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RELATIVE MOTION OF ORBITING PARTICLES UNDER THE INFLUENCE OF PERTURBING FORCES

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J. B. Eades, Jr.

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ANALYTICAL MECHANICS ASSOCIATES, INC.
10210 GREENBELT ROAD
SEABROOK, MARYLAND 20801

FOREWORD

This report was prepared under NASA Contract NAS1-11990. The work was conducted under the direction of J. W. Drewry, Space Application and Technology Division, Langley Research Center.

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RELATIVE MOTION OF ORBITING PARTICLES UNDER THE INFLUENCE OF PERTURBING FORCES

By J.B. Eades, Jr. *

SUMMARY

This report describes the mathematical developments carried out for this investigation. In addition to describing and discussing the solutions which were acquired, there are compendia of data presented herein which summarize the equations and describe them as representative trace geometries.

In this analysis the relative motion problems have been referred to two particular frames of reference; one which is inertially aligned, and one which is (local) horizon oriented. In addition to obtaining the classical initial values solutions, there are results which describe cases having applied specific forces serving as forcing functions. These forces are designated as having components parallel to the triads of the reference frames; hence the analytical results obtained are for motions referred to both frames and influenced by both forcing function systems. Also, in order to provide a complete state representation the speed components, as well as the displacements, have been described. These coordinates are traced on representative planes analogous to the displacement geometries. By this procedure a complete description of a relative motion can be developed; and, as a consequence range-rate as well as range information is obtained.

*Senior Analyst and Manager, Seabrook Office, Analytical Mechanics Associates, Inc.

SYMBOLS

a, b, c	Constants (see Appendix A)
A_a, A_s	Constants (see Eqs. II.25, II.26).
A_1, A_2	Constants (see Eqs. II.43).
$\mathcal{A}_a, \mathcal{A}_s$	Constants (see Eqs. II.83)
B, B_2, J_i	Special matrices (see Appendix B); $i = 1, 2, 3$.
$\bar{e}_r$	Unit radius vector.
$\bar{e}_x, \bar{e}_y, \bar{e}_z$	Unit vectors associated with the local horizon (rotating) frame of reference.
$\bar{e}_X, \bar{e}_Y, \bar{e}_Z$	Unit vectors associated with the inertially (fixed) frame of reference.
$\bar{F}$; F	General designation for force term.
G	Universal gravitational constant.
G_1, G_2	Constant terms (see definitions following Eq. II.71b).
I_j	Idem matrix ($j = 2, 3$: denoting rank).
K_o	Constant (see expressions following Eq. II.28).
K_c, K_s, K'_s	Constants (see Eqs. II.24, II.25).
K_1, K_2	Constants (see Eqs. II.45).
m	Orbiting mass.
M	Attracting mass.
O	Originating point (position).
P, Q	Designation for orbiting particles of interest.
$\bar{P}, \bar{Q}$	Position vectors, measured from M to particles (P, Q).
$\bar{r}$	Relative position vector, referred to the local horizon frame.

$\bar{R}$	Relative position vector, referred to the inertial frame.
$\bar{r}_r$	Relative position vector (locating "Q" from "P").
$\bar{\lambda}\,;\bar{\lambda}'$	Dimensionless relative position vector, referred to the local horizon frame of reference; dimensionless relative speed vector.
$\bar{\mathcal{R}};\bar{\mathcal{R}}'$	Dimensionless relative position vector, referred to the inertial (fixed) frame of reference; dimensionless relative speed vector.
s	Laplace transform variable.
t	Time.
$T(\theta^{\pm})_i$; $T(\varphi^{\pm})_i$	Transform matrices (see Appendix A); $i = 2, 3$.
$T(+)_i$; $T(-)_i$	Transform matrices (see Appendix A); $i = 2, 3$.
x, y, z	Cartesian coordinates, in the local horizon (rotating) frame of reference. (Origin at P).
X, Y, Z	Cartesian coordinates, in the inertially (fixed) frame of reference. (Origin at P).
Δ, δ	Denotes "increment of".
ξ, η, ζ; ξ', η', ζ'	Dimensionless coordinates (displacement; speed) in the local horizon frame.
Ξ, H, Z; Ξ', H', Z'	Dimensionless coordinates (displacement; speed) in the inertial frame.
μ	Gravitational parameter.
$\bar{\tau};(\bar{\tau}_I)$	Dimensionless specific force vector referred to the local horizon frame; (inertial frame).
φ	Angle of transfer, measured along an orbit arc from the $t = 0$ position.
$\dot{\varphi}$	Angular rate, referred to the base (circular) orbit.

Φ_τ, Φ'_τ	Constants (see Eqs. II.86, II.90).
Ψ_τ, Ψ''_τ	Constants (see expressions following Eq. II.28).

Subscripts

$(\ )_o$	Initial (reference) value.
$(\ )_I$	An inertially defined quantity.
$(\ )_{\xi,\eta,\zeta}$	Refers to component direction (dimensionless, local horizon frame).
$(\ )_{\Xi,H,Z}$	Refers to component direction (dimensionless, inertial frame).
$[\ ]_{trig}$	Refers to terms with trigonometric coefficients.
$[\ ]_{i.v.}$	Refers to initial-values problem.

Superscripts

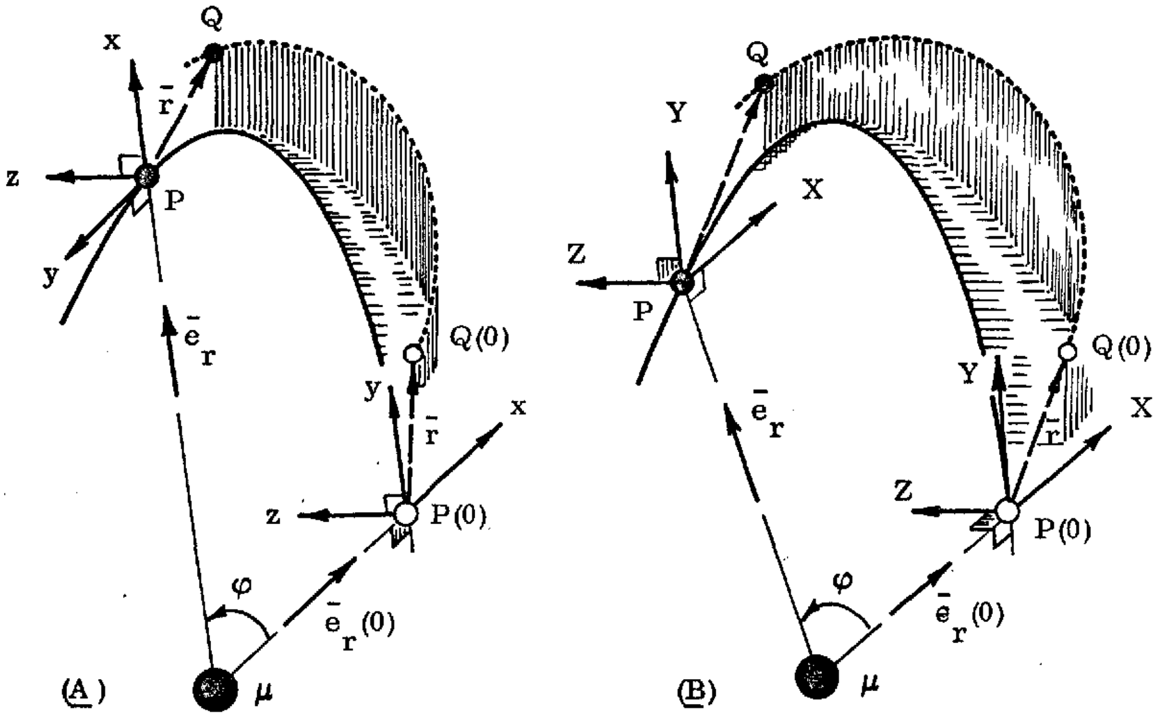
$(\ )^o$	Infers the (inertial) frame of reference.
$(\ )'$	Infers the (rotating) frame of reference; denotes differentiation with respect to φ.
$(\hat{\ })$	Unit vector quantity.
$(\bar{\ })$	Vector quantity.
$(\dot{\ }), (\ddot{\ })$	Time derivatives.
$(\ )^{-1}$	Inverse of ( ).
$(\ )^T$	Matrix transpose of ( ).

I. INTRODUCTION

Two bodies in motion, and in close proximity to one another during a space flight follow trajectories which are relatively independent of one another. Contrary to this, when physical connectors are in place, or "attractive forces" of sizeable magnitude are present, then these trajectories may be coupled so that the ensueing motions are dependent on one another.

The investigation reported here has considered several aspects of the relative motion problem. In particular, the influence of initial state (position and velocity), and the consequence of selected, applied force systems have been examined. The applied forces assumed in this study were designed to have specific orientations; the consequence and purpose for this will become evident as the results are brought forth and discussed. As a prelude to what follows herein, it is noted that these individual forces,which are presumed to be acting on one of the bodies, have components of fixed magnitude parallel to the reference triads. By this mechanism it is demonstrated that analytical solutions may be obtained for a "properly described" equation of the motion. Subsequently the relative motion descriptions which evolve will exhibit an influence from these particular forces. Also, in relation to this, the reader may recognize that a variety of simulated physical situations could be described in terms of these same actions. In this regard it is apparent that the results from this investigation represent a significant addendum to the present collection of information for the relative motion of orbiting particles.

As an aid to clarifying the physics of the present situation the sketches shown below indicate the two particular frames of reference to be utilized. In both drawings the motion commences with the reference triads parallel to one another. However, as time progresses the base particle (P) moves to a new location, as does the relatively moving particle (Q). Throughout this time the two masses may remain separated, as suggested by the relative displacement vector ($\bar{r}$). However, the main difference, as noted on these sketches, is that in one instance the displacement is referred to a moving frame of reference (x, y, z) - see Sketch (A) - while in the second (Sketch (B)) it is referred to an inertially aligned triad.



Incidentally, it is in reference to these triads that the applied forces are assumed to have their fixed magnitude components. And, it is these forces which are studied to learn how the relative state of motion is affected. In substance, then, it is apparent that the position and velocity, at any time, referred to either triad, is a consequence of three input quantities. One, the initial state of relative motion (this case is hereafter called the "initial values problem"); and secondly, the two separate applied force systems (this will, hereafter, be referred to as the "zero initial values problem").

In the discussions and descriptions which follow, here, there will emerge some definite patterns regarding the relative motion state. Most noticeable are the reactions produced by the applied force systems and by the initial state. In particular the consequences of these inputs on motions referred to the two triads of reference, are primarily seen in the in-plane displacements and velocities. As a general statement these in-plane relative motion components (displacements and speeds), when referred to a rotating frame of reference, trace out figures which are counter to the orbital motion. On the other hand, corresponding situations when referred to the inertially oriented frame of reference, trace out figures which are in the direction of the orbital motion. Secondly, the initial values problems are represented by different geometries in these two frames of reference. In particular, for general inputs the geometries which result show a divergence - as the time in motion progresses. However, the traces representing these various cases are not alike when described in terms of the various coordinates. The figures in a rotating frame are generally cycloidal and/or parabolic in character. On the other hand, the motions expressed in inertial (relative motion) coordinates are nominally spiral-like. This would seem to imply that the relative state of motion is not bounded, except as it is related to the central attracting mass. Fortunately this is not the case; there are means available which will provide a "closed" relative motion path. The conditions by which this may be brought about are described in the report under "non-secular" cases.

As a word description, illustrative of this last condition, consider the following situation. Suppose a sub-satellite is to be "launched" from an orbiting vehicle; and, also, suppose it is desired that this sub-satellite remain in the immediate vicinity of its parent. It can be shown that when the parent body is in a circular orbit, one means of attaining the desired relative motion is to impulsively launch the sub-satellite orthogonal to the parent's velocity vector. Of course, the geometric shape of the subsequent traces will appear differently on the planes for the two reference frames. For example, the in-plane trace (referred to the rotating frame of reference) would appear as an ellipse*. On the inertially oriented plane of reference the corresponding figure would trace as a limacon*. (Necessarily displacement traces alone do not fully describe the relative state of motion -- one needs the velocity diagram, or hodograph, as well. These are readily obtained and easily plotted; once the full state of motion is described the investigator has a complete graphical representation of the problem's actions).

One may wonder about the reasons behind the two reference frames. This is easily reconciled when it is recognized that some spacecraft may be Earth-oriented while others could be star-oriented; thus the two coordinate systems. In addition there is, occasionally, some question regarding the need for both the displacement and the hodograph traces of a given motion. Generally this is most easily explained by noting that the state of motion (absolute or relative) is comprised from both coordinates. Also, there may be a distinct need for range and range-rate information, especially in conjunction with maneuvering operations. Needless to say this information can be readily obtained from these relative motion coordinates; and, in addition, the graphical display of both can enhance the understanding of how contemplated flight operations might progress.

A few words regarding the out-of-plane motions would be worthwhile at this point. As a general observation it can be verified that this aspect of the relative motion, be it numerically defined, analytically described or graphically represented, is more complicated and less easily defined than the in-plane case(s). From an analytical point of view the main difficulty comes about from the over-all dependence of the representative equations on the full initial state and/or all the force vector components appearing in the input. As is demonstrated in the

*These situations are described in the body of the report.

report, it is indeed difficult to estimate (say) the geometry of these motion traces as an a priori piece of information. With some study, and a few example situations, it is reasonable to expect that the reader could acquire some "feel" for these problems; and he could develop a sense of prognostication with regard to the motion traces, but only in a general manner.

For those readers who are not too well acquainted with the ideas and applications of a relative motion study it would be worth the time and space needed to mention a few examples. Probably the first example to come to mind would be that of intercept and rendezvous between two orbiting vehicles. Whether the situation should consider a one active-one passive pair, or two active satellites, is of little consequence. When the two vehicles are cooperating, the rendezvous problem is straight forward and easily implemented (mathematically). In the event that one body is "uncooperative" (an "evade" situation) the rendezvous is more difficult to achieve, but remains a problem in relative motion - one which fits the overall concepts of this investigation. With the option, here, of adding "thrust" to the relative moving vehicle it is evident that (say) the rendezvous can be accomplished under this action rather than by impulsive means.

The use of sub-satellites, experiment packages, and other orbiting particles come directly under the relative motion problem concept when one seeks to track these bodies and/or predict where they might be at some subsequent time.

It should be remembered that in general the studies which could come under this classification need not be considered simply as "external" motions. That is, the equations and procedures developed for this investigation can be utilized to study the motion of particles <u>inside</u> a spacecraft as well as outside of the vehicle. By properly interpreting the physics of many problem situations, and adapting these results to the conditions at hand, the ideas set down here have a wealth of applications possibilities. One somewhat trivial but interesting example which comes to mind is that of a particle free to move about inside a spacecraft. Recognizing that the most significant force which would be acting (there) is that due to gravity gradient, then it is evident that the particle cannot remain motionless -- in a relative sense -- unless it is located at the dynamical center for the spacecraft-particle system. Actually the particle is most likely to be, and remain, in motion as a consequence of the gradient force. Necessarily, the closer the particle is to this center the smaller the acceleration acting on the body. It can be shown that the trajectories for such a particle, referred to a moving frame of reference, would be regular hyperbolae. The particle would tend to move toward* the dynamical center and in the gradient direction simultaneously.

Before leaving this section and moving to the mathematical developments, it would be well to mention some pertinent facts regarding the following sections. First, in the writing of many formulae in this report the matrix notations which appear may be somewhat unusual to the reader. In this regard it is suggested that those who may be interested in the details of developments, leading to and represented in these various expressions, should consult the Appendices as a first reading assignment. These will introduce the reader to the notation and make the reading of the mathematics much easier and more understandable. Also, the developments herein make ample use of matrix notation; and, the solutions, per se, are carried out by the method of Laplace transforms. Those who are unfamiliar with either or both of these notational operations may find some difficulty in following the mathematics.

Finally, in conjunction with discussions of the various types of motions, and those inputs which influence them, the reader will find compendia of results sections. These data presented collections of results which are typical of the various motion classifications, as described. The purpose in presenting

*This suggests an idealized situation wherein the spacecraft flies a circular orbit.

II. KINEMATICS AND PROBLEM DEFINITION

Kinematics (An Inertial Frame of Reference). In this section certain kinematical relations are developed to describe a relative motion situation as it would be viewed in an "inertially oriented" frame of reference.

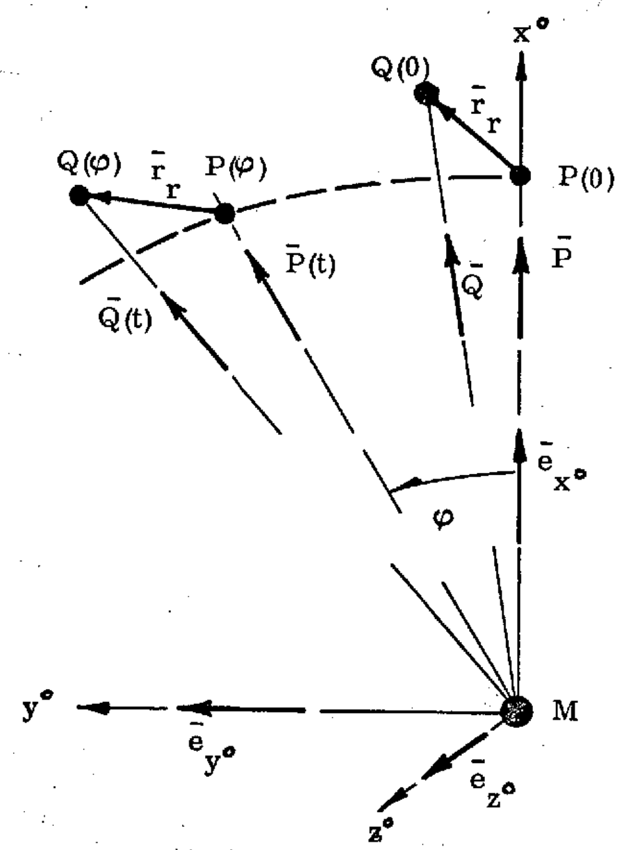


Fig. II.1 - Geometry of a relative motion. The vectors $\bar{P}$, $\bar{Q}$ locate the two particles (of interest) with respect to M (origin). The relative position vector ($\bar{r}$) locates "Q" with respect to "P".

It is assumed that the $(\sim)^{o}$-frame represents the inertial reference system. On the other hand the $(\sim)'$-frame undergoes angular displacements (φ) for the regular, orbital motion (particle P). Let an adjacent particle (Q) have its motion defined relative to P. Then, in this discussion, all motions are described (kinematically) in the inertial frame.

Accordingly, particle "Q" moves relative to "P", as both move about M (the attracting center). Here $\bar{r}_r$ locates "Q" with respect to "P", thus its position vectors is:

$$\bar{Q} = \bar{P} + \bar{r}_r; \qquad \text{(II.1)*}$$

which can be described in either frame of reference.

Assuming that $\bar{r}_r$ is measured in the local inertial frame, and since "P" (origin) moves along its own orbit, then the location of "Q" in the inertial frame is given by,

*$\bar{r}_r$ is a general designation for the relative position vector.

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$$\bar{Q}_I = \bar{R} + T(\varphi^-)\,\bar{P}. \tag{II.2}$$

(Herein $\bar{R}$ is used to denote the inertially defined relative position vector; $\bar{r}$ will be reserved as the description for $\bar{r}_r$ in a rotating frame of reference, one attached to "P". Also, $(\sim)_I$ will be attached to $\bar{P}$ and $\bar{Q}$ to denote their inertially defined values. The transformation matrix, $T(\varphi^-)$, transfers the vector $\bar{P}(\varphi)$, back to the inertial frame, $(\sim)^o$; see Appendix A for a discussion and development of the transforms).

Kinematic Definition of Velocity. From Eq. (II.2) above, it follows that the velocity of Q, defined in inertial space, is (generally) given as:

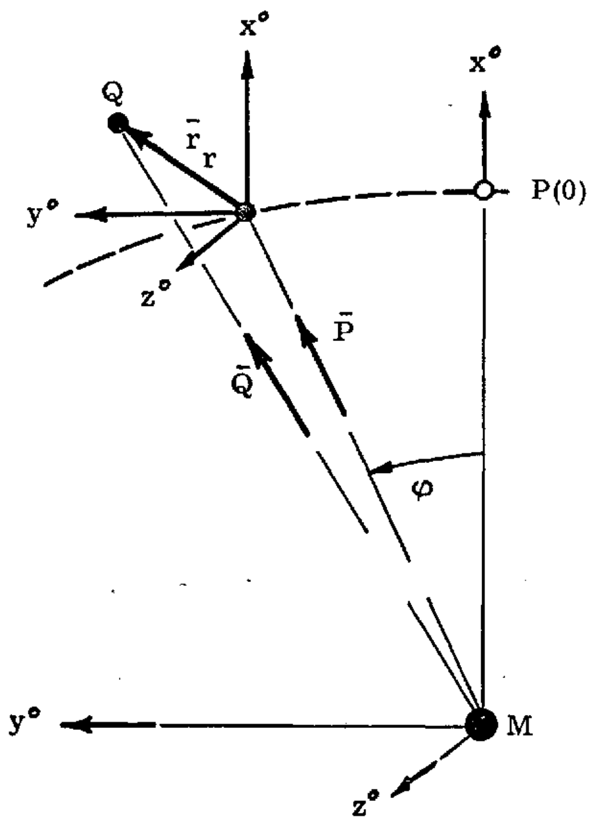

Fig. II.2. This figure is a sketch showing the inertial frame, attached to P, used to locate "Q" with respect to "P".

$$\dot{\bar{Q}}_I = \dot{\bar{R}} + (T_-)\,\dot{\bar{P}} + (\dot{T}_-)\,\bar{P}. \tag{II.3a)*}$$

But, since "P" is assumed to move on a circular path, and to have only planar rotation ($|\bar{P}|$ is not changing), then it follows that,

$$\dot{\bar{Q}}_I = \dot{\bar{R}} + (\dot{T}_-)\,\bar{P}, \tag{II.3b}$$

$$\text{where } (\dot{T}_-) = \dot{\varphi}\left[\begin{array}{cc|c} -\sin\varphi & -\cos\varphi & 0 \\ +\cos\varphi & -\sin\varphi & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\equiv \dot{\varphi}\left[\begin{array}{c|c} B(\dot{T}_-)_2 & 0 \\ \hline 0 & 0 \end{array}\right] \quad \text{(see Eq. (A.6a), Appendix A)}$$

Here $(\dot{T}_-)_2$ is a "reduced matrix", as defined, in the Appendix.

*(If $|\bar{P}|$ would be allowed to change, then this full expression would be used. Also; here, and elsewhere, the notation (T_-) means $T(\varphi^-)$; a(T_+) will be used (subsequently) to denote the transform, $T(\varphi^+)$).

Making use of this description for $\dot{T}$, then a more convenient form of Eq. (II.3b) can be had; substituting Eq. (A.6d) leads directly to,

▶ $$\dot{\bar{Q}}_I = \dot{\bar{R}} + \dot{\varphi}\,(BT_-)\,\bar{P}, \qquad \text{(II.3c)*}$$

wherein B is the constant matrix defined in Eq. (A.6c), Appendix A.

This last expression shows that the (kinematic) velocity of "Q" is composed of the relative (but inertially described) velocity ($\mathring{\bar{R}}$), plus the velocity of "P" transferred <u>back</u> to the inertial reference system.

<u>Kinematic Acceleration.</u> The "acceleration" for "Q" is readily evaluated from Eq. (II.3c), by differentiation. That is, as a general statement,

$$\ddot{\bar{Q}}_I = \ddot{\bar{R}} + \cancelto{0}{\ddot{\varphi}}\,(BT_-)\,\bar{P} + \dot{\varphi}\,[\cancelto{0}{\dot{B}}T_-)\,\bar{P} + \dot{\varphi}\,(BT_-)\cancelto{0}{\dot{\bar{P}}} + \dot{\varphi}\,[B\dot{T}_-]\,\bar{P};$$

here the deleted quantities are a consequence of the invariance of terms ($\dot{\varphi}$, $|\bar{P}|$, and B). Consequently, the applicable kinematic description is,

$$\ddot{\bar{Q}}_I = \ddot{\bar{R}} + \dot{\varphi}\,[B\dot{T}_-]\,\bar{P},$$

or, recalling that** $BB = -I_2$,

$$\ddot{\bar{Q}}_I = \ddot{\bar{R}} - \dot{\varphi}^2\,(I_2T_-)\,\bar{P}. \qquad \text{(II.4a)}$$

One should note that the (kinematic) acceleration for "Q" is composed of the relative value ($\ddot{\bar{R}}$) - for the displacement of "Q" with respect to "P" - plus a centripetal term, proportional to $\dot{\varphi}^2$.

A reorientation of Eq. (II.4a) shows the relative acceleration ($\ddot{\bar{R}}$), expressed in the inertial frame of reference, written as:

$$\ddot{\bar{R}} = \ddot{\bar{Q}}_I + \dot{\varphi}^2\,(I_2T_-)\,\bar{P}. \qquad \text{(II.4b)}$$

---

*The symbol "▶" denotes equations of importance.

**See Appendix A.

Here, the quantity ($\ddot{\bar{Q}}_I$) is symbolic of the "specific accelerations" arising from the forces acting on the particle (Q); while the second term is a centrifugal force component associated with the position of "P"!

Concluding Remarks. The expressions developed above describe a means for representing a particle's relative motion, as this is expressed in an inertial frame of reference. However, this case is restrictive in that it pertains to a regularly circulating base particle (P). Of course, the analytical operations introduced above are fundamentally geometric (or kinematic) in kind. In order to apply these relations to a particular case it would be necessary to introduce the "dynamical" (Newton mechanics) aspects of the situation - - that is, to describe the "forces" which are acting on the test particle (Q).

Kinematics, for a Rotating Frame of Reference. In the next developments, to follow, "motions" will be referred to a moving frame of reference - - one which has a leg parallel to the radius vector locating the base particle (P). In this rotating-frame vectors $\bar{P}$ and $\bar{r}$ will be defined relative to this moving triad; hence, in order to refer these vectors back to the inertial axis system, it is required that the transform $T(\varphi^-)$ be applied to both. (See the sketch below for a geometric representation of this case).

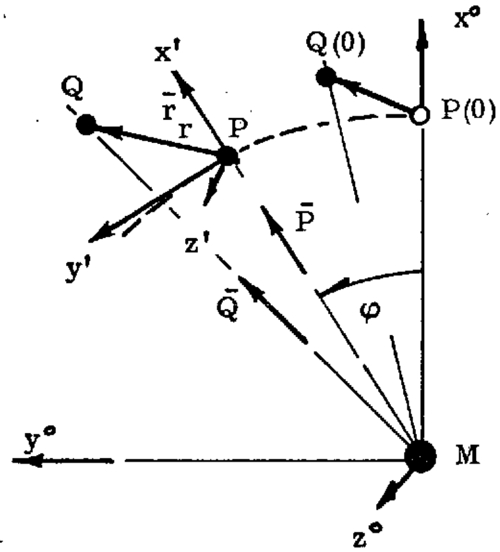

Fig. II.3. Sketch depicting the moving frame of reference.

Here $\bar{r}_r \equiv \bar{r}\,(x', y', z')$ and $\bar{P} \equiv |\bar{P}|\,\bar{e}_{x'}$; hence, the position vector

$\bar{Q}$ ($\equiv \bar{r}_r + \bar{P}$, in general)

is written as,

$$\bar{Q}_I = T(\varphi^-)\,\bar{r} + T(\varphi^-)\,\bar{P},$$

or

$$\bar{Q}_I = T(\varphi^-)(\bar{r} + \bar{P}). \qquad \text{(II.5)*}$$

*Here $\bar{r}$ describes the relative position vector; the lower case letter is used to distinguish from the previous development.

As before, the transformation matrix, (T_{-}), is applied to rotate the vectors $\bar{r}$, $\bar{P}$ back to the $(\bar{e}_i)^o$ triad.

It must be remembered that these vectors are expressed in the rotating frame; consequently, they relate to the moving triad.

Kinematic Velocity Definition. Differentiating Eq. (II.5) to obtain the velocity of "Q" - as it would be referred to the inertial frame - one obtains the general expression:

$$\dot{\bar{Q}}_I = (\dot{T}_{-})[\bar{r} + \bar{P}] + (T_{-})[\dot{\bar{r}} + \overset{0}{\cancel{\dot{\bar{P}}}}], \tag{II.6a}$$

wherein $(\dot{\bar{P}})$ vanishes due to the assumed invariance of $|\bar{P}|$. In addition, since $(\dot{T}_{-}) \equiv \dot{\varphi}\, B(T_{-})$ (see Eq. (A.6a), Appendix A) the velocity is most conveniently expressed as:

$$\dot{\bar{Q}} = (T_{-})[\dot{\bar{r}}] + \dot{\varphi} \cdot (BT_{-})[\bar{r} + \bar{P}] ; \tag{II.6b}$$

recall that (T_{-}) means $T(\varphi^{-})$. This is the transformation used to rotate vectors back to the $(\sim)^o$-frame of reference.

An examination of Eq. (II.6b) indicates that the "local" velocity $(\dot{\bar{r}})$, and the associated rotational component(s), are added vectorally (but transformed also!). It is worth noting that the rotational component is proportional to the orbit angular rate $(\dot{\varphi})$, and to the vector position of "Q", which is $(\bar{r} + \bar{P})$. Necessarily, to transform this part of the velocity back to the inertial frame the operator (BT_{-}), which is a geometric entity, is utilized.

In the next paragraphs the kinematic acceleration is defined.

Kinematic Acceleration. Carrying the differentiation one step further leads directly to the acceleration definition. From this operation the quantity is obtained as:

$$\ddot{\bar{Q}} = (T_-)[\ddot{\bar{r}}] + (\dot{T}_-)[\dot{\bar{r}}] + \overset{0}{\cancel{\ddot{\varphi}}}(BT_-)[\bar{r} + \bar{P}] + \dot{\varphi}\,(\overset{0}{\cancel{\dot{B}}}T_- + B\dot{T}_-)[\bar{r} + \bar{P}]$$

$$+ \dot{\varphi}\,(BT_-)[\dot{\bar{r}} + \underset{0}{\cancel{\dot{\bar{P}}}}],$$

wherein the vanishing components are due to those invariants previously mentioned.

Making substitutions, and collecting terms, one finds that:

$$\ddot{\bar{Q}}_I = (T_-)\,\ddot{\bar{r}} + 2\dot{\varphi}\,(BT_-)\,\dot{\bar{r}} - \dot{\varphi}^2\,(I_2 T_-)\,[\bar{r} + \bar{P}]. \qquad \text{(II. 7)}$$

In this expression the three components (shown) are the familiar quantities:

(a) $(\ddot{\bar{r}})$, the local acceleration,

(b) $(2\dot{\varphi}\,\dot{\bar{r}})$, the Coriolis acceleration,

and

(c) $(\dot{\varphi}^2\,[\bar{r} + \bar{P}])$, the centrifugal term. (II. 8)

Equation (II.7) defines the kinematic acceleration; this is a geometric description; however, in solving the <u>relative</u> <u>motion</u> problem it will be necessary to rearrange the expression in order to acquire an equation for $\ddot{\bar{r}}$. That is, (from Eq. (II.7)), one writes:

$$\ddot{\bar{r}} = (T_-^{-1})\,\ddot{\bar{Q}}_I - 2\dot{\varphi}\,(T_-^{-1}\,BT_-)\,\dot{\bar{r}} + \dot{\varphi}^2\,(T_-^{-1}\,I_2\,T_-)\,[\bar{r} + \bar{P}], \qquad \text{(II. 9a)}$$

wherein

$$(T_-)^{-1} = T(\varphi^-)^T = T(\varphi^+); \qquad \text{(see Eq. (A.5), Appendix A).}$$

Evaluating the matrix products in Eq. (II.9a), it is seen that $(T_+BT_-) = B_2$ and $(T_+I_2T_-) = I_2$; consequently,

$$\ddot{\bar{r}} = \{(T_+)\,\ddot{\bar{Q}}_I\} - 2\dot{\varphi}\,(B_2)\,\dot{\bar{r}} + \dot{\varphi}^2\,(I_2)\,[\bar{r} + \bar{P}]. \qquad \text{(II. 9b)}$$

In this last expression all quantities, except $\ddot{\bar{Q}}_I$, are referred to the local rotating triad. The parameter $(\ddot{\bar{Q}}_I)$ is a symbolic representation for the specific accelerations due to outside (or external) forces. E.g. for a central field attraction case $\ddot{\bar{Q}}_I$ would be replaced by the gravitational force, and/or the other forces which would influence the motion of "Q". These particulars will be discussed in more detail subsequently.

Motion of the Reference Particle (P). In order to examine the motion of "P", due to a central field of force, one should consider (first) the dynamical equation:

$$m_P \ddot{\bar{P}} = - \frac{(GM)\, m_P}{|\bar{P}|^3} \bar{P}, \tag{II.10a}$$

which assumes an inertial frame of reference.

Now, considering the kinematics of this motion, return to Eq. (II.9b); but, there, let $\bar{r} \Rightarrow \bar{0}$, thence "Q" → "P", and

$$\lim_{Q \to P} \left\{ \ddot{\bar{r}} = (T_+)\, \ddot{\bar{Q}}_I - 2\dot{\varphi}\, (B_2)\, \dot{\bar{r}} + \dot{\varphi}^2\, (I_2)\, [\bar{r} + \bar{P}] \right\} \Rightarrow$$

$$\left\{ (T_+)\, \ddot{\bar{P}}_I + \dot{\varphi}^2\, (I_2)\, \bar{P} \right\};$$

therefore:

$$(T_+)\, \ddot{\bar{P}} = - \dot{\varphi}^2\, (I_2)\, \bar{P}. \tag{II.10b}$$

(Note: here the quantity $(T_+)\, \ddot{\bar{P}}_I$ represents the transforming of $\ddot{\bar{P}}_I$ to the displaced position, and to the reference frame located by φ).

Now, using Eq. (II.10a) to describe $\ddot{\bar{P}}_I$ - - which applies solely to a simple field of attraction - - then Eq. (II.10b) leads to

$$\frac{GM}{|\bar{P}|^3} = \dot{\varphi}^2. \tag{II.10c)*$$

*It must be evident that $\dot{\varphi}$ represents a (constant) angular speed for the reference particle "P" as it moves on its circular orbit.

This last result is the well known expression for the angular rate ($\dot{\varphi}$) on a circular orbit ($|\bar{P}|$) due to the gravitational attraction (GM). This result states that the circular velocity, squared, is defined by the quotient GM/ $|\bar{P}|$.

<u>The Relatively Moving Particle (Q).</u> Having determined a motion for "P", the next situation to consider is that for "Q". Of course, this will quickly lead to a description of the <u>relative motion</u> as it develops under the influence of whatever force system(s) assumed. In this regard suppose (for the moment) that only a gravitational force is present; then,

$$m\ddot{\bar{Q}}_I = - \frac{(GM)m}{|\bar{Q}_I|^3} \bar{Q}_I , \tag{II.11a}$$

where m is understood to represent the mass of the particle located at "Q"; and where $\bar{Q}$ is its position vector (relative to M). Recalling that $\bar{Q}_I \equiv (T_-) [\bar{r} + \bar{P}]$, see Eq. (II.5), then it follows that

$$\ddot{\bar{Q}}_I = - \frac{(GM)}{|\bar{r} + \bar{P}|^3} (T_-) [\bar{r} + \bar{P}] . \qquad \text{(II.11b)*}$$

(This models the specific force, or acceleration, acquired from a mass attraction, M).

In attempting to evaluate this non-linear expression it is evident that an analytic solution cannot be had; however, a linearization (based on the smallness of $|\bar{r}|$ with respect to $|\bar{P}|$) will allow one to find an approximate analytical result. (It can be said that this formulation, acquired above, defines the "gravity gradient" effect for Q!)

An expansion for the denominator term, above, is outlined below:

Since

$$|\bar{r} + \bar{P}| \equiv \sqrt{(\bar{r} + \bar{P}) \cdot (\bar{r} + \bar{P})} ,$$

then

---

*The transformation matrix, (T_-), is not included in the denominator since it does not influence the magnitude of the vector (sum).

$$|\bar{r} + \bar{P}|^3 = |\bar{P}|^3 \left[1 + \left(\frac{|\bar{r}|}{|\bar{P}|}\right)^2 + 2\,\frac{\bar{r}\cdot\bar{P}}{\bar{P}^2}\right]^{3/2},$$

and

$$|\bar{r} + \bar{P}|^{-3} = |\bar{P}|^{-3}\left[1 - \frac{3}{2}\left(\frac{|\bar{r}|}{|\bar{P}|}\right)^2 - 3\,\frac{\bar{r}\cdot\bar{P}}{|\bar{P}|^2} + \ldots.\right]. \qquad \text{(II.11c)}$$

Now, making use of Eq. (II.11c), in Eq. (II.11b), it follows that:

$$\ddot{\bar{Q}}_I = -\frac{GM}{|\bar{P}|^3}(T_-)\left\{\left[(\bar{r}+\bar{P}) - 3\bar{P}\,\frac{\bar{r}\cdot\bar{P}}{|\bar{P}|^2}\right] \underbrace{- 3\bar{r}\,\frac{\bar{r}\cdot\bar{P}}{|\bar{P}|^2} - \frac{3}{2}(\bar{r}+\bar{P})\,\frac{r^2}{|\bar{P}|^2} + \ldots.}_{\text{H.O.T.}}\right\};$$

and, to first order in $(\bar{r})$, the approximate expression is:

$$\ddot{\bar{Q}}_I \cong -\frac{GM}{|\bar{P}|^3}(T_-)\left[(\bar{r}+\bar{P}) - 3\bar{P}\,\frac{\bar{r}\cdot\bar{P}}{|\bar{P}|^2}\right]. \qquad \text{(II.11d)}$$

(It should be noted that as $\bar{r} \to \bar{0}$ ($Q \Rightarrow P$), then

$$\ddot{\bar{Q}}_I \Rightarrow -\frac{GM}{|\bar{P}|^3}(T_-)\,\bar{P}.$$

This is equivalent to Eq. (II.10a). As before, this specifies a motion for "P" as it would be described by Newtonian mechanics. A consequence of this last result is that the two remaining terms in Eq. (II.11d) are "additions" to that influence attributed to the attraction of M).

Of the two remaining "force" components, in the equation above, the one quantity:

$$\Delta\ddot{\bar{Q}}_I \equiv -\frac{GM}{|\bar{P}|^3}(T_-)\,\bar{r} = -\dot{\varphi}^2\,(T_-)\,\bar{r}, \qquad \text{(II.11e)}$$

has a same <u>form</u> as before. Therefore it behaves as a "restoring force"; i.e., a force tending to return "Q" to the position of "P" during the motion.

However, the remaining quantity,

$$\Delta\ddot{\bar{Q}}_I \equiv +3\frac{GM}{|\bar{P}|^3}(T_-)\,\bar{P}\left[\frac{\bar{r}\cdot\bar{P}}{|\bar{P}|^2}\right] = 3\dot{\varphi}^2\,(T_-)\,\hat{P}\,[\bar{r}\cdot\hat{P}]\,, \qquad \text{(II. 11f)*}$$

has a different character (as compared to the others); this is a force which acts opposite to the "attraction" of gravity. Consequently, since it is radially directed ($\propto\bar{P}$), it behaves like a "reduction" in gravity and has the tendency to drive the particle away from the origin (at P).

These two partial results define the basic relative motion of "Q". It is apparent that one part of the motion is restorative while the other part tends to the opposite effect. This combination leads to an interesting physical interpertation of the motion, for Q, in the vicinity of P. Subsequently, a discussion of this will be made.

Equations of Motion. Having obtained a kinematic expression for the relative motion acceleration, Eq. (II.9b), and having developed a (linearized) description of those effects attributed to gravitational attraction, Eq. (II.11d) - then, a combination of effects* leads to a descriptive estimation of the relative motion. That is: following Eq. (II.9b),

$$\ddot{\bar{r}} = (T_+)\,\ddot{\bar{Q}}_I - 2\dot{\varphi}\,(B_2)\,\dot{\bar{r}} + \dot{\varphi}^2 I_2\,(\bar{r}+\bar{P}),$$

wherein, now,

$$(T_+)\,\ddot{\bar{Q}}_I \cong -\frac{GM}{|\bar{P}|^3}\,(T_+T_-)\left[(\bar{r}+\bar{P}) - 3\bar{P}\,\frac{\bar{r}\cdot\bar{P}}{|\bar{P}|^2}\right].$$

Taking into account that,

$$(T_+T_-) = I_3, \text{ and } GM/|\bar{P}|^3 = \dot{\varphi}^2 \text{ (see Eq. (II.10c))},$$

then $\ddot{\bar{r}}$ can be written as:

$$\ddot{\bar{r}} = \dot{\varphi}^2\,[(I_2 - I_3)(\bar{r}+\bar{P}) + 3\hat{P}\,(\bar{r}\cdot\hat{P})] - 2\dot{\varphi}\,B_2\,\dot{\bar{r}}\,. \qquad \text{(II. 12a)**}$$

*The description herein includes a force system aligned with the triad of reference.

** $\hat{P}$ is a unit vector corresponding to $\bar{P}$.

In this expression $\hat{P}$ is a unit vector pointing toward "P" from the center of attraction (M).

To be more precise with the format of Eq. (II.12a) it should be written as shown below. (Recall that in this equation $\bar{r} \equiv \bar{r}$ (x, y, z), where the scalar components are described in the moving frame of reference). As a consequence, Eq. (II.12a) is expressed as:

$$\ddot{\bar{r}} \equiv \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \cong \dot{\varphi}^2 \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} (\bar{r} + \bar{P}) + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{P}\,(\bar{r} \cdot \hat{P}) \right\}$$

$$-2\dot{\varphi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\bar{r}} ,$$

wherein $\bar{P} \equiv |\bar{P}|\ \bar{e}_{x'} = |\bar{P}|\ J_1$, and $(\bar{r} \cdot \hat{P}) = x$. Finally, then,

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \cong \dot{\varphi}^2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 2\dot{\varphi} \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} . \qquad \text{(II.12b)}$$

Eqs. (II.12) describe a (linearized) form of the equations for the (Q) particle's relative motion. This motion is with respect to an orbiting particle moving on a circular path about M. Of course, the equations are written in terms of "rotating" Cartesian coordinates; also they are approximations for the case wherein the test particle (Q) is acted on solely by the gravitational attraction of M.

In a "real life" situation the elementary mass (Q) would be acted on by forces other than gravity, also. In order to represent these additive effects, let any such specific accelerations be represented (symbolically) by a vector, $\bar{F}$. Of course, to be compatible with Eq. (II.12), it is necessary that $\bar{F}$ be described in the moving frame of reference. Consequently, a more complete description of Q's motion is given by the equation:

$$\ddot{\bar{r}} = \dot{\varphi}^2 \left[(I_2 - I_3)(\bar{r} + \bar{P}) + 3I_3 \hat{P} (\bar{r} \cdot \hat{P}) \right] - 2\dot{\varphi} B_2 \dot{\bar{r}} + \sum_j \bar{F}_j . \qquad \text{(II.12c)}$$

In Eq. (II.12c) the $\sum_j \bar{F}_j$ would account for all of the perturbative forces which could influence the relative motion of the test particle. For the present these forces are not described (other than symbolically); any particular descriptions will be left for the reader to describe, as needed.

Discussion on the Linearized Equations. Most likely the clearest descriptions of the motion for Q, as given in Eqs. (II.12) can be had by examining Eq. (II.12b), wherein the additive perturbations $(\sum \bar{F}_j)$ may or may not be included. However, when the perturbing quantities are included, they should be described in a manner such that they do not unduely complicate the solution procedure. Having gone to the trouble of "linearizing" the gravity gradient term, one should not model the perturbations in more than a linear format.

It might be expected that a general solution to these equations would evolve as an initial value problem; for such cases the results should be expressed in terms of elementary functions.

From Eq. (II.12b), it is seen that the in-plane (x, y) scalar differential equations are coupled through the Coriolis terms while the out-of-plane expression is independent of the other two. This suggests that the one scalar solution can be (conceptually) acquired independent of the others; however, the couple equations must be handled together and simultaneously. (One approach to a typical solution will be suggested subsequently). Of course, one of the simplest solutions to these expressions can be had by modeling the perturbations as constants.

In view of the separability of the in- and out-of-plane differential equations, it would be both practical and convenient to recast (say) Eq. (II.12c) as follows:

(a) For the in-plane components, write:

$$\ddot{\bar{r}}_{p\ell} \equiv \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \dot{\varphi}^2 \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \bar{r}_{p\ell} - 2\dot{\varphi}(B_2)\dot{\bar{r}}_{p\ell} + \Big(\sum_j \bar{F}_j\Big)_{p\ell} , \tag{II.12d}$$

wherein $\bar{r}_{p\ell} \equiv \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{\varphi}$ is the angular rate of the reference particle (P), and $(\sum \bar{F}_j)_{p\ell}$ would be composed of the perturbing force(s) in-plane components, only.

(b) The out-of-plane expression (a scalar) is given simply as:

$$\ddot{z} = -\dot{\varphi}^2 z + \Big(\sum_j F_j\Big)_z ; \tag{II.12e}$$

or, as an equivalent mathematical operation,

$$\bar{e}_{z'} \cdot \ddot{\bar{r}} = -\dot{\varphi}^2 (\bar{e}_{z'} \cdot \bar{r}) + \bar{e}_{z'} \cdot \Big(\sum_j \bar{F}_j\Big) .$$

Before outlining a solution procedure for the above expressions it is felt advisable to develop a set of corresponding, linearized differential equations, for the motion of Q, as it would be referred to the inertial frame of reference. Consequently, this is carried out in the following paragraphs.

The Relative Equations of Motion (for the Inertial Frame of Reference). Under the assumption that a relative position vector ($\bar{r}_r$) is known (and described) in the moving frame of reference, then this information can be "transferred" back to the inertial frame using the idea set down in Eq. (II.5); that is, by the application of:

$$\bar{R} = T(\varphi^{-})\,\bar{r}. \tag{II.13a)*}$$

On the other hand, having available the result set down in Eq. (II.12c) it is expedient to begin there and to develop the desired differential equations of motion, accordingly.

For present purposes rewrite Eq. (II.13) as

$$\bar{r} = T(\varphi^{-})^{-1}\,\bar{R};$$

or, as an equivalent expression,

$$\bar{r} = T(\varphi^{+})\,\bar{R}\,. \tag{II.13b}$$

From this last statement it is seen that the (kinematic) velocity is

$$\dot{\bar{r}} = (\dot{T}_{+})\,\bar{R} + (T_{+})\,\dot{\bar{R}}, \tag{II.14a}$$

where $(\dot{T}_{+})$ and (T_{+}) are given as Eqs. (A.6e) and (A.3a, or A.7c), respectively. Of course this can be simplified by introducing Eq. (A.6e); this leads to

$$\dot{\bar{r}} = -\dot{\varphi}\,(BT_{+})\,\bar{R} + (T_{+})\,\dot{\bar{R}}\,. \tag{II.14b}$$

Consequently, velocity, referred to the "moving frame", is expressed <u>in terms of</u> the inertially defined state vector(s).

Carrying the differentiation another step (forward), the kinematic description of acceleration is found to be:

$$\ddot{\bar{r}} = -\dot{\varphi}^{2}\,(I_{2}T_{+})\,\bar{R} - 2\dot{\varphi}\,(BT_{+})\,\dot{\bar{R}} + (T_{+})\,\ddot{\bar{R}}\,, \tag{II.14c}$$

after suitable substitutions have been made.

Now, when Eq. (II.14c) is solved for $\ddot{\bar{R}}$ (the inertial-frame's description of the acceleration), the result is:

*Recall that $\bar{R}$ is the representation for $\bar{r}_{r}$ in the inertial frame, while $\bar{r}$ is its designation in the "local" (rotating) frame.

$$\ddot{\bar{R}} = (T_+)^{-1} \ddot{\bar{r}} + \dot{\varphi}^2 \left[(T_+)^{-1}\right] \bar{R} + 2\dot{\varphi} \left[(T_+)^{-1} BT_+\right] \dot{\bar{R}} \, . \tag{II.14d}$$

This expression is "symbolic" of the motion; to account for the force systems which have (already) been assumed, it would be necessary to introduce Eq. (II.12c), for $\ddot{\bar{r}}$, into the result above. As an outline on how to proceed (from here), the following notes are included, with explanations:

(a) When evaluating $(T_+)^{-1} \ddot{\bar{r}}$ $(=T_- \ddot{\bar{r}})$, note that $(I_2-I_3)\bar{P}$ vanishes since $\bar{P} \propto \bar{e}_x$, and $(I_2-I_3)\bar{e}_x = \bar{0}$. Also, in the expression for $\ddot{\bar{r}}$ it is seen that both $\dot{\bar{r}}$ and $\bar{r}$ are present - these are obtained (for present purposes) from Eqs. (II.13b, and II.14b). Thus, with (T_-) replacing $(T_+)^{-1}$,

$$(T_-) \ddot{\bar{r}} = \dot{\varphi}^2 \left[(T_-)(I_2-I_3)(T_+\bar{R}) + 3(T_-)\, I_3\hat{P}\, (T_+\bar{R} \cdot \hat{P})\right]$$

$$-2\dot{\varphi}\, (T_-)\, B_2\left[-\dot{\varphi}\, (BT_+)\, \bar{R} + T_+\dot{\bar{R}}\right] + (T_-) \sum_j \bar{F}_j \, ,$$

$$= \dot{\varphi}^2 \left\{\left[(T_-)(I_2-I_3)(T_+)\right]\bar{R} + 3\left[(T_-)\, \hat{P}\, (T_+)\right]\bar{R} \cdot \hat{P}\right\}$$

$$+ 2\dot{\varphi}^2\, (T_-)\, B_2B\, (T_+)\bar{R} - 2\dot{\varphi}\, (T_-B_2T_+)\, \dot{\bar{R}} + (T_-)\sum_j \bar{F}_j .$$

Here, it can be shown that $(T_-B_2BT_+) = -(T_-I_2T_+) = -I_2$ and $(T_-B_2T_+) = B_2$; so

$$(T_-) \ddot{\bar{r}} = \dot{\varphi}^2 \left[(T_-)\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} (T_+)\, \bar{R}\right] - 2\dot{\varphi}^2 I_2 \bar{R} - 2\dot{\varphi} B_2 \dot{\bar{R}} + (T_-) \sum_j \bar{F}_j \, .$$

(b) Also, in Eq. (II.14d) there are transformation products which must be defined. In this regard it is noted that $(T_- I_2 T_+) = I_2$ and $(T_- BT_+) = B_2$, as before.

