrangian density, only a small number of the analyses were carried out by this method. These may be categorized by the plasma models used.

Cold Plasma: Using the cold plasma Lagrangian, Galloway and Crawford (1970) have obtained the coupling coefficients for transverse waves propagating at arbitrary angles to the static magnetic field. As mentioned in Section 4.1, a similar study has been presented by Boyd and

Table 5.1 Some derivations of wave coupling coefficients using (a) the averaged Lagrangian method and (b) the iterative method, with (c) related experimental work. The abbreviations used below represent, C: cold plasma, M: macroscopic warm plasma, and m: microscopic warm plasma. With respect to the static magnetic field, we have, \parallel : parallel propagation, \nmid : quasi-parallel propagation, \perp : perpendicular propagation, and X: oblique propagation. With respect to the directions of the phase velocities, we have, $\Theta = 0$: collinear waves, and $\Theta \neq 0$: noncollinear waves.

<u>Interacting Waves</u>	<u>References; Theoretical Assumptions</u>
Three ion-acoustic waves	(b) Litzenberger and Bekefi (1969); M, $\Theta = 0$ (b)(c) Ohnuma and Hatta (1970); M, $\Theta = 0$ (b)(c) Litzenberger, Mix, and Bekefi (1972); M, $\Theta = 0$
Two electron plasma waves and one ion-acoustic wave	(a) Suramlishvili (1964); m, $\Theta \neq 0$ (a) Kim (1972); M, $\Theta = 0$ (b) Oraevskii and Sagdeev (1963); M, $\Theta \neq 0$ (b) Lee and Su (1966); M, $\Theta \neq 0$ (b) Gratzl (1971); m, $\Theta \neq 0$
Three electron plasma waves	(a) Galloway and Kim (1971); m, $\Theta = 0$ (a) Kim (1972); m, \nmid , $\Theta \neq 0$
Two whistlers and one ion-acoustic wave	(b) Forslund, Kindel, and Lindman (1972); M and m, \parallel (b) Porkolab (1972); m, X, $\Theta \neq 0$ (c) Porkolab, Arunasalam, and Ellis (1972)

Table 5.1 (cont.)

<u>Interacting Waves</u>	<u>References; Theoretical Assumptions</u>
One long wavelength ordinary wave, one electron plasma wave, and one ion-acoustic wave	(a) Kim (1972); M (b) Nishikawa (1968); M (b) Lee and Su (1966); M (b) Harker (1971); M (b) Dubois and Goldman (1965); m (b) Silin (1965); m (c) Stern and Tzoar (1966b)
One extraordinary wave, one electron plasma wave, and one ion-acoustic wave	(b) Porkolab (1972); m, X, $\Theta \neq 0$
One cyclotron harmonic wave, one or two electron plasma waves, and one ion-acoustic wave	(b) Gratzl (1971); m, X, $\Theta \neq 0$
Two ion cyclotron, or two Alfvén waves, and one ion-acoustic wave	(a) Suramlishvili (1970); m, X, $\Theta \neq 0$ (b) Lee and Kaw (1972); M, m, (b) Hollweg (1971); M, (c) Belcher and Davis (1971) (c) Dubuoi and Fedyakov (1968)
Two (or one) lower hybrid waves and one ion cyclotron, or magnetosonic, wave (or two ion-acoustic waves)	(a) Suramlishvili (1971); m, X, $\Theta \neq 0$ (b) Kindel, Okuda, and Dawson (1972); m, X, $\Theta \neq 0$ (b) Fidone (1973); m, X, $\Theta \neq 0$ (b) Karney, Bers, and Kulp (1973); M, X, $\Theta \neq 0$ (c) Hooke and Bernarbei (1972)

Table 5.1 (cont.)

<u>Interacting Waves</u>	<u>References; Theoretical Assumptions</u>
Two cyclotron harmonic waves and one ion-acoustic wave	(b) Tzoar (1969); $m, \chi, \Theta \neq 0$ (c) Chang, Porkolab, and Grek (1972) (c) Keen and Fletcher (1971)
Three electromagnetic or right-handed circularly polarized waves	(a) Galloway and Crawford (1970); $C, \chi, \Theta \neq 0$ (a) Boyd and Turner (1972b); $C, \chi, \Theta \neq 0$ (a) Kim (1972); $m, \ddagger, \Theta \neq 0$ (b) Harker and Crawford (1969b); $C, \ddagger, \Theta \neq 0$
One transverse wave and two longitudinal (electron plasma and ion-acoustic) waves	(a) Suramlishvili (1965); $m, \Theta \neq 0$ (a) Kim (1965); $m, \ddagger, \Theta \neq 0$
Two whistlers and one Alfvén wave	(a) Suramlishvili (1967); $m, \chi, \Theta \neq 0$ (a) Harker, <u>et al</u> (1974); $C, \ddagger, \Theta \neq 0$
Two circularly polarized waves and one electron plasma wave	(a) Kim (1972); $m, \ddagger, \Theta \neq 0$ (a) Boyd and Turner (1972b); $m, \Theta = 0$ (b) Harker and Crawford (1970); M, \parallel (b) Sjölund and Stenflo (1967); M, \parallel (b) Montgomery (1965); M, \parallel (b) Kim, Harker, and Crawford (1971); $m, \ddagger, \Theta \neq 0$ (c) Stern and Tzoar (1966a)
Two ordinary waves and one extraordinary wave	(b) Etievant, Ossakow, Ozizmir, and Su (1968); $C, \perp, \Theta \neq 0$ (c) Cano, Fidone, and Zanfagna (1971)
Two electromagnetic waves and one cyclotron harmonic wave, or upper hybrid wave	(b) Boyd and Turner (1972a); $m, \perp, \Theta \neq 0$

Table 5.1 (cont.)

<u>Interacting Waves</u>	<u>References; Theoretical Assumptions</u>
Three cyclotron harmonic waves	(a) Kim (1972); $m, \perp, \Theta \neq 0$ (b) Harker and Crawford (1968); $m, \perp, \Theta \neq 0$ (c) Porkolab and Chang (1970) (c) Chang and Porkolab (1970)

Turner (1972b). The excitation of Alfvén waves by nonlinear interaction of whistlers in cold plasmas has been studied by Harker, Crawford, and Fraser-Smith (1974).

Microscopic warm plasma: Using the Low Lagrangian (1958), Suramlishvili (1964; 1965; 1967; 1970; 1971) has presented a series of studies concerning the nonlinear interactions between electron plasma and ion-acoustic waves, between one transverse and two longitudinal (electron plasma and ion-acoustic) waves, between Alfvén and whistler waves, between Alfvén and ion-acoustic waves, and among magnetosonic waves. Galloway and Kim (1971) have obtained the coupling coefficient for three collinear, longitudinal electron plasma waves in this model. Boyd and Turner (1972b), in a study mentioned above, have considered the interactions between two transverse waves and one electron plasma wave in a warm plasma. A comprehensive study by Kim (1972) has presented the coupling coefficients, in the microscopic warm electron plasma model, for interactions among three plasma waves, among one plasma wave and two circularly polarized waves, among three circularly polarized waves, and among two plasma waves and one circularly polarized wave, all of which propagate quasiparallel to the static magnetic field. For the case of perpendicular propagation, he has obtained the coupling coefficients for interactions among three longitudinal cyclotron harmonic waves, and among one longitudinal and two ordinary cyclotron harmonic waves.

Macroscopic warm plasma: In the work just mentioned, Kim (1972) has also examined wave interaction processes using the macroscopic Lagrangian of Section 4.2. His results include the coupling coefficients for interactions among electron plasma and ion-acoustic waves with

parallel propagation, and among an ordinary wave with perpendicular propagation, an electron plasma wave with quasiparallel propagation, and an ion-acoustic wave with quasiparallel propagation.

Most theories of wave-wave interaction have been carried out assuming the plasma to be homogeneous. In the experiments with which they have been compared, the plasmas have often been strongly inhomogeneous, rendering the wavenumber synchronism condition (4.48) difficult to realize experimentally. In these cases, comparison between theory and experiment will be impossible, unless the theory is improved by taking background inhomogeneity into account (Kino, 1960; Larsen, 1972). In mitigation, it should be pointed out that many of the experiments were aimed at studying the anomalous absorption of large amplitude waves, and the subsequent heating of the plasma (Porkolab, et al., 1972; Stern and Tzoar, 1966b; Dubuoi and Fedyakov, 1968; Hooke and Bernarbei, 1972; Chang, et al., 1972; Keen and Fletcher, 1971), rather than to produce ideal conditions for testing basic plasma theory.

Some experiments have, however, been carried out in effectively homogeneous plasmas with synchronism in both frequency and wavenumber realized. In particular, second harmonic generation caused by an ion-acoustic wave propagating along a plasma column have been observed by Litzenberger, et al. (1972). Three interacting ion-acoustic waves have been studied by Ohnuma and Hatta (1970). Three interacting cyclotron harmonic waves have been studied by Chang and Porkolab (1970). Parametric excitation of ion-acoustic waves by whistler waves has been observed by Porkolab, et al. (1972). To explain the observed wave behavior, it was found necessary to include the effects of wave damping. With the

exception of the case for harmonic generation by ion-acoustic waves (Litzenberger et al., 1972), only qualitative agreement between the solutions of the coupled mode equation and the waves observed has been achieved (Ohnuma and Hatta, 1970; Chang and Porkolab, 1970; Porkolab, et al., 1972). The purpose of this section, then, is to establish more nearly complete quantitative comparisons between some of these experiments and the theory developed in Section 4.

In this section we shall specialize the general results of Section 4 to two experimental situations. The first will be that of Ohnuma and Hatta (1970), for which we shall consider the collinear excitation of an ion-acoustic wave by two other ion-acoustic waves (Section 5.2), and by two longitudinal electron plasma waves (Section 5.3). The second will be that of Porkolab, et al. (1972), for which we shall consider the excitation of a collinear (Section 5.4), and a noncollinear (Section 5.5), ion-acoustic wave by a large amplitude whistler.

5.2 Nonlinear Interaction of Ion-Acoustic Waves

5.2.1 The Experiment and Interpretation

The experimental set-up used by Ohnuma and Hatta (1970) is shown schematically in figure 5.1. Their plasma had the typical parameters:

$$n_e \approx n_i \approx 10^9/\text{cm}^3, \quad T_e \approx 5 \times 10^4 \text{ }^\circ\text{K}, \quad T_i \approx 300^\circ \text{K},$$

$$P \approx 9 \text{ mTorr}, \quad \nu_{en} \approx 4.6 \times 10^7/\text{sec}, \quad \nu_{in} \approx 3.0 \times 10^5/\text{sec}, \quad (5.1)$$

where the effective electron- and ion-neutral momentum transfer collision frequencies have been calculated from (Ginzburg, 1970)

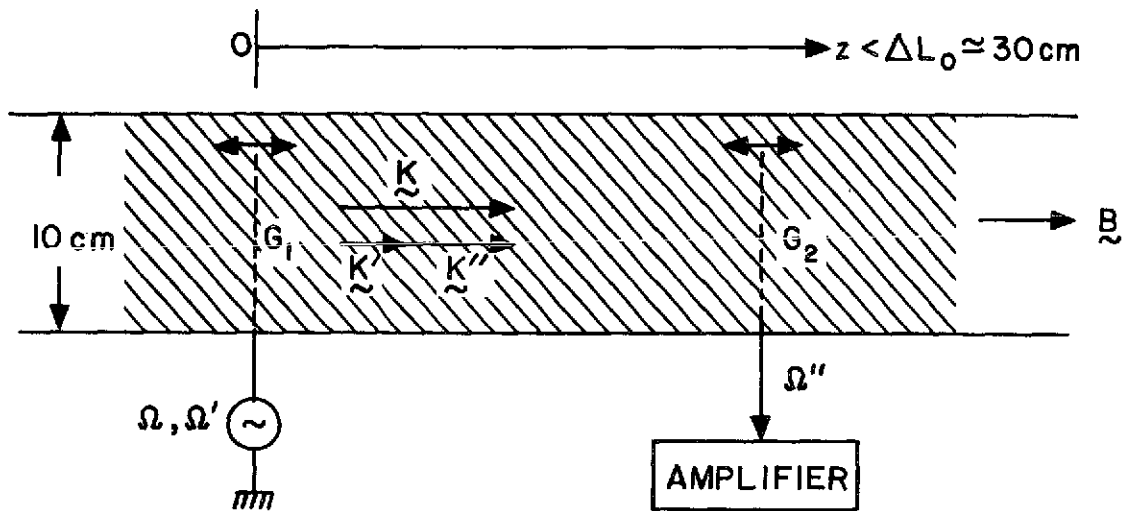


Figure 5.1 The argon ($\sigma \approx 7.4 \times 10^4$) plasma column used by Ohnuma and Hatta (1970). The pump (\underline{k}, Ω) and signal (\underline{k}', Ω') ion-acoustic waves were excited with grid G_1 . These waves and the idler ($\underline{k}'', \Omega''$) ion-acoustic wave were detected with grid G_2 .

$$v_{en} \approx 8.3 \times 10^5 (\pi a^2)_e n_n T_e^{1/2} \text{sec}^{-1} (T_e \text{ in } ^\circ\text{K}) ,$$

$$v_{in} \approx 1.1 \times 10^5 (\pi a^2)_i n_n \left(\frac{T_i}{m_i}\right)^{1/2} \text{sec}^{-1} (T_i \text{ in } ^\circ\text{K}) , \quad (5.2)$$

with $n_n (\text{cm}^{-3})$ being the neutral density and $m_i (\text{gm})$ the ion mass.

The averaged electron-neutral collision cross-section, $(\pi a^2)_e \approx 8.5 \times 10^{-16} \text{cm}^2$, has been obtained from the data given by Brown (1966), and the ion-neutral collision cross-section, $(\pi a^2)_i \approx 3.4 \times 10^{-15} \text{cm}^2$, has been assumed to be roughly $4(\pi a^2)_e$.

For nonlinear interaction among three ion-acoustic waves, a large amplitude pump, $\mathcal{E}_p(\underline{k}, \Omega)$, and a small amplitude signal, $\mathcal{E}_s(\underline{k}', \Omega')$, were excited by a grid immersed in the plasma. These waves and the parametrically amplified idler, $\mathcal{E}_i(\underline{k}'', \Omega'')$, were then measured by a moveable grid (figure 5.1). A typical example of the measured results is reproduced in figure 5.2, which shows that \mathcal{E}_s and \mathcal{E}_p (when sufficiently weak) decay exponentially, but that \mathcal{E}_i is first amplified substantially from noise before it decays. The measured decay rate of \mathcal{E}_s was found to be roughly the linear ion-acoustic wave damping rate due to ion-neutral collisions. But the decay rate of \mathcal{E}_p for $z \leq 4 \text{ cm}$ was found to increase with the exciting voltage, V_p , applied to the grid when $V_p > V_{th} \sim 2\text{V}$. This observed behavior of \mathcal{E}_p is reproduced in figure 5.3, and may be attributable to a combined result of wave-particle, and other wave-wave, interactions on the large amplitude pump wave, \mathcal{E}_p .

Interpretation of the observed evolution of \mathcal{E}_i : It was found in the experiment that $z_0 (\sim 2 \text{ cm})$ is roughly independent of V_p , and hence also independent of $\mathcal{E}_p(0)$ if we can assume $\mathcal{E}_p(0) \propto V_p$. To explain this observed feature of \mathcal{E}_i , Ohnuma and Hatta (1970) used a

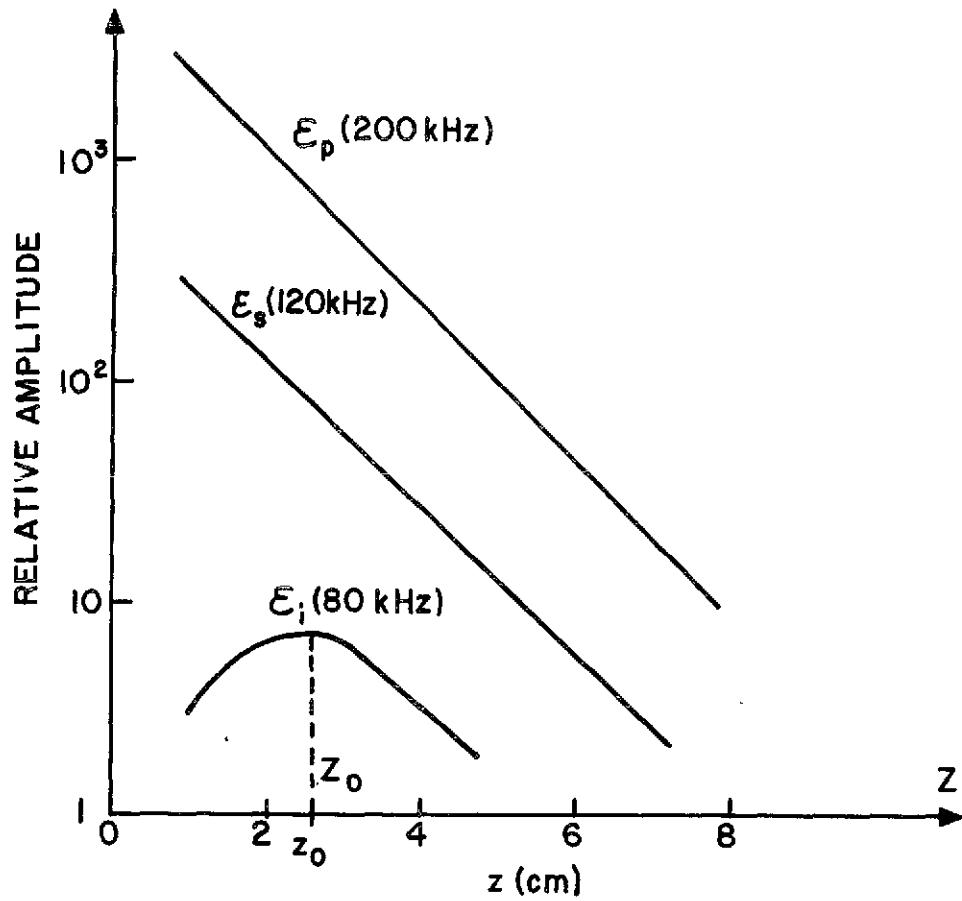


Figure 5.2 A typical observation by Ohnuma and Hatta (1970) of interaction among collinear ion-acoustic waves. The exciting grid for the pump, E_p , and signal, E_s , is located at $z = 0$. The peak of the idler, E_i , is located at z_0 .

