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RELATIVE MOTION OF ORBITING PARTICLES  
UNDER THE INFLUENCE OF  
PERTURBING FORCES

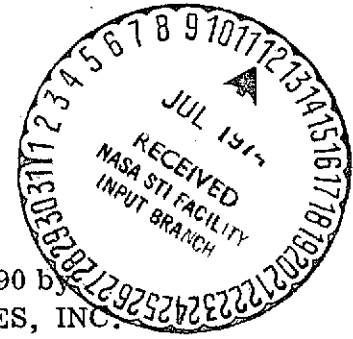
Volume I

(Summary)

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J.B. Eades, Jr.

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## FOREWORD

This report was prepared under NASA Contract NAS1-11990. The work was conducted under the direction of J. W. Drewry, Space Application and Technology Division, Langley Research Center.

The work done by Analytical Mechanics Associates, Inc. was under the direction of Dr. J. B. Eades, Jr.

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RELATIVE MOTION OF ORBITING PARTICLES  
UNDER THE INFLUENCE OF  
PERTURBING FORCES

By J. B. Eades, Jr.\*

SUMMARY

The relative motion for orbiting vehicles, under the influence of various perturbing forces, has been studied to determine what influence these inputs, and others, can have.

This report summarizes the work accomplished; it describes the analytical tasks, in general terms; it outlines the force types considered, modelled and simulated; and, it denotes the capabilities of the computer programs which have evolved in support of this work.

Subsequent reports give explicit information on the analytical and computational tasks completed. The purpose here is to outline the work effort and to present a summary of the accomplishments to date.

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## SYMBOLS

A	Reference area.
B	Special matrix (see Ref. [1]).
$C_r$	Reference value for solar radiation pressure.
D, $C_D$	Drag force; drag coefficient (dimensionless).
$\bar{e}_s$	Heliocentric unit radius vector.
F	General designation for force.
G	Universal gravitational constant.
$I_j$	Idem matrix (rank j).
$J_i$	Oblateness coefficients.
$k_r$	Coefficient of reflectivity.
M	Designation for central attracting mass; distance in meters.
m	Mass of a particle.
$\bar{n}$	Unit normal vector.
P, Q	Designation for bodies "P" and "Q", respectively.
$\bar{P}, \bar{Q}$	Same as $\bar{r}_P, \bar{r}_Q$ ; body position vectors.
q	Surface charge.
$\bar{r}_P, \bar{r}_Q$	Position vectors, for bodies P and Q.
$\bar{r}_r$	Relation position vector (usually $\bar{r}_r \equiv \bar{r}_Q - \bar{r}_P$ ).
$\Delta s$	Separation distance (see Eq. (I.12)).
$T(\varphi^\pm)$	The transformation matrix (relating the two frames of reference).
t	Time.

$\bar{V}$	Velocity vector.
X, Y, Z	Designation for relative position coordinates, referred to the inertially oriented frame of reference (centered at P); Z is normal to the nominal plane of motion.
x, y, z	Designation for relative position coordinates, referred to the local horizon frame of reference (centered at P); z is normal to the nominal plane of motion.
$\alpha$	Inclination angle for a velocity impulse (see Eq. (I.11)).
$\beta$	Ballistic coefficient (see Eq. (I.13)).
$\gamma$	Inclination angle for the orbit plane (see Eq. I.10)).
$\Gamma$	Speed ratio.
$\delta$	Geocentric latitude.
$\Delta$	Radii ratio, $ \bar{Q}  /  \bar{P} $ .
$\mu$	Gravitational parameter.
$\mu_0$	Magnetic permeability.
$\rho$	Atmosphere density.
$\varphi$	Position angle, referred to the circular, reference orbit; measured from the $t = 0$ position.
$\ddot{\varphi}$	Orbital angular rate, referred to the circular reference orbit.
$\Phi$	Gravitational potential function.
$\kappa$	A general coordinate designation.

### Subscripts

$(\sim)_I$	Referred to the inertially oriented frame of reference.
$(\sim)_0$	Initial (or reference) value.
$(\sim)_{SO}$	Impulse value.

### Superscripts

$(\hat{\sim})$  Denotes unit vector.

$(\dot{\sim})$  Time derivative.



## I. INTRODUCTION

Problems dealing with relative motion in celestial mechanics are not new; the most famous of past studies could be traced to Hill and his work. While the older investigations dealt with natural phenomenon present day studies are based on a somewhat different philosophy, dealing more directly with the ideas of controlling the motion. These concepts are somewhat analogous to the general differences usually ascribed to celestial mechanics and astrodynamics wherein the latter attempts to obtain "collisions" and the former seeks to avoid such.

In keeping with the classical notions of research a part of the present investigation was devoted to an analysis of a particular class of relative motion problems. For this work a mathematical solution was obtained, one which predicts motion consequences in response to various inputs.

The general formulation here shows the usual non-linearities associated with orbital problems. Thus these expressions were manipulated in order to describe a mathematically tractable set prior to seeking analytical solutions (see Ref. [1] and [2] for details). In keeping with these operations, and in the absence of disturbance forces, the equations to be solved were a classical Euler-Hill set. Necessarily these led directly to an initial-values solution; however, with a modification, by the addition of disturbance forces -- where these play the role of forcing functions -- a new family of solutions were acquired.

One shortcoming to problem solutions of this type, as is evidenced in the literature, is that they have been cast in terms of coordinates referred to a "local horizon" frame of reference. While this is certainly an acceptable representation, it does not go far enough. For example, consider the case of a vehicle which is Sun oriented. Observations from this spacecraft would not be seen in a rotating frame of reference, rather they would be inertially defined. It stands to reason, then, that when complete descriptions of relative motions are desired one should be able to view them in either of these frames of reference. One of the undertaking

in this investigation has been that of examining motions in both frames of reference by means of mathematical and numerical determinations. Not only have the pertinent initial-values problems been studied but the consequences from perturbing forces, referred to both frames of reference, have been ascertained as well. In this regard the work to be described here is complete and extensive in its representations.

As an aid to clarifying what is implied when one speaks of these two frames of reference, Fig. I.1 has been prepared. There one finds the two particles (P, Q) with Q playing the role of a test particle and P being the reference one. Here the reference particle is assumed to move along a fixed circular orbit; Q, however, is not constrained, but is free to move "about P" under the several influences discussed above. On the figure, sketch (1) indicates the initial ( $t = 0$ ) alignment of the two frame's triads; also shown is the initial separation vector ( $\bar{r}_P$ ) which can be described in either of the coordinate systems. Of course particle Q can have an initial relative velocity ( $\dot{\bar{r}}_P$ ); this has not been included for purposes of convenience. Sketch (2) is used to illustrate the inertially oriented frame of reference, while (3) shows the local horizon frame; note that both are centered at P. Throughout this investigation the role of P is that of a reference (baseline) particle subject to the influence of a central mass attraction only. Particle Q is the one studied to determine the various influences attributed to the several conditions imposed on these problems.

While the mathematical solutions developed here are instructive in understanding the relative motions, it must be remembered that these are linearized results and are based on the assumptions of small variations about the reference orbit. To a very significant degree this constraint is removed when the "solutions" are acquired numerically; however, the use of numerical integration takes away the generality achieved in the mathematical approach. This recognition is a sufficient argument for proceeding in the dual computational modes used for this study. Even though the forces are modelled simply, for tractability and for direct

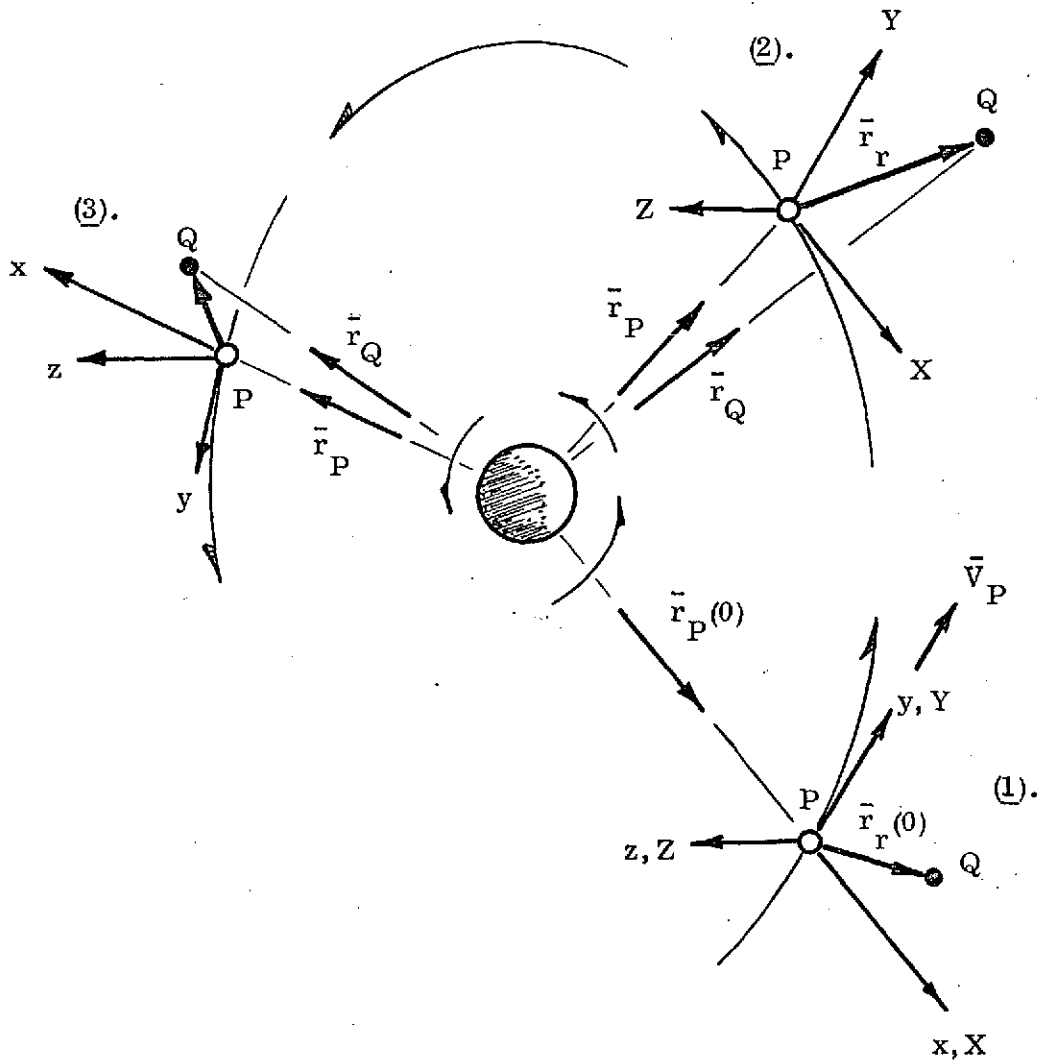


Fig. I.1. Sketch illustrating the principal frames of reference used for the relative motion problem. The local horizon frame  $(x, y, z)$  and an inertially oriented one  $(X, Y, Z)$  are shown here centered at the reference orbiting particle  $P$ . The test particle,  $Q$ , is located by the relative position vector  $\vec{r}_r$ . Position vectors  $(\vec{r}_P, \vec{r}_Q)$  locate the two particles with respect to the attracting center.

correlation of results, the imaginative reader will quickly recognize the many possibilities which can be simulated here. Since the forces are described in both frames of reference the possibilities for approximating a larger number of physical systems are obvious.

The particulars of the general mathematical results can be found in Refs. [1] and [2]; only a summary of these findings will be given herein. For those readers who are interested in analytical developments, it is recommended that they review the above documents and Ref. [6]. From these one finds that the solutions are expressed by sums of vectors, each vector describing a particular geometry. When the vectors are added they produce the several planar state traces which are typical to the solutions. Reference [6] is a special limited report describing this operation. On the other hand, Ref. [1] contains a compendium of results, forming a catalog of traces describing examples of the various problem types and special cases of interest. Some general comments on these results will be found in the body of this report.

From a practical, or operations, point of view the reader will find here a significant collection of information\* pertaining to a large number of space applications. The example used most frequently herein is that of the LDEF spacecraft. This is a "passive" payload scheduled for use with the Shuttle vehicle.

However, as will be noted later, the overall application of the present results are quite varied; they pertain to experiments outside (and inside) a spacecraft; to men and equipment located adjacent to the vehicle; to two-vehicle operations; and to general studies whereby perturbing force effects may be simulated and studied. Even though these various cases have not been pointed to specifically, it takes only a little imagination to see many of these and other possibilities.

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\*See, in particular, Section XI, Reference [1].

As an example, the results here point to locations, within a spacecraft, where zero-gee experiments should be located. Inside the spacecraft, it is noted that the gravity (gradient) accelerations are of the order of  $10^{-6}$  "gees" per meter of displacement away from the dynamical center of the spacecraft. Also; a direct means for defining how much thrust would be needed to overcome atmospheric drag and/or solar radiation effects can be ascertained here.

These are but a few simple uses which can be made to the results. However, the accuracies implied here must be judged in comparison with the assumptions introduced earlier. One added observation, and a reason for selecting more than one reference coordinate system, is that when external perturbing forces are small the local horizon frame may prove to be superior to the inertial one for studying motions. Thus particles located near the neutral points within the spacecraft would appear as stationary particles when referred to a local horizon system. The same particles, as seen in an inertial frame, would "rotate" if they were not at the dynamical center (origin), per se. This does not suggest that particles cannot be placed onto bounded orbits. Actually, bounded free paths do exist -- in both frames of reference -- but these are special cases; the general case (regardless of reference frame) produces a geometric divergence. In fact, the only closed free paths to be found, naturally, are those which evolve from a proper "set" of initial values; perturbed cases, with fixed force components, lead directly to divergences. It should be evident that through the judicious use of a controlled "action" the divergent motions could be turned into bounded traces, such as those needed for station-keeping and/or surveillance purposes. On the "other-side-of-the-coin" the alternate situation of providing a needed divergence comes about rather naturally. Here, however, one may wish to produce a divergence in some explicit fashion; consequently a knowledge of how to achieve these results is also available from this study.

As one reads through this document and its companions, it will be evident that the information described there represents a significant increment in the

practical knowledge pertaining to the "hows" and "whys" of relative motions. Not only will one find traces for the displacements, but hodograph figures are provided as well. These can play a significant role in studies of the control and alteration of motions. Very little use has been made of the hodograph figures in the past; yet, they (or some element of the velocity) enter into any analysis on corrective maneuvers, station-keeping, rendezvous, etc. For this reason, if no other, the graphing of a relative velocity can be as important as the graphing of a displacement. Also, where displacement diagrams yield range information, the velocity figures provide range-rate data.

In this report the general theme of the investigation is outlined. An overall look is taken at the most likely force types to be encountered by earth-orbiting satellites and some estimate of their influence on a relative motion is determined. Finally a discussion of the computer programs developed for this work is presented. This last section provides the reader with information concerning the capabilities, limitations and accuracies which could be expected from their use in the descriptions of relative motions. It should be mentioned that the output data from the computer primarily describes the state of a motion, but in many cases it also yields information on the accuracy of the theory as compared to numerical results.

In the final section of this report an attempt is made to summarize the overall efforts, and to point to those areas where added research and development work is needed. It is felt that a useful, preliminary design and study tool is available here; what remains to be done, now, is to apply this work to a real world situation.

## II. FORCE SYSTEMS

II.1 Introduction. The discussions which follow immediately are directed to describing and discussing the several force types which can be expected to influence the relative motion of a particle. Most of these forces are "natural" in origin; however, there are others to be considered which are best classified as mechanical, or applied, forces. The primary motivation for this section is, first, to identify the various forces and, second, to estimate the relative influence each will have on the problems to be studied here.

One should recognize, as an a priori condition, that the principal force to be encountered in this and all other orbital mechanics problems is that attributed to gravity. For the present study, like most others, this force is modelled by a simple and an immediately recognized statement. However, when the relative motion problem, per se, is set down in a formal mathematical statement it is found that the "gravity gradient" force assumes the major role of importance. Using this for a basic premise one can then proceed to study those motions which evolve, neglecting all other forces and concentrating on this single influence. Of course, when a better definition of the motion is desired it becomes necessary to examine the other force types which may be present, and to ascertain which of these are of consequence so that they may be added to the modelling.

The tasks which have been undertaken, and which are reported on in this section, constitute a study of these various force systems and an evaluation of their respective influences. It will be noted that each of these can have a varying degree of influence of a problem; this is dependent on those natural and/or imposed conditions which will be assumed to exist for the case at hand. Necessarily, then, when these forces are to be included in the model some provision should be made - explicitly or implicitly - whereby their associated influencing factors are accounted for as well.

