IFSM-73-45

LEHIGH UNIVERSITY

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Technical Report NASA TR-73-14

July 1973

National Aeronautics and Space Administration

Grant NGR-39-007-011

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Abstract. Using the filament model developed in the previous paper, the elastostatic interaction problem between a penny-shaped crack and a slender inclusion or filament in an elastic matrix is formulated. For a single filament as well as multiple identical filaments located symmetrically around the crack the problem is shown to reduce to a singular integral equation. The solution of the problem is obtained for various geometries and filament-to-matrix stiffness ratios, and the results relating to the angular variation of the stress intensity factor and the maximum filament stress are presented.

1. INTRODUCTION

In studies relating to the fracture of filament-reinforced composites and reinforced concrete in the presence of internal cracks, in practice two of the most important factors one needs to evaluate are the effect of the reinforcements on the stress state around the leading edge of the crack and the effect of the crack on the maximum stresses in the reinforcements. Basically the problem is that of interaction between an internal crack in an elastic matrix and elastic inclusions of different thermo-elastic properties and of given size, shape, orientation, and

* This work was supported by The National Science Foundation under the Grant GK-11977 and by the National Aeronautics and Space Administration under the Grant NGR-39-007-011.

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distribution which are imbedded in the matrix. The general problem is clearly intractable. However, if the medium fulfills certain conditions of symmetry with respect to the geometry and the external loads, under some simplifying assumptions it is possible to solve the problem within a reasonable degree of accuracy and with a reasonable amount of computational effort. Such a solution will be given in this paper.

Specifically, the following are the symmetry conditions which are assumed to be satisfied:

(a) the crack is a plane, circular (i.e., penny-shaped) internal crack;

(b) the filaments are circular cylinders imbedded in the matrix with their axes perpendicular to the plane of the crack, $z = 0$ (Figure 1);

(c) $z = 0$ is a plane of symmetry with respect to the geometry of the composite medium and the external loads; and

(d) the overall dimensions of the matrix are large in comparison with the local dimensions of the crack-filament region (i.e., the crack diameter, $2a$, the distance of the ith filament from the crack center, b_i, and the filament length, $2c_i$) so that in the analysis the matrix may be treated as being infinite.

The primary assumption to render the problem analytically tractable is that the Poisson's ratios of the filaments be equal to that of the matrix. Under this assumption it was shown in [1] that if the Green's functions (for a concentrated body force) of the matrix are available, the problem may be reduced to a system

of (two-dimensional) integral equations for layers of (fictitious) body forces imbedded along the filament-matrix interfaces. The following simplifying assumptions are then made in this paper in order to keep the analysis and the numerical work to solve the problem within manageable proportions:

(i) The filaments are identical, i.e., $c_1 = \ldots = c_N = c$, $r_1 = \ldots = r_N = r_o$, $E_1 = \ldots = E_N = E_f$, $\nu_1 = \ldots = \nu_N = \nu_f = \nu$, where $2c$ is the length, r_o is the radius, E_f is the Young's modulus, ν_f is the Poisson's ratio, and N is the number of filaments, and ν is the Poisson's ratio of the matrix (Figure 1);

(ii) c/r_o is sufficiently large so that a one-dimensional treatment of the auxiliary inclusion (having the elastic constants $E_f - E$ and ν, E being the modulus of the matrix) is justified (see [1]);

(iii) The filaments are distributed symmetrically (i.e., with equal angular spacings) on a circle in $z = 0$ plane concentric with the penny-shaped crack (i.e., $b_1 = \ldots = b_N = b$), they are parallel to the z axis, and their mid-points lie on $z = 0$ plane;

(iv) The filament radius r_o is sufficiently small in comparison with other dimensions of the medium so that the effect of the radial component R of the (fictitious) body force acting on the filament-matrix interface and the effect of the angular variation of the axial component Z of the body force may be neglected. It was shown in [1] that as a result of this assumption the effect of the end tractions (or the body forces distributed at the ends $z = \mp c$) may also be neglected; and

(v) The external load is the uniaxial stress $\sigma_{zz} = \sigma_o$ applied to the matrix away from the crack-filament region.

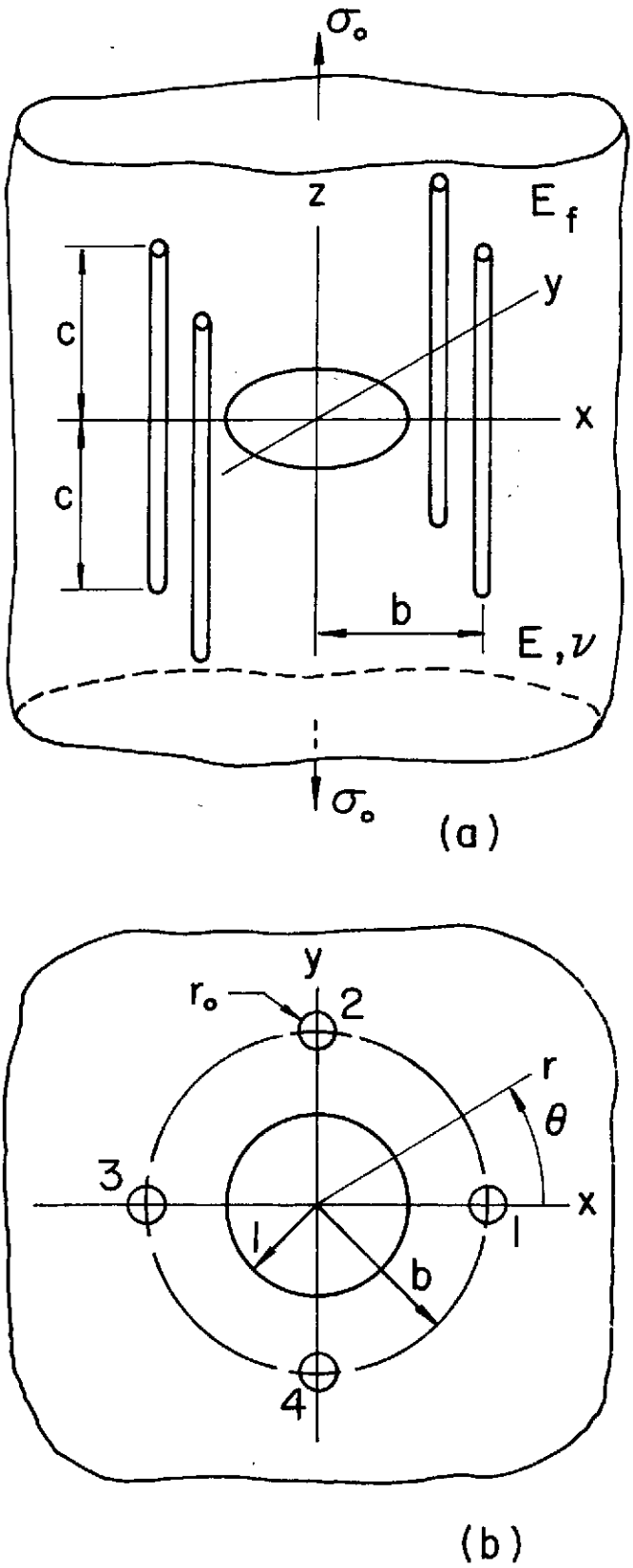

Figure 1. Geometry for the penny-shaped crack in an elastic matrix reinforced by symmetrically located identical filaments.

From the viewpoint of the elasticity solution, clearly the most drastic assumption is that regarding the distribution of the body forces. However, the results and some comparisons given in [1] indicate that the error introduced into the solution as a result of this assumption would be within an acceptable range.

The physical problem described in this paper was also considered in [2], where the problem was assumed to be essentially axisymmetric and the filaments were replaced by a discrete set of concentrated body forces acting in z-direction.