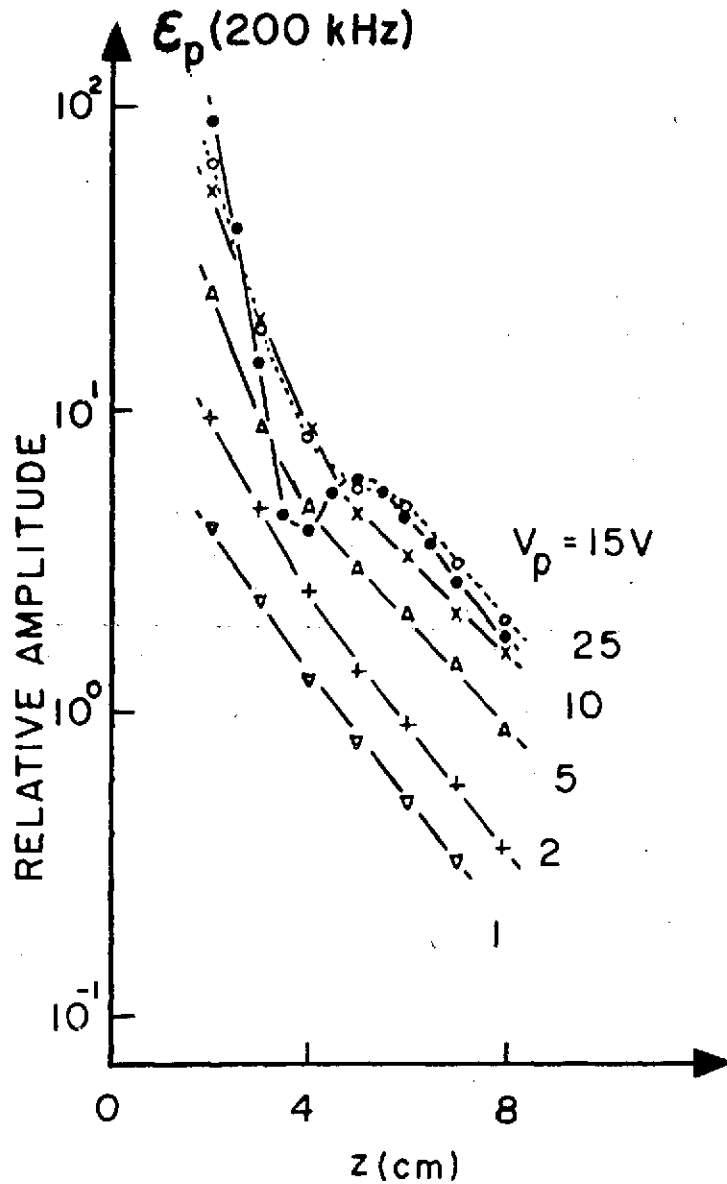


Figure 5.3 The measured behavior of ϵ_p for several exciting voltages, V_p , at the grid. For $z \leq 4$ cm, the spatial decay rate, μ_p , of ϵ_p was found to increase with V_p when $V_p > V_{th} \sim 2V$. [Ohnuma and Hatta (1970), figure 16].

set of coupled mode equations with identical wave damping rates, i.e.

$\mu_p = \mu_s = \mu_i = \mu$. Assuming (4.67), they were able to show that $Z_0 (\approx 0.7/\mu_p = 0.7/\mu)$ is independent of $\mathcal{E}_p(0)$, as was shown in the paragraph containing (4.67) in Section 4.4.2.

As indicated in figure 5.3, however, μ_p increases substantially with $\mathcal{E}_p(0)$ [assuming $\mathcal{E}_p(0) \propto V_p$]. Therefore, the theoretical behavior of \mathcal{E}_i is more suitably described by assuming $\mu_p \neq \mu_s = \mu_i = \mu$ in the coupled mode equations (4.64), whose solutions for this case are given by (4.66). Furthermore, because \mathcal{E}_p is damped nonlinearly, it is reasonable to assume that $\mathcal{E}_p(0) > \mathcal{E}_{th}$. Then, from (4.66), we have

$$M(Z_0) \sim M_0 = \frac{\mathcal{E}_p(0)\mu}{\mathcal{E}_{th}^{\mu_p}} \sim 1, \quad (5.3)$$

showing that the first condition of (4.67) fails. Therefore, the apparent independence of Z_0 in $\mathcal{E}_p(0)$ was not satisfactorily explained by the formula $Z_0 \approx 0.7/\mu_p$.

Since we can only assume $\mathcal{E}_s(0) \gg \mathcal{E}_i(0)$, use of (4.66) then shows that Z_0 should satisfy

$$\exp(-\mu_p Z_0) = \frac{\mu}{\mu_p M_0} \tanh \left\{ M_0 \left[1 - \exp(-\mu_p Z_0) \right] \right\}. \quad (5.4)$$

To determine Z_0 from (5.4), it is necessary to evaluate μ , and evaluate M_0 and μ_p as functions of $\mathcal{E}_p(0)/\mathcal{E}_{th}$. This calls for the examination of the linear properties and the nonlinear coupling coefficient of the ion-acoustic waves.

5.2.2 Ion-Acoustic Waves (I) and Coupling Coefficient: $\theta = 0$

In this case we have $\underline{\underline{K}} = K \hat{i}_{\underline{\underline{z}}}$, where $\hat{i}_{\underline{\underline{z}}}$ is the unit vector in the positive z direction, along which the static magnetic field is directed. The polarization constant, $\underline{\underline{M}}^s$, of (4.40) then reduces to

$$\begin{aligned} M_{xx}^s = M_{yy}^s &= \frac{1}{\Omega^2 - \Omega_s^2} \quad , \quad M_{zz}^s = \frac{1}{D_s} \quad , \quad M_{xy}^s = M_{yx}^{s*} = \frac{-i\Omega_s}{\Omega(\Omega^2 - \Omega_s^2)} \quad , \\ M_{xz}^s = M_{zx}^{s*} = M_{yz}^s = M_{zy}^{s*} &= 0 \quad , \quad D_s = \Omega^2 - \gamma_s v_s^2 K_z^2 \quad . \end{aligned} \quad (5.5)$$

For the experiment, we can assume

$$v_e^2 \gg \sigma v_i^2 \quad (T_e \gg T_i) \quad , \quad v_e^2 K_z^2 \ll 1 \quad , \quad (5.6)$$

where the second condition follows because $v_e \approx 2.9 \times 10^{-3}$ from (5.1), and $K_z \sim 50$ for a typical experiment wavelength of 0.21 cm.

Under the assumption of a low frequency longitudinal wave,

$$\underline{\underline{e}}^I = \{0, 0, 1\} \quad , \quad \Omega \ll 1 \quad , \quad (5.7)$$

the approximate ion-acoustic wave (I) dispersion relation can be obtained by using (5.5) and (5.6) in (4.42) and (4.43). For $v_e K_z \ll 1$, it is

$$\Omega^2 \approx v_a^2 K_z^2 \quad , \quad v_a^2 \approx \frac{1}{\sigma} \gamma_e v_e^2 \quad , \quad (5.8)$$

where v_a is the ion-acoustic speed. As illustrated in figure 5.4, the wave coupling synchronism conditions are automatically satisfied over the linear portion represented by (5.8).

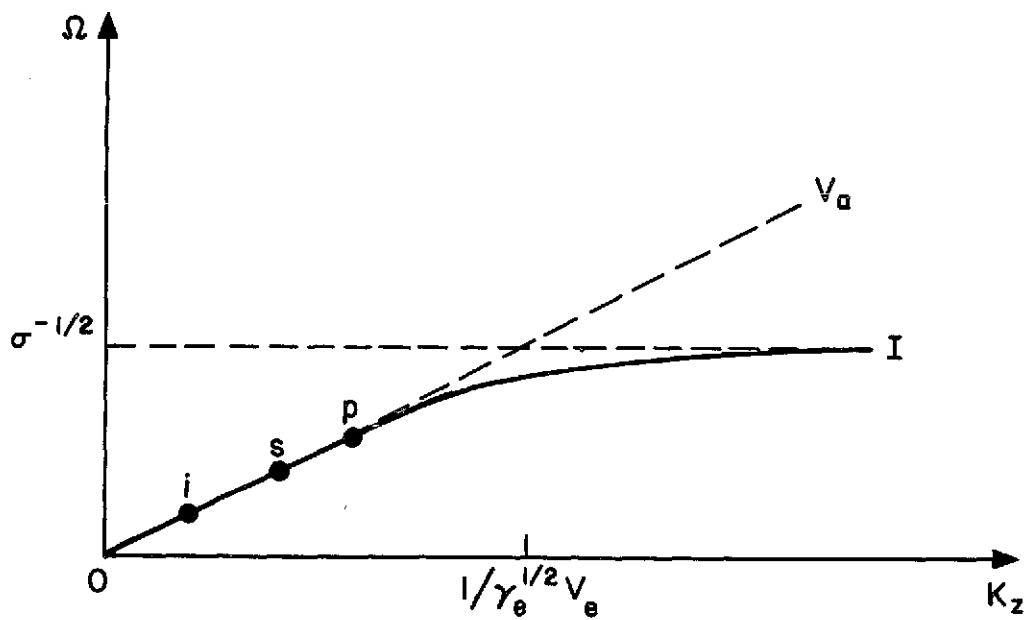


Figure 5.4 The dispersion curve for ion-acoustic waves (I) propagating parallel to the dc magnetic field under the conditions of (5.6). The synchronism conditions for nonlinear coupling of three ion-acoustic waves (p, s, and i) are always satisfied over its linear portion.

The corresponding ion-acoustic wave action, group velocity, and coupling coefficient can be obtained by using (5.5)-(5.8) in (4.46), (4.50), and (4.53), yielding

$$g_{K_z}^I \approx \frac{1}{\gamma_e v_e^2 \Omega K_z^2} \quad , \quad u_{K_z}^I \approx \hat{i}_z \frac{\Omega}{K_z} \quad ,$$

$$C_{\text{psi}} = \Lambda_{-K_z K_z' K_z''}^{\text{III}} \approx - \frac{\gamma_e + 1}{\gamma_e^2 v_e^4 K_z K_z' K_z''} \quad . \quad (5.9)$$

The significant contributions in C_{psi} come from the Λ_I^i , Λ_I^e , and Λ_T^e terms of (4.46). Using (5.9), the coupled mode equations (4.52) can be reduced to the form derived by Litzemberger and Bekefi (1969).

5.2.3 The Dependence of Z_0 on $\mathcal{E}_p(0)$

To determine the dependence of Z_0 on $\mathcal{E}_p(0)$, we now have to determine the temporal damping rate of an ion-acoustic wave due to ion-neutral collisions. This can be shown to be roughly $v_{\text{in}}/2$ (Ohnuma and Hatta, 1970). The plasma parameters given in (5.1) show that the ion Landau damping rate (Ginzburg, 1970),

$$\frac{\gamma_{\text{Li}}}{\omega} \approx \left(\frac{\pi}{8}\right)^{1/2} \left[\frac{1}{\sigma^{1/2}} + \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2T_i}\right) \right] \quad , \quad (5.10)$$

is much smaller than $v_{\text{in}}/2$, and negligible. Using (5.1), (5.2), (5.8), (5.9), and (4.63), we have the following values of the normalized parameters

$$v_e \approx 2.9 \times 10^{-3} \quad , \quad v_a \approx 1.4 \times 10^{-5} \quad , \quad \Gamma^I \approx 8.3 \times 10^{-5} \quad , \quad \mu^I \approx 5.9 \quad , \quad (5.11)$$

where we have taken $\gamma_e = 5/3$ as a convenient value.

It remains to obtain the dependence of μ_p/μ and M_0 on $\mathcal{E}_p(0)/\mathcal{E}_{th}$. From figure 5.3, we first plot the corresponding values of μ_p/μ and the pump wave exciting voltage, V_p , as shown in figure 5.5(a). Assuming $\mathcal{E}_p(0) \propto V_p$, figure 5.5(a) indicates the approximate relations between μ_p/μ and $\mathcal{E}_p(0)/\mathcal{E}_{th}$, and M_0 and μ_p/μ ,

$$\frac{\mu_p}{\mu} \approx 0.23 \frac{\mathcal{E}_p(0)}{\mathcal{E}_{th}} + 0.77 \quad , \quad M_0 \approx 4.3 - 3.3 \frac{\mu}{\mu_p} \quad , \quad (5.12)$$

where the second relation has been obtained from (4.66) by use of the first relation. Combination of (5.12) with (5.4) then determines the dependence of z_0 on $\mathcal{E}_p(0)/\mathcal{E}_{th}$, which is as shown in figure 5.5(b). It is seen that z_0 (≈ 2.5 cm) is in good agreement with the observed value of 2 cm, and its dependence on $\mathcal{E}_p(0)/\mathcal{E}_{th}$ according to (5.4) is much weaker than that according to $z_0 \approx 0.7/\mu_p$, which is obtained under the conditions (4.67) used by Ohnuma and Hatta (1970). Thus we conclude that, for $\mathcal{E}_p(0) > \mathcal{E}_{th}$, and with an externally applied signal wave $\mathcal{E}_s \gg \mathcal{E}_i$, the apparent independence of z_0 from $\mathcal{E}_p(0)$ observed by Ohnuma and Hatta is likely to be a result of the increase in μ_p with $\mathcal{E}_p(0)$.

Finally, it is of interest to examine the magnitude of \mathcal{E}_{th} for the typical pump field strength indicated by Ohnuma and Hatta, which is $E_p \sim 32$ V/cm, corresponding to a ratio of rf to dc charge densities of roughly 1/20. For the typical pump and signal frequencies of 200 and 120 kHz, we have, by use of (4.30), (5.1), and (5.8),

$$\Omega \approx 7.0 \times 10^{-4} \quad , \quad \Omega' \approx 4.2 \times 10^{-4} \quad , \quad K_z \approx 50 \quad , \quad K'_z \approx 30 \quad . \quad (5.13)$$

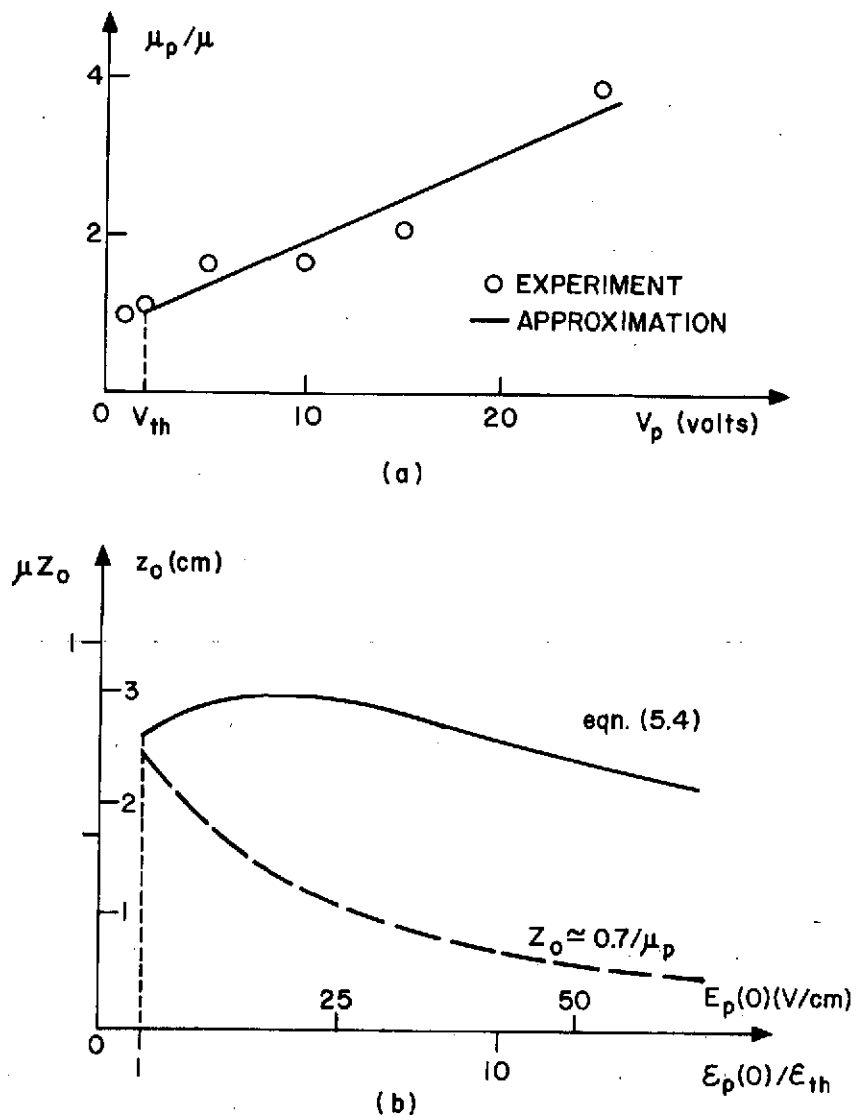


Figure 5.5 (a) The increase of μ_p with the grid exciting voltage, V_p , for the pump wave, as deduced from figure 5.3, and their approximate relation. (b) The dependence of the peak location, z_0 , of the idler, \mathcal{E}_i , on the boundary values of the pump, $\mathcal{E}_p(0)$, according to (5.4) and the relation, $z_0 \approx 0.7/\mu_p$ used by Ohnuma and Hatta (1970).

Then use of (4.61), (5.9), (5.11), and (5.13) gives $\mathcal{E}_{th} \approx 1.3 \times 10^{-4}$, corresponding to a pump threshold of 4.2 V/cm. This value confirms that the assumption $\mathcal{E}_p(0) > \mathcal{E}_{th}$ in connection with (5.3) and (5.4) is appropriate.

5.3 Excitation of an Ion-Acoustic Wave by Two Electron Plasma Waves

We will now examine the possibility of exciting an ion-acoustic wave by two electron plasma waves in the experimental plasma discussed in Section 5.2.1. Our purpose is to determine whether the nonlinearly excited ion-acoustic wave should be observable under practical conditions, so that a measurement of the coupling coefficient can be carried out.

5.3.1 Electron Plasma Waves (P) and Coupling Coefficient: $\theta = 0$

Use of (5.5) and (5.6) in (4.43) and (4.44), with the assumption of $\Omega \gg 1$, yields the dispersion relation for the electron plasma wave (P),

$$\Omega^2 \approx 1 + \gamma_e v_e^2 k_z^2, \quad \tilde{e}^P = \{0, 0, 1\}. \quad (5.14)$$

This P wave dispersion relation is plotted, together with that of the ion-acoustic wave (I), in figure 5.6, which shows that the interaction between the electron plasma (pump and signal) and ion-acoustic (idler) waves is of the contra-flow type according to the definition shown in figure 4.2(b). Because the slope of the P curve at $v_e k_z \sim 0.1$ is much larger than v_a , we have approximately $k_z'' \approx 2k_z \approx -2k_z'$.

The electron plasma wave action and group velocity can be obtained by using (5.5), (5.6), and (5.14) in (4.50), giving

$$\mathcal{J}_{k_z}^P \approx \Omega, \quad \tilde{u}_{k_z}^P \approx \hat{i}_z \gamma_e v_e^2 \frac{k_z}{\Omega}. \quad (5.15)$$

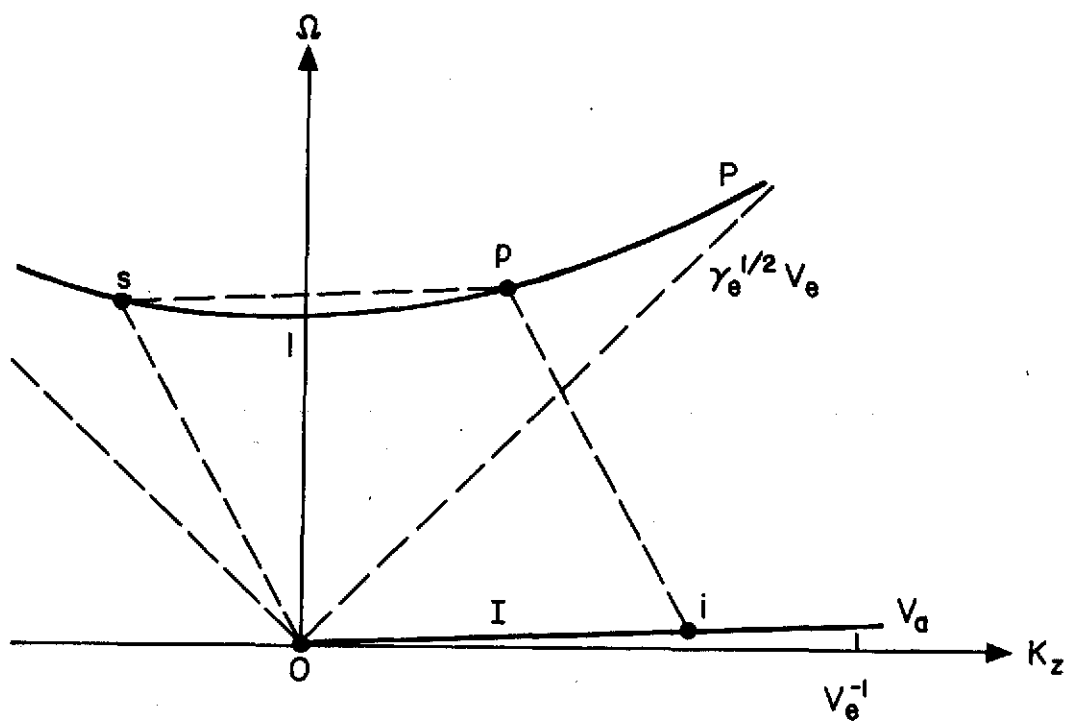


Figure 5.6 Dispersion curves and synchronism conditions for longitudinal electron plasma (pump and signal) waves (P) and ion-acoustic (idler) waves (I).