II.2 Gravity Gradient Effects. A particle in motion under the influence of a single mass attraction (M), can be simply described by the differential equation,

$$\ddot{\bar{P}} = - \frac{GM\bar{P}}{|\bar{P}|^3}, \quad (I.1)$$

wherein G is the universal gravitational constant and  $\bar{P}$  is the vector position of the particle with respect to M.

When there would be a second (or test) particle ("Q") present, located by the position vector,  $\bar{Q}$ , it too would have its motion described by a same equation provided that mutual "attraction" between particles could be neglected; and that there are no mechanical connector(s) between them. Hence the two "free" particles would move about the attracting center (M) on independent paths subject to a same force of attraction.

It should be recognized; however, that due to their separation ( $\bar{P} \neq \bar{Q}$ ) these paths are not identical, and, as a consequence the acceleration on "Q", as determined by an observer on "P", should exhibit some measurable difference. This difference in acceleration is what will be referred to herein as the gravity gradient effect.

In order to study this situation an example case is proposed. Consider that "Q" is an object located adjacent to "P". In addition, assume that "P" moves on its own path according to the relationship shown above. Now, if  $\bar{Q} \equiv \bar{P} + \bar{r}_r$ , where  $\bar{r}_r$  is the relative position vector of "Q" with respect to "P", then it can be shown that a linearized approximation for  $\ddot{\bar{Q}}$  is:

$$\ddot{\bar{Q}} \approx - \frac{GM}{|\bar{P}|^3} \left[ \left( \bar{r}_r + \bar{P} \right) - 3\bar{P} \frac{(\bar{r}_r \cdot \bar{P})}{|\bar{P}|^2} \right]. \quad (I.2)$$

(See, for example, Eq. (II.11d), Ref. 1; here the vectors are necessarily described in a same frame of reference).

Next, taking account of the fact that  $\bar{Q}$ , hence  $\ddot{\bar{Q}}$ , is vectorally constructed from  $\bar{P}$  and  $\bar{r}_r$ ; then it is easy to show that one representation of



the gravity gradient acceleration is:

$$\ddot{\vec{r}}_r \cong \frac{GM}{|\bar{P}|^3} \left[ -\vec{r}_r + 3\hat{P} (\vec{r}_r \cdot \bar{P}) \right], \quad (I.3)$$

with  $\hat{P}$  being the unit position vector corresponding to  $\bar{P}$ .

When this expression is described in a local horizon coordinate system [i.e.,  $\vec{r}_r \equiv \vec{r} = \vec{r}(x, y, z)$ ], it follows that:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} \cong \frac{GM}{|\bar{P}|^3} \left\{ - \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 3 \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \right\}. \quad (I.4)^*$$

For this formulation the origin of coordinates is located at ("P").

Looking into this result one can see that the first term represents a "restoring" action, while the second is indicative of a divergence. Since  $x$  is radially directed (here) it is apparent that in this direction the influence of the gradient force is to "pull" particles away from the origin (P). Yet, contrary to this, a test particle located in the  $(y, z)$ -plane is drawn toward "P". To more clearly see the consequences of this expression (on these accelerations) examine the sketches shown below, Fig. II.1(A) and II.1(B).

What one finds here in a graphical representation of the "gradient field" around "P". For the present approximation (deemed to be a reasonable representation for the immediate vicinity of "P") the "lines-of-force", in the  $(x, y)$ - and  $(x, z)$ -planes would appear as hyperbolae (see Fig. II.1(a)). It should be mentioned, also, that for positions away from the origin (P) the force magnitude is increased in comparison with the closer positions. On the  $(y, z)$ -plane, however, the situation is changed; there the "force field" shows an attraction toward "P"; one with magnitudes diminishing as the distances diminish (see Fig. II.1(B)).

\*The equation shown here has a special interpretation; it shows only the influence of gravity gradient, it does not include the kinematic effect of a moving frame of reference.

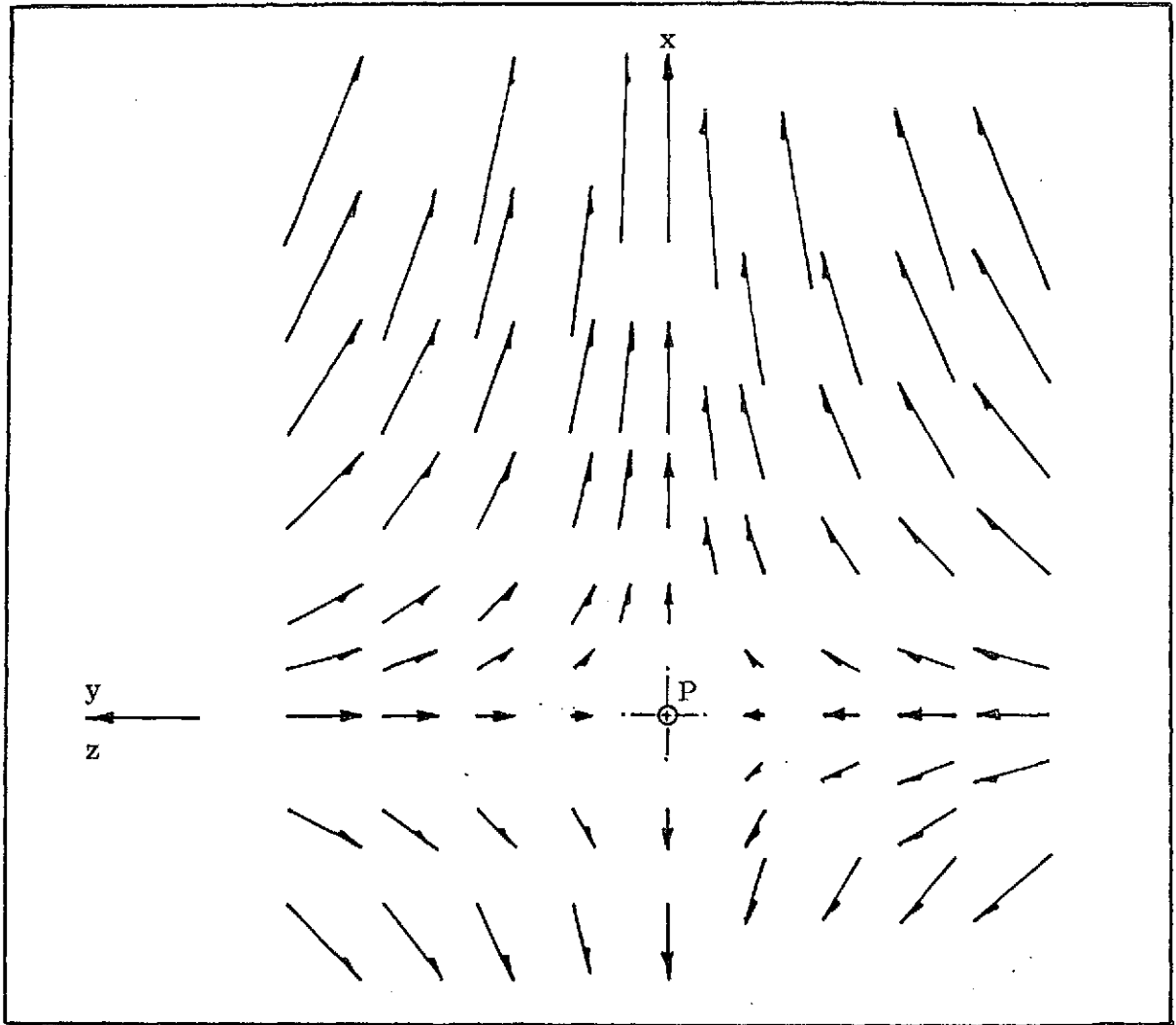


Fig. II.1. Sketch A. Graph depicting the linearized gravity gradient field about an origin P. Each arrow describes, in magnitude and direction, the acceleration associated with relative positions on the (x, y) and (x, z) planes. See Eq. (I.4).

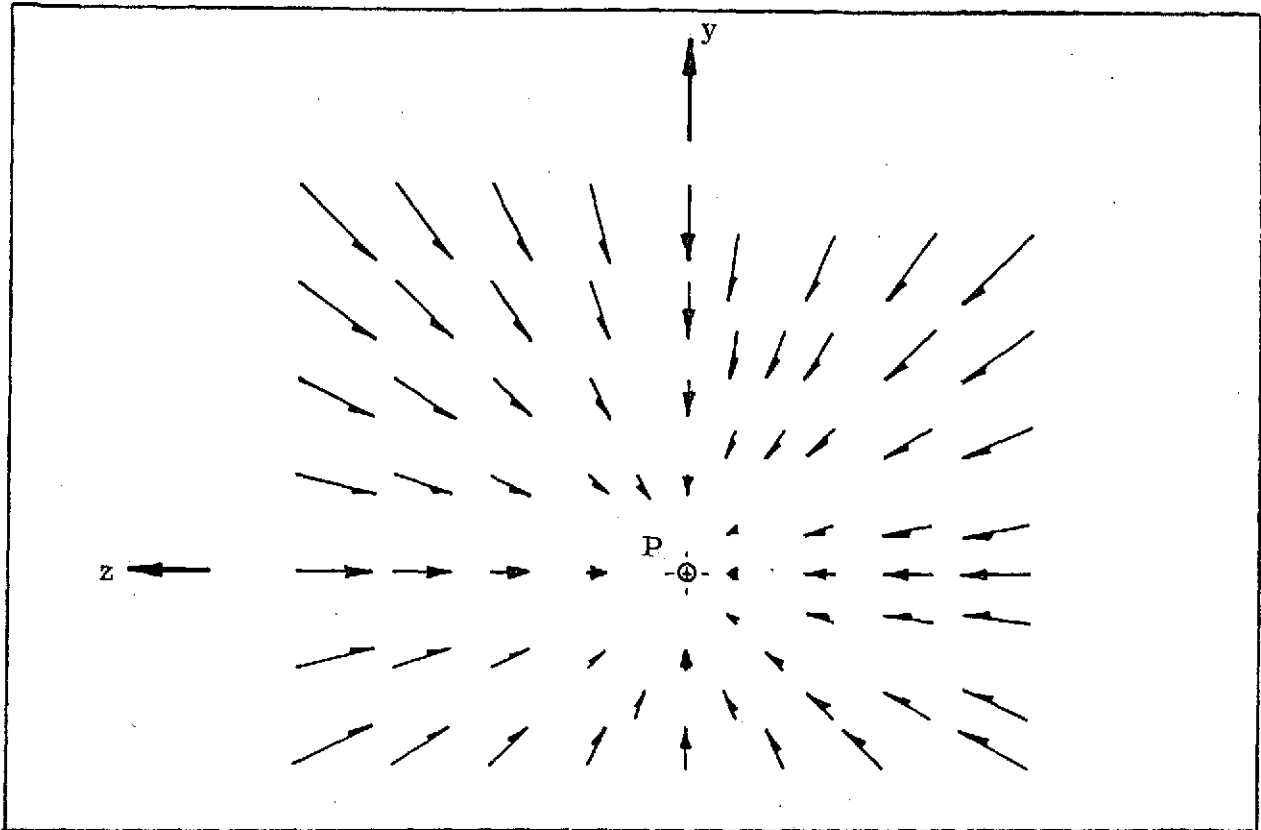


Fig. II.1. Sketch B. Graph depicting the linearized gravity gradient field, about  $P$ , for positions on the  $(y, z)$ -plane. Each vector describes the magnitude and direction of accelerations. See Eq. (I.4).

These illustrations imply that the gradient field remains essentially fixed in its relationship to the origin. Thus, at several loci, the test particle "Q" could experience the same gradient field accelerations provided its (relative) position is referred to a coordinate frame where  $\hat{x}$  corresponds to  $\hat{P}$ .

On the other hand, when the test particle is referred to an inertially oriented frame, the field's orientation will be shifted. That is, if "P" is moved to a different inertial location, so the x-axis does not remain aligned with  $\hat{P}$ , then the gradient field is rotated with respect to this reference frame.

II.3 Kinematic Effects. The gravity gradient influence on a test particle must be added to when one considers any orbital motion for both the test particle (Q) and the origin (P). That is, with these motions referred to a local horizon (or, rotating) frame of reference, there are the so-called "fictitious" accelerations present. These are the Coriolis and centripetal components. It should be remembered they are a direct consequence of the coordinate system chosen here.

To illustrate this phenomenon, consider a relative motion which is to be described in the local-rotating frame. In this situation the gravity gradient force (Eq.(I.4)) must be "corrected" for the kinematic consequences of the motion. These corrections arise because of the rotation imparted to the system. In particular, when they are included in with the gravity gradient accelerations, it is found that the Coriolis influence is subtractive and the centrifugal effect is additive. Mathematically this has been expressed in Eq. (II.9b), Ref. 1, as:

$$\ddot{\bar{r}} = \left\{ T(\varphi^+) \ddot{Q}_I \right\} - 2\dot{\varphi} [B] \dot{\bar{r}} + \dot{\varphi}^2 I_2 [\bar{r} + \bar{P}]. \quad (I.5)^*$$

Here  $\bar{r}$  and its derivatives are expressed in terms of coordinates referred to the rotating frame;  $T(\varphi^+)$  is the transformation matrix needed (here) to rotate the vector  $[\ddot{Q}_I]$  from the inertially-oriented to the rotating axis system. In addition,  $\dot{\varphi}$  is the rate of orbital motion, and  $B$  is a special unitary matrix\*\* associated with the transformation matrix,  $T$ .

\*See the Appendixes of Ref. 1 for a discussion of these various matrices.

\*\*Actually,  $B\dot{\varphi}$  can be shown to be equivalent to the vector operation " $\bar{\omega} \times$ ", when  $|\bar{\omega}| = \dot{\varphi}$ . The axis of rotation is normal to the plane in which  $B$  is composed.

Since the gradient field can be (instantaneously) described by Eq. (I.3); and with  $GM/|\bar{P}|^3 \equiv \dot{\varphi}^2$  for a circular orbit; then the lead term in Eq. (I.5) can be replaced by Eq. (I.2); and, consequently,

$$\ddot{\bar{r}} = \dot{\varphi}^2 \left[ \underbrace{\left\{ -I_3 (\bar{r} + \bar{P}) + 3 \hat{P} (\bar{r} \cdot \hat{P}) \right\}}_{\text{g.g.}} + \underbrace{I_2 (\bar{r} + \bar{P})}_{\text{cf}} \right] - 2\dot{\varphi} B \dot{\bar{r}}. \quad (\text{I.6a})^*$$

What is quite interesting, and somewhat unexpected, is that the centrifugal influence has nullified a part of the gravity gradient effect. Also, it is noted that the Coriolis term has come through unaltered and is indicative of a "change" in the accelerations measured in this frame of reference. It is readily seen and well known that this one of the acceleration components is directly dependent on the measured velocity ( $\dot{\bar{r}}$ ), and on the rotational rate.

When this last equation is cleared, one can show it reduces to:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \dot{\varphi}^2 \left\{ \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 2\dot{\varphi} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \right\}, \quad (\text{I.6b})^{**}$$

for a circular reference orbit. One influence, which is seen here, is that within a spacecraft (say) there are positions where test particles may be located free from acceleration. At least to the degree of approximation used here loci on the y-axis provide this environment (if, in addition,  $\dot{\bar{r}} = \vec{0}$ ). Ideally, then, the place to conduct zero-gee experiments is along the vehicle's axis; however, it must be recognized that the spacecraft is so oriented that it continually rotates as it moves along its orbit. In other words, this spacecraft is "local-horizon" aligned.

A next, and natural, question has to do with this same phenomenon when the spacecraft is inertially oriented. Necessarily some parts of the situation will be altered. Actually this problem has been examined, mathematically, in Ref. 1; there the following resultant was acquired. For a same set of orbit conditions the relative accelerations, referred now to a local-inertially-oriented frame,

\*The terms, here, are each identified: g.g. refers to "gravity gradient"; cf implies "centrifugal force"; and "Cor" denotes the Coriolis term.

\*\*See, also, Eqs. (II.12b), in Ref. 1.

are expressed by:

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{pmatrix} \equiv \dot{\varphi}^2 \left\{ T(\varphi^-) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} T(\varphi^+) \right\} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} . \quad (I.7)^*$$

It is quite evident that the velocity dependence (or, Coriolis effect) is gone. However, it is equally apparent that there are no "acceleration free" loci to be found in this spacecraft orientation scheme. Hence, now the space scientist could not be provided with a zero-gee environment, for his experiments, as he was in the previous case.

One result from this discussion is that there can be different g-environments made available for experimentation purposes, but these may be dependent on the "orientation" provided within the parent vehicle.