Introducing all of these results into Eq. (II.14d), it is found that:

$$\ddot{\bar{R}} = \dot{\varphi}^2 \left\{ (T_-) \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} (T_+) - 2I_2 \right\} \bar{R} \cancel{- 2\dot{\varphi} B_2 \dot{\bar{R}}} + (T_-) \sum_j \bar{F}_j + \dot{\varphi}^2 (I_2) \bar{R}$$

$$\cancel{+ 2\dot{\varphi} B_2 \dot{\bar{R}}} ;$$

or,

$$\ddot{\bar{R}} = \dot{\varphi}^2 \left\{ (T_-) \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} (T_+) - I_2 \right\} \bar{R} + (T_-) \sum_j \bar{F}_j \, . \tag{II.14e}$$

Before leaving these expressions, it should be noted that the matrix I_2 can be replaced by $(T_-)\, I_2\, (T_+)$; hence a contraction of Eq. (II.14e) follows, as:

▶ $$\ddot{\bar{R}} = \dot{\varphi}^2 \left\{ (T_-) \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} (T_+) \right\} \bar{R} + \left[(T_-) \sum_j \bar{F}_j \right] . \tag{II.14f}$$

For those cases where the displacements are known (implicitly or explicitly), then the linear solution for the motion is (in concept) capable of being determined. Of course here it should also be noted that the term "$(T_-) \sum \bar{F}_j$" is a symbolic representation of the perturbative force(s) as it would be <u>expressed in the inertial frame</u> of reference.*

<u>Summary</u>. To this point in these analytical developments, expressions have been obtained, with a linearization introduced, which (in theory) can be solved for the relative motion of "Q" with respect to "P". As described, these results are for both the <u>moving</u> frame of reference and the corresponding <u>inertial</u> frame. Both representations spring from the classical (Newtonian) equations governing the motion; and both contain symbolic descriptions for forces <u>other than</u> the gravitational attraction. It must be recognized that the appropriate representation for such forces <u>must</u> be made (in the described frame of reference) in order for the equations to be valid.

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*Any specific description of the forces, referred to this reference frame, would modify Eq. (II.14f) accordingly.

III. PROBLEM SOLUTIONS

In the next few paragraphs a method of solution, for an initial value problem, will be presented and discussed. For simplicity and conciseness an added simplification will be included, for these solutions, one which will reduce the numbers of manipulations which will need to be carried out.

A Solution for the Relative Motion. In the next several paragraphs a solution for the relative motion of "Q" with respect to "P" will be described. For present purposes this solution is acquired from Eqs. (II.12c); these equations describe the relative motion (vector $\bar{r}_r$) in reference to the moving frame of reference. Also, since the solution is to be obtained as an "initial value" problem the method of Laplace transforms is a natural and convenient mathematical tool to use.

Before entering into the mathematics of the solution it is worthwhile to study, somewhat critically, the governing differential equation, Eq. (II.12c). In this regard, looking at

$$\ddot{\bar{r}} = \dot{\varphi}^2 \left[(I_2 - I_3)(\bar{r} + \bar{P}) + 3I_3 \hat{P} \, (\bar{r} \cdot \hat{P}) \right] - 2\dot{\varphi} B_2 \dot{\bar{r}} + I_3 \sum_j \bar{F}_j \,, \qquad \text{(II.15a)}$$

It is evident that the variables ($\bar{r}$, $\bar{P}$, $\dot{\bar{r}}$ and $\bar{F}$) are all defined in the local (moving) frame of reference, $(\sim)'$. The remaining quantities (including $\dot{\varphi}$) are constants; generally, matrices.

Recalling that $\bar{P}$ (the position vector for "P") $\equiv |\bar{P}| \, \bar{e}_{x'}$, then it is apparent that:

(a) $(I_2 - I_3) \, \bar{P} = \bar{0}$;

(b) $I_3 \hat{P} = J_1$ (see Appendix B, for the definition of J_1);

and, (c) $\bar{r} \cdot \hat{P} = x$ (the radial relative motion coordinate).

Introducing other special matrices, described in Appendix B, and as required, it is easy to show that Eq. (II.15a) can be recast as:

$$\ddot{\bar{r}} = \dot{\varphi}^2 \, [3J_1 - J_3] \, \bar{r} - 2\dot{\varphi} \, (B_2) \, \dot{\bar{r}} + I_3 \sum_j \bar{F}_j \, . \qquad \text{(II.15b)}$$

(This is the general expression which will be manipulated for a solution, $\bar{r} = \bar{r}(t)$.)

The first operation to be undertaken in this solution is to transform Eq. (II.15b), by direct Laplace transforms*, wherein the only time-dependent variables are those noted above. Now, after transforming and collecting terms, the resulting expression is found to be:

$$\left[s^2 I_3 + 2\dot{\varphi} B_2 s - \dot{\varphi}^2 (3J_1 - J_3)\right] \bar{r}(s) = \left[sI_3 + 2\dot{\varphi} B_2\right] \bar{r}_o + I_3 \dot{\bar{r}}_o + \frac{I_3 \bar{F}}{s} \, , \qquad \text{(II.16a)}$$

wherein s is the transformation variable, and the vectors $\dot{\bar{r}}_o$, $\bar{r}_o$ represent an initial state of the relative motion. (The special matrixes, J_1, J_3, included here, are defined in Appendix B. Also, in these operations the perturbation acceleration term(s), $\bar{F}$, are assumed to be constant(s)).

In order to acquire a solution to Eq. (II.16a), the inverse transform of the above algebraic equation is needed. However, it must first be solved for $\bar{r}(s)$ explicitly.

Defining the coefficient of $\bar{r}(s)$ as a matrix, $A(s)$, then its inverse (A^{-1}) is needed since a symbolic form of the solution is expressed as:

$$\bar{r}(s) = A^{-1} \left[s I_3 + 2\dot{\varphi} B_2\right] \bar{r}_o + A^{-1} \left[I_3 \dot{\bar{r}}_o\right] + \frac{A^{-1}\left[I_3 \bar{F}\right]}{s} \, . \qquad \text{(II.16b)}$$

(Note that the matrix, A, has a same form as that described in Appendix B; consequently its inverse is of the form shown there. That is; since

$$A \equiv aJ_1 + bJ_2 + cJ_3 + eB_2 ; \qquad \text{(II.17a)}$$

then

$$A^{-1} = \frac{bJ_1 + aJ_2 - eB_2}{ab + e^2} + \frac{J_3}{c} , \qquad \text{(.17b)}$$

*The transforms used here are the more familiar ones (see most standard mathematical texts); the so-called "p-multiplied" transforms (see texts by L. Pipes) are not used in this work.

where the coefficients (a, b, c, e) are identified from Eq. (II.16a). In this case, when identifying the coefficients in A(s), recall that $I_3 = J_1 + J_2 + J_3$; then,

$$a \equiv s^2 - 3\dot{\varphi}^2, \quad b \equiv s^2, \quad c \equiv s^2 + \dot{\varphi}^2, \text{ and } \quad e \equiv 2\dot{\varphi} s). \tag{II.17c}$$

Introducing Eq. (II.17c) into Eq. (II.16b) and collecting terms on powers of s, an intermediate result is:

$$\bar{r}(s) = \left[\frac{s^3 I_3 + s^2 [2\dot{\varphi} (I_3 B_2 - B_2)] + s\dot{\varphi} (4I_2 - 3J_2 - 6\dot{\varphi}^3 J_2 B_2}{s^2 (s^2 + \dot{\varphi}^2)} \right] \bar{r}_o$$
$$+ \left[\frac{s^2 I_3 - s(2\dot{\varphi} B_2) - 3\dot{\varphi}^2 J_2}{s^2 (s^2 + \dot{\varphi}^2)} \right] \dot{\bar{r}}_o + \left[\frac{s^2 I_3 - s (2\dot{\varphi} B_2) - 3\dot{\varphi}^2 J_2}{s^3 (s^2 + \dot{\varphi}^2)} \right] \bar{F}, \tag{II.18a}$$

wherein $\bar{F}$ is understood to mean $\sum_j \bar{F}_j$ (sum of the perturbing specific forces).

A more meaningful collection of terms, from the above expression, in preparation for taking the inverse transform, is as follows: (here, the rationale applied is obvious):

$$\bar{r}(s) = \left(\frac{s}{s^2 + \dot{\varphi}^2} \right) I_3 \bar{r}_o + \frac{I_3 \dot{\bar{r}}_o}{(s^2 + \dot{\varphi}^2)} + \frac{\dot{\varphi}^2 (4I_2 - 3J_2) \bar{r}_o - 2\dot{\varphi} B_2 \dot{\bar{r}}_o + I_3 \bar{F}}{[s (s^2 + \dot{\varphi}^2)]}$$
$$- \left\{ \frac{6\dot{\varphi}^3 (J_2 B_2) \bar{r}_o + 3\dot{\varphi}^2 J_2 \dot{\bar{r}}_o + 2\dot{\varphi} B_2 \bar{F}}{[s^2 (s^2 + \dot{\varphi}^2)]} \right\} - \frac{3\dot{\varphi}^2 J_2 \bar{F}}{[s^3 (s^2 + \dot{\varphi}^2)]} . \tag{II.18b}$$

(In this expression obvious cancellations, etc. have been accounted for; also, prior to the next step, the reader may wish to consult Appendix B regarding the special matrixes used herein).

Recognizing that the solution (here) is to be expressed in terms of elementary functions; and, knowing the inverse transformation equivalents for these, it is prudent to "reduce" some of the algebraic functions above. Using partial fractions, the following equivalents are established:

$$(1) \quad \frac{1}{s(s^2+\dot{\varphi}^2)} = \frac{1}{\dot{\varphi}^2}\left(\frac{1}{s} - \frac{s}{s^2+\dot{\varphi}^2}\right);$$

$$(2) \quad \frac{1}{s^2(s^2+\dot{\varphi}^2)} = \frac{1}{\dot{\varphi}^2}\left(\frac{1}{s^2} - \frac{1}{s^2+\dot{\varphi}^2}\right);$$

and,

$$(3) \quad \frac{1}{s^3(s^2+\dot{\varphi}^2)} = \frac{1}{\dot{\varphi}^2}\left[\frac{1}{s^2} - \frac{1}{\dot{\varphi}^2}\left(\frac{1}{s} - \frac{s}{s^2+\dot{\varphi}^2}\right)\right]. \qquad \text{(II.19)*}$$

Relative Motion Displacements. Making use of the expressions above, transforming Eq. (II.18b) and clearing terms, a solution for $\bar{r}(t)$ may be expressed as:

$$\bar{r}(t) = \left[(4I_2 - 3J_2)\bar{r}_o - 2B_2\frac{\dot{\bar{r}}_o}{\dot{\varphi}} + I_3\frac{\bar{F}}{\dot{\varphi}^2} + \frac{3J_2\bar{F}}{\dot{\varphi}^2}\right] - \left[6\dot{\varphi}(J_2B_2)\bar{r}_o + 3J_2\dot{\bar{r}}_o + 2B_2\frac{\bar{F}}{\dot{\varphi}}\right]t - \frac{3J_2\bar{F}}{2}t^2 + \left[(I_3 - 4I_2 + 3J_2)\bar{r}_o + 2B_2\frac{\dot{\bar{r}}_o}{\dot{\varphi}} - (I_3 + 3J_2)\frac{\bar{F}}{\dot{\varphi}^2}\right]\cos(\dot{\varphi}t) + \left[6(J_2B_2)\bar{r}_o + (I_3 + 3J_2)\frac{\dot{\bar{r}}_o}{\dot{\varphi}} + \frac{2B_2\bar{F}}{\dot{\varphi}^2}\right]\sin(\dot{\varphi}t). \qquad \text{(II.20a)}$$

A more uniform representation of this result can be had by making use of the matrix equivalents mentioned in Appendix B. Thus, after some manipulation, Eq. (II.20a) can be recast as:

$$\bar{r}(t) = \left[(4J_1 + J_2)\bar{r}_o - 2B_2\frac{\dot{\bar{r}}_o}{\dot{\varphi}} + (J_1 + 4J_2 + J_3)\frac{\bar{F}}{\dot{\varphi}^2}\right] - \left[6J_2B_2\bar{r}_o + 3J_2\frac{\dot{\bar{r}}_o}{\dot{\varphi}} + 2B_2\frac{\bar{F}}{\dot{\varphi}^2}\right](\dot{\varphi}t) - \left[\frac{3}{2}J_2\frac{\bar{F}}{\dot{\varphi}^2}\right](\dot{\varphi}t)^2 + \left[(J_3 - 3J_1)\bar{r}_o + 2B_2\frac{\dot{\bar{r}}_o}{\dot{\varphi}} - (J_1 + 4J_2 + J_3)\frac{\bar{F}}{\dot{\varphi}^2}\right]\cos(\dot{\varphi}t)$$

(equation continued on next page)

*Some inverse transforms are noted here; $\mathcal{L}^{-1}(a/s) = a$ (constant); $\mathcal{L}^{-1}(1/s^2) = t$ (the independent variable); $\mathcal{L}^{-1}(2/s^3) = t^2$; $\mathcal{L}^{-1}[\dot{\varphi}/(s^2 + \dot{\varphi}^2)] = \sin(\dot{\varphi}t)$; $\mathcal{L}^{-1}[s/(s^2 + \dot{\varphi}^2)] = \cos(\dot{\varphi}t)$; and, $\mathcal{L}^{-1}[s\bar{r}(s) - r(0)] = \dot{\bar{r}}(t)$.

$$+\left[6J_2 B_2 \bar{r}_o + (J_1 + 4J_2 + J_3)\,\frac{\dot{\bar{r}}_o}{\dot{\varphi}} + 2B_2\,\frac{\bar{F}}{\dot{\varphi}^2}\right]\sin(\dot{\varphi}t). \qquad \text{(II.20b)*}$$

Eq. (II.20b) describes the relative motion of "Q" with respect to "P", in "local coordinates" (x, y, z), for the case where a central attraction and fixed perturbative forces $(\sum \bar{F}_j)$ are considered.

An expansion of the various matrices, etc. is instructive at this point to illustrate the composition of the various terms involved. For convenience in the resulting expansions, the specific force terms have been separated from those involving the initial state vectors. Thus:

$$\bar{r}(t) = \begin{bmatrix} 4 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \bar{r}_o + \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} - \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{r}_o + \begin{bmatrix} 0 & & 0 \\ & 3 & \\ 0 & & 0 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} \right\} (\dot{\varphi}\,t)$$

$$+ \left\{ \begin{bmatrix} -3 & & 0 \\ & 0 & \\ 0 & & 1 \end{bmatrix} \bar{r}_o + \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} \right\} \cos(\dot{\varphi}t) + \left\{ \begin{bmatrix} 0 & & 0 \\ 6 & 0 & \\ 0 & 0 & 0 \end{bmatrix} \bar{r}_o \right.$$

$$\left. + \begin{bmatrix} 1 & & 0 \\ & 4 & \\ 0 & & 1 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} \right\} \sin(\dot{\varphi}t) + \left\{ (1 - \cos\dot{\varphi}t) \begin{bmatrix} 1 & & 0 \\ & 4 & \\ 0 & & 1 \end{bmatrix} + \right.$$

$$\left. (\dot{\varphi}t - \sin\dot{\varphi}t) \begin{bmatrix} 0 & +2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{3}{2}(\dot{\varphi}t)^2 \begin{bmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{bmatrix} \right\} \frac{\bar{F}}{\dot{\varphi}^2}\,. \qquad \text{(II.20c)}$$

---

*Some of the quantities included here are in preparation for the non-dimensionalization to be introduced subsequently.

Here, the state vectors:

$\bar{r}(t) \equiv (x, y, z)^T$, $\bar{r}_o \equiv (x_o, y_o, z_o)^T$, $\mathring{\bar{r}}_o \equiv (\dot{x}_o, \dot{y}_o, \dot{z}_o)^T$, are (each) column vectors, as described. Incidentally, this result (above) may be checked with that given as Eqs. (A.29b), in Reference [1]*.

<u>The Relative Motion Velocity.</u> Equations (II.20) provide analytic expressions for approximating to the relative motion displacements of "Q" with respect to "P". However, the corresponding relative velocity is needed if one is to complete a description for the state of motion.

The simplest, and most direct, means for acquiring this latter information would be to differentiate the displacements given above. When this procedure has been carried out, on (say) Eq. (II.20), it is found that:

$$\frac{\mathring{\bar{r}}(t)}{\dot{\varphi}} = -\left\{3J_2\left[2B_2\bar{r}_o + I_2\frac{\mathring{\bar{r}}_o}{\dot{\varphi}}\right] + 2B_2\frac{\bar{F}}{\dot{\varphi}^2}\right\}$$
$$- 3J_2\frac{\bar{F}}{\dot{\varphi}^2}(\dot{\varphi}t)$$
$$-\left\{(J_3 - 3J_1)\bar{r}_o + 2B_2\frac{\mathring{\bar{r}}_o}{\dot{\varphi}} - (J_1 + 4J_2 + J_3)\frac{\bar{F}}{\dot{\varphi}^2}\right\}\sin(\dot{\varphi}t)$$
$$+\left\{6J_2B_2\bar{r}_o + (J_1 + 4J_2 + J_3)\frac{\mathring{\bar{r}}_o}{\dot{\varphi}} + 2B_2\frac{\bar{F}}{\dot{\varphi}^2}\right\}\cos(\dot{\varphi}t). \quad \text{(II.20d)}$$

For clarity, and continuity, an expanded form of Eq. (II.20d) is presented below with terms involving $\bar{F}$ separated from those described in terms of the initial state. (This is a companion expression to that given as Eq. (II.20c)).

*Reference [1]: "Relative Motion of Orbiting Satellites", by J.B. Eades, Jr. NASA CR 112113, dated July 1972.

$$
\frac{\dot{\bar{r}}(t)}{\dot{\varphi}} = -\begin{bmatrix} 0 & \diagdown & 0 \\ 6 & 0 & \diagdown \\ 0 & 0 & 0 \end{bmatrix} \bar{r}_o - \begin{bmatrix} 0 & \diagdown & 0 \\ \diagdown & 3 & \diagdown \\ 0 & \diagdown & 0 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} - \left\{ \begin{bmatrix} -3 & \diagdown & 0 \\ \diagdown & 0 & \diagdown \\ 0 & \diagdown & 1 \end{bmatrix} \bar{r}_o + \right.
$$

$$
\left. \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} \right\} \sin(\dot{\varphi}t) + \left\{ \begin{bmatrix} 0 & \diagdown & 0 \\ 6 & 0 & \diagdown \\ 0 & 0 & 0 \end{bmatrix} \bar{r}_o + \begin{bmatrix} 1 & \diagdown & 0 \\ \diagdown & 4 & \diagdown \\ 0 & \diagdown & 1 \end{bmatrix} \frac{\dot{\bar{r}}_o}{\dot{\varphi}} \right\} \cos(\dot{\varphi}t)
$$

$$
+ \left\{ (-1 + \cos\dot{\varphi}t) \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (\dot{\varphi}t - \sin\dot{\varphi}t) \begin{bmatrix} 0 & \diagdown & 0 \\ \diagdown & 3 & \diagdown \\ 0 & \diagdown & 0 \end{bmatrix} \right.
$$

$$
\left. + \sin\dot{\varphi}t \begin{bmatrix} 1 & \diagdown & 0 \\ \diagdown & 1 & \diagdown \\ 0 & \diagdown & 1 \end{bmatrix} \right\} \frac{\bar{F}}{\dot{\varphi}^2} \quad . \qquad \text{(II. 20e)}
$$

Here, $\frac{\dot{\bar{r}}(t)}{\dot{\varphi}} \equiv \left(\frac{\dot{x}}{\dot{\varphi}}, \frac{\dot{y}}{\dot{\varphi}}, \frac{\dot{z}}{\dot{\varphi}} \right)^T$, $\bar{r}_o \equiv (x_o, y_o, z_o)^T$, etc. By virture of the fact that this result is akin to that in Eq. (II. 20c), then these scalar expressions also may be checked against corresponding ones in Reference [1].

Non-dimensionalization. To this point in the development the state equations have been expressed in terms of dimensional quantities, see Eqs. (II.20). For many operations, and situations, it would be useful to have these expressed in a dimensionless form.

For purposes of compatibility with past work, and for the convenience of notation, etc., non-dimensionalization will be carried out as follows:

(a) All distances, such as $\bar{r}(t)$ ($\equiv \bar{r}(x, y, z)$, and the initial value $\bar{r}(0)$), are normalized using the reference particle's radius, $|\bar{P}|$.

(b) The velocity for "Q", and the associated initial value, will be non-dimensionalized by division with $|\bar{P}|\,\dot{\varphi}$ (the speed of the "P" particle).

(c) To retain compatibility, the perturbing forces are normalized by the quantity $|\bar{P}|\,\dot{\varphi}^2$ (the centrifugal force acting at the position of "P").

(d) Also, since $\dot{\varphi}$ is a fixed parameter (here), the time derivatives will be replaced by position derivatives; this is accomplished by writing,

$$(\dot{\varphi})^{-1} \frac{d}{dt} \equiv \frac{d}{d\varphi} \,.$$

In explanation of this procedure the following variables (and associated initial values) are introduced:

$$\bar{\lambda} \equiv \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \frac{1}{|\bar{P}|} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \frac{1}{|\bar{P}|\dot{\varphi}} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad \begin{bmatrix} \tau_\xi \\ \tau_\eta \\ \tau_\zeta \end{bmatrix} = \frac{1}{|\bar{P}|\dot{\varphi}^2} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \quad ; \qquad \text{(II.21)}$$

where $\frac{d}{d\varphi} \equiv (\sim)' = \frac{1}{\dot{\varphi}} \frac{d}{dt}$; here $\bar{\lambda}(\varphi)$, symbolically represents the dimensionless relative position vector; corresponding to this, $\bar{\lambda}'(\varphi)$, will be used to describe the associated dimensionless velocity vector.

<u>State Equations in Dimensionless Forms.</u> Making use of the definitions introduced above one can easily recast the solution equations into appropriate dimensionless forms. For instance, Eqs. (II.20b, II.20d) may be written as:

$$\bar{\lambda}(\varphi) = \left[(4J_1 + J_2)\,\bar{\lambda}_o - 2B_2\bar{\lambda}'_o + (J_1 + 4J_2 + J_3)\,\bar{\tau}\right]$$

$$- \left[3J_2\,(2B_2\bar{\lambda}_o + \bar{\lambda}'_o) + 2B_2\bar{\tau}\right]\varphi - \left[\frac{3}{2}J_2\bar{\tau}\right]\varphi^2$$

(equation continued on next page)

$$+ \left[(J_3 - 3J_1)\bar{\kappa}_o + 2B_2\bar{\kappa}'_o - (J_1 + 4J_2 + J_3)\,\bar{\tau} \right] \cos\varphi$$

$$+ \left[6J_2 B_2 \bar{\kappa}_o + (J_1 + 4J_2 + J_3)\bar{\kappa}'_o + 2B_2\,\bar{\tau} \right] \sin\varphi \,, \qquad \text{(II. 22a)}$$

and

$$\bar{\kappa}'(t) = - \left[3J_2\ (2B_2\bar{\kappa}_o + I_2\bar{\kappa}'_o) + 2B_2\bar{\tau} \right]$$

$$- 3\left[J_2\bar{\tau} \right] \varphi - \left[(J_3 - 3J_1)\bar{\kappa}_o + 2B_2\bar{\kappa}'_o - (J_1 + 4J_2 + J_3)\,\bar{\tau} \right] \sin\varphi$$

$$+ \left[6J_2B_2\bar{\kappa}_o + (J_1 + 4J_2 + J_3)\,\bar{\kappa}'_o + 2B_2\,\bar{\tau} \right] \cos\varphi \,. \qquad \text{(II. 22b)}$$

Due to the simplier notation introduced here, these expressions will be the ones manipulated hereafter (in these developments). Needless to say, the dimensional form(s) of these quantities may be recaptured by an application of the transforms shown in Eqs. (II. 21).

Prior to describing the state of motion, in terms of the inertial relative motion variables ($\bar{R}$, $\dot{\bar{R}}$), a more compact and revealing form of the position expression, above, will be developed. This particular representation combines the terms in Eq. (II. 22a), making a geometric description of the relative motion traces easier to "see". This development is outlined in the following paragraphs.

Rearranging the State Equations. In this rearrangement of the state equations (see Eqs. (II. 22), above) it is expedient to commence with the dimensionless displacement expression ($\bar{\kappa}$); and, to acquire the corresponding "velocity" expression by direct mathematical means. Also, since the out-of-plane displacements are uncoupled from the in-plane ones, then it is simplier to work with the corresponding separated expressions. Consequently, Eq. (II. 22a) are separated as seen below; see Appendix B for descriptions of the special matrices used here:

$$\bar{\lambda}(\varphi)(\equiv I_2\bar{\lambda}+J_3\bar{\lambda}) = \left[(4J_1 + J_2)\bar{\lambda}_o - 2B_2\bar{\lambda}'_o + (J_1 + 4J_2)\bar{\tau}\right]$$

$$-\left[3J_2(2B_2\bar{\lambda}_o + \bar{\lambda}'_o) + 2B_2\bar{\tau}\right]\varphi - \left[\frac{3}{2}J_2\bar{\tau}\right]\varphi^2$$

$$+\left[-3J_1\bar{\lambda}_o + 2B_2\bar{\lambda}'_o - (J_1 + 4J_2)\bar{\tau}\right]\cos\varphi$$

$$+\left[6J_2B_2\bar{\lambda}_o + (J_1 + 4J_2)\bar{\lambda}'_o + 2B_2\bar{\tau}\right]\sin\varphi$$

$$+\left\{J_3\left[\bar{\tau} + (\bar{\lambda}_o - \bar{\tau})\cos\varphi + \bar{\lambda}'_o\sin\varphi\right]\right\} . \qquad \text{(II.23)}$$

The first terms to be considered (here) are those involving the circular trigonometric functions. It is to be demonstrated that these can be related to the transform matrices, $T(\varphi^-)$ and $T(\varphi^+)$ - - but when they have been reduced to a second order form. Thus for the defined matrices $(T)_2$, write;

$$T(\varphi^-) = I_2\cos\varphi + B_2\sin\varphi + J_3 \equiv (T_-)_2 + J_3,$$

and

$$T(\varphi^+) = I_2\cos\varphi - B_2\sin\varphi + J_3 \equiv (T_+)_2 + J_3 . \qquad \text{(II.24)}$$

Now, in Eq. (II.23) define coefficients (K_c, K_s and/or K'_s) as shown below:

$$I_2K_c\cos\varphi \equiv \left\{-3J_1\bar{\lambda}_o + 2B_2\bar{\lambda}'_o - (J_1 + 4J_2)\bar{\tau}\right\}\cos\varphi ,$$

and

$$I_2K_s\sin\varphi \equiv \left\{6J_2B_2\bar{\lambda}_o + (J_1 + 4J_2)\bar{\lambda}'_o + 2B_2\bar{\tau}\right\}\sin\varphi . \qquad \text{(II.25a)}$$

A manipulation of the "sine" expression is needed here; in particular the "B_2" operator is removed (see Appendix B for the equivalent operations used below). For this purpose the following operations are described:

Since

$$I_2K_s\sin\varphi \equiv \left\{6(B_2J_1)\bar{\lambda}_o + (J_1 + 4J_2)(-B_2B_2)\bar{\lambda}'_o + 2B_2\bar{\tau}\right\}\sin\varphi$$

then the right side is manipulated so that it becomes:

$$\left\{B_2\,(6J_1\bar{k}_o + 2\bar{\tau}) - \left[4J_2 + J_1\right] B_2\,(B_2\bar{k}'_o)\right\} \sin\varphi$$

$$= B_2\left\{6J_1\bar{k}_o - \left[4J_1 + J_2\right] B_2\bar{k}'_o + 2\bar{\tau}\right\} \sin\varphi\,;$$

or, this is written as,

$$B_2K'_s \sin\varphi \equiv B_2\left\{6J_1\bar{k}_o - (4J_1 + J_2)\,B_2\bar{k}'_o + 2\bar{\tau}\right\} \sin\varphi. \qquad \text{(II.25b)}$$

Next, let it be assumed that:

$$I_2K_c \cos\varphi + B_2K'_s \sin\varphi \equiv (T_-)_2\,A_a + (T_+)_2\,A_s,$$

(where A_s and A_a are two matrices to be defined below). The present problem is to determine the quantities (A_a, A_s). By expanding the matrices "T_2", using Eq. (II.24) above,

$$I_2K_c \cos\varphi + B_2K'_s \sin\varphi = I_2\,(A_a + A_s) \cos\varphi + B_2\,(A_a - A_s) \sin\varphi\,;$$

then, by matching coefficients, it is found that:

$$\left.\begin{matrix} K_c \equiv A_a + A_s \\ K'_s \equiv A_a - A_s \end{matrix}\right\}, \text{ or, } \left.\begin{matrix} K_c + K'_s = 2A_a \\ K_c - K'_s = 2A_s \end{matrix}\right\}. \qquad \text{(II.25c)*}$$

Substituting from Eqs. (II.25b), it can be shown that

$$A_a = \frac{1}{2}\left\{3J_1\bar{k}_o + \left[J_2 - 2J_1\right] B_2\bar{k}'_o + \left[J_1 - 2J_2\right]\bar{\tau}\right\}, \qquad \text{(II.26a)}$$

and

$$A_s = \frac{3}{2}\left\{-3J_1\bar{k}_o + \left[2J_1 + J_2\right] B_2\bar{k}'_o - \left[J_1 + 2J_2\right]\bar{\tau}\right\}. \qquad \text{(II.26b)}$$

It is quite apparent that these two matrices have a marked similarity; they differ by a constant multiplier, and by a change in the sign of the matrix,

---

*Subscripts have been added to the matrix designation "A", indicating that they are acquired by "addition" and by "subtraction" of the constant matrices, K_c and K'_s.

J_1. (Note that the matrix operator altering the sign of J_1 while retaining the sign of J_2 is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1-2) & 0 \\ 0 & 1 \end{bmatrix} = (J_2 - J_1) = (I_2 - 2J_1);$$

hence

$$A_s = 3(J_2 - J_1) A_a.) \qquad \text{(II.26c)}$$

Therefore,

$$(I_2 K_c) \cos\varphi + (B_2 K_s') \sin\varphi = (T_-)_2 A_a + (T_+)_2 \left[3(J_2 - J_1)\right] A_a$$

$$= \left[(T_-)_2 + 3(T_+)_2 (J_2 - J_1)\right] A_a. \qquad \text{(II.27a)}$$

This expression, Eq. (II.27a), describes the contribution to $\bar{k}(\varphi)$ provided by the <u>trigonometric</u> terms. Hence, <u>this</u> partial solution is expressed as:

$$\left[\Delta\bar{k}(\varphi)\right]_{trig} \equiv \left[(T_-)_2 + 3(T_+)_2 (J_2 - J_1)\right] A_a;$$

or, since $(J_2 - J_1)$ is its own transpose;

$$\left[\Delta\bar{k}(\varphi)\right]_{trig} = \left[I_2 + 3(J_2 - J_1)\right] (T_-)_2 A_a, \qquad \text{(II.27b)}$$

where A_a is defined above, in Eq. (II.26a).

Next, consider the constant and first order secular terms in Eq. (II.23). First, however, a manipulation of the <u>secular</u> <u>term</u> is needed. For this, write

$$\varphi\left[6J_2 B_2 \bar{k}_o + 3J_2 \bar{k}'_o + 2B_2 \bar{\tau}\right] = B_2\left[6J_1 \bar{k}_o - 3J_1 (B_2 \bar{k}'_o) + 2\bar{\tau}\right]\varphi.$$

Now, when the constant term in $[I_2 \bar{k}(\varphi)]$ is multiplied by "$\frac{3}{2} J_1$",

$$\frac{3}{2} J_1\left\{(4J_1 + J_2)\bar{r}_o - 2(B_2\bar{r}'_o) + (J_1 + 4J_2)\bar{\tau}\right\} = 6J_1\bar{r}_o - 3J_1(B_2\bar{r}'_o) + \frac{3}{2} J_1\bar{\tau} ;$$

and, therefore, using these expressions, the <u>two</u> quantities can be combined as follows:

$$(4J_1 + J_2)\bar{r}_o - 2(B_2\bar{r}'_o) + B_2\left\{-\frac{3}{2} J_1\left[(4J_1 + J_2)\bar{r}_o - 2B_2\bar{r}'_o\right]\varphi\right\} + (J_1 + 4J_2)\bar{\tau} - 2B_2\bar{\tau}\varphi$$

$$= (I_2 - \frac{3}{2} B_2 J_1 \varphi)\left[(4J_1 + J_2)\bar{r}_o - 2B_2\bar{r}'_o\right] + \left[(J_1 + 4J_2) - 2\varphi B_2\right]\bar{\tau} .$$

Adding to this expression the remaining <u>secular</u> term, then the last part of the in-plane partial solution, for $I_2\bar{r}(\varphi)$, can be joined together as:

$$\Delta\bar{r}(\varphi) = (I_2 - \frac{3}{2}\varphi B_2 J_1)\left[(4J_1 + J_2)\bar{r}_o - 2B_2\bar{r}'_o\right] + \left\{(J_1 + 4J_2) - \varphi\left[2B_2 + \frac{3}{2} J_2\varphi\right]\right\}\bar{\tau} . \qquad \text{(II.27c)}$$

Finally, the zth component of this (positional) relative motion solution is seen to be:

$$J_3\bar{r}(\varphi) = J_3\left\{\bar{\tau} + (\bar{r}_o - \bar{\tau})\cos\varphi + \bar{r}'_o\sin\varphi\right\} . \qquad \text{(II.27d)}$$

Now, collecting the various partial solutions, then, Eq. (II.23) can be <u>symbolically</u> represented by:

$$\bar{r}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right](T_-)_2 A_a + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]K_o + \Psi_\tau\bar{\tau} + J_3\left\{\bar{\tau} + (\bar{r}_o - \bar{\tau})\cos\varphi\right.$$

$$\left. + \bar{r}'_o\sin\varphi\right\} ; \qquad \text{(II.28)}$$

wherein,

$$A_a \equiv \frac{1}{2}\left\{3J_1\bar{r}_o + \left[J_2 - 2J_1\right]B_2\bar{r}'_o + \left[J_1 - 2J_2\right]\bar{\tau}\right\} ,$$

$$K_o \equiv (4J_1 + J_2)\bar{r}_o - 2B_2\bar{r}'_o ,$$

and

$$\Psi_\tau \equiv (J_1 + 4J_2) - \varphi\left[2B_2 + \frac{3}{2} J_2 \varphi\right].$$

Note that the terms A_a and K_o, above, are constants - - they depend mainly on the initial values; however, the last term, Ψ_τ (the coefficient of $\bar{\tau}$), is secular in its composition. It is important to keep this in mind since the relative velocity will be determined (next) by differentiation. For this, Ψ_τ bears a derivative.

The Relative Velocity Equation. Rather than follow a procedure analogous to that above, for the description of relative velocity, a more direct approach is taken. That is, by differentiation, Eq. (II.28) yields the corresponding velocity equation.

The result of differentiating the displacement can be written, generally, as:

$$\bar{\kappa}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right](T_-)'_2 A_a - \frac{3}{2} B_2 J_1 K_o + \Psi'_\tau \bar{\tau} + J_3 \left\{\bar{\kappa}'_o \cos\varphi - (\bar{\kappa}_o - \bar{\tau}) \sin\varphi\right\}.$$

Of course, as shown in Appendix A, $(\dot{T}_-)_2 = \mathring{\varphi} B_2 (T_-)_2$; and hence, $(T_-)'_2 = B_2(T_-)_2$. Also, from the definition of Ψ_τ it is evident that:

$$\Psi'_\tau = -\left[2B_2 + 3J_2\varphi\right].$$

Therefore,

$$\bar{\kappa}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right]B_2 (T_-)_2 A_a - \frac{3}{2}(B_2 J_1) K_o + \Psi'_\tau \bar{\tau} + J_3 \left\{\bar{\kappa}'_o \cos\varphi - (\bar{\kappa}_o - \bar{\tau}) \sin\varphi\right\}, \quad \text{(II.29)}$$

with Ψ'_τ as shown above.

Summary. Eqs. (II.28, II.29) describe the relative motion of a test particle "Q" with respect to a reference particle "P" in terms of variables which are defined in the local (rotating) frame of reference. Here, the initial state parameters $(\bar{\kappa}, \bar{\kappa}')_o$ are dimensionless quantities as defined previously. Since this problem has been solved as an initial value situation, then the coefficients (A_a, K_o) are expressed in terms of the initial state parameters. Also, included here is a representation for a "fixed value" perturbing force system (i.e., $\bar{\tau}$, the dimensionless specific force vector).

The expressions noted above are merely rearrangements of the (similar) equations developed earlier (see Eqs. II.22). The purpose in presenting this last set is that they provide a simpler and more direct means of representing the relative motion state, geometrically.

In subsequent paragraphs a description of these geometries will be given; however, prior to this, a parallel development for the inertial description of these motions is to be obtained.

NOTE: Before moving on to these next developments it would be useful to write the state equations (II.28, II.29) in their expanded scalar format. For this, proceed as follows:

(a) First, from Eq. (II.28), the in-plane position variables are developed from:

$$I_2\bar{\lambda}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right](T_-)_2 A_a + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Psi_\tau \bar{\tau}. \qquad \text{(II.30a)}$$

In matrix form this can be written as:

$$I_2\bar{\lambda}(\varphi) = \begin{bmatrix} -\cos\varphi & \sin\varphi \\ +2\sin\varphi & 2\cos\varphi \end{bmatrix} \left\{ \begin{bmatrix} +3 & 0 \\ 0 & 0 \end{bmatrix} \bar{\lambda}_o + \begin{bmatrix} 0 & +2 \\ 1 & 0 \end{bmatrix} \bar{\lambda}'_o + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \bar{\tau} \right\}$$

$$+ \begin{bmatrix} 1 & 0 \\ -\frac{3\varphi}{2} & 1 \end{bmatrix} \left\{ \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \bar{\lambda}_o + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \bar{\lambda}'_o \right\} + \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} + \varphi \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \right.$$

$$\left. + \varphi^2 \begin{bmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} \right\} \bar{\tau} \quad ; \qquad \text{(II.30b)}$$

This equation, when separated into its scalar form, gives:

$$\xi(\varphi) = (-3\xi_o \cos\varphi + \xi_o' \sin\varphi - 2\eta_o' \cos\varphi - \tau_\xi \cos\varphi - 2\tau_\eta \sin\varphi) + (4\xi_o + 2\eta_o') + (\tau_\xi + 2\varphi\tau_\eta),$$

and

$$\eta(\varphi) = (6\xi_o \sin\varphi + 2\xi_o' \cos\varphi + 4\eta_o' \sin\varphi + 2\tau_\xi \sin\varphi - 4\tau_\eta \cos\varphi) - (6\xi_o\varphi - \eta_o + 2\xi_o' + 3\varphi\eta_o')$$

$$- \left[2\varphi\tau_\xi - (4 - \frac{3}{2}\varphi^2)\tau_\eta\right]. \qquad \text{(II.30c)}$$

Finally, the zth-equation is seen to be:

$$\zeta(\varphi) = \tau_\zeta (1 - \cos\varphi) + \zeta_o \cos\varphi + \zeta_o' \sin\varphi. \qquad \text{(II.30d)}$$

Next, the velocity equation(s) will be expanded, and presented.

Beginning with Eq. (II.29), and separating the in- and out-of-plane components, then the in-plane resultant is:

$$I_2 \bar{\lambda}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] B_2 (T_-)_2 A_a - \frac{3}{2}(B_2 J_1) K_o + \Psi_\tau' \bar{\tau}\ ; \qquad \text{(II.31a)}$$

this can be set down in matrix form as:

$$I_2 \bar{\lambda}' = \begin{bmatrix} \sin\varphi & \cos\varphi \\ 2\cos\varphi & -2\sin\varphi \end{bmatrix} \left\{ \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \bar{\lambda}_o + \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \bar{\lambda}_o' + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \bar{\tau} \right\} - \frac{3}{2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} *$$

$$\left\{ \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \bar{\lambda}_o + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \bar{\lambda}_o' \right\} + \left\{ \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 3\varphi \end{bmatrix} \right\} \bar{\tau}. \qquad \text{(II.31b)}$$

(It is recognized that the parameters $\bar{\lambda}_o$, $\bar{\lambda}_o'$, and $\bar{\tau}$ (here) are necessarily composed as the in-plane vectors only).

Next, when these matrix expressions are expanded into a scalar format one has:

$$\xi'(\varphi) = (3\xi_o \sin\varphi + \xi'_o \cos\varphi + 2\eta'_o \sin\varphi + \tau_\xi \sin\varphi) - 2\tau_\eta(\cos\varphi - 1), \tag{II.31c}$$

and

$$\eta'(\varphi) = (6\xi_o \cos\varphi - 2\xi'_o \sin\varphi + 4\eta'_o \cos\varphi + 2\tau_\xi \cos\varphi + 4\tau_\eta \sin\varphi) - (6\xi_o + 3\eta'_o) - (2\tau_\xi + 3\varphi\tau_\eta). \tag{II.31d}$$

Lastly, the zth velocity equation easily yields:

$$\zeta'(\varphi) = \zeta'_o \cos\varphi - (\zeta_o - \tau_\zeta)\sin\varphi . \tag{II.31e}$$

The companion dimensional expressions, to those shown above, are readily deduced from Eqs. (II.20c, and II.20e) presented earlier. Of course, one major difference between these two resultants is the reformation leading to the geometric interpretation provided by the latter expressions as compared to the earlier ones.

The development procedures for a relative motion, referred to a local frame of reference, are completed with this summary section. Following this, a similar description of this relative motion, referred to an inertially oriented frame, will be developed. This appears in the next few paragraphs.

<u>The Relative Motion, in Inertial Coordinates</u>. In determining the desired state equations, here, those which have been described (above) will be modified - - i.e. transformed - - according to results obtained earlier in the section(s) on kinematics. Accordingly, using Eq. (II.13a), the relative position vector, referred to the inertial frame, is related to the corresponding vector, described in the local rotating frame, by:

$$\bar{R} = T(\varphi^-)\,\bar{r}.$$

From this single relationship it appears that the transformation could be carried out quite simply (in concept). However, it must be remembered that the vector $(\bar{r})$ has been determined as an initial value problem; therefore, it will be necessary to transform the initial state vectors, also, in this development. This task will be undertaken first, below.

(Initial Values). According to Eqs. (II .13b), II .14a, and A.7), the variables of interest here are described, and related, as follows:

Since $\bar{r} = T(\varphi^{+})\bar{R}$,

then $\dot{\bar{r}} = \dot{T}(\varphi^{+})\bar{R} + T(\varphi^{+})\dot{\bar{R}}$,

wherein $T(\varphi^{+}) = I_2 \cos\varphi - B_2 \sin\varphi + J_3$. (II.32a)

Now, at the initial position (t = 0) these expressions are specialized to read:

$$\bar{r}(0) \equiv \bar{r}_o = T(0)\,\bar{R}_o,$$

and $\dot{\bar{r}}(0) \equiv \dot{\bar{r}}_o = \dot{T}(0)\bar{R}_o + T(0)\,\dot{\bar{R}}_o$. (II.32b)

Herein, $T(0) = I_2 + J_3 = I_3$; also, by definition, $\dot{T}(0) = \dot{\varphi}B_2I_3$. These results lead directly to:

$$\bar{r}_o = I_3\bar{R}_o,$$

and $\dot{\bar{r}}_o = I_3\dot{\bar{R}}_o - \dot{\varphi}B_2I_3\bar{R}_o$. (II.32c)

When these expressions are separated into in-plane and out-of-plane vectors, one obtains:

(in-plane) $I_2\bar{r}_o = I_2\bar{R}_o,\ I_2\dot{\bar{r}}_o = I_2\dot{\bar{R}}_o - \dot{\varphi}B_2\bar{R}_o$;

and,

(out-of-plane) $J_3\bar{r}_o = J_3\bar{R}_o,\ J_3\dot{\bar{r}}_o = J_3\dot{\bar{R}}_o$. (II.32d)

The corresponding dimensionless forms of these results are noted to be:

(in-plane) $I_2\bar{\mathcal{r}}_o = I_2\bar{\mathcal{R}}_o,\ I_2\bar{\mathcal{r}}'_o = I_2\bar{\mathcal{R}}'_o - B_2\bar{\mathcal{R}}_o$;

and,

(out-of-plane) $J_3\bar{\mathcal{r}}_o = J_3\bar{\mathcal{R}}_o,\ J_3\bar{\mathcal{r}}'_o = J_3\bar{\mathcal{R}}'_o$. (II.32e)*

*Here $\mathcal{R}$ and $\mathcal{R}'$ are the dimensionless (state) variables, referred to inertial axes: $\mathcal{R} \equiv \mathcal{R}(\Xi, H, Z)$ and $\mathcal{R}' \equiv (\Xi', H', Z')$, analogous to Eqs. (II.21).

(Equation Transformations). For the transformation from "local-coordinates" to "inertial" ones, the relationship to be applied is given in Eq. (II.13a); namely,

$$\bar{R}(\varphi) = T(\varphi^-)\,\bar{r}(\varphi); \tag{II.33a}$$

wherein, generally,

$$T(\varphi^-) = I_2 \cos\varphi + B_2 \sin\varphi + J_3.$$

For the actual use of Eq. (II.33a), above, the expression for $\bar{r}(\varphi)$, appearing as Eq. (II.28), will be employed. Consequently, in dimensionless format:

$$\bar{\mathcal{R}}(\varphi) \equiv T(\varphi^-)\,\bar{\hbar}(\varphi) = T_2(\varphi^-)\left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-)\, A_a + T_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o$$

$$+ T_2(\varphi^-)\,\Psi_\tau \bar{\tau} + J_3\left\{\bar{\tau} + (\bar{\mathcal{R}}_o - \bar{\tau})\cos\varphi + \bar{\mathcal{R}}'_o \sin\varphi\right\}. \tag{II.33b}$$

Before proceeding, a closer look at the lead term on the right of the equals sign is in order.

Expand the matrices shown there, as follows:

$$(T_-)_2\left[I_2 + 3(J_2 - J_1)\right](T_-)_2 = (T_-)_2 I_2 (T_-)_2 + 3(T_-)_2\,(J_2 - J_1)(T_-)_2 = (T_-)_2{}^*\,(T_-)_2$$

$$+ 3(T_-)_2\,(T_+)_2\,(J_2 - J_1),$$

since the matrix $(J_2 - J_1)$ is its own transpose. Thus, in view of the multiplication property noted in Eq. (A.7e);

$$(T_-)_2\left[I_2 + 3(J_2 - J_1)\right](T_-)_2 = T_2(2\varphi^-) + 3(J_2 - J_1) \tag{II.34a}$$

Next, making the required substitutions, from Eq. (II.32e), it is easy to show that*:

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*Subscripts $(\sim)_I$ are added to indicate that these matrices (now) infer a representation <u>in</u> the inertial frame.

$$A_{a_I} = \frac{1}{2}\left[I_2\bar{\Re}_o + (J_2 - 2J_1)\,B_2\bar{\Re}'_o + (J_1 - 2J_2)\,\bar{\tau}\right]$$

$$K_{o_I} = (2J_1 - J_2)\,\bar{\Re}_o - 2B_2\bar{\Re}'_o\,,$$

and

$$\Psi_{\tau_I} = \Psi_\tau \equiv (J_1 + 4J_2) - \varphi\left[2B_2 + \frac{3}{2}\,J_2\varphi\right]. \qquad \text{(II.34b)}$$

Introducing Eq. (II.34a) into Eq. (II.33b), and noting that the appropriate forms for A_a, K_o, Ψ_τ are those in Eqs. (II.34b), then

$$\bar{\Re}(\varphi) = \left[T_2(2\varphi^-) + 3(J_2 - J_1)\right]A_{a_I} + T_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]K_{o_I} + T_2(\varphi^-)\,\Psi_\tau\bar{\tau}$$
$$+ J_3\left\{\bar{\tau} + (\bar{\Re}_o - \bar{\tau})\cos\varphi + \bar{\Re}'_o\sin\varphi\right\}. \qquad \text{(II.35)}$$

Eq. (II.35) is the dimensionless, relative motion position vector referred to the inertial frame of reference. Note that $\bar{\tau}$ is the same vector as described before.