The coupling coefficient can be obtained by using (5.5)-(5.8) and (5.14) in (4.46) and (4.53), yielding

$$C_{\text{psi}} = \Lambda \frac{\text{PPI}}{-K_z K_z' K_z''} \approx \frac{1}{\gamma_e v_e^2 K_z''} \quad (5.16)$$

Equations (5.15) and (5.16) may be used in (4.54) to obtain the temporal amplification rate, Γ_0 , for the ion-acoustic wave as

$$\Gamma_0^2 \approx \frac{\Omega'' |\mathcal{E}_p|^2}{4\Omega \gamma_e v_e^2} \quad , \quad (5.17)$$

which agrees with the result given by Kadomtsev (1965), who considered the case $\gamma_e = 1$.

5.3.2 Evolution of \mathcal{E}_i and \mathcal{E}_s

As an example, let us consider an ion-acoustic wave excited with frequency 400 kHz and wavelength 1.1 cm, so that $\Omega'' \approx 1.4 \times 10^{-3}$ and $K_z'' \approx 100$. The corresponding pump wave number is then $K_z \approx 50$. The Landau damping rate, $\Gamma_{L_{pe}}^{\omega}$, for this electron plasma pump wave can then be obtained from (4.81). It shows that $\Gamma_{L_{pe}}^{\omega}$ is much smaller than the damping rate due to electron-neutral collisions, $\nu_{en}/2$, whose value is given by (5.2) (Ginzburg, 1970). Thus we have $\Gamma^P \approx 1.3 \times 10^{-2}$. By use of this result, and (4.63), (4.61), (5.9), (5.15), and (5.16), the parameters in (4.68) and (4.69) take the values

$$\mu_p = -\mu_s \approx 21 \quad , \quad \mu_i \approx 5.9 \quad , \quad z < 1.8 \quad , \quad \nu = 1.1 \quad ,$$

$$\mathcal{E}_{\text{th}} \approx 2.1 \times 10^{-4} \quad , \quad |M_0| \approx 2.1 \times 10^{-2} \quad , \quad \left(\frac{|u_s| g_s}{u_i \beta_i} \right)^{1/2} \approx 0.1 \quad . \quad (5.18)$$

Since $\chi \leq |M_0| \ll 1$, the approximate solution of (4.70) can be used.

When (5.18) is substituted, (4.70) becomes

$$\begin{aligned} \mathcal{E}_i &\approx \left\{ \mathcal{E}_i(0) \left[\exp(-2.1Z) - 10^{-3} \exp(2.1Z) \right] + 0.021 \mathcal{E}_s(0) \sinh(2.1Z) \right\} \exp(-3Z) , \\ \mathcal{E}_s &\approx \left[\mathcal{E}_s(0) \exp(23Z) - 0.0019 \mathcal{E}_i(0) \sinh(23Z) \right] \exp(-3Z) , \end{aligned} \quad (5.19)$$

where we have assumed $\mathcal{E}_p(0) \approx 5.8 \times 10^{-4}$, corresponding to a field strength of 17 V/cm.

The idler equation in (5.19) shows that if $\mathcal{E}_s(0) < 50 \mathcal{E}_{in}$, where $\mathcal{E}_i(0) = \mathcal{E}_{in}$ is the thermal noise level for some acceptable measuring frequency bandwidth, no observable $\mathcal{E}_i(Z)$ above \mathcal{E}_{in} is possible. Assuming a measurement bandwidth of 2 kHz, the thermal fluctuations in the electric field of the ion-acoustic wave branch may be estimated roughly to be 1.5×10^{-4} V/cm near 400 kHz (Bekefi, 1966). To excite \mathcal{E}_i to above this level near $Z \approx 0.2$, i.e. 3.5 cm from the pump source, with a 17 V/cm pump field, will require that $\mathcal{E}_s(0) > 2.5 \times 10^{-7}$, corresponding to a signal field strength of 7.5 mV/cm. According to the signal equation in (5.19), this requirement is equivalent to having $\mathcal{E}_s(0.3) > 10^{-4}$, i.e. injecting a 3 V/cm contra-flow signal wave at $z \approx 5$ cm.

It remains to determine whether these parameters represent an acceptable experiment to demonstrate the interaction between collinear electron plasma and ion-acoustic waves. According to (4.82), the condition to allow trapped electron effects on the pump wave to be neglected is $\mathcal{E}_p \leq 3.4 \times 10^{-6}$. From (4.83), the condition to allow electron heating effects due to the pump wave to be neglected is $\mathcal{E}_p \sim 2.6 \times 10^{-5}$. Also, from (4.85), the

plasma will not become significantly inhomogeneous due to spatial variation of the pump wave amplitude, provided that $\mathcal{E}_p < v_e \sim 2.9 \times 10^{-3}$. With the value of $\mathcal{E}_p \sim 5 \times 10^{-4}$, it is seen that, first, the assumption $\mathcal{E}_p \propto \exp(-\mu_p Z)$ is inappropriate because of the trapped-particle effects, thus rendering (5.19) also inappropriate, and second, according to (4.83), the background electron temperature may become roughly 100 times the value given in (5.1) when the pump wave is injected, thus significantly changing the linear wave properties and nonlinear coupling coefficient used in this subsection. Also, due to the temperature increase, the thermal fluctuations in the ion-acoustic wave branch may become much stronger than the previously indicated value of 1.5×10^{-4} V/cm. As a result, we conclude that it will be difficult to demonstrate the interaction between electron plasma and ion-acoustic waves in the experimental plasma defined in Section 5.2.1.

5.4 Excitation of an Ion-Acoustic Wave by Two Whistlers; Collinear Propagation

This nonlinear process has been observed in an experiment by Porkolab, Arunasalam, and Ellis (1972). A detailed quantitative comparison between the theoretical results of Section 4 and the observed parametric amplification process in this experiment will be given here. It will be shown that there is wide disagreement between theory and experiment. The comparison will suggest that important factors were overlooked in the theoretical model, and indicate possible directions in which the theory might be improved.

5.4.1 Experimental Observations

The experiment by Porkolab, et al.(1972) was carried out in a steady state magnetized helium plasma column, which is illustrated schematically in figure 5.7. The plasma has the following typical parameters:

$$n_e \approx n_i \approx 3 \times 10^{12}/\text{cm}^3, \quad T_e \approx 5 \text{ eV}, \quad T_i \approx 0.1 \text{ eV},$$

$$P \approx 2 \text{ mTorr}, \quad v_{en} \approx 9.7 \times 10^6/\text{sec}, \quad v_{in} \approx 4.9 \times 10^6/\text{sec}, \quad (5.20)$$

where v_{en} and v_{in} have been determined by use of (5.2). The effective electron-neutral collision cross section, $(\pi a^2)_e \approx 5 \times 10^{-16} \text{ cm}^2$, has been obtained from the data given by Brown (1966), and the ion-neutral collision cross section, $(\pi a^2)_i \approx 2 \times 10^{-15} \text{ cm}^2$, has been assumed to be $4(\pi a^2)_e$.

A large amplitude whistler pump wave, with a typical frequency of 2.45 GHz and a wavelength of 1.6 cm, was injected into the plasma with the dc electron cyclotron frequency set at 3.68 GHz. Their normalized values are then

$$\Omega \approx 0.16, \quad k_z \approx 1.4, \quad \Omega_c \approx 0.24. \quad (5.21)$$

The pump wave variation was observed to be nearly exponential with a damping length of roughly 20 cm, corresponding to a normalized damping length of $\mu_p \approx 0.016$. A typical interferometer output is reproduced in figure 5.8, for a frequency of 7.5 MHz, i.e. $\Omega'' \approx 4.9 \times 10^{-4}$. It shows that the excited ion-acoustic (idler) waves consisted of two wavelengths.

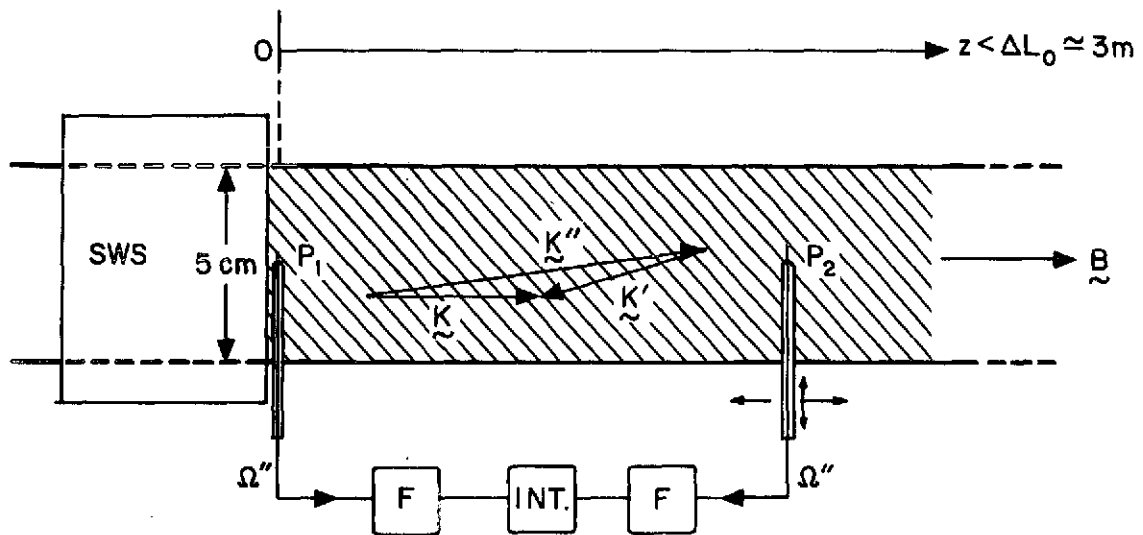


Figure 5.7 The experimental set-up used by Porkolab, *et al.* (1972), in which ion-acoustic (idler) waves parametrically excited by whistler (pump) waves were observed. The pump wave (\tilde{K}, Ω) was injected into the magnetized plasma column by a slow wave structure (SWS). The spatial behavior of the idler wave (\tilde{K}'', Ω'') was measured by use of two rf probes (P_1 and P_2), narrow-band-pass filters (F), and an interferometer (Int.).

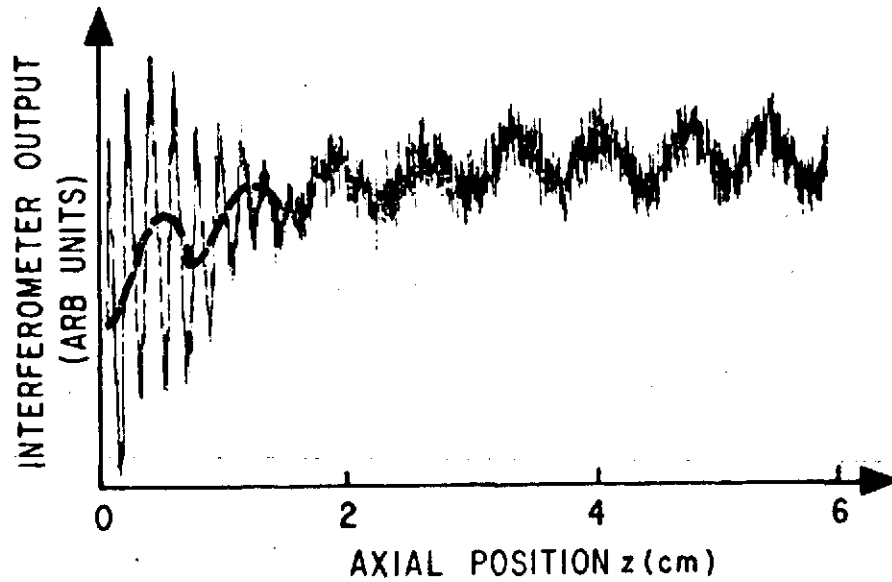


Figure 5.8 The interferometer output showing the ion-acoustic (idler) waves at 7.5 MHz, i.e. $\Omega'' \approx 4.9 \times 10^{-4}$, obtained by Porkolab, et al. (1972). The approximate behavior of the longer wavelength idler wave for $z \leq 2$ cm is shown dashed, and was obtained by subtracting the shorter wavelength idler wave from the trace.

Additional measurements indicated strongly that the longer wavelength wave propagated at $\theta'' \approx 20^\circ$ and the shorter at $\theta'' \approx 50^\circ$, where θ'' is the angle between \tilde{K}'' and the dc B field. The wavenumbers were found to satisfy $K'_z \approx -K_z$ and $K''_z \approx 2K_z$ in the first case, and $\tilde{K}' \approx -\tilde{K}''$ and $|\tilde{K}''| \gg K_z$ in the second case. We shall consider the first case under the assumption $\theta'' \approx 0^\circ$ here, and defer discussion of the second case with $\theta'' \approx 50^\circ$ to Section 5.5.

It should be pointed out here that only the ion-acoustic wave in the second case was found to satisfy the dispersion relation $\Omega'' \approx v_a K''$, so there is a reasonable doubt that the excited idler wave in the first case can indeed be considered as an ion-acoustic wave defined in an infinite homogeneous plasma background. We shall proceed by assuming that it is, and see whether the idler wave behavior shown in figure 5.8 agrees quantitatively with the corresponding theoretical results.

5.4.2 Whistler Waves (R) and the Coupling Coefficient: $\theta = 0$

Use of (5.5) and (5.6) in (4.43) and (4.44), with the assumption of $\Omega \sim \Omega_c \sim 1 \gg \Omega_i$, yields the dispersion relations and polarization vectors for the right-hand (R) and left-hand (L) polarized transverse waves,

$$K_z^2 \approx \Omega^2 - \frac{\Omega}{(\Omega \mp \Omega_c)} \quad , \quad \tilde{e}^{R,L} \approx 2^{-1/2} \{1, \mp i, 0\} \quad , \quad (5.22)$$

where the upper sign is associated with the R wave. For $\Omega < \Omega_c$, the R wave corresponds to the whistler, whose dispersion relation is plotted, together with that of the ion-acoustic wave (I), in figure 5.9, which shows that the interaction between the whistler (pump and signal) and ion-acoustic (idler) waves is of the contra-flow type. Because the slope

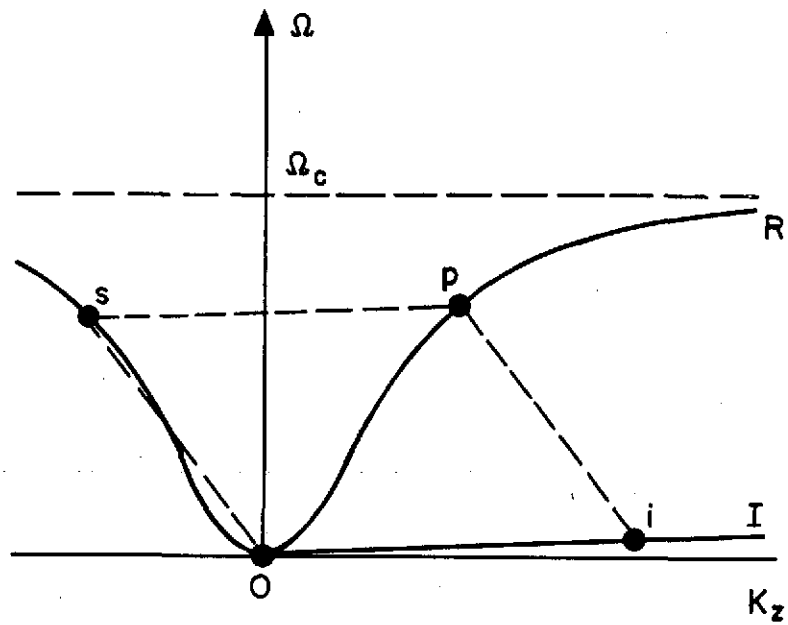


Figure 5.9 Dispersion curves and synchronism conditions for whistler (pump and signal) waves (R) and ion-acoustic (idler) waves (I).

of the R curve at $K_z \approx 1.4$ is much larger than V_a , we have approximately $K_z'' \approx 2K_z \approx -2K_z'$. Since this condition was observed experimentally, the collinear approximation to the wave interaction process is seen to be appropriate.

The whistler wave action and group velocity can be obtained by using (5.5), (5.6), and (5.22) in (4.50), giving

$$g_{K_z}^R \approx \frac{1}{2\Omega} \left[1 + \frac{K_z^2}{\Omega^2} + \frac{1}{(\Omega - \Omega_c)^2} \right], \quad \tilde{u}_{K_z}^R \approx \hat{i}_z \frac{K_z}{\Omega^2} \left(g_{K_z}^R \right)^{-1}. \quad (5.23)$$

The coupling coefficient can be obtained by using (5.5)-(5.8) and (5.22) in (4.46) and (4.53), yielding

$$C_{\text{psi}} = C_{K_z K_z' K_z''}^{\text{RRI}} = \Lambda_{-K_z K_z' K_z''}^{\text{LRI}} \approx \frac{1}{\gamma_e v_e^2 K_z''^2} \left[\frac{K_z'}{\Omega'(\Omega - \Omega_c)} - \frac{K_z}{\Omega(\Omega' - \Omega_c)} \right], \quad (5.24)$$

where we have used the relation, $\tilde{e}_{K_z}^R = \tilde{e}_{-K_z}^L$. In this expression, the factor $1/\gamma_e v_e^2 K_z''^2$ is due to the factor $1/D_e''$ in $M_{zz}^{e''}$ of (5.5). If the ion-acoustic wave (I) were replaced by an electron plasma wave (P), this denominator would be replaced by $\Omega''^2 - \gamma_e v_e^2 K_z''^2 \approx \Omega''^2 = 1$, by use of (5.5) and (5.6). The resulting coupling coefficient, $\Lambda_{-K_z K_z' K_z''}^{\text{LRP}}$, is then identical to that obtained by Harker and Crawford (1970) for the interaction between two right-hand polarized waves and an electron plasma wave.

5.4.3 Evolution of \mathcal{E}_i and \mathcal{E}_s

To determine the behavior of \mathcal{E}_i , we need to obtain the whistler and the ion-acoustic wave damping rates. The whistler wave is damped by electron-neutral collisions (Γ_c) and, and by cyclotron resonance (Γ_r) when

Ω is close to Ω_c . They are given by (Ginzburg, 1970)

$$\Gamma_c \approx \left(\frac{v_{en}}{\omega_{pe}} \right) \frac{\Omega}{2\Omega(\Omega - \Omega_c)^2 + \Omega_c} \quad , \quad \Gamma_r \approx \pi^{1/2} \frac{(\Omega - \Omega_c)^2}{v_e K_z} \exp \left[- \frac{(\Omega - \Omega_c)^2}{v_e^2 K_z^2} \right]. \quad (5.25)$$

Use of (5.20) and (5.21) in (5.25) and (5.10) shows that the damping of the whistler and the ion-acoustic waves is predominantly due to electron- and ion-neutral collisions, respectively. Then the first expression of (5.25), and the relation $\Gamma^I \approx v_{in}/2\omega_{pe}$, give

$$\Gamma^R \approx 6.7 \times 10^{-5} \quad , \quad \Gamma^I \approx 2.5 \times 10^{-5} \quad . \quad (5.26)$$

Use of (5.20), (5.21), and (5.23) gives the group velocities $u^R \approx 7.2 \times 10^{-2}$ and $u^I = v_a \approx 4.7 \times 10^{-5}$. Use of these results and (5.26) in (4.63) then gives

$$\mu_s \approx -9.2 \times 10^{-4} \quad , \quad \mu_i \approx 0.53 \quad , \quad (5.27)$$

where $\mu_s < 0$ because $\mu_s \propto -\hat{i}_z$. Note that $|\mu_s|$ is much smaller than the observed spatial damping rate for the whistler pump, $\mu_p \approx 0.016$, showing that the pump wave is heavily damped nonlinearly.