A word of caution should be given here. The above suggestions and statements may lead one to the premise that the equations (Eqs. (I.6), (I.7)) are indicative of real-world situations. This is not precisely true; rather, these equations, etc. are reasonable representations of actual conditions. Certainly the simplified model used here, for the gravity field, and the linearization imposed on the governing equations do not depict an exact situation. To this end there are errors present; but, it is known that these are small for the conditions implied. The most important fact to be drawn from these comparisons is that a "local orientation" of the vehicle (or some space within it) does have an influence on the observed state of a relative motion.

II.4 The Relative Motion Traces. One interesting consequence coming from a solution to the above equations is an ability to predict those motions which may occur. It is apparent that the paths followed by a particle, for either of these spacecraft orientations, would be circuitous and different in shape. In one instance above, accelerations were seen to be velocity dependent, explicitly; in

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\*Here, X, Y, Z are the relative position coordinates; and,  $T(\varphi^\pm)$  are the transformation matrices needed to correlate vectors in the two reference frames.

the other this dependence disappeared. Suffice it to say that when the two cases are examined, in some detail, it is found that the motions which occur (solely as a consequence of gravity effects) are markedly different. To a large extent what happens there is dependent on how much motion is originated\*.

In general the particles observed in a rotating frame of reference would describe cycloidal-type displacement traces in the plane\*\* of motion; ellipses, and s-shaped curves, for the out-of-plane diagrams\*\*\*.

The corresponding trace geometries, found on the inertially oriented planes, would be spiral-like curves (in the plane of motion), and "loops" for the out-of-plane representations.

The primary inference here is that in the usual case divergent motions (relative to the center, P) do occur. The consequence of this is that particles moving about the interior of a parent vehicle would soon collide with the walls -- an action quite likely to initiate a second divergent motion -- and so on and on. For a test particle (Q), located outside of the parent vehicle (P), the motions would also be divergent. However, there, it is likely that the test particle (Q) would soon be removed, or "lost", from the vicinity of the reference particle (P).

This discussion is not intended to imply that all is lost; that particles cannot remain close to the parent. To the contrary, there is a means of containing a motion within definable bounds. By properly adjusting the initial conditions (other than for the zero-g case noted earlier), the experimenter could "control" the region size in which free particles move<sup>§</sup>. For example the non-secular, in-plane trace, for a particle viewed in the rotating frame of reference, would be an ellipse. (Also, the associated out-of-plane traces are ellipses, too). However, these same particles, as viewed on inertially oriented planes, would describe different figures entirely. The in-plane traces there are found to be heart-

\*In Ref. 1, these cases would be identified as "initial-values" problems.

\*\*"in-the-plane motion" means in the local plane parallel to the plane of motion for the parent particle (P).

\*\*\*See the compendia of results in Ref. 1 for more complete descriptions, etc.

§In Ref. 1 these are identified as the "non-secular" case studies. Graphs, etc. for them may be found in the appropriate compendia.

shaped curves (limacons), while the out-of-plane ones appear as skewed closed curves (some look like lemniscates, while the others are figures of a more general nature).

In summary, then, it can be said that the relative motion of one particle with respect to another, moving on a circular orbit, can be "controlled". Controlled to the extent that it does not leave an immediate vicinity. Of course the reader must recognize that these assertions are based on the constraints mentioned before; it should not be surmised that the same motions would necessarily occur when other "applied" force systems are brought into play. More will be said regarding this, subsequently.

II.5 An Example. To illustrate how one might initiate a bounded motion consider the sample situation noted below.

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Suppose it is desired to establish a relative orbiting particle inside an LDEF experiment module after that module has been released from the space shuttle vehicle. For purposes of manipulation it is assumed that the module is in a circular orbit having an orbital rate ( $\dot{\varphi}$ ) of  $10^{-3}$  rad/sec. Also, for this example the LDEF is presumed to be rotating (itself) at a rate matching the orbital rate (thus the vehicle is fixed in the local (horizon fixed)\* frame of reference (see Sketch A)). For comparison purposes a second orientation is assumed, also; this has the LDEF in a gravity gradient stabilized position - thus, again, it is "fixed" in the rotating frame of reference (see Sketch B). Dimensionally the LDEF model is assumed to be 18 m long and 4.5 m in diameter; it is a cylindrical can.

It is known that a particle can be made to move in a bounded space if it is impulsively started from "the origin" (the cg, here) with a velocity directed normal to the vehicle's inertial velocity vector. For conciseness, here, it is assumed that this impulse is a  $\pm \dot{x}_0$  value. The result (desired) is to ascertain

---

\*This orientation is chosen since it appears to be a most probable one for the LDEF experiment.



a magnitude for  $\dot{x}_0$  so that the relative motion is a "free orbit" remaining inside the LDEF. Consequently the critical dimension of the spacecraft is its radius (2.25 m). In the example a dimension of 2 m is arbitrarily chosen as the value to work with.

Now, drawing from example 1. (Art. III.4, Ref. 2) it is found that for the condition stated this free orbit (in the local (x, y) frame) is an ellipse, passing through the c.g. (initial point); thus the trace is defined\* from:

$$x(t) = \frac{\pm \dot{x}_0}{\dot{\phi}} \sin(\dot{\phi} t), \quad (I.8a)$$

$$y(t) = \frac{-2[\pm \dot{x}_0]}{\dot{\phi}} [1 - \cos(\dot{\phi} t)]. \quad (I.8b)$$

Note that these equations predict the 1:2 ellipse;

$$\left( \frac{x}{|\dot{x}_0/\dot{\phi}|} \right)^2 + \left( \frac{y \pm 2\dot{x}_0/\dot{\phi}}{2|\dot{x}_0/\dot{\phi}|} \right)^2 = 1, \quad (I.9)$$

as sketched below on the figure.

### (Numerical Example)

In Sketch (A) the critical (interior) dimension is chosen as  $|x|_{\max} = 2 \text{ m}$ , while in Sketch (B) the y-dimension is the critical one. On solving for the allowable impulse ( $\dot{x}_0$ ) it is found that:

For Sketch (A)

$$|\dot{x}_0| \cong 2 \times 10^{-3} \text{ m/s}$$

since

$$x \begin{bmatrix} \max \\ \min \end{bmatrix} = \pm 2 \text{ m (given).}$$

For Sketch (B)

$$|\dot{x}_0| \cong \frac{1}{2} \times 10^{-3} \text{ m/s}$$

since

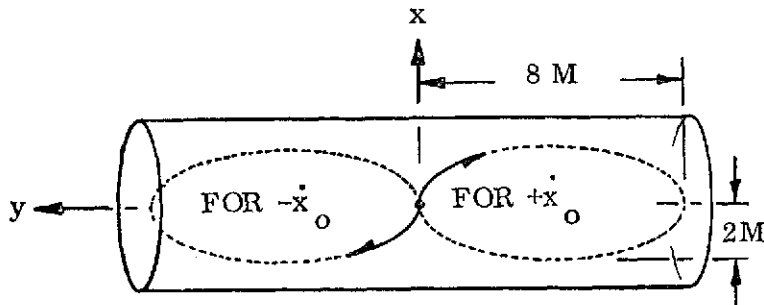
$$y \begin{bmatrix} \max \\ \min \end{bmatrix} = \pm 2 \text{ m (given).}$$

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\*These are the consequence of a linear analysis for the problem; since the distances involved here are very small (compared to the orbit dimensions) the present approximation is an excellent one.

Thus, for the trace:

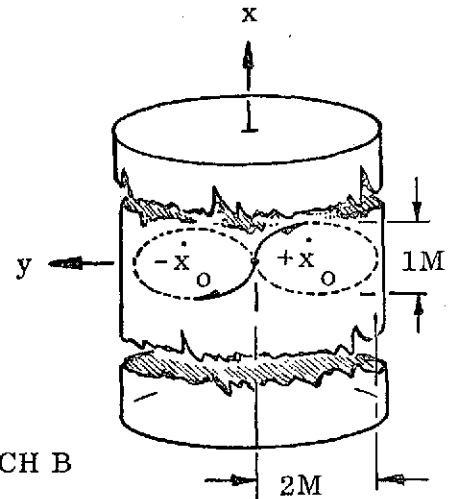
$$y \begin{bmatrix} \text{max} \\ \text{min} \end{bmatrix} = \pm 8 \text{ m.}$$



SKETCH A

Thus, for the trace:

$$x \begin{bmatrix} \text{max} \\ \text{min} \end{bmatrix} = \frac{1}{2} \text{ m.}$$



SKETCH B

If the result shown here is extended so that the initial impulse is defined by

$$\Delta \bar{V}_0 \equiv \Delta \bar{V}_0(\dot{x}_0, \dot{z}_0)$$

then it can be shown that the resulting traces are a combination of results (see Example IV, Sect. III.4, Ref. 2) leading to an ellipse, like the case above; but one described on a sloping plane (showing the influence of  $\dot{z}_0$ , here). The inclination of the plane (as measured from the x-axis) is described by an angle ( $\gamma$ ), where

$$\gamma \equiv \tan^{-1} \left( \frac{|\dot{z}_0|}{|\dot{x}_0|} \right), \quad (I.10)$$

in magnitude.

---

Prior to leaving this section it would be informative to address a few comments to the general relative motion trace geometries. First, it has been found that the in-plane traces, as seen in the two frames of reference, have one outstanding difference. That is, for figures in the rotation coordinate system the motion progresses in a direction opposite to the orbital motion. Thus, for a counterclockwise circular orbit, this relative motion trace is clockwise. On the other hand, when viewed in the inertially-oriented coordinates frame a relative motion follows that of the orbit. Thus, a same relative displacement trace progresses in opposite directions when followed in each of these reference frames.

In the example above it was presumed and stated that the means for maintaining a "bounded relative motion region" was to choose as initial values ( $\bar{r}_0 = \bar{0}$ ;  $\dot{y}_0 = 0$ ). In this regard the initial impulse was normal to the spacecraft's velocity vector. Any orientation, whereby the initial impulse was contained in the (x, y)-plane, would suffice. Actually, this is almost true (and for the example a very good approximation); however, a more precise analysis would indicate that a small adjustment is needed in order to have a bounded region, one in which the relative orbiting particle returns to the initial position once each orbital period.

As a matter of interest is found that the initial impulse vector ( $\equiv \bar{V}_{so}$ ) must lie on a cone whose axis is along the spacecraft's velocity vector; but with this vector inclined at an angle ( $\alpha$ ) relative to the axis. The required orientation for  $\bar{V}_{so}$  ( $\equiv \bar{V}_{so}(\dot{x}_0, \dot{z}_0)$ ) is given by:

$$\alpha = \cos^{-1} \left( \frac{-\Gamma_0}{2} \right), \quad (I.11)$$

where  $\Gamma_0 \equiv |\bar{V}_{so}| / |\bar{V}_P|$ ,  $\bar{V}_P$  representing the spacecraft's velocity, (see the sketch below).

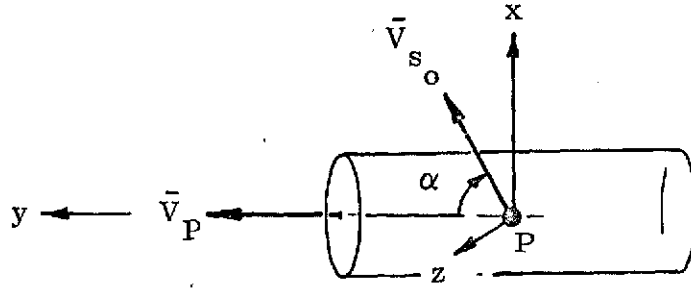


Fig. II.2. Sketch of the added impulse ( $\bar{V}_{s_o}$ ), the velocity of the Body ( $\bar{V}_P$ ), and the inclination angle ( $\alpha$ ) of the impulse. It should be recalled that the coordinates ( $x, y, z$ ) constitutes a "local-horizon" frame of reference.

Normally  $\Gamma_o \ll 1$ ; therefore the (linear) approximation of  $\alpha = \pi/2$  is quite reasonable. Nonetheless, for accuracy of definition one should be aware of this discrepancy.

It must be evident that for those instances where  $\bar{V}_{s_o}$  does not acquire the proper orientation ( $\alpha$ ), there would be a separation distance ( $\Delta s$ ) between the positions for particle (Q) and the parent (P) at the end of one period of motion. This can be estimated, also, to within a high degree of accuracy; it has been found that this separation displacement is:

$$\Delta s \cong 2\pi |\bar{P}| \left| \frac{\Gamma_o}{2} + \cos \alpha \right| \Gamma_o, \quad (I.12)$$

wherein  $|\bar{P}|$  is the radius to the (circular) base orbit.

II.6 The Hodograph. There is yet another coordinate representation for these motions; one which plays an important role in these studies, but which is somewhat esoteric to this discussion. For purposes of identification and information these other traces are the hodographs\* corresponding to the

\*A hodograph is that geometry for a motion which is traced out on the velocity planes rather than on the displacement planes. In this regard the hodographs are descriptive of range-rate information while the displacement traces provide relative "range" data.

relative displacements. Generally speaking these diagrams would be of more interest to the operations analyst, particularly in estimating maneuvers and transfers. It should be recognized that a full motion state requires both position and velocity information for its description. One without the other leaves the investigator with an incomplete picture of what is happening throughout a given maneuver study.

In order to provide some idea of the levels of accuracy to be expected from this analysis, the following remarks are offered. In general, for relative displacements on the order of ten kilometers, and speeds of the order of 10's of m/sec, the agreement between numerically defined state variables and the analytical ones is to better than one-percent for a one orbit description. What this means is that from the model used here the analytical results may be considered as completely descriptive of the relative motion, in this time span. Thus, for near earth orbits, and for one orbital time period, it is not necessary to resort to numerical studies if one desires to plan and evaluate a relative motion situation. Since the mathematical problems studied here were based on the idea of small variations (small with regard to the reference, circular orbit), then deviations of a few km and a few meters/sec are well within this framework.

As indicated earlier, this section has not discussed the ramifications of "applied" forces in regards to a relative motion; it has been concerned only with the influences derived from "local" kinematics and from the gradient of the gravitational attraction. In the next sections there will be mention of some specific applied force systems.

II.7 Forces other than Gravity. In space flight operations the spectrum of forces which can be assumed to act on a spacecraft is fairly large. Fortunately, however, there are many instances where such a listing can be weighed and pared

according to the mission or operation which is contemplated. For example, in the present situation, where near earth orbiting vehicles are being considered, the primary natural force systems of consequence are most likely those due to: (1) atmospheric reactions - aerodynamic drag and lift; (2) forces of solar influence and origin -- mainly solar radiation effects; (3) added gravitational influences -- due to earth's unsymmetrical mass distribution; and (4) lunar and solar gravitational perturbations.

Of course, there is also a distinct possibility of having mechanical forces present; these, too, can affect the state of motion for space vehicles. A few representative possibilities are: (1) applied reaction units, such as jets, rockets, etc. (momentum transfer devices, both controlled and uncontrolled); (2) mechanical connectors such as lines and rods, and other structural members -- these may be additionally classified as flexible or rigid, and weightless or otherwise; (3) natural or applied impulse reactions (e.g., those arising from the impacting of debris, or other objects), "bumps" from an adjacent craft or a protrudence; or, possibly, from an "intentional act" where the level of force could be large. Needless to say, this is not an all inclusive list; with some thought the reader can easily ascertain other disturbances which might affect the "free motion" of a spacecraft.

In conjunction with the above listing of possible forces, there is to follow a description for some of these. Also there will be an estimate of their probable effect on the motion.

II.8 Reactions with the Atmosphere. Certainly the force having a largest influence on near earth satellites is that due to gravity; and, it is likely that for relative motion considerations, the gravity gradient influence would be a most significant consideration for sub-satellites, adjacent mass particles and the like.

In all likelihood the next most influential action would be that associated with atmospheric reactions. Even though the earth's "atmosphere" is very "thin" at satellite altitudes it is, nonetheless, a force system to be reckoned

with -- one which should not be ignored and/or passed by lightly. Of course, there are certain physical considerations which must be kept in mind when these forces are being examined.

It is beyond the scope of this investigation to critically discuss the subject of earth's upper atmospheric physics; however, it is essential that the present investigators have some knowledge and understanding of this topic. For instance, it is a well known fact that solar activity has a marked influence on upper atmospheric density. Actually, some changes occur with a marked degree of regularity, on a day to day basis; while others, principally those triggered by an active sun, occur sporadically and are almost totally unpredictable. Generally, the analyst who deals with reentry and high altitude flight will have a good understanding of the "art" associated here, and consequently will be able to make allowances and adjustments so that a modeling of these phenomena can be constructed.

