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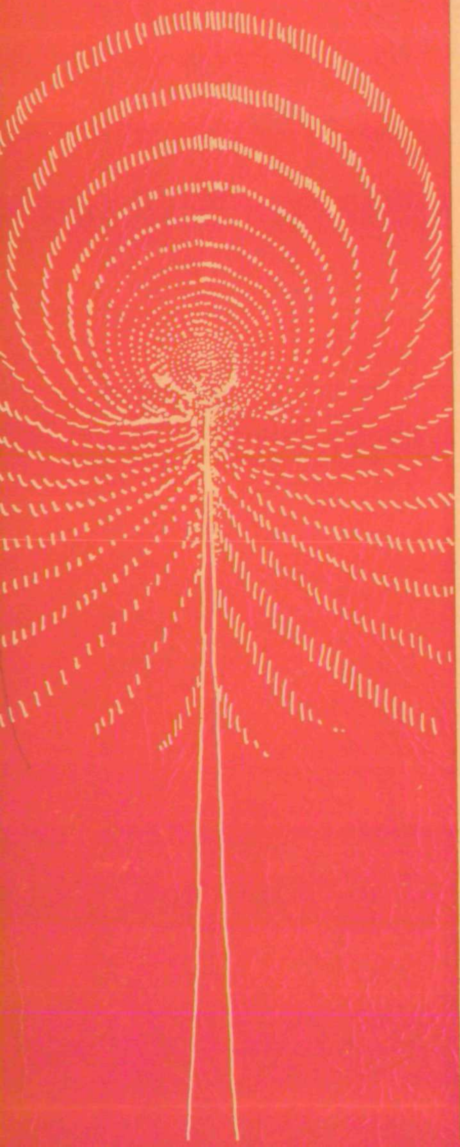

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TRANSVERSE CRACKS IN A STRIP WITH REINFORCED SURFACES*

by

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Abstract. The symmetrical problem of two transverse cracks in an elastic strip with reinforced surfaces is formulated in terms of a singular integral equation. The special cases of one central crack or two edge cracks are discussed. Numerical methods for solving the problems with internal cracks are outlined and stress intensity factors are presented for various geometries and degrees of surface reinforcement.

1. INTRODUCTION

In a recent paper [1] Gupta and Erdogan have treated the problem of symmetrical transverse cracks in an elastic strip by means of singular integral equations. References to alternative approaches to similar problems are given in [1]. In this paper we consider transverse cracks in an elastic strip with reinforced surfaces. The bending stiffness of the surface reinforcement is neglected, and the effect of the reinforcement is accounted for by a modification of the boundary conditions for the elastic material. The problem is formulated as a singular integral equation with the derivative of the normal crack surface displacement as the unknown function. A numerical solution is carried out for several geometries and degrees of reinforcement, and the stress intensity factors are given.

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2. THE INTEGRAL EQUATIONS

Consider the strip shown in Figure 1. The geometry and the external loads are symmetric with respect to $x=0$ and $y=0$. The sides of the strip are reinforced with membranes with finite stretching stiffness T. The bending stiffness of the membranes is neglected. By using symmetry considerations it is seen that the corresponding perturbation problem can be formulated with the following boundary conditions:

$$u(0,y) = 0 , \qquad \sigma_{xy}(0,y) = 0 , \qquad 0 \le y < \infty; \qquad (1.a,b)$$

$$\sigma_{xx}(h,y) = 0 , \qquad \sigma_{xy}(h,y) = T \frac{\partial^2}{\partial y^2} v(h,y) , \qquad 0 \le y < \infty; \qquad (2.a,b)$$

$$\sigma_{yy}(x,\infty) = 0 , \qquad \sigma_{xy}(x,\infty) = 0 , \qquad 0 \le x \le h; \qquad (3.a,b)$$

$$\sigma_{xy}(x,0) = 0 , \qquad 0 \le x \le h; \qquad (4)$$

$$\sigma_{yy}(x,0) = - \sigma(x) , \qquad a < x < b,$$

$$\frac{\partial}{\partial x} v(x,0) = 0 , \qquad 0 \le x < a, \quad b < x \le h; \qquad (5.a,b)$$

$$\int_a^b \frac{\partial}{\partial x} v(x,0)\,dx = 0 ; \qquad (6)$$

u and v are the x- and y-components of the displacement vector.