It is also necessary to determine the values C_{psi} and \mathcal{E}_{th} before the theoretical behavior of \mathcal{E}_i can be predicted. Using (5.21), (5.23), (5.24), (5.26), and (4.61) then gives

$$C_{psi} \approx 1.8 \times 10^6 \quad , \quad \mathcal{E}_{th} \approx 1.0 \times 10^{-5} \quad , \quad (5.28)$$

where the corresponding unnormalized pump threshold is 16 V/cm. We expect this \mathcal{E}_{th} to be smaller than $\mathcal{E}_p(0)$, which can be determined approximately from the input power and cross-sectional area of the plasma

at the pump wave exciting structure (Lisitano, Fontanesi, and Sindoni, 1970; see figure 5.7). Taking 0.8 kW and 20 cm², respectively, and using (5.23), we find $\mathcal{E}_p(0) \leq 7.2 \times 10^{-6}$, which is smaller than \mathcal{E}_{th} of (5.28). Therefore, if both v_{en} and v_{in} given in (5.20) were not overestimated, the theory predicts no parametric excitation of the ion-acoustic wave.

To remove this discrepancy, let us assume that $\Gamma^R = \Gamma_s \approx 3.4 \times 10^{-5}$, as opposed to (5.26). Then the threshold field is reduced to $\mathcal{E}_{th} \approx 7 \times 10^{-6}$. Use of this \mathcal{E}_{th} , the reduced values of μ_s ($\approx -4.6 \times 10^{-4}$), (5.26), and the observed value of μ_p (≈ 0.016), in (4.69) then gives

$$v \approx 17 \quad , \quad |M_0| \approx 2.0 \quad . \quad (5.29)$$

Because $v \gg |M_0| \geq \chi$, the approximate solutions of (4.72) can be used. With $(|u_s| \beta_s / u_i \beta_i)^{1/2} \approx 0.14$ for this case, (4.72) becomes,

$$\begin{aligned} \mathcal{E}_i(z) \approx & \mathcal{E}_i(0) \left[\exp(-0.53 z) - 0.0038 \exp(-0.016 z) \right] \\ & + 0.0085 \mathcal{E}_s(0) \left[\exp(-0.016 z) - \exp(-0.53 z) \right] \quad . \end{aligned} \quad (5.30)$$

The observed ion-acoustic (idler) wave has a nearly monotonic variation from 0 to 6 cm from the pump wave exciter. According to figure 5.8, the ratio of \mathcal{E}_i to the apparent noise amplitude, \mathcal{E}_{in} , changed approximately from $\mathcal{E}_i/\mathcal{E}_{in} = 1.5$ at $Z = 0$ to $\mathcal{E}_i/\mathcal{E}_{in} = 2.0$ at $Z \approx 19$. It is clear that this observed \mathcal{E}_i behavior is substantially different from that indicated by (5.30).

The discrepancy between theory and experiment for this case may be due to the effects of the finite plasma column size, the noncollinear wave propagation (i.e. $\theta'' \approx 20^\circ$), and some other nonlinear processes, which were neglected in the foregoing theoretical model. In a more realistic model, the parameters used in (4.68) and (4.69) may be modified by these effects. It will be noted, in particular, that because $\mu_i \gg \mu_p \gg |\mu_s|$ in the case of interest here, $\mathcal{E}_i(Z)$ depends most strongly on μ_i (through ν). If we reduce μ_i by a factor of 4 from (5.27) to $\mu_i \approx 0.13$ and $\Gamma_i \approx 0.6 \times 10^{-5}$, with $\mu_s (\approx -4.6 \times 10^{-4})$ already reduced by a factor of 2, then we have $\mathcal{E}_p(0) \approx 2\mathcal{E}_{th}$, $\nu \approx 4.7$, and $|M_0| \approx 2.0$. Using these parameters, (4.72) can be used to estimate \mathcal{E}_i and \mathcal{E}_s as

$$\begin{aligned} \mathcal{E}_i(Z) &\approx \mathcal{E}_i(0) \left[\exp(-0.13 Z) - 0.061 \exp(-0.016 Z) \right] \\ &\quad + 0.038 \mathcal{E}_s(0) \left[\exp(-0.016 Z) - \exp(-0.13 Z) \right] , \\ \mathcal{E}_s(Z) &\approx \mathcal{E}_s(0) \exp(4.6 \times 10^{-4} Z) . \end{aligned} \quad (5.31)$$

Equation (5.31) gives a good match to the observed \mathcal{E}_i with $\mathcal{E}_i(0) : \mathcal{E}_s(0) : \mathcal{E}_{in}$ given approximately as 1:80:1. It is also seen from (5.31) that for $Z \leq 19$, we have $\mathcal{E}_s(Z) \approx \mathcal{E}_s(0) \approx 80 \mathcal{E}_{in}$. To determine if this modification in μ_i is appropriate, it would be necessary to measure $\mathcal{E}_s(Z)/\mathcal{E}_{in}$ in the corresponding experiment.

5.5 Excitation of an Ion-Acoustic Wave by Two Whistlers: Oblique Propagation

In Section 5.4 it was shown that the observed behavior of the idler ion-acoustic wave with wavelength 0.8 cm does not correspond to the collinear three-wave interaction theory in an infinite homogeneous plasma

background. Here, we shall examine the theoretical behavior of \mathcal{E}_1 corresponding to the second observed idler ion-acoustic wave with wavelength roughly equal to 0.16 cm (see figure 5.8). Typical wave-numbers of the idler ion-acoustic and signal whistler waves were found to be

$$k'' \approx 15 \quad , \quad \theta'' \approx 48^\circ \quad , \quad k' \approx 14 \quad , \quad \theta' \approx \pi + 52^\circ \quad . \quad (5.32)$$

Since obliquely propagating waves are involved in this wave interaction process, the linear characteristics of these waves may be very different from their corresponding waves with parallel propagation. Therefore, we shall proceed with a discussion of these obliquely propagating waves.

5.5.1 Obliquely Propagating Ion-Acoustic Waves (I''), $\underline{k}'' = \{k_x'', 0, k_y''\}$

In the hydromagnetic region ($\Omega'' \ll \Omega_c \approx 3.3 \times 10^{-5}$), the linear ion-acoustic wave dispersion properties may be significantly affected by the presence of a sufficiently strong dc magnetic field. For the experimental plasma of interest here, the Alfvén speed, $V_A \approx (\Omega_i \Omega_c)^{1/2} \approx 2.8 \times 10^{-3}$, is much larger than the ion-acoustic speed, $V_a \approx 4.7 \times 10^{-5}$. The ion-acoustic wave dispersion relation, and the polarization vector in the hydromagnetic region, are then (Ginzburg, 1970)

$$\Omega''^2 \approx V_a^2 k''^2 \quad , \quad \underline{e}'' \approx \{0, 1, 0\} \quad . \quad (5.33)$$

Here \underline{e}'' is linearly polarized perpendicular to \underline{k}'' and \underline{B} .

The ion-acoustic wave observed in the experiment, however, had a frequency $\Omega'' \approx 4.9 \times 10^{-4}$, and therefore does not fall into the hydromagnetic region. It is necessary to examine the properties of the wave

in more detail. Under the condition (5.6), and with $\Omega_c^2 \gg \Omega''^2 \gg \Omega_1^2$, we have from (4.40)

$$\begin{aligned}
M_{xx}^{i''} &= M_{yy}^{i''} = M_{zz}^{i''} \approx \frac{1}{\sigma \Omega''^2} , & M_{xy}^{i''} &= M_{yx}^{i''*} \approx \frac{-i\Omega_1}{\sigma \Omega''^3} , \\
M_{xx}^{e''} &\approx \frac{-u^2}{\Omega_c^2} , & M_{yy}^{e''} &\approx -\frac{u^2 + a^2 \sin^2 \theta''}{\Omega_c^2 \cos^2 \theta''} , \\
M_{zz}^{e''} &\approx \frac{1}{\Omega_c^2} \left[a^2 - (a^2 + \tan^2 \theta'')(u^2 + a^2) \right] , & M_{xy}^{e''} &= M_{yx}^{e''*} \approx \frac{-iu^2}{\Omega'' \Omega_c} , \\
M_{xz}^{e''} &= M_{zx}^{e''*} \approx \frac{u^2 + a^2}{\Omega_c^2} \tan \theta'' , & M_{yz}^{e''} &= M_{zy}^{e''*} \approx \frac{-i \tan \theta''}{\Omega'' \Omega_c (u^2 + a^2)} , \\
a^2 &= \frac{\Omega''^2}{\gamma_e v_e^2 K_z''^2} , & b^2 &= \frac{\Omega''^2}{\Omega_c^2} , & u^2 &= 1 + \frac{b^2}{\cos^2 \theta''} . \quad (5.34)
\end{aligned}$$

Although $a^2, b^2 \ll 1$ in (5.34), they are included because the zero order terms are found to cancel each other in the derivation of the dispersion relation and the polarization vector. Substitution of (5.34) in (4.42)-(4.44), we then have

$$\Omega''^2 \approx \frac{K''^2}{(1/v_a^2) - (K_x''^2/v_A^2)} \approx v_a^2 K''^2 , \quad \underline{e}'' \approx \left\{ \sin \theta'' , \frac{i\Omega_1}{\Omega''} \sin \theta'' , \cos \theta'' \right\} , \quad (5.35)$$

revealing a nearly linearly polarized longitudinal wave, since $\Omega'' \sim 15 \Omega_1$. The approximation $\Omega''^2 \approx v_a^2 K''^2$, is appropriate because $1/v_a^2 \sim 30 K_x''^2/v_A^2$. When $\Omega'' \ll \Omega_1$, we note that \underline{e}'' reduces to (5.33) of the hydromagnetic case.

We may now substitute (5.34) and (5.35) in (4.50) to obtain

$$\underline{u}_i \approx \frac{1}{\gamma_e v_e^2 \Omega'' K''^2} , \quad \underline{u}_i \approx v_a \left\{ \sin \theta'' , 0 , \cos \theta'' \right\} . \quad (5.36)$$

When $\theta'' = 0$, (5.36) reduces to the corresponding expressions in (5.9).

Also, we have $\Psi_i = \theta''$ since \underline{u}_i is parallel to \underline{k}'' .

5.5.2 Obliquely Propagating Whistlers (R'), $\underline{k}' = \{k'_x, 0, k'_z\}$

Reduced forms of the plasma polarization constant, $\underline{M}^{s'}$, can be obtained from (5.5) by use of (5.6), and the condition, $\Omega'^2 \gg \gamma_e v_e^2 k'^2$,

$$\underline{M}^{i'} \approx \frac{\underline{I}}{\Omega'^2}, \quad \underline{M}^{e'} \approx \frac{1}{\Omega'^2 - \Omega_c^2} \begin{bmatrix} 1 & i\Omega_c/\Omega' & 0 \\ -i\Omega_c/\Omega' & 1 & 0 \\ 0 & 0 & 1 - \Omega_c^2/\Omega'^2 \end{bmatrix}. \quad (5.37)$$

The dispersion relation and the polarization vector for the oblique whistler wave may be obtained from (4.42)-(4.44) by use of (5.6), (5.37), and the conditions $\Omega' < \Omega_c$ and $\Omega'^2 \Omega_c^2 \sin^4 \theta' \ll 4(\Omega'^2 - 1)^2 \cos^2 \theta'$ (see for example, Helliwell, 1965). We have

$$k'^2 = \Omega'^2 - \frac{\Omega'}{\Omega' + \Omega_c \cos \theta'} \quad \left(\frac{3\pi}{2} > \theta' > \frac{\pi}{2} \right),$$

$$\underline{e}' \approx \left\{ \sin \theta' - \frac{\Omega'}{\Omega_c k'^2 \tan \theta'}, \frac{i\Omega'}{\Omega_c k'^2 \sin \theta'}, \cos \theta' + \frac{\Omega'}{\Omega_c k'^2} \right\}. \quad (5.38)$$

When $\theta' = 0$, \underline{e}' reduces to \underline{e}^R of (5.22). Since $k'^2 \Omega_c \gg \Omega'$, we have $\underline{e}' \approx \{\sin \theta', 0, \cos \theta'\}$, which represents a linearly polarized longitudinal wave, corresponding to a whistler wave that is near the oblique cyclotron resonance $\Omega' \approx -\Omega \cos \theta' \quad (3\pi/2 > \theta' > \pi/2)$.

Use of (5.37) and (5.38) in (4.50) now gives

$$g_s \approx \frac{\Omega'}{2(\Omega'^2 - \Omega_c^2)^2} \left[\left(1 + \frac{\Omega_c^2}{\Omega'^2}\right) \sin^2 \theta' + \left(1 - \frac{\Omega_c^2}{\Omega'^2}\right) \cos^2 \theta' \right] \approx \frac{1}{\Omega_c^2 \Omega' \sin^2 \theta'}$$

$$\underline{g}_s \underline{u}_s \approx \frac{2}{\Omega_c \Omega' K' \sin \theta'} \{-\cos \theta', 0, \sin \theta'\} \quad (5.39)$$

Here, the second equation for \underline{g}_s is obtained by replacing Ω' by $-\Omega_c \cos \theta' (1 - 1/K'^2)$, from (5.38), before eliminating terms involving $1/K'^2$ as small compared with unity. From (5.39), we see that the direction of \underline{u}_s is perpendicular to \underline{K}' and oriented so that $\psi_s = \theta' - \pi/2$.

5.5.3 Synchronism Conditions and Coupling Coefficient

Synchronism conditions: Since the signal and idler waves propagate obliquely, the synchronism conditions become less obvious than those cases shown in figures 5.4, 5.6, and 5.9. The synchronism conditions for this case are shown schematically in figure 5.10. It contains the dispersion curve R from (5.22), for whistler waves with parallel propagation; the dispersion curve R' from (5.38), for whistler waves with oblique propagation near $\theta' \approx \pi + 52^\circ$, and the dispersion curve I'' from (5.35) (plotted with respect to a displaced origin O''), for ion-acoustic waves with oblique propagation near $\theta'' \approx 48^\circ$. Since the whistler signal on the R' curve is nearly at the oblique cyclotron resonance, we have

$$\Omega = \Omega' \approx \Omega_c |\cos \theta'| \gg \Omega'' \quad (5.40)$$

Coupling Coefficient: To obtain the coupling coefficient, $C_{\underline{K}\underline{K}'\underline{K}''}^{RR'I''}$, (4.53) shows that we have to evaluate $\Lambda_{-\underline{K}\underline{K}'\underline{K}''}^{LR'I''}$, since $\underline{e}_{\underline{K}}^R = \underline{e}_{\underline{K}}^L$. This

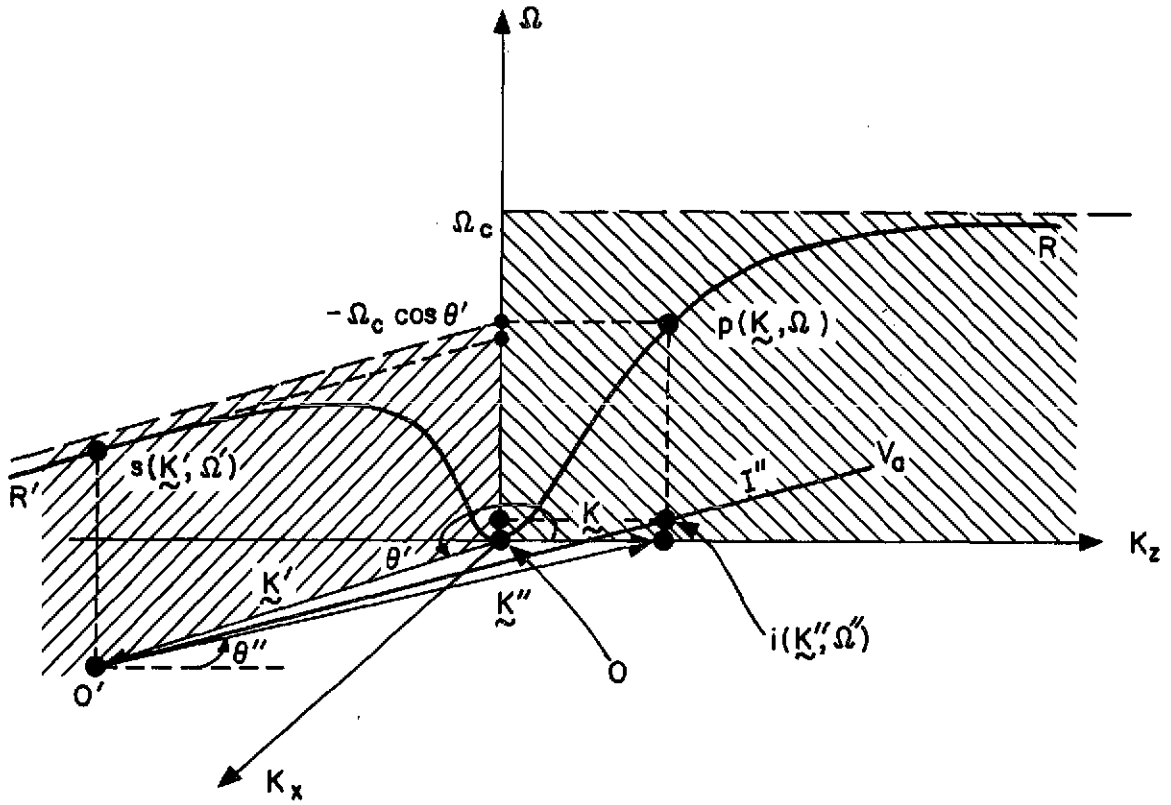


Figure 5.10 The synchronism conditions for interaction among a whistler (R) with parallel propagation as the pump (\underline{k}, Ω), an obliquely propagating whistler (R') as the signal (\underline{k}', Ω'), and an obliquely propagating ion-acoustic wave (I'') as the idler ($\underline{k}'', \Omega''$). The I'' curve is plotted with respect to the displaced origin, O' , to show that the conditions, $\underline{k} = \underline{k}' + \underline{k}''$ and $\Omega = \Omega' + \Omega''$, are satisfied.

requires \tilde{M}^i and \tilde{M}^e for a left-handed circularly polarized wave, which can be obtained from (5.36) as

$$\tilde{M}^i = \tilde{M}^{i*} \Big|_{\Omega' \rightarrow \Omega}, \quad \tilde{M}^e = \tilde{M}^{e*} \Big|_{\Omega' \rightarrow \Omega}. \quad (5.41)$$

Then use of (5.22), (5.34), (5.35), (5.37), (5.38), and (5.41) reduces the expression for $\Lambda_{\tilde{K}\tilde{K}'}^{\text{LR}'} \tilde{I}''$ from (4.46) to a simpler form. With the parameters given in (5.20), (5.32), and (5.40), the significant terms are, in order of decreasing importance

$$\begin{aligned} \Lambda_{\tilde{K}\tilde{K}'}^{\text{LR}'} \tilde{I}'' &\sim -\frac{\Omega}{\Omega'} (\tilde{e}' \cdot \tilde{M}^e \cdot \tilde{e}) (\tilde{K}'' \cdot \tilde{M}^{e'} \cdot \tilde{e}') - \frac{\Omega'}{\Omega} (\tilde{e}'' \cdot \tilde{M}^{e'} \cdot \tilde{e}') (\tilde{K}' \cdot \tilde{M}^e \cdot \tilde{e}) \\ &\quad - \frac{\Omega}{\Omega'} (\tilde{e}' \cdot \tilde{M}^e \cdot \tilde{e}) (\tilde{K}' \cdot \tilde{M}^{e''} \cdot \tilde{e}''). \end{aligned} \quad (5.42)$$

The corresponding coupling coefficient, C_{psi} , is then

$$C_{\text{psi}} = \Lambda_{-\tilde{K}\tilde{K}'}^{\text{RR}'} \tilde{I}'' = \frac{2^{1/2} k'' \sin \theta}{\Omega(\Omega_c - \Omega)} \left[\frac{\Omega_i \Omega_c \sin^2 \theta}{\Omega''^2 (\Omega_c^2 - \Omega^2)} - \frac{1}{\gamma_e v_e^2 k''^2} \right], \quad (5.43)$$

where we have made the additional simplifications to $\Lambda_{-\tilde{K}\tilde{K}'}^{\text{RR}'} \tilde{I}''$, appropriate to the experimental conditions of interest, $k'' \approx k' \gg k_z$, $\theta \approx \theta'' = \theta' - \pi$, $\Omega \approx \Omega' \gg \Omega''$, and $\Omega^2 \gg \gamma_e v_e^2 k''^2$. The first term in the square brackets of (5.43) is the combined contribution of the first two terms of (5.42). Note that (5.43) cannot be reduced to (5.24), for the coupling coefficient among whistlers and ion-acoustic waves with parallel propagation, by making $\theta = 0$. This is because the polarization vector, \tilde{e}' , in (5.38) for the obliquely propagating whistlers, is nearly a linearly polarized longitudinal wave, rather than a circularly polarized transverse wave of (5.22).