The Inertially Described Relative Velocity. Following the procedure used for the development of Eq. (II.29), then the velocity vector, here, referred to an inertial frame of reference, will be obtained by differentiation. (It should be apparent that this is equivalent to the differentiation of Eq. (II.13a), and the subsequent substitution for $\bar{r}$ and $\dot{\bar{r}}$ (or their equivalent dimensionless form(s)).

Taking the direct mathematical approach, that of acquiring the velocity ($\bar{\Re}'$), from the position vector ($\bar{\Re}$), then (symbolically);

$$\bar{\Re}'(\varphi) = 2\left[T'_2(2\varphi^-)\right]A_{a_I} + T'_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]K_{o_I} - \frac{3}{2}\left[T_2(\varphi^-)B_2 J_1\right]K_{o_I}$$
$$+ T'_2(\varphi^-)\,\Psi_\tau\bar{\tau} + T_2(\varphi^-)\left[\Psi'_\tau\right]\bar{\tau} + J_3\left\{\bar{\Re}'_o\cos\varphi - (\bar{\Re}_o - \bar{\tau})\sin\varphi\right\}. \qquad \text{(II.36)}$$

For this expression it should be recognized that (see Appendix A),

$$T'(\varphi^-) = B_2T_2(\varphi^-),\ \text{and}\ T'_2(2\varphi^-) = 2B_2T_2(2\varphi^-)\,. \qquad \text{(II.37a)}$$

Also, Ψ_τ, being the secular coefficient of $\bar{\tau}$, bears a derivative; namely,

$$\Psi'_\tau = -\left[2B_2 + 3J_2\varphi\right] \equiv \left[\Psi'_\tau\right]_I . \qquad \text{(II.37b)}$$

When these various quantities are inserted into Eq. (II.36), and some manipulations are carried out, it is found that (one form of) the velocity vector is:

$$\bar{\mathcal{R}}'(\varphi) = B_2\left\{\left[I_2\bar{\mathcal{R}}(\varphi)\right] + \left[T_2(2\varphi^-) - 3(J_2 - J_1)\right]A_{a_I} - T_2(\varphi^-)\left[\frac{3}{2}(J_1K_{o_I}) + 2(I_2\bar{\tau})\right]\right\}$$

$$- 3T_2(\varphi^-)\left[J_2\varphi\right]\bar{\tau} + J_3\left\{\bar{\mathcal{R}}'_o \cos\varphi - (\bar{\mathcal{R}}_o - \bar{\tau}) \sin\varphi\right\} . \qquad \text{(II.38)}$$

In this expression the vector $\bar{\mathcal{R}} \equiv \bar{\mathcal{R}}(\varphi)$ is the relative position vector given by Eq. (II.35); also the coefficients (A_{a_I}, K_{o_I}) are those quantities noted as Eqs. (II.34b).

Summary. The expressions for $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$, given by Eqs. (II.35, and II.38), represent a linearized solution for the stated relative motion problem when it is referred to an inertially oriented frame of reference. These results describe the same problem as that in Eqs. (II.28 and II.29); the difference between these two solutions is the frame of reference implied by each. As in the earlier solution, the external force system,here, has the same fixed components as before; that is, each component is parallel to one of the axes for the "local rotating frame of reference".

There is a second exception for these results. Here, the parameter coefficients (A_a, K_o) have been altered (necessarily) to describe the proper set of initial values, those required for the inertial frame's representation.

In order to see the more definitive representation for the inertial frame's results, the state equations are rewritten in the expanded format shown below. That is, from Eq. (II.35) write:

(a) for the in-plane displacement solution:

$$I_2\bar{\mathfrak{R}}(\varphi) \equiv \begin{bmatrix} \Xi \\ H \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cos 2\varphi - 3 & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi + 3 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Xi_o \\ H_o \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Xi'_o \\ H'_o \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix}\right\} + \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{3\varphi}{2} & 1 \end{bmatrix}\left\{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \Xi_o \\ H_o \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} \Xi'_o \\ H'_o \end{bmatrix}\right\} + \left\{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} 1 & 2\varphi \\ -(2\varphi) & (4 - \frac{3\varphi^2}{2}) \end{bmatrix}\right\}\begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} \; ; \qquad \text{(II.39a)}$$

and; (b), for the out-of-plane displacement:

$$J_3\bar{\mathfrak{R}}(\varphi) \equiv \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\begin{bmatrix} \tau_\xi \\ \tau_\eta \\ \tau_\zeta \end{bmatrix} + \begin{bmatrix} \Xi_o & -\tau_\xi \\ H_o & -\tau_\eta \\ Z_o & -\tau_\zeta \end{bmatrix}\cos\varphi + \begin{bmatrix} \Xi'_o \\ H'_o \\ Z'_o \end{bmatrix}\sin\varphi\right\}. \qquad \text{(II.39b)}$$

In place of the relative velocity (Eq. (II.38)), a more convenient expression, for manipulation, is:

$$\bar{\mathfrak{R}}'(\varphi) = B_2\left\{\left[2T_2(2\varphi^-)\right]A_{a_I} + T_2(\varphi^-)\left[\left(I_2 - \frac{3}{2}\varphi B_2 J_1\right)K_{o_I} + \Psi_\tau\bar{\tau}\right]\right\} + T_2(\varphi^-)\left[\Psi'_\tau\bar{\tau}\right.$$

$$\left. - \frac{3}{2}(B_2 J_1) K_{o_I}\right] + J_3\left\{\bar{\mathfrak{R}}'_o \cos\varphi - (\bar{\mathfrak{R}}_o - \bar{\tau})\sin\varphi\right\}. \qquad \text{(II.40a)}$$

Written in matrix notation this becomes:

(a) for the in-plane solution:

$$I_2\bar{\mathfrak{R}}'(\varphi) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Xi' \\ H' \end{bmatrix} \equiv \begin{bmatrix} \Xi' \\ H' \end{bmatrix} = \begin{bmatrix} -\sin 2\varphi & -\cos 2\varphi \\ \cos 2\varphi & -\sin 2\varphi \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Xi_o \\ H_o \end{bmatrix}\right.$$

(continued on next page)

$$+\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Xi'_o\\ H'_o\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}\tau_\xi\\ \tau_\eta\end{bmatrix}+\begin{bmatrix}-\sin\varphi & -\cos\varphi\\ \cos\varphi & -\sin\varphi\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\ -\frac{3}{2}\varphi & 1\end{bmatrix}\left(\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\Xi_o\\ H_o\end{bmatrix}\right.\right.$$

$$\left.\left.+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}\Xi'_o\\ H'_o\end{bmatrix}\right)+\begin{bmatrix}1 & 2\varphi\\ -2\varphi & 4-\frac{3}{2}\varphi^2\end{bmatrix}\begin{bmatrix}\tau_\xi\\ \tau_\eta\end{bmatrix}\right\}+\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix} *$$

$$*\left\{\begin{bmatrix}0 & 2\\ -2 & -3\varphi\end{bmatrix}\begin{bmatrix}\tau_\xi\\ \tau_\eta\end{bmatrix}+\begin{bmatrix}0 & 0\\ -\frac{3}{2} & 0\end{bmatrix}\left(\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\Xi_o\\ H_o\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}\Xi'_o\\ H'_o\end{bmatrix}\right)\right\}; \qquad \text{(II.40b)}$$

and, (b), the out-of-plane solution:

$$J_2\bar{\Re}'(\varphi) \equiv \begin{bmatrix}0\\ 0\\ Z'\end{bmatrix} \equiv \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\left\{\begin{bmatrix}\Xi'_o\\ H'_o\\ Z'_o\end{bmatrix}\cos\varphi - \begin{bmatrix}\Xi_o & -\tau_\xi\\ H_o & -\tau_\eta\\ Z_o & -\tau_\zeta\end{bmatrix}\sin\varphi\right\} . \qquad \text{(II.40c)}$$

Equations (II.39 and II.40) are a more descriptive set of expressions for this solution to the relative motion problem. Here, as previously, the displacements and speeds depend on the initial state quantities ($\bar{\Re}_o$, $\bar{\Re}'_o$) and on the applied (fixed) specific force ($\bar{\tau}$). It is seen, and expected, that these force components are the same as those used earlier since these equations describe the same relative motion problem but as it would be seen in the inertial frame of reference.

With one aim of this investigation being that of describing a geometry for these motion traces, that task will be undertaken in the following sections. For the illustrations to follow, both solution descriptions - i.e., the one represented in the "local rotating" frame of reference, and the other referred to an inertial triad - will be examined. Subsequently, the geometric figures which these describe on the various coordinate planes will be described graphically and analytically.

IV. GRAPHICAL DESCRIPTIONS

Geometric Representations. In the following descriptions certain relative motion traces will be examined. Those which are selected have been chosen so that comparisons may be made between representations found for the two reference frames.

The first geometries discussed will be those for in-plane displacements. For convenience and conciseness all discussions and graphical representations are cast in terms of dimensionless variables.

Only a limited number of problem situations are described, in detail, below; however, a compendium of these results is included herein for reference purposes. The details for constructing the various trace geometries is contained in another report*. The first case study is that for:

(a) In-Plane Displacements; Rotating Triad. The general expression to be investigated here was previously given in Eq. (II.30a); that result was:

$$I_2 \bar{\lambda}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) A_a + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Psi_\tau \bar{\tau}. \qquad \text{(II.41)}$$

From this equation one sees the displacements expressed as three vectors; one, each, to describe contributions from the parameters A_a, K_o and $\bar{\tau}$, respectively**. In order to describe a geometry which these produce, and to do so in a systematic fashion, each vector statement will be examined separately; and, then, in combination with the others.

The first component vector (that due to A_a) is noted below. It is recalled that $A_a \equiv A_a(\bar{\lambda}_o, \bar{\lambda}_o', \bar{\tau})$; however, at present it is more convenient to let A_a be expressed as a general two component vector; i.e., $A_a \equiv A_a(A_1, A_2)$. Then this contribution to $\bar{\lambda}(\varphi)$ is described as:

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* "Construction of Relative Motion Traces" (an Interim Report), by J.B. Eades, Jr., AMA Report No. 73-39, August 1973.

**This composition is "mathematical" in its make-up; a more "realistic" construction will be obtained in the following section. There the problem is recast. to separate initial values from the effects of an "applied force".

$$\delta\left[I_2\,\bar{h}\,(\varphi)\right]_1 \equiv \begin{bmatrix}\delta\xi\\ \delta\eta\end{bmatrix}_1 = \begin{bmatrix}-\cos\varphi & \sin\varphi\\ 2\sin\varphi & 2\cos\varphi\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix}, \tag{II.42}$$

where these two scalars (A_i), in terms of the initial values, are:

$$A_1 = 3\xi_o + 2\eta_o' + \tau_\xi ,$$

and (II.43)

$$A_2 = \xi_o' - 2\tau_\eta .$$

It is evident that the expression above describes a (2:1) ellipse on the (ξ, η)-plane. This ellipse is centered at the coordinate origin and described by the parametric equation

$$\frac{(\delta_1\xi)^2}{\left(\sqrt{A_1^2 + A_2^2}\right)^2} + \frac{(\delta_1\eta)^2}{\left(2\sqrt{A_1^2 + A_2^2}\right)^2} = 1. \tag{II.44}$$

A general sketch of the trace is shown on Fig.II.4 below*.

The second vector contribution, arising from the K_o term, involves only initial value quantities; i.e., $K_o \equiv K_o(\bar{h}_o, \bar{h}_o')$. However, for convenience, this partial solution is expressed as:

$$\delta\left[I_2\,\bar{h}\,(\varphi)\right]_2 \equiv \begin{bmatrix}\delta\xi\\ \delta\eta\end{bmatrix}_2 = \begin{bmatrix}1 & 0\\ -\frac{3\varphi}{2} & 1\end{bmatrix}\begin{bmatrix}K_1\\ K_2\end{bmatrix}, \text{ wherein } \begin{cases}K_1 \equiv 2(2\xi_o + \eta_o')\\ K_2 \equiv (\eta_o - 2\xi_o')\end{cases} . \tag{II.45a}$$

The parametric equation for this trace is readily determined as:

$$\left[\delta_2\eta\right] - K_2 = -\frac{3\varphi}{2}\left[\delta_2\xi\right]. \tag{II.45b}$$

The equation describes a line on the (ξ, η) plane which is parallel to the η-axis. Note that the line originates from $\delta\eta(0) = K_2$ and that the expression is <u>linear</u> in the independent variable $\varphi\ (\equiv \dot{\varphi}t)$.

*This is a familiar result for a relative motion described on this plane; see Reference [1]. Also, note the direction of motion about this trace geometry.

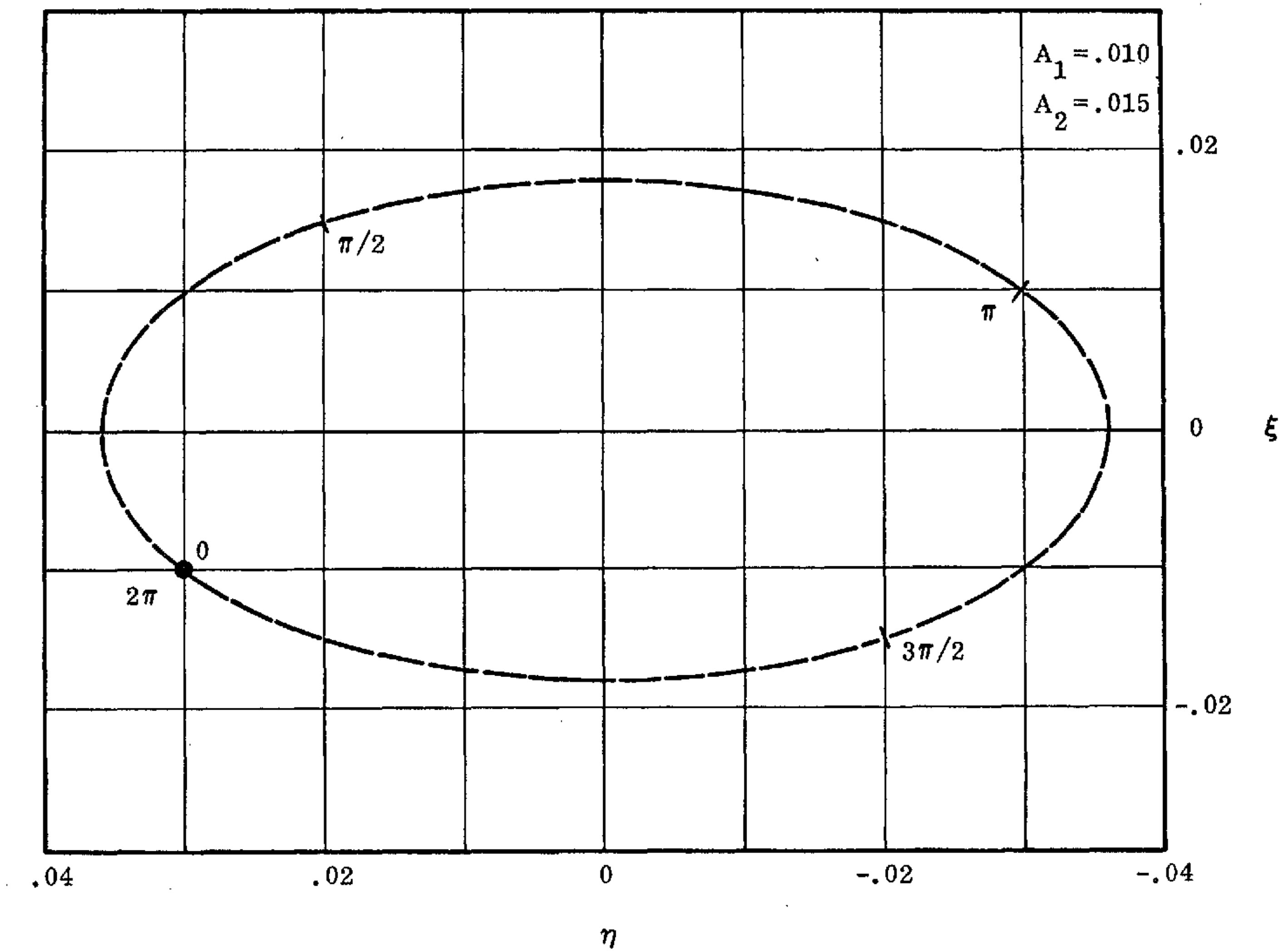

FIG. II.4. Sketch showing an in-plane trace produced by the coefficient, A_a. The figure is described in the local, rotating frame of reference. Note that the trace's initial point (0), and selected angle (φ) positions as indicated. (Refer to Eqs. (II .42) for a mathematical description of this figure).

The secular influence for the in-plane trace is eliminated when K_1 vanishes. This condition could be achieved with either of the starting conditions: $\xi_o = \eta_o' = 0$, or $\eta_o' = -2\xi_o$. (Such constraints on the initial state parameters may occur in either an "intentional" or "chance" situation).

The third component vector is expressed in terms of the applied force $(\bar{\tau})$, where $\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta)$.

The trace for this part of the solution is obtained from:

$$\delta\left[I_2 \bar{\lambda}(\varphi)\right]_3 \equiv \begin{bmatrix} \delta\xi \\ \delta\eta \end{bmatrix}_3 = \begin{bmatrix} 1 & (2\varphi) \\ (-2\varphi) & 4-\frac{3}{2}\varphi^2 \end{bmatrix} \begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} . \qquad \text{(II.46a)}$$

A descriptive equation for the geometry is:

$$\delta_3\eta - 4\tau_\eta\left[1 + \frac{2}{3}\left(\frac{\tau_\xi}{2\tau_\eta}\right)^2\right] = -\frac{3}{8\tau_\eta}\left[\delta_3\xi + \frac{\tau_\xi}{3}\right]^2 ; \qquad \text{(II.46b)}$$

which describes a parabola on the (ξ, η) plane; but, with its apex not at the origin. The figure opens in the negative η-direction*.

The full in-plane trace for this problem is constructed by adding the three partial solutions above. It is apparent that the general figure would be an "ellipse" which moves away from the origin at a "linear-plus-parabolic rate".

Instead of examining these various figures in detail, now, a physically more realistic study of this problem will be undertaken. This considers, as separate entities, those influences which arise as a consequence of the initial values $(\bar{\lambda}_o, \bar{\lambda}_o')$, and those due to the applied force system $(\bar{\tau})$. The individual effects are described below.

The influence of initial values $(\bar{\lambda}_o, \bar{\lambda}_o')$ on the in-plane trace is obtained from A_a and K_o, collectively. In matrix format this partial solution is:

*This description assumes all $\bar{\tau}$-components are positive valued.

$$\Delta\left[I_2\,\bar{\lambda}\,(\varphi)\right]_1 \equiv \begin{bmatrix}\Delta\xi\\ \Delta\eta\end{bmatrix}_1 = \begin{bmatrix}-\cos\varphi & \sin\varphi\\ 2\sin\varphi & 2\cos\varphi\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ -\frac{3\varphi}{2} & 1\end{bmatrix}\begin{bmatrix}K_1\\ K_2\end{bmatrix} \quad ; \qquad \text{(II.47)}$$

wherein, now,

$$A_1 \equiv (3\xi_o + 2\eta_o') = K_1 - \xi_o; \quad A_2 = \xi_o' ;$$

and

$$K_1 \equiv 2(2\xi_o + \eta_o'); \qquad K_2 = \eta_o - 2\xi_o' ,$$

as before.

The geometric figure describing the equation above is a "wandering ellipse"; one whose center is continually moving from the coordinate origin*. (This is typical of initial value problems; see Reference [1]). A quadric equation for the trace is:

$$\frac{(\Delta_1\xi - K_1)^2}{\left(\sqrt{A_1^2 + A_2^2}\right)} + \frac{[(\Delta_1\eta - K_2) + \frac{3}{2}K_1\varphi]^2}{\left(2\sqrt{A_1^2 + A_2^2}\right)^2} = 1. \qquad \text{(II.48a)}$$

Note that when the secular term is eliminated (i.e., $K_1 \equiv 0$) the figure becomes a closed locus. It is the (2:1) ellipse with its center not at the coordinate origin. An appropriate cartesian equation for this figure is:

$$\frac{(\Delta_1\xi)^2}{(\xi_o^2 + \xi_o'^2)} + \frac{(\Delta_1\eta - K_2)^2}{4(\xi_o^2 + \xi_o'^2)} = 1. \qquad \text{(II.48b)**}$$

(See Fig. II.5 for the traces).

The effects of an applied specific force (τ) produces an in-plane trace all its own. For this study the initial state values are zeroed ($\bar{\lambda}_o = \bar{\lambda}_o' \equiv 0$); consequently, this partial solution is acquired from terms in A_a and from the $\bar{\tau}$ solution noted earlier. The equation for this case is found to be:

*A more definitive name for this figure would be cycloid.

**Due to the linearization which has been imposed for these solutions, the closed figures have a periodicity which matches that of the circular (base) orbit. It must be remembered that this is an approximation to the more realistic case.

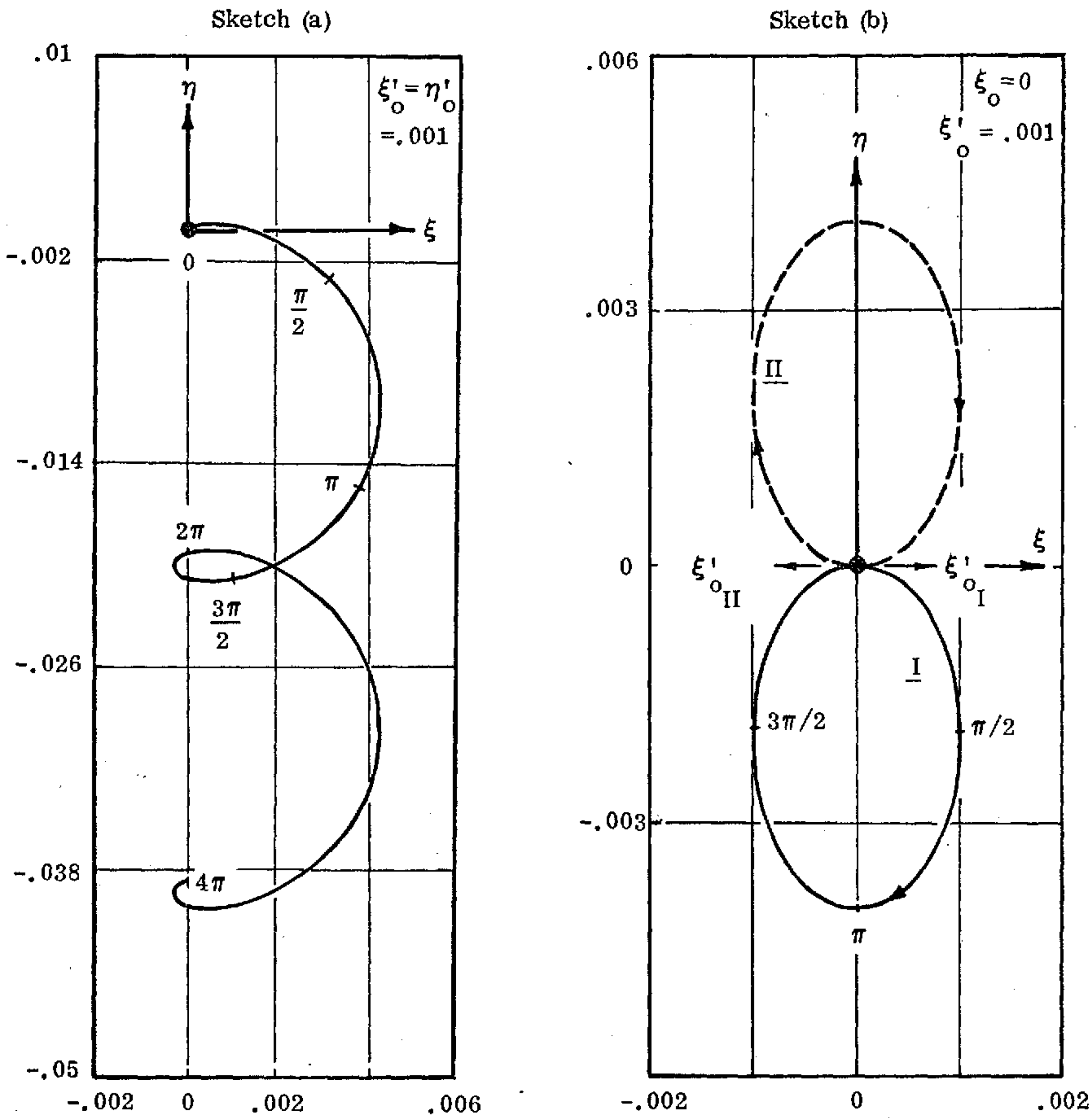

FIG. II.5. Graphs to illustrate the influence of initial values on the in-plane relative motion displacements, as referred to a rotating frame of reference. Sketch (a) describes the more general case (see Eqs. (II.48a)), wherein both ξ_o' and η_o' are present. Sketch (b) illustrates the consequence of removing the secular influence (i.e., $\eta_o' \equiv 0$). The two ellipses (sketch (b)) describe the influence of "$+\xi_o'$" (curve I) and a "$-\xi_o'$" (curve II) on the motion traces. Note: All curves here originate from the origin ($\mathcal{R}_o = 0$).

$$\Delta\left[I_2 \bar{\imath}(\varphi)\right]_2 \equiv \begin{bmatrix} \Delta\xi \\ \Delta\eta \end{bmatrix}_2 = \begin{bmatrix} -\cos\varphi & \sin\varphi \\ 2\sin\varphi & 2\cos\varphi \end{bmatrix} \begin{bmatrix} \tau_\xi \\ -2\tau_\eta \end{bmatrix} + \begin{bmatrix} 1 & (2\varphi) \\ (-2\varphi) & (4-\frac{3}{2}\varphi^2) \end{bmatrix} \begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} . \quad \text{(II.49)}$$

This expression suggests: (1), a (2:1) ellipse; couplied with (2), the parabolic figure found for the $(\Psi_\tau \bar{\tau})$-terms. (Note that this geometry is not a closed figure; and that there are multiple secular influences present).

It is seen here that when $\tau_\eta = 0$, the degree of divergence diminishes. This constraint leads to a cycloid for the motion's trace; or:

$$\Delta_2\xi = \tau_\xi (1 - \cos\varphi),$$

and (II.50a)

$$\frac{\Delta_2\eta}{2} = -\tau_\xi (\varphi - \sin\varphi).$$

On the other hand, when τ_ξ vanishes the figure is akin to both the cycloid and a parabola. The describing equations for this case are:

$$\frac{\Delta_2\xi}{2} = \tau_\eta (\varphi - \sin\varphi),$$

and (II.50b)

$$\frac{1}{4}\left(\Delta_2\eta + \frac{3\varphi^2}{2}\tau_\eta\right) = \tau_\eta (1 - \cos\varphi).$$

(A general quadric for this particular in-plane trace is:

$$\frac{(\Delta_2\xi)^2}{\tau_\xi^2 + 4\tau_\eta^2} + \frac{(\Delta_2\eta + \frac{3}{2}\tau_\eta\varphi^2)^2}{4(\tau_\xi^2 + 4\tau_\eta^2)} = [(1-\cos\varphi)^2 + (\varphi - \sin\varphi)^2]) . \quad \text{(II.50c)*}$$

(b) In-Plane Displacements; Referred to an Inertial Triad. In this section the motion's displacement traces are referred to an inertially-oriented triad.

The general analytical expression pertinent to this part of the study has appeared previously as Eq. (II.35); its in-plane part is:

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*Geometrically this may be thought of as a "wandering ellipse", but, one which has a time dependent coefficient, as shown on the right.

$$I_2\bar{\mathfrak{R}}(\varphi) = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right]A_{a_I} + T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}(B_2J_1)\varphi\right]K_{o_I} + \Psi_\tau \bar{\tau}\right\}. \qquad \text{(II.51)}$$

It should be noted that here the displacement vector may be separated into three component vectors, also; one for each of the coefficients: A_{a_I}, K_{o_I} and $\bar{\tau}$. Each of the components is described below so that the reader can become acquainted with how each adds to the overall geometry. For reference purposes the coefficient vectors (A_{a_I}, K_{o_I} and $\bar{\tau}$) are repeated below. Thus:

$$A_{a_I} \equiv \frac{1}{2}\left[I_2\bar{\mathfrak{R}}_o + (J_2 - 2J_1)B_2\bar{\mathfrak{R}}'_o + (J_1 - 2J_2)\bar{\tau}\right],$$

$$K_{o_I} \equiv (2J_1 - J_2)\bar{\mathfrak{R}}_o - 2B_2\bar{\mathfrak{R}}'_o, \qquad \text{(II.52a)}$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$

On close examination one finds that the A_a and K_o quantities are not precisely those noted above. This is because the initial state parameters have been transformed into a form compatible with the present reference triad. However, the force vector ($\bar{\tau}$) (and its constant scalar components) are unchanged since these are not "rotated" to the new frame of reference.

The first of the vector components is expressed in terms of A_{a_I} ($\equiv A_{a_I}(A_1, A_2)$; with:

$$A_1 \equiv \frac{1}{2}\left[\Xi_o + 2H'_o + \tau_\xi\right], \text{ and } A_2 \equiv \frac{1}{2}\left[H_o + \Xi'_o - 2\tau_\eta\right].\ ) \qquad \text{(II.52b)}$$

This particular partial solution is given by:

$$\delta\left[I_2\bar{\mathfrak{R}}(\varphi)\right]_1 = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right]A_{a_I};$$

and, may be set down as the matrix equation:

$$\delta\left[I_2\bar{\mathfrak{R}}(\varphi)\right]_1 \equiv \begin{bmatrix}\delta\Xi \\ \delta H\end{bmatrix}_1 = \left\{\begin{bmatrix}-3 & 0 \\ 0 & 3\end{bmatrix} + \begin{bmatrix}\cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi\end{bmatrix}\right\}\begin{bmatrix}A_1 \\ A_2\end{bmatrix}. \qquad \text{(II.53)}$$

The present displacement trace, on the (Ξ, H) plane, is seen to be composed of: (1), a constant vector, $3(J_2 - J_1)A_{a_I}$; plus (2), a circular locus of radius $\left(\sqrt{A_1^2 + A_2^2}\right)$. The full figure is a circle of double orbit frequency $(2\dot{\varphi})$ with its center located at $(\Xi, H)_c \equiv (-3A_1, 3A_2)$. Analytically, the figure can be described by:

$$(\delta_1\Xi + 3A_1)^2 + (\delta_1 H - 3A_2)^2 = (A_1^2 + A_2^2). \qquad \text{(II.54)}$$

This particular geometry is analogous to the ellipse found for the (ξ, η)-plane, a trace which evolved as a consequence of the coefficient, A_a.

The second contribution is acquired from the constant, K_{o_I}. The corresponding partial solution is:

$$\delta\left[I_2\bar{R}(\varphi)\right]_2 \equiv T_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi\,(B_2J_1)\right]K_{o_I};$$

or, as a matrix equation:

$$\delta\left[I_2\bar{R}(\varphi)\right]_2 \equiv \begin{bmatrix}\delta\Xi\\ \delta H\end{bmatrix}_2 = \left\{\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix} + \frac{3\varphi}{2}\begin{bmatrix}\sin\varphi & 0\\ -\cos\varphi & 0\end{bmatrix}\right\}\begin{bmatrix}K_1\\ K_2\end{bmatrix}. \qquad \text{(II.55)}$$

The scalars K_i (above) are defined from:

$$K_{o_I} \equiv K_{o_I}(K_1, K_2),$$

where, in terms of initial values:

$$K_1 \equiv 2(\Xi_o + H_o'), \text{ and } K_2 \equiv -(H_o + 2\Xi_o').$$

An analytical parametric equation for this trace is:

$$(\delta_2\Xi)^2 + (\delta_2 H)^2 = K_1^2 + \left(\frac{3\varphi}{2}K_1 - K_2\right)^2. \qquad \text{(II.56)}$$

Geometrically the figure is a spiral originating at a locus away from the coordinate origin.

On the (ξ, η)-plane the trace due to K_o was a line parallel to the ξ-axis. Here, one could show that the present trace is merely a transformation of the uniformly moving point (on the (ξ, η)-plane) into the spiral (on the (Ξ, H)-plane).

Once more it is seen that a secular influence developes from one component of K_o, namely K_1. When this quantity vanishes the resulting geometry is a closed figure.

The applied force system, $\bar{\tau}$ $(\equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta))$ contributes to the partial trace solutions as follows:

$$\delta\left[I_2\bar{R}(\varphi)\right]_3 \equiv T_2(\varphi^-)\ \Psi_\tau \bar{\tau}. \tag{II.57a}$$

This expression is recognized to be (simply) a transformation of the earlier solution (see Eq. (II.46a)). In a matrix format the expression is written as:

$$\delta\left[I_2\bar{R}(\varphi)\right]_3 \equiv \begin{bmatrix}\delta\Xi\\ \delta H\end{bmatrix}_3 = \left\{\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}1 & 2\varphi\\ -2\varphi & (4-\frac{3\varphi^2}{2})\end{bmatrix}\right\}\begin{bmatrix}\tau_\xi\\ \tau_\eta\end{bmatrix}, \tag{II.57b}$$

which can be expanded to:

$$\delta_3\Xi(\varphi) = \tau_\xi\left[\cos\varphi + 2\varphi\sin\varphi\right] + \tau_\eta\left[2\varphi\cos\varphi - (4-\frac{3\varphi^2}{2})\sin\varphi\right],$$

and

$$\delta_3 H(\varphi) = \tau_\xi\left[\sin\varphi - 2\varphi\cos\varphi\right] + \tau_\eta\left[2\varphi\sin\varphi + (4-\frac{3\varphi^2}{2})\cos\varphi\right]. \tag{II.57c}$$

Apparently the specific force $(\bar{\tau})$ leads to a divergent (or secular) condition which cannot be circumvented without removing the force, per se.

A quadric expression for the trace is:

$$(\delta_3\Xi)^2 + (\delta_3 H)^2 = \left[\tau_\xi + 2\varphi\tau_\eta\right]^2 + \left[2\varphi\tau_\xi - (4-\frac{3\varphi^2}{2})\,\tau_\eta\right]^2. \tag{II.58}$$

This expression describes the spiral-like influence typical to these figures. Actually each force component, alone, leads directly to a spiral (form) as its contribution to the overall geometry.

The general traces produced here, as in-plane displacement geometries, are a composite of the figures generated from the partial solutions above. Thus the curve found on the (Ξ, H) plane would be a form of spiral, generally, but one incorporating the circle described through A_{a_I}.

These vector descriptions provide a convenient, mathematical means to represent this solution; however, this is not a scheme which is always consistent with the physics of a problem. For instance, to show the influence of initial values on this solution, one should include influences from both A_{a_I} and K_{o_I}. To clarify this situation, and that arising from the specific force ($\bar{\tau}$), the next discussions are included.

The effect of initial state ($\bar{\Re}_o$, $\bar{\Re}'_o$), alone, on this solution is obtained by setting $\bar{\tau} = \bar{0}$ in the analytical results. For this case a partial solution would appear as:

$$\Delta\left[I_2\bar{\Re}(\varphi)\right]_1 = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right]\left[A_{a_I}\right]_{i.v.} + T_2(\varphi^-)\left[I_2 - \frac{3}{2}(B_2J_1)\varphi\right]K_{o_I}, \qquad \text{(II. 59a)}$$

wherein $\left[A_{a_I}\right] \equiv A_{a_I}(A_1, A_2)$, but with:

$$A_1 \equiv \frac{1}{2}\left[\Xi_o + 2H'_o\right], \quad A_2 \equiv \frac{1}{2}\left[H_o + \Xi'_o\right].$$

As before, it is assumed that

$$K_{o_I} \equiv K_{o_I}(K_1, K_2), \text{ (see Eqs. (II .55)).}$$

In matrix form Eq. (II .59a) is:

$$\Delta\left[I_2\bar{\Re}(\varphi)\right]_1 \equiv \begin{bmatrix}\Delta\Xi \\ \Delta H\end{bmatrix}_1 = \begin{bmatrix}-3 + \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & 3 + \cos 2\varphi\end{bmatrix}\begin{bmatrix}A_1 \\ A_2\end{bmatrix}$$

$$+ \begin{bmatrix}\cos\varphi - \frac{3\varphi}{2}\sin\varphi & -\sin\varphi \\ \sin\varphi - \frac{3\varphi}{2}\cos\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}K_1 \\ K_2\end{bmatrix}. \qquad \text{(II. 59b)}$$

This expression suggest a geometry composed of: (1), a circle, displaced from the origin, due to A_{a_I}; plus (2), a spiral, from the K_{o_I} term. Here, again, the figure's divergence develops as a consequence of the scalar, K_1. Therefore, if that term is removed the trace becomes a closed curve with a period matching that of the circular base orbit.

The non-secular trace geometry, defined for $K_1 = 0$, is described by:

$$\Delta\left[I_2 \Re(\varphi)\right]_1 \equiv \begin{bmatrix} \Delta\Xi(\varphi) \\ \Delta H(\varphi) \end{bmatrix}_1 = \frac{1}{2}\begin{bmatrix} -3+\cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & 3+\cos 2\varphi \end{bmatrix}\begin{bmatrix} H'_o \\ H_o + \Xi'_o \end{bmatrix}$$

$$+ \begin{bmatrix} \cos\varphi - \frac{3\varphi}{2}\sin\varphi & -\sin\varphi \\ \sin\varphi - \frac{3\varphi}{2}\cos\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} 0 \\ K_2 \end{bmatrix} \quad . \qquad \text{(II.59c)}$$

Generally speaking, this equation describes a limacon.

A special case of the limacon arises when $H'_o \equiv 0$. For this condition, added to $K_1 = 0$, a symmetric cardioid results; one which is symmetric about the H-axis.

The in-plane displacements associated with the applied force ($\bar{\tau}$), are described in terms of A_{a_I} and the transformed vector, $T_2(\varphi^-)\ \Psi_\tau \bar{\tau}$. The matrix expression for this partial solution is:

$$\Delta\left[I_2 \bar{\Re}(\varphi)\right]_2 \equiv \begin{bmatrix} \Delta\Xi(\varphi) \\ \Delta H(\varphi) \end{bmatrix}_2 = \frac{1}{2}\begin{bmatrix} -3+\cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & 3+\cos 2\varphi \end{bmatrix}\begin{bmatrix} \tau_\xi \\ -2\tau_\eta \end{bmatrix}$$

$$+ \begin{bmatrix} \cos\varphi + 2\varphi\sin\varphi & 2\varphi\cos\varphi - (4 - \frac{3\varphi^2}{2})\sin\varphi \\ \sin\varphi - 2\varphi\cos\varphi & 2\varphi\sin\varphi + (4 - \frac{3\varphi^2}{2})\cos\varphi \end{bmatrix}\begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} \quad . \qquad \text{(II.60)}$$

Here the first matrix describes a displaced circle (of double orbit frequency, and radius $\equiv \sqrt{\tau_\xi^2 + 4\tau_\eta^2}$). The second term represents a spiral-like figure; thus, the combination is a spiral-like trace typical of inertial, in-plane displacements.

This is the last description for in-plane traces referred to the two reference triads. The next section describes some of the companion hodographs, for the situations noted above. Since these discussions are restricted to the in-plane cases, all considerations of out-of-plane traces are deferred until later.

(c) <u>In-Plane Hodograph; Rotating Frame.</u> The hodographs, or velocity diagrams, for this relative motion problem are obtained from Eq. (II.29). The in-plane portion of that expression is:

$$I_2 \bar{\hbar}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] B_2 T_2(\varphi^-) A_a - \frac{3}{2}(B_2 J_1) K_o + \Psi_\tau' \bar{\tau}, \tag{II.61a}$$

where the constant parameters A_a, K_o and $\bar{\tau}$ have been set down in Eqs. (II.43) and (II.45a).

The geometric figures to be described here are obtained by a procedure analogous to that used for the displacements. That is, selected situations are described to show how the various parameters affect the motion and its traces.

A first contribution to be examined here is that due to the quantity A_a. Recalling that this term is acquired from the trigonometric parts of the displacement solution, then making use of the same definitions as before one finds the partial solution:

$$\delta\left[I_2 \bar{\hbar}'(\varphi)\right]_1 = \begin{bmatrix} \delta\xi' \\ \delta\eta' \end{bmatrix}_1 = \begin{bmatrix} \sin\varphi & \cos\varphi \\ 2\cos\varphi & -2\sin\varphi \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} . \tag{II.61b}$$

Here, also,

$$A_a \equiv A_a(A_1, A_2).$$

It is seen that this expression describes a (2:1) ellipse centered at the coordinate origin. A cartesian equation for that figure is:

$$\frac{(\delta_1 \xi')^2}{\left(\sqrt{A_1^2 + A_2^2}\right)^2} + \frac{(\delta_1 \eta')^2}{\left(2\sqrt{A_1^2 + A_2^2}\right)^2} = 1. \tag{II.62}$$

(This is an expected result also; its equivalent has been obtained and discussed in Ref. [1]).

The second component vector equation is, from Eq. (II.47):

$$\delta\left[I_2 \bar{\lambda}'(\varphi)\right]_2 \equiv -\frac{3}{2}(B_2 J_1) K_o = \begin{bmatrix} \delta\xi' \\ \delta\eta' \end{bmatrix}_2 = \begin{bmatrix} 0 & 0 \\ -\frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} 2(2\xi_o + \eta'_o) \\ \eta_o - 2\xi'_o \end{bmatrix} . \tag{II.63a}$$

This plots as a single point on the (ξ', η') hodograph plane. Obviously, this expression defines the motion rate associated with Eq. (II.45a).

Next, the influence of $\bar{\tau}$ on these hodographs is found from:

$$\delta\left[I_2 \bar{\lambda}'(\varphi)\right]_3 = \Psi'_\tau \bar{\tau} \equiv \begin{bmatrix} \delta\xi' \\ \delta\eta' \end{bmatrix}_3 = \begin{bmatrix} 0 & 2 \\ -2 & -3\varphi \end{bmatrix} \begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} . \tag{II.64a}$$

Here the trace is a line parallel to the η'-axis; it represents a point moving, continually, in a specified (±) direction. An equation for this line is easily found as:

$$\frac{\delta_3 \eta'}{2} + \tau_\xi = -\left(\frac{3\varphi}{4}\right) \delta_3 \xi'. \tag{II.64b}$$

In summary, these three influences produce a trace which is composed from an ellipse moving away from the origin (in an η'-direction), and which has an offset central point. The offset is due to the combined influence of K_o and $\bar{\tau}$.

As noted earlier these component expressions, taken independently, do not define a physically real situation. In order to clarify this and to illustrate a more realistic problem the equations are reformulated as shown below.

The contribution of a prescribed initial state to the relative motion hodograph can be expressed as:

$$\Delta\left[I_2 \bar{\mathcal{R}}'(\varphi)\right]_1 \equiv \begin{bmatrix} \Delta\xi' \\ \Delta\eta' \end{bmatrix}_1 = \begin{bmatrix} \sin\varphi & \cos\varphi \\ 2\cos\varphi & -2\sin\varphi \end{bmatrix} \begin{bmatrix} 3\xi_o + 2\eta'_o \\ \xi'_o \end{bmatrix} + \begin{bmatrix} 0 \\ -3(2\xi_o + \eta'_o) \end{bmatrix}. \quad \text{(II.65)}$$

Examining this result the figure is seen to be a (2:1) ellipse not centered at the coordinate origin. If the secular factor is removed (i.e. $K_1 \equiv 0$) the hodograph trace reduces to the (2:1) ellipse,

$$\frac{(\Delta_1\xi')^2}{\left(\sqrt{\xi_o^2 + \xi'_o}\right)^2} + \frac{(\Delta_1\eta')^2}{\left(2\sqrt{\xi_o^2 + \xi'_o}\right)^2} = 1. \quad \text{(II.66)}$$

(Necessarily, the quantities describing this ellipse may be varied according to how K_1 is made to go to zero).

Finally, to determine the influence of $\bar{\tau}$ alone, on the hodograph, consider the matrix equation:

$$\Delta\left[I_2 \bar{\mathcal{R}}'(\varphi)\right]_2 \equiv \begin{bmatrix} \Delta\xi' \\ \Delta\eta' \end{bmatrix}_2 = \begin{bmatrix} \sin\varphi & \cos\varphi \\ 2\cos\varphi & -2\sin\varphi \end{bmatrix} \begin{bmatrix} \tau_\xi \\ -2\tau_\eta \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & -3\varphi \end{bmatrix} \begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix}. \quad \text{(II.67a)}$$

Here, the first matrix is obtained from A_a, while the second comes directly from terms in $(\Psi'_\tau \bar{\tau})$. The geometry associated with the first part of this expression is, again, a (2:1) ellipse while that defined from the second is a line locus paralleling the η'-axis. Consequently, the composite geometry would appear as a "wandering ellipse", or cycloidal curve. An analytical expression for the trace is:

$$\tau_\eta(\Delta_2\eta') - \tau_\xi(\Delta_2\xi') = (4\tau_\eta^2 - \tau_\xi^2)\sin\varphi - (3\tau_\eta^2)\varphi . \qquad \text{(II.67b)}$$

Physically, one should recognize that each of the $\bar{\tau}$ scalars affects the hodograph differently. For instance, suppose that $\tau_\xi \equiv 0$; then,

$$\Delta_2\xi' - 2\tau_\eta = -2\tau_\eta\cos\varphi,$$

and (II.67c)

$$\Delta_2\eta' + 3\varphi\tau_\eta = 4\tau_\eta\sin\varphi .$$

These equations describe a regular cycloid on the hodograph plane.

Next, if $\tau_\eta = 0$, the hodograph's trace is found to be the ellipse:

$$\frac{(\Delta_2\xi')^2}{\tau_\xi^2} + \frac{(\Delta_2\eta' + 2\tau_\xi)^2}{(2\tau_\xi)^2} = 1. \qquad \text{(II.67d)}$$

(Sketches of these curves are found on Fig. II.6 below). Note that the secular character of the hodograph vanishes when τ_η is zero.

This completes a discussion on this hodograph plane's geometry. In the next paragraphs a parallel study, for the inertial triad's hodograph, will be presented.

(d) <u>In-Plane Hodograph; Inertial Frame.</u> A general expression for the relative velocity referred to an inertial frame of reference has appeared earlier in Eq. (II.40a).

Since only the in-plane geometry is of interest here, then the applicable equation is:

$$I_2\bar{R}'(\varphi) \equiv \begin{bmatrix} \Xi'(\varphi) \\ H'(\varphi) \end{bmatrix} = B_2\left\{\left[2T_2(2\varphi^-)\right]A_{a_I} + T_2(\varphi^-)\left[\left(I_2 - \frac{3}{2}\varphi B_2 J_1\right)K_{o_I} + \Psi_\tau\bar{\tau}\right]\right\}$$

$$+ T_2(\varphi^-)\left[\Psi_\tau'\bar{\tau} - \frac{3}{2}(B_2 J_1)K_{o_I}\right]. \qquad \text{(II.68)}$$

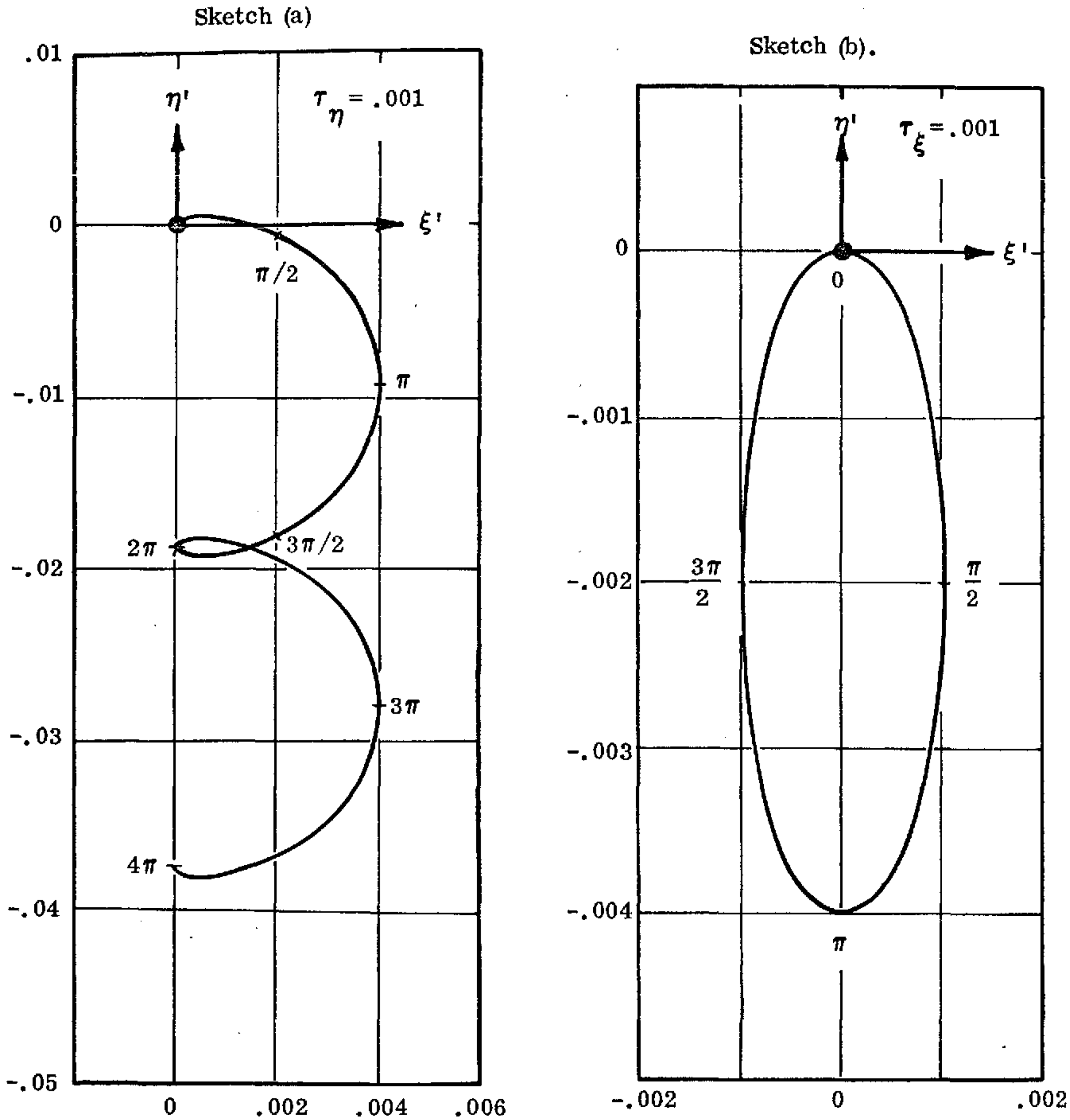


FIG. II.6. Graphs illustrating the effect of $\bar{\tau}$ on the relative motion hodographs. Sketch (a) describes an influence due to τ_η, while sketch (b) shows the consequence of τ_ξ, applied separately (see Eqs. (II .67)). Note: These curves originate from the coordinate origin ($\bar{h}_o' \equiv \bar{0}$).