2. FORMULATION OF THE PROBLEM

Consider now the problem described in Figure 1 subject to the symmetry conditions (a)-(d) and the simplifying assumptions (i)-(v) outlined in the previous Section. Since the filaments are identical and symmetrically located, the body forces $Z(z)$ imbedded in different filament-matrix interfaces will be the same. Hence if one can obtain the Green's functions for a cracked matrix, it is possible to formulate the problem in terms of a single integral equation with $Z(z)$ as the unknown function. This integral equation may be obtained by matching either the z-components of the displacement vector or the strains ε_{zz} in the cracked matrix subjected to the external loads σ_o and $Z(z)$ and that in the auxiliary filaments subjected to the external loads $-Z(z)$. In using the equality of the strains in the derivation the single-valuedness condition of the displacements must separately be satisfied. The solution for the cracked matrix may be obtained by superimposing four solutions described in Figure 2. Note that

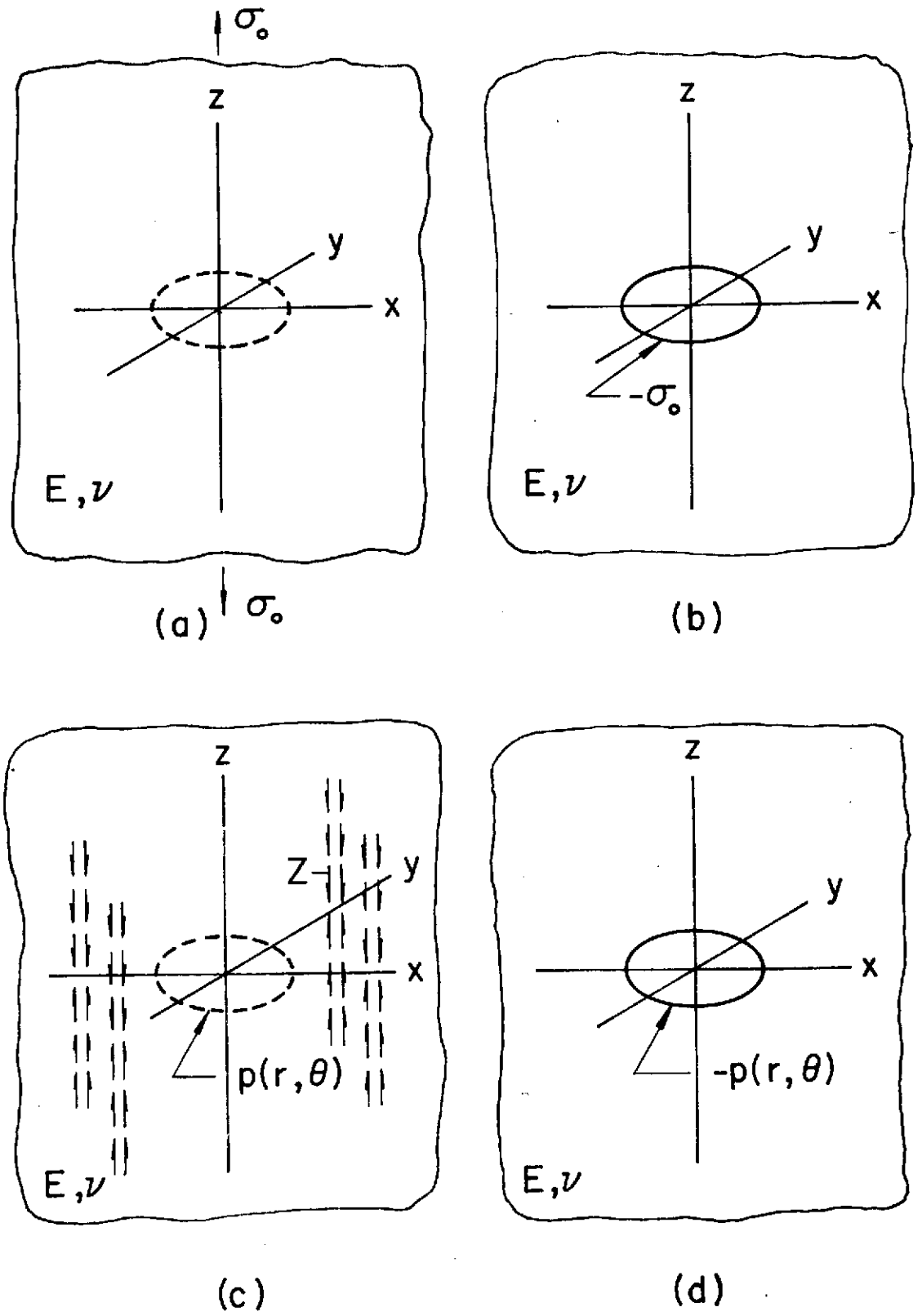

Figure 2. Superposition for the stresses in the matrix.

there are no cracks in problems described by Figures 2a and 2c.

Let the filament 1 be located on the x axis (i.e., at $r = b$, $\theta = 0$) and let (δ,ϕ,z) be the cylindrical coordinates referred to the center of filament 1. Let $\varepsilon^a_{zz},..,\varepsilon^d_{zz}$ be the strains corresponding to problems described by Figures 2a,..,2d, and ε^f_{zz} be the axial strain in the auxiliary filament. From [1] it follows that

$$\varepsilon^a_{zz} + \varepsilon^b_{zz} + \varepsilon^c_{zz} + \varepsilon^d_{zz} = \varepsilon^f_{zz}(z) \ , \qquad (\delta = r_o, \ \phi = \phi_o, \ -c < z < c), \tag{1}$$

where r_o is the radius, 2c is the length of the filament, and ϕ_o is an appropriate angle describing the generator of the cylinder along which the strains are matched (Note that the body force $Z(r_o,\phi,z)$ is assumed to be independent of ϕ). The filament strain is given by [1]

$$\varepsilon^f_{zz}(z) = - \frac{2}{r_o(E_f - E)} \int_z^c Z(t)dt \ , \qquad (-c < z < c). \tag{2}$$

Within the confines of the simplifying assumptions made in the previous section regarding the body forces R and Z, the strains ε^a_{zz} and ε^c_{zz} were obtained in [1] as follows:

$$\varepsilon^a_{zz} = \frac{\sigma_o}{E} \ , \tag{3}$$

$$\varepsilon^c_{zz}(z) = \frac{1}{EC_1} \left[\int_{-c}^{c} \frac{Z(t)dt}{t-z} + \int_{-c}^{c} m_{22}(z,t)Z(t)dt \right] , \qquad (-c < z < c), \tag{4}$$

$$m_{22}(z,t) = \frac{2r_o}{\rho_o}\left(\frac{\partial K(k)}{\partial z} - \frac{1}{t-z}\right) + \frac{2r_o - \rho_o}{\rho_o(t-z)} + \frac{2r_o(t-z)}{\rho_o^3} K(k)$$

$$+ \frac{2r_o\gamma}{\rho_o}\frac{\partial E(k)}{\partial z} + \frac{2r_o\gamma(t-z)}{\rho_o^3} E(k)$$

$$+ \pi r_o(t-z) \sum_{i=2}^{N} \frac{1}{[d_i^2 + (t-z)^2]^{3/2}} (1 - 2\gamma$$

$$+ \frac{3\gamma(t-z)^2}{d_i^2 + (t-z)^2}) ,$$

$$C_1 = \frac{4\pi(1-\nu)}{(1+\nu)(3-4\nu)} , \qquad \gamma = \frac{1}{3-4\nu} ,$$

$$\rho_o^2 = 4r_o^2 + (t-z)^2 , \qquad k = \frac{2r_o}{\rho_o} ,$$

$$d_i = b[2(1 - \cos\frac{2\pi(i-1)}{N})]^{1/2} , \qquad (i = 2,..,N), \tag{5}$$

where K(k) and E(k) are complete elliptic integrals, N is the number of filaments, and b is the distance from the center of the crack to the midpoint of the filament (Figure 1).

To obtain ε_{zz}^b and ε_{zz}^d the solutions of the crack problems shown in Figures 2b and 2d (where the only external loads are the self-equilibrating crack surface tractions) are needed. In Figure 2b the crack surface tractions are

$$\sigma_{zz} = -\sigma_o , \qquad \sigma_{rz} = 0 = \sigma_{\theta z} . \tag{6}$$

This is an axisymmetric problem the solution of which is known (see, e.g., [4]). In the problem described by Figure 2d the crack surface tractions are equal in magnitude and opposite in sign to the stresses at z=0 plane obtained from the solution of the problem given by Figure 2c. In Figure 2c, because of symmetry, the

shear stresses at z=0 plane will again be zero. Hence for the problem 2d the crack surface tractions will be

$$\sigma_{zz} = - p(r,\theta) , \qquad \sigma_{rz} = 0 = \sigma_{\theta z} , \tag{7}$$

where $p(r,\theta)$ is the normal stress at z=0 plane in Figure 2c.

$p(r,\theta)$ may easily be calculated in terms of $Z(z)$ by using the basic solution given in [3] (see also [1]). However, since $r_o << c$ and $r_o << a$ (where r_o is the radius, 2c is the length of the filament and a is the crack radius), to calculate $p(r,\theta)$ for simplicity it will be assumed that the concentrated body forces $2\pi r_o Z(z)$ act along the lines ($r = b$, $-c < z < c$, $\theta_i = 2\pi i/N$), $i = 1,2,..,N$ (i.e., along the axes of the filaments), rather than $Z(z)$ acting along the cylindrical surfaces ($\delta = r_o$, $-c < z < c$, $0 \leq \phi < 2\pi$). Thus, following [3] we find

$$p(r,\theta) = \frac{r_o}{4(1-\nu)} \int_{-c}^{c} Z(t)t \sum_{i=1}^{N} \frac{1}{\rho_i^3} \left(1 - 2\nu + \frac{3t^2}{\rho_i^2}\right) dt ,$$

$$\rho_i^2 = r^2 - 2br \cos\left(\theta - \frac{2\pi i}{N}\right) + b^2 + t^2 . \tag{8}$$

For the solution of the crack problem it will be necessary to express $p(r,\theta)$ in product form. Since p is periodic (and even) in θ this may easily be done, giving

$$p(r,\theta) = \sum_{n=0}^{\infty} b_n \left(\frac{r}{a}\right)^{Nn} \cos(Nn\theta) , \quad (0 \leq r < a,\ 0 \leq \theta < 2\pi), \tag{9}$$

where

$$b_o = \frac{1}{\pi a^2} \int_0^a \int_0^{2\pi} p(r,\theta) r\, dr\, d\theta ,$$

$$b_n = \frac{2(nN+1)}{\pi a} \int_0^a \int_0^{2\pi} p(r,\theta) \left(\frac{r}{a}\right)^{nN+1} \cos(nN\theta)\, dr\, d\theta ,$$

$$(n = 1,2,\ldots). \tag{10}$$

Assuming that crack radius a is the length unit for the medium, without any loss in generality for the remainder of this paper we will let a = 1.