5.5.4 Evolution of \mathcal{E}_i and \mathcal{E}_s

For interaction of obliquely propagating waves, the coupled mode equations of (4.79) are required. We make $X = 0$ because only \mathcal{E}_i along the axis of the plasma column were measured (Porkolab, et al., 1972). After substituting the appropriate values from (5.20), (5.21), and (5.32), for $\mu_p (\approx 0.016)$ and $\theta (\approx 50^\circ)$, and using the experimental pump wave amplitude, $\mathcal{E}_p(0) \approx 7.2 \times 10^{-6}$, we obtain from (5.36), (5.39), (5.43), and (4.69),

$$g_s u_{sz} \approx -3.7 \quad , \quad g_i u_{iz} \approx 24 \quad , \quad c = 4.3 \times 10^5 \quad ,$$

$$|M_0| \approx \frac{c \mathcal{E}_p(0)}{\mu_p (g_s |u_{sz}| g_i |u_{iz}|)^{1/2}} \approx 21 \quad . \quad (5.44)$$

The expression for $|M_0|$ is modified from (4.69) by replacing u_s and u_i by u_{sz} and u_{iz} , respectively, following the definitions of (4.77).

To obtain μ_{iz} , we must use (5.27) and $\Psi_i = \theta'' \approx 50^\circ$ in (4.77).

To obtain μ_{sz} , an expression for the damping of an obliquely propagating whistler is required. It is (Ginzburg, 1970)

$$\Gamma_s = \frac{v_{en}}{\omega_{pe}} \frac{\Omega'}{2\Omega'(\Omega_c \cos\theta' + \Omega')^2 + \Omega_c} \quad \left(\frac{3\pi}{2} > \theta' > \frac{\pi}{2} \right) \quad , \quad (5.45)$$

which reduces to $\Omega' v_{en} / \omega_{pe} \Omega_c$ at the cyclotron resonance ($\Omega' \approx -\Omega_c \cos\theta'$). Then use of (5.45), (5.39), and (4.78) yields the value for μ_{sz} . We have

$$\mu_{iz} \approx 0.34 \quad , \quad \mu_{sz} \approx -6.8 \times 10^{-3} \quad . \quad (5.46)$$

By use of (5.46), the value of $\mu_p (\approx 0.016)$, and (4.69), we find

$$\nu = \frac{1}{2\mu_p} (\mu_p - \mu_{sz} + \mu_{iz}) \approx 12 \quad (5.47)$$

According to figure 5.8, we are interested in the region where $z \leq 2$ cm, which corresponds to $Z \leq 5$, and hence $\mu_p Z \ll 1$. Therefore, from (5.44) and (5.47), we have $\chi \approx |M_0| > \nu \gg 1$, so (4.74) can be used to approximate \mathcal{E}_i and \mathcal{E}_s . Substitution of (5.44), (5.46), and (5.47) in (4.74) then gives

$$\begin{aligned} \mathcal{E}_i(Z) &\approx \left[\mathcal{E}_i(0) \cos(0.34 Z) + 0.39 \mathcal{E}_s(0) \sin(0.34 Z) \right] \exp(-0.17 Z) \quad , \\ \mathcal{E}_s(Z) &\approx \left[\mathcal{E}_s(0) \cos(0.34 Z) - 0.39 \mathcal{E}_i(0) \sin(0.34 Z) \right] \exp(-0.17 Z) \quad , \quad (5.48) \end{aligned}$$

which have been plotted in figure 5.11. Note that good agreement of \mathcal{E}_i with the experimental observations has been obtained without changing the value of μ_i given in (5.27). This indicates that this value of μ_i is acceptable as the collisional damping for short wavelength ion-acoustic waves ($K'' \approx 15$). For $K'' \approx 3$, however, the results of Section 5.4 suggest that the long wavelength ion-acoustic wave may have been strongly modified by those factors in the experiment not included in the wave theory of an infinite homogeneous plasma background. Finally, to make the agreement more satisfactory, $\mathcal{E}_s(Z)$ and \mathcal{E}_{in} should be measured in the experiment to see whether the theoretical result in (5.48) is also acceptable.

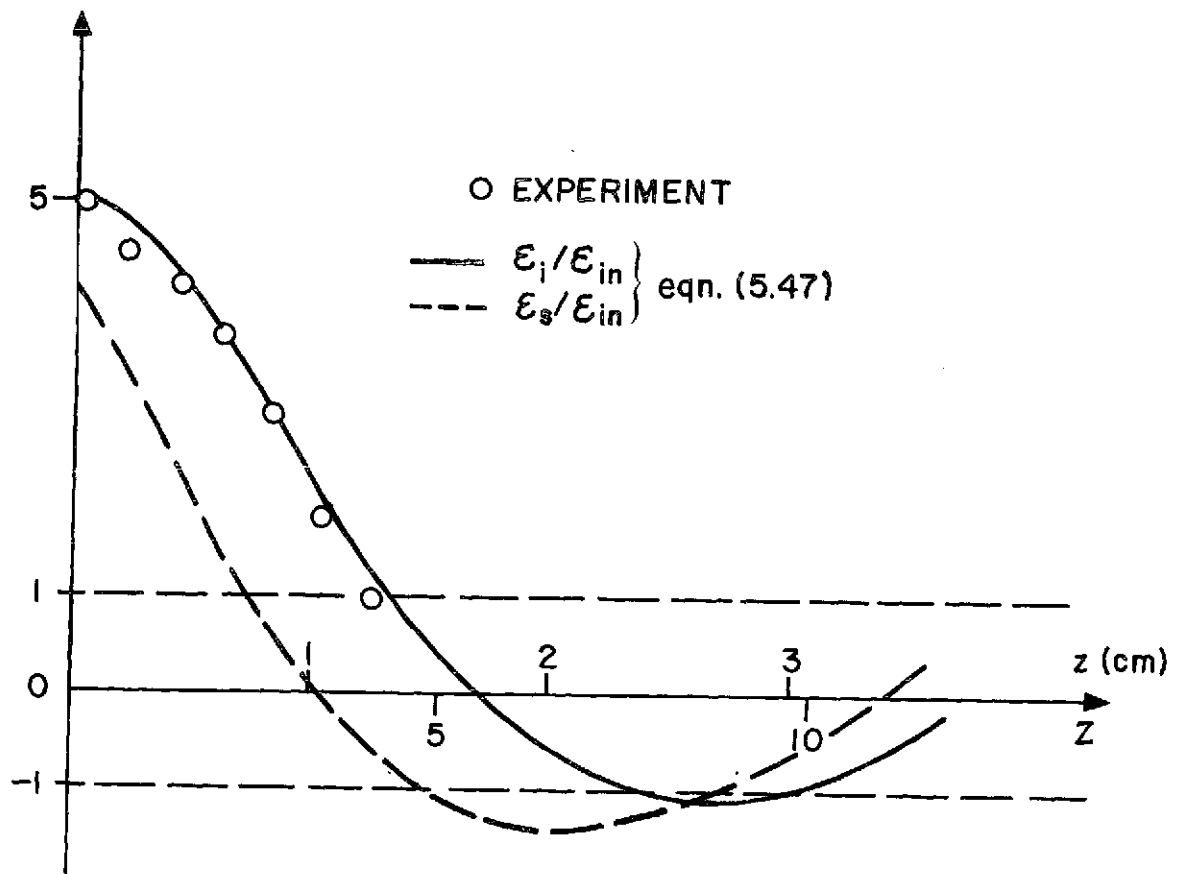


Figure 5.11 Comparison between the observed evolution of the short wavelength idler ion-acoustic wave, deduced from a result obtained by Porkolab, et al. (1972) (figure 5.8), and the theoretical evolution of $\mathcal{E}_i(z)$, according to (5.48), with $\mathcal{E}_i(0) : \mathcal{E}_s(0) : \mathcal{E}_{in}$ given approximately as 5:4:1. The theoretical evolution of the signal whistler wave amplitude, $\mathcal{E}_s(z)$, is subject to future experimental verification.

5.6 Discussion

The verification of linear plasma wave theory imposes only the necessity for verifying the dispersion relation, i.e. measuring \tilde{K} and Ω . In the case of nonlinear wave-wave interaction, it is necessary to establish not only that the linear waves satisfy the dispersion relation, but that also that the synchronism conditions are satisfied and coupling coefficients follow theory. This requires more extended series of measurements, providing details of the amplitude variations of the pump, signal, and idler waves with distance. Few satisfying comparisons between theory and experiment have been made so far.

We have tried to remedy this deficiency to some extent by predicting, in Section 4, the wave amplitude behavior in space, taking into account such practical factors as damping of the pump, signal, and idler waves. In this section we have tested this theory against available experimental results (Ohnuma and Hatta, 1970; Porkolab, et al., 1972). Unfortunately, these experiments do not provide all the data required for a complete quantitative comparison with theory, since they are confined to a qualitative presentation of the idler wave behavior without specifying the observed values of pump threshold, noise intensity, and, in one case (Porkolab, et al., 1972), the signal wave behavior.

The four examples treated have all involved the excitation of ion-acoustic waves, either by other ion-acoustic waves, electron plasma waves, or whistlers. In Section 5.2 we have shown satisfactorily that the observed independence of the location of the idler wave peak from the ion-acoustic pump wave amplitude is due to the increased pump wave damping

with amplitude. In Section 5.3, we have examined the difficulty of exciting an ion-acoustic wave to observable amplitude by injecting two counter-streaming electron plasma waves into a weakly ionized plasma when appreciable collisional damping occurs. In Section 5.4, the predicted idler wave behavior has disagreed with the experimental results. We have shown that the discrepancy can be explained if the effective ion-acoustic wave damping was substantially lower than that due to electron-neutral collisions. Further theoretical and experimental work is necessary to determine whether this discrepancy is caused by inhomogeneous plasma background, oblique wave propagation, or by other nonlinear wave processes. Good agreement has been obtained in Section 5.5, where the excitation of an obliquely propagating ion-acoustic wave of short wavelength was considered. The results have served to demonstrate the need for extensive and detailed measurements to be taken, in which wave and noise amplitudes are thoroughly documented, so that detailed comparisons can be made against theories such as those of Section 4.

It will be realized by the reader that the averaged Lagrangian method is more efficient and versatile than the conventional iterative method for the description of nonlinear wave-wave interaction processes. The results of Sections 4 and 5 can be readily applied to wave interaction processes other than those studied in this section.

6. CONCLUSIONS

The main aim of this thesis has been the establishment and application of Lagrangian methods to magnetoplasmas described by macroscopic equations. This involves derivation of a Lagrangian of which the given macroscopic equations are the Euler-Lagrange equations obtained by applying Hamilton's principle. We have taken two approaches to this problem. In Section 2, we considered the general mathematical inverse problem of the calculus of variations, i.e. the derivation of Lagrangian densities from an arbitrary set of equations. We were able to establish sufficient conditions for systems of first and second order, quasilinear, differential equations, and used these conditions to transform equations apparently not in Euler-Lagrange form to Euler-Lagrange form. As examples, appropriate Lagrangians were obtained for a linear resistive transmission line, and for a linear, collisional, one-dimensional, warm plasma.

For reasons summarized in Section 2.5, it was found that substantial further development is necessary before this general mathematical approach can be used to obtain Lagrangians for plasmas. Consequently, in Chapter 3, a suitable Lagrangian density was obtained for the macroscopic plasma description through energy considerations, taking account of the dual roles played by the dynamic variables, i.e. the macroscopic plasma cell position and electromagnetic potentials. The effects of viscosity, heat conduction, and elastic collisions were included by energy balance arguments. To obtain the corresponding Hamiltonian, the canonical momentum conjugate to the plasma cell position was defined in Lagrangian coordinates, while the canonical momentum conjugate to the electromagnetic vector potential was defined in Eulerian coordinates. The resulting

Hamiltonian was shown to equal the appropriate macroscopic plasma energy at $t = 0$. For completeness, the Hamiltonian corresponding to the Low Lagrangian (Low, 1958), appropriate to the microscopic description of plasmas, was obtained, and shown to equal the appropriate microscopic plasma energy at time t . It was found that, for a collisionless (Vlasov) plasma, the macroscopic Lagrangian could not be obtained by a simple velocity integration of the Low Lagrangian. This fact was traced to the loss of particle discreteness in the macroscopic approximation.

In Section 4, the macroscopic Lagrangian was expanded in terms of small perturbations for the case of scalar pressure and adiabatic compression, and used to study nonlinear three-wave interactions in a homogeneous magnetoplasma. The averaged Lagrangian method was applied to obtain the coupled mode equations. These were extended to include phenomenologically the effects of wave damping. Solutions of the coupled mode equations were given describing the spatial variations of the signal and idler waves under the assumption of a strong but damped pump wave.

In Section 5, these solutions were specialized to experimental conditions involving parametric excitation of ion-acoustic waves by other ion-acoustic waves, by electron plasma waves, and by whistlers. Quantitative comparison with available experimental data demonstrated good agreement for situations in which the propagation characteristics of the uncoupled waves could be measured accurately. Suggestions were given in Section 5.6 for improved theory and experiments to reduce the most significant discrepancies between them.

Although Lagrangian techniques can be used to obtain complete solutions to plasma wave problems, such solutions may not always be necessary. For

example, in determining the oscillation properties of a bounded plasma, the variational properties of the action integral can be used to obtain the resonance frequencies, using only approximate trial functions for the electromagnetic fields and plasma dynamics. We have treated one such problem in Appendix B. The quadratic Lagrangian obtained in Section 4 was specialized to the problem of electrostatic resonances in an inhomogeneous plasma column. The Rayleigh-Ritz procedure was applied directly to the Lagrangian. For a low pressure positive column, it was found that accurate frequencies and eigenfunctions could be obtained efficiently for the first few resonances, provided that appropriate coordinate functions were defined. In contrast to numerical solutions to this problem obtained by Parker, Nickel, and Gould (1964), the variational approach was found to be applicable for the entire range of the ratio (column radius/electron Debye length) without incurring serious numerical instability in the calculation of the first few resonance frequencies.

Some extensions to the work described in this thesis have already been discussed in the individual sections. Among, and in addition to, these are the following:

First, the general mathematical approach of the inverse problem of the calculus of variations was found to be less effective in practice than the more intuitive approach by energy considerations used in Section 3. One of the reasons for this is that the mathematical approach lacks an independent definition of the dynamic variables that represent the degrees of freedom in the corresponding physical problem. It would be of interest to consider a problem in which the dependent (generalized) variables are defined by the physical degrees of freedom before employing the general results of the inverse problem. We have not treated the question of

subsidiary constraints. When these are imposed, the results of Section 2 must be modified to include them.

Second, one of the long-standing difficulties associated with variational principles is that of including energy dissipation effects in a physical system (see for example, Goldstein, 1959). We have only partially succeeded in eliminating this drawback: in Section 2, because of the difficulties associated with solving first order, nonlinear partial differential equations, the examples on dissipative systems were limited to linear equations; in Section 3, by closing the system of energy transfer in the macroscopic plasma, the effects of viscosity, heat conduction, and elastic collisions, were included in the Lagrangian. These effects had to be discarded in Section 4.2, however, because of difficulties involved in obtaining the corresponding perturbation approximations. The phenomenological approach used in Section 4.4.2 includes the wave damping effects in the coupled mode equations, rather than the Lagrangian leading to them. It would be of interest to see if dissipation effects could be included in the Rayleigh-Ritz procedure used in Appendix B. A possible procedure has been suggested by Mikhlin (1964).

Third, the results of Section 4.3 were written completely in terms of quantities representing the linear wave properties: the polarization vector \underline{e} , the plasma polarization constant \underline{M}^s , the frequency Ω , and the wavenumber \underline{K} . In the case of nonlinear wave-wave interactions, the averaged Lagrangian method is equivalent to expressing the approximate Lagrangians in terms of the linear eigenfunctions, $\underline{e} \exp[i(\Omega T - \underline{K} \cdot \underline{X})]$, of the homogeneous plasma. The Manley-Rowe relations, and the synchronism conditions, can then be viewed as results of the orthogonality property of these eigenfunctions. Extensions of the averaged Lagrangian method

to inhomogeneous plasmas should then be straightforward: we would start with the general energy conservation theorem derivable from the approximate Lagrangians obtained in Section 4. A formal substitution of the eigenfunctions, in place of the dynamic variables, could then be made. The orthogonality property of the eigenfunctions should lead to selection rules for the appropriate eigenvectors, and a power balance relationship between the interacting linear modes. In those cases where the exact eigenfunctions are difficult to obtain, the approximate eigenfunctions could be obtained by the Rayleigh-Ritz method used in Appendix B.

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APPENDIX A: SOLUTION OF ULTRAHYPERBOLIC EQUATIONS

To obtain the solution of the homogeneous part of (2.26),

$$\frac{\partial^2 \mathcal{J}}{\partial U_1^\alpha \partial U_j^\beta} + \frac{\partial^2 \mathcal{J}}{\partial U_j^\alpha \partial U_i^\beta} = 0 \quad , \quad (\text{A.1})$$

when $\alpha \neq \beta$ and $i \neq j$, we first convert it to ultrahyperbolic form (Koshlyakov, Smirnov, and Gliner, 1964),

$$\frac{\partial^2 \mathcal{J}}{\partial s_1^2} + \frac{\partial^2 \mathcal{J}}{\partial s_2^2} - \frac{\partial^2 \mathcal{J}}{\partial t_1^2} - \frac{\partial^2 \mathcal{J}}{\partial t_2^2} = 0 \quad , \quad (\text{A.2})$$

by the substitutions,

$$U_i^\alpha = s_1 + t_1 \quad , \quad U_j^\beta = s_1 - t_1 \quad , \quad U_j^\alpha = s_2 + t_2 \quad , \quad U_i^\beta = s_2 - t_2 \quad ,$$

$$\mathcal{J}(U_i^\alpha, U_j^\beta, U_j^\alpha, U_i^\beta) = \mathcal{J}(s_1, s_2, t_1, t_2) \quad . \quad (\text{A.3})$$

Equation (A.2) is similar to the wave equation in a homogeneous two-dimensional medium, but with two-dimensional time coordinates. Its general solution can readily be obtained by separation of variables.