It should be remembered that A_{a_I} and K_{o_I} are primarily written in the initial, inertial state of the motion $(\bar{\mathcal{R}}_o, \bar{\mathcal{R}}'_o)$, though A_{a_I} does include the applied force. (See Eqs. (II.34b), through (II.37) for descriptions of these and other quantities of interest in this expression).

As a first part to this study consider those contributions made by the three parameters, A_{a_I}, K_{o_I} and $\bar{\tau}$.

The vector A_{a_I} influences the velocity trace in the following manner:

$$\delta\left[I_2\mathcal{R}'(\varphi)\right]_1 \equiv \begin{bmatrix}\delta\Xi' \\ \delta H'\end{bmatrix}_1 = 2B_2\left[T_2(\varphi^-)\right]A_{a_I}$$

$$= 2\begin{bmatrix}-\sin 2\varphi & -\cos 2\varphi \\ \cos 2\varphi & -\sin 2\varphi\end{bmatrix}\begin{bmatrix}A_1 \\ A_2\end{bmatrix} . \qquad \text{(II.69a)}$$

Here, $A_1 \equiv \frac{1}{2}\left[\Xi_o + 2H'_o + \tau_\xi\right]$, and $A_2 \equiv \frac{1}{2}\left[H_o + \Xi'_o - 2\tau_\eta\right]$.

This geometry is recognized as a circle centered at the coordinate origin, but having a motion frequency which is twice that of the base orbit. (The quadric equation for this figure is:

$$(\delta_1\Xi')^2 + (\delta_1 H')^2 = A_1^2 + A_2^2). \qquad \text{(II.69b)}$$

By comparison, the corresponding trace on the (ξ', η') plane was an ellipse, traced over once during each orbit.

The second partial solution for this hodograph is due to the constant, K_{o_I}. With K_{o_I} defined as $K_{o_I}(K_1, K_2)$ then:

$$\delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right]_2 \equiv \begin{bmatrix}\delta\Xi' \\ \delta H'\end{bmatrix}_2 = \left\{\left[B_2T_2(\varphi^-)\right]\left[I_2 - \frac{3}{2}\varphi B_2J_1\right] - \frac{3}{2}T_2(\varphi^-)\left[B_2J_1\right]\right\}K_{o_I},$$

or,

$$\begin{bmatrix} \delta\Xi' \\ \delta H' \end{bmatrix}_2 = \begin{bmatrix} \frac{1}{2}(\sin\varphi + 3\varphi\cos\varphi) & -\cos\varphi \\ \frac{1}{2}(-\cos\varphi + 3\varphi\sin\varphi) & -\sin\varphi \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \quad ; \tag{II.70a}$$

wherein, $K_1 \equiv 2(\Xi_o + H'_o)$ and $K_2 \equiv -(H_o + 2\Xi'_o)$.

The secular terms in this expression produce a trace which is the involute of a circle; however, the trigonometric terms alone describe a circle. When the components of Eq. (II .70a) are joined they lead to the following quadric equation:

$$(\delta_2\Xi')^2 + (\delta_2 H')^2 = \left(\frac{\sqrt{1+9\varphi^2}}{2} K_1\right)^2 + K_2^2 \ . \tag{II.70b}$$

It is noted here that when K_1 vanishes (a condition removing the secular influence for the displacement geometry) the resulting figure is again a circle:

$$(\delta_2\Xi')^2 + (\delta_2 H')^2 = K_2^2 \ . \tag{II.70c}$$

This figure is centered at the origin of coordinates and has a radius = K_2.

For the converse situation, that of having K_2 vanish, the figure reduces to a spiral originating away from the coordinate origin.

The last quantity encountered in the relative velocity vector is the specific force ($\bar{\tau}$). The particular solution describing its influence is:

$$\delta\left[I_2\bar{\mathfrak{R}}'(\varphi)\right]_3 \equiv \begin{bmatrix} \delta\Xi' \\ \delta H' \end{bmatrix}_3 = \left\{\left[B_2 T_2(\varphi^-)\right]\Psi_\tau + T_2(\varphi^-)\Psi'_\tau\right\}\bar{\tau} \ ;$$

or, in an expanded format:

$$\begin{bmatrix} \delta\Xi' \\ \delta H' \end{bmatrix}_3 = \begin{bmatrix} [\sin\varphi + 2\varphi\cos\varphi] & \left[\left(\frac{3\varphi^2}{2} - 2\right)\cos\varphi + \varphi\sin\varphi\right] \\ [-\cos\varphi + 2\varphi\sin\varphi] & \left[\left(\frac{3\varphi^2}{2} - 2\right)\sin\varphi - \varphi\cos\varphi\right] \end{bmatrix} \begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} . \tag{II.71a}$$

An examination of this result quickly shows divergence is present here. (The scalar equations here can be cast into the equivalent form:

$$(\delta_3\Xi')^2 + (\delta_3 H')^2 = (\tau_\xi)^2 - 6(\tau_\xi \tau_\eta)\varphi + [(2\tau_\xi)^2 + 6(\tau_\xi \tau_\eta)\varphi - 5(\tau_\eta)^2]\varphi^2 + \left[\frac{3\varphi^2}{2}\tau_\eta\right]^2. \tag{II.71b}$$

This expression is composed of the force constants plus various ordered secular terms).

Due to the explicit dependence of these equations on φ it is obvious that the resulting geometry is a spiral, also.

Recalling that A_{a_I} and K_{o_I} lead to a circle and a spiral, respectively, on the hodograph plane, it is obvious that the general figure will be spiral-like too.

The next task is that of determining the consequence of a prescribed initial state $(\bar{\mathcal{R}}_o, \bar{\mathcal{R}}'_o)$, and applied specific force $(\bar{\tau})$, on the trace geometry, when these are independently applied. The two situations are examined below.

The consequence of initial values on the hodograph figure is described by specializing Eqs. (II.49). Grouping the state parameters as: $G_1 \equiv (\Xi_o + H'_o)$ and $G_2 \equiv (H_o + \Xi'_o)$, then for present definitions:

$$\Delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right]_1 \equiv \begin{bmatrix}\Delta\Xi' \\ \Delta H'\end{bmatrix}_1 = -\begin{bmatrix}\sin 2\varphi & \cos 2\varphi \\ -\cos 2\varphi & \sin 2\varphi\end{bmatrix}\begin{bmatrix}G_1 + H'_o \\ G_2\end{bmatrix} + \begin{bmatrix}\frac{1}{2}(\sin\varphi + 3\varphi\cos\varphi) & -\cos\varphi \\ \frac{1}{2}(-\cos\varphi + 3\varphi\sin\varphi) & -\sin\varphi\end{bmatrix}\begin{bmatrix}2G_1 \\ -(G_2 + \Xi'_o)\end{bmatrix}. \tag{II.72a}$$

Obviously the first matrix represents the (now familiar) circle of double frequency, while the latter describes a spiral. Once more the resulting trace is largely influenced by the spiral's divergent characteristics.

Since secular effects are suppressed when $G_1 = 0$ (the condition introduced for the displacement case), the corresponding hodograph geometry becomes the closed figure described as:

$$(\Delta_1 \Xi')^2 + (\Delta_1 H')^2 = (H'_o)^2 + 2(G_2 + \Xi'_o)[G_2(1 - \cos\varphi) - H'_o \sin\varphi] . \qquad \text{(II.72b)}$$

This parametric equation defines a limacon.

In addition to the constraint noted above, when $H'_o \equiv 0$ also, an asymmetry for the limacon is removed. Finally if $G_2 \equiv \Xi'_o$ (only) the classic cardioid is found as a hodograph trace. This is noted from a specialization of Eq. (II.72a); i.e., when the conditions indicated above are included, the equations become:

$$\Delta_1 \Xi'(\varphi) = -\Xi'_o \cos 2\varphi + 2\Xi'_o \cos\varphi ,$$

and (II.72c)

$$\Delta_1 H'(\varphi) = -\Xi'_o \sin 2\varphi + 2\Xi'_o \sin\varphi .$$

(These expressions may be combined to yield the quadric:

$$(\Delta_1 \Xi')^2 + (\Delta_1 H')^2 = (2\Xi'_o)^2 \left[\frac{5}{4} - \cos\varphi\right] .) \qquad \text{(II.72d)}$$

Next, the influence of $(\bar{\tau})$ alone on the hodograph's trace is examined. Since components of $\bar{\tau}$ appear in A_{a_I}, the proper partial solution here is:

$$\Delta\left[I_2 \bar{\mathfrak{R}}'(\varphi)\right]_2 \equiv \begin{bmatrix} \Delta\Xi' \\ \Delta H' \end{bmatrix}_2$$

$$= \begin{bmatrix} -\sin 2\varphi + \sin\varphi + 2\varphi\cos\varphi & 2\cos 2\varphi + \left(\frac{3\varphi^2}{2} - 2\right)\cos\varphi + \varphi\sin\varphi \\ \cos 2\varphi - \cos\varphi + 2\varphi\sin\varphi & 2\sin 2\varphi + \left(\frac{3\varphi^2}{2} - 2\right)\sin\varphi - \varphi\cos\varphi \end{bmatrix} \begin{bmatrix} \tau_\xi \\ \tau_\eta \end{bmatrix} . \qquad \text{(II.73)}$$

Once again the explicit presence of φ, seen here, eliminates the possibility of acquiring a non-divergent trace. In fact the figure for this hodograph is akin to a spiral; however, this one originates at the coordinate origin; and, it does have a monotonically increasing radius.

<u>Summary</u>. Trace geometries for the initial values problem and the zero-initial values problem have been examined briefly in the paragraphs above. There

displacement and hodograph loci were described as they would appear on representative planes in the local rotating and inertially aligned frames of reference.

One limiting constraint imposed on these solutions was that of a constant disturbance vector ($\bar{\tau}$). This vector has fixed components parallel to the coordinate axes of the rotating triad. In order to examine an analogous case, but one with fixed components parallel to the inertial triad's directions, a second problem is formulated and solved next. In that solution one finds that only the disturbance itself contributes new information about the motion traces. Hence, an examination of these trace geometries will be a much simpler task. Therefore, in the next section a formulation and solution of the problem will be carried out first; then the trace geometries will be described, and discussed briefly.

Descriptions in the foregoing paragraphs have been brief, and the illustrations there were limited in number. Recognizing that interested readers would like to see typical graphs for each of these various contributions, a compendium of results has been prepared. This catalogue of traces, for the various in-plane cases, is attached below. What one will find there is a set of typical traces; each one obtained for positive valued parameters (components, etc. from the vectors A_a, K, and $\bar{\tau}$). In most cases the matrix equations (only) are noted; however, the corresponding scalar expressions are easily developed from these; or, they may be found by looking into the volume describing trace constructions*.

Before leaving this section it should be mentioned that a similar collection of trace data (a compendium) is found behind the next section also. There, data are presented for the solution involving the specific force ($\bar{\tau}_I$). Also, to complete the catalogue one will find another data collection for the out-of-plane traces. This is found attached to the other descriptive materials.

*Interim Report, "Construction of Relative Motion Traces", by J.B. Eades, Jr., AMA Report No. 73-39.

V. DATA SUMMARY

A Compendium of Data for In-Plane Trace Geometries. The collected data, which follows, illustrates in-plane motion trace geometries typical of those found in both frames of reference. These curves describe both the initial-values problem and the zero-initial values case where the disturbance is due to $\bar{\tau}(\equiv\bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta))$.

Trace geometries appear in the following order:

(1). Displacements for the Rotating Frame of Reference

(2). Displacements for the Inertially oriented Reference Frame

(3). Hodographs referred to the Rotating Frame

(4). A Non-Secular Case, in the Moving Frame Notation

(5). Hodographs referred to the Inertial Frame

(6). A Non-Secular Case, in the Inertially oriented Frame

For these various representations, the individual mathematical components are shown first (Cases 1, 2, 3). Next, the initial-values problem and the zero-initial values case (due to $\bar{\tau}$) are shown. These more realistic cases are numbered 4 and 5. The last situation, 6, illustrates the full trace geometry -- that due to both the initial values and the disturbance, $\bar{\tau}$.

Following the hodograph sections the reader will find a case study illustrating a non-secular condition for each of the reference frames.

It should be remembered, throughout, that all of these geometries result from an arbitrary selection of initial value parameters $(\xi, \eta, \zeta)_o$ and $(\xi', \eta', \zeta')_o$ and disturbance components $(\tau_\xi, \tau_\eta, \tau_\zeta)$. In all instances these quantities were chosen as positive valued constants.

In-Plane Displacements, referred to the Rotating Frame of Reference

In-Plane Displacement Diagram; Local, Rotating Frame of Reference.

The describing equation:

$$I_2 \bar{k}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) A_a + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Psi_\tau \bar{\tau}.$$

Partial Solutions.

1. The construction due to A_a (the "Trigonometric" Solution Part).

The expression for this partial solution is:

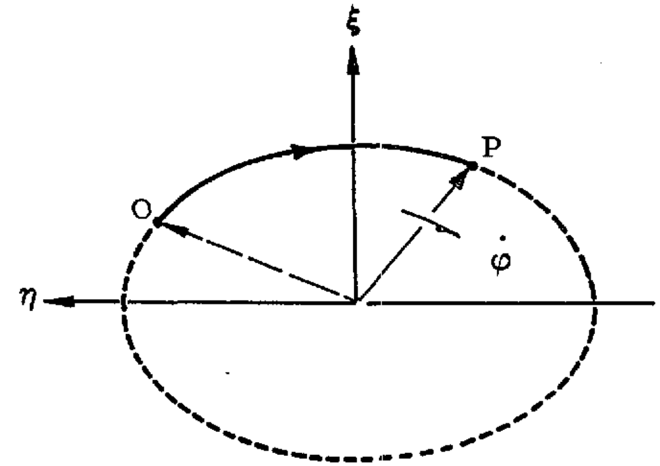
$$\delta\left[I_2 \bar{k}(\varphi)\right] = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) A_a = 2\left[2J_2 - J_1\right] T_2(\varphi^-) A_a;$$

wherein

$$A_a \equiv \frac{1}{2}\left[3J_1 \bar{k}_o + (J_2 - 2J_1) B_2 \bar{k}'_o + (J_1 - 2J_2)\bar{\tau}\right],$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$



1. This trace geometry is an Ellipse (2:1) centered at coordinate origin. Rate of motion over this trace is $\dot{\varphi}$.

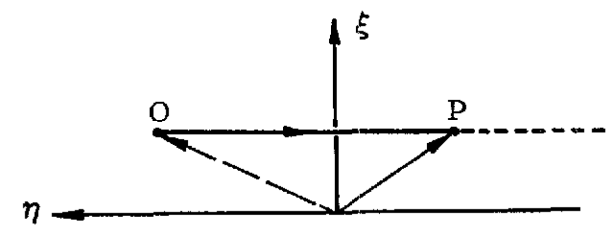
2. Construction due to the K_o-term (the Constant Initial Values).

This partial solution is defined by:

$$\delta\left[I_2 \bar{k}(\varphi)\right] = \left[I_2 - \frac{3}{2}\varphi (B_2 J_1)\right] K_o,$$

wherein

$$K_o \equiv (4J_1 + J_2)\bar{k}_o - 2B_2 \bar{k}'_o.$$



2. This trace is a straight line, parallel to the η-axis.

---

*O describes the $\varphi = 0$ locus;
P is a general point on the trace.

In-Plane Displacement Diagram (continued)

3. The construction due to $\Psi_\tau \bar{\tau}$ (an influence due to "Applied Thrust" alone).

This partial solution is obtained from:

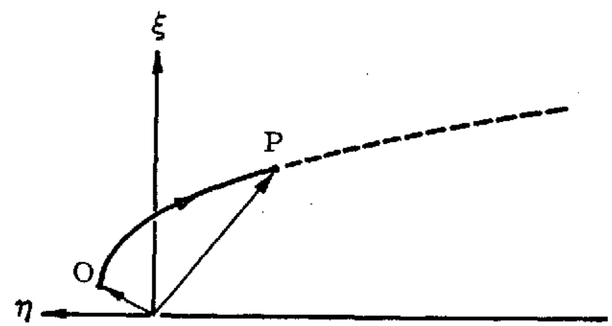
$$\delta\left[I_2 \bar{\lambda}(\varphi)\right] = \Psi_\tau \bar{\tau},$$

wherein

$$\Psi_\tau \equiv (J_1 + 4J_2) - (2B_2 + \frac{3}{2}\varphi J_2)\varphi,$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$



3. This geometry shows a parabolic trace on the (ξ, η) plane of motion.

4. The Initial Values Solution ($\bar{\tau} \equiv \bar{0}$); described by:

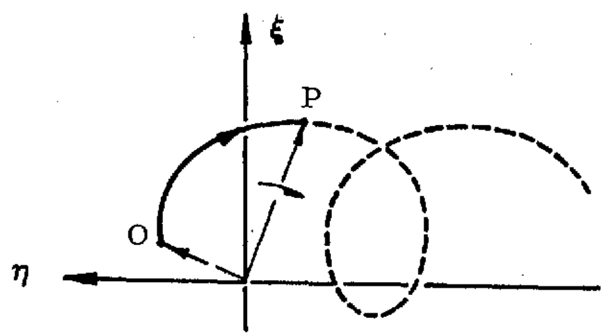
$$\Delta\left[I_2 \bar{\lambda}(\varphi)\right] = \left[I_2 + 3(J_2 - J_1)\right](A_a)_{i.v.} + \left[I_2 - \frac{3}{2} B_2 J_1 \varphi\right] K_o,$$

wherein

$$(A_a)_{i.v.} \equiv \frac{1}{2}\left[3J_1 \bar{\lambda}_o + (J_2 - 2J_1) B_2 \bar{\lambda}'_o\right]$$

and

$$K_o \equiv (4J_1 + J_2)\bar{\lambda}_o - 2B_2 \bar{\lambda}'_o .$$



4. This trace is a combination of 1. and 2. above.

*O is the initial trace locus; P is a general point on the curve.

In-Plane Displacement Diagram (continued)

5. The Applied Force Solution (with $\bar{\hbar}_o$, $\bar{\hbar}'_o$, $\equiv \bar{0}$) is obtained from:

$$\Delta\left[I_2\bar{\hbar}(\varphi)\right] = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-)(A_a)_\tau + \Psi_\tau \bar{\tau};$$

wherein

$$(A_a)_\tau \equiv \frac{1}{2}\left[J_1 - 2J_2\right]\bar{\tau},$$

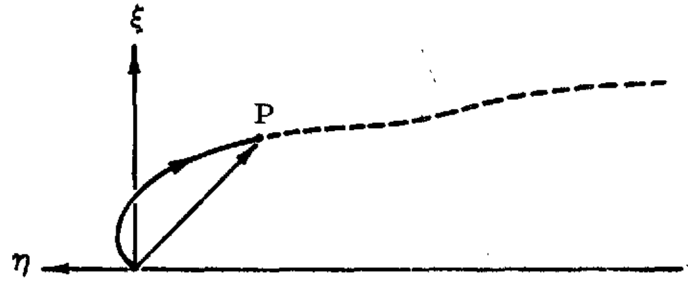
$$\Psi_\tau \equiv (J_1 + 4J_2) - \left[2B_2 + \frac{3}{2}\varphi J_2\right]\varphi,$$

and

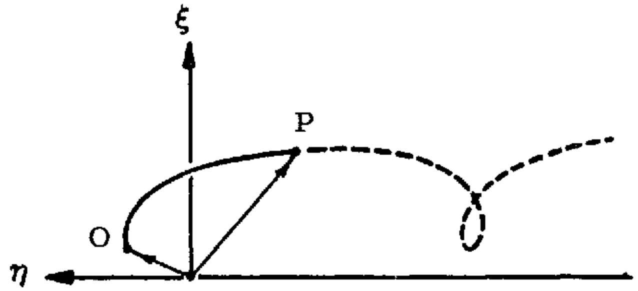
$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$

6. A Full Solution (i.e., a combination of 4. and 5. above), is described from:

$$I_2\bar{\hbar}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] A_a + \left[I_2 - \frac{3}{2}B_2 J_1 \varphi\right] K_o + \Psi_\tau \bar{\tau}.$$



5. (The trace geometry here is a description from 1. and 3. above).



6. This figure is obtained as a combination of 1, 2, and 3, or 4 and 5, above.

---

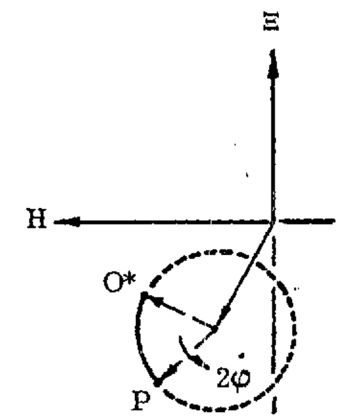
*O is the initial point for the trace; P is a general position on each curve.

In-Plane Displacements, referred to the Inertially Oriented Frame of Reference

In-Plane Displacement Diagram; Inertial Frame of Reference

The describing equation:

$$I_2\bar{R}(\varphi) = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right] A_{a_I} + T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}(B_2 J_1)\varphi\right] K_{o_I} + \Psi_\tau \bar{\tau}\right\}.$$



1. The trace is a circle with its center offset from the coordinate origin. Offset is due to fixed terms in A_a; the circle is a consequence of the transform, T.

Partial Solutions.

1. The construction due to A_{a_I} (the "Trigonometric" part of the solution).

The descriptive expression for this partial solution is:

$$\delta\left[I_2\bar{R}(\varphi)\right] = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right] A_{a_I},$$

wherein

$$A_{a_I} \equiv \frac{1}{2}\left[I_2\bar{R}_o + B_2(J_1 - 2J_2)\bar{R}'_o + (J_1 - 2J_2)\bar{\tau}\right],$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$

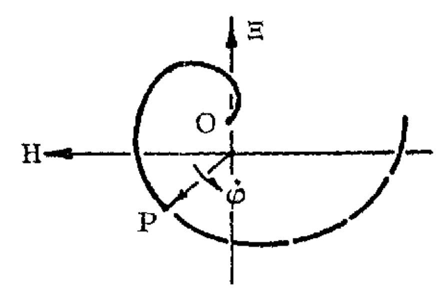
2. Construction due to the K_{o_I} (the Constant Initial Values quantity).

This partial solution is defined as:

$$\delta\left[I_2\bar{R}(\varphi)\right] = T_2(\varphi^-)\left[I_2 - \frac{3}{2}(B_2 J_1)\varphi\right] K_{o_I},$$

wherein

$$K_{o_I} \equiv (2J_1 - J_2)\bar{R}_o - 2B_2\bar{R}'_o.$$



2. This trace is a spiral-like curve; it is a consequence of the secular form of the expression. A displaced initial locus is due to I_2K_o.

---

*O is the initial point for this trace; P is a general locus.

In-Plane Displacement Diagram (continued)

3. A Construction for the $\Psi_\tau \bar{\tau}$ Solution.

This partial solution is obtained from:

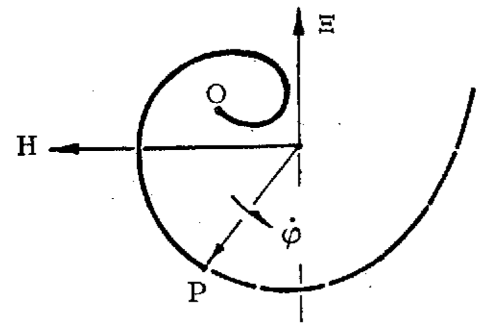
$$\delta\left[I_2 \bar{\mathcal{R}}(\varphi)\right] = T_2(\varphi^-)\left[\Psi_\tau \bar{\tau}\right]$$

wherein

$$\Psi_\tau \equiv (J_1 + 4J_2) - (2B_2 + \frac{3}{2}\varphi J_2)\varphi$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$



3. A spiral-like trace primarily due to the secular nature of Ψ_τ.

4. The Initial Values Solution ($\bar{\tau} \equiv \bar{0}$), described by:

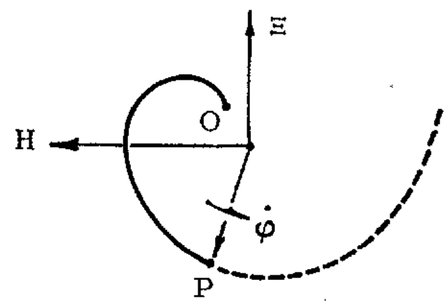
$$\Delta\left[I_2 \bar{\mathcal{R}}(\varphi)\right] \equiv \left[3(J_2 - J_1) + T_2(2\varphi^-)\right](A_{a_I})_{i.v.} + T_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi(B_2 J_1)\right]K_{o_I}$$

wherein

$$(A_{a_I})_{i.v.} \equiv \frac{1}{2}\left[I_2 \bar{\mathcal{R}}_o + B_2(J_1 - 2J_2)\bar{\mathcal{R}}'_o\right]$$

and

$$K_{o_I} \equiv (2J_1 - J_2)\bar{\mathcal{R}}_o - 2B_2 \bar{\mathcal{R}}'_o.$$



4. This spiral trace is a combination of the types shown as 1. and 2. above. Origin offset relates to initial state.

*O defines the locus for $\varphi = 0$; P is a general trace point.

In-Plane Displacement Diagram (continued)

5. The Applied Force Solution (with $\bar{\mathcal{R}}_o = \bar{\mathcal{R}}'_o = \bar{0}$);

defined by

$$\Delta\left[I_2 \bar{\mathcal{R}}(\varphi)\right] = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right](A_{a_I})_{\tau} + T_2(\varphi^-)\left[\Psi_\tau \bar{\tau}\right]$$

wherein

$$(A_{a_I})_{\tau} \equiv \frac{1}{2}(J_1 - 2J_2)\,\bar{\tau};$$

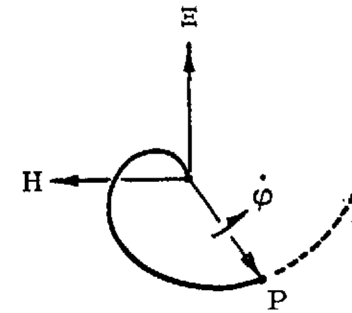
$$\Psi_\tau \equiv (J_1 + 4J_2) - (2B_2 + \frac{3}{2}J_2\varphi)\,\varphi,$$

and

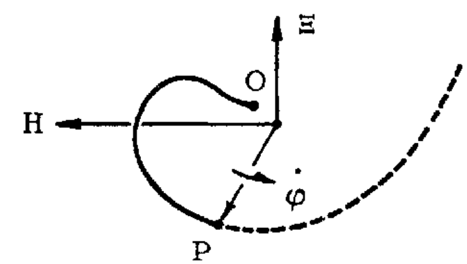
$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$

6. A Complete Solution (i.e. Combination of 4. and 5. above).

$$I_2 \bar{\mathcal{R}}(\varphi) = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right]A_{a_I} + T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}\varphi(B_2 J_1)\right]K_{o_I} + \Psi_\tau \bar{\tau}\right\}$$



5. This spiral trace is related to the geometries described as 1. and 3. above. Note the curve originates at the coordinate origin; this is due to the null initial state.



6. (The special case of motion beginning at the origin (Particle Ejection Problem) is not shown here. This figure is, also, the combined solutions for 1., 2., and 3. above).

Hodographs, In-Plane, referred to the Rotating Frame of Reference

In-Plane Hodograph Diagram; Local, Rotating Frame of Reference.

The general hodograph equation here is:

$$I_2\bar{\hbar}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right]\left[B_2 T_2(\varphi^-)\right]A_a - \frac{3}{2}\left[B_2 J_1\right]K_o + \Psi'_\tau \bar{\tau}$$

$$= B_2\left\{\left[I_2 + 3(J_1 - J_2)\right]T_2(\varphi^-)A_a - \frac{3}{2}\left[J_1\right]K_o\right\} + \Psi'_\tau \bar{\tau}.$$

Partial Solutions

1. Trace geometry due to A_a (the "Trigonometric" Solution).

The equation describing this trace is:

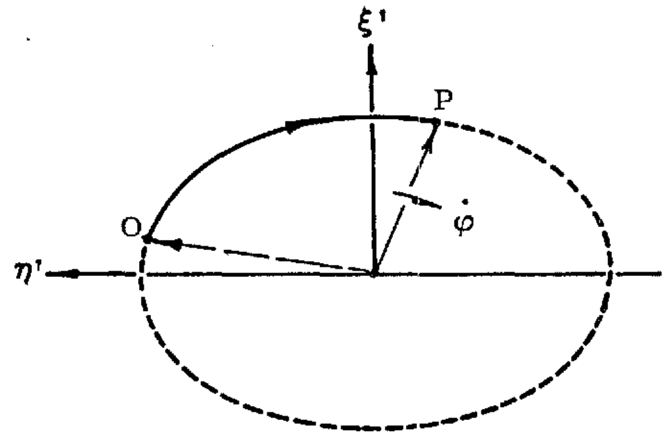
$$\delta\left[I_2\bar{\hbar}'(\varphi)\right] = \left[I_2 + 3(J_2 - J_1)\right]B_2 T_2(\varphi^-)A_a,$$

wherein

$$A_a \equiv \frac{1}{2}\left[3J_1\bar{\hbar}_o + (J_2 - 2J_1)B_2\bar{\hbar}'_o + (J_1 - 2J_2)\bar{\tau}\right],$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$



1. This trace is a 2:1 ellipse, centered at the coordinate origin. The motion rate over the trace is $\dot{\varphi}$.

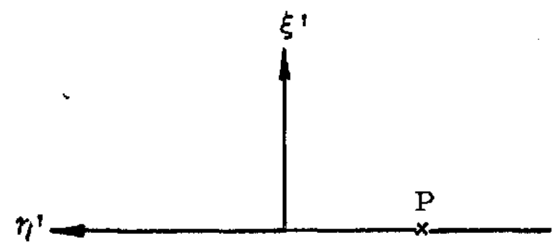
2. The Contribution for K_o (the Constant Initial Values term).

This partial solution is given by:

$$\delta\left[I_2\bar{\hbar}'(\varphi)\right] = -\frac{3}{2}B_2 J_1 K_o,$$

wherein

$$K_o \equiv (4J_1 + J_2)\bar{\hbar}_o - 2B_2\bar{\hbar}'_o.$$



2. The solution produces a single point on the hodograph plane. (Note that the point lies <u>on</u> the η'-axis).

*O is an initial point for the trace; P is a general locus.

Hodograph Construction (continued)

3. The Construction for the term $\Psi'_\tau \bar{\tau}$.

The describing equation is:

$$\delta\left[I_2 \bar{\lambda}'(\varphi)\right] = \Psi'_\tau \bar{\tau},$$

wherein

$$\Psi'_\tau \equiv -(2B_2 + 3J_2\varphi),$$

and

$$\bar{\tau} \equiv \bar{\tau}\,(\tau_\xi,\ \tau_\eta,\ \tau_\zeta).$$

4. The Initial Values Solution (for $\bar{\tau} \equiv \bar{0}$) is described from:

$$\Delta\left[I_2 \bar{\lambda}'(\varphi)\right] = B_2\left\{\left[I_2 + 3(J_1 - J_2)\right]T_2(\varphi^-)\left[A_a\right]_{i.v.} - \frac{3}{2}J_1 K_o\right\},$$

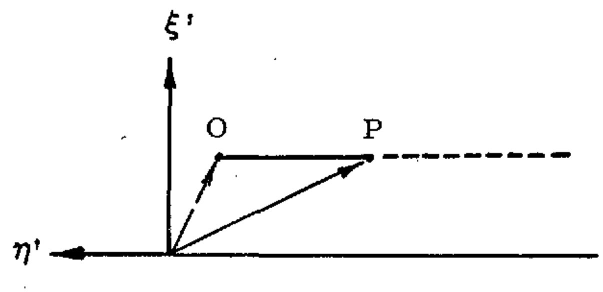
wherein

$$\left[A_a\right]_{i.v.} \equiv \frac{1}{2}\left[3J_1\bar{\lambda}_o + (J_2 - 2J_1)B_2\bar{\lambda}'_o\right]$$

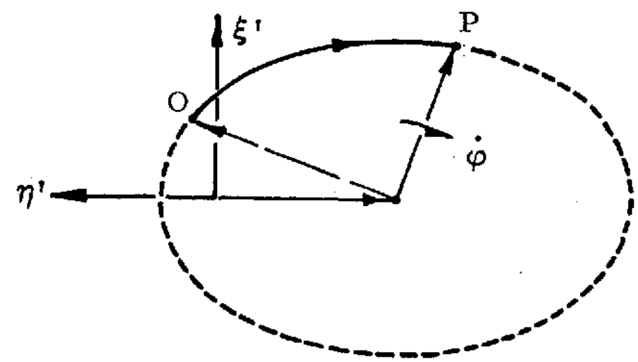
and

$$K_o \equiv (4J_1 + J_2)\bar{\lambda}_o - 2B_2\bar{\lambda}'_o.$$

*O is the initial point for a trace; P is a general locus.



3. This trace appears as a line, parallel to η'-axis; trace motion is as shown, typically.



4. This trace is a 2:1 ellipse, with center shifted due to K_o-term. Motion and rate, over hodograph are as shown. (This geometry is a combination of 1. and 2. above).

Hodograph Construction (continued)

5. The Applied Force Solution ($\bar{\lambda}_o, \bar{\lambda}'_o = \bar{0}$) has its trace defined by:

$$\Delta\left[I_2 \bar{\lambda}'(\varphi)\right] = T_2(\varphi^-)\left[I_2 + 3(J_2 - J_1)\right]B_2\left[A_a\right]_\tau + \Psi'_\tau \bar{\tau},$$

wherein

$$\left[A_a\right]_\tau \equiv \frac{1}{2}(J_1 - 2J_2)\bar{\tau},$$

$$\Psi'_\tau \equiv -(2B_2 + 3J_2\varphi),$$

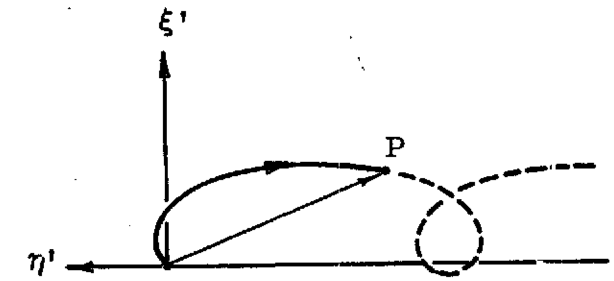
and

$$\bar{\tau} \equiv \bar{\tau}\,(\tau_\xi, \tau_\eta, \tau_\zeta).$$

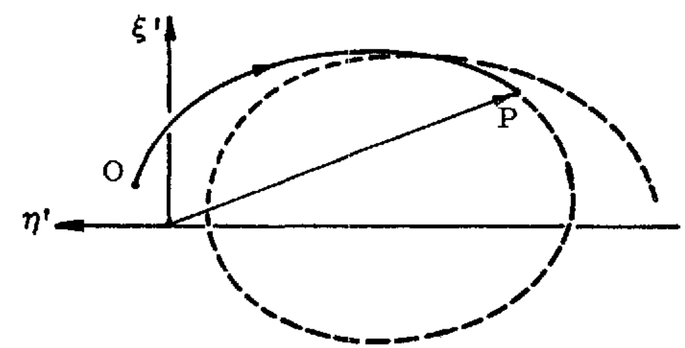
6. A Full Solution (i.e., the combination of 4. and 5. above). is defined from:

$$I_2 \bar{\lambda}'(\varphi) = B_2\left\{\left[I_2 + 3(J_1 - J_2)\right]T_2(\varphi^-)\,A_a - \frac{3}{2}J_1 K_o\right\}$$

$$+ \Psi'_\tau \bar{\tau}.$$



5. The trace is now a meandering (2:1) ellipse; the center is shifted and moving in the negative η'-direction. (This is a combination of 1. and 3. above).



6. The geometry is also a combination of 1, 2, and 3, above.

*O is the initial point for the trace; P is a general position on the curve.

An In–Plane Non–Secular Case for the Rotating Frame of Reference

Non-Secular Traces; Local, Rotating Frame of Reference

To remove the divergent nature of these in-plane displacements it is necessary for $(2\xi_o + \eta'_o \equiv 0)$.

The resultant defining equation is:

$$I_2\bar{k}(\varphi) \equiv \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) A_a^{(1)} + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o^{(1)},$$

wherein

$$A_a^{(1)} \equiv \frac{1}{2}\left[-J_1\bar{k}_o + J_2 B_2 \bar{k}'_o\right],$$

and

$$K_o^{(1)} \equiv J_2\bar{k}_o - 2B_2 J_1 \bar{k}'_o = J_2\left[\bar{k}_o - 2B_2\bar{k}'_o\right].$$

(These are reduced forms for A_a and K_o, needed to eliminate the secular influence).

The Hodograph corresponding to the non-secular requirement(s) noted above is obtained from the expression shown below:

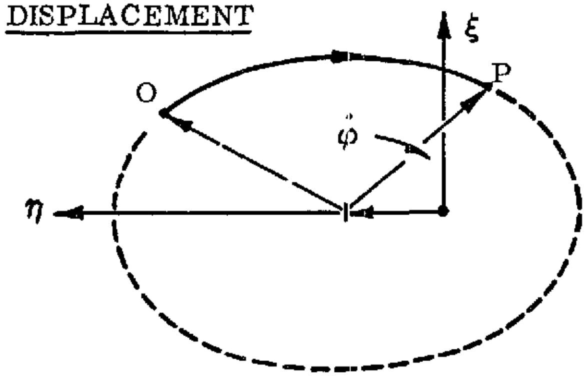
$$I_2\bar{k}'(\varphi) = B_2\left\{\left[I_2 + 3(J_1 - J_2)\right] T_2(\varphi^-) A_a^{(1)} - \frac{3}{2} J_1 K_o^{(1)}\right\},$$

wherein

$A_a^{(1)}$, $K_o^{(1)}$ are as defined above.

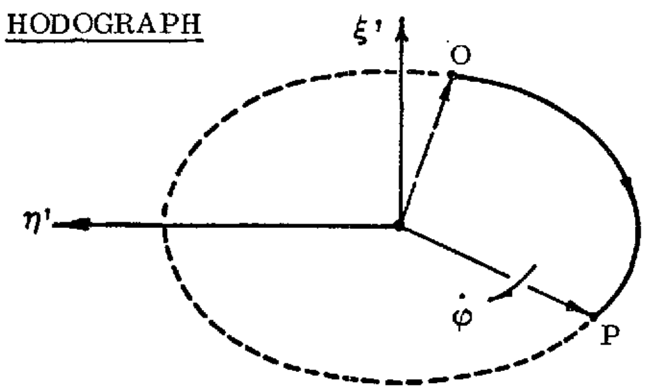
*O is the initial locus in this plane; P is a general point on the trace.

DISPLACEMENT



This trace is a 2:1 ellipse with a shifted center (shift due to $I_2K_o^{(1)}$). Motion on the figure is as shown.
(A special combination of the displacement constructions 1. and 2. (or 4.) above. Another geometry for this case arises when $\xi = \eta'_o = 0$).

HODOGRAPH



This trace is also a 2:1 ellipse but centered at the coordinate origin.
(This is a special combination of the hodograph construction 1. and 2. (or 4.) above.

Hodographs, In-Plane, referred to the Inertially Oriented Frame of Reference

In-Plane Hodograph Diagrams; Inertial Frame of Reference

The general hodograph equation is:

$$I_2\bar{\mathcal{R}}'(\varphi) = 2\left[B_2 T_2(2\varphi^-)\right] A_{a_I} + B_2 T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_{o_I} + \Psi_\tau \bar{\tau}\right\} + T_2(\varphi^-)\left\{\Psi'_\tau \bar{\tau} - \frac{3}{2}\left[B_2 J_1\right] K_{o_I}\right\}.$$

Partial Solutions.

1. Trace construction due to A_{a_I} ("Trigometric" Solution)

The defining equation for this solution is:

$$\delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right] = 2\left[B_2 T_2(2\varphi^-)\right] A_{a_I};$$

wherein

$$A_{a_I} \equiv \frac{1}{2}\left[I_2\bar{\mathcal{R}}_o + B_2(J_1 - 2J_2)\bar{\mathcal{R}}'_o + (J_1 - 2J_2)\bar{\tau}\right]$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta).$$

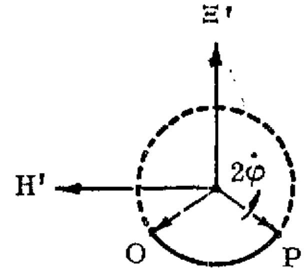
2. The construction for K_{o_I} (the constant initial value partial solution).

This solution is described by:

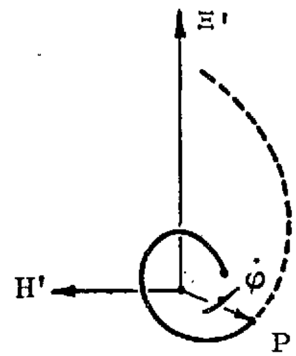
$$\delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right] = T_2(\varphi^-)\left\{B_2\left[I_2 - \frac{3}{2}\varphi(B_2 J_1)\right] - \frac{3}{2}(B_2 J_1)\right\} K_{o_I}$$

wherein

$$K_{o_I} = (2J_1 - J_2)\bar{\mathcal{R}}_o - 2B_2\bar{\mathcal{R}}'_o.$$



1. Trace for this partial solution describes a circle with center at the coordinate origin. The trace motion is at double orbit frequency.



2. This hodograph trace produces a spiral-like curve. The divergence is linked to the secular coefficient $(B_2 J_1 \varphi)$.

---

*O is the initial locus point, while P is a general point on the trace.

In-Plane Hodograph Diagrams (continued)

3. A Construction due to the $\bar{\tau}$-terms alone (Explicit Applied Force Terms).

The describing equation for this partial solution is:

$$\delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right] = T_2(\varphi^-)\left\{B_2\Psi_\tau + \Psi'_\tau\right\}\bar{\tau};$$

wherein

$$\Psi_\tau = (J_1 + 4J_2) - (2B_2 + \frac{3}{2}J_2\varphi)\varphi,$$

$$\Psi'_\tau = -(2B_2 + 3J_2\varphi),$$

and

$$\bar{\tau} \equiv \bar{\tau}(\tau_\xi,\ \tau_\eta,\ \tau_\zeta)$$

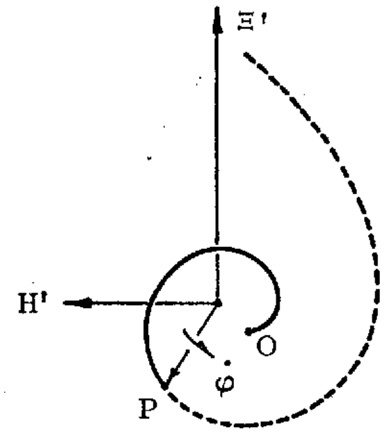
4. The Initial Values Solution ($\bar{\tau} \equiv \bar{0}$) is obtained as:

$$\Delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right] = B_2\left\{2\left[T_2(2\varphi^-)\right]\left(A_{a_I}\right)_{i.v.} + T_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]K_{o_I}\right\} - \frac{3}{2}T_2(\varphi^-)\left[B_2 J_1\right]K_{o_I}.$$

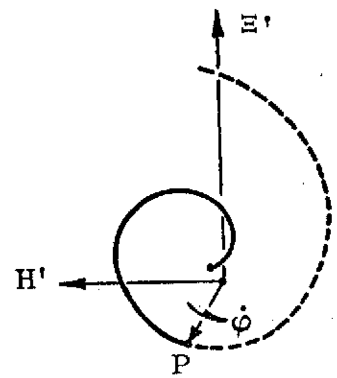
wherein

$$\left(A_{a_I}\right)_{i.v.} \equiv \frac{1}{2}\left[I_2\bar{\mathcal{R}}_o + B_2(J_1 - 2J_2)K_{o_I}\right] \text{ and } K_{o_I} \equiv (2J_1 - J_2)\bar{\mathcal{R}}_o - 2B_2\bar{\mathcal{R}}'_o.$$

* O is the initial locus point, while P is a general point on the trace.



3. This trace produces a spiral; the divergence here results from terms in both Ψ_τ and Ψ'_τ.



4. The trace appears as a form of the archimedian spiral coupled with circles.

<u>In-Plane Hodograph Construction</u> (continued)

5. The Applied Force Solution ($\bar{\mathfrak{R}}_o$, $\bar{\mathfrak{R}}'_o \equiv \bar{0}$), where the motion trace is described by:

$$\Delta\left[I_2\bar{\mathfrak{R}}'(\varphi)\right]=\left[B_2\left\{2\left[T_2(2\varphi^-)\right]\left(A_{a_I}\right)_\tau+T_2(\varphi^-)\,\Psi_\tau\right\}+T_2(\varphi^-)\,\Psi'_\tau\right]\bar{\tau},$$

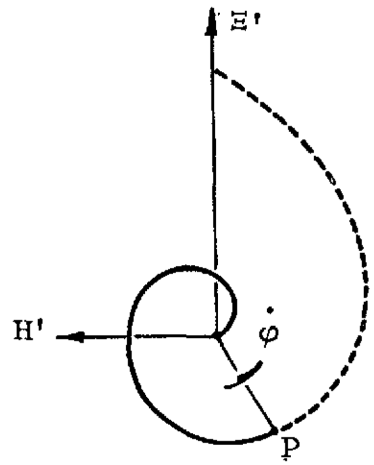
wherein

$$\left(A_{a_I}\right)_\tau \equiv \frac{1}{2}\,(J_1-2J_2) \quad ; \quad \Psi_\tau=(J_1+4J_2)-\left[2B_2+\frac{3}{2}\,J_2\varphi\right]\varphi,$$

$$\Psi'_\tau=-\left[2B_2+3J_2\varphi\right],$$

and

$$\bar{\tau}\equiv\bar{\tau}(\tau_\xi,\ \tau_\eta,\ \tau_\zeta).$$



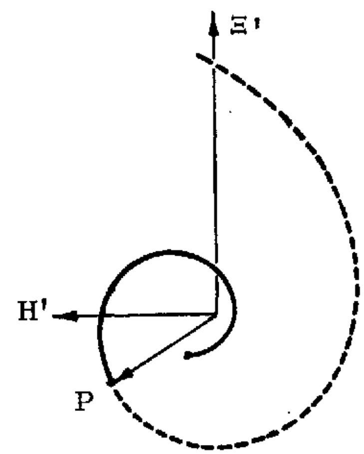
5. This trace is produced as a spiral-like geometry. Secular terms appear in both Ψ_τ and Ψ'_τ.

6. A Full Solution (or, the combination of 4. and 5. above).

The hodograph is obtained from:

$$I_2\bar{\mathfrak{R}}'(\varphi)=B_2\left\{2T_2(2\varphi^-)\,A_{a_I}+T_2(\varphi^-)\left[\left(I_2-\frac{3}{2}\,\varphi\,B_2J_1\right)K_{o_I}+\Psi_\tau\bar{\tau}\right]\right\}$$

$$+\,T_2(\varphi^-)\left[\Psi'_\tau\bar{\tau}-\frac{3}{2}\,(B_2J_1)\,K_{o_I}\right].$$



6. This geometry is composed from traces in 4. and 5. above; or, a combination of 1., 2., and 3.

An In-Plane Non-Secular Case, for the Inertial Frame of Reference

Non-Secular Traces; Inertial Frame of Reference

To remove the divergent nature of the in-plane displacement trace it is necessary for $(\Xi_o + H'_o) \equiv 0$. The resultant defining equation is:

$$I_2\bar{\mathcal{R}}(\varphi) \equiv \left[3(J_2 - J_1) + T_2(2\varphi^-)\right]A_{a_I}^{(1)} + T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]K_{o_I}^{(1)}\right\};$$

wherein

$$A_{a_I}^{(1)} \equiv \frac{1}{2}J_2\left[\bar{\mathcal{R}}_o + B_2\bar{\mathcal{R}}'_o\right], \text{ and } K_{o_I}^{(1)} \equiv -\left[J_2\bar{\mathcal{R}}_o + 2B_2 J_1 \bar{\mathcal{R}}'_o\right].$$

(These coefficients are reduced forms of the like expressions noted above).