The crack problem described in Figure 2d may thus be considered as a mixed boundary value problem in elasticity for a half space $z > 0$ subject to the following boundary conditions:

$$\sigma_{zz}(r,\theta,0) = -p(r,\theta) , \qquad (0 \le r < 1,\ 0 \le \theta < 2\pi),$$

$$u_z(r,\theta,0) = 0 , \qquad (1 < r < \infty,\ 0 \le \theta < 2\pi),$$

$$\sigma_{rz}(r,\theta,0) = 0 = \sigma_{\theta z}(r,\theta,0) , \qquad (0 \le r < \infty,\ 0 \le \theta < 2\pi). \tag{11.a-c}$$

Similar mixed boundary value problems have been treated in [5] - [7]. As shown in, for example, [8], because of (11.c) the problem may be formulated in terms of a single harmonic function ϕ, which may be expressed as

$$\phi(r,\theta,z) = \sum_{n=0}^{\infty} \cos(nN\theta) \int_0^{\infty} \frac{1}{\alpha} f_n(\alpha) e^{-z\alpha} J_{nN}(r\alpha)\, d\alpha , \tag{12}$$

where f_n is the unknown function. Using (12) and the relations [8]

$$2\mu u_z = 2(1-\nu) \frac{\partial \phi}{\partial z} - z \frac{\partial^2 \phi}{\partial z^2} ,$$

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} - z \frac{\partial^3 \phi}{\partial z^3} , \tag{13.a,b}$$

from (9), (11.a) and (11.b) it is found that

$$\int_0^{\infty} \alpha f_n(\alpha) J_{nN}(r\alpha)\, d\alpha = -b_n r^{nN} , \qquad (0 \le r < 1),$$

$$\int_0^{\infty} f_n(\alpha) J_{nN}(r\alpha)\, d\alpha = 0 , \qquad (1 < r < \infty),\ (n = 0,1,2,\ldots). \tag{14}$$

After some manipulations the solution of the dual integral equations (14) may be expressed as [4, 9]

$$f_n(\alpha) = -\sqrt{\frac{2}{\pi}}\, b_n a_n \alpha^{-1/2} J_{nN+3/2}(\alpha) ,$$

$$a_n = 2^{-2nN} \sum_{k=0}^{nN} \binom{2nN+1}{k} \frac{(-1)^{n-k}}{2n-2k+1} . \qquad (15)$$

Substituting now (12) and (15) into (13.a) the desired strain may now be expressed as

$$\varepsilon^d_{zz}(r,\theta,z) = -\frac{1+\nu}{E}\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} a_n b_n \cos(nN\theta)[(1-2\nu)F_n(r,|z|) + |z|G_n(r,|z|)] , \qquad (|z| \geq 0), \qquad (16)$$

where

$$F_n(r,z) = \int_0^{\infty} \alpha^{1/2} J_{nN}(r\alpha) J_{nN+3/2}(\alpha) e^{-\alpha z} d\alpha ,$$

$$G_n(r,z) = \int_0^{\infty} \alpha^{3/2} J_{nN}(r\alpha) J_{nN+3/2}(\alpha) e^{-\alpha z} d\alpha , \quad (z > 0). \quad (17.a,b)$$

The functions F_n and G_n may be evaluated by using Filon's integration formula [9]. However, a simpler way to evaluate these integrals would be to use the series expansions given in [10, p. 400]*.

The problem described by Figure 2b is a special case of Figure 2d, σ_o replacing $p(r,\theta)$ in (11.a). The strain for this case may be expressed as

$$\varepsilon^b_{zz}(r,z) = -\frac{1+\nu}{E}\sqrt{\frac{2}{\pi}}\, \sigma_o[(1-2\nu)F_o(r,|z|) + |z|G_o(r,|z|)] , \qquad (-\infty < z < \infty), \qquad (18)$$

where F_o and G_o are also given by (17).

*The details may be found in [11].

By substituting now from (2), (3), (4), (16), and (18) into (1) we obtain a singular integral equation for the unknown function $Z(z)$. Z again has integrable singularities at $z = \mp c$, the index of the integral equation is +1, and the general solution contains one arbitrary constant which is accounted for by the single-valuedness condition of the displacement u_z. In the numerical solution of the problem if one assumes that $Z(z) = -Z(-z)$, because of the symmetry properties of the kernels, the single-valuedness condition is automatically satisfied [1]. Note that in the derivation given in this section the elastic moduli appear only through E/E_f.

3. THE FILAMENT STRESS

Once the body force $Z(z)$ is obtained the shear stress on the filament-matrix interface and the axial stress in the actual filament may be obtained by using the general results given in [1]. Thus, from (24) or (48) of [1] the interface shear may be expressed as

$$\sigma_{\delta z}(r_o,z) = - \frac{E_f}{E_f - E} Z(z) , \qquad (-c < z < c). \tag{19}$$

Similarly the filament stress $\sigma_{fzz}(z)$ may be evaluated by using (27) or (49) of [1], which may be expressed as

$$\sigma_{fzz}(z) = - \frac{2}{r_o} \int_z^c Z(t)dt + \sigma_o + \sigma_{zz}^b(b,z) + \sigma_{zz}^c(b,z) + \sigma_{zz}^d(b,z) , \qquad (-c < z < c), \tag{20}$$

where on the right-hand side of (20) the first term is the stress in the auxiliary filament, the second and the subsequent terms correspond to the matrix stresses obtained from the problems

described by Figures 2a - 2d. The matrix stresses may either be averaged over the area ($0 \leq \delta < r_o$, $0 \leq \phi < 2\pi$) or be approximated by the stress σ_{zz} along the filament axis, ($r = b$, $-c < z < c$). Following [1] these stresses may be expressed as

$$\sigma^c_{zz}(b,z) = \frac{r_o}{4(1-\nu)} \int_{-c}^{c} \frac{Z(t)(t-z)}{[r_o^2 + (t-z)^2]^{3/2}} \left(1 - 2\nu + \frac{3(t-z)^2}{r_o^2 + (t-z)^2}\right) dt$$

$$+ \frac{r_o}{4(1-\nu)} \int_{-c}^{c} Z(t)(t-z)dt \sum_{i=2}^{N} \left[\frac{1}{[d_i^2 + (t-z)^2]^{3/2}} \left(1\right.\right.$$

$$\left.\left. - 2\nu + \frac{3(t-z)^2}{d_i^2 + (t-z)^2}\right)\right] , \qquad (21)$$

(where d_i is given by (5));

$$\sigma^b_{zz}(b,z) = - \sigma_o \sqrt{2/\pi} \left[F_o(b,|z|) + |z| G_o(b,|z|)\right] , \qquad (22)$$

$$\sigma^d_{zz}(b,z) = - \sqrt{2/\pi} \sum_{n=0}^{\infty} a_n b_n \left[F_n(b,|z|) + |z| G_n(b,|z|)\right] , \qquad (23)$$

(with (17), (8), (10) and (15)).

4. THE STRESS INTENSITY FACTOR

Because of symmetry on the z=0 plane $\sigma_{rz} = \sigma_{\theta z} = 0$ in the matrix, and the stress state in the close neighborhood of and in the plane normal to the leading edge of the crack is plane strain. Thus the stress intensity factor may be defined by

$$K(\theta) = \lim_{r \to a} \sqrt{2\pi(r-a)}\ \sigma_{zz}(r,\theta,0) , \quad (r > a,\ 0 \leq \theta < 2\pi), \qquad (24)$$

or

$$K(\theta) = - \lim_{r \to a} \frac{E}{2(1-\nu^2)} \sqrt{2\pi(a-r)}\ \frac{\partial}{\partial r} u_z(r,\theta,+0) ,$$

$$(0 \leq r < a,\ 0 \leq \theta < 2\pi), \qquad (25)$$

where $u_z(r,\theta,+0)$ is the crack surface displacement on $z > 0$ side of the medium and it is assumed that $a = 1$. From (13.a) and the solution given in Section 2 the crack surface displacement in the matrix may be expressed as

$$u_z(r,\theta,+0) = u_z^b(r,+0) + u_z^d(r,\theta,+0)$$

$$= \frac{2(1-\nu^2)}{E}\sqrt{2/\pi}\,[\sigma_o H_o(r,0) + \sum_{n=0}^{\infty} a_n b_n \cos(nN\theta) H_n(r,0)]\,,$$

$$(0 \leq r < 1,\ 0 \leq \theta < 2\pi), \qquad (26)$$

where

$$H_n(r,z) = \int_0^{\infty} \alpha^{-1/2} J_{nN}(r\alpha) J_{nN+3/2}(\alpha) e^{-\alpha z} d\alpha\,, \qquad (z > 0). \qquad (27)$$