In addition to the variations just noted the aerodynamics of this flight regime offer some interesting problems. At the altitudes considered here the atmosphere no longer behaves as a continuum. Flight conditions now are such that this gas dynamic phenomenon is better examined on a particle basis; and, as a consequence, the "usual" variabilities in the aerodynamics of bodies no longer apply. Suffice it to say that the "drag coefficient", in this regime, takes on a (more or less) constant value -- nominally selected at some value in the vicinity of "two". In effect, then, the aerodynamic drag becomes relatively insensitive to "body shape" and is more dependent on density variations. Of course, once the spacecraft ceases to fly as an orbiting vehicle, and begins its plunge into the atmosphere, then the entire concept of this flight operation is altered; and, a new and different problem must be considered -- generally this is referred to as "reentry physics". Here, also the technology and understanding needed to discuss the problem on a truly scientific basis is beyond the scope of this work.

Rather than diverge into discussions and studies, attempting to reach the depth needed for these problem areas, a simplified approach to the orbital aspects of this situation has been followed. In this regard one should simulate the aerodynamics for these flights, recognizing that "effects" are the essential results to be obtained. It is reasonable to assume that with a knowledge of the physical problem to be studied the analyst can "model" the situation, with an acceptable degree of accuracy, and acquire results which are descriptive of the consequences. For example, a vehicle of large mass density and small volume, without extensive perturbances, will be largely influenced by drag. On the other hand, one with large extended panels might be viewed as a lifting configuration, and quite possibly one which will be acted on by significant aerodynamics torques. For most purposes, here and elsewhere, the concern, insofar as aerodynamic reactions are to be considered, will be in relation to "slowly changing" force levels, at most.

As an example of what might be expected in describing this problem area, let us assume a typical case for drag, as a force to be considered in addition to the gravitational attraction. Allowing for the usual description of drag ( $D \equiv \frac{1}{2} C_D \rho V^2 A$ ); a force which is quadratic\* in the vehicle's speed (V); and where  $C_D$  is a "descriptive aerodynamic constant",  $\rho$  the atmospheric density, and A the "representative area"\*\*, one finds this specific force most conveniently written as (a scalar):

$$\frac{D}{M} \equiv \left[ \frac{C_D A}{2m} \right] \rho V^2 \equiv \beta [\rho V^2]. \quad (1.13)$$

Here  $\beta$  is used to define the Ballistic Coefficient, a characteristic number for the vehicle. In the flight regime presumed here ( $C_D \cong 2$ ), this coefficient ( $\beta$ ) is, therefore, descriptive of the "area to mass ratio" for the spacecraft. Being at a nearly fixed altitude, and moving at almost constant speed, then the specific

\*This assumption of  $V^2$  dependence has been questioned by some investigators, for satellite speeds and altitudes. The other contention is that a more appropriate dependence, of Drag on V, is given by a first power law (i.e.,  $D(V)$ ).

\*\*For a tumbling, nonspherical vehicle, a suggested replacement for A (the area normal to motion direction) has been given as  $S/4$ , where S is the total exposed surface area of the vehicle [Refs. 3, 4].



force term ( $D/m$ ) is quite close to a fixed value. By definition, this force is opposed to the velocity, hence it has a direct retarding (energy dissipating) influence. Most likely, with due regard to the vehicle types which are descriptive of sub-satellites, etc., the drag is a dominant aerodynamic reaction, and the most significant force after gravity gradient. (If, however, the lift force is considered to be of significance, it too could be modelled, similar to the drag, and approximated as a fixed magnitude reaction. One must recognize, of course, that this force is directed normal to the velocity!)

To illustrate the influence which drag plays on a relative motion problem, consider the following circumstances: A subsatellite, or test particle, is referred to the standard (reference) particle for its relative motion. This reference particle travels a circular orbit under the influence of gravity alone; thus the relative motion problem can be expressed by the following equation format:

$$\ddot{\mathbf{x}} = -\frac{\mu}{[\sum \kappa]^{3/2}} - \beta \rho V \dot{\mathbf{x}} . \quad (\text{I.14})^*$$

Here  $\kappa \equiv X, Y, Z$  (inertially oriented, relative motion coordinates), in turn. Also, the two influence, gravity ( $\propto \mu$ ) and drag ( $\propto \beta \rho V$ ), are clearly distinguished. Of course, in reference to a (near) circular orbit these two influences act on the test particle quite differently (regardless of "signs" shown here). The gravity force "pulls" on the particle, inducing a radial acceleration toward the center of rotation. The drag, however, retards the vehicle's motion, pulling it back from "ideal orbit" conditions.

II.9 Example. To provide some indication as to how drag can affect the relative motion of a satellite vehicle the following situation is examined:

Suppose a circular orbit, at 222 km altitude, for an LDEF type vehicle; assume that the drag coefficient is "2". Estimating the LDEF to have an A/m (value) of  $9.9 \times 10^{-3}$  ( $\text{m}^2/\text{kg}$ ), then at the stated altitude,

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\*Here,  $\rho$  is the local atmospheric density, hence dependent on  $|\bar{Q}|$ ; and  $v$  is the test vehicle's speed, generally assumed to be written as "relative to the atmosphere".

$$V \cong 7.77 * 10^3 \text{ m/s, and } \rho \cong 2.4 * 10^{-10} \text{ kg/m}^3.$$

Now, suppose that the vehicle's mass is estimated at  $m = 8170 \text{ kg}$ . As a consequence of these conditions it is found that the specific drag force ( $F/m$ ) is:

$$F/m \cong 1.456 * 10^{-4} \text{ m/s}^2.$$

(Here, it has been assumed that the LDEF is gravity gradient stabilized, hence it is flying "broad-side-on").

For an extreme in comparison purposes the specific drag force acting on an ECHO type satellite is obtained at these orbit conditions; and, for a ballistic coefficient ( $\beta$ ) of  $15 \text{ m}^2/\text{kg}$ . Thus, the specific drag "acceleration" is:

$$F/m \cong 0.22 \text{ m/s}^2.$$

A look at these two specific forces is a clear indication of how "large" the variance may be for this "acceleration". Obviously, as seen here, the ratio of these forces has a considerable magnitude.

For additional reference, the gravitational acceleration ( $\mu/r^2$ ), at this altitude, is easily found to be:

$$\mu/r^2 = 9.15 \text{ m/sec}^2.$$

This latter value is (necessarily) much the larger of all values defined above. Of course, such is to be expected; the number is simply given to keep the other values in proper perspective.

Unfortunately Eq. (I.14) is not the most representative (and usually used) expression for this problem. As noted earlier, it is more useful to recast the equation so that the "relative motion" is expressed (and solved) in terms of the relative displacement coordinates ( $x, y, z$ ); those referred to a "local horizon" frame of reference\*.

In order to see the consequence of such a coordinate transformation (the kinematics involved, and the inclusion of a drag force, as defined above), the

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\*Recall that Eq. (I.6b) is the "ideal orbit" relative motion expression cast in these same coordinates (but for the linearized case).

following equation is given. In this the linearization has not been applied (hence the variable,  $\Delta$ , which is shown); however, each term is identified, by name, so that its generic origin is known. Thus, the differential equation for the motion is:

$$\begin{aligned}
 \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = & \overbrace{\dot{\varphi}^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x + |\bar{P}| \\ y \\ z \end{pmatrix}}^{\text{(Centrifugal term)}} - \overbrace{2\dot{\varphi} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}}^{\text{(Coriolis effect)}} - \overbrace{\frac{\mu}{(|\bar{P}| \Delta)^3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x + |\bar{P}| \\ y \\ z \end{pmatrix}}^{\text{(gravity gradient)}} \\
 & - \overbrace{\beta \rho v \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} + \dot{\varphi} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x + |\bar{P}| \\ y \\ z \end{pmatrix} \right\}}^{\text{(atmospheric drag, as modeled)}} .
 \end{aligned} \tag{I.15}*$$

Here,  $\Delta \equiv |\bar{Q}| / |\bar{P}|$ , a ratio of the radii to "P" and "Q". The reader should recognize that if: (1) drag is neglected; and, (2) the linearization (of  $\Delta^3$ ) is invoked, then the equation reverts to that form given in Eq. (I.6b)

If, now, an analytic solution to this last result would be sought for, it would be required that; (1) the non-linear nature of the above expression be removed; or, (2) that some other approximations be introduced. (One approach to this situation occurs if the force is presumed to remain approximately constant -- then, after applying this "fixed disturbance", a mathematical description may be acquired. Such an approach has been used in the current analytical study; some typical results will be discussed subsequently.

One of the more interesting consequences of these formulations is that the out-of-plane equation ( $\ddot{z}$ ) is coupled to the in-plane coordinates only through the gravity gradient term, when drag is neglected. It is additively coupled (through the velocity) when drag is included. By inspection, one can see that

\*The dependence of drag on  $(x, y, z; \dot{x}, \dot{y}, \dot{z})$  arises from the kinematic description of velocity in this frame of reference.

coupling between the in-plane coordinate, per se, is far more involved than that between them and the z-coordinate. Thus, a linearization (here) is readily "justified" and the subsequent sources of error are known.

One note of caution: for developments leading to Eq. (I.6b), linearization, and the assumption of a circular base orbit were introduced. The consequence (there) was that the centrifugal term and the gradient term combined to produce the lead matrix in that equation. For similar conditions Eq. (I.15) would show these same reductions.

In attempting to estimate the "size" of the aerodynamic influence, and in particular, that due to drag, one should reexamine the statement given as Eq. (I.15). There, it is essential to compare and weigh the terms proportional to  $\beta$  (drag) and gravity gradient (also, one should include the centrifugal force, here). By comparing these quantities it is reasonable to expect that drag may, under some conditions, overshadow the gradient effect. In particular, for those situations where  $\beta$  is large and the altitude is low ( $\rho$  large), the drag force can be the larger force for small separation distances (on the order of a few meters). It should be evident that much of what is stated here (as speculation) depends directly on what the physical situation may be for the problem at hand.

For instance, the atmospheric density is known to vary during each day; and to be even more variable during various periods of solar activity. (For reference purposes, typical variations are shown on the graph below, Fig. II.3. There one can see that during intense solar storms the density could be increased two orders of magnitude - in the upper elevations, as shown. Also, on the figure, for reference purposes, is an estimate of the magnitude of the unit area atmospheric impact pressure (drag pressure) for some of the representative altitudes. The numbers (in parentheses), at selected positions (+), indicated on the graph, are a measure of this pressure. Note that the appropriate altitude (in KM) is indicated below the curve(s) at the marked points. For example, the

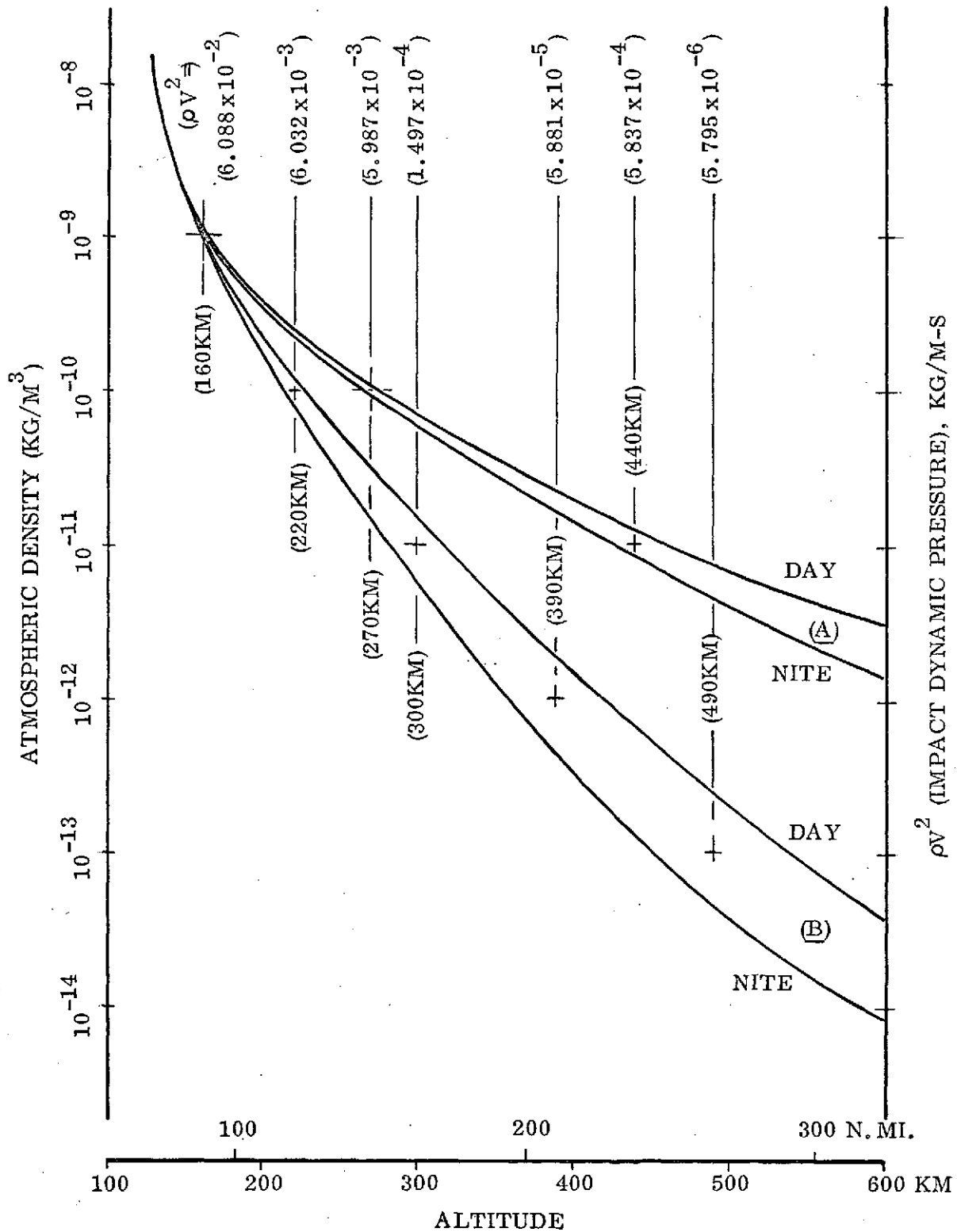


Fig. II.3. Variation of atmospheric density ( $\rho$ ) with altitude for High Solar Activity (A) and Minimum Activity (B). Also, typical values of impact pressure for selected altitudes are indicated (+). Note that impact pressure values ( $\rho V^2$ ) are presented in parens across graph top.

drag pressure experience on a circular orbit of 390 KM altitude, during minimal solar activity periods, is estimated to be  $5.881 * 10^{-5} \text{ kg/m-s}^2$ ).

It was mentioned earlier that the magnitude of the Ballistic Coefficient ( $\beta$ ) can have a marked influence on this force's contribution to a given relative motion. Obviously this physical parameter is dependent on the constructed makeup of the vehicle (satellite, etc.). For present approximations  $\beta$  is a measure of the (impact) area to (body) mass ratio. In order to provide some "feel" as to how this quantity may vary, by shape and by construction, Table I has been included, below. There a spherical shape (solid and shell construction) is described, as well as a cylindrical shape (solid and shell). The cylinder is depicted in two aerodynamic flight configurations; one where the body is flying "broad-side-on", and another when it is flying "end-on".

To illustrate levels of  $\beta$  values which might be described for these typical shapes, one should consider the number results shown in the Table II. There, dimensions approximate to the size of LDEF are used. A material and metal thickness were arbitrarily chosen so that a comparison of A/M numbers\* could be made. These results are stated in the table.

What should be most apparent from this listing is that one can adjust the A/M numbers, over a rather wide range of values, by construction and configuration details (note that the sphere's values range over more than three orders of magnitude, for the extremes shown). This alone can make a most significant difference in the acceleration levels attributed to this force system.

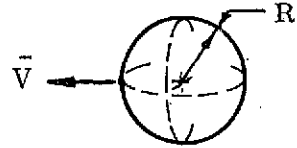
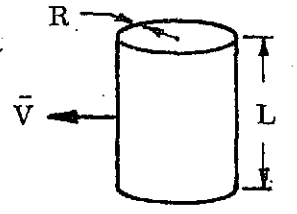
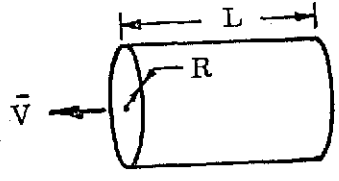
Needless to say, under usual circumstances, one would not attempt to launch a solid metal sphere, of such size and weight, as shown, into orbit. Yet, the implication here is evident -- the spacecraft (subsatellite, or whatever) can be aerodynamically sized (its A/M varied) to meet most requirements, as desired.