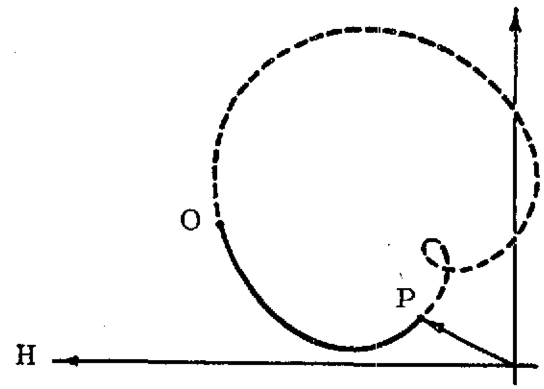
The hodograph, corresponding to the non-secular requirement(s) indicated above, is defined as:

$$I_2\bar{\mathcal{R}}'(\varphi) \equiv 2\left[B_2 T_2(2\varphi^-)\right]A_{a_I}^{(1)} + T_2(\varphi^-)\left\{B_2\left[I_2 - \frac{3}{2}(B_2 J_1)\varphi\right] - \frac{3}{2}(B_2 J_1)\right\}K_{o_I}^{(1)},$$

wherein

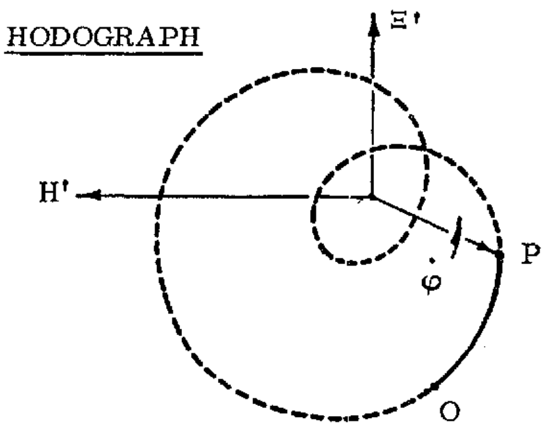
$A_{a_I}^{(1)}$ and $K_{o_I}^{(1)}$ are as defined above.

DISPLACEMENT



(A General Construction; Special cases not shown).

HODOGRAPH



(A General Construction; Special cases not shown).
These curves are limacons, in general; the special cases are noted to be varied forms of the limacon (e.g., the cardioid).

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*On these traces O is the initial point while P is a general position on the curve.

VI. PROBLEM SOLUTION

An Inertially Aligned Force System. In the foregoing sections of this report the relative motion problem was solved for the case of a disturbing force with components parallel to the local rotating frame of reference. The study which follows, below, will differ from the former one in that now the force is assumed to have components in the inertial frame of reference. As with the former case, each of these components is assumed to be fixed in value. This will allow for an analytical solution to be obtained, directly, and will form a basis of comparison with the previous solution.

Return, for the moment, to the differential equation for this problem (Eq. (II.12c)); there the relative motion's acceleration vector is expressed by:

$$\ddot{\bar{r}} = \dot{\varphi}^2\left[(I_2 - I_3)(\bar{r}+\bar{P}) + 3I_3\hat{P}\,(\bar{r}\cdot\bar{P})\right] - 2\dot{\varphi}\,B_2\dot{\bar{r}} + \Sigma_j\,\bar{F}, \tag{II.74}$$

wherein the summation on $\bar{F}$ represents all the external forces which may be applied to the test particle (Q). Also, here, $\bar{r}$ is the relative position vector (for Q with respect to P); and, $\hat{P} \equiv \bar{e}_X$, is the unit position vector for the base particle (P). Other terms are as defined for Eq. (II.12c).

The force system of interest now is represented in the inertial frame, hence:

$$\Sigma_j\,\bar{F} \equiv \bar{F}_I \equiv \bar{F}_I(F_X, F_Y, F_Z)\,, \tag{II.75}$$

where the F_i scalars are the projections of $\bar{F}$ onto the inertial triad of reference. Recognizing that Eq. (II.74) can be written (for solution) in terms of local, rotating coordinates, then it will be necessary to relate $\bar{F}_I$ to the same triad of reference. This is most easily done by means of the transforms described in Appendix A. There, Eq. (A.3a) relates the rotating triad $(\bar{e}'_j)$ to the inertial one $[(\bar{e}_i)^o]$; accordingly, an inertial vector is transformed back to the local frame by this transformation. As a consequence one finds that the components of $\bar{F}_I$, which are "seen" in the rotating triad's frame, are:

$$(\bar{F})' = [T(\theta^+)]\,\bar{F}_I\,, \tag{II.76a}$$

where $T(\theta^+) \equiv T(\varphi^+)$, since φ is the transfer angle here. Therefore, Eq. (II.76a) is expanded to:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \equiv \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix}\,. \tag{II.76b}$$

Probably a more meaningful statement for $T(\varphi^+)$ is that noted in Eq. (A.7c), Appendix A; that is:

$$T(\varphi^+) \equiv I_2 \cos\varphi - B\sin\varphi + (I_3 - I_2). \tag{II.77}^*$$

<u>A Solution for the Relative Motion.</u> The solution which is found here follows from Eq. (II.74) <u>after</u> Eq. (II.76a), or Eq. (II.76b), has been included. Making use of the simplifying interpretations to terms in the differential equation (see the statements following Eq. (II.15a)), it is apparent that a solution is to be obtained for the expression:

▶ $$\ddot{\bar{r}} = \dot{\varphi}^2\left[3(J_1 - J_3)\,\bar{r}\right] - 2\dot{\varphi}\,B_2\,\dot{\bar{r}} + T(\varphi^+)\,\bar{F}_I\,. \tag{II.78}$$

It should be evident that this equation differs from that in the previous case <u>only</u> in the last term -- the disturbance function $\bar{F}$. Here, as noted above, the scalars describing $\bar{F}_I$ are constants; however, the disturbance term itself is a variable through $T(\varphi^+)$.

Before the formal solution is attempted it should be noted that for this case the relative motion displacement ($\bar{r}$) is referred to a triad moving uniformly ($\dot{\varphi}$ = constant) with the base particle (P). The constant matrices (J_i, B) are employed to properly "locate" the associated vectors in the triad space.

---

*It should be recalled that $(I_3 - I_2) \equiv J_3$; this is one of the special diagonal matrices used here.

The state vectors $(\bar{r}, \dot{\bar{r}})$ are those of primary interest; however, the initial state $(\bar{r}_o, \dot{\bar{r}}_o)$ is arbitrarily set so that the full solution is obtained as a combined "initial-values problem" and a solution for a forcing function (hereafter called the "zero-initial-value problem").

Again, applying the method of Laplace Transforms, Eq. (II.78) is cast into the following transformed format:

$$s^2 \bar{r}(s) - I_3\left[s\bar{r}_o + \dot{\bar{r}}_o\right] = \dot{\varphi}^2\left[3J_1 - J_3\right]\bar{r}(s) - 2\dot{\varphi}B\left\{I_3\left[s\bar{r}(s) - \bar{r}_o\right]\right\}$$
$$+\left[\frac{I_2 s}{s^2+\dot{\varphi}^2} - \frac{B_2\dot{\varphi}}{s^2+\dot{\varphi}^2} + \frac{J_3}{s}\right]\bar{F}_I ; \qquad \text{(II. 79a)*}$$

or, in a more concise algebraic form:

$$\left[I_3 s^2 - \dot{\varphi}^2(3J_1 - J_3) + 2\dot{\varphi}B_2 s\right]\bar{r}(s)\left\{\equiv A(s)\bar{r}(s)\right\} = \left[I_3 s + 2\dot{\varphi}B_2\right]\bar{r}_o + I_3\dot{\bar{r}}_o$$
$$+\left[\frac{I_2 s}{s^2+\dot{\varphi}^2} - \frac{B_2\dot{\varphi}}{s^2+\dot{\varphi}^2} + \frac{J_3}{s}\right]\bar{F}_I . \qquad \text{(II. 79b)}$$

Formally, the solution for $\bar{r}(s)$ is recognized to be:

$$\bar{r}(s) = \left[A(s)\right]^{-1}\left[I_3 s + 2\dot{\varphi}B_2\right]\bar{r}_o + \left[A(s)\right]^{-1}\left[I_3\dot{\bar{r}}_o\right] + \left[A(s)\right]^{-1}\left[\frac{I_2 s}{s^2+\dot{\varphi}^2} - \frac{B_2\dot{\varphi}}{s^2+\dot{\varphi}^2} + \frac{J_3}{s}\right]\bar{F}_I, \qquad \text{(II. 79c)}$$

wherein $\left[A(s)\right]^{-1}$ is the inverse of the matrix A(s) defined in Eq. (II.79b). (This inverse matrix is set down in Eqs. (II .17)).

After manipulating and rearranging, and finding appropriate inverse transforms, a solution for $\bar{r}(t)$ is found as:

$$\bar{r}(t) = \left\{I_3\cos\dot{\varphi}t + \left[4J_1 + J_2\right](1-\cos\dot{\varphi}t) - 6\left[J_2B_2\right](\dot{\varphi}t - \sin\dot{\varphi}t)\right\}\bar{r}_o$$
$$+\left\{I_3\sin\dot{\varphi}t - 3J_2(\dot{\varphi}t - \sin\dot{\varphi}t) - 2B_2(1-\cos\dot{\varphi}t)\right\}\frac{\dot{\bar{r}}_o}{\dot{\varphi}}$$

(equation continued on next page)

*For reference purposes, one should recognize that $T(\varphi^{\pm})$ is transformed as: $\mathcal{L}\{T(\varphi\pm)\} = \mathcal{L}\{I_2\cos\varphi \mp B_2\sin\varphi + J_3\} = \frac{I_2 s}{s^2+\dot{\varphi}^2} \mp \frac{B_2\dot{\varphi}}{s^2+\dot{\varphi}^2} + \frac{J_3}{s}$, wherein "s" is the transformed variable.

$$+\left\{J_3\,(1-\cos\dot{\varphi}t)+\frac{1}{2}\,I_2(\dot{\varphi}t\sin\dot{\varphi}t)-\frac{3}{2}\,B_2\,(\sin\dot{\varphi}t-\dot{\varphi}t\cos\dot{\varphi}t)\right.$$

$$\left.-\left[2J_1+5J_2\right](1-\cos\dot{\varphi}t-\frac{1}{2}\dot{\varphi}t\,\sin\dot{\varphi}t)+\frac{3}{2}\,J_2B_2(2\dot{\varphi}t+\dot{\varphi}t\cos\dot{\varphi}t-3\sin\dot{\varphi}t)\right\}\frac{\bar{F}_I}{\dot{\varphi}^2}. \qquad \text{(II. 80)}$$

Introducing the dimensionless variables shown in Eq. (II.21), and regrouping the above expression in terms of its in- and out-of-plane components, a more useful form of this result is:

$$\bar{k}\,(\varphi)=\left\{\left[4J_1+J_2\right]\bar{k}_o-2B_2\bar{k}'_o\right\}-\left\{3J_1\bar{k}_o-2B_2\bar{k}'_o\right\}\cos\varphi+\left\{6\left[B_2J_1\right]\bar{k}_o\right.$$

$$\left.-B_2\left[J_2+4J_1\right]B_2\bar{k}'_o\right\}\sin\varphi-3B_2J_1\left\{2I_2\bar{k}_o-B_2\bar{k}'_o\right\}\varphi+\left\{-\left[2J_1+5J_2\right]\right.$$

$$-3B_2J_1(\varphi-\sin\varphi)+\frac{3}{2}\left[J_1+2J_2\right]\left[I_2+\varphi B_2\right]\left[I_2\cos\varphi-B_2\sin\varphi\right]$$

$$\left.+\frac{1}{2}\left[J_1+4J_2\right]\cos\varphi\right\}\bar{\tau}_I+J_3\left\{\bar{k}_o\cos\varphi+\bar{k}'_o\sin\varphi+\bar{\tau}_I(1-\cos\varphi)\right\} \qquad \text{(II. 81)}$$

In this expression it must be remembered that φ plays the role of the independent variable ($\varphi \equiv \dot{\varphi}t$), and that all state and disturbance parameters are now in non-dimensional form.

Eq. (II.81) is rather difficult to study and interpret due to its complicated form. Thus, it will be presented in a more compact format; the scheme followed here is directly analogous to that noted by Eqs. (II.24) through (II.27).

First, the trigonometric terms having initial values as coefficients are consolidated, and grouped, using the matrices $T(\varphi^{\pm})$. Secondly, the constants and the first ordered secular terms are separately grouped and named.

(1). For the trigonometric terms, write:

$$I_2\bar{k}_c\cos\varphi+B_2\bar{k}_s\sin\varphi\equiv T_2(\varphi^-)\left[\mathscr{a}_a\right]+T_2(\varphi^+)\left[\mathscr{a}_s\right]$$

$$=I_2\left[\mathscr{a}_a+\mathscr{a}_s\right]\cos\varphi+B_2\left[\mathscr{a}_a-\mathscr{a}_s\right]\sin\varphi\,, \qquad \text{(II. 82)}$$

where $\bar{k}_c$ and $\bar{k}_s$ are representations for the cosine and sine coefficients in Eq. (II.81), respectively. Recognizing that, now, $2\mathcal{A}_a = \bar{k}_c + \bar{k}_s$, and $2\mathcal{A}_s = \bar{k}_c - \bar{k}_s$, then it is easy to show that

$$\mathcal{A}_a = \frac{1}{2}\left\{3J_1 \bar{\lambda}_o + \left[J_2 - 2J_1\right] B_2 \bar{\lambda}'_o\right\},$$

and

$$\mathcal{A}_s = \frac{3}{2}\left\{-3J_1 \bar{\lambda}_o + \left[J_2 + 2J_1\right] B_2 \bar{\lambda}'_o\right\}. \tag{II.83a)*}$$

On examining these coefficients it is found that:

$$\mathcal{A}_s = 3(J_2 - J_1)\mathcal{A}_a; \tag{II.83b}$$

here the matrix $[J_2 - J_1]$ provides a proper change in signs, while the three (3) is a needed multiplier.

Incorporating Eqs. (II.83) into Eq. (II.82) it is easy to show that:

$$\begin{aligned} I_2\bar{k}_c \cos\varphi + B_2\bar{k}_s \sin\varphi &= \left\{T_2(\varphi^-) + 3T_2(\varphi^+)\left[J_2 - J_1\right]\right\}\mathcal{A}_a, \\ &= \left\{I_2 + 3\left[J_2 - J_1\right]\right\}\left[T_2(\varphi^-)\right]\mathcal{A}_a, \end{aligned} \tag{II.84)**}$$

after some manipulations.

(2). Next, when the first term (the constant parameter) in Eq. (II.81) is multiplied by $\frac{3}{2}J_1$ the result assumes a form which is very similar to the secular coefficient in the equation. Consequently, these two quantities can be combined as follows:

$$\begin{aligned} &\left\{\left[4J_1 + J_2\right]\bar{\lambda}_o - 2B_2\bar{\lambda}'_o\right\} - 3B_2J_1\left[2I_2\bar{\lambda}_o - B_2\bar{\lambda}'_o\right]\varphi \\ &= \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]\left\{\left[4J_1 + J_2\right]\bar{\lambda}_o - 2B_2\bar{\lambda}'_o\right\}. \end{aligned} \tag{II.85}$$

(3). Finally, for conciseness and uniformity of notation, the coefficient of $\bar{\tau}_I$ is replaced by (the equivalent set of terms):

---

*When this result is compared to Eqs. (II.26), it is seen that $(\mathcal{A}_a, \mathcal{A}_s) \equiv (A_a, A_s)_{i.v.}$; i.e., the previous definitions, but now only initial value parameters are contained.

**When this result is related to the previous one it is noted that the two are identical if in A_a the $\bar{\tau}$ term is removed (see Eq. (II.27)). Hence, here, the $\mathcal{A}_a$ term is replaced by $[A_a]_{i.v.}$, signifying this constraint on A_a.

$$\left[2J_1+5J_2\right]\left[T_2(\varphi^+)-I_2\right]+\frac{3\varphi}{2}\left[J_1+2J_2\right]\left[J_2B_2+B_2T_2(\varphi^+)\right]+\frac{1}{4}\left[J_1-2J_2\right]\left[T_2(\varphi^-)\right.$$

$$\left.-T_2(\varphi^+)\right]. \tag{II.86}$$

When due account is given to all of the results noted above then the solution, Eq. (II.81), can be replaced by:

$$\bar{r}(\varphi)=\left[I_2+3(J_2-J_1)\right]T_2(\varphi^-)\left[A_a\right]_{i.v.}+\left[I_2-\frac{3}{2}\varphi B_2J_1\right]K_o+\Phi_\tau\bar{\tau}_I$$

$$+J_3\left\{\bar{r}_o\cos\varphi+\bar{r}_o'\sin\varphi+\bar{\tau}_I(1-\cos\varphi)\right\}, \tag{II.87a}$$

wherein

$$\left[A_a\right]_{i.v.}\equiv\frac{1}{2}\left[3J_1\bar{r}_o+(J_2-2J_1)B_2\bar{r}_o'\right],$$

$$K_o\equiv\left[4J_1+J_2\right]\bar{r}_o-2B_2\bar{r}_o', \tag{II.88}$$

and Φ_τ is the quantity given by Eq. (II.86). This solution should be compared to Eq. (II.30a), the solution for $\bar{r}(\varphi)$ when $\bar{\tau}$ (the specific force, referred to the rotating triad) was the "disturbing function". A comparison of these two mathematical statements will show that the only new information provided here is associated with the $\bar{\tau}_I$ terms. Hence, the initial value problem has remained intact (as would be expected) while the "zero-initial-value problem" is altered accordingly.

The Relative Velocity Equation. The equation in question, now, is most readily obtained by the differentiation of (say) Eq. (II.87). One must be careful, noting that: (1), the $T_2(\varphi^{\pm})$ terms bear derivatives, as does the coefficient, Φ_τ. In fact it is only the trigonometric quantities which are involved in this operation. Now, after differentiating and clearing, the resulting (velocity) equation is:

$$\bar{\lambda}'(\varphi) = \left\{I_2 + 3(J_2 - J_1)\right\}\left[B_2 T_2(\varphi^-)\right]\left[A_a\right]_{i.v.} - \frac{3}{2}\left[B_2 J_1\right]K_o + \Phi'_\tau \bar{\tau}_I$$

$$+ J_3\left\{\bar{\lambda}'_o \cos\varphi + (\bar{\tau}_I - \bar{\lambda}_o)\sin\varphi\right\}, \qquad \text{(II.89a)*}$$

wherein

$$\Phi'_\tau \equiv -\left[2J_1 + 5J_2\right]B_2 T_2(\varphi^+) + \frac{3}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2 + (I_2 - \varphi B_2) B_2 T_2(\varphi^+)\right]$$

$$+ \frac{1}{4}\left[J_1 - 2J_2\right]B_2\left[T_2(\varphi^-) + T_2(\varphi^+)\right]. \qquad \text{(II.90)}$$

<u>Summary.</u> The descriptive equations for $\bar{\lambda}(\varphi)$ and $\bar{\lambda}'(\varphi)$ above denote a time history of the state of relative motion for the particle (Q), relative to the base particle (P). Here "P" is assumed to move on a circular orbit about a simple attracting center (μ). There are no mutually reacting forces between the particles; however, Q is presumed to be acted upon by an external force which has constant valued projections referred to the inertial frame of reference.

The state equations are given in a dimensionless form; nevertheless the dimensionality of the problem can be recaptured by noting and applying proper multipliers (see Eqs. (II.21)). Also, the expressions above define the relative motion <u>in</u> the local, rotating frame. Consequently, when the motion is to be described in an inertially oriented frame of reference it will be necessary to take a proper accounting of the transformation which can accomodate this result. (This is the task to be undertaken in the next section; comments are deferred until that time).

A study of the expressions thus far developed shows that the separating of these results into an initial-value problem and a zero-initial-value problem is a natural division. In fact, there is no coupling between these solution types; hence, any new information which is to be gained here must arise as a consequence of the applied force ($\bar{\tau}_I$) alone. This reduces the number of situations to

*See Eqs. (A.6), Appendix A, for a description of the derivative forms of $T(\varphi^{\pm})$.

be discussed, now, and allows problem solutions to be added to one another for a more complete definition and accumulation of effects and simulations. By combining the results here (for $\bar{\tau}_I$) with the previous cases it is possible to imply a variety of applied force situations, in addition to the initial values effects. Of course, the reliability of these answers cannot be expected to extend too far, into physical space around the base particle, or into time, since the analytical solutions are a consequence of the linearization imposed on the governing differential equation. This is a constraint on the problem; nevertheless it is obvious that these formulae do provide an excellent insight into and understanding of the several situations studied herein. In this regard the reader can become much better acquainted with these classes of relative motions than he would otherwise have through (say) discreet numerical examples.

In order to provide a better view of these solution equations the primary results, Eqs. (II.87a) and (II.89a) are written below in matrix format. Recalling that the initial-values solution, now, is the same as before then the principal added effects are to be found in terms of the $\bar{\tau}_I$. Nonetheless, to avoid confusion the full expressions are written here.

(a). From Eq. (II.87a) the in-plane displacements are given by:

$$I_2 \bar{\lambda}(\varphi) \equiv \begin{bmatrix} \xi(\varphi) \\ \eta(\varphi) \end{bmatrix} = \begin{bmatrix} -\cos\varphi & \sin\varphi \\ 2\sin\varphi & 2\cos\varphi \end{bmatrix} \begin{bmatrix} 3\xi_o + 2\eta'_o \\ \xi'_o \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -\frac{3\varphi}{2} & 1 \end{bmatrix} \begin{bmatrix} 4\xi_o + 2\eta'_o \\ \eta_o - 2\xi'_o \end{bmatrix}$$

$$+ \left\{ \begin{bmatrix} -2(1-\cos\varphi) & \frac{3}{2}\sin\varphi \\ -6\sin\varphi & -5(1-\cos\varphi) \end{bmatrix} + \frac{3\varphi}{2} \begin{bmatrix} \sin\varphi & -\cos\varphi \\ 2(1+\cos\varphi) & (2\sin\varphi) \end{bmatrix} \right\} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix}. \quad \text{(II.87b)}$$

(b). The out-of-plane coordinate is:

$$J_3 \bar{\lambda}(\varphi) \equiv \zeta(\varphi) = \zeta_o \cos\varphi + \zeta'_o \sin\varphi + \tau_z (1-\cos\varphi). \quad \text{(II.87c)}$$

(c). From Eq. (II.89a) the in-plane velocity is described as:

$$I_2 \bar{\kappa}'(\varphi) \equiv \begin{bmatrix} \xi'(\varphi) \\ \eta'(\varphi) \end{bmatrix} = \begin{bmatrix} \sin\varphi & \cos\varphi \\ 2\cos\varphi & -2\sin\varphi \end{bmatrix} \begin{bmatrix} 3\xi_o + 2\eta'_o \\ \xi'_o \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} 4\xi_o + 2\eta'_o \\ \eta_o - 2\xi'_o \end{bmatrix}$$

$$+ \left\{ \begin{bmatrix} -\frac{1}{2}\sin\varphi & 0 \\ -3\cos\varphi & -2\sin\varphi \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2}\varphi\cos\varphi & \frac{1}{2}\varphi\sin\varphi \\ (1-\varphi\sin\varphi) & (\varphi\cos\varphi) \end{bmatrix} \right\} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} \quad . \tag{II.89b}$$

(d). The out-of-plane component is:

$$J_3 \bar{\kappa}'(\varphi) \equiv \zeta'(\varphi) = \zeta'\cos\varphi + (\tau_z - \zeta_o)\sin\varphi. \tag{II.89c}$$

In the next section the case just studied will be reexamined but, then, with the motion referred to an inertially oriented frame of reference. For that analysis the state variables in use will be a dimensionless set; these are to be acquired by the same scheme as was used in the foregoing case study.

The Relative Motion in Inertial Coordinates. In this section a solution to the relative motion problem is developed with the state variables described in reference to an inertial frame of reference. The method used to acquire these results is identical to that employed earlier; that is, the solution expressed in a rotating frame is transformed to the inertial frame. It should be remembered that the initial values appearing in the various parameter coefficients are also subject to the same transformations. (This operation has been carried out previously (see Eqs. (II.32))). The basic operation in this transformation is carried out according to:

$$\bar{R}(t) = T(\varphi^-)\,\bar{r}(t), \tag{II.91}^*$$

where

$$\bar{R}(t) \equiv \bar{R}(X, Y, Z).$$

Here $\bar{R}(t)$ is the inertially described displacement vector while $\bar{r}(t)$ is the vector in a local coordinate representation. The operator, $T(\varphi^-)$, is the transform matrix needed to obtain the desired results.

*It should be noted that $\bar{R}$ and $\bar{r}$, here, are not dimensionless; for symbolic representations the physical coordinates are used. Thus time is the independent variable, etc.

Once this transformation is completed, and the position vector is properly expressed, then it is a simple task to develop the relative velocity equation. Mathematically this is accomplished by differentiation; symbolically this follows from Eq. (II.91), above, as:

$$\dot{\bar{R}}(t) = T(\varphi^-)\,[\dot{\bar{r}} + \dot{\varphi}\, B\, \bar{r}]\,, \tag{II.92}$$

having taken into account the derivative of Eq. (A.6d), Appendix A. These expressions represent the transformed state equations in dimensional form; the corresponding equations, expressed in dimensionless variables, are:

$$\bar{\mathcal{R}}(\varphi) = T(\varphi^-)\,\bar{\imath}(\varphi),$$

and

$$\bar{\mathcal{R}}'(\varphi) = T(\varphi^-)\,[\,\bar{\imath}\,' + B\,\bar{\imath}\,]\,. \tag{II.93)*$$

Recall, in these formulae, that $\bar{\mathcal{R}}$ and $\bar{\imath}$, and their derivatives, ($\bar{\mathcal{R}}'$, $\bar{\imath}\,'$) are written in the previous dimensionless form; the derivative here is with respect to φ (the independent variable for the dimensionless notation).

<u>The Displacement Equation.</u> A direct application of the first of Eqs. (II.93) leads to the following expressions:

$$I_2\bar{\mathcal{R}}(\varphi) = \left\{T_2(2\varphi^-) + 3\left[J_2 - J_1\right]\right\}\left[A_{a_I}\right]_{i.v.} + T_2(\varphi^-)\left\{I_2 - \frac{3}{2}\varphi\left[B_2J_1\right]\right\}K_{o_I} + T_2(\varphi^-)\Phi_\tau\bar{\tau}_I;$$

wherein $\left[A_{a_I}\right]_{i.v.}$, K_{o_I} and Φ_τ may be deduced and/or are defined in Eqs. (II.34b), and (II.86), respectively; and, for the third component:

$$J_3\bar{\mathcal{R}}(\varphi) = J_3\left\{\bar{\mathcal{R}}_o\cos\varphi + \bar{\mathcal{R}}'_o\sin\varphi + \bar{\tau}_I(1 - \cos\varphi)\right\}. \tag{II.94}$$

For quick reference the coefficient matrices are repeated below:

$$\left[A_{a_I}\right]_{i.v.} \equiv \frac{1}{2}\left\{I_2\bar{\mathcal{R}}_o + \left[J_2 - 2J_1\right]B_2\bar{\mathcal{R}}'_o\right\},$$

$$K_{o_I} \equiv \left[2J_1 - J_2\right]\bar{\mathcal{R}}_o - 2B_2\bar{\mathcal{R}}'_o\,,$$

and

*The nomenclature here is such that $\bar{\imath}(\varphi) \equiv \bar{\imath}(\xi, \eta, \zeta)$; $\bar{\imath}\,'(\varphi) \equiv \bar{\imath}\,'(\xi', \eta', \zeta')$; and $\bar{\mathcal{R}}(\varphi) \equiv \bar{\mathcal{R}}(\Xi, H, Z)$; $\bar{\mathcal{R}}'(\varphi) \equiv \bar{\mathcal{R}}'(\Xi', H', Z')$.

$$\Phi_\tau \equiv \left[2J_1 + 5J_2\right]\left[T_2(\varphi^+) - I_2\right] + \frac{3\varphi}{2}\left[J_1 + 2J_2\right]\left[J_2B_2 + B_2T_2(\varphi^+)\right]$$

$$+ \frac{1}{4}\left[J_1 - 2J_2\right]\left[T_2(\varphi^-) - T_2(\varphi^+)\right]. \qquad \text{(II.95)*}$$

(The reader should remember that the previous A_a and K_o matrices have been revised by including a proper set of initial values; the consequence of those manipulations is noted above as A_{a_I} and K_{o_I}. Also, for this analysis the $\bar{\tau}$ terms of A_a are absent; the new matrix contains only initial values (i.v.) - - thus the subscript).

Equations (II.94) are to be used for the development of in-plane and out-of-plane trace geometries. Subsequently there will be some descriptions of these presented herein.

The Inertially Described Relative Velocity. Following from Eqs. (II.94) one can obtain the velocity definition for the present problem situation. Once again the analytical expressions may be arranged so that there is a logical separation, within the equations, to describe both the Initial-Values Problem and the Zero-Initial-Values Problem. By the application of Eq. (II.93), or by differentiation, it can be shown that one form of the dimensionless velocity equation, referred to the inertial frame, is:

(a), the in-plane component:

$$I_2\bar{\mathcal{R}}'(\varphi) = B_2\left\{2\left[T_2(2\varphi^-)\right]\left[A_{a_I}\right]_{i.v.} + \frac{1}{2}T_2(\varphi^-)\left[2J_2 - J_1 - 3B_2J_1\varphi\right]K_{o_I} + T_2(\varphi^-)\left[\Phi_\tau\right]\bar{\tau}_I\right\} + T_2(\varphi^-)\left[\Phi'_\tau\right]\bar{\tau}_I\,;$$

and, (b), the out-of-plane coordinate:

$$J_3\bar{\mathcal{R}}'(\varphi) = J_3\left\{\bar{\mathcal{R}}'_o\cos\varphi - \bar{\mathcal{R}}_o\sin\varphi + \bar{\tau}_I\sin\varphi\right\}. \qquad \text{(II.96)}$$

*See Eqs. (A.7), Appendix A, for the general form of the equations $(T(\varphi^\pm)$. The subscript $(\sim)_2$, attached to these, infers the two-dimensional form of T (e.g., $T_2 \equiv I_2T$).

The coefficient matrices A_{a_I}, K_{o_I} and Φ_τ are noted in Eqs. (II.95). The Φ'_τ matrix, a derivative of Φ_τ, is found to be:

$$\Phi'_\tau = -\left[2J_1 + 5J_2\right] B_2 T_2(\varphi^+) + \frac{3}{2}\left[J_1 + 2J_2\right]\left\{J_2 B_2 + \left[I_2 - \varphi B_2\right] B_2 T_2(\varphi^+)\right\}$$

$$+ \frac{1}{4}\left[J_1 - 2J_2\right] B_2 \left[T_2(\varphi^-) + T_2(\varphi^+)\right]. \qquad \text{(II.97a)}$$

Summary. Equations (II.95) and (II.96) represent an analytical solution to the relative motion problem for that case when the particle of interest (Q) is acted upon by the central-inverse-square gravitational attraction field of force, and by an externally applied force. This external force is presumed to have fixed components in an inertial frame of reference. Also the state variables here are defined in this inertial frame.

The previously studied problem parallels the present case, except that there the force had its fixed components referred to the rotating frame of reference. When these two situations are compared, directly, one notes that the initial-value problem (i.e. the solution in terms of $\bar{\mathfrak{R}}_o$ and $\bar{\mathfrak{R}}'_o$) is found in both. Some reflection on this will lead to the conclusion that this is a natural consequence,and a logical happening, in view of the methodology applied for both situations. The conclusion then, is that here the only new information which is provided is directly traced to the effects found for terms in $\bar{\tau}_I$. (Remember that $\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_Z)$). Once again the ability to separate the initial-values problem from the forcing function solution (Zero-Initial-Values Problem) is found to be of advantage. An obvious conclusion reached is that now one has the ability to consider the influence of initial values disturbances in the rotating frame, and disturbances in the inertial frame simultaneously or separately. This enhances the investigator's chances to simulate a larger variety of conditions and situations in his analysis. The added ability to combine influences and to examine multi-case situations will lead to a much better understanding of the overall relative motions problem. Of course the specifics of various cases may not be in evidence, but the intuitive grasp of the problem can be significantly increased.

As an aid to understanding the makeup of these solution equations the general formulae, Eqs. (II.95) and (II.96), are expanded and presented below in matrix format. There each coefficient matrix is expressed in terms of appropriate initial-values. (Recall that A_{a_I} and K_{o_I} do not involve the $\bar{\tau}_I$ components. Also, it should be remembered that the initial-values solution here is the same as that for the former case (see Eqs. (II.39) and (II.40); these are repeated for completeness)).

(a). The in-plane displacements may be expressed as:

$$I_2\bar{\mathfrak{R}}(\varphi) \equiv \begin{bmatrix} \Xi(\varphi) \\ H(\varphi) \end{bmatrix} = \begin{bmatrix} \cos 2\varphi - 3 & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi + 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(\Xi_o + 2H'_o) \\ \frac{1}{2}(H_o + \Xi'_o) \end{bmatrix}$$

$$+ \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ -\frac{3\varphi}{2} & 1 \end{bmatrix} \begin{bmatrix} 2(\Xi_o + H'_o) \\ -(H_o + 2\Xi'_o) \end{bmatrix} \right) + \left\{ \begin{bmatrix} 4 - 2\cos\varphi & 5\sin\varphi \\ -2\sin\varphi & \frac{13}{4} - 5\cos\varphi \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} -2\cos 2\varphi & -\frac{7}{4}\sin 2\varphi \\ -2\sin 2\varphi & \frac{7}{4}\cos 2\varphi \end{bmatrix} + 3\varphi \begin{bmatrix} -(\sin\varphi + \frac{1}{4}\sin 2\varphi) & \frac{1}{4}(\cos 2\varphi - 3) \\ \cos\varphi + \frac{1}{4}(\cos 2\varphi + 3) & \frac{1}{4}\sin 2\varphi \end{bmatrix} \right\} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} . \quad \text{(II.97b)}$$

(b). The out-of-plane component is:

$$J_3\bar{\mathfrak{R}}(\varphi) \equiv Z(\varphi) = Z_o \cos\varphi + Z'_o \sin\varphi + \tau_z (1 - \cos\varphi). \quad \text{(II.97c)}$$

(c). The in-plane velocity components are, from Eq. (II.96):

$$I_2\bar{\mathfrak{R}}'(\varphi) \equiv \begin{bmatrix} \Xi'(\varphi) \\ H'(\varphi) \end{bmatrix} = \begin{bmatrix} -\sin 2\varphi & -\cos 2\varphi \\ \cos 2\varphi & -\sin 2\varphi \end{bmatrix} \begin{bmatrix} \Xi_o + 2H'_o \\ H_o + \Xi'_o \end{bmatrix}$$

(equation continued on next page)

$$
+ \begin{bmatrix} -\sin\varphi & -\cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ -\frac{3\varphi}{2} & 1 \end{bmatrix} \begin{bmatrix} 2(\Xi_o + H'_o) \\ -(H_o + 2\Xi'_o) \end{bmatrix} \right)
$$

$$
+ \left\{ \frac{1}{4} \begin{bmatrix} 13\sin 2\varphi & -(9 + 11\cos 2\varphi) \\ (9 - 13\cos 2\varphi) & -(11\sin 2\varphi) \end{bmatrix} + \begin{bmatrix} -\sin\varphi & 5\cos\varphi \\ \cos\varphi & 5\sin\varphi \end{bmatrix} \right.
$$

$$
\left. -3\varphi \begin{bmatrix} \cos\varphi + \frac{1}{2}\cos 2\varphi & \sin 2\varphi \\ \sin\varphi + \frac{1}{2}\sin 2\varphi & -\cos 2\varphi \end{bmatrix} \right\} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} \quad . \tag{II.97d}
$$

(d). The out-of-plane component is given as:

$$
J_3 \tilde{\mathcal{R}}'(\varphi) \equiv Z'(\varphi) = Z'_o \cos\varphi + (\tau_z - Z_o) \sin\varphi . \tag{II.97e}
$$

Following the pattern established earlier, a next task is that of describing geometries which represent motions on the various displacement and hodograph planes. In the next section this is done; there, graphs and mathematical definitions will be displayed.

VII. GRAPHICAL DESCRIPTIONS

Geometric Representations. For the previously described traces, the relative motion figures represented a particular family of general solutions. There the complementary function had added to it results acquired by allowing the specific (perturbative) force play the role of a forcing function. For that case, however, the applied force term was represented by a system of components which were aligned with the local horizon frame of reference.

In this, the present case, the only new information which is to be acquired arises solely from the forcing function -- the applied force which is to be considered. Necessarily the new solution is altered in view of the fact that the "force" is represented, now, by components parallel to the inertially aligned frame of reference. (It is recognized that the complementary solution is not altered; thus, the initial values problem retains its singular identity for both cases).

Assuming that the forcing function is representative of the inertially aligned disturbing force, then the consequences of this, on the geometrically described traces for the motion, will be ascertained. Of course, one must recognize that this quantity will affect motion traces in both frames of reference; therefore, a full investigational expose will evolve and be discussed in the next few paragraphs. Here, then, will be descriptions of the displacement traces and the hodographs, for both frames of reference; but, only the consequence of the disturbing force will appear. Generally speaking, these descriptions will be presented primarily in an analytical format; nevertheless, a few figures are shown to indicate the trends which have been found. For a more complete graphical description the reader is referred to the compendia of results found at the end of this section.

It should be mentioned that for the present only in-plane results are discussed and described. The out-of-plane motions, traces, etc. are deferred until later; at that time all of the out-of-plane motions and geometries will be presented, in one section.

(a). <u>In-Plane Displacements; Rotating Triad</u>. The general analytical equation for this problem is:

$$I_2 \bar{\lambda}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) \left[A_a\right]_{i.v.} + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Phi_\tau \bar{\tau}_I . \qquad \text{(II.98)}$$

However, here, only the partial solution dealing with $\bar{\tau}_I$ is of consequence; therefore interest is focused on:

$$\Delta\left[I_2 \bar{\lambda}(\varphi)\right] \equiv \Phi_\tau \bar{\tau}_I ; \qquad \text{(II.99a)}$$

wherein

$$\Phi_\tau \equiv \left[2J_1 + 5J_2\right]\left[T_2(\varphi^+) - I_2\right] + \frac{3\varphi}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2 + B_2 T_2(\varphi^+)\right]$$

$$+ \frac{1}{4}\left[J_1 - 2J_2\right]\left[T_2(\varphi^-) - T_2(\varphi^+)\right] ;$$

and

$\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_Z)$, with each scalar being a constant.

Examining Eq. (II.99a) one finds that, geometrically, the trace is conveniently constructed as a four vector sum. One vector for loci on an ellipse; two describing line positions; and, one, an archimedian spiral. When the full geometry is displayed it is found that the figure looks as "spiral-like"; hence, it will be classed as such. A description of this system of vectors is developed below:

(1). The first partial solution (vector) will be described by:

$$\delta_1\left[I_2 \bar{\lambda}(\varphi)\right] \equiv \left[2J_1 + 5J_2\right]\left[T_2(\varphi^+) - I_2\right] \bar{\tau}_I . \qquad \text{(II.100a)}$$

Expanded into a matrix format it becomes:

$$\begin{bmatrix} \delta\xi(\varphi) \\ \delta\eta(\varphi) \end{bmatrix}_1 = - \begin{bmatrix} 2(1-\cos\varphi) & -2\sin\varphi \\ 5\sin\varphi & 5(1-\cos\varphi) \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} . \qquad \text{(II.100b)}$$

It is an easy exercise to show that this represents a (2:5) ellipse with an origin at $(\xi, \eta)_c \equiv (-2\tau_{\Xi}, -5\tau_H)$.

An analytic equation for this figure is expressed by the quadric equation:

$$\left(\frac{\delta_1 \xi + 2\tau_{\Xi}}{2\sqrt{\tau_{\Xi}^2 + \tau_H^2}}\right)^2 + \left(\frac{\delta_1 \eta + 5\tau_H}{5\sqrt{\tau_{\Xi}^2 + \tau_H^2}}\right)^2 = 1. \qquad \text{(II.100c)}$$

Necessarily this in-plane trace originates at the coordinate origin; however, the figure size is explicitly dependent on the force magnitude ($|\bar{\tau}_I|$); and, it has a periodicity matching that of the reference orbit.

(2). One of the "line" loci found here comes from:

$$\delta_2\left[I_2 \bar{\kappa}(\varphi)\right] \equiv \frac{3\varphi}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2\right]\bar{\tau}_I . \qquad \text{(II.101a)}$$

As a matrix equation the expression is expanded to:

$$\begin{bmatrix} \delta\xi(\varphi) \\ \delta\eta(\varphi) \end{bmatrix}_2 = \begin{bmatrix} 0 & 0 \\ 3\varphi & 0 \end{bmatrix}\begin{bmatrix} \tau_{\Xi} \\ \tau_H \end{bmatrix} . \qquad \text{(II.101b)}$$

This is a line parallel to the η-axis; its direction is controlled by sgn (τ_H); and, the rate of displacement along the line is $|3\tau_H|$.

(3). A second "line" trace is developed from the partial solution:

$$\delta_3\left[I_2 \bar{\kappa}(\varphi)\right] \equiv \frac{1}{4}\left[J_1 - 2J_2\right]\left[T_2(\varphi^-) - T_2(\varphi^+)\right]; \qquad \text{(II.102a)}$$

or:

$$\begin{bmatrix} \delta\xi(\varphi) \\ \delta\eta(\varphi) \end{bmatrix}_3 = - \begin{bmatrix} 0 & \frac{\sin\varphi}{2} \\ \sin\varphi & 0 \end{bmatrix}\begin{bmatrix} \tau_{\Xi} \\ \tau_H \end{bmatrix} . \qquad \text{(II.102b)}$$

These points (ξ, η) describe a line passing through an origin once each half orbit. The amplitude of oscillation here is:

$$|\delta_3|_{max} \equiv \sqrt{(\tau_{\Xi})^2 + \left(\frac{\tau_H}{2}\right)^2} .$$

(4). The remaining vector from Eq. (II.99a) is:

$$\delta_4\left[I_2\bar{\lambda}(\varphi)\right] \equiv \frac{3\varphi}{2}\left[J_1+2J_2\right]\left[B_2T_2(\varphi^+)\right]\bar{\tau}_I . \qquad \text{(II.103a)}$$

As a matrix, this equation can be recast as:

$$\begin{bmatrix} \delta\xi(\varphi) \\ \delta\eta(\varphi) \end{bmatrix}_4 = \frac{3\varphi}{2}\begin{bmatrix} \sin\varphi & -\cos\varphi \\ 2\cos\varphi & 2\sin\varphi \end{bmatrix}\begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} . \qquad \text{(II.103b)}$$

Geometrically the equation is represented by $r = k\varphi$, when a new set of coordinates $(\xi, \eta/2)$ is used. That is:

$$|\bar{\delta}_4| \equiv \sqrt{(\delta_4\xi)^2+\left(\frac{\delta_4\eta}{2}\right)^2} = \frac{3}{2}\sqrt{\tau_\Xi^{\,2}+\tau_H^{\,2}}\;\varphi . \qquad \text{(II.103c)}$$

It is seen that, here, the spiral's parameter (k) is $\frac{3}{2}\,|\bar{\tau}_I \text{ (in-plane)}|$; the figure completes a circuit once each orbit; and, this solution describes a divergence for the full trace geometry. It is evident from Eq. (II.103b) that the secular influence of $\bar{\tau}_I$ cannot be removed, without eliminating $\bar{\tau}_I$ itself.

When these partial solutions are added the complete in-plane trace, due to $\bar{\tau}_I$, is obtained. Obviously, as the particle moves under this $(\bar{\tau}_I)$ action the secular influence plays the more dominant role. Therefore, in time, the overall trace becomes "spiral-like", and ultimately it will exhibit this characteristic* alone. (See Fig. II.7 for an illustrative sketch).

(b). <u>In-Plane Displacements; Referred to an Inertial Triad.</u> When the above problem is referred to an inertially oriented frame of reference the in-plane trace geometry is found to be significantly altered. In order to examine this in some detail a solution corresponding to (see Eq. (II.94)) must be developed. Thus, the partial solution to be studied next is:

$$\Delta\left[I_2\bar{\mathfrak{R}}(\varphi)\right] \equiv T_2(\varphi^-)\left[\Phi_{\tau}\bar{\tau}_I\right]. \qquad \text{(II.104a)}$$

*A more detailed discussion of these figures can be found in Reference [2]. There it is shown that the individual components of $\bar{\tau}_I$ leads to spiral-like figures also.

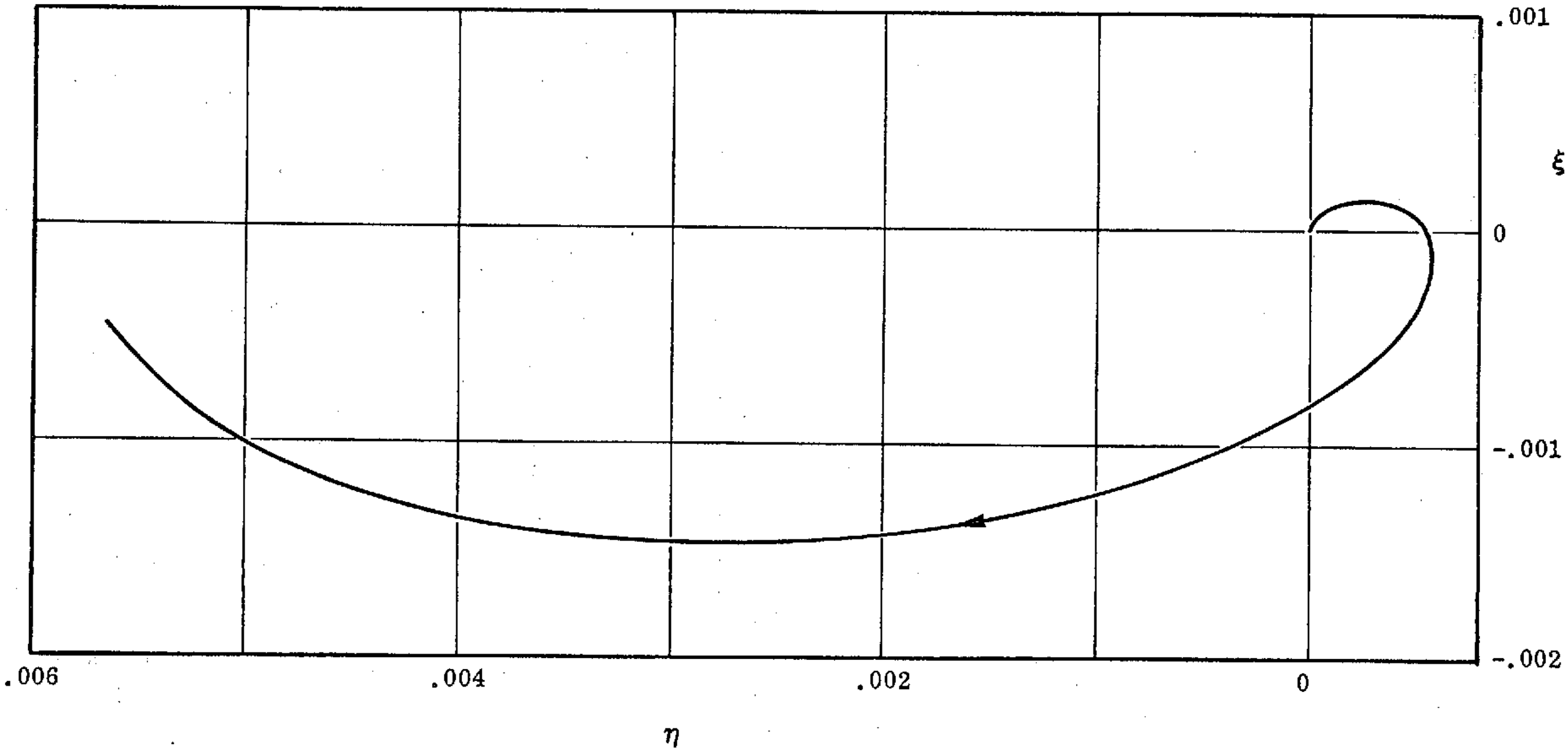


FIG. II. 7. A typical curve, depicting an in-plane displacement trace as produced by the specific force $\bar{\tau}_I$. The sketch is for one orbit of motion; it originates from the coordinate origin; and is spiral-like in character. (See Eqs. (II.103)).