Following [10]

$$H_n(r,+0) = \begin{cases} \dfrac{r^{nN}(1-r^2)^{1/2}}{\sqrt{2}\,\Gamma(3/2)}\,, & (0 \leq r < 1), \\ 0\,, & (1 < r < \infty), \end{cases} \qquad (28)$$

which gives

$$u_z(r,\theta,+0) = \frac{4(1-\nu^2)}{\pi E}(1-r^2)^{1/2}[\sigma_o + \sum_{n=0}^{\infty} a_n b_n r^{nN}\cos(nN\theta)]\,,$$

$$(0 \leq r < 1,\ 0 \leq \theta < 2\pi), \qquad (29)$$

and

$$K(\theta) = \frac{2}{\sqrt{\pi}}\,[\sigma_o + \sum_{n=0}^{\infty} a_n b_n \cos(nN\theta)]\,, \qquad (0 \leq \theta < 2\pi). \qquad (30)$$

5. NUMERICAL RESULTS

In deriving the integral equation for $Z(z)$ it was pointed out in Section 2 (of this paper) that the strain ϵ_{zz}^f in the auxiliary filament has to be matched with the strain in the matrix

along an appropriate generator of the cylinder ($\delta = r_0$, $0 \leq \phi < 2\pi$, $-c < z < c$), i.e., along the line ($\delta = r_0$, $\phi = \phi_0$, $-c < z < c$). The effect of this selection on the stress intensity factor for a single filament is shown in Figure 3. The three curves given in the figure correspond to selecting this generator at the outermost ($\phi_0 = 0$ or $r' = b+r_0$), at the innermost ($\phi_0 = \pi$ or $r' = b-r_0$), and at an intermediate point ($\phi_0 \simeq \pi/2$ or $r' = b$) of the filament cross-section. The differences appear to be rather insignificant. Hence, for the remainder of the numerical analysis in this paper the strains are matched along the line ($r = b$, $-c < z < c$). The solution of the integral equation is of the form [12]

$$Z(z) = f(z)\sqrt{c^2 - z^2}\ , \qquad (-c < z < c), \tag{31}$$

where $f(z) = -f(-z)$ is a bounded function which is obtained numerically [13, 14]. In the examples given in this paper the crack radius was assumed to be the length unit, i.e., $a = 1$. For $a \neq 1$ the stress intensity factor ratio given in the figures should be read as $K_I/(2\sigma_0\sqrt{a/\pi})$.

The effect of the modulus ratio E_f/E and filament distance b in the case of single filament is shown in Figure 4. As expected the stress intensity factor decreases for increasing E_f/E, decreasing b, and decreasing $|\theta|$, where the angle θ along the crack front is measured from the radius going through the filament center.

Figure 5 shows the effect of the filament radius r_0 and filament length 2c on the angular variation of the stress intensity factor for a single filament.

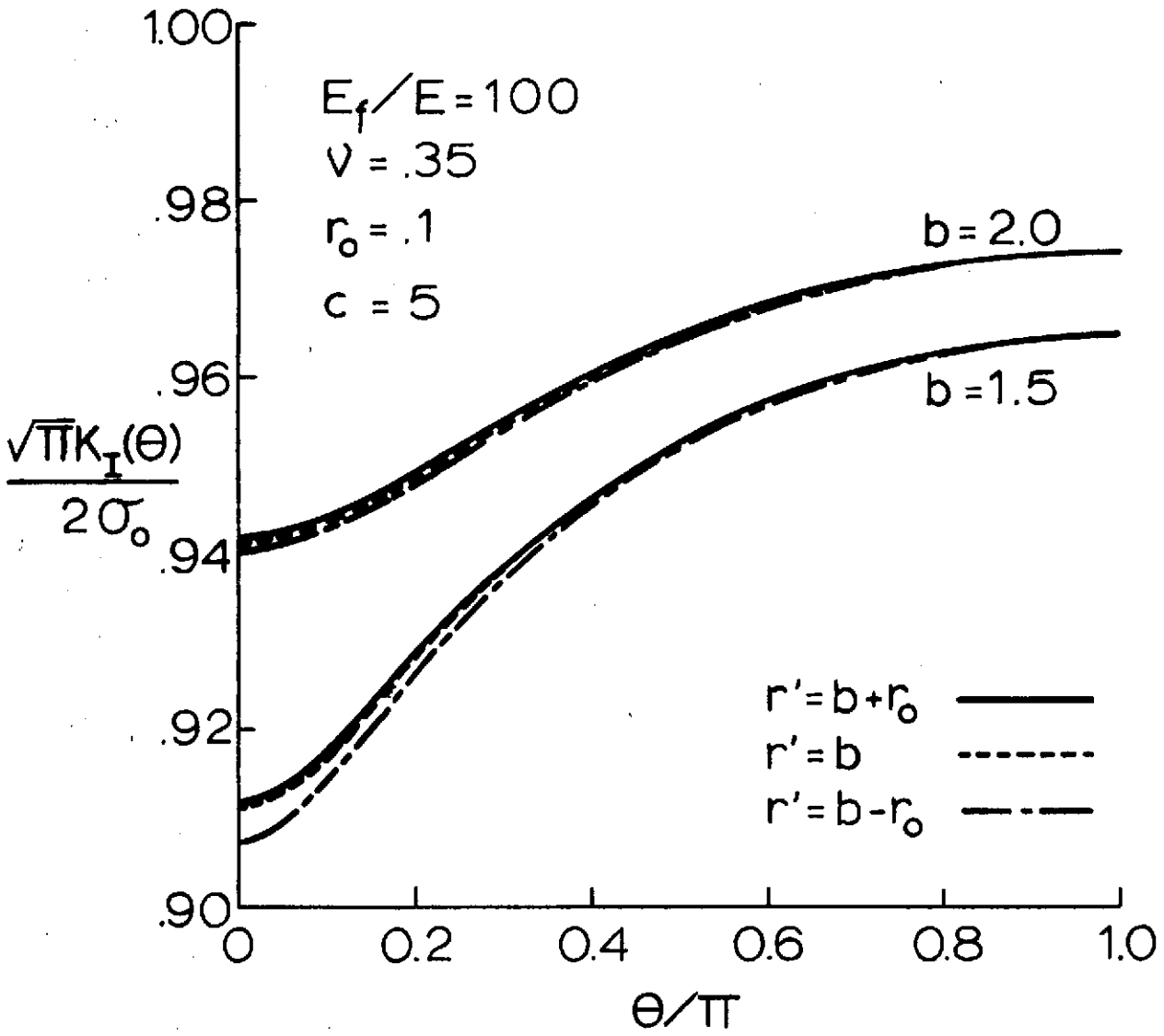

Figure 3. The effect of the location of the strain matching line on the stress intensity factor.