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\*Remember that  $\beta \cong A/M$  at the altitudes assumed herein.

TABLE I. AREA-TO-MASS RATIOS FOR SIMPLE SATELLITE SHAPES

A/M VALUES (2, 3)

Shape	SKETCH (indicating flight dir.)	SOLID BODY	(SHELL* CONSTRUCTION)
Sphere (shell)		$\frac{A}{M} = \frac{0.75}{\rho R}$	$\frac{A}{M} = \frac{0.25}{\rho t}$
Cylinder (can)** [flying broad-on]		$\frac{A}{M} = \frac{0.6366}{\rho R}$	$\frac{A}{M} = \frac{0.3183}{\rho t (1 + \frac{R}{L})}$
Cylinder (can)** [flying end-on]		$\frac{A}{M} = \frac{R}{L} \left( \frac{1}{\rho R} \right)$	$\frac{A}{M} = \frac{0.5}{\rho t (1 + \frac{L}{R})}$

(1) The  $\vec{V}$  is an assumed velocity vector; to indicate orientation for aerodynamic impact area (A).

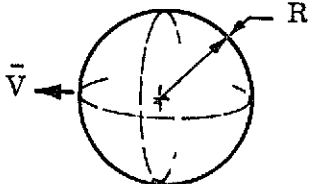
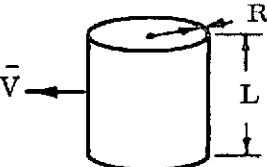
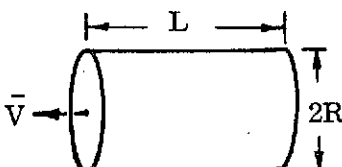
(2) A/M is the (aerodynamic impact) area-to-body-mass ratio based on a homogeneous material used for construction ( $\rho \equiv$  mass density in  $\text{kg}/\text{m}^3$ , or  $\text{lb}/\text{ft}^3$ , etc.); R is the body radius.

(3) Units for A/M will be  $\text{m}^2/\text{kg}$ ,  $\text{ft}^2/\text{lb}$ , (etc.). A is body cross-sectional area; its unit vector parallels  $\vec{V}$ .

\* Shell construction assume a constant thickness (t) skin for any of the bodies.

\*\* Formula accounts for both "ends" in the can.

TABLE II. TYPICAL A/M VALUES<sup>(1)</sup>

SHAPE	SKETCH	PHYSICAL CONSTRUCTION	
		SOLID* BODY	SHELL CONSTRUCTION*
Sphere		$\frac{A}{M} = 4.204 \times 10^{-5}$	$\frac{A}{M} = 1.261 \times 10^{-2}$
Cylinder (broad-on)		$\frac{A}{M} = \frac{N}{A}$	$\frac{A}{M} = 1.427 \times 10^{-2}$
Cylinder (end-on)		$\frac{A}{M} = \frac{N}{A}$	$\frac{A}{M} = 2.803 \times 10^{-3}$

(1) See Table I for formulae.

\*For this example: the material is assumed to have a specific weight ( $\rho$ ) of  $7929.14 \text{ kg/m}^3$ ; the thickness ( $t$ ) is chosen as  $1/4 \text{ cm.}$ , and, the body dimensions are  $L \cong 18 \text{ m}$ ,  $R$  (for all bodies)  $\cong 2.25 \text{ m}$ .



For instance, to match a sub-satellite's A/M to that of its parent vehicle, the construction and/or equipment content could be "juggled" to satisfy this condition. Or, the counter situation could be made to exist! (Most likely this latter case is a more probable one for such as the LDEF/shuttle experiment(s)).

Thus, the implications which may be drawn from the above paragraphs is that the effect of external (natural) forces, and especially drag, can be varied extensively. Not only will these variations be controlled (as through A/M) but they may be "natural" in origin. Recall that Fig. II.3 illustrates a marked "change" in the atmospheric density from periods of a quiet sun to an active one. To a lesser extent the daily variations, due to solar heating, convection, and other natural causes, are of significance.

The discussions, so far in this section, have been directed toward sub-satellites, test particles, etc. which are presumed to be in the atmosphere and directly affected by drag. Contrary to this there is a countering action, due to drag, for those test particles which are located inside the parent vehicle. As a means for studying this situation, the "drag force" may be given a change in sign and then applied to the test particle. By this device the drag becomes a "relative force", one which is acting on the parent vehicle and not directly on the particle. The interpretation, or simulation, here, leads to the indirect effect of drag; that whereby the parent is "slowed" in its motion while allowing the test particle to move "freely" inside the "decelerating" spacecraft.

The action of drag, on a satellite, is to cause a slightly eccentric orbit to be circularized\*, first, followed by an orbit which decays into a reentry spiral. This latter action is a consequence of energy dissipation. These conditions, seen as relative motion for a subsatellite particle, are referred to a circular reference orbit. In this regard the vehicle would appear to lose altitude and to lose transverse position, simultaneously. This can be predicted through the analytical (linear) results. There, these actions would appear as a wavering

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\*In addition to the circularization, it is found that the speed of the affected vehicle is increased (a secular influence). This, at first glance, is a paradoxical situation; however, a closer analysis will show this to be reasonable and expected. This phenomenon is well known and understood now.

trace, somewhat exponential in character. If, in addition to this dissipative action, the test particle would be initially projected forward (from the parent particle) then after an appropriate time lapse drag will overcome this input energy (state), and the subsequent action will be that due to the "dissipative force!".

Assuming that the initial input was a "forward directed impulse" then, by an energy analysis, one could estimate the time required for drag to overcome this input. Presuming a fixed drag force, including a fixed density, it can be shown that this time ( $\Delta t_{\text{drag}}$ ) can be estimated from:

$$\Delta t_{\text{drag}} = \frac{\Gamma_o (\Gamma_o + 2 \cos \alpha)}{2 \beta \rho V_P} \cdot \quad (I.16a)$$

In this expression  $\Gamma_o \equiv \frac{V_{s_o}}{V_P}$  ( $\bar{V}_{s_o}$  is the relative velocity of the launched particle),  $\alpha$  is the angle between  $\bar{V}_{s_o}$  and  $\bar{V}_P$  ( $\bar{V}_P$  being the circular velocity of the reference particle "P"); while  $\beta$  and  $\rho$  are the Ballistic Coefficient and atmospheric density, respectively. (Note that when the test particle is "launched" ahead of the parent (P), then  $\alpha = 0$  and the time expression reduces to

$$\Delta t_{\text{drag}} \cong \left( \frac{\Gamma_o (\Gamma_o + 2)}{2 \beta \rho V_P} \right)^* \cdot \quad (I.16b)$$

In the foregoing paragraphs it was indicated that a linearized analytic solution could be used to predict the overall effects due to atmospheric drag. To a marked extent these indicators are a consequence of the model used for the force. On the supposition that the ballistic coefficient has fixed value, and that the density is essentially constant, then any variance which appears would arise solely through the drag related velocity description. Presuming that this velocity is defined as "relative to the atmosphere", and assuming that the atmosphere "rotates" about the primary mass, then the drag term will show an additional degree of complexity. (Incidentally, this addition does not materially

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\*With  $V_P \equiv |\bar{P}| \dot{\phi}$ , these expressions may be altered so that time is replaced by transfer angle ( $\phi$ ). Here  $\Delta \phi \equiv \dot{\phi} \Delta t$ , thus the result implied.

affect the general effects of drag -- the consequences still come through in the analysis, implying the same results as above).

When a study of the orbit's motion is included in the analysis, the two principal influences found are the change in orbital inclination and the variation in nodal position. A first order analysis for these effects can be obtained through the definitions of momentum and energy (changes), due to a disturbing force (here, drag). When such an analysis is undertaken, the consequences found are: (1), that drag causes a reduction in inclination; but, (2) the node is unaffected. A more accurate and in-depth study would show these descriptions to be, in the large, the same as those indicated by the simpler evaluation.

Before closing this section on aerodynamic forces some comment should be made regarding the lift force. Normally this component of the force system is not mentioned since: (1) it is usually of lesser consequence; and/or, (2) it is not apparent due to the near spherical shape of satellites. However, if the test particle is representative of (say) a vehicle with large solar panels, or one with a shape which is considerably removed from spherical-type symmetry, then aerodynamic lift may be significant. In such cases this force component might be modelled as a fixed force also; or, for a more accurate evaluation, one could resort to an aerodynamic analysis typical to such as hypersonic-particle impact theory. Regardless of what approach might be suggested, this topic is not examined herein; it is deemed to be beyond the scope of this investigation. Interested readers should consult the extensive literature on hypersonic (and/or free molecule flow) aerodynamic theory.

II.10 Radiation Pressure (and Solar Wind). Solar radiation falling into a reflective surface produces a force, which though small, could have a magnitude comparable with some of the other force systems described herein. In particular, for large area to mass ratio vehicles, this reaction might easily be a source of "disturbing acceleration" worthy of consideration\*.

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\*This phenomenon was first noted as a situation worth considering when ECHO satellites were put into orbit. It is recalled that ECHO was a passive balloon type spacecraft having a large exposed area, but small mass.

Generally speaking this disturbing acceleration can be described in terms of parametric quantity:

$$\frac{C_r A}{m} \quad (I.17a)$$

Here  $C_r$  is a reference level of solar radiation pressure (the value at a solar distance of 1 AU); A, is the "affected area" for the satellite, and m is the vehicle mass. (An approximate value for  $C_r$  is  $4.6 \times 10^{-5}$  dynes/cm<sup>2</sup>, at one AU). Since this radiation is assumed to vary according to an inverse-square law, then the ideal specific radiation force may be expressed as:

$$\frac{C_r A}{m} \frac{\bar{r}}{r^3} \quad (I.17b)$$

where  $\bar{r}$  is the position vector relative to the sun. (Note: For a perfect reflector this value is doubled, to account for both incident and reflected photons).

A more realistic description of this specific force - one which accounts for the body's reflective characteristics and surface orientation - is given by:

$$\frac{C_r A}{m} \frac{\bar{r}}{r^3} (\bar{e}_s \cdot \bar{n}) \left[ 2k_r (\bar{e}_s \cdot \bar{n}) \bar{n} + (1 - k_r) \bar{e}_s \right] \quad (I.17c)$$

Here  $\bar{e}_s$  is the solar radial unit vector, and  $\bar{n}$  is the unit normal vector to an element of the illuminated surface. In addition,  $k_r$  is the coefficient of reflectivity (a ratio for a number of reflected photons to the number of incident photons).

In Eq. (I.17c) the "2k<sub>r</sub>" term is an indication of "incidence and reflection", while the second quantity represents the "absorbed energy" for the body.

Coupled with this interesting natural phenomenon is the added complication which arises as a consequence of periods of "sunlight" and "darkness" during an orbit. When the test particle is shadowed by earth (in the umbra) it is free from this disturbance; however, when it is only partially shadowed (in the penumbra) then it is only partially affected. Needless to say, the modelling

of such an orbital disturbance presents interesting problems. In the simplest of analyses this perturbation could be accepted as a fixed level of force which has a fixed direction in inertial space.

To estimate the level of force which should be accounted for, the investigator must ascertain what "size" area will be facing the sun ( $\equiv A \bar{n}$ ); then assuming the surface characteristics are known, a value of  $k_r$  is found. Finally, some assumption regarding the type of body reflection must be made, so that with a value for  $C_r$ , a reasonable estimate of the disturbing force can be ascertained.

Once the physical and material measurements for a proposed spacecraft are in hand it is a routine matter to determine the contribution which solar radiation adds to the disturbing accelerations. There are, of course, some natural occurrences which may suggest other considerations in regard to this evaluation. Two factors which first come to mind, and which are related, would be the influence of solar activity and/or solar wind. Since these two are related it is plausible to expect that their individual influences could and should be lumped together. Consequently it would not be unreasonable to include both in the radiation constant ( $C_r$ ); thus the overall effect might be modelled with a "multiplier" used to adjust this parameter. (Even though the solar wind is not truly radial in direction, the assumption that it "flows radially" is not a serious inaccuracy at earth distances. Most likely such an approximation is no worse than the uncertainty present in other quantities).

The reader is reminded that, here, as in the discussion on atmospheric reactions, the magnitude of the disturbance is proportional to an  $A/m$  (area to mass ratio) for the vehicle. Once more, one is led directly to a same "degree of uncertainty", as with the drag force; that is, until the physical measurements for a vehicle are known it is improbable that a well defined magnitude can be given for the force. It is recalled that for large  $A/m$  values the perturbing influence can be considerable (as was the case for ECHO satellites); but for the converse situation this disturbance would, in most instances, be unnoticed. It

is quite likely that those vehicles which have a large density are more affected by gravitational anomalies than they are by these forces - especially at low orbit altitudes.

For quick estimation purposes one could consider this force as fixed in magnitude, especially at earth-sun distances. This is argued on the basis that the gradient of the force (force per unit area) is, in ratio to its own magnitude:

$$\frac{\Delta F_{\text{rad}}}{F_{\text{rad}}} \approx -2 \frac{\Delta r}{r} , \quad (\text{I.18})$$

where  $r$  is distance measured from the sun. Consequently, a one percent change in magnitude would imply a change in distance ( $\Delta r$ ) comparable in size to a lunar orbit diameter. Similarly a displacement normal to the earth-sun line, but equal to the lunar orbit radius, implies an equally insignificant variation in the force. Thus, for satellites operating in the earth-moon reaches of space it is acceptable, except for refined analysis, to treat the radiation as a fixed force (fixed in direction and magnitude).

Perhaps the largest "natural" consequence, associated with this force, is the shadowing effect of the earth. For those situations where the spacecraft is eclipsed by earth, the force vanishes, and does not reappear until the vehicle emerges from "behind" the planet. Here, in order to estimate shadow effects, one could model the eclipse as a cylinder with its axis parallel to the earth-sun line, and a diameter matching that of earth.

For the case of a nominal circular orbit, with the satellite and sun occupying the same plane, it can be shown that the influence of solar pressure (as a perturbing force) arises as a secular influence\*. In particular the radial relative displacement will grow, showing periodic increments, inward and outward, relative to the base orbit. Likewise, the transverse coordinate exhibits secular growth as well.

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\*This long-time secular influence, due to solar radiation, is a well known phenomenon, particularly from ECHO satellite orbit measurements. There it was found that the perigee height varied directly with the shadow effect, due to earth.

In addition to these conditions, if one seeks to learn what influence solar radiation has on the orbit's geometry, also he will find that (nominally) the inclination is reduced and the node is advanced.

II.11 The Influence of Secondary Massive Bodies. The many-body problem and/or the celebrated Problem of Three Bodies enters into these discussions for perturbative effects. Satellites, subsatellites, etc. are adversely influenced by the other massive bodies in the solar system. Certainly the two most influential of these, for earth orbiters, would be the Moon and the Sun. Following, in order, would be the other planets -- not the least of which is Venus, then Jupiter, with the remaining ones following in their order of influence.

The modelling of this phenomenon is generally carried out as a modification of the classical two-body problem. The inclusion of a "disturbance function", in reality a potential function, added to the classical formulation is sufficient to illustrate these consequences. This potential describes those influences which can be attributed to the influence produced planets, each of which then plays a role analogous to the gravitational effect from the attracting primary. When this function is used to define the "disturbing force" one can show that the perturbing influence, as compared to the primary attraction, may be represented by:

$$\frac{m_d}{m_E} \left( \frac{r}{r_d} \right)^3, \quad (I.19)$$

where  $m_d/m_E$  defines the mass ratio ("disturbing" to "primary" mass); and,  $r/r_d$  is the distance ratio, for the satellite, from the disturbing planet and the primary one.

Playing a "numbers game" with this quantity, it is found that the predominant influences, for earth-based orbits, are due to the sun and moon. As a matter of some interest the results shown in Table III below are indicative of these results. There a close earth circular orbit (at 222 km alt), and a synchronous orbit, are evaluated and compared. As a point of interest, and for comparison, similar data for the planet Jupiter are included, in the tabulation.

TABLE III.  
DISTURBANCES FROM CELESTIAL BODIES

Perturbing Body	Radius $r_d$ (km)	$(m_d/m_E)(r/r_d)^3$	
		(close-in orbit)	(synchronous orbit)
Moon	$4.166 \times 10^5$	$4.89 \times 10^{-8}$	$0.862 \times 10^{-5}$
Sun	$1.495 \times 10^8$	$2.865 \times 10^{-8}$	$5.05 \times 10^{-6}$
Jupiter	$6.285 \times 10^8$	$3.683 \times 10^{-13}$	$6.488 \times 10^{-11}$

Now, the effect felt by a subsatellite, originally moving with the parent vehicle on the circular orbit, is to alter its relative motion. Oddly, this disturbing force produces an influence somewhat reminiscent of Hooke's Law. Here the relative radial displacement is an elongation in the direction normal to the disturbing body's position, and a foreshortening in the direction of the perturbing mass. The transverse relative displacement exhibits a secular growth, from orbit to orbit; but, in a cumulative sense, due to the "rotation" of the perturbing mass, this secular input should tend to vanish.