Following [1] and [2] the solution is given in the form

$$u(x,y) = - \frac{2}{\pi} \int_0^\infty \frac{m(p)}{p} \left(\frac{\kappa-1}{2} - py\right) e^{-py} \sin(px)\,dp$$

$$- \frac{2}{\pi} \int_0^\infty \left\{\frac{1}{s} \left[f(s) - \frac{\kappa-1}{2} g(s)\right] \sinh(sx)\right.$$

$$\left. + x g(s) \cosh(sx) \right\} \cos(sy)\,ds$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{m(p)}{p}\left(\frac{\kappa+1}{2} + py\right)e^{-py}\cos(px)\,dp$$

$$+ \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{s}\left[f(s) + \frac{\kappa+1}{2}g(s)\right]\cosh(sx)\right.$$

$$\left. + xg(s)\sinh(sx)\right\}\sin(sy)\,ds \qquad (7.a,b)$$

When the solution is expressed in this form the field equations and the conditions (1), (3) and (4) are satisfied. The material parameter κ is defined by $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, where ν is Poisson's ratio. The functions $m(p)$, $f(s)$ and $g(s)$ will be determined from the remaining boundary conditions (2) and (5). From (7) the stresses are found to be

$$\frac{1}{2\mu}\sigma_{xx} = -\frac{2}{\pi}\int_0^\infty m(p)(1-py)e^{-py}\cos(px)\,dp$$

$$-\frac{2}{\pi}\int_0^\infty [f(s)\cosh(sx) + sxg(s)\sinh(sx)]\cos(sy)\,ds \ ,$$

$$\frac{1}{2\mu}\sigma_{yy} = -\frac{2}{\pi}\int_0^\infty m(p)(1+py)e^{-py}\cos(px)\,dp$$

$$+\frac{2}{\pi}\int_0^\infty \{[f(s) + 2g(s)]\cosh(sx)$$

$$+ sxg(s)\sinh(sx)\}\cos(sy)\,ds \ ,$$

$$\frac{1}{2\mu}\sigma_{xy} = -\frac{2}{\pi}\int_0^\infty ypm(p)e^{-py}\sin(px)\,dp$$

$$+\frac{2}{\pi}\int_0^\infty \{[f(s) + g(s)]\sinh(sx)$$

$$+ sxg(s)\cosh(sx)\}\sin(sy)\,ds \qquad (8.a\text{-}c)$$

where μ is the shear modulus. By transforming (2) and (8) the following equations are obtained

$$f(s)\cosh(sh) + shg(s)\sinh(sh) = -\frac{4s^2}{\pi}\int_o^\infty \frac{pm(p)}{(p^2+s^2)^2}\cos(ph)\,dp\,,$$

$$f(s)\sinh(sh) + g(s)[\sinh(sh) + (1+\Phi)sh\cosh(sh)]$$

$$= \frac{4s}{\pi}\int_o^\infty \frac{p^2 m(p)}{(p^2+s^2)^2}\sin(ph)\,dp + \Phi sh\,\frac{2}{\pi}\int_o^\infty \frac{pm(p)}{p^2+s^2}\cos(ph)\,dp$$

(9.a,b)

where the degree of reinforcement Φ is defined by

$$\Phi = \frac{(\kappa+1)}{4\mu h}T = \begin{cases} 2\,\dfrac{T}{Eh} & \text{for plane stress} \\ 2\,\dfrac{(1-\nu^2)T}{Eh} & \text{for plane strain} \end{cases} \tag{10}$$

Straightforward use of the mixed boundary condition (5) for the determination of the last unknown function m(p) would lead to a set of dual integral equations. Here the method from [1] is used. A new unknown function G(x) is defined as

$$G(x) = \frac{\partial}{\partial x}v(x,0)\,, \qquad 0 \le x \le h \tag{11}$$

By use of (5.b) and (7.b) the following relation is obtained

$$m(p) = -\frac{2}{\kappa+1}\int_a^b G(t)\sin(pt)\,dt \tag{12}$$

Substituting for m(p) by means of (12) in (9) and (8.b) the last boundary condition (5.a) takes the form

$$\int_a^b H(x,t)G(t)\,dt = -\frac{1+\kappa}{4\mu}\pi\sigma(x)\,, \qquad a < x < b \tag{13}$$