Assuming

$$\mathcal{J} = v_1(s_1) v_2(s_2) w_1(t_1) w_2(t_2) \quad (\text{A.4})$$

leads to the system of ordinary differential equations,

$$\frac{d^2 v_1}{ds_1^2} = a_1 v_1 \quad , \quad \frac{d^2 v_2}{ds_2^2} = a_2 v_2 \quad , \quad \frac{d^2 w_1}{dt_1^2} = b_1 w_1 \quad ,$$

$$\frac{d^2 w_2}{dt_2^2} = b_2 w_2 \quad , \quad a_1 + a_2 = b_1 + b_2 \quad , \quad (A.5)$$

where a_1 , a_2 , b_1 , and b_2 are constants. The explicit solutions of v_1 , etc. are sinusoidal functions. Because no boundary conditions or initial values restrict the solution for \mathcal{J} , the general solution of (A.2) takes the form,

$$\mathcal{J} = \int da_1 da_2 db_1 C(a_1, a_2, b_1) v_1(a_1) v_2(a_2) w_1(b_1) w_2(a_1 + a_2 - b_1) \quad , \quad (A.6)$$

where the weighting function, $C(a_1, a_2, b_1)$, will be restricted by the sufficient conditions of (2.31), (2.32), and (2.34).

APPENDIX B:
VARIATIONAL CALCULATIONS FOR RESONANCE OSCILLATIONS
OF INHOMOGENEOUS PLASMAS

by
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NASA Grant NGL 05-020-176

SU-IPR Report No. 548
November 1973

Institute for Plasma Research
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ABSTRACT

In this paper, the electrostatic resonance properties of an inhomogeneous plasma column are treated by application of the Rayleigh-Ritz method. In contrast to Parker, Nickel, and Gould (1964), who carried out an exact computation, we have used a description of the rf equation of motion and pressure term that allows us to express the system of equations in Euler-Lagrange form. The Rayleigh-Ritz procedure is then applied to the corresponding Lagrangian to obtain approximate resonance frequencies and eigenfunctions. An appropriate set of trial coordinate functions is defined, which leads to frequency and eigenfunction estimates in excellent agreement with the work of Parker, et al. (1964).

1. INTRODUCTION

This paper is concerned with the use of the Rayleigh-Ritz procedure to estimate the electron resonance frequencies of a warm inhomogeneous plasma column. This procedure has been extensively applied to single self-adjoint equations with great success (Mikhlin, 1964). For a system of equations, however, theoretical extensions have been noted only for the case of elliptic equations (Mikhlin, 1965). For the equations to be used here, which are not elliptic, it will be shown that accurate resonance frequencies can be predicted, provided that a certain set of coordinate functions is defined.

Previous theoretical treatments of the electron resonance problem give predictions which agree well with experimental data. However, these approaches have encountered difficulties stemming from the inhomogeneous electron density profile. For example, Parker, Nickel, and Gould (1964) solved numerically an appropriate fourth-order differential equation for the rf potential in a cylindrical positive column. Because of the exponential nature of the solutions in the cutoff region, their calculations were limited by the condition, $a^2/\lambda_D^2 \leq 4500$, where a is the column radius, $\lambda_D^2 (= \epsilon_0 kT/n_0 e^2)$ is the mean-squared Debye length, and \bar{n}_0 is the mean electron density in the column. Baldwin (1969) used a kinetic model in the low-temperature approximation to obtain the external admittance of a cylindrical plasma capacitor. The appropriate differential equation was solved after using inner-outer expansions connected through the resonance region. A WKB description was used for the inner region, where resonant waves are essentially evanescent, while a travelling wave description was used for the outer region, where Landau damping is important. Because the wave nature of the solutions was assumed in the outer region,

this theory is appropriate only for higher order resonances. Similar difficulties have been shown to occur in the simpler one-dimensional model (Harker, Kino, and Eitelbach, 1968; Miura and Barston, 1971; Peratt and Kuehl, 1972).

Variational methods offer an attractive alternative to these treatments. They have been used previously to estimate plasma resonance frequencies with simplified trial functions. Resonances of a cold inhomogeneous plasma were treated by Crawford and Kino (1963). Using the variational principle established by Sturrock (1958), Barston (1963) approximated the dispersion relations for wave propagation along an infinite cold plasma slab, and along the interface between two semi-infinite, counter-streaming cold plasmas. Some general features of the guided waves on a cold, transversely inhomogeneous plasma column in an axial magnetic field were studied by Briggs and Paik (1968). These papers (Crawford and Kino, 1963; Barston, 1963; Briggs and Paik, 1968) show that, with appropriate variational principles and judicious choices of trial functions, useful results can be obtained with relative ease by the variational approach.

A theoretical variational formulation for the electrostatic resonance oscillations of a warm, inhomogeneous plasma column in a dc electric or magnetic field of arbitrary direction was presented by Barston (1965) with the adiabatic index, γ , taken as unity. The variational principle to be presented here, however, is not restricted in the values of γ . One important feature in Barston's (1965) analysis is that the rf electric potential was treated as the solution of the rf Poisson equation, with the rf electron density considered given. It will be seen that the coordinate functions to be used here are defined in a similar fashion. A variational method of the Rayleigh-Ritz type has

been applied successfully by Dorman (1969) to a one-dimensional, warm, and field-free plasma with arbitrary dc density profile. A single second order differential equation for the electric field was obtained, and shown to have hermitian operators. The variational principle to be used here differs from Dorman's (1969) in that we are dealing directly with a system of Euler-Lagrange equations. In so doing, we can keep down the order of the equations, and are able to consider warm inhomogeneous plasmas in more than one dimension.

In this paper, we shall show that by appropriate definitions of the rf equation of motion and pressure term, the equations of the hydrodynamic model used by Parker, et al. (1964) become Euler-Lagrange equations. This will enable us to demonstrate the effectiveness of the Rayleigh-Ritz procedure in estimating the resonance frequencies of an inhomogeneous plasma. The associated numerical method mainly involves evaluations of definite integrals and solutions of finite algebraic eigenvalue equations, and is applicable over the entire range of $a^2/\lambda_D^2 \gg 1$ for estimating the first few resonance frequencies.

In most of the papers that deal with the electrostatic resonance problem (Crawford and Kino, 1963; Parker et al., 1964; Harker et al., 1968; Baldwin, 1969; Dorman, 1969; Miura and Barston, 1971; Peratt and Kuehl, 1972), it is assumed that the rf plasma current normal to the glass wall is zero. However, in the low temperature limit, $a^2/\lambda_D^2 \rightarrow \infty$, the main resonance frequency seems to agree with that of cold plasma theory, in which the normal rf plasma current is retained. We consider this problem and show that the resulting difference in predicted resonance frequencies is negligibly small for low pressure positive columns.

In §2, we present the basic equations, the corresponding Lagrangian, and the procedure to be applied in the variational approach. In §3, the numerical methods are explained before comparing computations with those of Parker et al. (1964). The paper concludes with a brief discussion in §4.

2. THEORY

For a low pressure positive column, moment equations with scalar pressure and negligible heat conduction are appropriate when the wave phase velocity is much larger than the thermal speed. For the first few electrostatic resonances, the wave phase velocity may be scaled to $\omega_p a$, where $\omega_p [= (e^2 n_0(0)/m\epsilon_0)^{1/2}]$ is the axial plasma frequency. Thus we require $a^2/\lambda_D^2 \gg 1$. A stationary ion background will be assumed, since we are interested only in electron resonances. Dissipation due to collisions, and Landau damping, will be neglected. Our analysis will consequently be valid only for the first few resonances. Also, the analysis will be quasi-static. Apart from some differences in definition of the rf equation of motion and pressure term, the equations we shall use are essentially those used by Parker, et al. (1964). The equations are generalized here to include dc magnetic field, \underline{B}_0 , and electron drift velocity, \underline{v}_0 . We have,

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (n\underline{v}) &= 0, & \epsilon_0 \nabla \cdot \underline{E} + e(n - n_I) &= 0, \\ mn \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) + \nabla P + en (\underline{E} + \underline{v} \times \underline{B}) &= 0 \quad \left(\text{at } \underline{r}_0 + \underline{\xi} \right). \end{aligned} \quad (1)$$

Specialized to small perturbations, these reduce to the dc equations,

$$\begin{aligned} \nabla \cdot (n_{0-0} \underline{v}_{0-0}) &= 0, & \epsilon_0 \nabla \cdot \underline{E}_{0-0} + e(n_{0-0} - n_I) &= 0, \\ mn_{0-0} \underline{v}_{0-0} \cdot \nabla \underline{v}_{0-0} + \nabla P_{0-0} + en_{0-0} (\underline{E}_{0-0} + \underline{v}_{0-0} \times \underline{B}_{0-0}) &= 0 \quad (\text{at } \underline{r}_0), \end{aligned} \quad (2)$$

and rf equations,

$$\begin{aligned} \frac{\partial n_1}{\partial t} + \nabla \cdot (n_{0-1} \underline{v}_{0-1} + n_{1-0} \underline{v}_{1-0}) &= 0, & \epsilon_0 \nabla \cdot \underline{E}_1 + en_1 &= 0, \\ mn_{0-0} \ddot{\underline{\xi}} + (\nabla \cdot \underline{\xi}) \nabla P_{0-0} + \nabla P_1 + \underline{\xi} \cdot \nabla (\nabla P_{0-0}) \\ + en_{0-0} (\underline{\xi} \cdot \nabla \underline{E}_{0-0} + \underline{E}_1 + \underline{\xi} \times \underline{B}_{0-0} + \underline{v}_{0-0} \times \underline{B}_1) &= 0 \end{aligned} \quad (3)$$

where $\dot{\xi} = d\xi/dt = \partial\xi/\partial t + \underline{v}_0 \cdot \nabla \xi$. In these equations, m and $-e$ are the electron mass and charge; n and P are the electron density and pressure; \underline{E} is the electric field; n_I is the ion density; ϵ_0 is the vacuum permittivity; and ξ is the perturbation displacement for the electrons (Figure 1).

The magnitude of \underline{v}_0 is relatively small in the plasma region, but increases in the sheath region from the ion-acoustic speed to roughly the electron thermal speed at the glass wall (Self, 1963; Parker, 1963). We shall consequently neglect it in our analysis. However, due to the presence of non-zero \underline{v}_0 , n_0 , and an rf electric field at the wall, a non-zero rf normal current term arises, and hence an rf surface charge term. The inclusion of this surface charge term, in the cases where the electron rf excursion exceeds the Debye length, is equivalent to the use of the dielectric model for a cold plasma column (Crawford, 1965). Further discussion of this surface charge term will be given in §3.4. With \underline{v}_0 neglected, the following relations become appropriate

$$\begin{aligned} n_0(\underline{r}) &= n_0(0)f(\underline{r}), & f(\underline{r}) &= \exp[-e\varphi_0(\underline{r})/\kappa T_e], \\ n_1 &= -\nabla \cdot (n_0 \underline{\xi}), & \underline{v}_1 &= \partial \underline{\xi} / \partial t, \end{aligned} \quad (4)$$

where κ is the Boltzmann constant, $\varphi_0(\underline{r})$ is the dc column potential,

$$\underline{E}_0(\underline{r}) = -\nabla \varphi_0(\underline{r}), \quad (5)$$

and the first equation of (3) has been used to obtain n_1 . The rf electron pressure, P_1 , is determined by the adiabatic equation of state,

$$P(\underline{r}_0 + \underline{\xi}) / P_0(\underline{r}_0) = n(\underline{r}_0 + \underline{\xi})^\gamma / n_0(\underline{r}_0)^\gamma, \quad P_0 = n_0 \kappa T_e. \quad (6)$$

The form of (6) can be understood by reference to Figure 1, and follows from the fact that when a cell is displaced from \underline{r}_0 to $\underline{r}_0 + \underline{\xi}$,

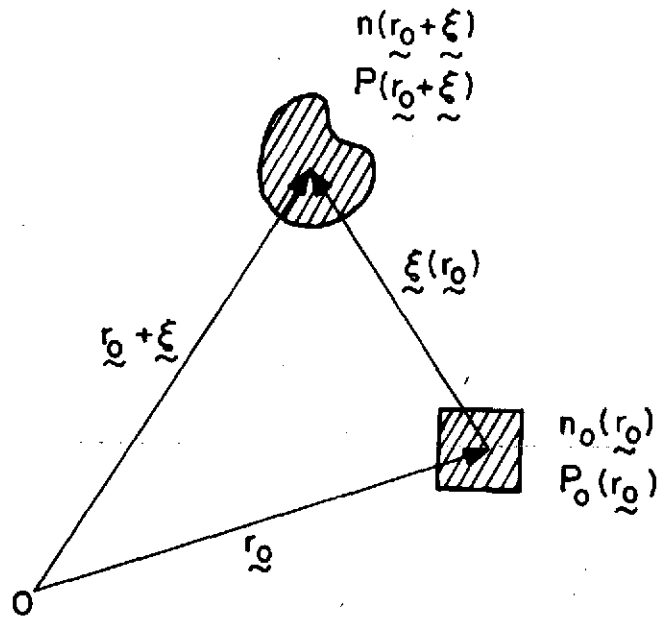


FIG. 1. Definition of plasma perturbation.

P is defined for the given cell, rather than for the given position, \underline{r}_0 . The adiabatic equation of state must consequently be applied to the same cell, before and after the displacement. Using the usual definition of perturbations,

$$P(\underline{r}) = P_0(\underline{r}) + P_1(\underline{r}), \quad n(\underline{r}) = n_0(\underline{r}) + n_1(\underline{r}), \quad (7)$$

and (4), we obtain

$$P_1 = -\gamma P_0 \nabla \cdot \underline{\xi} - \underline{\xi} \cdot \nabla P_0. \quad (8)$$

The rf force law in (3) is obtained by comparing the force laws in (1) and (2) in the same fashion (Newcomb, 1962).

It is now straightforward to use (4), (5), and (8) to rewrite (3) in terms of only $\underline{\xi}$ and the rf potential, φ_1 ($-\nabla\varphi_1 = \underline{E}_1$),

$$\begin{aligned} mn_0 \ddot{\underline{\xi}} - (\gamma-1) \nabla P_0 \nabla \cdot \underline{\xi} - \nabla \underline{\xi} \cdot \nabla P_0 - \gamma P_0 \nabla (\nabla \cdot \underline{\xi}) \\ - en_0 (\nabla\varphi_1 + \underline{\xi} \cdot \nabla \nabla\varphi_0 - \underline{\xi} \times \underline{B}_0) = 0, \\ \epsilon_0 \nabla^2 \varphi_1 + e \nabla \cdot (n_0 \underline{\xi}) = 0. \end{aligned} \quad (9)$$

Equations in (9) can be Fourier-transformed, normalized, and expressed in a cylindrical coordinate system, (r, θ) , for a column of cylindrical symmetry,

$$\begin{aligned} \Omega^2 f \zeta_r - \Omega_c \Omega f \zeta_\theta + \Lambda_D^2 \left[\gamma \left(\zeta_r' + \frac{1}{R} \zeta_r - \frac{\ell}{R} \zeta_\theta \right) + \frac{1}{R} (\ell \zeta_\theta - \zeta_r) \right] f' \\ + \Lambda_D^2 \gamma f \left(\zeta_r' + \frac{1}{R} \zeta_r - \frac{\ell}{R} \zeta_\theta \right)' + f (\Phi_0'' \zeta_r + \Phi_1') = 0, \end{aligned} \quad (10)$$

$$\begin{aligned} \Omega^2 f \zeta_\theta - \Omega_c \Omega f \zeta_r + \Lambda_D^2 f' \frac{1}{R} (\ell \zeta_r - \zeta_\theta) + \Lambda_D^2 \gamma f \left(\zeta_r' + \frac{1}{R} \zeta_r - \frac{\ell}{R} \zeta_\theta \right) \\ + f \left(\frac{1}{R} \Phi_0' \zeta_\theta + \frac{\ell}{R} \Phi_1 \right) = 0, \end{aligned} \quad (11)$$

$$(R\hat{\phi}_1')' - \frac{\ell^2}{R} \hat{\phi}_1 + (Rf\zeta_r)' - \frac{\ell}{R} f \zeta_\theta = 0, \quad (12)$$

where the derivative with respect to R is denoted by $(')$ and

$$\begin{aligned} \hat{\phi}_0(0) &= 0, \quad \underline{\zeta} = \hat{i}_r \zeta_r + \hat{i}_\theta (i\zeta_\theta), \\ \hat{\phi}_1(R, \theta, T) &= \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} d\Omega \hat{\phi}_1'(R) \exp i(\Omega T + \ell \theta), \end{aligned} \quad (13)$$

with \hat{i}_1 denoting unit vectors. The normalized quantities are defined as

$$\begin{aligned} R &= r/a, \quad T = \omega_p t, \quad \Omega = \omega/\omega_p, \\ \Omega_c &= eB_0/m\omega_p, \quad \underline{\zeta} = \underline{\xi}/a, \\ \Lambda_D^2 &= \lambda_D^2/a^2 = \epsilon_0 \kappa T_e / n_0(0) e^2 a^2, \quad \hat{\phi} = \varphi \epsilon_0 / n_0(0) e a^2, \end{aligned} \quad (14)$$

and a static axial magnetic field, B_0 , has been included.

2.1 Lagrangian Density

The forms of the force law in (3), and the rf pressure in (7), represent the important differences from the paper by Parker, et al. (1964) in that they make (9), as well as (10)-(12), systems of Euler-Lagrange equations without having to restrict the values of γ (Barston, 1965). In one of the models used by Dorman (1969), an appropriate pressure term similar to (7) was used without the benefit of the rf force law in (3). As a result, he was able to establish the variational principle only for the one-dimensional case.

The Lagrangian corresponding to (10)-(12) can be shown to be the following, by straightforward application of Hamilton's variational principle,

$$I_2 = \sum_{\ell=-\infty}^{\infty} \int_0^{\infty} d\Omega \mathcal{L}_2(\Omega, \ell) , \quad \mathcal{L}_2(\Omega, \ell) = \Omega^2 A - \Omega_c \Omega B + H ,$$

$$A = \int_0^1 f R dR \left(|\zeta_r|^2 + |\zeta_\theta|^2 \right) , \quad B = \int_0^1 f R dR \left(\zeta_r \zeta_\theta^* + \zeta_r^* \zeta_\theta \right) ,$$

$$H = -\Lambda_D^2 (\gamma T + T') - V_0 S - I + F + F' < 0 ,$$

$$T = \int_0^1 f R dR \left| \zeta_r' + \frac{1}{R} \zeta_r - \frac{\ell}{R} \zeta_\theta \right|^2 + \frac{1}{2} [f \zeta_r^* (\zeta_r' + \zeta_r - \ell \zeta_\theta) + \text{c.c.}]_{R=1} ,$$

$$T' = \int_0^1 f R dR \left[\frac{1}{R} \zeta_r' (\ell \zeta_\theta^* - \zeta_r^*) + \frac{1}{R} \zeta_\theta' (\ell \zeta_r^* - \zeta_\theta^*) + \text{c.c.} \right]$$

$$+ \frac{1}{2} [f \zeta_r^* (\ell \zeta_\theta - \zeta_r) + f \zeta_\theta^* (\ell \zeta_r - \zeta_\theta) + \text{c.c.}]_{R=1} ,$$

$$S = \int_0^1 f R dR \left(\phi_0'' |\zeta_r|^2 + \frac{1}{R} \phi_0' |\zeta_\theta|^2 \right) ,$$

$$I = - \int_0^1 f R dR \left(\zeta_r \phi_1'^* + \frac{\ell}{R} \zeta_\theta \phi_1^* + \text{c.c.} \right) , \quad F = \int_0^1 R dR \left(|\phi_1'|^2 + \frac{\ell^2}{R^2} |\phi_1|^2 \right)$$

$$F' = \int_0^{R_b} \epsilon_g R dR \left(|\phi_1'|^2 + \frac{\ell^2}{R^2} |\phi_1|^2 \right) + \int_{R_b}^{R_c} R dR \left(|\phi_1'|^2 + \frac{\ell^2}{R^2} |\phi_1|^2 \right) ,$$

$$R_b = \frac{b}{a} , \quad R_c = \frac{c}{a} , \quad (15)$$

where c.c. denotes complex conjugate. Equation (15) is appropriate to the configuration shown in Figure 2 of a concentric metal cylinder surrounding a glass tube, of relative permittivity ϵ_g , that contains the plasma column. In the expressions for T and T' , the boundary terms are included to modify the natural boundary conditions on ζ_γ and ζ_θ (see for example, Courant and Hilbert, 1953) that would otherwise be unphysical. In the expressions for H and S , $\phi_0(r)$ and V_0 are defined as

$$\phi_0(r) = -\Phi_0(r)/V_0, \quad V_0 = -\Phi_0(a).$$

Substitution of the exact solutions of (10)-(12) would make $\mathcal{L}_2(\Omega, \ell)$ zero. We see from (15) that, for negligible Ω_c , Λ_D , and V_0 , the resonance frequencies are determined essentially by the values of I , F , and F' . Since V_0 is approximately proportional to Λ_D (Self, 1963; Parker, 1963), the effect of higher electron temperature is to raise each resonance frequency. When $\Omega_c \neq 0$, the roots of $\mathcal{L}_2(\Omega, \ell) = 0$ are

$$\Omega_{1,2} = \Omega_c B/2A \pm [(\Omega_c B/2A)^2 - H/A]^{1/2}. \quad (16)$$

Since Ω and $-\Omega$ are indistinguishable in experimental observations, we see that all the resonance frequencies are predicted by (16) to split in two, in agreement with the theoretical results of Barston (1965), and Vandenplas and Messiaen (1965). For sufficiently small Ω_c , when the values of B and A are not greatly affected by the presence of an axial static magnetic field, the amount of the split, $|\Omega_c B/A|$, will be proportional to Ω_c . This is in agreement with the observed splitting character of the main dipole resonance frequencies (Crawford, Kino, and Cannara, 1963; Messiaen and Vandenplas, 1962). Furthermore, since $A \geq |B|$, where the equal sign applies when $\zeta_\gamma \equiv \zeta_\theta$, this split is predicted to be always less than or equal to Ω_c .