A study of Table III will indicate that these particular influences are rather minimal for low altitude orbits. As a consequence it is generally not worth the effort to study this particular input in any detail. Most likely, at the lower altitudes, there are other perturbing influences which would overshadow this one almost completely.

The influence of this disturbance on the orbital geometry, per se, is of some interest. The most obvious effect is the nodal regression which is brought to light here. Secondly, the orbit's inclination is decreased, by this input, though this effect is most apparent over a longer time period; one that is comparable with the motion of the disturbing mass about the primary. It is seen



that in general this influence is one which tends to vanish. It is also noted that, in general, the magnitude of the change(s) in both inclination and node are directly related to the parameter of significance in Table III. Thus, the mass ratio and the distance ratio both play a large role in regards to this effect.

II.12 Unsymmetrical Mass Distribution. The classical problem of two bodies, both traveling along free orbits, is formulated under the assumption that each is replaced by a "weighted particle". This treatment presumes a mass distribution which is symmetric about the mass center. If the distribution is not symmetric then there will be disturbing forces present, due to asymmetry, which can perturb the orbits.

The usual procedure used to account for unsymmetrical\* mass effects is that of modifying the "potential function"; that is, to add terms to the central body term, as a means of representing the mass distribution. Normally this is accomplished by the addition of zonal and tesseral harmonics. The zonal harmonics arise because of meridional ellipticity, while the tesserals are a consequence of longitudinal variations in the shape of the earth (or massive body of interest). One form of the potential ( $\Phi$ ), which includes the first few added terms (the more significant ones), is:

$$\Phi = -\frac{\mu}{r} \left[ 1 + \frac{J_2}{2} \left(\frac{R}{r}\right)^2 (1 - 3 \sin^2 \delta) + \frac{J_3}{2} \left(\frac{R}{r}\right)^3 (3 \sin \delta - 5 \sin^3 \delta) \sin \delta + \frac{J_4}{8} \left(\frac{R}{r}\right)^4 (3 - 30 \sin^2 \delta + 35 \sin^4 \delta) + \dots \right]. \quad (I.20)$$

Here  $\mu/r$  is the "central mass potential";  $R$ ,  $\delta$  are the radius and geocentric latitude for the primary body; and,  $r$  is the geocentric radius to the satellite or test particle. The  $J_i$  coefficients, seen in Eq. (I.20), are adjusted to best represent the mass asymmetry in the primary.

One set of values, for the constants appearing in Eq. (I.20), are:

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\*One of the first "space" studies conducted in this area lead to the (now famous) pear-shaped earth. Subsequent examinations of satellite orbits have lead to some modifications in the potential constants, and to an extensive array of the tesseral and zonal coefficients. Also, applications to lunar orbits have resulted in the discovery of "mascons" for the moon.

$$\begin{aligned}
R &= 6.33818 \times 10^6 \text{ m (earth's equatorial radius)} \\
J_2 &= 1.0826 \times 10^{-3} \\
J_3 &= -2.5 \times 10^{-6} \\
J_4 &= 1.8 \times 10^{-6}
\end{aligned}
\tag{I.21}$$

Since the attractive force, as produced by a massive body, and its perturbations, arise by the differentiation of this function then it should be apparent what the ratio of the various terms lead to with regard to the disturbances. For instance, the largest effect (obviously) comes from  $J_2$ . Thus, the specific force due to the central mass is  $\mu/r^2$ ; and, that due to the  $J_2$ -term is  $J_2 R^2 / 2r^2 (\mu/r^2)$ ; therefore the ratio of these terms is

$$\frac{J_2 R^2}{2r^2} .
\tag{I.22}$$

This quantity is, at most, of the order of  $J_2$ . Consequently the perturbing acceleration, due to  $J_2$ , is at best only  $10^{-3}$  times that due to the primary term. As an immediate influence this acceleration is small; and, subsequent  $J_i$ -terms are even smaller. This remark does not suggest that  $J_2$  effects are negligible; certainly the long time influence cannot be ignored, but the short time effect is demonstrably small. When the force is defined and applied, one will find that  $J_2$  "pulls" on the orbiting subsatellite, and on the parent as well, tending to "reduce" the relative motion.

The more classic influence from  $J_2$  is found in its effect on the position of the orbit, per se. From a first order analysis one can show that this is the cause of the familiar regression of nodes; and, as is equally well known,  $J_2$  has no considered influence on the inclination of the orbital plane. It should be remembered, however, that inclination plays a role in the nodal regression expression.

As an estimate of how orbital altitude enters into this situation an example case is presented below (Table IV). There an inclination of 30 degrees has been assumed; and two circular orbit altitudes have been selected (222 km and 2220 km, respectively). Both the regression, and the rate of regression are estimated and noted in the tabulation.

TABLE IV.

J<sub>2</sub> - EFFECT ON NEAR-EARTH ORBITS

Alt. (km)	Period (sec)	$\dot{\Omega}_{av}$ (rad/sec)	$\dot{\Omega}_{av}$ ( $^{\circ}$ /orbit)	No. of days for regression of $2\pi$
222	5336	$-1.55 * 10^{-6}$	-.474	$\cong 47$ da.
2220	7937	$-6.14 * 10^{-7}$	-.279	$\cong 118.5$ da.

In an analysis which would extend beyond the J<sub>2</sub>-effect, one would most likely include those terms indicated in Eq. (1.20), and possibly should include the J<sub>22</sub> term, as well. This last quantity is suggested for inclusion in view of its "magnitude" when compared to the other terms already noted above.

II.13 Other Natural Forces. The remaining perturbative force systems to be considered here are those which arise as a consequence of the electrical charge which has accumulated on a spacecraft. In one instance the disturbance comes about because of an interaction between the charge and earth's magnetic field. In the second case the body charge acts to attract (or possibly repel) small particles which, themselves, have acquired a charge.

For the first situation noted above, a rough estimate of the force could be acquired by assuming that the earth has a magnetic dipole situated along its axis. (This is known, today, to be a rather crude approximation; however, the complexity required for a more elaborate analysis is not warranted here). In this regard, then, the force which acts of the vehicle is dependent on the spacecraft's velocity, the dipole induction (hence the magnetic moment of the earth), the charge on the vehicle, etc. It can be demonstrated that the force size, in ratio to the mass attraction ( $\mu/r^2$ ) force is:

$$\frac{q \mu_0 M v}{4\pi m \mu r} \quad (I.23)$$

Here,  $q$  is the charge;  $M$  is earth's magnetic moment;  $\mu_0$ , the permeability of space; while  $v$  and  $r$  describe the vehicle's motion (state), and  $m$  is its mass. When appropriate values are introduced (for near earth satellites) it is found that this ratio (Eq. I.23) is of the order  $10^{-10}$ . Therefore, this effect is most probably of negligible size. When the force is examined to ascertain how it may influence a relative motion, one finds that it could bring about a loss in the radial displacement and a secular incrementing of the transverse one. Examining for the consequence which the force might have on the orbit proper, it is found that there is practically no effect on the inclination, but that there is a tendency to cause some nodal regression.

For the second case indicated above, the influence which a vehicle's surface charge ( $q$ ) might have on other particles, free to move in the vicinity of the spacecraft, is briefly noted below. Here one will find that for small separation distances it is necessary, in general, to describe the force by the evaluation of a surface integral. This means that it is essential for one to know the external geometry of the vehicle. Hence, except for an idealized situation, it would be most difficult to define this force accurately, in a generalized formulation. A reasonable, alternate approach to this would be to model the "attraction" as an inverse square law force, and to estimate the consequences accordingly.

II.14 Mechanical Forces. In the previous section the most probable natural force types were introduced and discussed. Now, in the next few paragraphs, the discussions will be directed to other systems with can, either intentionally or unintentionally, bring about changes in a spacecraft's motion. Earlier it was noted that the most likely mechanical devices to be encountered would be those operating on a momentum transfer principal; and, possibly, some type of physical connector. However, as additional candidates, one should not overlook those actions which are identified with the impulsive state changes they can produce.

One of the general constraints implied throughout this investigation has been that wherein all motions were considered to be particle motions. In this regard any and all energies utilized to change a state of motion have gone to alter position and linear velocity; none was expended in producing body rotations. That is, the motion of a body about its own mass center has been ignored, here.

A second restriction placed on the study has to do with the use of physical connectors between the bodies of interest. Herein, "free orbit" motions have been the topic of interest and physical "links" between particles have not been allowed. Even though this is a topic of real significance and interest, in a variety of relative motion studies, it has not been included in this investigation.

II.15 Momentum Transfer Devices. The mechanical fixtures which immediately come to mind now are rockets and jets; and, in particular those which can be utilized to "power" a spacecraft from one state of motion to another. Of course, for present purposes, these units are not those employed in the launching of a space vehicle; or even those generally utilized to inject a spacecraft into interplanetary or, even, translunar flight. At most, the devices of interest here are those with a capability to provide for intercept and rendezvous operations between spacecraft on adjacent orbits. It is not prudent, at this point, to place stringent limits on the size of these units since the "variety" of operations fitting this classification is numerous. For purposes of mathematical tractability and simplicity any

mass efflux which would be applicable to the use of these devices is neglected.

It should not be surmised that all situations which fit into this operational category are represented by "controlled conditions". Such undesirable contingencies as "leaking" jets, "out-gassing" and "overboard dumping" can be considered as candidates for this general classification of devices.

Summarily, then, insofar as size of a momentum transfer is concerned, this may extend over several orders of magnitude; yet, without regard to what purpose each device may serve, its operation is presumed to be one which does alter the test particle's motion (specifically its relative motion).

In the pursuit of information from the operation of these various units it has been assumed, throughout, that the observed spacecraft is "controlled". That is, it maintains a prescribed orientation with respect to one of the "frames of reference". (It should be remembered that the two reference frame orientations used in this study are: (1) the "local horizon" system; and, (2) the inertially oriented one. In the local-horizon scheme, one axis is always radially directed, while another is normal to the base plane of motion). The main reason for prescribing forces which have fixed direction components is that this most conveniently allows for the analytical solutions. Also, it provides a reasonable approximation for simulating a variety of real world situations.

In the simulations one can consider (say) that the observer maintains an inertial orientation. However, the observed craft may be operating in either of the frames of reference; and, would have its "thrusters" aligned accordingly. Consequently, the relative motion being viewed is "seen" in the inertial frame (now) even though the relatively moving vehicle may be "located" in either of the reference systems. What this amounts to is that there are two force disturbance orientations for each of the observer's frames of reference. This, then, allows for the combinations of "thruster" alignment and "referenced" relative motions particular to the two selected spacecraft orientations.

In addition to having determined analytical solutions for these several combinations; these same problems have been "solved" numerically. From this operation more precise and longer timed motion sequences may be described. Also, with the magnitude of each force system open to arbitrary selection, it is quite possible to simulate and study a wider range of problem situations.

II.16 Impulsive Reactions. A second scheme for simulating mechanical disturbances, and that one most frequently encountered in the literature, is the "impulse". Here, one can assert that the reaction occurs in response to any "strong", but short timed action. Nominally the impulse is represented as an instantaneous change in velocity, impressed on the system without a change in position -- hence only a partial change in state is apparent.

Without being specifically concerned with "how" such inputs arise, the subsequent results are made known through a time history of the relative motion. Analytically these results may be classed as "initial-values" problems; and, the corresponding numerical results would be so classed, also. Since a problem of this type is explicitly dependent on the magnitudes of the inputs, then here also is a wide range of simulation possibilities which may be treated in a simple manner.

II.17 Motion Traces. For any of the situations just described the resulting motion traces -- for both displacements and velocities -- are of interest. From these one obtains a graphical time history of a motion; and, as a consequence, with this in hand it is easy to prescribe how the motion should be altered to accomodate some operational maneuver. In addition, for mission planning purposes; or for rescue and retrieval; and for contingency situations, it is essential that this sort of information be readily available to the planners.

One of the earliest uses made of the (linearized) relative motion trace equations was that of defining an intercept and rendezvous maneuver (see Ref. 5). In that study an in-plane intercept mode was described. For that operation

it was assumed that an impulsive velocity change would be used to accomodate the intercept. Of course, now, with present results at hand, it is a simple task to apply a same set of criteria and to develop a "thrusting" intercept, as well. (These various case situations have been studied as a part of the present investigation; but, additionally, the non-linear (numerically integrated) problem is "solved" as well. Of course this one solution type must be obtained by an interactive procedure).

There are obviously many other problem areas which may be investigated and studied under the classifications noted above. In nearly all of these, and certainly for those which require some degree of control, the need for a well defined trace is evident.

As an aid to understanding how these initial values, and fixed levels of thrust, may influence a relative motion, Ref. 1 contains a compendium of results showing examples. Included with the various sketches, there, are the descriptive equations for each case. (One word regarding these trace results: Each sketch, shown, was developed from positive valued input quantities -- either state parameters or "thrust" components. Also, and as an exercise, it is not difficult to ascertain what changes occur on the traces when one of these parameter's signs is changed). The most difficult task for the investigator is to guess, as a priori information, what shape a trace may acquire for arbitrarily selected inputs. Therefore, one of the most valuable uses to be made of the "compendium of results" is that of learning how the various input parameters affect the trace geometries. This knowledge leads directly to an intuitive "feel" for all problem types; and, it makes it possible to estimate what actions should be taken in order to develop a "strategy" for a variety of operational situations.

II.18 General Comments. Some general comments regarding these traces would be in order here. First; it should be mentioned that the following statements are based on those analytical results developed and noted in References 1 and 2. There, the linearized differential equations, referred to a circular



base orbit, was solved to obtain expressions showing how the state variables changed during a given relative motion situation. From these equations "initial-values" problems, and fixed thrust cases (as noted earlier), may be studied. Also, these algebraic expressions are cast into forms which can predict those motions which appear on (both) the inertially-oriented and local-horizon frames of reference.

By examining these (linearized) traces one finds that the in-plane coordinates are coupled while the out-of-plane coordinate is independently defined. Consequently, in-plane traces are quite easily represented in a concise and precise mathematical format. Unfortunately, those for the out-of-plane motion are more difficult to define, analytically.

After studying typical geometries one finds that: (1) in general, most situations show a divergence; that is, the traces indicate an almost monotonically increasing separation distance between the (circulating) parent and the test particle. This is true regardless of the frame of reference used. (2) The in-plane traces, found for the local-horizon frame of reference, depict test particles moving in a direction opposite to that of the base orbital motion. However, the counter situation is found for traces described on the inertially oriented planes. (3) In order to obtain closed, or non-divergent, in-plane trace curves, there is a unique requirement stipulated for the initial state parameters (a particular requirement is set for both frames of reference; however, these are not identical for both coordinate systems). (4) The "thrusting\* problems" do not develop closed and bounded trace geometries. All of the thrusting situations examined here show a divergence of some sort; and, there have been no relationships found between components, which would point to traces which became bounded on an orbit-by-orbit basis. (5) In general, trace geometries found for the in-plane cases, referred to the local horizon coordinate system, where the disturbances were given fixed in-plane components, were either parabolic and/or cycloidal in shape. For the inertially-oriented frame the corresponding geometries

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\*Those explicit in fixed-force-components.

were spirals. When the applied force system was stipulated to have fixed inertially-oriented components, the in-plane traces (referred to the local horizon frame) became spiral-like, also. In addition, the corresponding figures, referred to the inertial frame, were seen to be spiral-like as well.

Carrying the analytical investigation one step further, it was found that all of the various in-plane geometries could be simply and easily represented by a sum of position vectors. In this regard, during the investigation, each case was examined and a system of (rotating) vectors was obtained. These were added together so that descriptive trace points could be located on the planes of motion. (This task is described in a special document, an interim report, Ref. 6). For the geometries scribed onto planes in the local-horizon frame the vectors were found to be composed of fixed (constant) and rotating vectors; the latter were noted to have a rate of rotation matching that of the base orbit. However, in the inertial-system of coordinates some of the rotations were found to have a double orbit rate.

All in all, it is an easy task to sum these vectors, and to develop a trace on both the displacement and velocity planes. What does present some difficulty is the setting up of the various vectors from the analytical expressions. For most case studies undertaken in this investigation, the "building" of descriptive geometries is described in the reference above noted; interested readers should consult that document for any additional information.