Noting that the first term of (8.b) gives rise to a Cauchy-type singularity, (13) may be expressed as

$$\int_a^b \left[\frac{1}{t-x} + \frac{1}{t+x} + k(x,t) \right] G(t)dt = -\frac{1+\kappa}{4\mu} \pi\sigma(x) ,$$

$$a < x < b , \qquad (14)$$

where

$$k(x,t) = \int_0^\infty [K(x,t,s)e^{ts} - K(x,-t,s)e^{-ts}]ds$$

$$\begin{aligned} K(x,t,s) = \{ & [(1-\Phi sh)(sh-2) - (2sh+\Phi sh-3)s(h-t) \\ & - ((1-\Phi sh)(sh+2) - (1-\Phi sh)s(h-t))e^{-2sh}]\cosh(sx) \\ & - [1-\Phi sh - 2s(h-t) + (1-\Phi sh)e^{-2sh}]sx\sinh(sx) \} \\ & / [\sinh(2sh) + 2sh + 2\Phi sh\cosh^2(sh)] \end{aligned} \qquad (15.a,b)$$

The index of the singular integral equation is +1, and the necessary extra condition is given by (6),

$$\int_a^b G(x)dx = 0 \qquad (16)$$

For $0 < a < b < h$ the kernel $k(x,t)$ is bounded. Thus (14) is an ordinary singular integral equation with the fundamental function

$$w(x) = [(x-a)(b-x)]^{-1/2} \qquad (17)$$

A numerical solution can then be obtained by the method described in [3] and [4].

For $a = 0$, $b < h$ the problem reduces to a single central crack. No additional condition is present in this case. The term $1/(t+x)$ now becomes unbounded for $x,t \to 0$. This influences the fundamental function. $k(x,t)$ is bounded and the fundamental function may be determined in a straightforward manner by variable transformation

[5] or reflection in the line $x=0$ [1]*. The result is

$$w(x) = (b^2 - x^2)^{-1/2} \tag{18}$$

3. EDGE CRACKS

For $a > 0$, $b = h$ the character of the integral equation changes as the function $k(x,t)$ becomes unbounded for $x,t \to h$. In the following calculations there is an essential difference between the cases $\Phi = 0$ and $\Phi \neq 0$. This is caused by the denominator of (15.b), where the term of highest order for $s \to \infty$ is multiplied by Φ. As the case $\Phi = 0$ has been treated in detail in [1], only $\Phi \neq 0$ will be considered here. In (15.b) terms which for $x,t \to h$ become of order less than $1/s$ for $s \to \infty$ are extracted and integrated analytically. This is done by use of the formula [6]

$$\int_0^\infty s^m e^{-s(2h-t)} \begin{Bmatrix} \sinh(sx) \\ \cosh(sx) \end{Bmatrix} ds = \frac{d^m}{dt^m} \int_0^\infty e^{-s(2h-t)} \begin{Bmatrix} \sinh(sx) \\ \cosh(sx) \end{Bmatrix} ds$$

$$= \frac{d^m}{dt^m} \left[\frac{1}{(2h-t)^2 - x^2} \begin{Bmatrix} x \\ 2h - t \end{Bmatrix} \right]$$

$$= -\frac{1}{2} \frac{d^m}{dt^m} \left[\frac{1}{t - (2h-x)} \mp \frac{1}{t - (2h+x)} \right] \tag{19}$$

It appears that only the terms which originally contained the factor Φ contribute to the singular part of the kernel. Only these terms are extracted in the following formulation.

$$\int_a^h \left[\frac{1}{t-x} - \frac{1}{t-(2h-x)} + k_f(x,t) \right] G(t) = -\frac{\kappa+1}{4} \pi\sigma(x) \ ,$$

$$a < x < h, \tag{20}$$

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*The reflection technique applied in [1] requires a slight modification of the kernel, which would otherwise be defined by a divergent integral.

where

$$k_f(x,t) = \frac{1}{t+x} - \frac{1}{t-(2h+x)}$$

$$+ \int_0^\infty [K(x,t,s)e^{ts} - K(x,-t,s)e^{-ts} - K_\infty(x,t,s)e^{ts}]ds$$

$$K_\infty(x,t,s) = 2e^{-sh}\{[-sh + 2 - s(h-t)]\cosh(sx) + sx\sinh(sx)\}$$

(21.a,b)

The first two terms in the kernel of (20) constitute a generalized Cauchy kernel. It is noted that this part of the kernel — and thereby the form of the singularities — is independent of Φ, provided $\Phi \neq 0$. Because of the simple form of the generalized Cauchy kernel the fundamental function can be determined directly by change of variables [5]. The result is similar to (18).