When the coefficient function, $T_2(\varphi^-)[\Phi_\tau]$, is examined it is found that a convenient description of this trace geometry is acquired as a sum of six vectors*. These vectors are developed from an expanded form of Eq. (II.104a). The combinations, which are suggested here, are as follows:

(a). First, taking into account the form of Φ_τ, one part of the solution is:

$$\delta\left[I_2\bar{R}(\varphi)\right] \equiv T_2(\varphi^-)\left\{\left[2J_1+5J_2\right]\left[T_2(\varphi^+)-I_2\right]+\frac{1}{4}\left[J_1-2J_2\right]\left[T_2(\varphi^-)-T_2(\varphi^+)\right]\right\}\bar{\tau}_I. \qquad \text{(II.104b)}$$

This equation leads to a three vector sum. Deleting details for expanding to obtain these, a convenient grouping of terms leads directly to the following three vectors:

$$\delta_i\left[I_2\bar{R}(\varphi)\right] \equiv \begin{bmatrix} \delta\Xi(\varphi) \\ \delta H(\varphi) \end{bmatrix}_{i=1,2,3} \equiv \left\{\begin{bmatrix} 4 & 0 \\ 0 & \frac{13}{4} \end{bmatrix} + \begin{bmatrix} -2\cos 2\varphi & -\frac{7}{4}\sin 2\varphi \\ -2\sin 2\varphi & +\frac{7}{4}\cos 2\varphi \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} -2\cos\varphi & 5\sin\varphi \\ -2\sin\varphi & -5\cos\varphi \end{bmatrix}\right\}\begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} . \qquad \text{(II.104c)}$$

From this expression one can see that $\delta_1[I_2\bar{R}(\varphi)]$ describes a fixed position in the plane of motion. The second component solution represents a circle; $(\delta_2\Xi)^2+(\delta_2 H)^2=(2\tau_\Xi)^2+(\frac{7}{4}\tau_H)^2$; and, the last partial result suggests a second circle. Here, then, one has a circle whose center is shifted from the origin, plus a second circular locus to complete the trace. (This grouping, now, actually represents a two-vector sum).

Yet to be considered is the remainder of Eq. (II.104a). That part of the solution has the secular coefficient; it is:

$$\delta\left[I_2\bar{R}(\varphi)\right] = \frac{3\varphi}{2}\left[T_2(\varphi^-)\right]\left\{\left[J_1+2J_2\right]\left[J_2B_2+B_2T_2(\varphi^+)\right]\right\}\bar{\tau}_I . \qquad \text{(II.104d)}$$

Written in matrix format, this expression expands into:

---

*As seen following Eq. (II.104c) the first three vectors condense to a two-vector system; consequently the full solution is described by a five-vector sum.

$$\delta_i\left[I_2\bar{R}(\varphi)\right] \equiv \begin{bmatrix} \delta\Xi(\varphi) \\ \delta H(\varphi) \end{bmatrix}_{i=4,5,6} = \frac{3\varphi}{4}\left\{\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -4\sin\varphi & 0 \\ 4\cos\varphi & 0 \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} -\sin 2\varphi & \cos 2\varphi \\ \cos 2\varphi & \sin 2\varphi \end{bmatrix}\right\}\begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} \quad . \tag{II.104e}$$

Examining this result one finds that the first ($\bar{\delta}_4$)-vector (which makes use of the constant matrix) describes a line-locus, one which moves monotonically from an origin. The second and third parts of the solution represent spirals; i.e.:

$$|\bar{\delta}_5| \equiv \left[\left(\frac{\delta_5\Xi}{\tau_\Xi}\right)^2 + \left(\frac{\delta_5 H}{\tau_\Xi}\right)\right]^{1/2} = 3\varphi \tag{II.104f}$$

and

$$|\bar{\delta}_6| \equiv \sqrt{(\delta_6\Xi)^2 + (\delta_6 H)^2} = \frac{3}{4}\sqrt{\tau_\Xi^{\,2} + \tau_H^{\,2}}\ \varphi . \tag{II.104g}$$

Unfortunately the spiral $|\bar{\delta}_5|$ is not defined in the coordinate space, per se, but the modified space shown by Eq. (II.104f).

The last trace geometry ($\bar{\delta}_6$) is an archimedian spiral, with a parameter which is proportional to the magnitude of $\bar{\tau}_I$ (in-plane).

The figure for the full solution here is found to be spiral-like, in the large; however, it does exhibit some unusual characteristics near the origin of motion. (A more detailed study of these partial solutions will identify which component (vector) is responsible for this)*. As an example of the trace produced here Fig. II.8 shows a sketch for this case, but with the τ_i-scalars chosen as positive numbers. Incidentally, it can be shown that if $\tau_\Xi \equiv 0$ the resulting trace is a spiral-like curve without the "wiggles" found otherwise present.

(c). In-Plane Hodograph; Referred to a Rotating Triad. An analytical solution for the in-plane relative velocity, referred to a rotating frame of reference, has been obtained as:

$$I_2\bar{r}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right]B_2 T_2(\varphi^-)\left[A_a\right]_{i.v.} - \frac{3}{2}\left[B_2 J_1\right]K_o + \Phi'_\tau \bar{\tau}_I . \tag{II.105}$$

*See Reference [2] for more details on the construction of these traces.

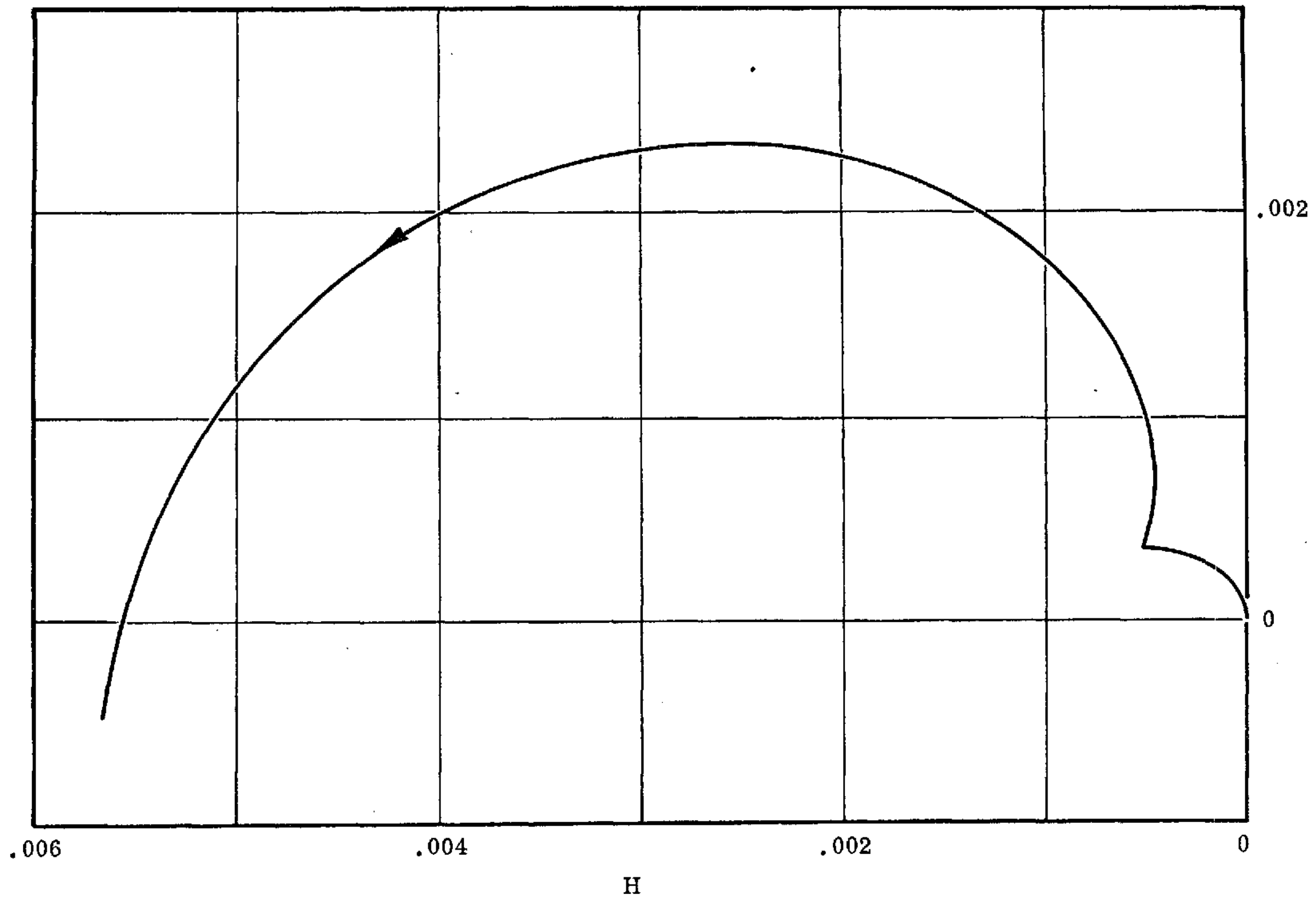


FIG. II.8. An in-plane, relative motion figure in an inertially oriented frame of reference. These displacements result from the application of a specific force, $\bar{\tau}_I$. See Eqs. (II.104) for a description of this case.

However, the partial solution pertaining to $\bar{\tau}_I$ is all that is of interest here; thus, this investigation will examine only the result:

$$\Delta\left[I_2 \bar{k}'(\varphi)\right] \equiv \Phi'_\tau \bar{\tau}_I ; \qquad \text{(II.106)}$$

wherein,

$$\Phi'_\tau = -\left[2J_1 + 5J_2\right] B_2 T_2(\varphi^+) + \frac{3}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2 + (I_2 - B_2 \varphi) B_2 T_2(\varphi^+)\right]$$

$$+ \frac{1}{4}\left[J_1 - 2J_2\right] B_2 \left[T_2(\varphi^-) + T_2(\varphi^+)\right].$$

An analysis of this expression finds that the trace, on the hodograph plane, is composed of a four vector sum. The individual vectors have been selected so that each can be described by a simple geometric figure. The addition of these vectors defines the full trace on the (ξ', η')-plane. The selected vectors are described below:

(1). A first vector, extracted from Eq. (II.106), locates a fixed point on the hodograph plane; i.e.,

$$\delta_1\left[I_2 \bar{k}'(\varphi)\right] \equiv \frac{3}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2\right] \bar{\tau}_I ; \qquad \text{(II.107a)}$$

or, as a matrix expression:

$$\begin{bmatrix} \delta\xi'(\varphi) \\ \delta\eta'(\varphi) \end{bmatrix}_1 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} .$$

It is seen that this fixed locus lies on the η'-axis at a distance $3|\tau_\Xi|$ from the origin.

(2). A second function locus, obtained here, represents an oscillating line point of bounded magnitude which passes through the origin. The partial resultant is:

$$\delta_2\left[I_2 \bar{k}'(\varphi)\right] \equiv \frac{1}{4}\left[J_1 - 2J_2\right] B_2 \left[T_2(\varphi^-) + T_2(\varphi^+)\right] \bar{\tau}_I . \qquad \text{(II.107b)}$$

Expanded into matrix form, it can be shown the equivalent equation is:

$$\begin{bmatrix} \delta\xi'(\varphi) \\ \delta\eta'(\varphi) \end{bmatrix}_2 = \begin{bmatrix} 0 & -\frac{\cos\varphi}{2} \\ -\cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} .$$

(3). Next in complexity is the expression:

$$\delta_3\left[I_2\bar{\lambda}'(\varphi)\right] \equiv \left\{-\left[2J_1 + 5J_2\right] + \frac{3}{2}\left[J_1 + 2J_2\right]\right\}\left[B_2 T_2(\varphi^+)\right]\bar{\tau}_I ; \qquad \text{(II.107c)}$$

or, in a matrix form:

$$\begin{bmatrix} \delta\xi'(\varphi) \\ \delta\eta'(\varphi) \end{bmatrix}_3 = - \begin{bmatrix} \frac{\sin\varphi}{2} & -\frac{\cos\varphi}{2} \\ 2\cos\varphi & 2\sin\varphi \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} .$$

The point loci, defined here, describe a (4:1) ellipse whose principal axes parallel the hodograph axes. Also, it should be noted that if the two vectors, $\bar{\delta}_1$ and $\bar{\delta}_3$, are combined the resulting geometry defines the same ellipse, but with its center shifted to the point located by $\bar{\delta}_1$.

(4). The remaining partial solution, from Eq. (II.106), is:

$$\delta_4\left[I_2\bar{\lambda}'(\varphi)\right] \equiv \left\{\frac{3}{2}\varphi\left[J_1 + 2J_2\right]T_2(\varphi^+)\right\}\bar{\tau}_I , \qquad \text{(II.107d)}$$

which can be rewritten as:

$$\begin{bmatrix} \delta\xi'(\varphi) \\ \delta\eta'(\varphi) \end{bmatrix}_4 = 3\varphi \begin{bmatrix} \frac{\cos\varphi}{2} & \frac{\sin\varphi}{2} \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} .$$

A quadric equation developed from this is found to be:

$$\sqrt{[2(\delta_4\xi')]^2 + [\delta\eta']^2} \equiv |\delta_4| = 3\varphi\sqrt{\tau_\Xi^2 + \tau_H^2} ; \qquad \text{(II.107e)}$$

it describes an archimedian spiral in the $(2\xi', \eta')$-space.

If the four vectors (above) are summed they will describe the overall hodograph trace on the (ξ', η')-plane. Basically this curve is a spiral type (as shown on Fig. II. 9, below). It is seen that the secular influence is apparent only in the $\bar{\delta}_4$-vector; however, this divergent character cannot be removed without eliminating the in-plane force, per se. Such a condition should not be unexpected in view of the fact that the force acts continually, in "fixed" directions; and, the divergence is obvious.

(d). <u>In-Plane Hodograph; Inertial Frame of Reference</u>. A description of the (Ξ', H')-hodograph trace is obtained from the analytical expression:

$$\delta\left[I_2\bar{\mathfrak{R}}'(\varphi)\right] \equiv T_2(\varphi^-)\left[B_2\Phi_\tau + \Phi'_\tau\right]\bar{\tau}_I , \qquad \text{(II.108)}$$

wherein Φ_τ and Φ'_τ are to be found in expressions following Eqs. (II.99a) and (II.106), respectively.

After studying the full equation above it has been ascertained that this hodograph trace is most conveniently defined by a four vector sum, also. The vectors selected for these representations are set down and discussed below:

(1). The first vector used here defines loci on a circle; one whose center is removed from the coordinate origin. The circle is found in the partial solution:

$$\begin{bmatrix} \delta\Xi'(\varphi) \\ \delta H'(\varphi) \end{bmatrix}_1 = \frac{1}{4}\begin{bmatrix} 13\sin 2\varphi & -(9+11\cos 2\varphi) \\ (9-13\cos 2\varphi) & -11\sin 2\varphi \end{bmatrix}\begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} . \qquad \text{(II.109)}$$

(Note that the circle's center is shifted to a position $(\Xi', H')_c = (-\frac{9}{4}\tau_H, \frac{9}{4}\tau_\Xi)$; its radius is $\sqrt{\left(\frac{13}{4}\tau_\Xi\right)^2 + \left(\frac{11}{4}\tau_H\right)^2}$).

(2). A second circle, of single orbit frequency, is also acquired from the general expanded form of Eq. (II.108). This one can be shown to be:

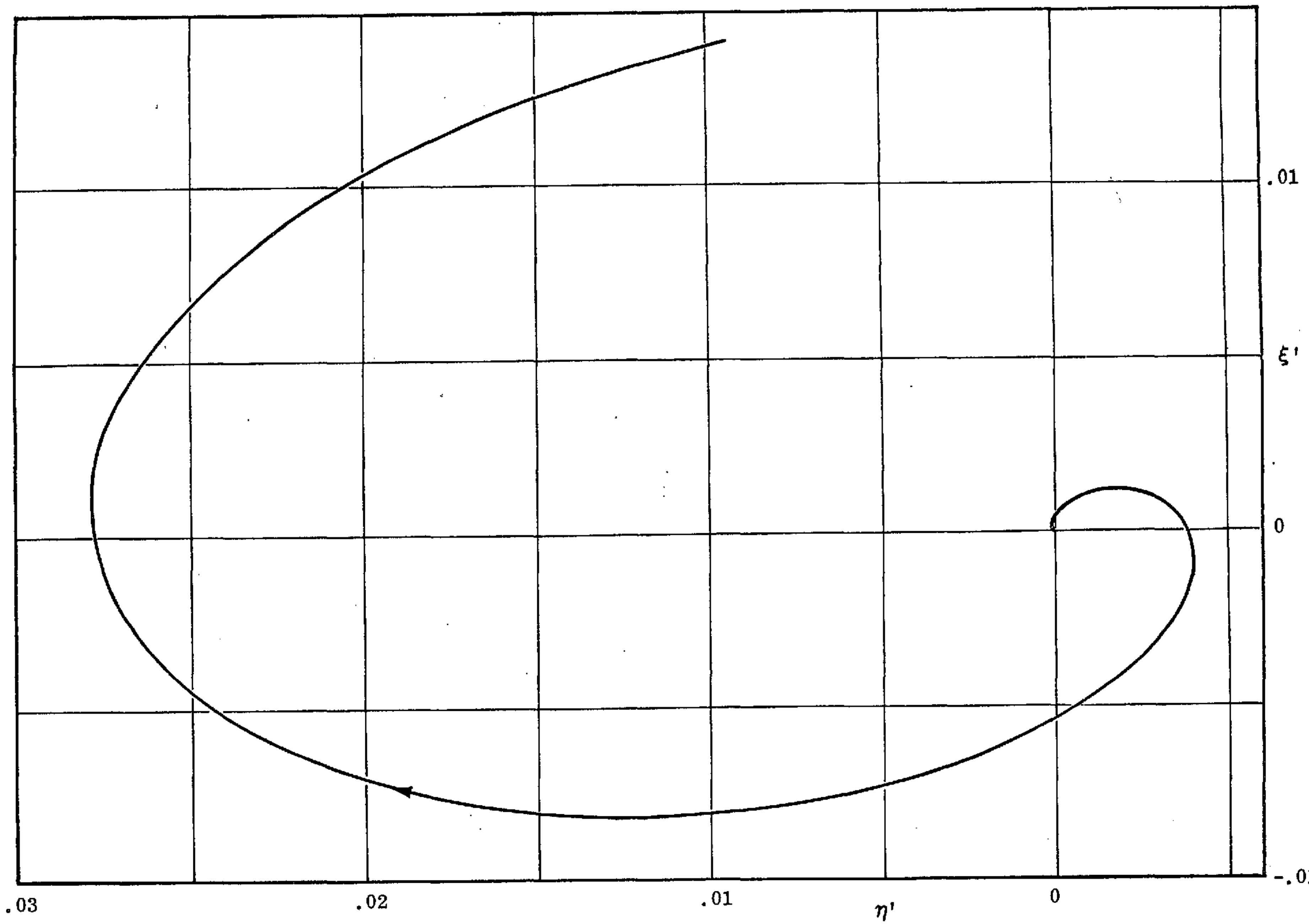


FIG. II. 9. The hodograph (ξ', η') produced from the application of $\bar{\tau}_I$. See Eqs. (II .107).

$$\begin{bmatrix} \delta \Xi'(\varphi) \\ \delta H'(\varphi) \end{bmatrix}_2 = \begin{bmatrix} -\sin\varphi & 5\cos\varphi \\ \cos\varphi & 5\sin\varphi \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} \quad ; \tag{II.110}$$

it describes a circle of radius, $\sqrt{\tau_\Xi^2 + (5\tau_H)^2}$.

The secular aspect of the more general equation will produce two spiral geometries. The simplest of these is:

$$\begin{bmatrix} \delta \Xi'(\varphi) \\ \delta H'(\varphi) \end{bmatrix}_3 = -3\varphi \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} \quad ; \tag{II.111}$$

which can be reshaped into the archimedian spiral equation:

$$|\bar{\delta}_3| \equiv 3\tau_\Xi \varphi .$$

Finally, the remaining terms of the solution equation can be set down as:

$$\begin{bmatrix} \delta \Xi'(\varphi) \\ \delta H'(\varphi) \end{bmatrix}_4 = -\frac{3\varphi}{2} \begin{bmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{bmatrix} \begin{bmatrix} \tau_\Xi \\ \tau_H \end{bmatrix} \quad . \tag{II.112}$$

It should be pointed out that this spiral has double orbit frequency; also, the spiral's parameter (k) is determined to be:

$$k \equiv \frac{3}{2} |\bar{\tau}_I \text{ (in-plane)}| .$$

Summing the four vectors (above) it can be shown that the hodograph's geometry is "spiral-like", generally, but has a frequency of traverse which is double orbit frequency. (See Fig. II.10, below. Since the vector summing introduces considerable geometric complexity, the interested readers should consult Reference [2] for more details on the construction). An observation worth noting is that the trace for $\tau_\Xi = 0$ is of single frequency and has the appearance of a regular spiral. However, the converse situation ($\tau_H = 0$) produces a spiraling trace also, but one of <u>double</u> orbit frequency.

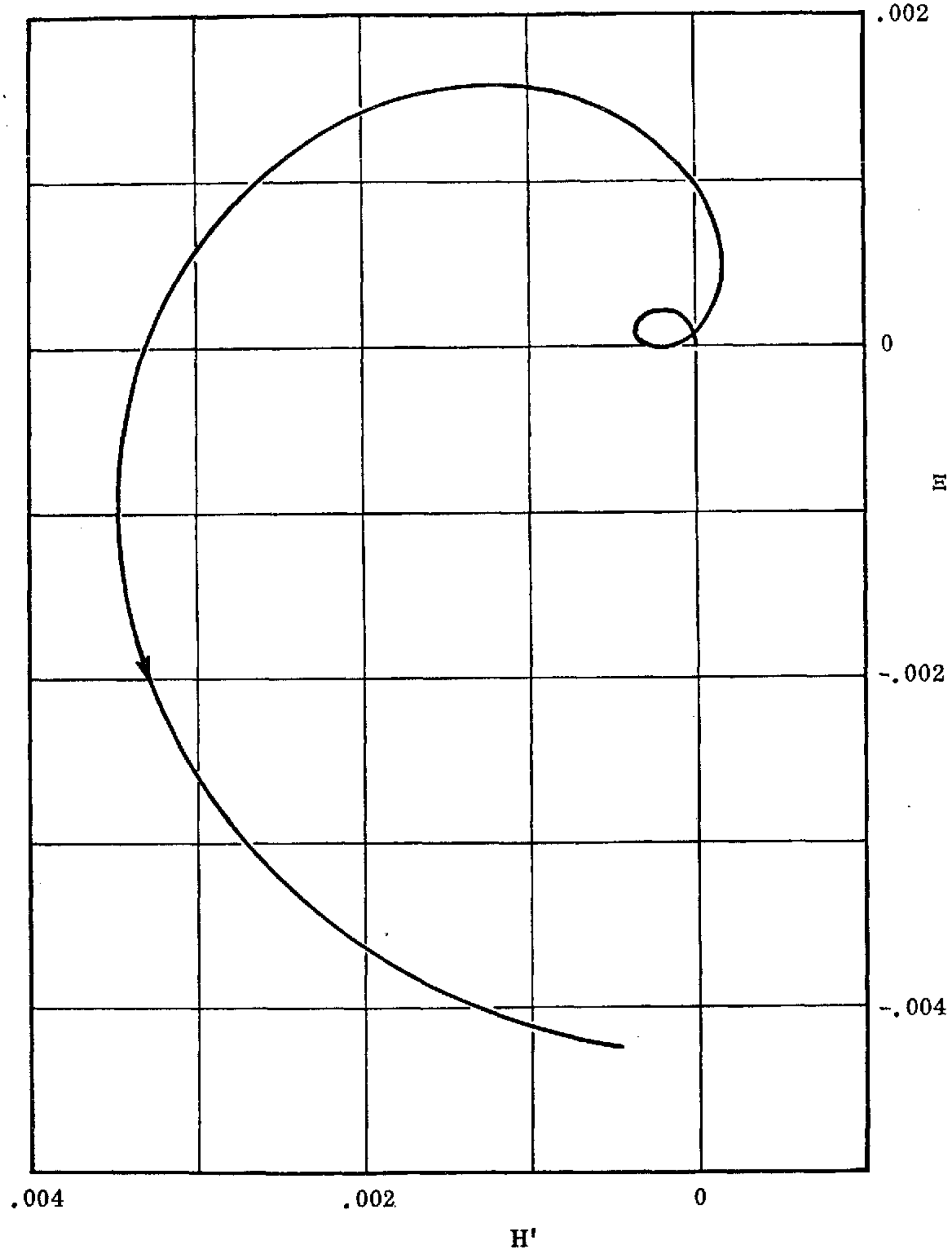


FIG. II. 10. The hodograph, referred to an inertial frame of reference, resulting from the specific force, $\bar{\tau}_I$. See Eqs. (II .109) through (II .112) for a description of this double frequency spiral-like figure.

Summary. The material presented in this section has been given to the description of how the specific force $\bar{\tau}_I$ affects the in-plane state of a relative motion. Primarily, the information set down above has been descriptive, yet the graphical representation of these results is limited. In order to expand this aspect of the investigation, a compendium of results (equations and figures) is presented below. There the reader will find typical traces constructed using positive valued coefficients. This collection of data should provide a more complete and comprehensive understanding of this aspect of the overall problem.

In the section to follow, some comments on out-of-plane traces will be made and some sample situations illustrated and discussed. Following that descriptive material another collection of data will be found; these will provide a compendium for that phase of the motion study.

VIII. DATA SUMMARY

A Compendium of Data for In-Plane Trace Geometries. The following illustrations and information describe the influence of a disturbance $\bar{\tau}_I \, (\equiv \tau_I \, (\tau_\Xi, \tau_H, \tau_z))$ on the in-plane trace geometries. These graphs describe typical situations only; they are the consequence of a set of arbitrarily chosen positive valued parameters. Shown herein are: (1) the general case; and (2) special situations wherein each $\bar{\tau}_I$ component is applied separately.

These data appear in the following order:

(1). In-Plane Displacements, referred to the Rotating Frame of Reference

(2). In-Plane Displacements, referred to the Inertially Oriented Frame

(3). In-Plane Hodographs, referred to the Rotating Frame

(4). In-Plane Hodographs, referred to the Inertial Frame.

In-Plane Displacements

(a) Referrcd to the Rotating Frame of Reference

(b) Referred to the Inertially Oriented Frame of Reference.

In-Plane Displacement Diagram; Local, Rotating Frame of Reference.

The describing equation:

$$I_2 \bar{\lambda}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) \left[A_a\right]_{i.v.} + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Phi_\tau \bar{\tau}_I .$$

Partial Solutions

1. The constructions due to $\left[A_a\right]_{i.v.}$ and K_o are identical to those obtained and described earlier. These are not reported here.

2. The Applied Force Solution (with $\bar{\lambda}_o = \bar{\lambda}'_o = 0$)* is obtained from:

$$\Delta\left[I_2 \bar{\lambda}(\varphi)\right] \equiv \Phi_\tau \bar{\tau}_I ,$$

wherein

$$\Phi_\tau \equiv \left[2J_1 + 5J_2\right]\left[T_2(\varphi^+) - I_2\right] + \frac{3\varphi}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2 + B_2 T_2(\varphi^+)\right] + \frac{1}{4}\left[J_1 - 2J_2\right]\left[T_2(\varphi^-) - T_2(\varphi^+)\right],$$

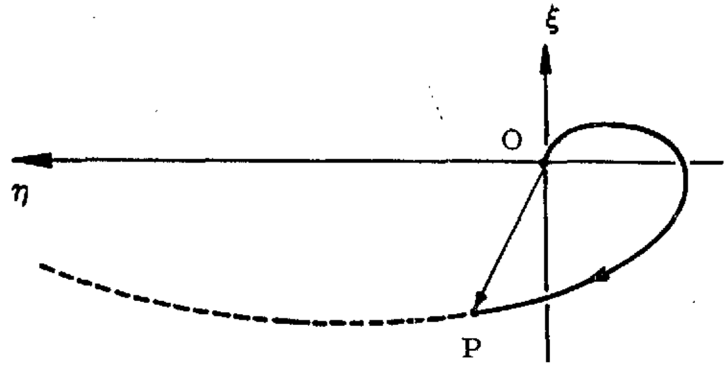
and

$$\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi , \tau_H , \tau_z).$$

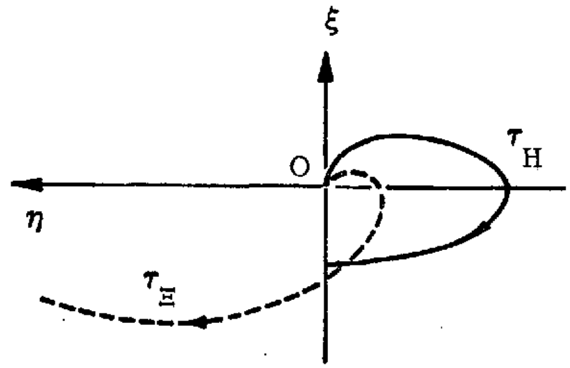
(a). Special Cases.

The situations implied here result in traces due to τ_Ξ and to τ_H alone. Each component is applied separately, producing a trace on the displacement plane. (See sketch 2a).

*The motion traces originate at the coordinate origin.



2. A typical in-plane trace, due to $\bar{\tau}_I$.



2a. Typical traces, in the (ξ, η) plane, due to τ_Ξ and τ_H, respectively.

In-Plane Displacement Diagram; Inertial Frame of Reference.

The describing equation:

$$I_2\bar{R}(\varphi) = \left[T_2(2\varphi^-) + 3(J_2 - J_1)\right]\left[A_{a_I}\right]_{i.v.} + T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}\varphi B_2 J_1\right]K_{o_I} + \Phi_\tau \bar{\tau}_I\right\}.$$

Partial Solutions

1. Constructions due to $\left[A_{a_I}\right]_{i.v.}$ and K_{o_I} have been obtained previously. These are unaltered, in this problem, and do not appear here.

2. The Applied Force Solution (with $\bar{R}_o = \bar{R}'_o = 0$)*.

This partial solution is described from:

$$\Delta\left[I_2\bar{R}(\varphi)\right] \equiv T_2(\varphi^-)\left[\Phi_\tau \bar{\tau}_I\right],$$

wherein:

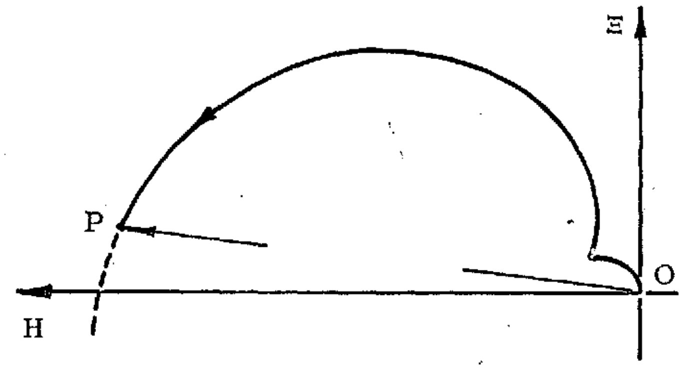
$$\Phi_\tau \equiv \left[2J_1 + 5J_2\right]\left[T_2(\varphi^+) - I_2\right] + \frac{3\varphi}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2 + B_2 T_2(\varphi^+)\right] + \frac{1}{4}\left[J_1 - 2J_2\right]\left[T_2(\varphi^-) - T_2(\varphi^+)\right],$$

and

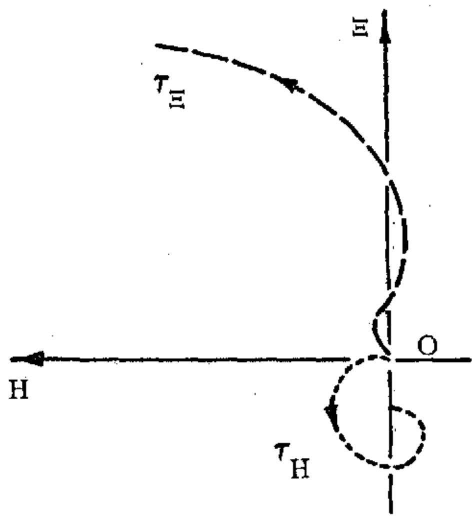
$$\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi,\ \tau_H,\ \tau_z).$$

(a) Special Cases.

The cases examined here lead to traces produced by the components τ_Ξ and τ_H, considered separately. (See sketch 2a).



2. Displacement diagram for τ_Ξ and τ_H applied simultaneously; this is the combined trace from below.



2a. Examples of the traces, due to τ_Ξ and τ_H applied separately.

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*The traces originate at the coordinate origin.

In-Plane Hodographs

(a) Referred to the Local, Rotating Frame of Reference

(b) Referred to the Inertially Oriented Frame of Reference.

In-Plane Hodograph Diagram; Local, Rotating Frame of Reference.

This general hodograph equation is:

$$I_2 \bar{\lambda}'(\varphi) = \left\{\left[I_2 + 3(J_2 - J_1)\right] B_2 T_2(\varphi^-)\right\} A_{a_{i.v.}} - \frac{3}{2}\left[B_2 J_1\right] K_o + \Phi'_\tau \bar{\tau}_I .$$

Partial Solutions

1. Hodographs due to $A_{a_{i.v.}}$ and K_o have been described previously. The contribution due to Φ'_τ is the only new information to be obtained here.

2. The Applied Force Solution ($\bar{\lambda}_o = \bar{\lambda}'_o = 0$)* has its trace described from:

$$\Delta\left[I_2 \bar{\lambda}'(\varphi)\right] \equiv \Phi'_\tau \bar{\tau}_I ,$$

wherein

$$\Phi'_\tau \equiv -\left[2J_1 + 5J_2\right] B_2 T_2(\varphi^+) + \frac{3}{2}\left[J_1 + 2J_2\right]\left\{J_2 B_2 + \left[I_2 - B_2\varphi\right] B_2 T_2(\varphi^+)\right\} + \frac{1}{4}\left[J_1 - 2J_2\right]\left[B_2 T_2(\varphi^-) + B_2 T_2(\varphi^+)\right],$$

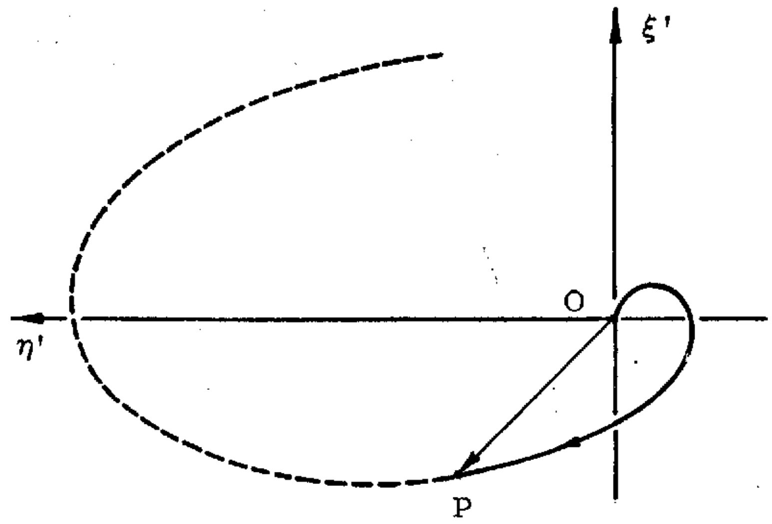
and

$$\bar{\tau}_I \equiv \bar{\tau}_I\,(\tau_\Xi\,,\ \tau_H,\ \tau_z).$$

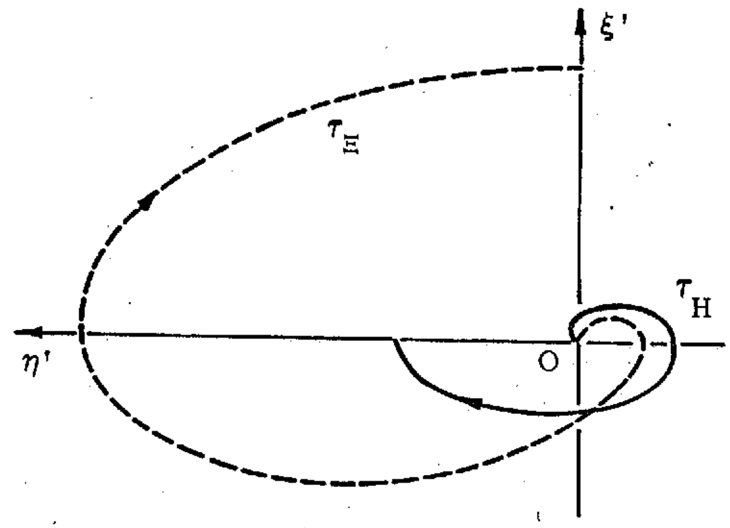
(a). Special Cases.

These situations lead to traces due to τ_Ξ and τ_H as applied separate forces. Typical trace sketches are shown on the right.

*The traces originate at the coordinate origin.



2. Typical hodograph trace, due to $\bar{\tau}_I$.



2a. Hodographs for τ_Ξ and τ_H applied separately.

In-Plane Hodograph Diagrams; Inertial Frame of Reference.

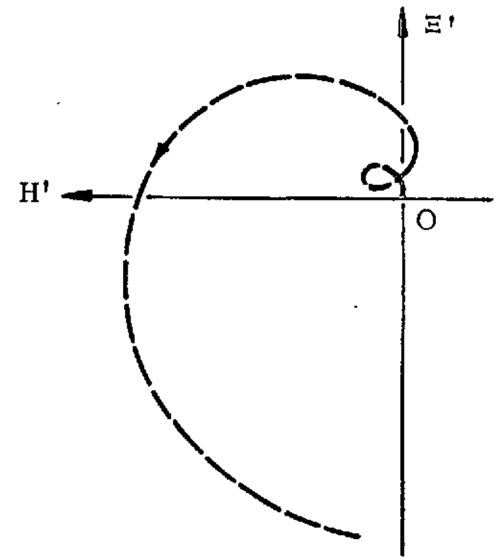
The general hodograph equation is:

$$I_2\bar{\mathcal{R}}'(\varphi) = B_2\left\{\left[2T_2(2\varphi^-)\right]\left[A_{a_I}\right]_{i.v.} + \frac{1}{2}T_2(\varphi^-)\left[2J_2 - J_1 - 3\varphi B_2 J_1\right]K_{o_I} + T_2(\varphi^-)\left[\Phi_\tau \bar{\tau}_I\right]\right\} + T_2(\varphi^-)\left[\Phi'_\tau \bar{\tau}_I\right].$$

Partial Solutions

1. Hodographs due to $\left[A_{a_I}\right]_{i.v.}$ and K_{o_I} have been described previously. These are not altered in this solution; they are not repeated here.

2. The Applied Force Solution (for $\bar{\mathcal{R}}_o = \bar{\mathcal{R}}'_o = \bar{0}$)*, has a trace described from:



2. Hodograph described for $\bar{\tau}_I$ applied to the test particle.

$$\Delta\left[I_2\bar{\mathcal{R}}'(\varphi)\right] \equiv T_2(\varphi^-)\left[B_2\Phi_\tau + \Phi'_\tau\right]\bar{\tau}_I\,,$$

wherein:

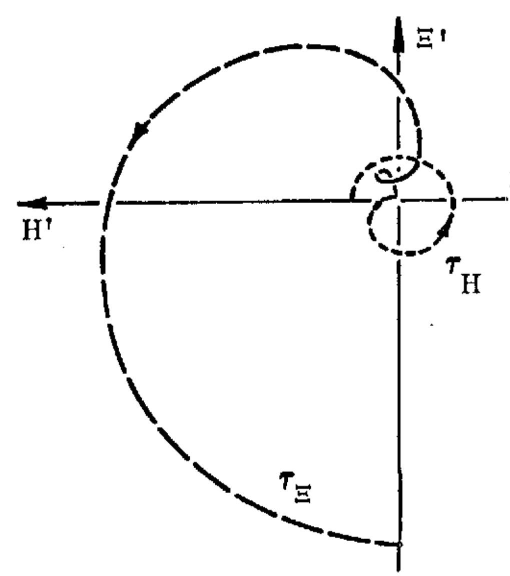
$$\Phi_\tau \equiv \left[2J_1 + 5J_2\right]\left[T_2(\varphi^+) - I_2\right] + \frac{3\varphi}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2 + B_2 T_2(\varphi^+)\right]$$
$$+ \frac{1}{4}\left[J_1 - 2J_2\right]\left[T_2(\varphi^-) - T_2(\varphi^+)\right],$$

$$\Phi'_\tau \equiv -\left[2J_1 + 5J_2\right]B_2 T_2(\varphi^+) + \frac{3}{2}\left[J_1 + 2J_2\right]\left[J_2 B_2\right.$$
$$\left. + (I_2 - B_2\varphi)B_2 T_2(\varphi^+)\right] + \frac{1}{4}\left[J_1 - 2J_2\right]\left[B_2 T_2(\varphi^-) + B_2 T_2(\varphi^+)\right],$$

and $\quad \bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_z).$

(a) Special Cases.

These cases lead to traces produced by the separate actions of τ_Ξ and τ_H. Sketches are shown at the right.



2a. Separate effects of τ_Ξ and τ_H on hodograph.

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*The traces originate at the coordinate origin.

IX. OUT-OF-PLANE TRACE DESCRIPTIONS

Out-of-Plane Motion Traces. So far, in these discussions, the relative motion have been restricted to the in-plane cases. Now, in this section, the corresponding out-of-plane representations will be examined and illustrated. Once again the motions, as seen in both frames of reference and for the various problem types previously noted, will be discussed.

Earlier in this investigation it was noted that the in-plane coordinates were coupled, hence there was a convenient and compact method of notation available to describe those relative motions. For the situations to be examined here, however, the coordinates and results are not coupled; this leads to an inconsistency in notation, one which adds a degree of complexity to the present case studies. In this regard it is not so conveient, now, to describe the geometric traces as simple vector sums; nor is it convenient to represent these in terms of elementary curve forms. Thus, only a selected number of cases will be detailed below; for a more comprehensive discussion of these geometries the interested reader should consult Reference [2].

After a review of the various situations described earlier it seems that a best grouping, for present purposes, would be according to physical conditions. That is, the initial-values-problems will be considered separate from the force disturbance cases. In addition, within each of these categories the graphical representations in the two reference frames will be described. Finally, included in with these will be discussions of the displacement and hodograph traces which develop for each.

(1a). Initial-Values Problem Displacements; Rotating Frame of Reference. The analytical solution for this problem type can be acquired directly from the general results, Eqs. (II.22) or (II.28). Since here, $\bar{\tau} = \bar{0}$, necessarily, it follows that the out-of-plane traces retain those general characteristics which are typical of the in-plane cases. That is, these traces will exhibit the secular character found earlier; however, this effect can be deleted by imposing the same restrictions as before. In this regard, if $K_1 = 0$ (i.e.; $2J_1\bar{\ell}_o - B_2\bar{\ell}'_o \equiv 0$), by either of the obvious means, then the geometric divergence is removed and closed figures result. Then, if an added constraint is imposed, one whereby motion originates at the origin ($\bar{\ell}_o = \bar{0}$), the resulting out-of-plane curves are found to be line traces* and ellipses*. On the other hand, if the non-secularity restriction is relaxed, here, but the condition of $\bar{\ell}_o \equiv \bar{0}$ is retained then the line trace becomes an s-shaped on the (η, ζ)-plane and the divergence reappears.

The several traces described above are not shown below in graphic form. Rather, the interested reader should examine the compendium of results, located at the end of this section, or he may consult either Reference [1] and/or [2]. These documents are more explicit regarding the geometric construction of figures which evolve from the various situations noted above.

(1b). Initial-Value Problem Displacements; Inertial Frame of Reference. This study also exhibits a secular characteristic for the out-of-plane traces. As was mentioned in the earlier discussions the geometric divergence can be removed if a particular combination of initial values is introduced. In terms of the solution format which has been developed, this secular influence vanishes for $(K_1)_I \equiv 0$ (i.e., when $J_1\bar{\mathfrak{R}}_o - J_1B_2\bar{\mathfrak{R}}'_o \equiv 0$). To see the consequence of this a bit more clearly, consider the situation of a relative motion originating at the origin ($\bar{\mathfrak{R}}_o = \bar{0}$). For this situation the geometry's divergence is removed when $H'_o \equiv 0$. Physically, this implies that the test particle (Q) is set into motion by an impulsive action applied normal to the motion of the main (or reference) particle. The subsequent displacement diagram traced out by the test particle will be a closed curve, one which returns to the origin periodically.

Analytical expressions describing these various out-of-plane traces, for those conditions indicated above, are acquired from the general solution, Eqs. (II.35); this particular non-secular case, for $H'_o = 0$, is expressed by:

*From a broad point of view this motion may be considered as being represented by two ellipse-like traces; one a true ellipse, the other (the line) being a limit case for the ellipse.

$$\Xi(\varphi) = 2\Xi'_0 (\sin\varphi - \frac{1}{4}\sin 2\varphi),$$

$$H(\varphi) - \frac{3}{2}\Xi'_0 = 2\Xi'_0 \left(\frac{1}{4}\cos 2\varphi - \cos\varphi\right), \qquad (II.113)$$

and

$$Z(\varphi) = Z'_0 \sin\varphi.$$

(Note that the speed components which appear here satisfy those non-secular initial conditions mentioned above. Sketches developed from these expressions are found on Fig. II.11, below).

(1c). Initial Values Problem Hodograph; Rotating Frame of Reference. The hodographs for this motion bears the same constraints and restrictions as did the displacement diagrams in paragraph (1a) above. The general expressions for this case (with $\bar{\tau} \equiv \bar{0}$) are given by Eqs. (II.22) or (II.29); consequently the restrictions imposed on this system will modify these equations accordingly.

The general trace geometries which evolve here are found to be skewed ellipses. However, the non-secular cases are found to be ellipses symmetric to the coordinate axes.

Following with the comments noted in paragraph (1a), above, curves for these traces are not included below. This graphical information is also found in References [1] and [2]; however, for reference purposes, there are sketches in the data compendium attached to this section.

(1d). The Initial Values Problem Hodograph; Inertial Frame of Reference. Equation (II.36) is the general expression for the relative speeds; that expression includes $\bar{\tau}$ as well as the initial state parameters. Of course, in this, the present case, $\bar{\tau} = \bar{0}$.

Hodographs for the initial values problem generally exhibit a divergent shape. Nevertheless, as indicated in paragraph (1b) this characteristic can be removed by setting $(K_1)_I = 0$. Since these traces are not too well known, a sample set is included below, for references purposes*. Interestingly, the

*Also, see Reference [2], and the data compendium at the end of this section.

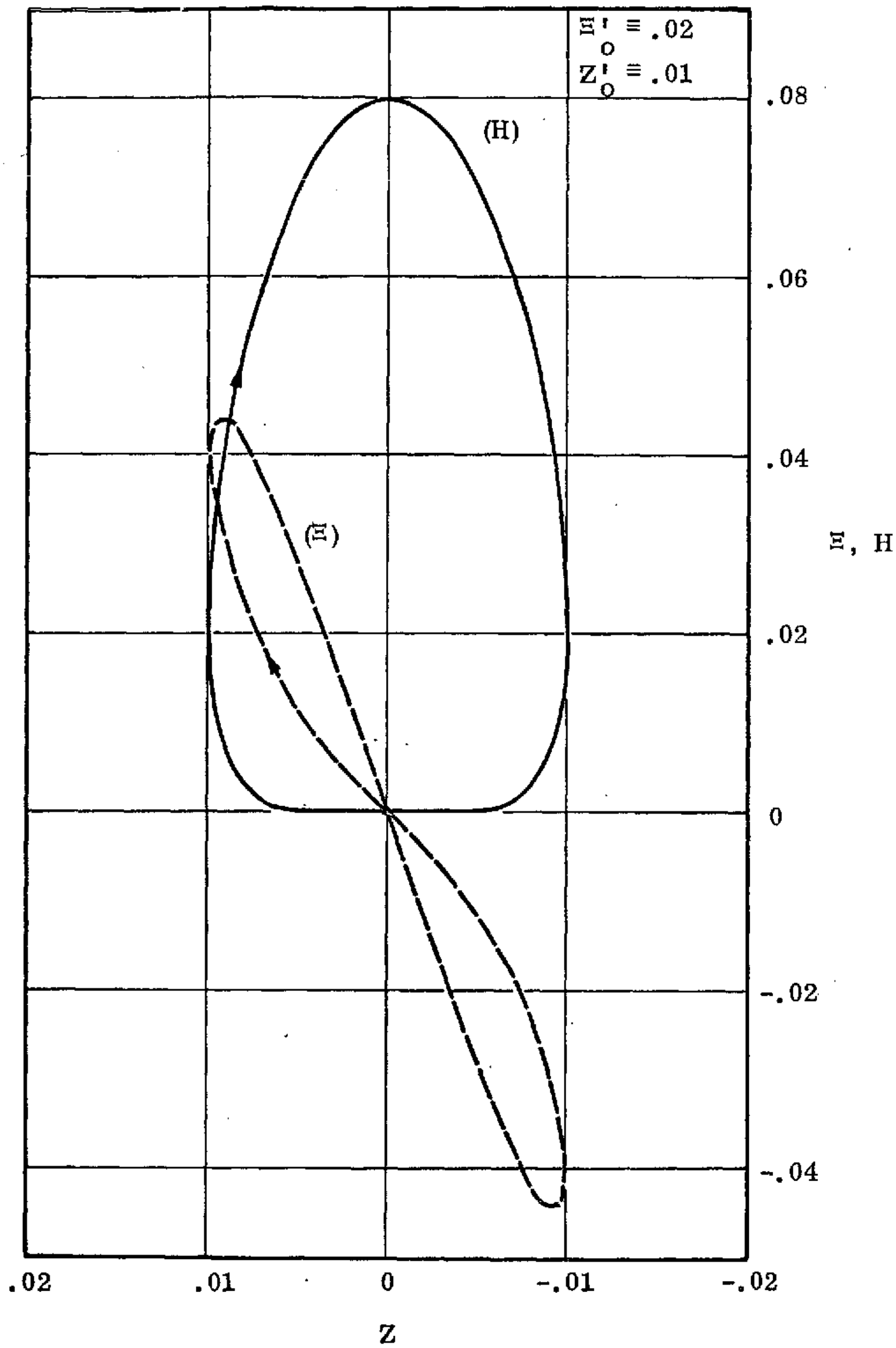


FIG. II.11a. A sketch showing out-of-plane displacements (Ξ, Z; H, Z) for a non-secular situation with motion originating from the origin.