The out-of-plane traces -- those incorporating the coordinate normal to the reference plane of motion -- are much more difficult to describe and to predict, as a priori information. One of the major problems arising here stems from the fact that the linearization procedure, as applied to the differential equations of motion, separated this last coordinate (and its solution) from the in-plane problem. Consequently there is no neat and systematic notation, obtained here, similar to that found for the in-plane traces.

As for the situations noted above the general conditions show that a geometric divergence is present for the trace figures. However, the possibility of acquiring some bounds for these graphs, on an orbit-by-orbit basis, is apparent here too. The initial value coordinates which cause the divergence are those in the direction of the orbital motion. When they are eliminated the boundedness reappears and the traces are constrained, accordingly. Contrary to this, all traces are (again) divergent in character when the disturbance appears as any fixed magnitude "thrust" component. (Even though the out-of-plane geometries are more difficult to describe, they too are commented upon in Reference 4, but not treated in as much depth as the in-plane situations. Once again the interested reader is urged to consult that reference for a more illuminating description).

In summary; the reader is urged to study the geometries which have been presented as a part of this investigation. The information to be gained there is most worthwhile and enlightening. For the first time, in the open literature, one has available a ready reference showing typical curves and their analytical descriptions for a large number of relative motion problem types. These traces illustrate the time history for each motion; also, they are indicative of what actions should be undertaken to initiate and accomplish a required "maneuver" from any point on a particular relative motion path. This information is the means for knowing what immediate and proper response should be prescribed, as a correction, to avoid (say) an unpleasant contingency; or what "means should be provided" to bring about a rescue; or, the action to be taken for a host of other decisions bearing on changes in state in order to meet desired end conditions.

It is for the "initial-values problem", wherein only the initial state serves as an input, that the opportunity to attain closed and bounded traces exists. Here, by proper inputs\*, the geometries can be constrained to be periodic (of period matching that of the base orbit). As a generality, it has been ascertained that (for linearized cases) only when impulses are directed normal to the base velocity

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\*See discussions in References 1, 2, and 4 on "non-secular traces".

vector can this requirement be met. When fixed magnitudes of thrust are introduced (as disturbances) the opportunity to develop bounded, periodic traces does not exist. At best there may be some pseudo-periodicity indicated; but the boundedness is not there.

One last remark concerning the closed, bounded relative orbits: Earlier in this report, it was noted that the true orientation of an impulse, to produce closed traces, was not normal to the base velocity vector. Instead, there is some "small" deviation from this orthogonal direction (See Eq. I.11)); however, this deviation tends to vanish as the magnitude of the impulse vanishes. Consequently, the application of nearly normal impulses does lead to closed trace figures about the reference particle. When the impulse is directed away from the normal direction then one finds that the state has a geometric divergence; (see the various compendia, Ref. 1, for typical figures).

### III. COMPUTATIONAL PROGRAMS

III.1 Introduction. The sections prior to this have been given to a discussion of various perturbative force systems which can affect space flight; and, to describing results pertaining to these and other influences. In the next paragraphs an overview of computer programs, developed to obtain a history of these various relative motion states, will be presented. Primarily, here, the program capabilities will be outlined; however, some indications will be given regarding limitations and suggested useage. In this manner the reader will be made aware of overall conditions rather than specifics. As a consequence it will be made evident that these programs have a wider range of utility than what may be surmised on a first examination basis.

The two computational programs which are to be discussed here have been developed to describe various relative motion situations which arise in response to the disturbance forces noted earlier. In the one case a general type computational program is described; however, the second one is more specific in its operation, and leads to a more sophisticated description of relative motions. First, a description of the more general program will be outlined and discussed. Later the more formal system will be commented upon.

III.2 The Relative Motion Program, RELMOT. A general, relative motion program (RELMOT)\* has been developed so that results typical to the analytical studies, noted in References [1] and [2], can be displayed. Of course the algorithms making up this program are not restricted to analytical results, per se; others have been included to provide corresponding numerical results and to allow for special case studies.

Since a primary aim of this program was to provide data obtained from the analytical results, described earlier, it was necessary to include in the formulations those overall, general characteristics needed for that analysis. Briefly, the pertinent conditions and assumptions are noted below.

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\*RELMOT is the name given to this particular machine program. Its operation is more fully described in a companion report, Ref. [7].

III.3 General Considerations. In concurrence with the mathematical analysis the computer algorithms are developed around a base (reference) circular orbit\*. This orbit, and all others to be considered here, must come about mainly in response to an inverse square attractive force field. Also, all of the principal mathematical statements are cast in terms of relative position and velocity components. Since the main intent in this study is to provide information about the state of a motion, all bodies are viewed as fixed mass particles. Consequently any body motions about the respective centers of mass are disregarded.

For purposes of convenience and compatibility (with the analytical formulations) all internal program operations are carried out using the dimensionless quantities. This leads to a compact notation and provides more general results. Introducing the base orbit parameters, as reference values, the nondimensionalization follows naturally through a normalization of the differential equations. During this conversion it is found convenient to replace time with the transfer angle as an independent variable. Also, included in with this normalization process, the applied specific force components are nondimensionalized by the specific centrifugal force from the base orbit. Through these conversions a simpler set of differential equations (and, consequently, solution expressions) are obtained. These are the equations programmed and exercised in the computational subroutines.

Because of the need to retain an identity in formulating the equations from the analytical results and for the numerical evaluations, the perturbing force systems -- those added to the central field influence -- are described by having fixed components. That is, each specific force component (in a specified coordinate direction) is assigned a fixed value, one which it retains throughout a particular problem situation. Thus, for compatibility in results, this assigning of fixed components is maintained for both the mathematical (linearized, etc.) and numerical (integrated) study situations.

Recognizing that there are two preferred vehicle orientations normally employed in near earth orbit situations, the solutions acquired here reflect both

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\*It should be remembered that this is included to make the mathematical analysis tractable and more easily adapted to the several case studies to be outlined later.

of these. Specifically these two most frequently used systems are those associated with: (1) an inertial ("sun oriented") frame; and, (2) a local horizon (rotating) frame of reference. In reference to the circular reference orbit the local horizon frame has a constant rotational rate, with respect to the inertial one, when a relative motion is to be described. This is the philosophy of representation which has been adopted throughout the investigation.

Return for a moment to the applied specific force systems: It should be remarked that some of those forces have been referred to one or the other of the reference frames. Thus, some results are acquired from forces designated to act along the inertial axes, while other problems have forces directed in the local horizon directions. By this means the investigator has a freedom to choose and pick the "kinds" of forces which he wishes to simulate in the program. For example: drag is nominally taken opposite to the velocity while radiation pressure retains a fixed inertial direction. Thus, in simulating these two "force" conditions one would choose the local horizon frame to describe drag and let the solar radiation influence be aligned with the inertial frame.

One of the prime requisities denoted for this study was an ability to describe a relative motion in terms of either local horizon coordinates or inertial state coordinates. The importance of this is immediately apparent when one realizes that a measured relative state could be acquired from a parent vehicle aligned with either of these frames. As a consequence, the computer program can define the state of motion in either frame. (This is nominally the situation for both the analytical and numerical results).

The system of coordinates employed in defining these relative motions, regardless of the frame of reference, is nominally the cartesian. (See Fig. I.1 for a sketch). There is, however, an exception (or addition) to this statement. For the local horizon system the "usual" coordinates are cartesian; however the in-plane, or two-dimensional, solution can be described in "shell" coordinates

if desired (see Fig. B.1, Ref. [2] for a sketch). This system expresses the relative altitude by the difference in radii, and describes the angular position difference by an arc of separation measured along the base orbit. It is interesting to note that when the linearized relative motion problem is solved, and cast into a dimensionless format, the solution in shell coordinates is in precisely the same form as that for the dimensionless cartesian system. In this regard one finds that no new information (about the relative motion) is obtained by using the shell coordinate representation. For this reason the discussions to follow will not dwell extensively on this one solution type.

III.4 Program Options. The program, RELMOT, operates with five options, or relative motion problem situations, which can be "called" for solution results. For some of these the tabulated results provide information from both analytical and numerical integrations; while in others only one data type is available. These differences and other more definitive information will be found in the descriptions below.

III.4.1 OPTION 1. This first solution type, developed from RELMOT, is used to represent the more "general" relative motion situations. That is, this option provides a time history of the relative motion state which is acquired for the initial-values problem and for the perturbing force(s) case(s). Since each case study will define a motion for the "perturbed" body, relative to the reference (circular orbiting) particle, then these solutions describe the most general relative motion situations of all. Here an initial relative state  $(\vec{r}_0, \vec{V}_0)$  and/or disturbing specific force components are inputs. Internally these data are used to develop the information on the resulting state of motion from both analytical results and a numerical integration. (With regard to the disturbing forces, both coordinate frame representations may be implied in input -- i. e., the force(s) may be defined in both of the reference frames).

With the analytical results described by solutions from a set of linearized differential equations then it is expected that these data would not match, precisely,



those acquired from the numerical integration. In order to ascertain the "error" attributed to the linearization an internal algorithm computes an error difference between the two state predictions. These errors are given directly as a percentage of the integrated state. That is to say the magnitude difference of the linear and integrated relative position and velocity, taken in ratio to the integrated magnitude, is output as a percent number.

This option provides information on the relative motion state up to some (input) final time; hence the output data (tabulated state components and plotted motion traces) let the investigator "see" the influence of the various inputs on the solution. Nominally this option is a most useful one for general studies purposes.

In addition to describing the motion as it progresses from some initial state, and/or is being driven by some perturbing force system, this option can yield information on one other special situation. This is an integrated solution to describe the relative motion of a charged particle which is attracted to the parent vehicle by some surface charge. (This particular solution under OPTION 1 has no linearized-analytical counterpart built into the program; only numerical results can be obtained for this case).

Under OPTION 1 the program will accept any input data which defines: (1) an initial relative state ( $\bar{r}_0, \bar{V}_0$ ); (2) a perturbative force aligned with the inertial frame of reference; and/or, (3) a perturbative force aligned with the local horizon frame. Using these, separately or collectively, the output, from both linear theory and numerical integration, appears as indicated earlier. For convenience the investigator may select to have the resulting motion output as state values referred to either frame of reference. In addition, from the in-plane initial-values problem, there is an output option which allows one to ask for these data to appear in "shell" coordinates.

It should be mentioned here, also, that the initial state parameters may be input with respect to either frame of reference. Special triggers are set for this purpose.

For those readers who have an interest in the mathematical statements manipulated under this option, the tabulation below points to the several appropriate expressions:

<u>Problem Type and Solution</u>	<u>Eqn. #</u>	(in) <u>Ref. #</u>	<u>Output Referred to (Frame):</u>
<b>A. <u>Linearized Solution</u></b>			
1. Initial Values Problem	A.33 and A.34	[2]	Rotating Frame
2. Perturbing forces (aligned with Rotating Frame's Triad)	{ II.30 and II.31 II.39 and II.40	{ [1] [1]	Rotating Frame* Inertial Frame*
3. Perturbing forces (aligned with Inertial Frame's Triad)	{ II.87 and II.89 II.94 and II.96	{ [1] [1]	Rotating Frame* Inertial Frame*
<b>B. <u>Numerical Solution</u></b>			
1. Initial Values Problem	VI.1 (with $\tau_j=0$ )	[2]	Rotating Frame**
2. Perturbing forces (aligned with Rotating Frame's Triad)	VI.1 (with $\tau_j \neq 0$ )	[2]	Rotating Frame**
3. Perturbing forces (aligned with Inertial Frame's Triad)	II.76a	[1]	Rotating Frame**
4. Charged Particle's Motion.	See I.17b	[herein]	(Note 1, below)

The analytical solution for the initial values problem (A.1, above) is acquired from the famous Euler-Hill equations; these are noted by Eqs. (A.17), Ref. [2] (shown in dimensional relative coordinate form). The more general equations, e.g. those used for the numerical evaluation, are the set seen in Eq. (A.15b), Ref. [2]. The dimensionless expressions manipulated in RELMOT are noted as Eqs. (VI.1), Ref. [2]; as indicated in the table above, these expressions

\*These expressions reduce to the appropriate initial-values problem when the perturbing force components ( $\tau_j$ ) are nulled.

\*\*To describe the state variables in the Inertial Frame of Reference, the transforms shown in Eq. (II.13a), Ref. [1], may be applied (see, also, Eq. (II.14b) for the velocity relationship).

Note 1. A simplest model of a charged particle is analogous to the relationship for solar radiation (but with  $\bar{r}$  being the relative position vector). A more complete discussion is found in Section II.12 herein.

contain dimensionless force components referred to a local-horizon frame of reference. For the linearized case the solution to Eqs. (D.1), Ref. [2], was developed and programmed into RELMOT. The analogous situation, for forces aligned with the inertial frame of reference, would be achieved by replacing the  $T_j$  components in Eqs. (D.1) with appropriate components described from Eqs. (II.76a), Ref. [1]. The analytical solution to that set of equations produces the results given as Eqs. (II.87) and (II.89), in Ref. [1].

III.4.2 OPTION 2. This particular computational mode represents a special interest situation for an intercept maneuver. First, the program calculates, from linear theory, a thrust vector needed to produce an intended intercept. Incidentally, the active vehicle here is the relatively positioned one; the passive spacecraft is the reference orbiter. The controlling input parameters here are the initial state of the interceptor and the time to intercept.

Once the necessary thrusting action has been determined\* its components are introduced as an additional input; then the program proceeds to calculate a time history of the relative motion for the problem. Here, as before, program algorithms develop state information from both linear theory results and from a numerical integration.

The output format for this problem is identical to that used by OPTION 1; hence, "error" values appear with the tabulated relative position and speed output. One of the motivations for this option was the need to provide some indication of how good, or how badly, the linear estimate of thrust could be determined for an intercept. This particular information, of course, is implied in both the tabulated\*\* and graphical data from this solution.

From the comparatively few numbers of sample situations studied under this option some generalizations have been made regarding the accuracy and behavior of the analytical result: First, and foremost, as the size of the various

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\*This is a problem situation analogous to the well known Clohessey-Wiltshire (Ref. [5]) impulse case.

\*\*The error data are not plotted, these appear in tabular form only.

input quantities grows (this includes the size of the thrust needed for intercept) the tendency for error increases. The word "tendency" is used here to imply that not all components of the initial inputs have a same degree of influence on outputs from the program. For example, not all components of an initial relative position, or an initial velocity, etc. have a same influence on the calculated time history of the relative motion. (This statement applies also to the combinations of these same parameters). In this regard the magnitudes of the various inputs have a varying degree of effect on the output time histories. In addition, and as expected, the time to intercept plays a role of some significance regarding the accuracy of the interceptor's track (from theory, as compared to the numerical evaluations). Generally the intercept is well defined so long as the "time to intercept" does not much exceed half an orbital period for the maneuver. Also, these predictions are quite accurate when the separation distances are of the order of a few tens of kilometers; and so long as the relative speeds do not exceed a few tens of meters per second.

It is of some interest to note that the calculated "error" in position and speed does not grow continually with time. In some cases the error cycles (numerically and with sign); in other cases it remains relatively "flat" until near the end of an operation. Of course, the last position error (here) will reach 100 percent since the linear theory must indicate an intercept while the numerical result will, most likely, not do so. In addition, the last values in the output will show the trend of heading toward this hundred percent value. This occurs because the linear solution is closing the separation distance while the integrated result is simply continuing to compute a relative motion track.

All things considered equal, the linear results are better behaved than one might at first believe; especially for intercept cases which have fairly large relative "separation values", and those which are designated to have an intercept sometime after the half-period. (Another indication of the influence of "time" can be seen from the results given in Section III.11, Ref. [2]).

As an example of how well this solution works consider a typical representative case: Suppose an intercept is to occur at roughly two-tenths of an orbit transfer. Let the initial position of the interceptor be 80 km below and 80 km ahead of the target; however, suppose that the active vehicle has no relative velocity. The base (target) orbit is at an altitude of approximately 480 km.

Except for the last few computational steps, here, the "error" in position is less than one-percent; and, the speed "error" remains in the neighborhood of one and two percent throughout. This, one should note, is an intercept achieved by means of a continuous, fixed thrusting action. Though it is a relatively quick intercept maneuver, the results are indicative of those generally found.

III.4.3 OPTION 3. The problem situation represented by the output from this option is that of second-order corrections to the linear theory initial-values cases noted under OPTION 1. Here the input parameters accepted by the program are: (1), an initial relative motion state ( $\bar{r}_0, \bar{V}_0$ ); (2), reference orbit conditions; (3), a final time and various triggers to denote the output format, etc. This particular case study does not accept any "thrusting" inputs, hence the computations will describe only a refined, free, relative motion track. Of course, the output from this option may be described in either frame of reference (by coordinate selection); and, a comparison with integrated results is available for an estimate of "error" due to this solution. (The meaning of "error" here is the same as that described under OPTION 2. There is the one additional error computation provided from this study; herein the difference between second-order corrected results and numerical values is output as an "error" calculation).