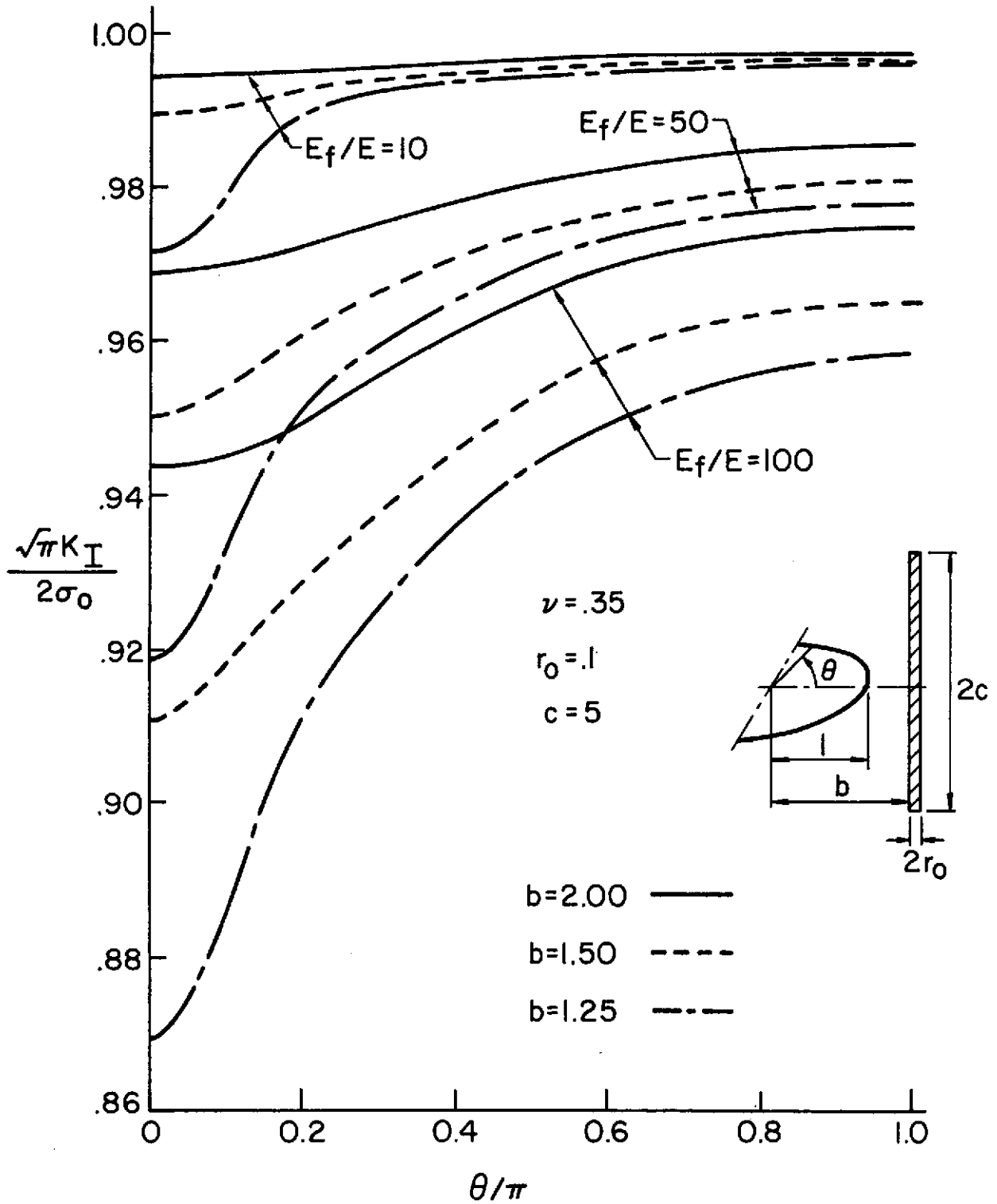

Figure 4. The effect of the filament-to-matrix modulus ratio E_f/E and the filament-to-crack center distance b on the angular distribution of the stress intensity factor.

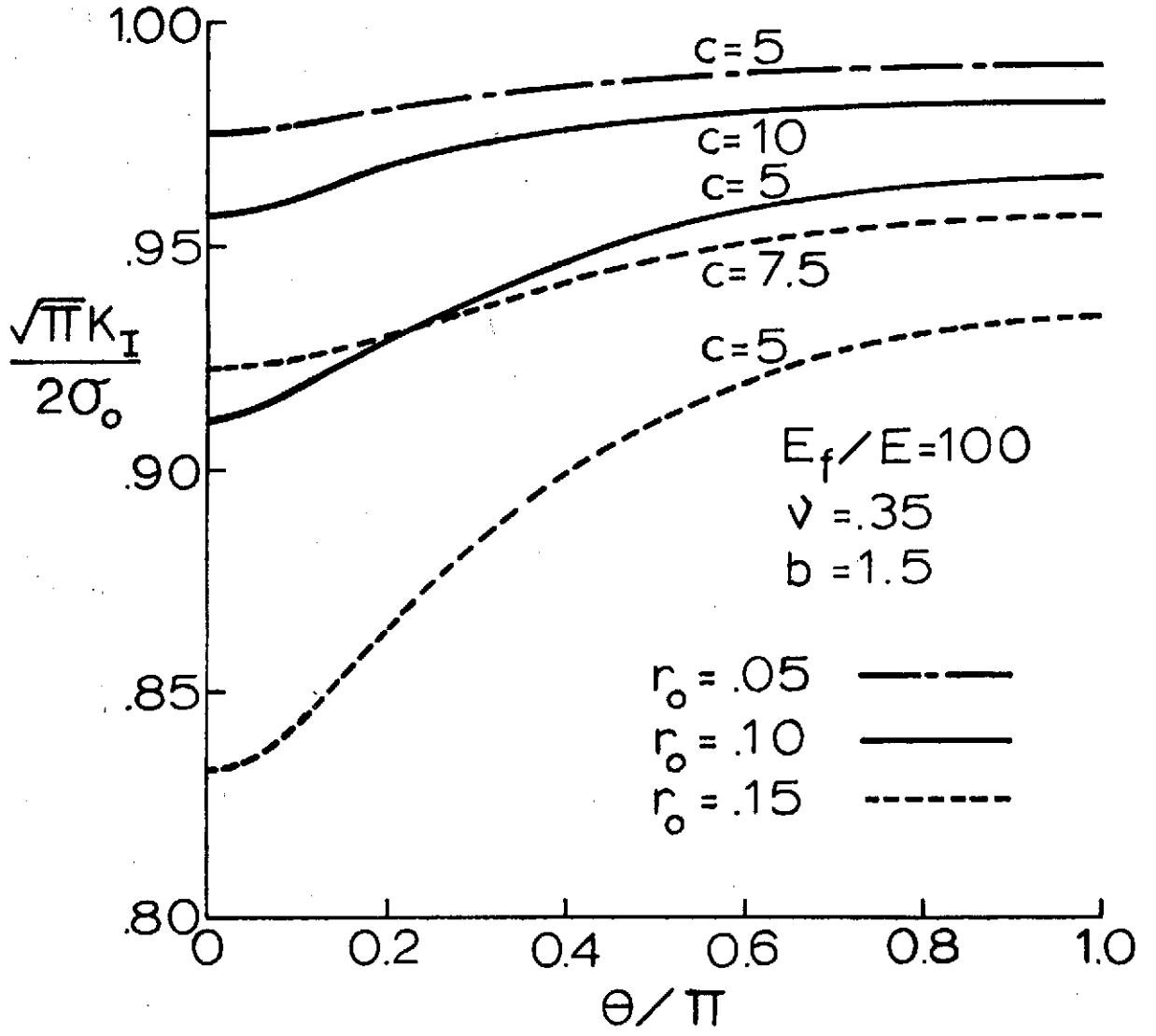

Figure 5. The effect of the filament dimensions r_o and c on the stress intensity factor.

The effect of the number of filaments, N, is shown in Figure 6. It may be observed that for $N \geq 4$ in the case of $b/a = 2$ and for $N \geq 6$ in the case of $b/a = 1.5$ (and for $c/a = 5$, $r_o/a = 0.1$, and $E_f/E = 100$) the stress intensity factor will be independent of θ and the problem may be simplified considerably by assuming axial symmetry (and by treating the filaments as "shells" of equivalent stiffness which are concentric with the crack).

The effect of the modulus ratio E_f/E together with N is shown in Figure 7.

Figures 8 and 9 show the "stress concentration" effect in the filament resulting from the existence of the crack. Here, $\sigma^c_{fzz}(0)$ is the axial stress in the filament at $z = 0$ obtained from (20) in the presence of the crack for a single filament whereas $\sigma^o_{fzz}(0)$ is the filament stress at $z = 0$ calculated from (20) in the absence of the crack (i.e., obtained by ignoring σ^b_{zz} and σ^d_{zz}). Figure 8 shows the effect of the modulus ratio and the distance b, Figure 9 shows the effect of the filament dimensions r_o and c, and the distance b.

Figures 10 and 11 show the effect of the modulus ratio E_f/E on the actual filament stress at $z = 0$ for a single filament in the cracked matrix. The result for $E_f/E = 1$ (i.e., the case of no filament) is obtained from the simple penny-shaped crack solution (e.g., [4]) as the stress σ_{zz} at $(r = b, z = 0)$. Note that as b/a increases, the filament stress at $z = 0$ approaches E_f/E asymptotically. This is the known result in the uncracked matrix for the case of reinforcement by fibers or long filaments.