$$w(x) = [(a-h)^2 - (x-h)^2]^{-1/2} \tag{22}$$

In the case where a strip with edge cracks is loaded with stresses not acting on the crack surfaces, the problem formulated here corresponds to membranes broken at $y = 0$. This is seen in the following way. First the boundary value problem for the uncracked strip is solved. This solution yields the normal stress $\sigma(x)$ on the crack surfaces. For reasons of continuity the membrane force at $y = 0$ is

$$S = \frac{T}{\mu}\frac{\kappa+1}{8}\lim_{x\to h}\sigma(x) \tag{23}$$

When the perturbation problem is solved with the load $-\sigma(x)$ on the crack surfaces, continuity requires the membrane to be loaded with the force $-S$. Thus the resulting membrane force is zero at $y = 0$

and no interaction takes place between the membrane for $y > 0$ and $y < 0$.

4. NUMERICAL METHOD

The objective is to determine G(t) from the singular integral equation (14). First we consider the case $0 < a < b < h$. The kernel then has a normal Cauchy singularity and we normalize the variables by defining τ and ξ

$$x = (b+a)/2 + \xi(b-a)/2$$

$$t = (b+a)/2 + \tau(b-a)/2 \tag{24.a,b}$$

Furthermore we introduce the notation

$$G(t) = g(\tau)(1-\tau^2)^{-1/2} \tag{25}$$

where we have used the fundamental function (17). (14) may now be expressed as

$$\int_{-1}^{1} M(\xi,\tau)g(\tau)(1-\tau^2)^{-1/2}d\tau = -\frac{1+\kappa}{4\mu}\pi\sigma(\xi), \quad -1 < \xi < 1 \tag{26}$$

$\sigma(\xi)$ is used for convenience for $\sigma(x)$ in connection with (24.a).

$$M(\xi,\tau) = (\tau-\xi)^{-1} + (\tau+\xi)^{-1} + \frac{b-a}{2}k(x,t) \tag{27}$$

(16) takes the form

$$\int_{-1}^{1} g(\tau)(1-\tau^2)^{-1/2}d\tau = 0 \tag{28}$$

Based on the assumption that $g(\tau)$ can be approximated by an (n-1) order polynomial we obtain [3]

$$\sum_{j=1}^{n} M(\xi_i,\tau_j)g(\tau_j) = -\frac{1+\kappa}{4\mu}n\sigma(\xi_i)$$

$$\sum_{j=1}^{n} g(\tau_j) = 0$$

$$\xi_i = \cos(\frac{i\pi}{n}) \ , \qquad i = 1,2,\ldots,n-1$$

$$\tau_j = \cos(\frac{2j-1}{2n}\pi) \ , \qquad j = 1,2,\ldots,n \qquad (29.a\text{-}d)$$

(29) gives n linear equations for the determination of the n values $g(\tau_j)$. Special importance is attached to the values $g(1)$ and $g(-1)$ giving the stress intensity factors in dimensionless form. As proved in [4], $g(1)$ is determined by

$$g(1) = \frac{1}{n} \sum_{j=1}^{n} \frac{\sin[\frac{2n-1}{4n}(2j-1)\pi]}{\sin[\frac{2j-1}{4n}\pi]} g(\tau_j) \qquad (30)$$

$g(-1)$ is found from a similar formula with $g(\tau_j)$ exchanged by $g(\tau_{n+1-j})$.

The case $a = 0$, $b < h$ is solved by a similar method. We now define τ and ξ by

$$x = b\xi \ , \qquad t = b\tau \qquad (31.a,b)$$

(25) remains valid, and the integral equation (14) becomes

$$\int_0^1 N(\xi,\tau)g(\tau)(1-\tau^2)^{-1/2}d\tau = -\frac{1+\kappa}{4\mu}\pi\sigma(\xi) \ , \qquad 0 < \xi < 1 \qquad (32)$$

$$N(\xi,\tau) = (\tau-\xi)^{-1} + (\tau+\xi)^{-1} + bk(x,t) \qquad (33)$$

By using the fact that $g(\tau)$ and $N(\xi,\tau)$ are odd functions of τ we may use the formulas (29) to evaluate the integral in (32). We use 2n instead of n in the formulas (29) and thereby obtain:

$$\sum_{j=1}^{n} N(\xi_i,\tau_j)g(\tau_j) = -\frac{1+\kappa}{4\mu} 2n\sigma(\xi_i)$$

$$\xi_i = \cos(\frac{i}{2n}\pi) , \qquad i = 1,2,\ldots,n$$

$$\tau_j = \cos(\frac{2j-1}{4n}\pi) , \qquad j = 1,2,\ldots,n \qquad (34.a-c)$$

$g(1)$ is obtained from (30).