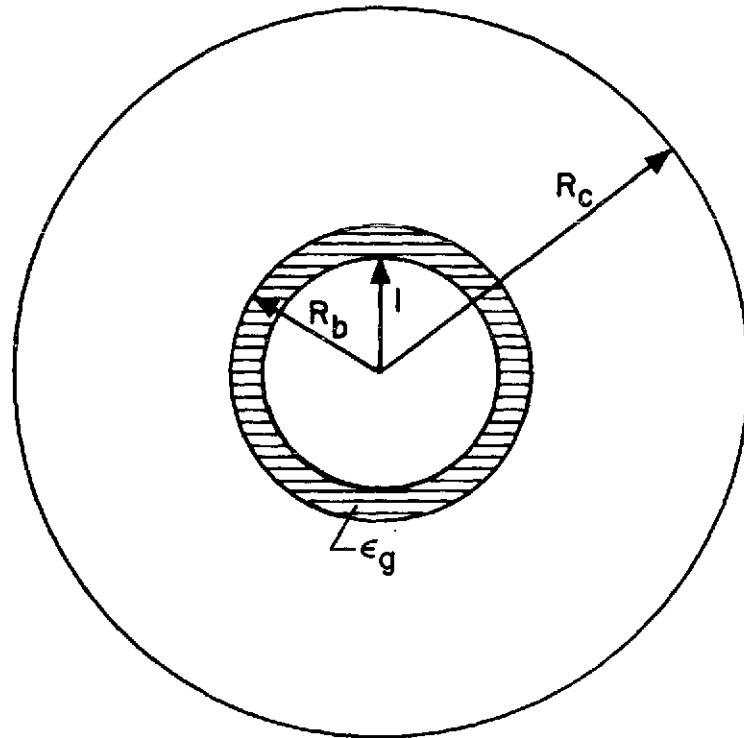


FIG. 2. Plasma column geometry.

2.2 Rayleigh-Ritz Procedure and Coordinate Functions

For a single linear Euler-Lagrangian equation, the Rayleigh-Ritz procedure is efficient in obtaining an approximate solution, by use of a weighted summation of a set of judiciously chosen coordinate functions. These coordinate functions must be linearly independent and complete, and satisfy the boundary conditions specified by the problem. The weighting coefficients that appear in the approximated Lagrangian are varied independently. This results in a system of algebraic equations that take the place of the original equation. Theoretically, better approximations can be obtained by using more coordinate functions. When eigenvalues are involved, the approximate eigenvalues always converge to the exact values from above (see for example, Mikhlin, 1964).

Much less attention has been given to the analogous problem for a system of linear Euler-Lagrangian equations, e.g. (10)-(12), though the theoretical extension of the variational method to a system of second order elliptic equations has been mentioned by Mikhlin (1965): the way to set up the corresponding coordinate functions is similar to that for the single equation case, i.e. the coefficients of each dependent variable are varied independently.

For our problem, in which (10)-(12) are not in elliptic form, the coordinate functions and the coefficients must be more restricted. In the Appendix to the paper it is shown that acceptable estimates of resonance frequencies and eigenfunctions can be obtained for our problem provided that the coordinate functions chosen for each dependent variable are related by (10)-(12).

By expanding ζ_r , ζ_θ , and ϕ_1 in power series of R , and substituting in (10)-(12), we see that for small R , $\zeta_r \propto \zeta_\theta \propto R^{\ell-1}$, and $\phi_1 \propto R^\ell$.

Thus, for $\ell \geq 1$, the solutions are well behaved at $R = 0$. If we also choose even functions for $f(R)$ and $\phi_0(R)$, then ζ_r and ζ_θ are even in R , and ϕ_1 is odd in R , when ℓ is odd, and vice versa. Since there are no other singularities in (10)-(12), polynomials in R constitute appropriate coordinate functions for our problem. For convenience, the coordinate functions chosen for ζ_r will be

$$\zeta_{rj} = R^{\ell-1} - R^{\ell+2j-1} \quad (j = 1, 2, \dots), \quad (17)$$

which conform to the usual assumption of zero normal rf current, since $\zeta_{rj}(1) = 0$.

Rather than choosing $\zeta_{\theta j}$ and $\phi_{1,j}$ independently, we must determine them via the original differential equations and (17). After eliminating ϕ_1 in (10) and (11), it follows by using the second equation of (4) that

$$R\Omega^2 (R\zeta'_\theta + \zeta_\theta - \ell\zeta_r) = [R\Omega_c^2 + \ell(\gamma-1)\phi_0] (R\zeta'_r + \zeta_r - \ell\zeta_\theta). \quad (18)$$

With a given expression for $\phi_0(r)$, and an assigned value of Ω , e.g. $\Omega = 1$, $\zeta_{\theta j}$ can then be easily determined for any given ζ_{rj} .

An immediate question arises concerning the dependence of the resulting variational estimate of resonance frequencies on the size of Ω chosen arbitrarily here. We have found that the first few resonance frequencies do not change by more than $10^{-4} \omega_p$ when Ω in (18) changes from 0.4 to 1 for all the values of $1/\Lambda_D^2$ used in this paper. If this were not the case, an iterative procedure would have to be used, i.e. the resulting variational estimate of Ω would have to be used in (18) to obtain a new set of $\zeta_{\theta j}$. These would be used in turn to obtain improved frequency estimates, and so on.

The corresponding coordinate function, $\phi_{1,j}$, is obtained by solving (12) (Barston, 1965), with conditions of continuity of potential and normal displacement across the boundaries defined in Figure 2. According to the

discussion given in the Appendix, identical coefficients are assigned to each set of coordinate functions,

$$\zeta_r = \sum_{j=1}^N a_j \zeta_{rj}, \quad \zeta_\theta = \sum_{j=1}^N a_j \zeta_{\theta j}, \quad \phi_1 = \sum_{j=1}^N a_j \phi_{1,j}, \quad (19)$$

before substitution in the Lagrangian, $\mathcal{L}_2(\Omega, \ell)$, of (15). The resulting Lagrangian then gives the algebraic Euler-Lagrange equation below,

$$\sum_{j=1}^N (\Omega^2 A_{ji} - \Omega \Omega_c B_{ji} + H_{ji}) a_j = 0 \quad (i = 1, 2, \dots, N),$$

$$A_{ij} = \int_0^1 r R dr (\zeta_{ri} \zeta_{rj} + \zeta_{\theta i} \zeta_{\theta j}), \quad (20)$$

where A_{ij} , B_{ij} , and H_{ij} are the matrix elements of the integrals, A, B, and H, given in (15), respectively, and are obtained by substituting the coordinate functions in a fashion given by the above A_{ij} expression. Equation (20) can be transformed into a generalized eigenvalue problem (Dorman, 1969)

$$\sum_{j=1}^{2N} (\bar{\Omega} A_{ij} - \bar{B}_{ij}) b_j = 0 \quad (i = 1, 2, \dots, 2N),$$

$$b_j = (a_i, \Omega a_i) \quad (j = 1, 2, \dots, 2N, \quad i = 1, 2, \dots, N),$$

$$\bar{A} = \begin{pmatrix} -H & 0 \\ 0 & A^+ \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 & -H \\ -H^+ & \Omega_c B^+ \end{pmatrix}, \quad (21)$$

with superscript + signifying the transposition of a matrix. Equation (21) is now solvable by standard computer codes.

3. NUMERICAL METHODS AND RESULTS

The computing procedure is straightforward:

- (i) read in physical and computational parameters,
- (ii) compute coordinate functions $\zeta_{\theta j}$, $\phi_{i,j}$, and all other functions appearing in the Lagrangian, $\mathcal{L}_2(\Omega, \ell)$, at intervals Δr ,
- (iii) compute $\mathcal{L}_2(\Omega, \ell)$ by Simpson's rule to obtain A_{ij} , B_{ij} , and H_{ij} , and
- (iv) solve (21) for Ω and a_j , and compute relevant eigenfunctions for N , $N-1$, and $N-2$ coordinate functions.

3.1. Approximate DC Density Profile

A density profile which approximates Parker's results (Parker, 1963), and is convenient for both analytical and numerical manipulation, is given by

$$f(R) = \exp[-\eta_w \phi_0(R)] , \quad \phi_0(R) = \beta R^2 + (1-\beta)R^h , \quad (22)$$

where $h(> 2)$ is an even integer, $\beta < 1$, and $\eta_w [= v_0/\Lambda_D^2]$ is the wall value of the potential function, $\eta(r)$, used by Self (1963) and Parker (1963). The particular form of (22) is used to justify the choice of coordinate functions of (17). Furthermore, with the use of (22), $\zeta_{\theta j}$ and $\phi_{1,j}$ can now be solved analytically in terms of power series in R , in addition to the numerical solutions of (12) and (18). Comparison of the solutions by the two methods will provide an estimate of the degree of accuracy achieved in obtaining $\zeta_{\theta j}$ and $\phi_{1,j}$.

The values of β and h are varied until $f(r)$ best approximates, by least-square deviation, the profile given by Parker for specified values of Λ_D and η_w . The resulting profiles are shown in Figure 3, and the corresponding values of β and h are given in Table 1. It will be seen that as $1/\Lambda_D^2$ increases in value, (22) decreases in accuracy because

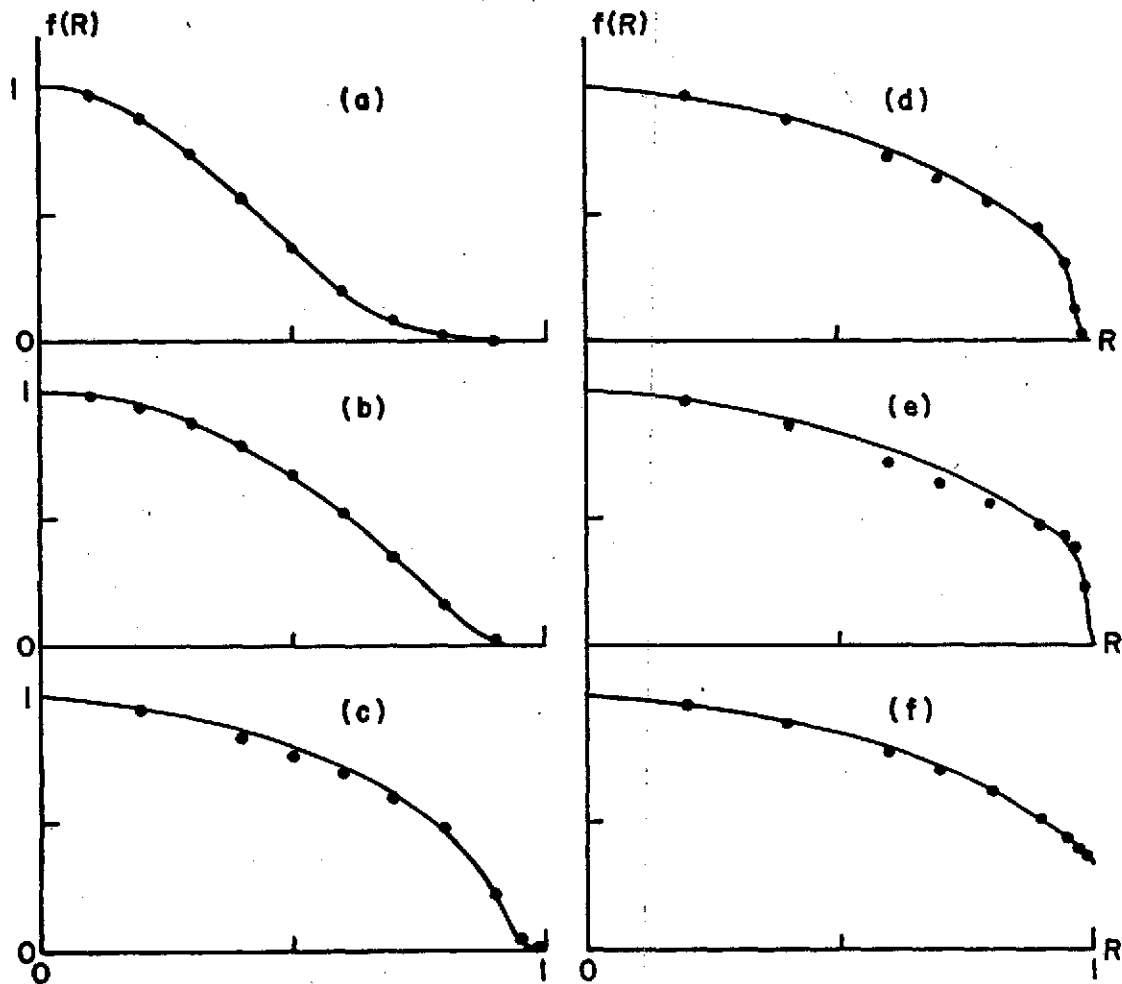


FIG. 3. Comparison between (22) (indicated by dots), and Parker's density profile, for the parameters given in Table 1.

Table 1

Parameters used in (22), and Figure 3, for a mercury-vapor positive column. The conversion between $1/\Lambda_D^2$ and $\overline{1/\Lambda_D^2}$ is obtained by the calculations of Parker (1963).

	$1/\Lambda_D^2$	$\overline{1/\Lambda_D^2}$	η_w	β	h	$^* \Sigma \Delta^2, \dagger \Sigma \Lambda^2 R$
a	3.4×10^2	7.2×10^1	6.72	0.447	4	$^* 6.2 \times 10^{-4}$
b	1.3×10^3	5.1×10^2	6.60	0.221	8	$^* 1.3 \times 10^{-4}$
c	8.2×10^3	4.5×10^3	6.52	0.159	20	$\dagger 6.2 \times 10^{-4}$
d	6.7×10^4	4.3×10^4	6.44	0.142	54	$\dagger 2.9 \times 10^{-3}$
e	6.2×10^5	4.2×10^5	6.40	0.141	156	$\dagger 7.5 \times 10^{-3}$
f	∞	∞	1.08	0.644	12	$\dagger 1.3 \times 10^{-4}$

of the increasing degree of steepness displayed by the density profile in the sheath region. At zero temperature, the sheath is omitted and an accurate approximation can again be obtained. The approximation of (22), however, was found to be sufficient to give accurate frequency and eigenfunction estimates for all of the values of $1/\Lambda_D^2$ listed in Table 1.

3.2. Solutions for $\zeta_{\theta j}$ and $\phi_{1,j}$

By the use of (14), (15), and (19), $\zeta_{\theta j}$ can be put in the form of a rapidly converging series,

$$\zeta_{\theta j} = R^{\ell-1} - \sum_{i=1}^{\infty} \eta_{j+1,i} R^{\ell+2j+(i-1)(h-2)-1},$$

$$\eta_{j,1} = \frac{\Omega^2 + (\ell+2j-2) \left[\frac{\Omega}{\ell} - 2\beta(\gamma-1)V_0 \right]}{\frac{\Omega^2}{\ell}(\ell+2j-2) + \Omega_c \Omega - 2\beta(\gamma-1)\ell V_0},$$

$$\eta_{j,2} = \frac{(\gamma-1)(1-\beta)hV_0 [\ell a_{j,1}^{-(\ell+2j-2)}]}{\frac{\Omega^2}{\ell^2}(\ell+2j+h-4) + \Omega_c \Omega - 2\beta(\gamma-1)\ell V_0},$$

$$\eta_{j,i} = \frac{\ell^2(\gamma-1)(1-\beta)hV_0}{i\Omega^2(h-2)} \eta_{j,i-1} \quad (i \geq 3). \quad (23)$$

Since the maximum value of V_0 of interest is roughly 0.02 and $\Omega \sim 1$, one needs at most five or six terms in (23) to attain a precision of 10^{-8} for $\zeta_{\theta j}$.

To solve for $\phi_{1,j}$, the predictor-corrector method of Adams-Bashforth (Fox, 1962) has been used on (12), which can be reduced to the form,

$$y_1' = y_2/R, \quad y_2' = g(R) + \ell^2 y_1/R,$$

$$y_1 = \phi_{1,j}, \quad y_2 = R\phi_{1,j}',$$

$$g(R) = f(-R\eta_w \phi_0' \zeta_{rj} + R\zeta_{rj}' + \zeta_{rj} - \ell\zeta_{\theta j}). \quad (24)$$

Since the complementary solution of (24) is $\phi_{1,j}^N = y_1^N = R^\ell$, and its particular solution, y_1^P , is proportional to $R^{\ell+2j}$ near $r = 0$, $y_1^{P'}(0)$ and $y_2^{P'}(0)$ are both equal to zero. So the starting values of y_1^P and y_2^P are well-behaved, and easily obtained by Taylor series expansion near $R = 0$. The total solution of $\phi_{1,j}$ can then be written as

$$\phi_{1,j} = c_j R^\ell + \phi_{1,j}^P, \quad (25)$$

where c_j is determined by imposing the boundary conditions of ϕ_1 . By making the interval $\Delta r = 0.01$, and using double precision, $\phi_{1,j}$ can be calculated to within 10^{-8} . This is arrived at, first, by comparing results that use different values of Δr , and secondly, checking against solutions of (12) obtained by power-series expansions in R .

3.3 Numerical Instability

In the process of solving the algebraic equation, (21), the size of N is limited by the inaccuracy involved in obtaining A_{ij} , etc. This inaccuracy introduces a numerical instability whenever the coordinate functions are not orthogonal functions with respect to the differential operators of (10)-(12) (Mikhlin, 1971, Chap. 2). The situation is best illustrated by an example in which the dipole resonance frequencies corresponding to Figure 3(b) are calculated for different values of N while holding the size of Δr constant at 0.05.