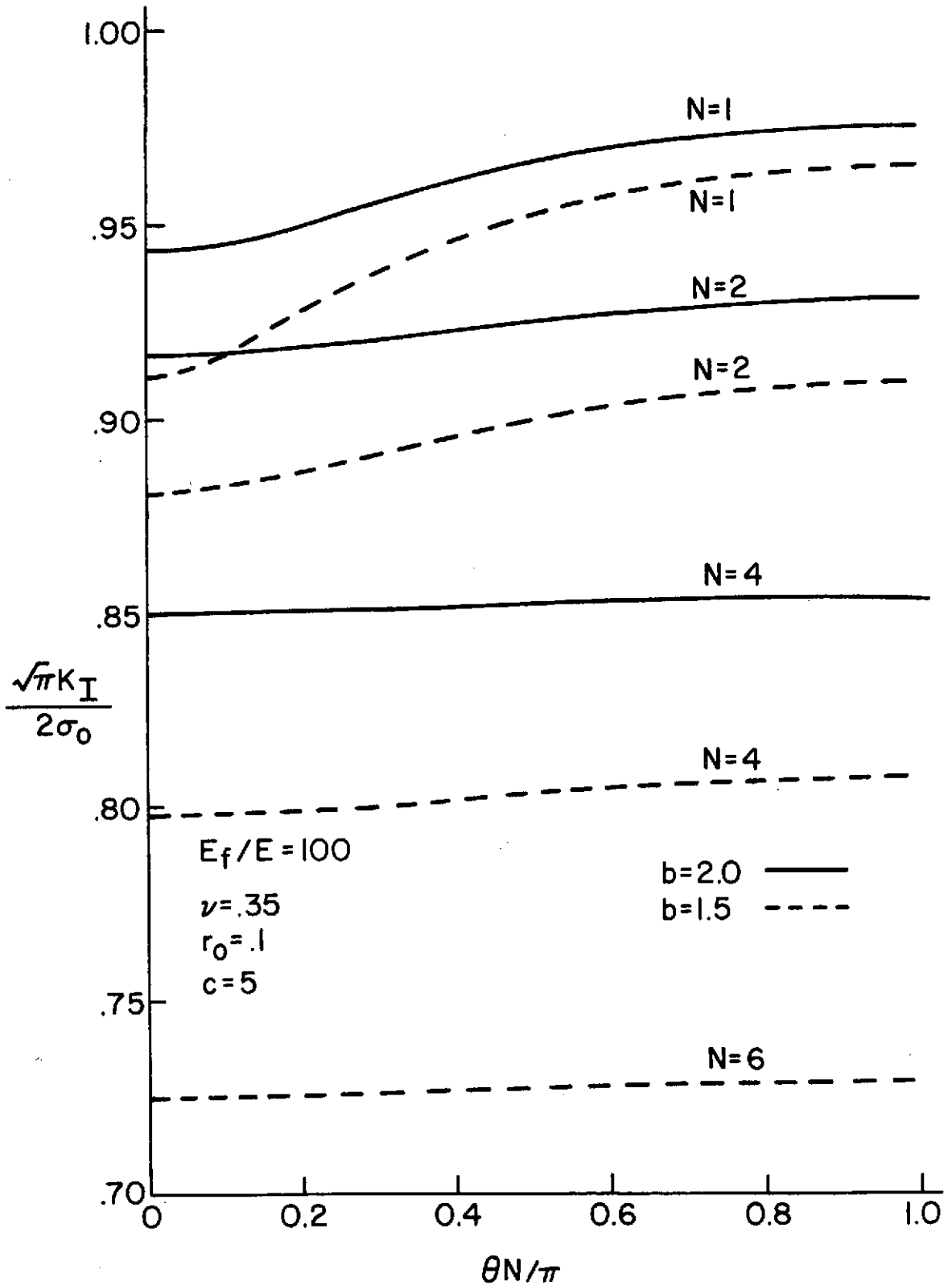

Figure 6. The effect of the number of filaments N on the stress intensity factor.

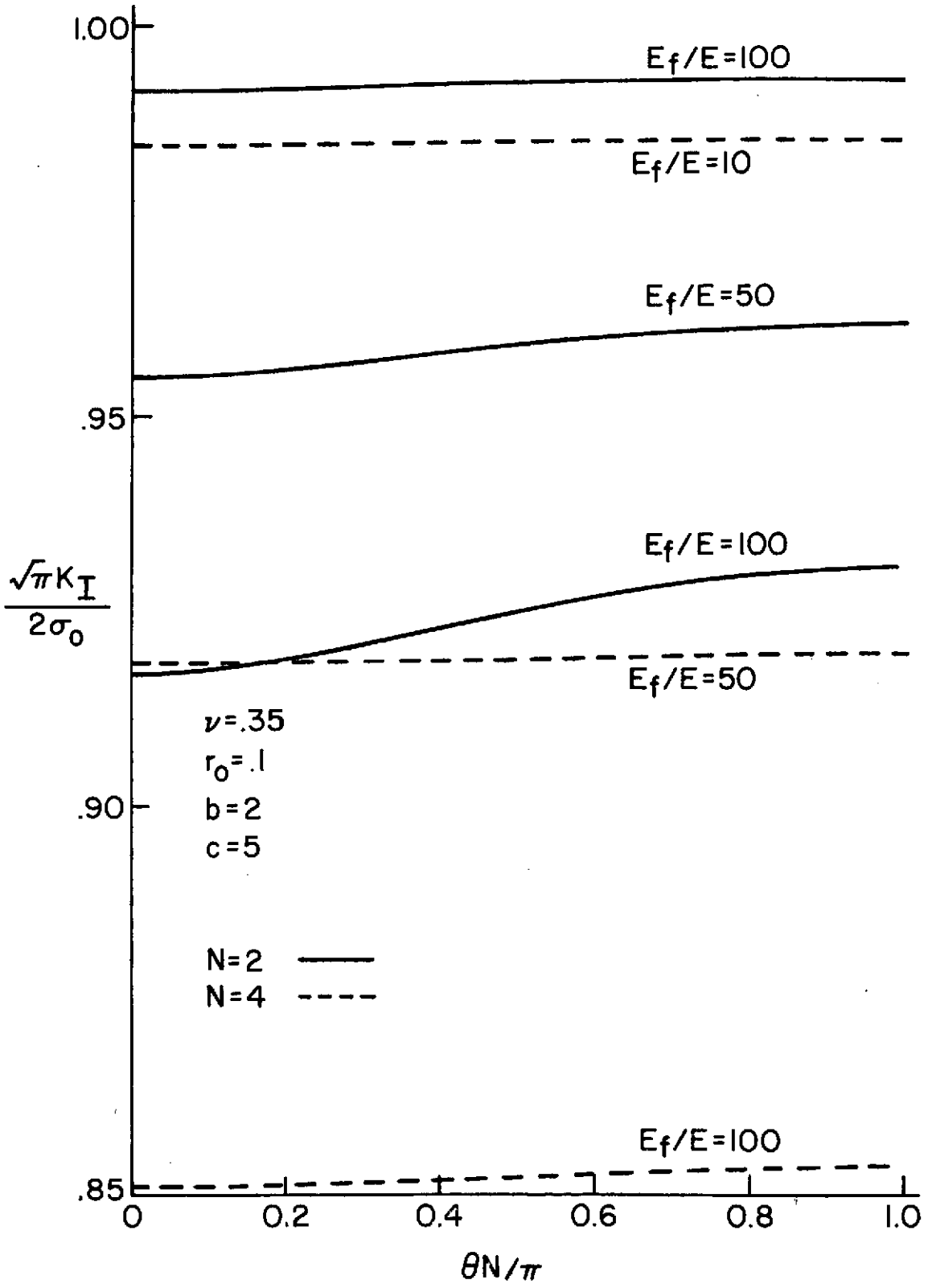

Figure 7. The effect of the modulus ratio E_f/E_N on the stress intensity factor.

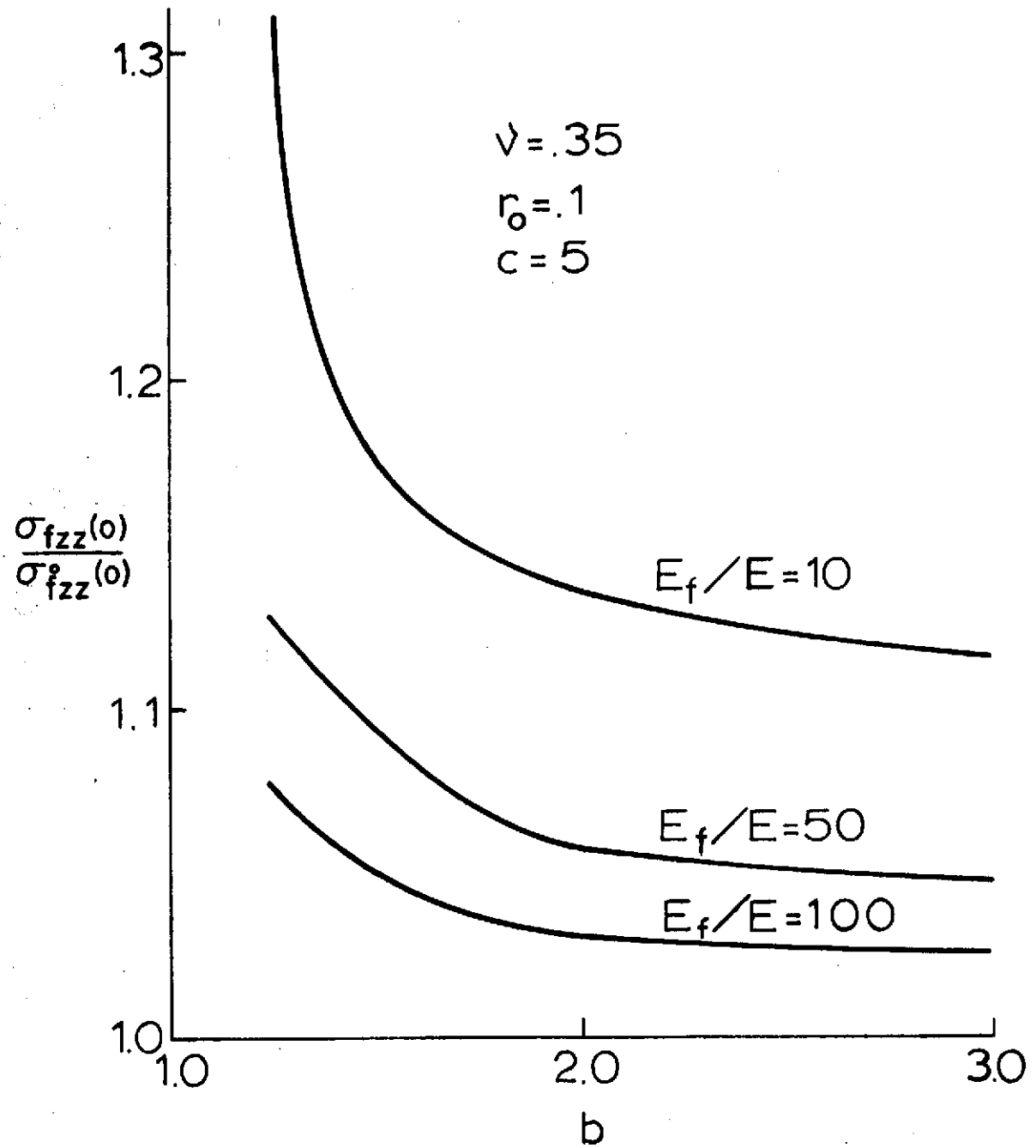

Figure 8. The ratio of the maximum filament stress in a cracked matrix to that in an uncracked matrix for a single filament.