Output from the second order correction mode is analogous to that acquired from the two previous options; e.g. the data are tabulated and graphically presented. The format for the tabulated data is the same as that in the foregoing modes; however, the second order state - and its associated error values - are added into each of the format statements which denote the linear and integrated values. As a clarification it should be mentioned that for the output the second

order corrections are added to the linear predicted state (values); this gives a "revised" state prediction which is printed (and plotted) for each formatted step interval. Without the influence of thrusting (perturbing forces) these theoretical results relate to the initial-values problem(s) alone.

A rather significant number of case studies have been examined under this option. In general the second order corrections do improve the predictability of theory. Also, the "time interval" for a maneuver can be extended here; i. e., the final time can be increased in comparison to linear theory, above, before unacceptable error numbers are encountered.

An interesting observation in regard to this computation has been noted: That is, the second order corrections tend to overcompensate initially. However, as the time for the operation is extended the predictions become increasingly better so, that as a general conclusion, it is worth the effort involved to calculate these corrections.

It should be mentioned that, as indicated above, the degree of error committed by the various inputs is not constant and/or consistent. Once again the parameters themselves affect the various error values; and the final time, per se, does have an influence on computational accuracy. Rather than attempt to describe the numerous causes and affects which have been observed, the advice offered is that of "try it and see!". Most likely this will prove to be the more illuminating procedure and the one which will best describe the consequences.

In regard to this solution a few statements concerning the mathematics are given below: First, a rather complete description of the development can be found in Appendix E, of Ref. [2]. The general equations to be "solved" are the set, Eq. (E. 6)\*; these lead to the companion solutions seen in Eqs. (E. 29, E. 30, E. 31 and E. 32). There, Eqs. (E. 29 and E. 30) are the dimensionless results from the Euler-Hill expressions. However, Eqs. (E. 31) and (E. 32) describe the

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\*The first and second-order solutions are acquired from Eqs. (E. 8) and (E. 9), respectively. In Eq. (E. 9) one sees that the second-order results depend explicitly on the first order solution.

second-order corrections\* (additive to the linear results) which have evolved. Necessarily these latter expressions are quadratic in the initial parameters; and, therefore, are indicative of the expected error committed by the linearization, per se.

One primary advantage of the "correction" solution is that it extends the time range validity for analytical results (with a given error bound) in addition to aiding in the overall accuracy of most all of the analytical solutions types.

III.4.4 OPTION 4. Under this computational mode the program describes an intercept maneuver provided for by a "thrust vector" whose components are aligned with the local horizon frame. For the calculations, use is made of the thrust vector developed from linear theory (as in OPTION 2); it is employed as a first estimate. However the procedure calls for this quantity to be corrected, iteratively, so that the final thrust vector is the one guaranteeing the intercept. Also the relative motion track, under this option, is obtained by the numerical integrator. Thus, this maneuver is described as an "exact" thrusting intercept between the maneuvering particle (active vehicle) and the target (or base particle flying the reference, circular orbit).

The input for this problem situation is sufficiently flexible so that any desired initial state could be utilized; however, the one parameter which "controls" the results is the final time. That is, this time specifies how long the maneuver is to take; hence it "controls", implicitly, the level(s) of thrust which must be produced by the interceptor. Since the first, or initial estimate, of this thrust is acquired from linear theory; and, since the final (or output) trajectory is determined by the iterated value; then, the investigator has a direct means to study the accuracy of these theoretical predictions. (Both sets of values, from theory and from the iterator, are tabulated in the output).

In addition to the tabulated state\*\* values appearing as output, the program

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\*The several constants ( $K_i$  and  $C_j$ ) appearing in these results are evaluated, and listed; see Eqs. (E.33).

\*\*The output, under this option, can be called for in terms of coordinates from either of the frames of reference.

can be asked to provide graphical (planar) plots of these quantities. Consequently the full intercept maneuver can be carefully studied and examined.

Mathematically this problem is described from the general expressions denoted as Eqs. (D.1), Appendix D, Ref. [2]. There the thrusting action plays the role of a "forcing function". By systematically varying this set of parameters the desired solution to this set of equations could be obtained. Obviously this would be a time consuming and unsophisticated approach to the result. To speed the operation, a first guess of the required thrust is obtained from linear theory (see Eqs. (D.15a), Appendix D, Ref. [2]). Following from this a systematic determination of the thrust vector, needed for intercept, is obtained directly. This is achieved by means of an iterator (sub-routine) contained in the program.

III.4.5 OPTION 5. This problem situation is analogous to that in OPTION 4 with one exception. Here, the intercept is achieved by a velocity impulse rather than by means of a thrusting action.

The logic employed for these solutions is similar to that for the thrusting case. Time to intercept "controls" the size of impulse vector in much the same manner as it defined the size of the thrust vector, previously. Also, the program begins the search for an "exact" solution by using the linear theory result as a first guess. Following this a systematic search for the correct  $\Delta \bar{V}$ , the one which leads to the intended intercept, is carried through. It is in this manner that the two intercept schemes are alike\*.

As with other options, the active vehicle is that one for which the relative motion is described. The passive particle, or, in this case, the one intercepted is that one which flies the reference orbit.

This particular scheme makes use of both cartesian and shell coordinates when the solution is referred to the local horizon frame of reference. Of course

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\*The input and output, for both computational methods, are very much alike.

Also, the two systems may be triggered to write the output in terms of either of the reference frame's coordinates.



when the inertial frame is called for shell coordinates are not available; however, when cartesian coordinates are used the program can accept the initial state expressed in terms of inertial quantities.

Expecting that this mode would receive more use than (say) OPTION 4, the program has been designed to execute for two consecutive intercept operations and then to manipulate both data sets to describe an alternate maneuver. By these actions the reference particle is eliminated from the analysis and an intercept between two non-circular orbiting particles is described. This last option is automatic within the program and occurs whenever the two sequentially called intercepts are asked for.

The output usually provided here is much the same as that for a single case. That is the relative motion appears in tabulated format; and data graphs (planar plots) may be made available through a proper setting of program triggers.

As an added explanation; the program is designed so that this solution is acquired by differencing two impulsive intercepts. Also, here, the differencing is set up so that the relative motion described is that for the second vehicle (particle) relative to the first.

With regard to error computations and accuracies; these are not topics of real concern here (or in OPTION 4) since both modes produce numerically integrated results only. Linearized solutions\* have not been programmed as a part of these operations; therefore the previously discussed error computations are not needed. With respect to accuracy of predictions; what should be noted here is that these results are no better (and no worse) than what could be expected from a fourth-order Runge-Kutta (symmetrical) integrator scheme. Because of other assumptions made in the modelling of this problem, it is felt that a more sophisticated integrator is not justified; yet at the same time a lesser one may not achieve the levels of accuracy which one would hope for.

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\*The linearized case (here), referred to the local horizon frame of reference, would match the neo-classic results of Clohessey and Wiltshire, Ref. [5].

III.4.6 Remarks. Earlier in these sections some mention was made of the expected and experienced accuracies for the linear theory predictions. There it was noted that, generally, linear theory results should not be relied upon, as a means of acquiring an accurate state description, for much more than approximately half an orbit. Of course, it was also implied that this time interval could be extended if the region of interest (coordinate-wise) was confined to that space immediately adjacent to the reference particle. (This, too, points to a need for the interested reader to examine cases and to ascertain how well, or badly, the theory does hold). With the integrated solutions this time constraint can be relaxed completely; certainly the inaccuracies which evolve here are due to more subtle effects and are much more difficult to ascertain and predict. The most likely sources of error now would be due (first) to the modelling of the problems and (second) to the inherent behavior of the integrator, per se. This last error type is too involved and too far removed from the theme of this work to be discussed in any detail here. In much the same context it would be redundant to reexamine the assumptions and philosophy of the mathematical set up for this problem, at this point. In view of these comments the only observation to be made is that the accuracy of prediction, in this program, is no better or worse than the accuracy of the model used to describe the various problem situations. Consequently, the system (numerically integrated) should provide good accuracy for at least a few orbits; and these would certainly be enhanced if the relative motion could be contained within a small spatial region adjacent to the reference particle.

III.4.7 Summary. As a means to summarize the results which may be developed by the program, RELMOT, a tabulation is presented below. There one will find notations on the problem types which can be manipulated by the various options, as well as the coordinate frames which may be used to represent results.

TABLE V.

SOLUTION SUMMARY FOR RELMOT

I. COORDINATE SYSTEMS

CARTESIAN (Rotating Frame)  
(Inertial Frame)  
SHELL (2-D only)

II. SOLUTION DIMENSIONS

3-DIMENSIONAL  
2-DIMENSIONAL

III. SOLUTION TYPES

A. Analytical (Linear Theory)

1. Gen. Initial-Values Prob.\*
2. Gen. Thrust (Rot. Fr.) Case
3. Gen. Thrust (Inert. Fr.) Case
4. Impulsive Intercept
5. Thrust (Rot. Fr.) Intercept
6. Thrust (Inert. Fr.) Intercept

B. Analytical (Second Order)

1. Gen. Initial-Values Prob.\*
2. Thrust (Rot. Fr.) Case
3. Thrust (Inert. Fr.) Case

C. Numerically Integrated

1. Gen. Initial-Values Prob.\*
2. Gen. Thrusting (Rot. Fr.) Prob.
3. Gen. Thrusting (Inert. Fr.) Prob.
4. Impulsive Intercept

	OPTION 1	OPTION 2	OPTION 3	OPTION 4	OPTION 5
	x	x	x	x	x
		x	x	x	x
		x			x
	x	x	x	x	x
	x	x	x	x	x
	x	x			
	x	x			
	x	x			
	(1)	(1)			
	(1)	(1)	x	x	
	(1)	(1)			
			x	x	
	x	x	x		
	x	x			
	x	x			
					x
					x
					x

\*Initial Values Inputs can be expressed in coords. for either FRAME OF REF.

(1)Using externally defined inputs, solutions are obtained.

TABLE V. (continued)

III. SOLUTION TYPES (continued)

C. Numerically Integrated

- 5. Thrusting (Rot. Fr.) Intercept
- 6. Thrusting (Inert. Fr.) Intercept
- 7. Charged Particle Prob.
- 8. Intercept (non-circular orbits)

IV. OUTPUT DATA REPRESENTATIONS

A. TABULATED

- 1. Cartesian Coords. (Rot. Fr.)
- 2. Cartesian Coords. (Inert. Fr.)
- 3. Shell Coords.

B. GRAPHICAL

- 1. Cartesian Coords. (Rot. Fr.)
- 2. Cartesian Coords. (Inert. Fr.)
- 3. Shell Coords.

OPTION 1		OPTION 2		OPTION 3		OPTION 4		OPTION 5	
		(2)	(2)			X	X		
X	X								
								X	X
X	X			X	X	X	X	X	X
X	X			X	X	X	X	X	X
		X							X
X	X			X	X	X	X	X	X
X	X			X	X	X	X	X	X
		X							X

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(2) These solutions use linear theory thrust required as an input.

III.5 The ITEM Program. One requirement stipulated for the computer programs associated with this investigation was that there be a capability to predict relative motions in excess of one hundred orbits. This prediction was to be carried out in the presence of the basic perturbing forces which were assumed to be acting on the relative moving body. It was decided, finally, that the most expedient manner by which this requirement could be satisfied was to modify one version of the ITEM program for relative motion studies. These modifications were introduced into that version of ITEM presently existing on the Langley Research Center's machine; they are noted below.

For those readers not familiar with this collection of computational algorithms, a brief discussion is given in the next few paragraphs. The ITEM\* (Interplanetary Trajectory Encke Method) program is one of general purpose designed to provide trajectory predictions having maximum accuracy without incurring prohibitive penalties in machine time. The basic computational scheme is based on the Encke Method; however this has been modified to rid it of some objectionable features found in the classical format. The program, per se, has gravitational effects from all planets, the Sun and moon, as well as a set of zonal and tesseral harmonics for earth's gravity model. It also has the perturbative influences from atmospheric drag and solar radiation pressure (including shadow effects).

Since these are the more usual disturbing perturbations to be expected for near earth orbiters, then the primary problem which was undertaken here was that of modifying the program so that relative motions could be acquired.

Suffice it to say that the needed modifications have been carried out; and as a consequence the program output now provides the necessary relative motion state. In addition to the (usual) inertial coordinates, an output block has been added so that relative motions referred to the local rotating frame of reference also appear. In order to provide for a needed degree of compatibility with the RELMOT program there has been a "thrusting" computation added

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\*See Ref. [8] for a complete description of this program.

into ITEM. The present system accepts an inertially described specific "thrust"; this particular influence has been added to the normal input for ITEM, it is called by means of a parameter loading in ID=24 (input block 24). There the "thrust vector" is specified by a right ascension and declination (for direction), a magnitude value (number) and a magnitude for the gravitational acceleration. The two vehicles (passive parent and active interceptor) are supplied with an initial state, thus each can fly its own trajectory in a near earth region of space. (Each vehicle is identified, in the input statements, by means of a number assigned to the flag NC(42). A nine (9) in this position designates the parent; an eight (8) denotes the active particle).

For the output data blocks both particles have their inertial state (referred to the geocenter) printed as well as the osculating orbit elements at each indicated print time. Also the relative motion state (from both frames of reference) as well as the range and range-rate information are tabulated in block format (see Ref. [7] for a more complete description).

The addition of ITEM to the computational package greatly extends the capabilities for this study effort. Specifically, ITEM presents a more explicit modelling of the perturbative effects; it meets the requirement of "extended time" predictions; and, it is more precise in definition of the two particle orbits. The output from ITEM can be called for a number of modes: there are selected planet, or heliocentric, reference centers; two systems of units, and other options of interest to the investigator. For a more complete discussion of the options available under the general ITEM operations, the reader should consult Ref. [8].

III.6 Concluding Remarks. The two computer programs developed and adopted for use here have been described, rather briefly, in the foregoing paragraphs. It should be evident that RELMOT is a special purpose computational system developed in direct response to the analysis and problems of interest to this

study. The adaptation of ITEM, noted above, adds a significant dimension to the predictions provided in studying a relative motion. However, it does not take one long to recognize that there are other areas of interest and concern which have not been included. In addition, when one peruses the sections on analysis which have accompanied the numerical efforts, it is again apparent that there are situations, extensions and topics which would be of interest (theoretically and operationally).

From an analysis point of view one can tell that there are situations and problem cases which have not been developed, but which should be, to complete the "causes and effects" studies which have already been initiated. A study of TABLE V will indicate most of those areas inferred by this remark. In addition, an analysis of step type forcing functions, and some of the simpler variations for forces, would shed light on the "causes and effects" aspects of the investigation. Also, there are other areas to be examined, under the present results, in order for a more complete understanding of the initial-values problems and the disturbances type problems to be acquired. Many of these areas will have an impact on several of the operation modes which will arise in connection with presently planned space flights. The Shuttle and its many payloads; the astronauts who will work around the Shuttle and its experiments; and even some studies to be conducted inside the Shuttle bay have a direct relationship to relative motions. It would be a most worthwhile project to continue this work, but to do so in conjunction with that planning which will go into the next generation of earth-orbiting vehicles. By extending these efforts one will soon find that there is a "control" problem (or problems) to be studied and solved in connection with these next flight operations. Stationkeeping, position predictions, methods for retrieval (and/or rescue) and many other problem areas are candidates for study under the general heading of "relative motion".

In the past only a few study efforts have taken advantage of motion traces which have evolved from a relative motion - these are most useful in determining "how" and "when" corrective actions should be undertaken. Or, in the event they

do occur, unavoidably, what are the consequences? And, last of all, there does not seem to be any evidence that past investigators have made any use at all of the relative motion hodographs. Here lies the key to "what" needs to be done, and "when"; this information is as important as the motion traces themselves when one begins to examine the ramifications of a relative motion situation. All things considered, the general analysis which has been undertaken here provides an excellent framework for further study as well as the means to provide answers for a variety of operational problems.

It would be remiss on our part to let this opportunity pass and not mention that here, in this study, one has a first time knowledge of relative motion traces as these exist (and are related) in the two primary frames of reference. The compendium of results, Ref. [1], depicts this; the accompanying written material is indicative of how these various trace geometries are developed (at least in a linear sense). Not only are these figures studied here from a "mathematical" point of view, but they are evolved from a physical standpoint as well. Certainly, nowhere else in the open literature does it appear that one can locate this complete collection of information; and nowhere else can one find it developed from a single unified and simple mathematical approach.



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