As shown in Table 2, as N is increased from 2, the first few resonance frequencies are approached from above with rapidly stabilized estimates. When N is increased beyond 8, undesirable fluctuations larger than 10^{-4} , and clearly erratic changes in the values of Ω , start to appear. In the case $N = 9$, for example, one would obtain an erroneous

Table 2

Dipole resonance frequency estimates obtained with $\Delta r = 0.05$ for the case of Figure 3(b). Significant numerical instability sets in when $N \geq 8$. The 'best estimates' are obtained with $\Delta r = 0.01$ and $N = 9$.

N	Main	First	Second	Third	Fourth
Best estimates:	0.4447	0.7099	0.8746	0.9924	1.080
2	0.4703	0.8813			
3	0.4486	0.7446	0.9250		
4	0.4450	0.7155	0.8818	1.022	
5	0.4448	0.7119	0.8757	0.9958	1.084
6	0.4448	0.7117	0.8753	0.9956	1.084
7	0.4448	0.7104	0.8751	0.9947	1.082
8	0.4448	0.7098	0.8748	0.9921	1.081

fundamental resonance frequency. Characteristic of the variational nature of the Lagrangian, $\mathcal{L}_2(\Omega, \ell)$, more serious errors are found in the approximate eigenfunctions, than in the resonance frequencies. The optimal combination of N and Δr , that produces acceptable results in the shortest computation time, can be obtained by trial and error. Repeated solution of (21) for a few adjacent values of N thus becomes an economical technique; this requires computation of the matrices A_{ij} , etc. only once, and offers safeguards against obtaining erroneous results due to numerical instabilities.

3.4 Computer Results

As a practical example, the approximate density profiles given in Table 1 have been used to predict dipole resonances for Tube No. 1 used by Parker, et al. (1964) ($\lambda = 1$; $a = 0.5$ cm; effective relative permittivity at the surface of the column $K_{\text{eff}} = \epsilon_1' / \lambda \epsilon_1 = 2.1$). The computation time varies roughly as N^2 . With $N = 10$, a typical calculation takes about 40 seconds, and requires a core space of less than 100K bytes in an IBM 370/67 machine.

The resulting approximate solutions for $\xi_r, i\xi_\theta, n_1, \phi_1$, and ϕ_1' are plotted in Figures 4-6. The density and the radial electric field solutions, $n_1(R)$ and ϕ_1' resemble very closely those given by Parker, et al. (1964), and Parbhakar and Gregory (1971), respectively. The relative amplitudes of ξ_r and ξ_θ shown in these figures are retained, revealing that as $1/\Lambda_D^2$ increases, ξ_r progressively dominates over ξ_θ . For $1/\Lambda_D^2 > 4500$, it will be seen that the perturbations should be progressively compressed toward the sheath region as $1/\Lambda_D^2$ increases in value. These

solutions are not reproduced here because they also exhibit undesirable oscillations with wavenumber equal to N , an expected characteristic when we try to approximate rapidly varying functions with truncated polynomials. Since only a moderate computer storage is used for $N \leq 10$, there is room to increase N , and decrease Δr , to obtain better approximate solutions. However, this is considered unimportant for our purpose, since we are able to obtain good frequency estimates for this region with $N \leq 10$, as Figure 7 reveals.

The corresponding estimates of resonance frequencies are shown in Figure 7. Since the electron temperature corresponding to the experimental resonance data of Parker et al. (1964) was adjusted to fit their theoretical spectrum, it would be reasonable for us to make a similar adjustment. As is evident from Figure 7, however, no such adjustment is necessary. Indeed, our result seems to be in slightly better agreement with the $T_e = 3$ eV data. The minor differences between the two theoretical results probably come from the differences in the rf equation of motion and pressure term used in the two treatments

Similar to other papers (Crawford, 1964; Parker et al. 1964; Harker et al. 1968; Baldwin, 1969; Dorman, 1969; Miura and Barston, 1971; Peratt and Kuehl, 1972), we have assumed zero normal rf plasma current density at the glass wall, through the form of ζ_{rj} in (17). This is appropriate when the plasma is sufficiently warm that the electron excursion velocity is much smaller than the thermal speed, and $f(1) \ll 1$. This assumption, however, is inconsistent with the dielectric model for a cold plasma column, where normal rf plasma current must be included. It is of interest to ask why the main resonance frequency of a warm plasma column,

RESONANCES

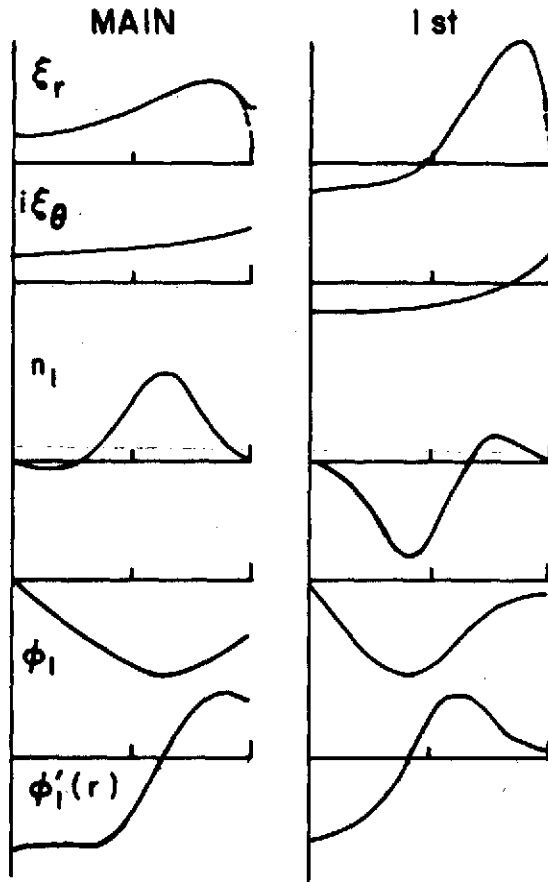


FIG. 4. Approximate solutions for $1/\lambda_D^2 = 72$, $\ell = 1$.

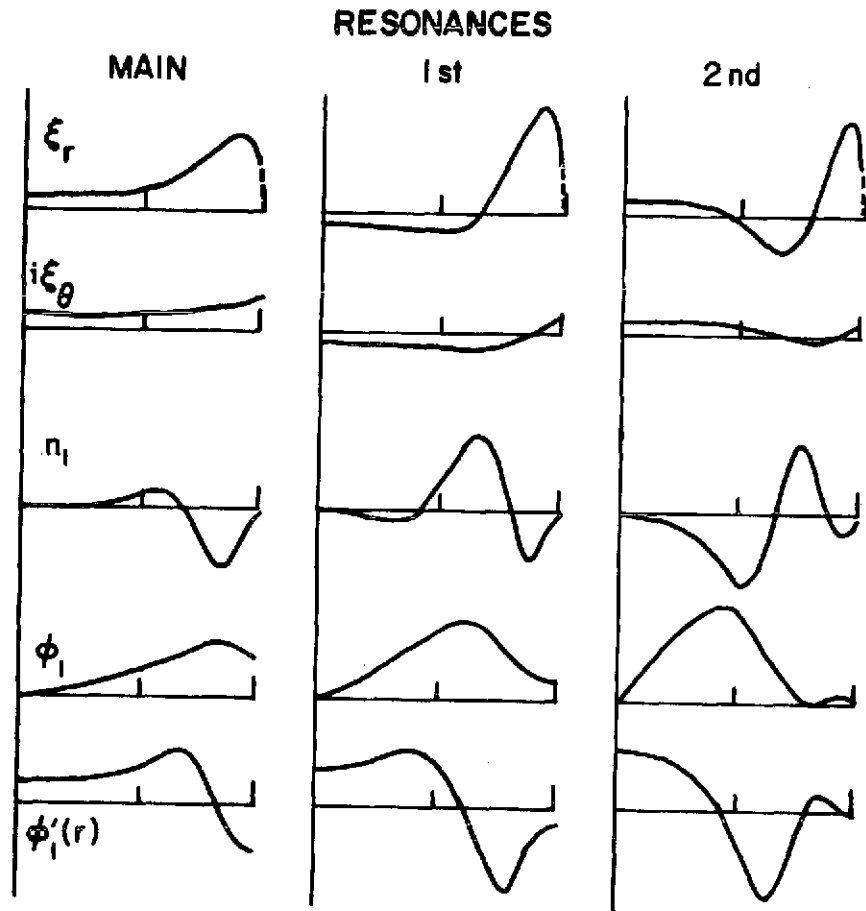


FIG. 5. Approximate solutions for $1/\lambda_D^2 = 510$, $\ell = 1$.

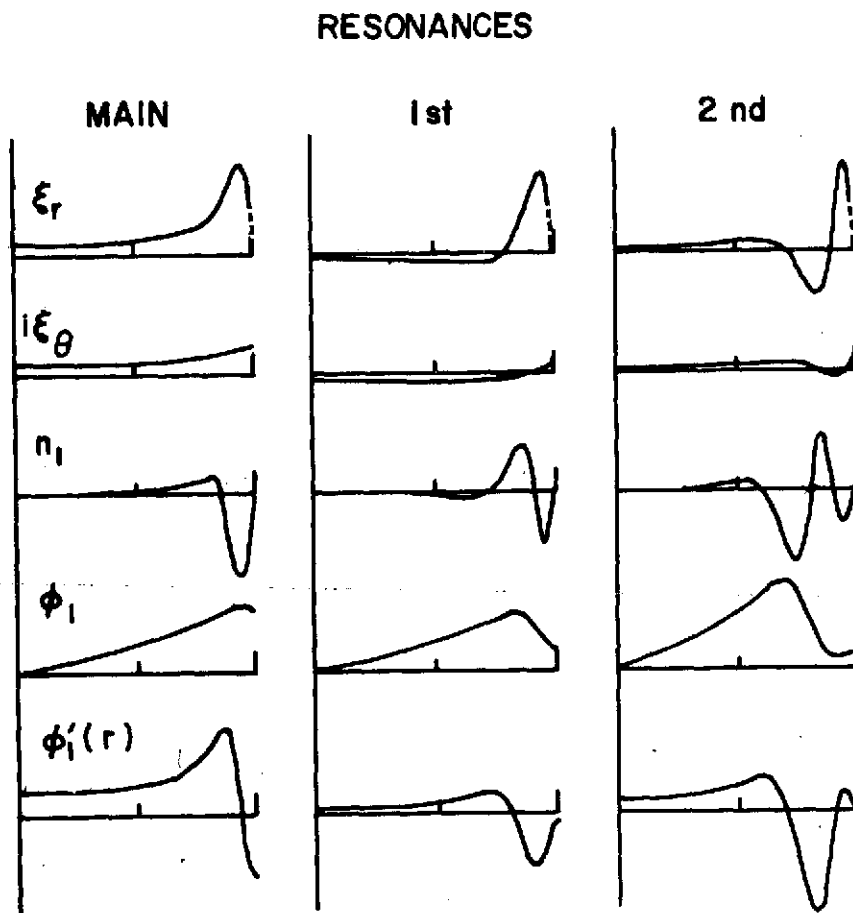


FIG. 6. Approximation solution for $1/\lambda_D^2 = 4500$, $\ell = 1$.

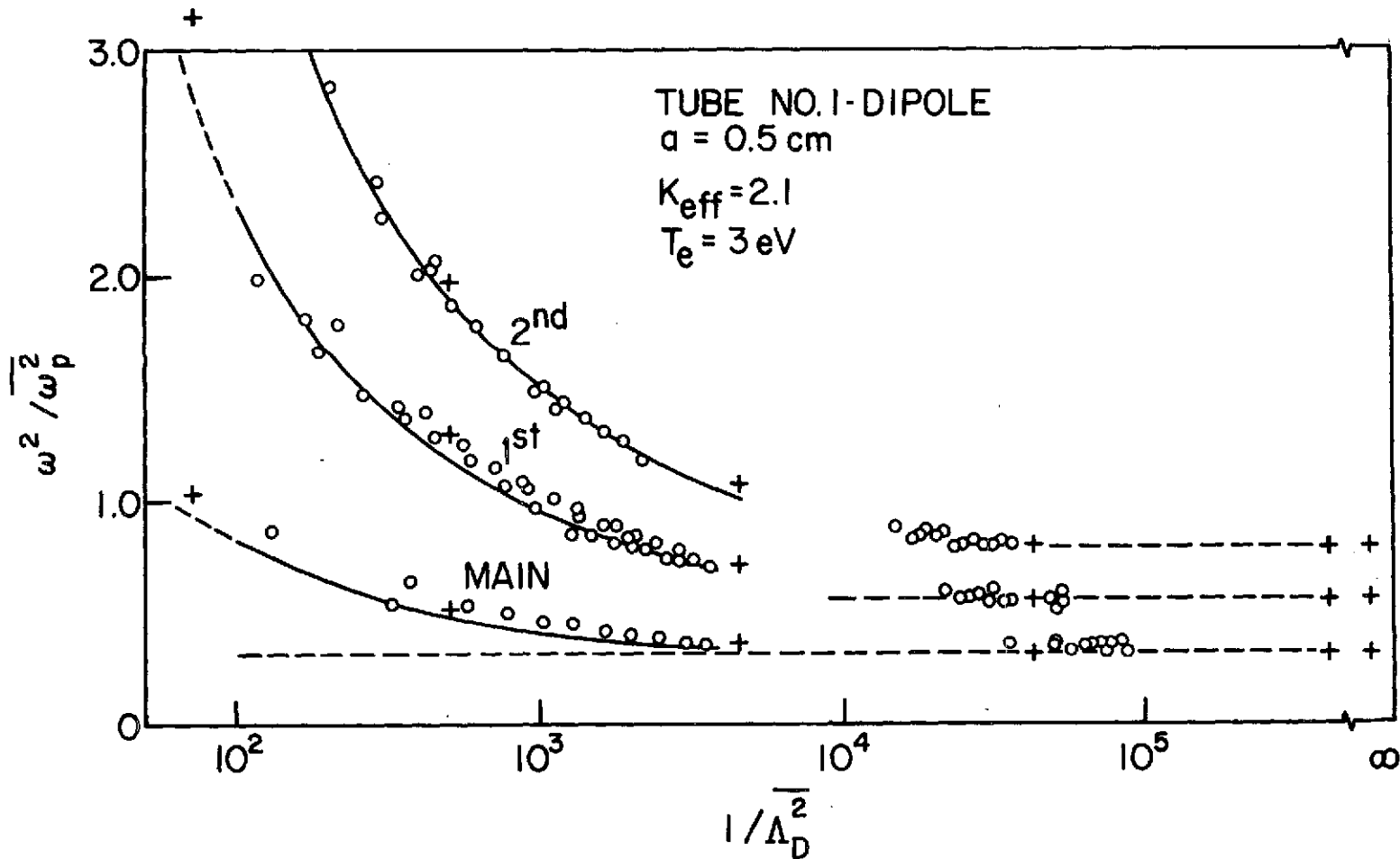


FIG. 7. Dipole resonance spectrum of Tube No. 1 (Parker et al.), compared with estimates by the variational method (indicated by crosses).

in the limit of low temperature, approaches the principal resonance of a cold plasma column. To answer this question, we need only change (17) to

$$\zeta_{rj} = R^{\ell-1} + R^{\ell+2j-1} \quad (j = 1, 2, \dots), \quad (26)$$

and impose the requirement of continuity of normal displacement in the form

$$\phi_1'(1^-) + f(1)\zeta_r(1) = \epsilon_g \phi_1'(1^+) \quad (27)$$

The resulting solutions of ζ_r are found to be only slightly different from the previous case near $R = 1$ [Figures 4-6, where the dashed lines correspond to the use of (17)]. Furthermore, the main resonance is lowered by less than 1 per cent for all of the values of $1/\Lambda_D^2$ used here, including the case $1/\Lambda_D^2 \rightarrow \infty$. This is well within the experimental errors.

4. DISCUSSION

In this paper, we have applied the Rayleigh-Ritz procedure to a system of three Euler-Lagrangian equations that describe the electron resonances of a nonuniform warm plasma column. It is shown that accurate frequencies for the first few resonances can be obtained for the entire range of $1/\Lambda_D^2 \gg 1$. Results which agree closely with those of Parker, et al. (1964) have been obtained

Contrary to the case of a system of elliptic equations, where the coefficients are assigned independently to each dependent variable (Mikhlin, 1971), we have found that for (10)-(12), the same coefficient must be assigned to each set of coordinate functions, e.g. (19). In addition to the usual requirements, that the coordinate functions must be linearly independent and complete, we have chosen that they be set up in accordance with (10)-(12).

The present method can be easily modified to include the effects of electron dc drift, dc magnetic field, and ion motion. With the axial dimensions and rf magnetic field included, this procedure would be efficient in solving travelling wave problems in a nonuniform plasma waveguide.

ACKNOWLEDGMENTS

The authors are indebted to Drs. K. J. Harker and H. Kim for many useful discussions. The work was supported by the National Aeronautics and Space Administration.

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APPENDIX

Here we shall show that the coordinate functions, ζ_{rj} , $\zeta_{\theta j}$, and $\phi_{1,j}$, must be assigned the same coefficients, a_j , for the Rayleigh-Ritz procedure applied to $\mathcal{L}_2(\Omega, \ell)$ of (15) to be successful: it will be shown that, by restricting these coordinate functions according to (10)-(12), the appropriate Rayleigh-Ritz procedure can be established.

A system of second order elliptic equations can be written as

$$(D_{jk} U_k')' + \lambda E_{jk} U_k = 0, \quad D_{jk} = D_{kj}, \quad E_{jk} = E_{kj}, \quad (\text{A.1})$$

where U_k is the k^{th} dependent variable, the matrices D_{jk} and E_{jk} are functions of the independent variable x , the eigenvalue is $\lambda (\geq 0)$, and the summation convention has been used. Ellipticity demands that

$$D_{jk} \alpha_j \alpha_k > 0, \quad E_{jk} \alpha_j \alpha_k \geq 0, \quad (\text{A.2})$$

for any real non-zero vector α_j . The solutions of (A.1) then admit of variational estimates, as outlined by Mikhlin (1965).

When we apply the Rayleigh-Ritz procedure to the Lagrangian for (A.1),

$$L = \int_0^1 dx (D_{jk} U_j' U_k' - \lambda E_{jk} U_j U_k), \quad (\text{A.3})$$

the coefficients preceding the coordinate function for each U_k can be varied independently. Suppose a set of legitimate trial functions, $U_k = C_k V_k$ (summation convention not used here) with coefficient C_k , are used in (A.3). Because of (A.2), we can obtain crude estimates of λ larger than the lowest eigenvalue, even if only one of the C_k is non-zero.

If the same procedure is applied to the Lagrangian in (15), we will obtain erroneous estimates of Ω . Consider the case of a cold, uniform plasma column with $B_0 = 0$, so that $H = -I + F + F'$. Suppose C_r ,

C_θ , and C_ϕ are the coefficients for the coordinate functions ζ_{rj} , $\zeta_{r\theta}$, and $\phi_{i,j}$, respectively. Then if $C_\phi = 0$, the value of the Ω estimate will be zero, and fail the requirement of the Rayleigh-Ritz sequence.

Consequently, to make $(-H/A)^{1/2}$ from (15) at least non-zero, we must use a single coefficient for each set of the coordinate functions ζ_{rj} , $\zeta_{\theta j}$, and $\phi_{1,j}$. Merely making $C_r = C_\theta = C_\phi = a_j$ is insufficient to produce a legitimate Rayleigh-Ritz sequence, because the value of $\phi_{1,j}$, for example, can be arbitrarily small in comparison with ζ_{rj} and $\zeta_{\theta j}$, making the Ω estimates also arbitrarily small. We see that restricting these coordinate functions according to (10)-(12), is sufficient to reduce the resulting approximated Lagrangian to a single Euler-Lagrange equation. The appropriateness of the resulting Rayleigh-Ritz procedure can then be guaranteed. It is conceivable, of course, that there may be less restrictive choices of appropriate coordinate functions corresponding to the Lagrangian in (15), but we have not chosen to pursue this point.