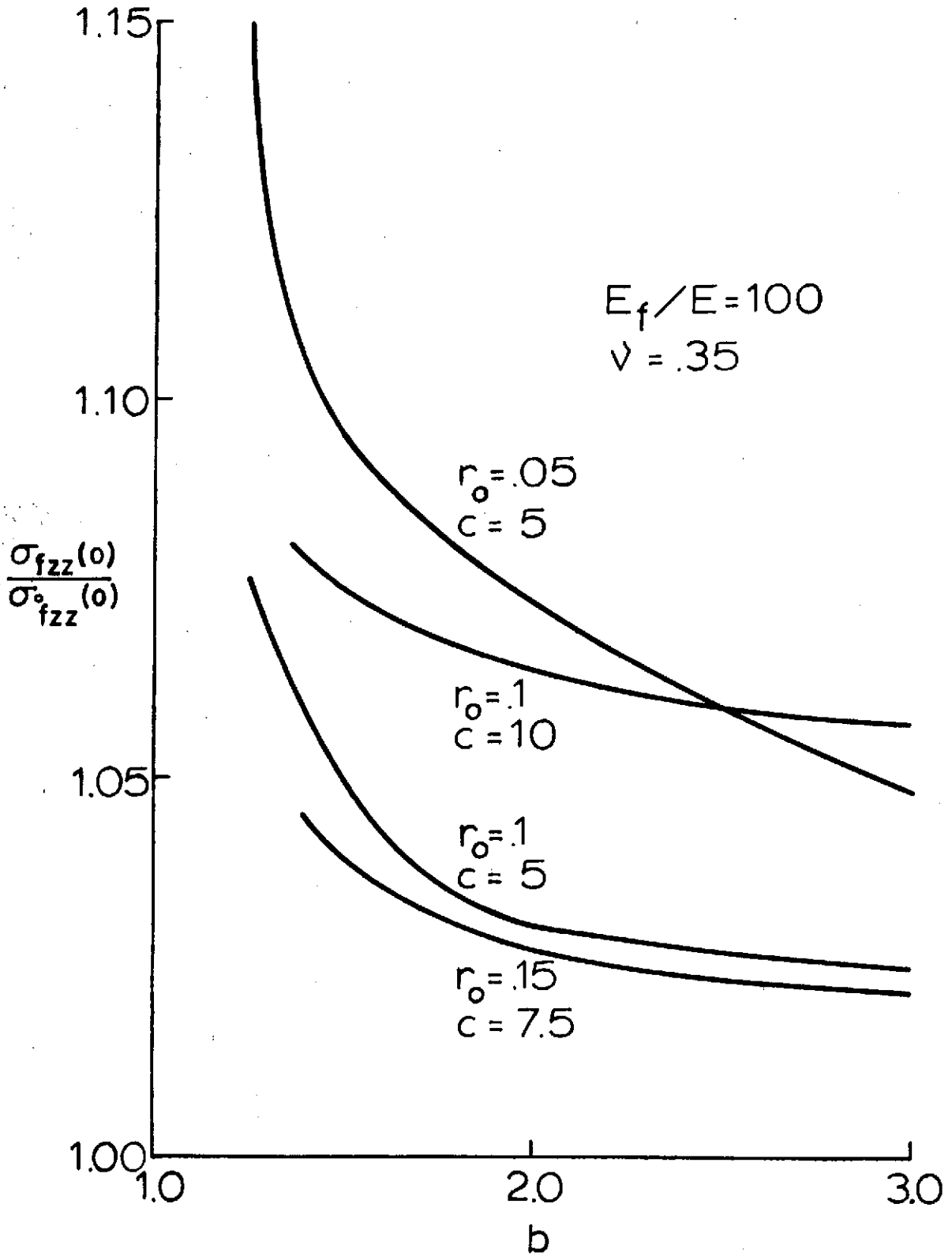

Figure 9. The ratio of the maximum filament stress in a cracked matrix to that in an uncracked matrix for a single filament, the effect of filament dimensions.

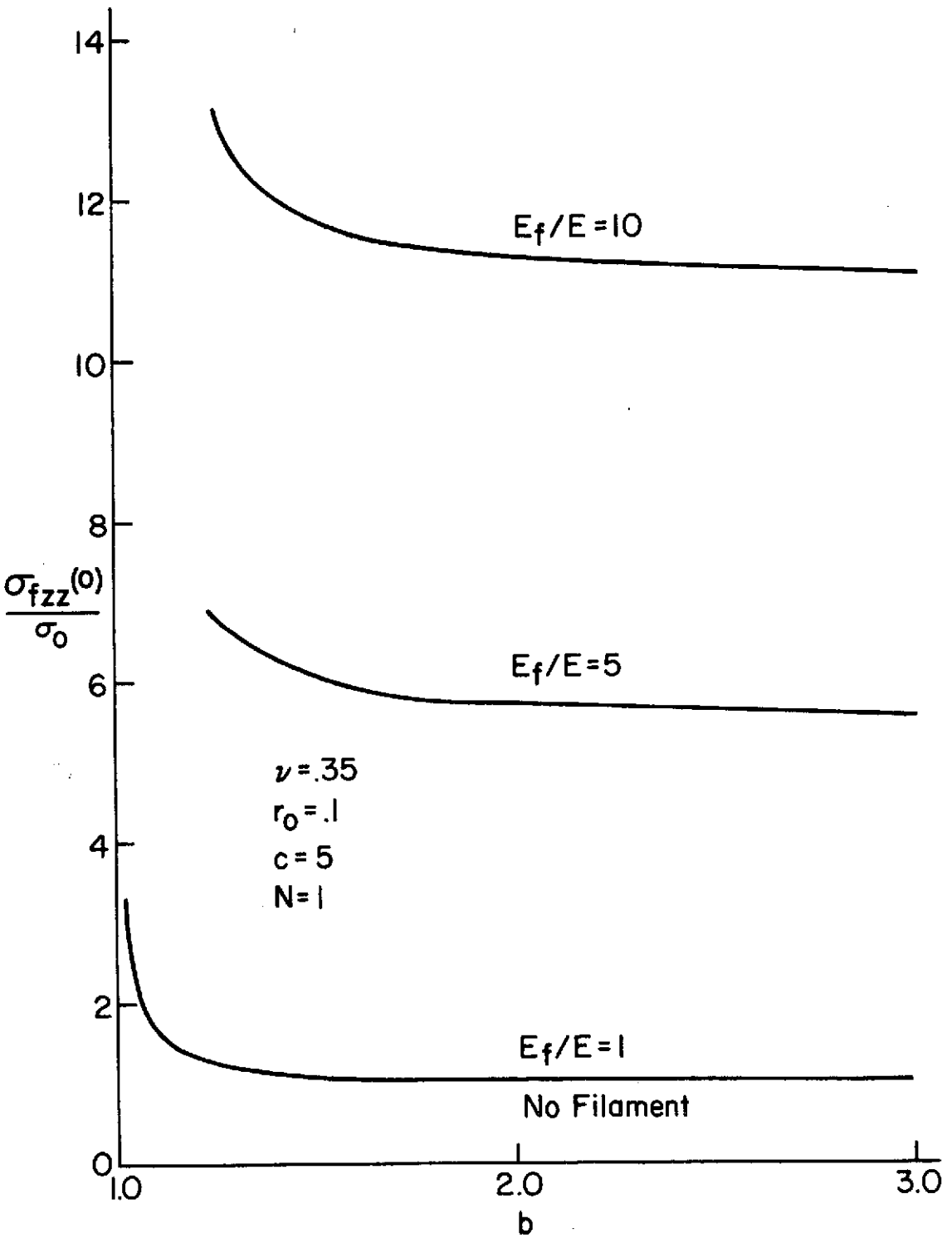

Figure 10. Maximum filament stress for a single filament in the cracked matrix.