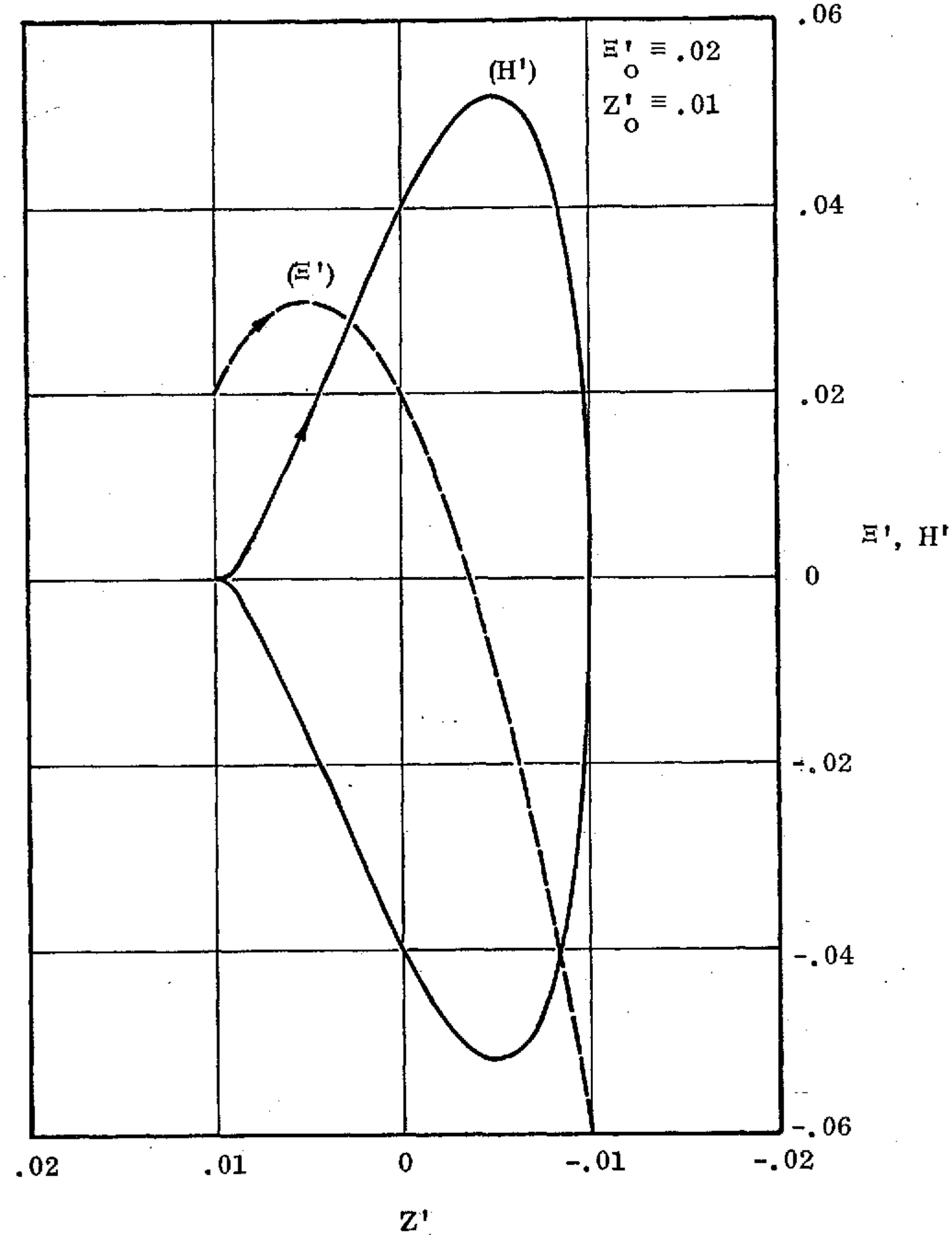


FIG. II.11b. Sketch showing the out-of-plane hodographs (Ξ', Z'; H', Z') corresponding to the non-secular displacements on the preceding figure.

graphs exhibit a definite skewed appearance, for the parameter inputs used. (It is mentioned that these initial inputs were arbitrarily chosen as positive quantities; obviously, varying these terms would introduce a change in the geometry of these figures).

2. Zero Initial-Values Problems. The problem types referred to here are a consequence of the forcing quantities, $\bar{\tau}$ and $\bar{\tau}_I$. It should be recalled that these vectors represent specific (dimensionless) force systems which are aligned with the rotating and inertial triads, respectively.

(a). In this first situation $\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta)$. The analytical expressions illustrating the influence of this parameter on a relative motion are acquired from the same general equations noted earlier. As a result, the general statements set down below follow accordingly.

First, the effect of $\bar{\tau}$ (alone) on the motion traces, that is both displacement and hodograph traces, is one of a secular nature. There are no non-trivial force component relationships which can be introduced to remove this divergence. Likewise the curve forms which result here are not simple ones. Therefore, it is quite difficult to postulate, a priori, the shape of the various curves which are to be found*. (Incidentally, the comments above are not directed to either frame of reference, in particular; these generalizations hold for both).

In view of the geometric complexities which arise for this class of problems, it is not advisable to attempt an in depth analysis of the traces. However, as an alternate, typical curves are included in the data section which follows this; also, some generalizations may be gathered from a study of those figures.

(b). The specific force system aligned with the inertial reference triad, $\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_Z)$ has much the same influences on the relative motion as does

*See the data compendium for representative sketches.

$\bar{\tau}$. That is, the displacements and speeds are found to exhibit a secular character when either or both in-plane components are present. Therefore, the only means available to nullify this influence is to eliminate both τ_{Ξ} and τ_{H} simultaneously.

(As noted before, these comments apply to both coordinate reference systems; also, the trace geometries*, for both, are affected accordingly).

From the discussions above it should be evident that these disturbance force systems lead to rather complicated relative motion figures. Consequently no concerted effort has been made, herein, to codify and classify them. Instead, as mentioned elsewhere in these paragraphs, a collection of representative sketches has been made; these are found in the data compendium.

In the event that a reader would like a more in-depth discussion and description of these traces he should consult Reference [2]. That document contains added details on these constructions.

Summary. In the foregoing paragraphs the various out-of-plane traces have been broadly referred to. Generally, it is known that these figures are more complicated, in makeup, than the companion in-plane curves. Also, mathematically, the functional form for these graphs is more complex than that for the in-plane cases. However, when one studies these it is found that there is a geometric pattern present; and that some estimation of a particular trace may be made, a priori. Unfortunately, it would be rather cumbersome to attempt to develop a sense, or feeling, for these by means of written descriptions. Consequently, the reader is advised to study the attached sketches and to acquaint himself with their overall particulars.

*See the data compendium for typical traces.

X. DATA SUMMARY

A Compendium of Data for the Out-of-Plane Relative Motion Cases. The information presented in the following section is a condensation and illustration of the trace geometries which could appear as the out-of-plane displacement and hodograph diagrams. Since the case studies provided here include the full range of problem types considered in this investigation, there is a fairly large number of figures, etc. which appear. As an aid toward clarifying the arrangement of materials presented here, a brief indexing is included below:

(a). The several subsections are arranged in the following order:

1. Displacements referred to the Rotating Frame of Reference.
2. Displacements referred to the Inertially Oriented Frame of Reference.
3. Hodographs for the Rotating Reference Frame.
4. Hodographs for the Inertial Reference Frame.

Within each of the above subsections the material is arranged as follows:

1. Initial Values Problems ($\bar{\tau}$, $\bar{\tau}_I = 0$).
2. Zero Initial Values Problems ($\bar{\tau} \neq 0$; $\bar{\tau}_I = 0$; Initial relative motion state set to zero.)
3. Cases for Non-Secular Displacements.
4. The influence of $\bar{\tau}_I$ (alone) on the relative motion. (Some special situations are noted.)

For all of the illustrations shown below, the input parameter (initial state values, $\bar{\tau}$ and $\bar{\tau}_I$ components) were arbitrarily selected as positive valued constants. In this regard the "typical trace geometries" will have characteristics related to this selection of quantities. However, it is not too difficult a task to examine the pertinent analytical expressions and ascertain the effect of (say) a particular negative valued constant. What does present difficulties is a determination of how the magnitude of the various inputs would influence the traces. In many cases it would be necessary to examine the output, in detail, to find an adequate answer to these conditions. Nevertheless one should be aware that the figures found herein are typical of these various influences; and that these general shape characteristics are representative of the various cases studied. One final comment regarding the illustrations: in all instances the figures represent the relative motion for one orbit of the reference particle, only. Also, the various trace geometries are not reproduced here to a consistent scale; the primary motivation for this collection of data is to present, in concise format, illustrative information, primarily.

Out-of-Plane Displacements, for the Rotating Frame of Reference

Out-of-Plane Displacement Diagrams; Rotating Frame of Reference

The expressions defining these displacement diagrams are: (a) the in-plane equation, and (b) the third component relationship.

(a) $$I_2 \bar{r}(\varphi) = \left[4J_2 - 2J_1\right] T_2(\varphi^-) A_a + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Psi_\tau \bar{\tau}.$$

(b) $$J_3 \bar{r}(\varphi) = J_3\left[\bar{r}_o \cos\varphi + \bar{r}'_o \sin\varphi + \bar{\tau}\,(1 - \cos\varphi)\right].$$

Partial Solutions:

1. Construction for Initial State Values ($\bar{\tau} = \bar{0}$).

The scalar displacement equations are:

$$\xi(\varphi) = 2\,(-A_1 \cos\varphi + A_2 \sin\varphi) + K_1,$$

$$\eta(\varphi) = 4(A_1 \sin\varphi + A_2 \cos\varphi) - \frac{3}{2}\varphi K_1,$$

$$\zeta(\varphi) = \zeta_o \cos\varphi + \zeta'_o \sin\varphi\ ;$$

wherein

$$A_1 \equiv \frac{1}{2}\left[K_1 - J_1 \bar{r}_o\right];\quad A_2 \equiv \frac{1}{2}\left[J_2 B_2 \bar{r}'_o\right];$$

$$K_1 \equiv 2\left[2J_1 \bar{r}_o - J_1 B_2 \bar{r}'_o\right];\quad K_2 \equiv J_2 \bar{r}_o - 2J_2 B_2 \bar{r}'_o.$$

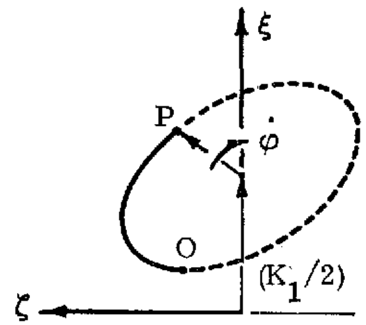
Out-of-Plane Displacement Diagrams; Rotating Frame of Reference (continued)

Quadric equations for these planar traces are:

(a) For the (ξ, ζ)-plane;

$$\left[\frac{\xi(\varphi)-K_1}{2}\right]^2 + [\zeta(\varphi)]^2 = \frac{1}{2}\left[(A_1^2+\zeta_o^2)+(A_2^2+\zeta_o'^2)\right] + \frac{1}{2}\left[(A_1^2+\zeta_o^2)-(A_2^2+\zeta_o'^2)\right]\cos 2\varphi + \left[\zeta_o\zeta_o' - A_1A_2\right]\sin 2\varphi$$

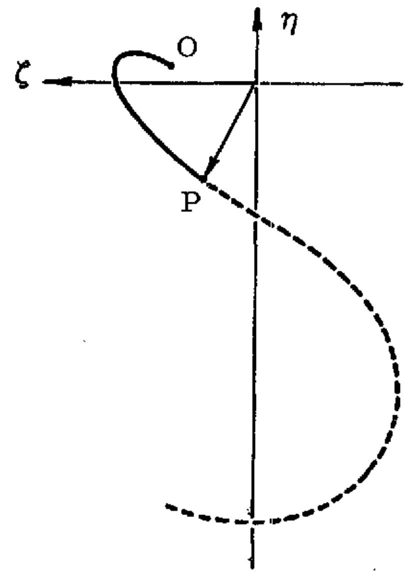
This trace is a skewed ellipse with its geometric center not at the coordinate origin. The principal axes of the figure do not coincide with the coordinate axes.



(1a). O is the initial locus of the trace;
P is a general position on the curve!

(b) For the (η, ζ)-plane;

$$\left[\frac{\eta(\varphi)-K_2}{4} + \frac{3}{8}K_1\varphi\right]^2 + [\zeta(\varphi)]^2 = \frac{1}{2}\left[(A_1^2+\zeta_o'^2)+(A_2^2+\zeta_o^2)\right] + \frac{1}{2}\left[(A_2^2+\zeta_o^2)-(A_1^2+\zeta_o'^2)\right]\cos 2\varphi + \left[A_1A_2+\zeta_o\zeta_o'\right]\sin 2\varphi.$$



(1b). This trace is not a closed curve due to the secular effect of η. The figure produced is a "moving" ellipse, skewed with respect to the coordinate axes. Note that the secular influence vanishes if $K_1 = 0$.

2. Traces for the Zero Initial State $(\bar{\lambda}_o = \dot{\bar{\lambda}}_o' = 0;\ \bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta))$, are defined from:

$$\xi(\varphi) = \tau_\xi(1-\cos\varphi) + 2\tau_\eta(\varphi - \sin\varphi),$$

$$\eta(\varphi) = 4\tau_\eta(1-\cos\varphi) - 2\tau_\xi(\varphi-\sin\varphi) - \frac{3\varphi^2}{2}\tau_\eta,$$

$$\zeta(\varphi) = \tau_\zeta(1-\cos\varphi).$$

Equations for the Planar Geometries

(a) Traces for (τ_ξ, τ_ζ) only. (τ components > 0) defined from:

$$\xi(\varphi) = \tau_\xi (1 - \cos\varphi),$$

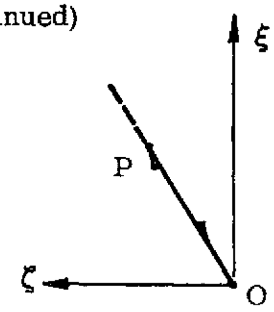
$$\eta(\varphi) = -2\tau_\xi (\varphi - \sin\varphi),$$

$$\zeta(\varphi) = \tau_\zeta (1 - \cos\varphi);$$

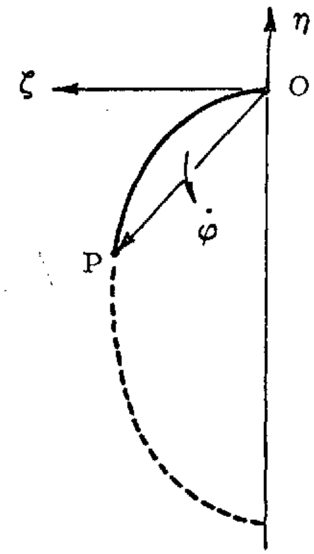
hence,

(a,1) $$\frac{\xi(\varphi)}{\tau_\xi} = \frac{\zeta(\varphi)}{\tau_\zeta}\ ; \text{ and } \frac{\eta(\varphi)}{2\tau_\xi} = -(\varphi - \sin\varphi),$$

(a,2) $$\frac{\zeta(\varphi)}{\tau_\zeta} = 1 - \cos\varphi.$$



(a,1). This trace is:
A line, originating at the origin; frequency of motion is equal to orbit frequency.



(a,2). This trace is:
A cycloid, originating at the origin, incrementing in the negative η-direction.

(b) Traces for (τ_η, τ_ζ) only. (τ components > 0)

$$\xi(\varphi) = 2\tau_\eta (\varphi - \sin\varphi),$$

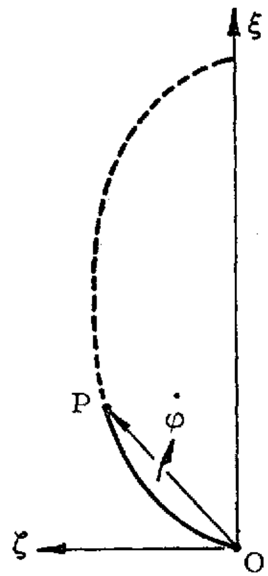
$$\eta(\varphi) = 4\tau_\eta \left[(1 - \cos\varphi) - \frac{3\varphi^2}{8}\right],$$

$$\zeta(\varphi) = \tau_\zeta (1 - \cos\varphi);$$

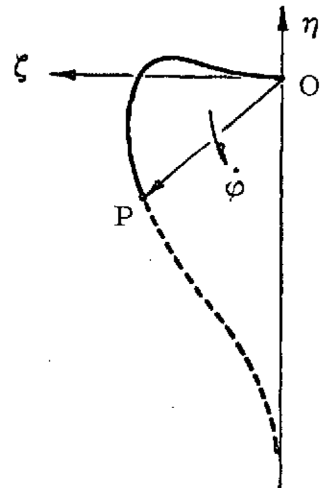
hence, $$\frac{\xi(\varphi)}{2\eta_\tau} = \varphi - \sin\varphi, \qquad \frac{\eta(\varphi)}{2\tau_\eta} = \frac{\zeta(\varphi)}{\tau_\zeta} - \frac{3\varphi^2}{8}\ ; \text{ or,}$$

and $$\frac{\zeta(\varphi)}{\tau_\zeta} = 1 - \cos\varphi. \qquad \text{and} \qquad \frac{\eta(\varphi)}{4\tau_\eta} + \frac{3\varphi^2}{8} = \frac{\zeta(\varphi)}{\tau_\zeta}.$$

(b,1) (b,2)



This is a:
(b,1). Cycloidal trace, beginning at origin; its frequency equals the orbit frequency.



The trace is:
(b,2). A positive sloped line, with an accelerated incrementing locus, in the negative η-direction.

Out-of-Plane Displacement Diagrams; Rotating Frame of Reference (continued)

(c) Displacement Geometries for $(\tau_\xi, \tau_\eta, \tau_\zeta)$.

Note: The resulting traces (here) are combinations of the effects exhibited in (a) and (b) above.

On the (ξ, ζ) plane, the displacement curve is described as:

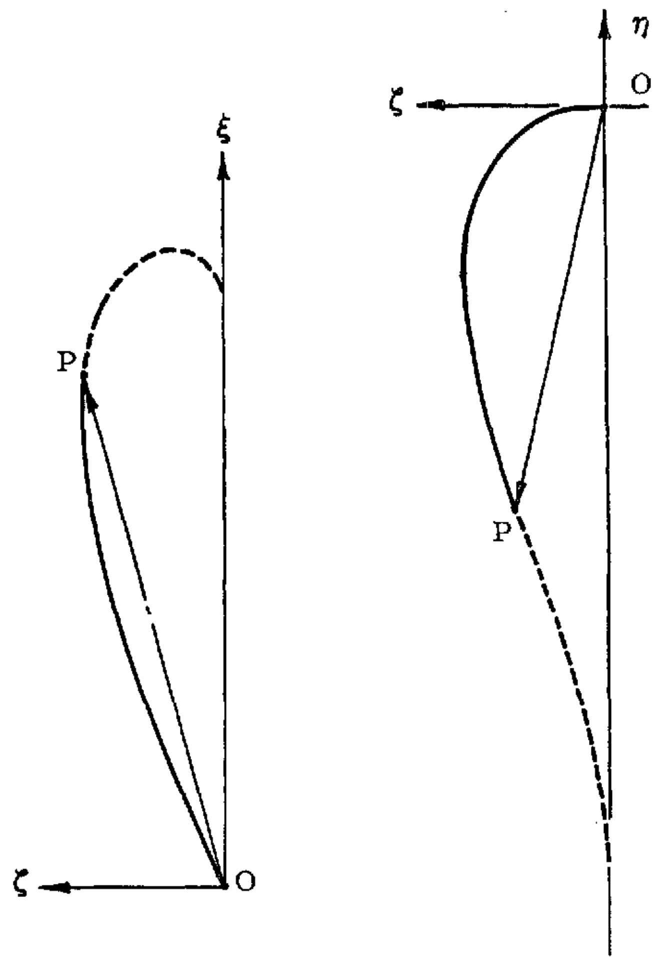
$$\text{(c,1)} \quad \frac{\xi(\varphi)}{\tau_\xi} - \frac{2\tau_\eta}{\tau_\xi}(\varphi - \sin\varphi) = \frac{\zeta(\varphi)}{\tau_\zeta} \; .$$

The straight line trace here is modified by the cycloid (due to τ_η); consequently, the ξ-symmetry of the cycloid is skewed (as shown).

On the (η, ζ) plane, the displacement curve is described by:

$$\text{(c,2)} \quad \frac{\eta(\varphi)}{2\tau_\xi} = -1\,(\varphi - \sin\varphi) + \left[\frac{2\zeta(\varphi)}{\tau_\zeta} - \frac{3\varphi^2}{4}\right]\frac{\tau_\eta}{\tau_\xi} \; .$$

Here, the cycloid (from τ_ξ) is modified (by the line-with-accelerating secular effects) to produce the complex geometry shown at right.



(c,1). (c,2).

Relative motion displacements, traced for one orbit, illustrating a typical situation as produced by a fixed specific force, $\bar{\tau}$.

3. Traces for Displacements, without Secular Terms;
Modification of the Initial Values Solution ($\bar{\tau} \equiv \bar{0}$).

Note: For the elimination of secular effects K_1, of $K_o \equiv K_o(K_1, K_2)$, must vanish. Since $K_1 \equiv 2[2J_1 \bar{\lambda}_o - J_1 B_2 \bar{\lambda}'_o]$, there are two means by which $K_1 = 0$; namely, $J_1 \bar{\lambda}_o = J_1 B_2 \bar{\lambda}'_o = 0$, or, $J_1 B_2 \bar{\lambda}'_o = 2J_1 \bar{\lambda}_o$.

The case to be illustrated below sets K_1 to zero without regard to how this occurs; i.e., K_1 is eliminated from all coefficients, per se. Hence: (see part 1), $A_1 \equiv -\frac{1}{2}[J_1 \bar{\lambda}_o]$, $A_2 = \frac{1}{2}[J_2 B_2 \bar{\lambda}'_o]$, $K_2 = J_2 \bar{\lambda}_o - 2J_2 B_2 \bar{\lambda}'_o$; and consequently:

$$\xi(\varphi) = 2\left[\frac{1}{2}\xi_o \cos\varphi + \frac{1}{2}\xi'_o \sin\varphi\right],$$

$$\eta(\varphi) = 4\left[-\frac{1}{2}\xi_o \sin\varphi + \frac{1}{2}\xi'_o \cos\varphi\right] + K_2 ,$$

$$\zeta(\varphi) = \zeta_o \cos\varphi + \zeta'_o \sin\varphi .$$

Out-of-Plane Displacement Diagrams; Rotating Frame of Reference (continued)

Quadric Equations for the Planar Traces.

(a) For the Geometry on the (ξ, ζ)-plane.

$$\left[\frac{\xi(\varphi)}{2}\right]^2 + [\zeta(\varphi)]^2 = \frac{1}{2}\left\{\left[\left(\frac{1}{2}\xi_o'\right)^2 + (\zeta_o')^2\right] + \left[\left(\frac{1}{2}\xi_o\right)^2 + (\zeta_o)^2\right]\right\}$$

$$+ \frac{1}{2}\left\{\left[\left(\frac{1}{2}\xi_o\right)^2 + (\zeta_o)^2\right] - \left[\left(\frac{1}{2}\xi_o'\right)^2 + (\zeta_o')^2\right]\right\}\cos 2\varphi$$

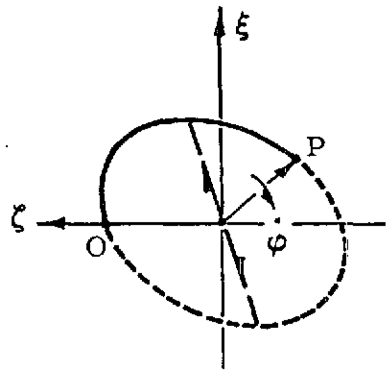
$$+ \left\{\frac{\xi_o\xi_o'}{4} + \zeta_o\zeta_o'\right\}\sin 2\varphi .$$

(b) For the Geometry on the (η, ζ)-plane.

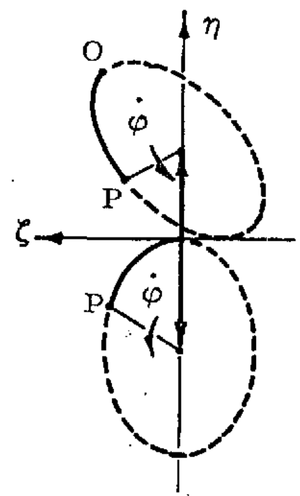
$$\left[\frac{\eta(\varphi) - K_2}{4}\right]^2 + [\zeta(\varphi)]^2 = \frac{1}{2}\left\{\left[\left(\frac{1}{2}\xi_o\right)^2 + (\zeta_o')^2\right] + \left[\left(\frac{1}{2}\xi_o'\right)^2 + (\zeta_o)^2\right]\right\}$$

$$+ \frac{1}{2}\left\{\left[\left(\frac{1}{2}\xi_o'\right)^2 + (\zeta_o)^2\right] - \left[\left(\frac{1}{2}\xi_o\right)^2 + (\zeta_o')^2\right]\right\}\cos 2\varphi$$

$$+ \left\{\zeta_o\zeta_o' - \frac{\xi_o\xi_o'}{4}\right\}\sin 2\varphi .$$



(a). The (ξ, ζ) trace is a skewed ellipse, as shown. Its frequency of motion is the same as the base (circular) orbit. If the motion is constrained to pass through the origin ($\bar{A}_o \equiv 0$), the ellipse reduces to the line shown on this plane.



(b). The non-secular (η, ζ) trace is a skewed ellipse, beginning at O, with a displaced center (K_2). If the motion is further constrained to pass through the origin ($\bar{A}_o \equiv 0$), the skewed ellipse is replaced by the symmetric one, beginning at the origin. This ellipse has its center located at ($\eta_c = -K_2$).

Out-of-Plane Displacement Diagrams; Rotating Frame of Reference; $\bar{\tau}_I$ Influence.

The general expressions describing the displacements are:

(a) the in-plane equation, and (b) the third component expression.

(a) $I_2 \bar{\lambda}(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] T_2(\varphi^-) \left[A_a\right]_{i.v.} + \left[I_2 - \frac{3}{2}\varphi B_2 J_1\right] K_o + \Phi_\tau \bar{\tau}_I$

(b) $J_3 \bar{\lambda}(\varphi) = J_3 \left[\bar{\lambda}_o \cos\varphi + \bar{\lambda}'_o \sin\varphi + \bar{\tau}_I (1 - \cos\varphi)\right].$

Partial Solutions

1. The only partial solutions which will provide new information are those due to the Φ_τ-term(s). Only traces involving $\bar{\tau}_I$ are discussed below.

2. Traces for Zero Initial State ($\bar{\lambda}_o = \bar{\lambda}'_o = 0$; $\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_z)$),* are defined from:

$$\xi(\varphi) = 2\tau_\Xi \left[\frac{3\varphi}{4}\sin\varphi - (1 - \cos\varphi)\right] + \frac{3\tau_H}{2}\left[\sin\varphi - \varphi\cos\varphi\right],$$

$$\eta(\varphi) = 3\tau_\Xi \left[\varphi(1 + \cos\varphi) - 2\sin\varphi\right] + 5\tau_H \left[\frac{3\varphi}{5}\sin\varphi - (1 - \cos\varphi)\right],$$

$$\zeta(\varphi) = \tau_z (1 - \cos\varphi).$$

Equations for the Planar Geometries

(a) Traces for (τ_Ξ, τ_z) only (τ components > 0), defined from:

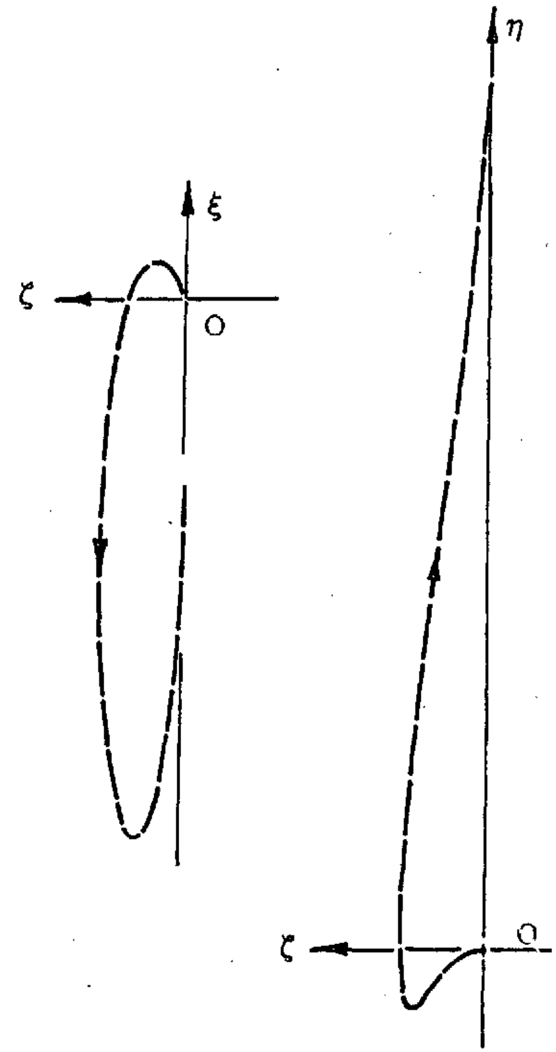
$$\xi(\varphi) = 2\tau_\Xi \left[\frac{3\varphi}{4}\sin\varphi - (1 - \cos\varphi)\right],$$

$$\eta(\varphi) = 3\tau_\Xi \left[\varphi(1 + \cos\varphi) - 2\sin\varphi\right],$$

$$\zeta(\varphi) = \tau_z (1 - \cos\varphi);$$

hence, $\frac{\xi(\varphi)}{2\tau_\Xi} + \frac{\zeta(\varphi)}{\tau_z} = \frac{3\varphi}{4}\sin\varphi$, and $\frac{\eta(\varphi)}{3\tau_\Xi} + \frac{\zeta(\varphi)}{\tau_z}\varphi = 2(\varphi - \sin\varphi)$.

*The motion traces originate at the coordinate origin.



2a. Example of out-of-plane traces, illustrating effects of τ_Ξ, τ_z as applied force(s).

Out-of-Plane Displacement Diagrams; Rotating Frame of Reference (continued)

(b) Traces for (τ_H, τ_z) only (τ components > 0), defined from:

$$\xi(\varphi) = \frac{3\tau_H}{2}\left[\sin\varphi - \varphi\cos\varphi\right],$$

$$\eta(\varphi) = 5\tau_H\left[\frac{3\varphi}{2}\sin\varphi - (1-\cos\varphi)\right],$$

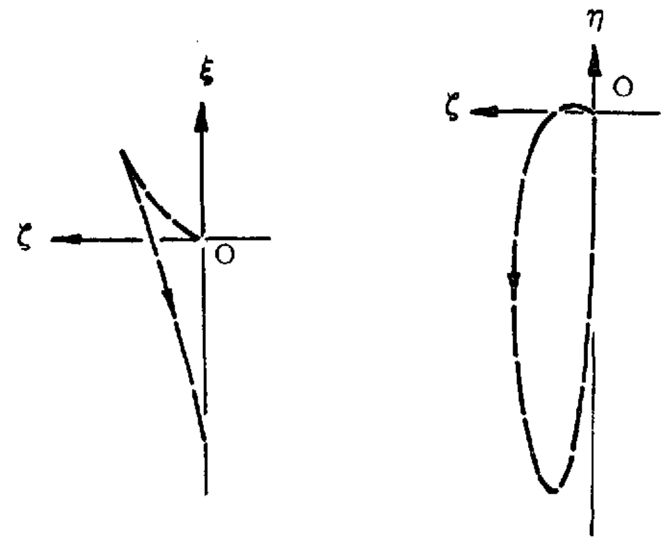
$$\zeta(\varphi) = \tau_z\,(1-\cos\varphi);$$

hence,

$$\frac{2\xi(\varphi)}{3\tau_H} - \frac{\zeta(\varphi)}{\tau_z}\varphi = -(\varphi - \sin\varphi),$$

and

$$\frac{\eta(\varphi)}{5\tau_H} + \frac{\zeta(\varphi)}{\tau_z} = \frac{3\varphi}{5}\sin\varphi.$$

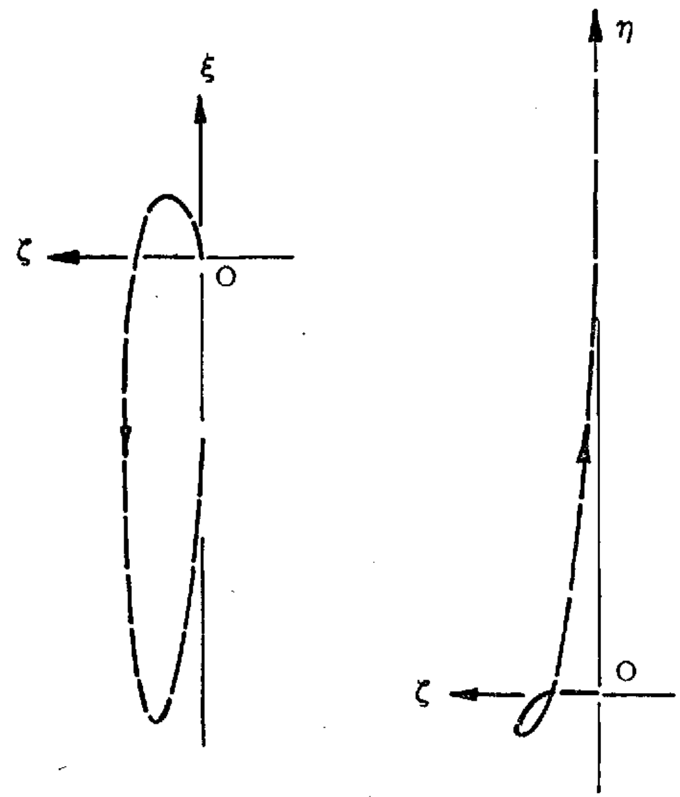


(2b). Traces for τ_H, τ_z.

(c) Displacement Geometries for $\bar{\tau}_I$.

Note: These traces are described by the summing of coordinates in (a). and (b). above.

Sketches depicting the trend, for the present situation, are included at right.



(2c). Sketch of combined effects, due to $\bar{\tau}_I$.

Out-of-Plane Displacements, for the Inertially Oriented Frame of Reference

Out-of-Plane Displacement Diagrams; Inertial Frame of Reference

These general solution equations are:

$$I_2\bar{\mathfrak{R}}(\varphi) = \left[3(J_2 - J_1) + T_2(2\varphi^-)\right]A_{a_I} + T_2(\varphi^-)\left\{\left[I_2 - \frac{3}{2}\varphi(B_2J_1)\right]K_{o_I} + \Psi_\tau \tau\right\}$$

$$J_3\bar{\mathfrak{R}}(\varphi) = J_3\left\{\bar{\mathfrak{R}}_o\cos\varphi + \bar{\mathfrak{R}}'_o\sin\varphi + \bar{\tau}(1-\cos\varphi)\right\}.$$

Partial Solutions.

1. For the Initial State Values $(\bar{\tau} = \bar{0})$ Problem; the displacements are:

$$\Xi(\varphi) + 3A_1 = (A_1\cos 2\varphi - A_2\sin 2\varphi) + (K_1\cos\varphi - K_2\sin\varphi) + \frac{3\varphi}{2}K_1\sin\varphi,$$

$$H(\varphi) - 3A_2 = (A_1\sin 2\varphi + A_2\cos 2\varphi) + (K_1\sin\varphi + K_2\cos\varphi) - \frac{3\varphi}{2}K_1\cos\varphi,$$

$$Z(\varphi) = Z_o\cos\varphi + Z'_o\sin\varphi;$$

wherein

$$A_1 \equiv \frac{1}{2}\left[K_1 - J_1\bar{\mathfrak{R}}_o\right],\quad A_2 \equiv \frac{1}{2}\left[J_2\bar{\mathfrak{R}}_o + (J_2B_2)\bar{\mathfrak{R}}'_o\right];$$

$$K_1 \equiv 2\left[J_1\bar{\mathfrak{R}}_o - J_1B_2\bar{\mathfrak{R}}'_o\right],\quad K_2 \equiv -\left[J_2\bar{\mathfrak{R}}_o + 2J_2B_2\bar{\mathfrak{R}}'_o\right]$$

Quadric equations for the planar traces are:

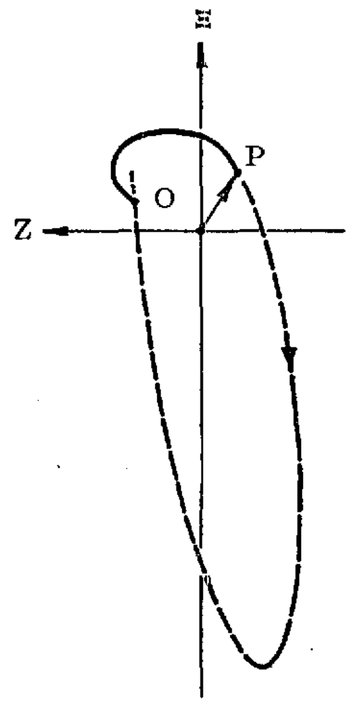
(a) For the (Ξ, Z)-plane.

$$\left(\Xi + 3A_1 - \frac{3K_1}{2}\varphi\sin\varphi\right)^2 + Z^2 = (A_2^2 + K_2^2 + Z_o'^2) + (A_1^2 - A_2^2)\cos^2 2\varphi$$

$$+ (K_1^2 - K_2^2 + Z_o^2 - Z_o'^2)\cos^2\varphi - A_1A_2\sin 4\varphi + (Z_oZ'_o - K_1K_2)\sin 2\varphi$$

$$+2\left[A_1K_1\cos 2\varphi\cos\varphi - A_1K_2\cos 2\varphi\sin\varphi - A_2K_1\sin 2\varphi\cos\varphi\right.$$

$$\left. + A_2K_2\sin 2\varphi\sin\varphi\right].$$



A typical Result

(1a). This trace is secular (though appearing to be closed). Complex nature of the quadric equation does not lend to generalization and individual parameter influence descriptions.

Note: O locates the initial position for each trace.

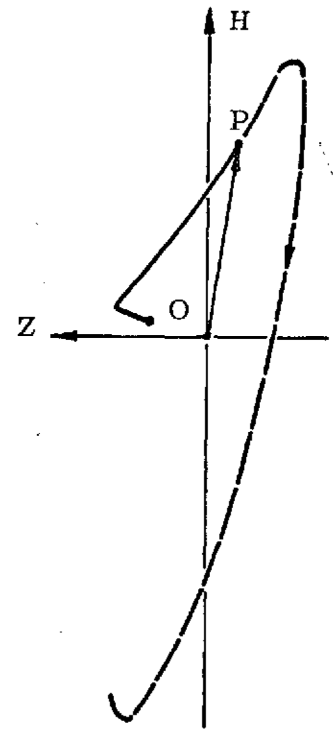
(b) For the (H, Z)-plane.

$$\left(H-3A_2+\frac{3K_1}{2}\varphi\cos\varphi\right)^2+Z^2=(A_1^2+K_1^2+Z_o'^2)+(A_2^2-A_1^2)\cos^2 2\varphi$$

$$+(K_2^2-K_1^2+Z_o^2-Z_o'^2)\cos^2\varphi+A_1A_2\sin 4\varphi+(K_1K_2+Z_oZ_o')\sin 2\varphi$$

$$+2\Big[A_1K_1\sin 2\varphi\sin\varphi+A_1K_2\sin 2\varphi\cos\varphi+A_2K_1\cos 2\varphi\sin\varphi$$

$$+A_2K_2\cos 2\varphi\cos\varphi\Big].$$



(1b). A secular, open trace largely complicated and influenced by the Initial Values present.

2. The Zero Initial State Solution

$(\bar{\mathcal{R}}_o=\bar{\mathcal{R}}'_o=\bar{0};\ \bar{\tau}\equiv\bar{\tau}(\tau_\xi,\tau_\eta,\tau_\zeta))$ is expressed by:

$$(\Xi(\varphi)+\frac{3}{2}\tau_\xi)=\tau_\xi\left[\cos\varphi+\frac{1}{2}\cos 2\varphi\right]-\tau_\eta\left[4\sin\varphi-\sin 2\varphi\right]$$

$$+2\varphi\left\{\tau_\xi\sin\varphi+\tau_\eta\left[\cos\varphi+\frac{3\varphi}{4}\sin\varphi\right]\right\},$$

$$(H(\varphi)+3\tau_\eta)=\tau_\xi\left[\sin\varphi+\frac{1}{2}\sin 2\varphi\right]+\tau_\eta\left[4\cos\varphi-\cos 2\varphi\right]$$

$$+2\varphi\left\{\tau_\eta\left[\sin\varphi-\frac{3\varphi}{4}\cos\varphi\right]-\tau_\xi\cos\varphi\right\},$$

$$(Z(\varphi)-\tau_\zeta)=-\tau_\zeta\cos\varphi.$$

Note: O is the initial locus for each trace.

Out-of-Plane Displacement Traces; Inertial Frame of Reference (continued)

Equations of the Planar Traces:

(a) The Influence of (τ_ξ, τ_ζ) only (with τ components > 0):

$$(\Xi(\varphi) + \frac{3}{2}\tau_\xi) = \tau_\xi\left[\cos\varphi + \frac{1}{2}\cos 2\varphi + 2\varphi\sin\varphi\right]$$

$$H(\varphi) = \tau_\xi\left[\sin\varphi + \frac{1}{2}\sin 2\varphi - 2\varphi\cos\varphi\right]$$

$$(Z(\varphi) - \tau_\zeta) = -\tau_\zeta\cos\varphi .$$

Hence the describing expressions:

$$\Xi + \tau_\xi\left(\frac{3}{2} - 2\varphi\sin\varphi\right) = \tau_\xi\left\{\left[\frac{1}{2} - \left(\frac{Z - \tau_\zeta}{\tau_\zeta}\right)\right]^2 - \frac{3}{4}\right\}$$

and $$H + 2\tau_\xi(\varphi\sin\varphi) = \tau_\xi\sin\varphi\left[1 - \frac{Z - \tau_\zeta}{\tau_\zeta}\right].$$

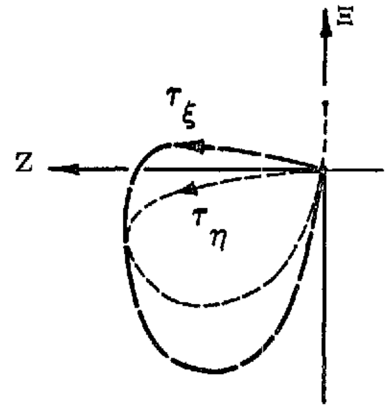
(b) Traces for (τ_η, τ_ζ) only (τ components > 0).

$$\Xi(\varphi) = -\tau_\eta\left[4\sin\varphi - \sin 2\varphi\right] + 2\tau_\eta\left[\cos\varphi + \frac{3\varphi}{4}\sin\varphi\right]\varphi$$

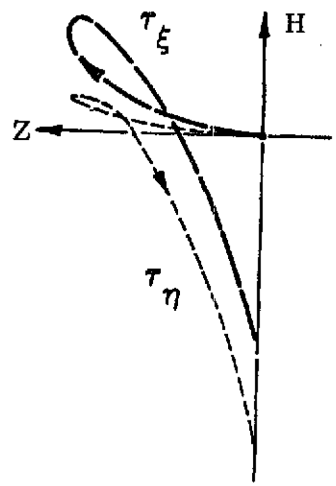
$$H(\varphi) + 3\tau_\eta = \tau_\eta\left[4\cos\varphi - \cos 2\varphi\right] + 2\tau_\eta\left[\sin\varphi - \frac{3\varphi}{4}\cos\varphi\right]\varphi$$

$$Z(\varphi) - \tau_\zeta = -\tau_\zeta\cos\varphi .$$

(c) Combined effect of $(\tau_\xi, \tau_\eta, \tau_\zeta)$ would be represented by the additive influences shown above. The general geometries would be amplified, in divergence, as is easily seen here.



(2a,b). The secular nature of the describing equations appear to indicate closed curves; however, the divergence is evident (by inspection).



(2a,b). The divergent nature of these traces is quite evident from the descriptive expressions.

3. Traces, with Secular Terms Removed; Initial Value Solution ($\tilde{\tau} = \bar{0}$).

Note: To eliminate the divergent (secular) nature of these traces it is only necessary to nullify K_1, where $K_1 \equiv 2[J_1 \bar{\mathfrak{R}}_o - J_1 B_2 \bar{\mathfrak{R}}'_o]$. Obviously this can be accomodated by either: (1) letting $J_1 \bar{\mathfrak{R}}_o = J_1 B_2 \bar{\mathfrak{R}}'_o = 0$; or, (2) by setting $J_1 \bar{\mathfrak{R}}_o = J_1 B_2 \bar{\mathfrak{R}}'_o$.

(a) For illustration the case depicted below merely sets $K_1 = 0$ without stipulating how this was accomplished. As a consequence:

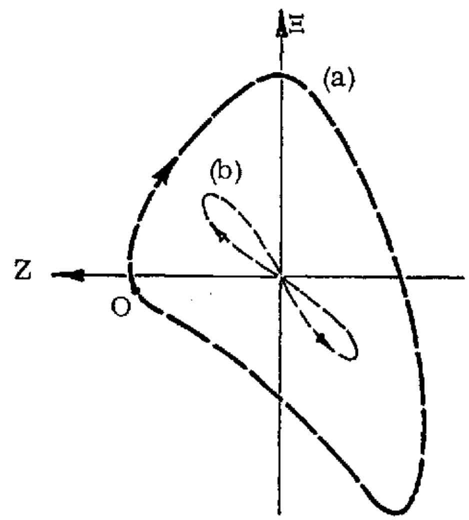
$A_1 \equiv -\frac{1}{2}(J_1 \bar{\mathfrak{R}}_o)$; $K_1 = 0$, but K_2 and A_2 remain as before; and,

$$\Xi(\varphi) - \frac{3}{2}\Xi_o = -\frac{1}{2}\left[\Xi_o \cos 2\varphi + (H_o + \Xi'_o)\sin 2\varphi\right] + (H_o + 2\Xi'_o)\sin\varphi$$

$$H(\varphi) - \frac{3}{2}(H_o + \Xi'_o) = \frac{1}{2}\left[-\Xi_o \sin 2\varphi + (H_o + \Xi'_o)\cos 2\varphi\right] - (H_o + 2\Xi'_o)\cos\varphi$$

$$Z(\varphi) = Z_o \cos\varphi + Z'_o \sin\varphi.$$

In general, these are not easy expressions to discuss, geometrically; they do represent closed curves on the coordinate planes; but they are nominally complex (of single and double frequency), with an initial point not at the coordinate origin.



Cases 3(a) and 3(b) are shown, sketched on the (Ξ, Z)-plane. Case (b) describes a non-secular situation; motion is initiated at the origin.

(A typical situation)

Out-of-Plane Displacement Diagrams, Inertial Frame of Reference (continued)

(b) If, in addition to the elimination of the secular influence, the motion commences at the coordinate origin ($\bar{R}_o \equiv \bar{0}$), then the trace equations reduce to:

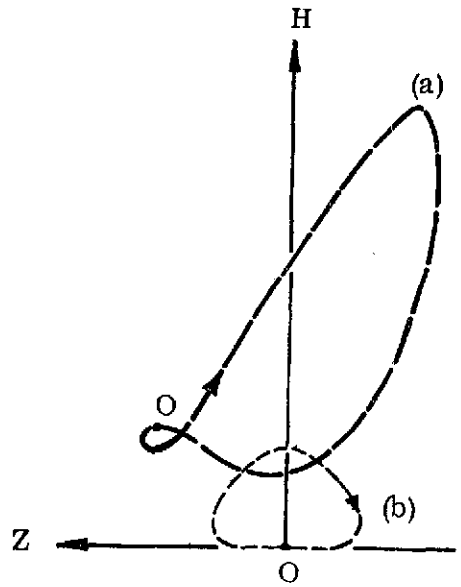
$$\Xi(\varphi) = \Xi_o' \sin\varphi\,[2 - \cos\varphi]$$

$$H(\varphi) - \Xi_o' = \Xi_o' \cos\varphi\,[\cos\varphi - 2]$$

$$Z(\varphi) = Z_o' \sin\varphi\ ;$$

or, the descriptive equations for the planar traces:

$$\frac{\Xi(\varphi)}{\Xi_o'} = \frac{Z(\varphi)}{Z_o'}(2 - \cos\varphi); \quad \frac{H(\varphi) - \Xi_o'}{\Xi_o'} = 1 - 2\cos\varphi - \left(\frac{Z(\varphi)}{Z_o'}\right)^2.$$



3. Typical displacement traces for non-secular state conditions.

Out-of-Plane Displacement Diagrams; Inertial Frame of Reference; $\bar{\tau}_I$ Influence.