5. RESULTS

First we consider the case shown in Figure 1. The value of a is kept constant and b and Φ are varied. The Tables 1 and 2 contain the stress intensity factors for constant σ. When comparing results for different values of b it must be kept in mind that the stress intensity factors are normalized with respect to different values of ℓ. For $a = 0.2h$, $b = 0.6h$ and $\Phi = 0$ the following results are given in [1]:

$$k(a)/\sigma\sqrt{\ell} = 1.1102 , \qquad k(b)/\sigma\sqrt{\ell} = 1.0961$$

Table 3 contains results for a central crack. For $\Phi = 0$ the results may be compared with similar data given in [8] and [9]. With the exception of $b = 0.9h$ there is agreement to within one unit of the last digit. For $\Phi \neq 0$ curves are given in [8]. Within the error of this kind of comparison the agreement is good.

Table 1. The stress intensity factor at a for a = 0.2h.

a/h=0.2	$k(a)/\sigma\sqrt{\ell}$, ℓ = (b-a)/2				
b/h	Φ = 0	Φ = 0.2	Φ = 1.0	Φ = 2.0	Φ = 1000
0.3	1.009	1.008	1.007	1.006	1.005
0.4	1.031	1.029	1.024	1.020	1.013
0.5	1.064	1.058	1.046	1.039	1.020
0.6	1.109	1.097	1.073	1.059	1.025
0.7	1.169	1.148	1.105	1.081	1.025
0.8	1.253	1.213	1.140	1.102	1.017
0.9	1.38	1.30	1.18	1.12	1.00

Table 2. The stress intensity factor at b for a = 0.2h.

a/h=0.2	$k(b)/\sigma\sqrt{\ell}$, ℓ = (b-a)/2				
b/h	Φ = 0	Φ = 0.2	Φ = 1.0	Φ = 2.0	Φ = 1000
0.3	1.008	1.007	1.006	1.005	1.004
0.4	1.026	1.024	1.018	1.015	1.007
0.5	1.053	1.047	1.034	1.026	1.008
0.6	1.094	1.080	1.053	1.037	1.001
0.7	1.156	1.127	1.074	1.046	0.984
0.8	1.263	1.197	1.094	1.045	0.946
0.9	1.51	1.31	1.09	1.00	0.85

Table 3. Stress intensity factor at b for a = 0.

a/h = 0	$k(b)/\sigma\sqrt{b}$				
b/h	Φ = 0	Φ = 0.2	Φ = 1.0	Φ = 2.0	Φ = 1000
0.1	1.006	1.005	1.003	1.002	0.988
0.2	1.025	1.021	1.011	1.006	0.992
0.3	1.058	1.048	1.026	1.013	0.981
0.4	1.110	1.089	1.047	1.023	0.965
0.5	1.187	1.150	1.075	1.035	0.942
0.6	1.304	1.237	1.112	1.048	0.911
0.7	1.488	1.366	1.156	1.058	0.867
0.8	1.817	1.567	1.201	1.055	0.802
0.9	2.57	1.89	1.21	1.00	0.68

REFERENCES

[1] Gupta, G.D. and Erdogan, F.
"The problem of edge cracks in an infinite strip", Lehigh University, Report IFSM-73-38, April 1973 (to be published).

[2] Sneddon, I.N. and Srivastav, R.P.
"The stress field in the vicinity of a Griffith crack in a strip of finite width", Int. J. Engng. Sci., 1971, Vol. 9, pp. 479-488.

[3] Erdogan, F. and Gupta, G.D.
"On the numerical solution of singular integral equations", Quart. Appl. Math., 1972, Vol. 30, pp. 525-534.

[4] Krenk, S.
"A note on the use of the interpolation polynomial for solutions of singular integral equations", Lehigh University, Report IFSM-73-48, August 1973 (to appear in Quart. Appl. Math.)

[5] Spence, D.A.
"An eigenvalue problem for elastic contact with finite friction", Proc. Camb. Phil. Soc., Vol. 73, 1973, pp. 249-268. See Appendix A.

[6] Erdelyi, A., ed.
Tables of integral transforms, Vol. 1, McGraw-Hill, New York, 1953.