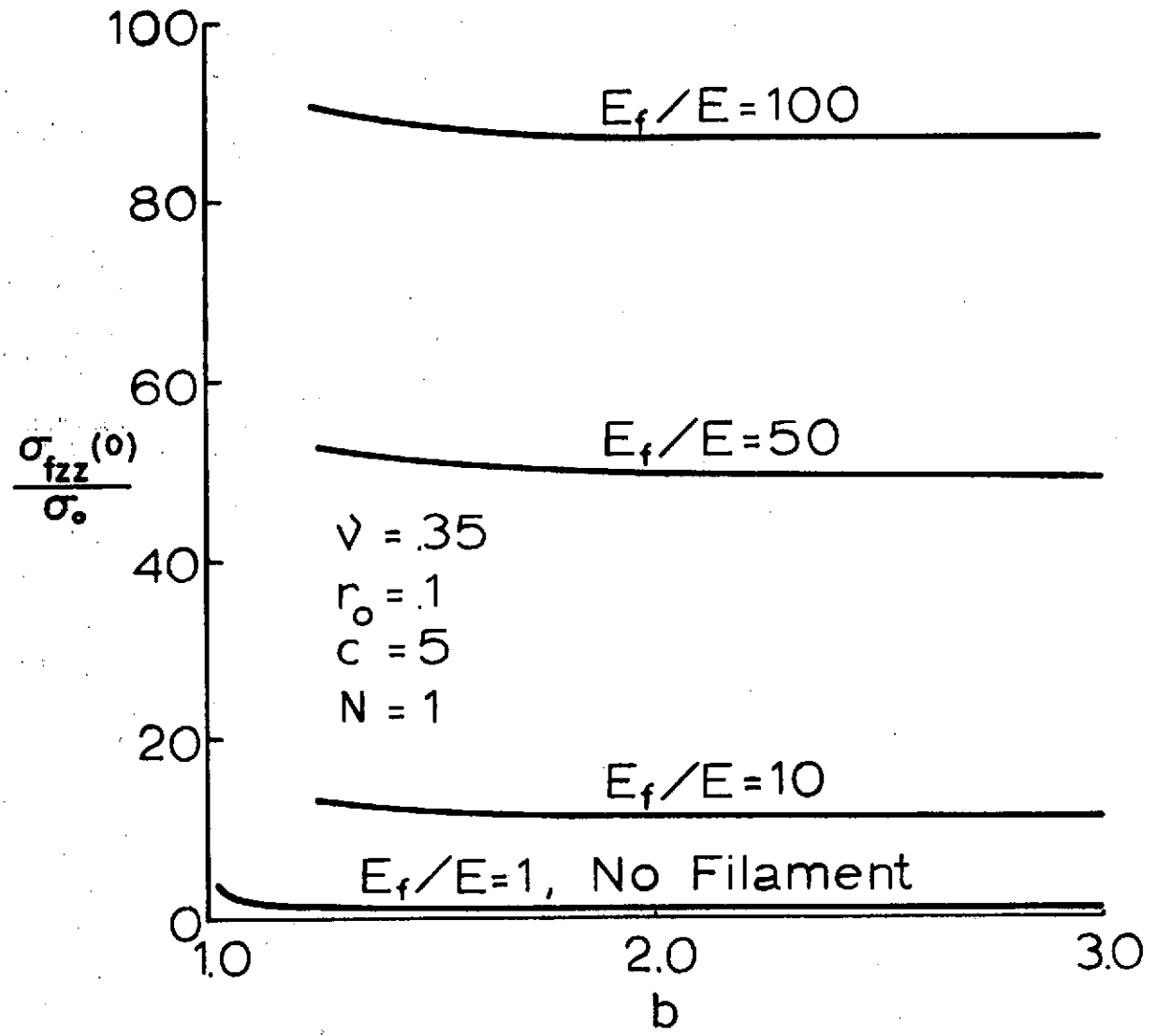

Figure 11. Maximum filament stress for a single filament in the cracked matrix, the effect of the modulus ratio E_f/E.

Finally, Figure 12 shows the effect of the number of filaments N and the modulus ratio E_f/E on the filament stress at $z = 0$ in the presence and in the absence [1] of the crack. The "stress ratio" given in the figure is the ratio of the filament stress $\sigma_{fzz}(0)$ to the filament stress at $z = 0$ obtained from the solution of a single filament in the uncracked matrix.

The analysis given in this paper may easily be extended to filaments with different dimensions, locations, and elastic moduli provided they are oriented perpendicular to the plane of the crack and this plane is a plane of symmetry with respect to the external loads and the geometry of the composite medium. In this case, the formulation of the problem would lead to a system of singular integral equations in the body forces $Z_1(z),\ldots,Z_N(z)$ imbedded in the filament-matrix interface, N being the number of filaments. In the case of symmetrically located large number of filaments, the problem may be treated as being axisymmetric with unknown body forces imbedded along cylindrical surfaces concentric with the crack. These extensions will be considered in the next paper which deals with a somewhat more practical problem of infinite fibers rather than finite filaments.

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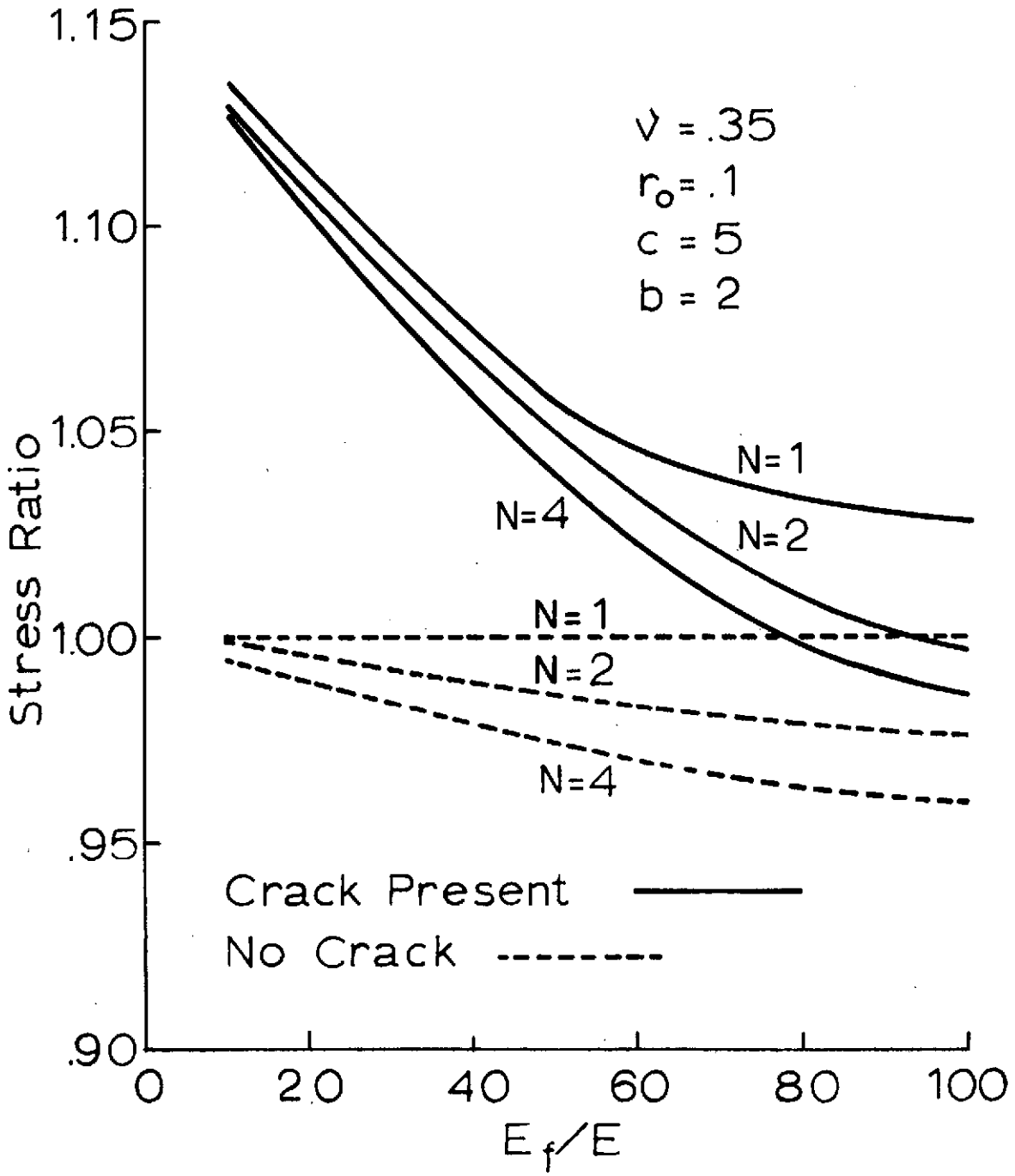

Figure 12. The ratio of the maximum filament stress in a cracke and in an uncracked matrix to that for a single filament in an uncracked matrix.

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