[7] Erdogan, F., Gupta, G.D. and Cook, T.S.
"Numerical solution of singular integral equations", in Methods of analysis and solutions of crack problems, G.C. Sih, ed., Noordhoff International Publishing, Leyden, 1973.

[8] Isida, M.
"Laurant series expansion for internal crack problems", in Methods of analysis and solutions of crack problems, G.C. Sih, ed., Noordhoff International Publishing, Leyden, 1973.

[9] Krenk, S.
"The problem of an inclined crack in an elastic strip", Lehigh University, Report IFSM-73-52, December 1973 (to be published).

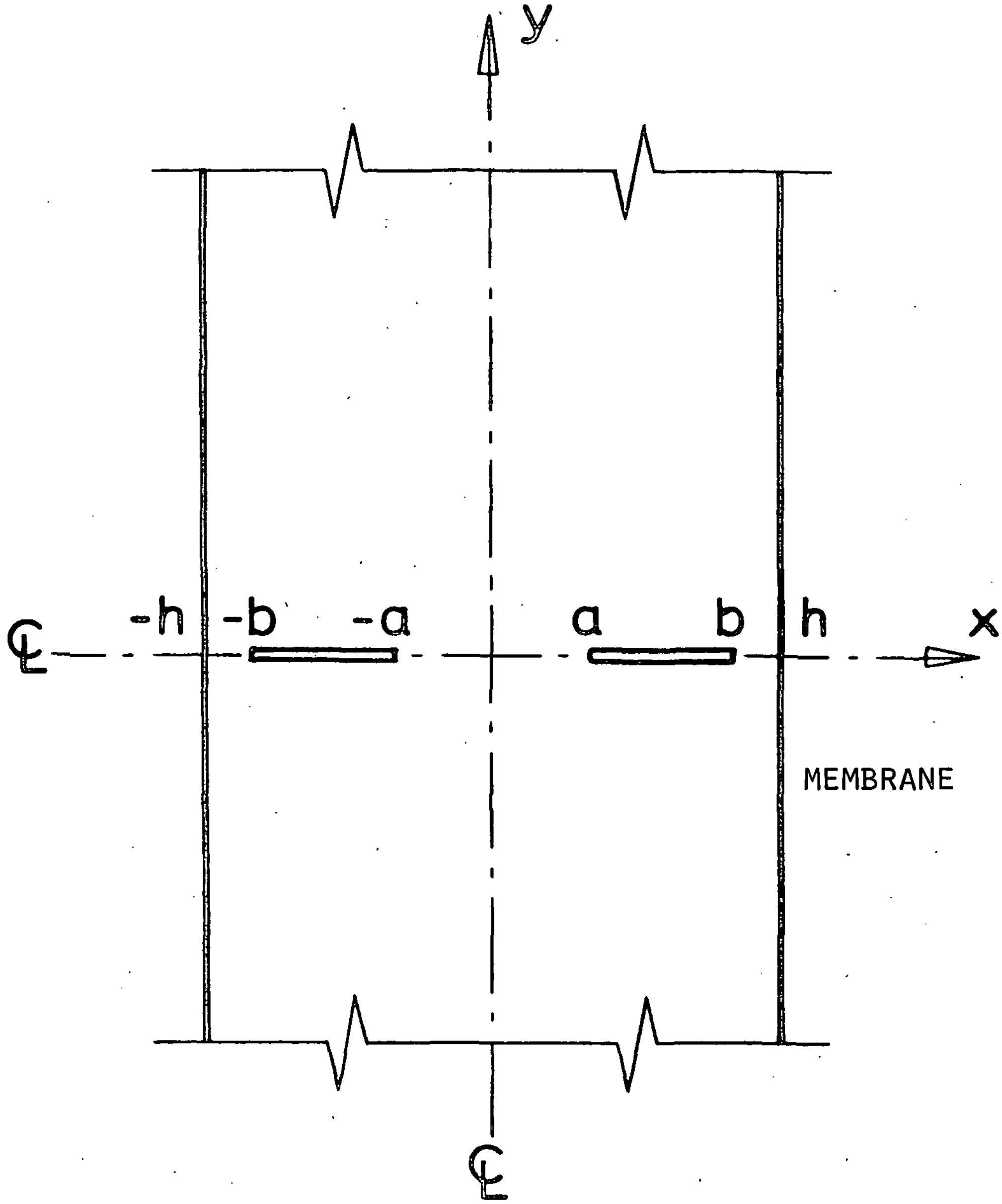

Figure 1. Infinite strip with membrane reinforced surfaces.