General expressions describing the displacements are:

(a) the in-plane equation, and (b) the third component expression.

(a) $I_2\bar{R}(\varphi) = \left\{T_2(2\varphi^-) + 3\left[J_2 - J_1\right]\right\}\left[A_{a_I}\right]_{i.v.} + T_2(\varphi^-)\left[I_2 - \frac{3}{2}\varphi(B_2 J_1)\right]K_{o_I} + T_2(\varphi^-)\left[\Phi_\tau \bar{\tau}_I\right]$,

and (b) $J_3\bar{R}(\varphi) = J_3\left[\bar{R}_o \cos\varphi + \bar{R}'_o \sin\varphi + \bar{\tau}_I(1 - \cos\varphi)\right]$.

Partial Solutions.

1. The only partial solutions which provide new information are those due to the Φ_τ-terms. Only traces involving $\bar{\tau}_I$ are discussed below.

2. Traces for Zero-Initial State ($\bar{R}_o = \bar{R}'_o = 0$; $\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_z)$*), are defined from:

$$\Xi(\varphi) = \tau_\Xi\left[2(1-\cos\varphi) + 2(1-\cos 2\varphi) - 3\varphi(\sin\varphi + \frac{1}{4}\sin 2\varphi)\right]$$
$$+ \tau_H\left[5\sin\varphi - \frac{7}{4}\sin 2\varphi - \frac{3\varphi}{4}(3 - \cos 2\varphi)\right],$$

$$H(\varphi) = \tau_\Xi\left[\frac{3\varphi}{4}(4\cos\varphi + 3 + \cos 2\varphi) - 2(\sin\varphi + \sin 2\varphi)\right]$$
$$+ \tau_H\left[5(1-\cos\varphi) - \frac{7}{4}(1-\cos 2\varphi) + \frac{3\varphi}{4}\sin 2\varphi\right],$$

and $Z(\varphi) = \tau_z(1 - \cos\varphi)$.

Equations for Planar Geometries.

(a) Traces for (τ_Ξ, τ_z) only (τ components > 0), defined from:

$$\Xi(\varphi) = \tau_\Xi\left[2(1-\cos\varphi) + 2(1-\cos 2\varphi) - 3\varphi(\sin\varphi + \frac{1}{4}\sin 2\varphi)\right],$$

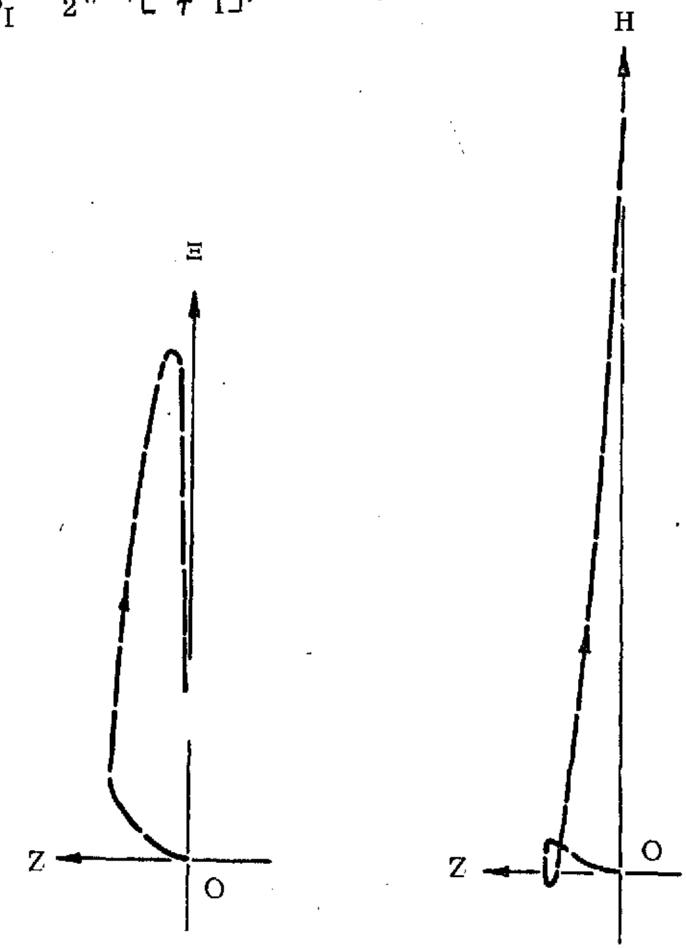
$$H(\varphi) = \tau_\Xi\left[\frac{3\varphi}{4}(3 + 4\cos\varphi + \cos 2\varphi) - 2(\sin\varphi + \sin 2\varphi)\right],$$

and $Z(\varphi) = \tau_z(1 - \cos\varphi)$;

hence, the expressions:

(continued on next page)

*The motion traces originate at the coordinate origin.



2a. Examples of trace geometry, on the displacement planes (Ξ, z) and (H, Z) due to applied force(s) τ_Ξ, τ_z.

Out-of-Plane Displacement Diagrams; Inertial Frame of Reference (continued)

(a) Traces for (τ_Ξ, τ_z) only (continued)

$$\frac{\Xi(\varphi)}{\tau_\Xi} - 2\left(\frac{Z(\varphi)}{\tau_z}\right) - 4\left(\frac{Z(\varphi)}{\tau_z}\right)^2 = -3\varphi\left(\sin\varphi + \frac{1}{4}\sin 2\varphi\right),$$

$$\frac{H(\varphi)}{\tau_\Xi} - \frac{3\varphi}{2}\left[\left(\frac{Z(\varphi)}{\tau_z}\right)^2 + 4\left(1 - \frac{Z(\varphi)}{\tau_z}\right)\right] = -2(\sin\varphi + \sin 2\varphi).$$

(b) Traces for (τ_H, τ_z) only (τ components > 0), defined from:

$$\Xi(\varphi) = \tau_H\left[5\sin\varphi - \frac{7}{4}\sin 2\varphi - 3\varphi\left(1 - \frac{1}{2}\cos^2\varphi\right)\right],$$

$$H(\varphi) = \tau_H\left[5(1-\cos\varphi) - \frac{7}{2}(1-\cos^2\varphi) + \frac{3\varphi}{4}\sin 2\varphi\right],$$

and $Z(\varphi) = \tau_z(1-\cos\varphi)$;

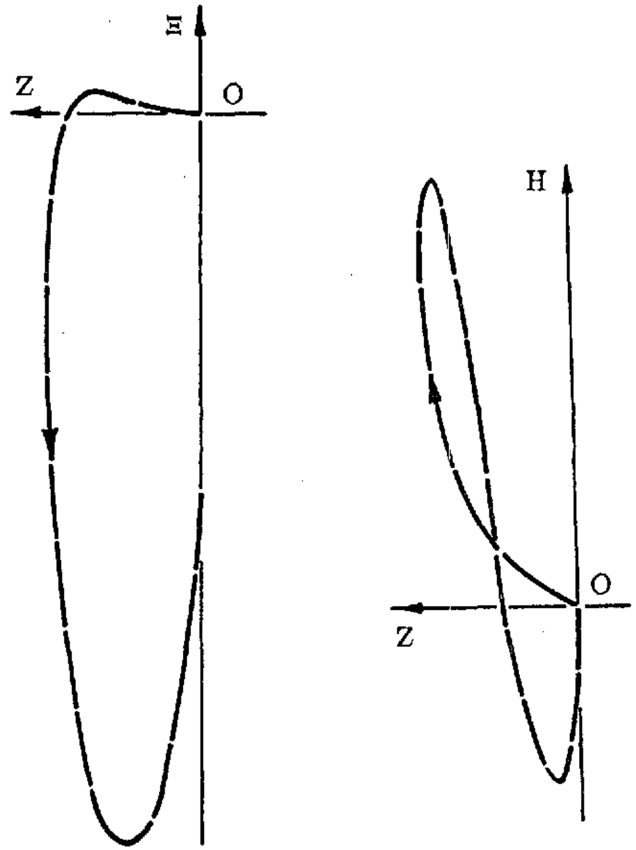
hence, the expressions:

$$\frac{\Xi(\varphi)}{\tau_H} + \frac{3\varphi}{2}\left[2 + \left(\frac{Z(\varphi)}{\tau_z}\right)^2\right] = 5\sin\varphi - \frac{7}{4}\sin 2\varphi,$$

and $$\frac{H(\varphi)}{\tau_z} - 5\,\frac{Z(\varphi)}{\tau_z} + \frac{7}{2}\left(\frac{Z(\varphi)}{\tau_z}\right)^2 = \frac{3\varphi}{4}\sin 2\varphi.$$

(c) Displacement Geometries for $\bar{\tau}_I$.

Note: These traces are obtained by summing coordinates from (a). and (b). above.



2b. Trace geometries illustrating effects due to τ_H, τ_z.

Out-of-Plane Hodographs, for the Rotating Frame of Reference

Out-of-Plane Hodograph Diagrams; Rotating Frame of Reference

The general expressions are:

$$I_2 \bar{\lambda}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] B_2 T_2(\varphi^-) A_a - \frac{3}{2}\left[B_2 J_1\right] K_o + \Psi_\tau \bar{\tau},$$

and

$$J_3 \bar{\lambda}'(\varphi) = J_3\left[\bar{\lambda}'_o \cos\varphi + (\bar{\tau} - \bar{\lambda}_o)\sin\varphi\right].$$

Partial Solutions.

1. Traces for the Initial State Solution ($\bar{\tau} = \bar{0}$), may be acquired from:

$$\xi'(\varphi) = 2(A_1 \sin\varphi + A_2 \cos\varphi)$$

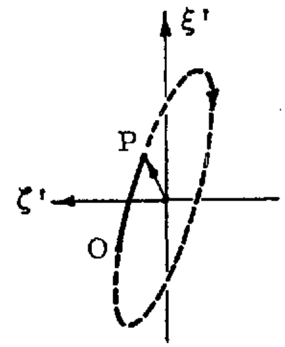
$$\eta'(\varphi) = 4(A_1 \cos\varphi - A_2 \sin\varphi) - \frac{3}{2} K_1$$

$$\zeta'(\varphi) = \zeta'_o \cos\varphi - \zeta_o \sin\varphi.$$

wherein

$$A_1 \equiv \frac{1}{2}\left[K_1 - J_1 \bar{\lambda}_o\right]; \quad A_2 \equiv \frac{1}{2}\left[J_2 B_2 \bar{\lambda}'_o\right]$$

$$K_1 \equiv 2\left[2J_1 \bar{\lambda}_o - J_1 B_2 \bar{\lambda}'_o\right]; \quad K_2 = \left[J_2 \bar{\lambda}_o - 2J_2 B_2 \bar{\lambda}'_o\right].$$



(1, a). This trace is an ellipse, symmetrically disposed about the origin. Its principal geometric axes do not align with the coordinate axes.

Quadric Equations for the Planar Traces.

(a) Trace for the (ξ', ζ')-plane.

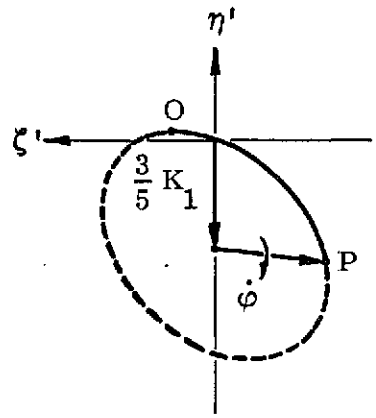
$$\left[\frac{\xi'(\varphi)}{2}\right]^2 + [\zeta'(\varphi)]^2 = \frac{1}{2}\left\{\left[A_1^2 + \zeta_o^2\right] + \left[A_2^2 + \zeta_o'^2\right]\right\}$$

$$+ \frac{1}{2}\left\{\left[A_2^2 + \zeta_o'^2\right] - \left[A_1^2 + \zeta_o^2\right]\right\}\cos 2\varphi + \left[A_1 A_2 - \zeta_o \zeta'_o\right]\sin 2\varphi.$$

(b) Trace for the (η', ζ')-plane.

$$\left[\frac{\eta'(\varphi)}{4} + \frac{3}{8}K_1\right]^2 + [\zeta'(\varphi)]^2 = \frac{1}{2}\left\{\left[A_2^2 + \zeta_o^2\right] + \left[A_1^2 + \zeta_o'^2\right]\right\}$$

$$+ \frac{1}{2}\left\{\left[A_1^2 + \zeta_o'^2\right] - \left[A_2^2 + \zeta_o^2\right]\right\}\cos 2\varphi - \left[A_1 A_2 + \zeta_o \zeta'_o\right]\sin 2\varphi.$$



(1, b). The trace here is an ellipse with its center displaced to $\eta_c \equiv -\frac{3}{2}K_1$. Note, this ellipse is also skewed.

2. Trace for Zero Initial State ($\bar{r}_o = \bar{r}'_o = \bar{0}$; $\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta)$)

The scalar expressions are:

$$\xi'(\varphi) = 2\tau_\eta(1 - \cos\varphi) + \tau_\xi(\sin\varphi)$$
$$\eta'(\varphi) = -2\tau_\xi(1 - \cos\varphi) + 4\tau_\eta(\sin\varphi) - 3\varphi\tau_\eta$$
$$\zeta'(\varphi) = \tau_\zeta(\sin\varphi).$$

Equations for the Planar Geometries.

(a) Traces for (τ_ξ, τ_ζ) only, (τ components > 0)

are acquired from:

$$\xi'(\varphi) = \tau_\xi \sin\varphi,$$
$$\eta'(\varphi) = -2\tau_\xi(1 - \cos\varphi),$$
$$\zeta'(\varphi) = \tau_\zeta \sin\varphi;$$

hence

$$\frac{\xi'(\varphi)}{\tau_\xi} = \frac{\zeta'(\varphi)}{\tau_\zeta}; \text{ and, } \left(\frac{\eta'(\varphi)}{2\tau_\xi} + 1\right)^2 + \left(\frac{\zeta'(\varphi)}{\tau_\zeta}\right)^2 = 1.$$

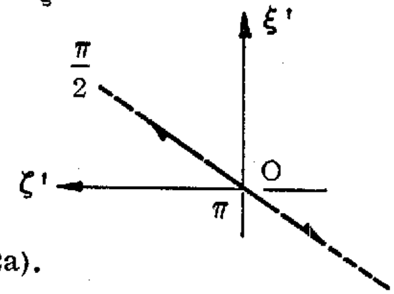
(b) Traces for (τ_η, τ_ζ) only, (τ components > 0)

defined by:

$$\xi'(\varphi) = 2\tau_\eta(1 - \cos\varphi),$$
$$\eta'(\varphi) = 4\tau_\eta(\sin\varphi) - 3\varphi\tau_\eta,$$
$$\zeta'(\varphi) = \tau_\zeta \sin\varphi;$$

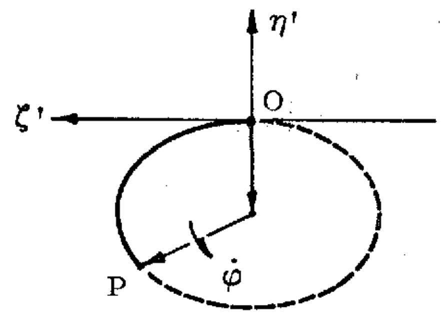
hence, the quadrics:

$$\left(\frac{\xi'(\varphi)}{2\tau_\eta} - 1\right)^2 + \left(\frac{\zeta'(\varphi)}{\tau_\zeta}\right)^2 = 1; \text{ and, } \frac{\eta'(\varphi)}{4\tau_\eta} + \frac{3}{4}\varphi = \frac{\zeta'(\varphi)}{\tau_\zeta}.$$

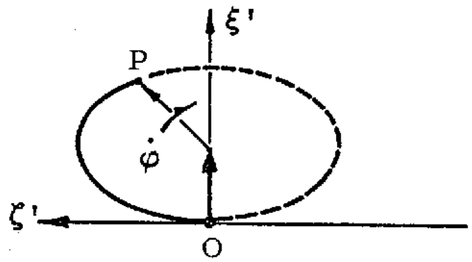


(2a).

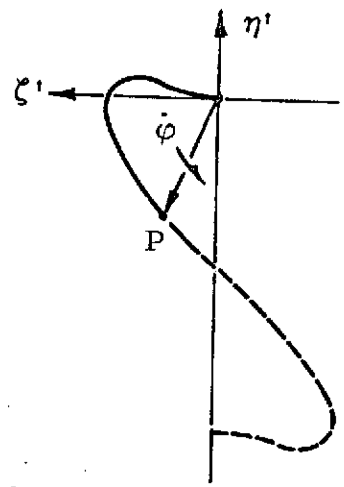
This trace is a line, symmetric about ξ' axis, with frequency matching that of circular orbit.



(2a). The trace here is:
An ellipse with a displaced center ($\eta_c = -2\tau_\xi$). Motion on the ellipse has a frequency equal to the orbit frequency.



(2b). This trace is an ellipse, with center at $\xi'_c = 2\tau_\eta$. Motion's frequency is the same as that of the base orbit.



(2b). This S-shaped trace is produced by a "line" modified by the secular η-influence, $(3\varphi\tau_\eta)$.

Out-of-Plane Hodograph Diagrams; Rotating Frame of Reference (continued)

(c) The Hodograph Geometries, for $(\tau_\xi, \tau_\eta, \tau_\zeta)$ combined.

Note: These traces are due to the combined effects shown in (a). and (b). above.

On the (ξ', ζ')-plane the displacement curve can be expressed by the quadric:

(c, 1). $$[\xi'(\varphi) - 2\tau_\eta]^2 + [\zeta'(\varphi)]^2 = (\tau_\xi^2 + \tau_\zeta^2) + \frac{1}{2}\left[4\tau_\eta^2 - (\tau_\xi^2 + \tau_\zeta^2)\right](1 + \cos 2\varphi) - 2\tau_\xi \tau_\zeta \sin 2\varphi.$$

Here, the ellipse (due to τ_η, τ_ζ) and the line (from the τ_ξ-effect) combine to produce the skewed ellipse shown at right.

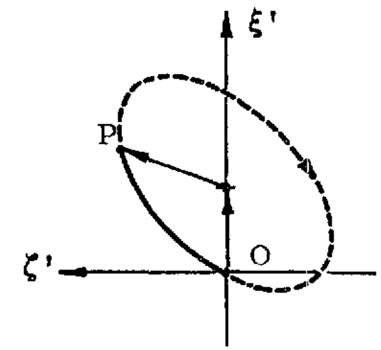
On the (η', ζ')-plane, the equation of the curve seen there can be written as:

(c, 2). $$\left[\frac{\eta'(\varphi)}{4\tau_\eta} + \frac{3}{4}\varphi\right] = \frac{\zeta'(\varphi)}{\tau_\zeta} - \frac{\tau_\xi}{2\tau_\eta}(1 - \cos\varphi).$$

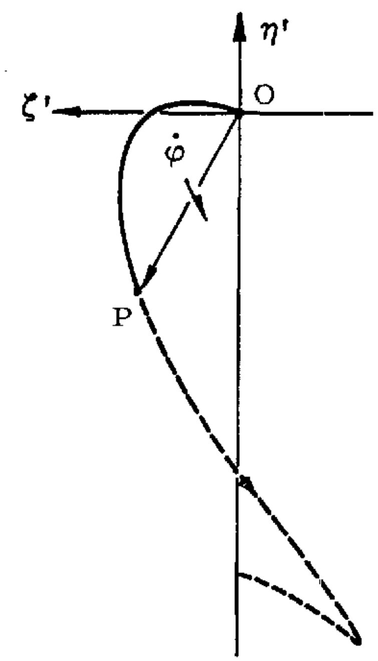
Consequently, the ellipse (due to τ_ξ, τ_ζ) is coupled with the s-shaped curve (from τ_η) leading to the trace shown at the right.

3. Traces for Secular Terms Eliminated; a Modified Initial Value Solution $(\bar{\tau} \equiv \bar{0})$

Note: It has been noted that the secular (divergent) influence can be eliminated by setting $K_1 = 0$ (see discussion with displacement traces).



(c, 1).



(c, 2).

Typical hodograph traces developed for the fixed thrust, $\bar{\tau}$.

(a) When K_1 vanishes it is found that one set of hodograph equations is:

$$\xi'(\varphi) = \xi'_o \cos\varphi - \xi_o \sin\varphi,$$
$$\eta'(\varphi) = -2[\xi'_o \sin\varphi - \xi_o \cos\varphi],$$
$$\zeta'(\varphi) = \zeta'_o \cos\varphi - \zeta_o \sin\varphi;$$

these lead to the following quadric hodograph expressions:

$$[\xi'(\varphi)^2 + \zeta'(\varphi)^2] = \frac{1}{2}\left\{\left[\left(\xi_o^2 + \zeta_o^2\right) + \left(\xi_o'^2 + \zeta_o^2\right)\right] + \left[\left(\xi_o'^2 + \zeta_o'^2\right) - \left(\xi_o^2 + \zeta_o^2\right)\right]\cos 2\varphi\right\} - \left[\xi_o \xi'_o + \zeta_o \zeta'_o\right]\sin 2\varphi,$$

and

$$\left[\frac{\eta'(\varphi)^2}{4} + \zeta'(\varphi)\right] = \frac{1}{2}\left\{\left[\left(\xi_o'^2 + \zeta_o^2\right) + \left(\xi_o^2 + \zeta_o'^2\right)\right] + \left[\left(\xi_o^2 + \zeta_o'^2\right) - \left(\xi_o'^2 + \zeta_o^2\right)\right]\cos 2\varphi\right\} + \left[\xi_o \xi'_o - \zeta_o \zeta'_o\right]\sin 2\varphi$$

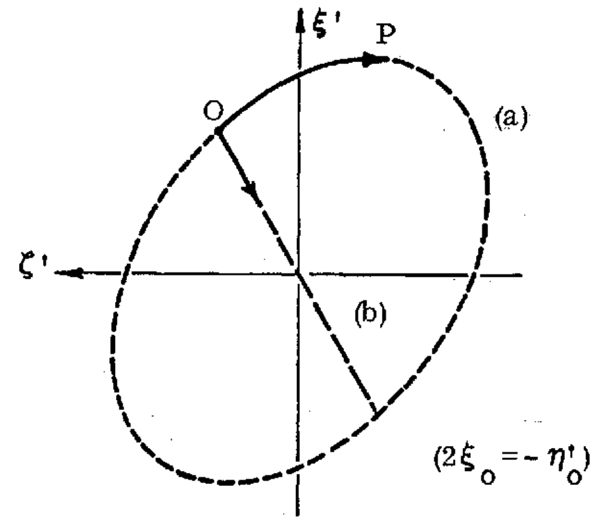
Both traces (here) suggest ellipses, but these figures have their principal axes not parallel to the coordinate axes (i.e., skewed figures).

(b) For the added specialization of motion commencing at the origin ($\bar{\lambda}_o \equiv \bar{0}$), the (above) equations reduce to the following set:

$$\frac{\xi'(\varphi)}{\xi'_o} = \frac{\zeta'(\varphi)}{\zeta'_o}$$ (a line passing through the hodograph's origin)

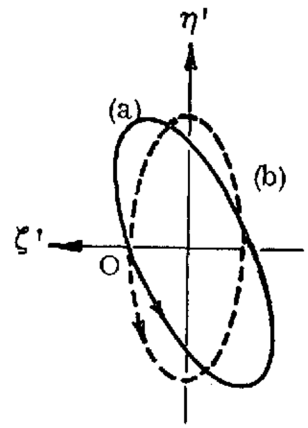
and

$$\left(\frac{\eta'(\varphi)}{2\xi'_o}\right)^2 + \left(\frac{\zeta'(\varphi)}{\zeta'_o}\right)^2 = 1$$ (an ellipse, aligned with the hodograph's axes)



$(2\xi_o = -\eta'_o)$

O is the initial locus on these planes.



$(\xi_o = \eta'_o = 0)$

Typical traces depicting a non-secular solution.

Out-of-Plane Hodograph Diagrams; Rotating Frame of Reference.

The general solution equations are; (a) for the in-plane components, and (b) for the third component:

(a) $I_2 \bar{h}'(\varphi) = \left[I_2 + 3(J_2 - J_1)\right] B_2 T_2 (\varphi^-) \left[A_a\right]_{i.v.} - \frac{3}{2}\left[B_2 J_1\right] K_o + \Phi'_\tau \bar{\tau}_I$,

and (b) $J_3 \bar{h}'(\varphi) = \left[\bar{h}'_o \cos\varphi + (\bar{\tau}_I - \bar{h}_o) \sin\varphi\right]$.

Partial Solutions.

1. Only the partial solutions expressed in terms of the $\bar{\tau}_I$ components provide new information. Consequently, only these are described below.

2. Traces for Zero Initial State ($\bar{h}_o = \bar{h}'_o = 0$; $\bar{\tau}_I \equiv \bar{\tau}_I(\tau_\Xi, \tau_H, \tau_z)$)*, are described by:

$$\xi'(\varphi) = \frac{\tau_\Xi}{2}\left[3\varphi\cos\varphi - \sin\varphi\right] + \frac{3\tau_H}{2}\left[\varphi\sin\varphi\right],$$

$$\eta'(\varphi) = 3\tau_\Xi\left[(1 - \cos\varphi) - \varphi\sin\varphi\right] + 2\tau_H\left[\frac{3\varphi}{2}\cos\varphi - \sin\varphi\right],$$

and $\zeta'(\varphi) = \tau_z \sin\varphi$.

Equations for the Planar Geometries

(a) Traces for (τ_Ξ, τ_z) only, (τ components > 0).

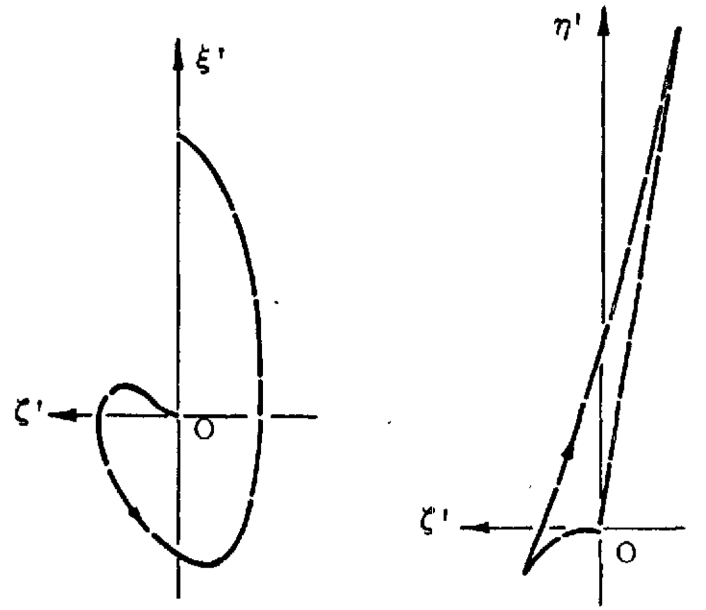
$$\xi'(\varphi) = \frac{1}{2}\tau_\Xi\left[3\varphi\cos\varphi - \sin\varphi\right],$$

$\eta'(\varphi) = 3\tau_\Xi\left[(1 - \cos\varphi) - \varphi\sin\varphi\right]$, and $\zeta'(\varphi) = \tau_z \sin\varphi$;

hence, the parametric equations:

$$\frac{2\xi'(\varphi)}{\tau_\Xi} + \frac{\zeta'(\varphi)}{\tau_z} = 3\varphi\cos\varphi,$$

and $$\frac{\eta'(\varphi)}{3\tau_\Xi} + \frac{\zeta'(\varphi)}{\tau_z}\varphi = (1 - \cos\varphi) .$$



2a. Typical hodograph traces due to τ_Ξ and τ_z.

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*The motion traces originate at the coordinate origin.

Out-of-Plane Hodograph Diagrams; Rotating Frame of Reference (continued)

(b) Traces for (τ_H, τ_z) only, (τ components > 0).

$$\xi'(\varphi) = \frac{3\tau_H}{2}\varphi\sin\varphi,$$

$$\eta'(\varphi) = 2\tau_H\left[\frac{3\varphi}{2}\cos\varphi - \sin\varphi\right],$$

and $$\zeta'(\varphi) = \tau_z \sin\varphi;$$

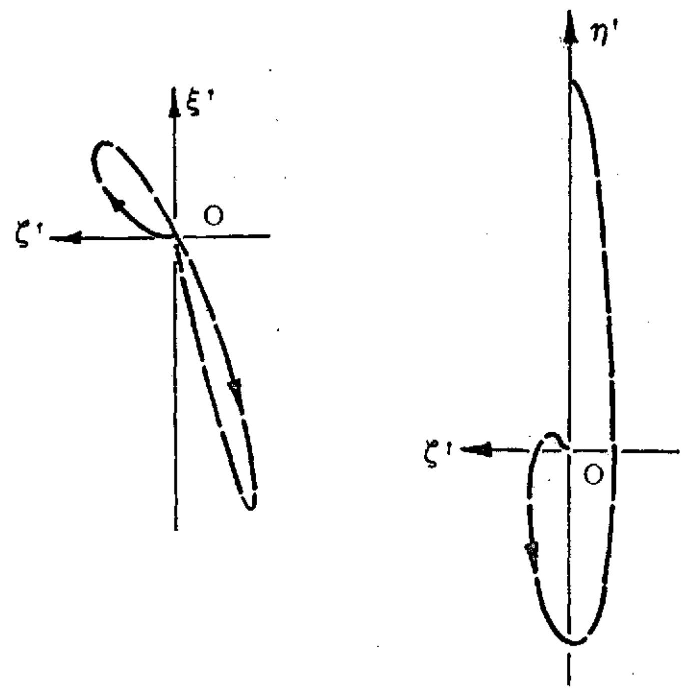
hence, the parametric equations:

$$\frac{2\xi'(\varphi)}{3\tau_H} - \frac{\zeta'(\varphi)}{\tau_z}\varphi = 0,$$

and $$\frac{\eta'(\varphi)}{2\tau_H} + \frac{\zeta'(\varphi)}{\tau_z} = \frac{3\varphi}{2}\cos\varphi.$$

(c) Hodograph Geometries for $\bar{\tau}_I$.

Note: These traces are the composite of those diagrams described for (a). and (b). above.



2b. Typical traces for the influence of τ_H, τ_z.

Out-of-Plane Hodographs, for the Inertially Oriented Frame of Reference

Out-of-Plane Hodograph Traces; Inertial Frame of Reference

General equations for this solution type are:

$$I_2\bar{\mathfrak{R}}'(\varphi) = B_2\left\{\left[2T_2(2\varphi^-)\right]A_{a_I} + T_2(\varphi^-)\left[I_2 - \frac{3}{2}J_1 - \frac{3}{2}(B_2 - J_1)\varphi\right]K_{o_I}\right.$$

$$\left. + T_2(\varphi^-)\,\Psi_\tau\bar{\tau}\right\} + T_2(\varphi^-)\,\Psi'_\tau\,\bar{\tau}$$

$$J_3\bar{\mathfrak{R}}'(\varphi) = J_3\left\{\bar{\mathfrak{R}}'_o\cos\varphi + (\bar{\tau} - \bar{\mathfrak{R}}_o)\sin\varphi\right\}.$$

Partial Solutions

1. Traces for the Initial State Problem ($\bar{\tau} = \bar{0}$), are obtained from:

$$\Xi'(\varphi) = -2\left[A_1\sin 2\varphi + A_2\cos 2\varphi\right] + \frac{1}{2}K_1\sin\varphi - K_2\cos\varphi + \frac{3\varphi}{2}K_1\cos\varphi,$$

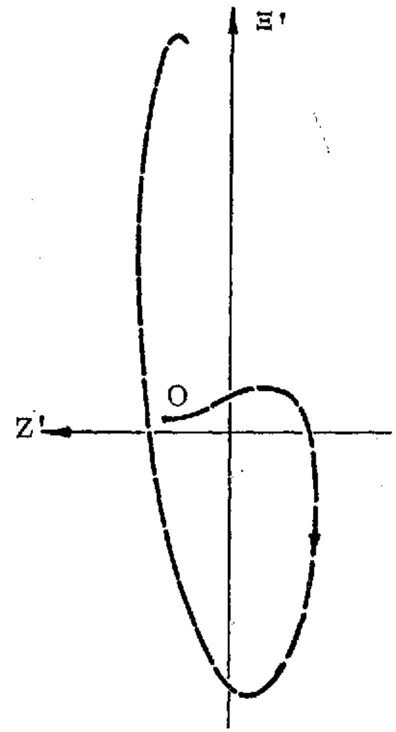
$$H'(\varphi) = 2\left[A_1\cos 2\varphi - A_2\sin 2\varphi\right] - \frac{1}{2}K_1\cos\varphi - K_2\sin\varphi + \frac{3\varphi}{2}K_1\sin\varphi,$$

$$Z'(\varphi) = Z'_o\cos\varphi - Z_o\sin\varphi;$$

wherein:

$$A_1 \equiv \frac{1}{2}\left[K_1 - J_1\bar{\mathfrak{R}}_o\right];\quad A_2 \equiv -\frac{1}{2}\left[K_2 + J_2B_2\bar{\mathfrak{R}}'_o\right]$$

$$K_1 \equiv 2\left[J_1\bar{\mathfrak{R}}_o - J_1B_2\bar{\mathfrak{R}}'_o\right];\quad K_2 \equiv -\left[J_2\bar{\mathfrak{R}}_o + 2J_2B_2\bar{\mathfrak{R}}'_o\right].$$



Quadric Equations for the Planar Traces.

(a) For a trace on the (Ξ', Z')-plane.

$$\left(\Xi' - \frac{3\varphi}{2}K_1\cos\varphi\right)^2 + (Z')^2 = (4A_1^2 + \frac{1}{4}K_1^2 + Z_o^2) + 4(A_2^2 - A_1^2)\cos^2 2\varphi$$

$$+ (K_2^2 - \frac{1}{4}K_1^2 + Z_o'^2 - Z_o^2)\cos^2\varphi + 4A_1A_2\sin 4\varphi - \left(\frac{1}{2}K_1K_2 + Z_oZ'_o\right)\sin 2\varphi$$

$$- 4\left[\frac{1}{2}A_1K_1\sin 2\varphi\sin\varphi - A_1K_2\sin 2\varphi\cos\varphi + \frac{1}{2}A_2K_1\cos 2\varphi\sin\varphi - A_2K_2\cos 2\varphi\cos\varphi\right].$$

(1a). These traces are necessarily secular, and complex, due to the make-up of the scalar equations forming the quadrics. This complexity eliminates the generalizing and prediction otherwise expected.

Note: O is the initial point for each trace.

Out-of-Plane Hodograph Trace, Inertial Frame of Reference (continued)

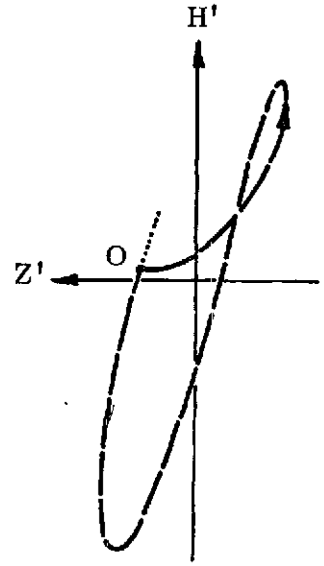
(b) For the trace on the (H', Z')-plane

$$\left(H' - \frac{3\varphi}{2} K_1 \sin\varphi\right)^2 + (Z')^2 = (4A_2^2 + K_2^2 + Z_o^2) + 4(A_1^2 - A_2^2)\cos^2 2\varphi$$

$$+ \left[\frac{1}{4} K_1^2 - K_2^2 + Z_o'^2 - Z_o^2\right]\cos^2\varphi + \left(\frac{1}{2} K_1 K_2 - Z_o Z_o'\right)\sin 2\varphi$$

$$- 4A_1A_2 \sin 4\varphi - 4\left[\frac{1}{2} A_1 K_1 \cos 2\varphi \cos\varphi + A_1 K_2 \cos 2\varphi \sin\varphi\right.$$

$$\left. - \frac{1}{2} A_2 K_1 \sin 2\varphi \cos\varphi - A_2 K_2 \sin 2\varphi \sin\varphi\right] .$$



(1b). This secular curve passes through the initial point at $\varphi = 2\pi$ - (this is due to the parametric form of the expressions).

2. Trace for Zero Initial State ($\bar{\Re}_o = \bar{\Re}'_o = \bar{0}$; $\bar{\tau} = \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta)$).

$$\Xi'(\varphi) = \tau_\xi(\sin\varphi - \sin 2\varphi) - 2\tau_\eta(\cos\varphi - \cos 2\varphi) + \varphi\left[2\tau_\xi \cos\varphi\right.$$

$$\left. + \tau_\eta\left(\sin\varphi + \frac{3\varphi}{2}\cos\varphi\right)\right] .$$

$$H'(\varphi) = -\tau_\xi(\cos\varphi - \cos 2\varphi) - 2\tau_\eta(\sin\varphi - \sin 2\varphi) + \varphi\left[2\tau_\xi \sin\varphi\right.$$

$$\left. - \tau_\eta\left(\cos\varphi - \frac{3\varphi}{2}\sin\varphi\right)\right] .$$

$$Z'(\varphi) = \tau_\zeta \sin\varphi .$$

Note: O is the initial point for each trace.

Out-of-Plane Hodograph Trace, Inertial Frame of Reference (continued)

Parametric Equations for the Planar Traces

(a) Traces for (τ_ξ, τ_ζ) only (τ components > 0).

$$\Xi'(\varphi) = \tau_\xi \left[\sin\varphi - \sin 2\varphi + 2\varphi\cos\varphi \right]$$

$$H'(\varphi) = -\tau_\xi \left[\cos\varphi - \cos 2\varphi - 2\varphi\sin\varphi \right]$$

$$Z'(\varphi) = \tau_\zeta \sin\varphi .$$

(b) Traces for (τ_η, τ_ζ) only (τ components > 0).

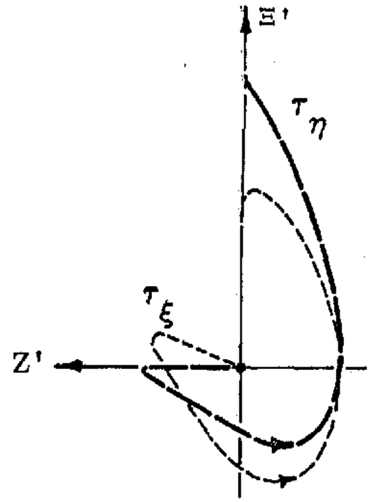
$$\Xi'(\varphi) = -\tau_\eta \left[2(\cos\varphi - \cos 2\varphi) - \varphi(\sin\varphi + \frac{3\varphi}{2}\cos\varphi) \right]$$

$$H'(\varphi) = -\tau_\eta \left[2(\sin\varphi - \sin 2\varphi) + \varphi(\cos\varphi - \frac{3\varphi}{2}\sin\varphi) \right]$$

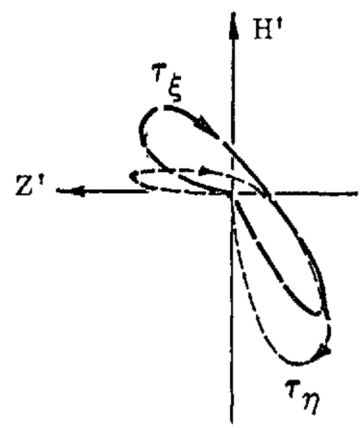
$$Z'(\varphi) = \tau_\zeta \sin\varphi .$$

(c) The combined effects of the full three components of the specific force $\bar{\tau}$ is obtained by adding ordinates on the traces depicted here. Due to the close similarity of the curves it is quite apparent that the geometries would not significantly be altered by this composition.

Note: O is the initial position for each trace geometry.



(2a, b). The secular influence of these expressions requires a divergent geometry. (Combined effects of τ_ξ, τ_η, τ_ζ would be provided by adding displacements on the curves).



(2a, b). The additive influence of total thrusting is found by combining the ordinates of the curves. Secular influences are predominant here.

3. Traces, with Secular Terms Removed; Initial Value Solution ($\bar{\tau} = \bar{0}$)

Note: The general requirement for removal of secular effects was found to be satisfied by nullifying $K_1 \ (\equiv 2[J_1 \bar{\mathcal{R}}_o - J_1 B_2 \bar{\mathcal{R}}'_o])$. It has been noted that this may be carried out in either of two ways.

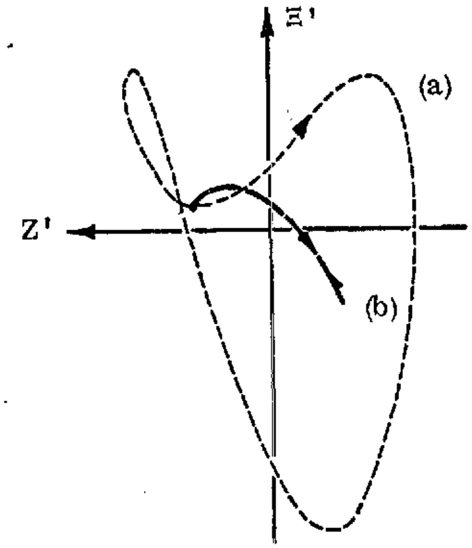
(a) Again, for illustration purposes $K_1 = 0$, solely; then $A_1 = -\frac{1}{2} J_1 \bar{\mathcal{R}}_o$; and, consequently:

$$\Xi'(\varphi) = \left[\Xi_o \sin 2\varphi - (H_o + \Xi'_o) \cos 2\varphi\right] + (H_o + 2\Xi'_o) \cos\varphi$$

$$H'(\varphi) = -\left[\Xi_o \cos 2\varphi + (H_o + \Xi'_o) \sin 2\varphi\right] + (H_o + 2\Xi'_o) \sin\varphi$$

$$Z'(\varphi) = Z'_o \cos\varphi - Z_o \sin\varphi .$$

As before, these expressions are not readily visualized for their geometric interpretation. These equations represent closed curves involving functions with single and double frequencies.



3. Typical traces produced for the general and special non-secular state conditions.

Out-of-Plane Hodograph Diagrams, Inertial Frame of Reference (continued)

(b) When these non-divergent relations are further specialized to pass through the origin, the equations reduce to:

$$\Xi'(\varphi) = \Xi'_o \left[1 + 2\cos\varphi\,(1 - \cos\varphi)\right]$$

$$H'(\varphi) = \Xi'_o \left[2\sin\varphi\,(1 - \cos\varphi)\right]$$

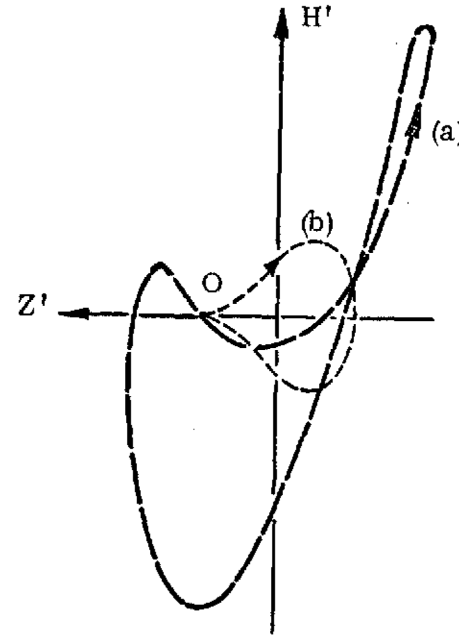
$$Z'(\varphi) = Z'_o \cos\varphi$$

and, the representative planar trace equations are recast as:

$$\frac{\Xi'(\varphi)}{\Xi'_o} = 1 + 2\,\frac{Z'(\varphi)}{Z'_o}\left(1 - \frac{Z'(\varphi)}{Z'_o}\right)$$

and

$$\frac{H'(\varphi)}{\Xi'_o} = 2\left(1 - \frac{Z'(\varphi)}{Z'_o}\right)\sin\varphi .$$



3. Typical hodograph traces for the non-secular state conditions noted in (a) and (b), at left.

Out-of-Plane Hodograph Diagrams; Inertial Frame of Reference.

The general solution equations are: (a) for the in-plane components, and (b) for the third component.

$$\text{(a)}\quad I_2\bar{\mathfrak{R}}'(\varphi)=B_2\left\{2\left[T_2(2\varphi^-)\right]\left[A_{a_I}\right]_{i.v.}+\frac{1}{2}T_2(\varphi^-)\left[2J_2-J_1-3\varphi(B_2J_1)\right]K_{o_I}+T_2(\varphi^-)\left[\Phi_\tau\bar{\tau}_I\right]\right\}+T_2(\varphi^-)\left[\Phi'_\tau\bar{\tau}_I\right],$$

and $$\text{(b)}\quad J_3\bar{\mathfrak{R}}'(\varphi)=J_3\left\{\bar{\mathfrak{R}}'_o\cos\varphi-\bar{\mathfrak{R}}_o\sin\varphi+\bar{\tau}_I\sin\varphi\right\}.$$

Partial Solutions.

1. Only those partial solutions expressed in terms of the $\bar{\tau}_I$ components provide new information. Consequently, only these are described below.

2. Traces for Zero-Initial State $(\bar{\mathfrak{R}}_o=\bar{\mathfrak{R}}'_o=0;\ \bar{\tau}_I\equiv\bar{\tau}_I(\tau_\Xi,\tau_H,\tau_z))$*, are described by:

$$\Xi'(\varphi)=\tau_\Xi\left[\frac{13}{4}\sin 2\varphi-\sin\varphi-3\varphi\left(\cos\varphi+\frac{1}{2}\cos 2\varphi\right)\right]$$
$$+\tau_H\left[5\cos\varphi-\frac{11}{4}\cos 2\varphi-\frac{3\varphi}{2}\sin 2\varphi-\frac{9}{4}\right],$$
$$H'(\varphi)=\tau_\Xi\left[\cos\varphi+\frac{1}{4}(9-13\cos 2\varphi)-3\varphi\left(\sin\varphi+\frac{1}{2}\sin 2\varphi\right)\right]$$
$$+\tau_H\left[5\sin\varphi-\frac{11}{4}\sin 2\varphi+\frac{3\varphi}{2}\cos 2\varphi\right],$$

and $$Z'(\varphi)=\tau_z\sin\varphi.$$

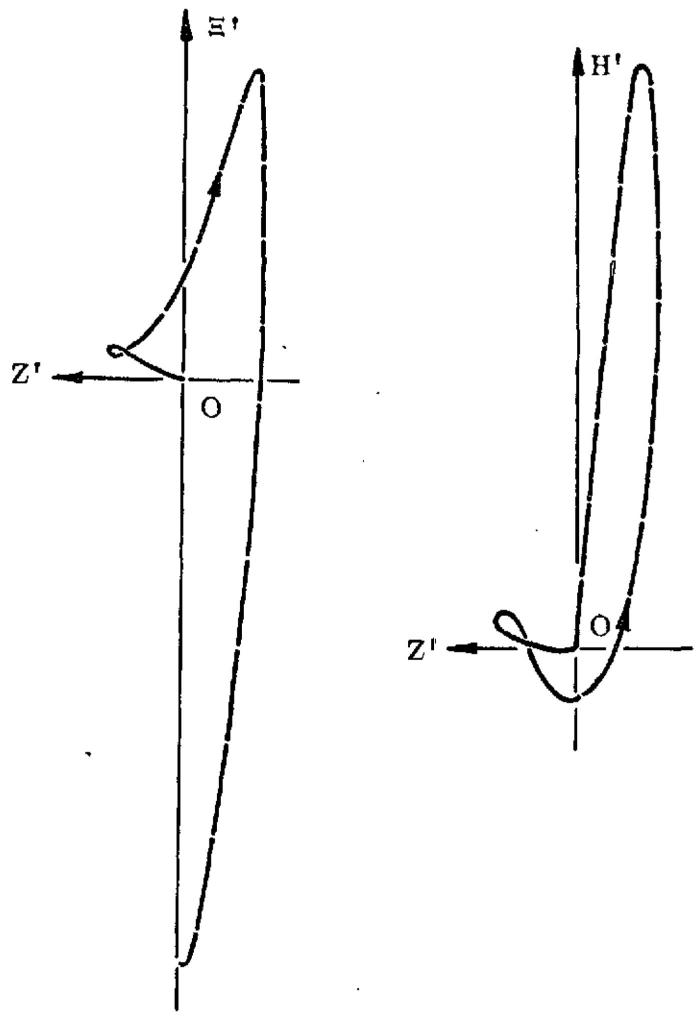
Equations for the Planar Geometries.

(a) Traces for (τ_Ξ, τ_z) only (τ components > 0), defined from:

$$\Xi'(\varphi)=\tau_\Xi\left[\sin\varphi\left(\frac{13}{2}\cos\varphi-1\right)-3\varphi\left(\cos\varphi+\frac{1}{2}\cos 2\varphi\right)\right],$$
$$H'(\varphi)=\tau_\Xi\left[\frac{13}{4}(1-\cos 2\varphi)-(1-\cos\varphi)-3\varphi\sin\varphi(1+\cos\varphi)\right],$$

and $$Z'(\varphi)=\tau_z\sin\varphi;$$

*The traces originate at the coordinate origin.



2a. Hodographs illustrating the trace geometry due to τ_Ξ, τ_z.

Out-of-Plane Hodograph Diagrams; Inertial Frame of Reference (continued)

(a) Traces for $(\tau\ ,\tau_z)$ only (continued)

hence the parametric equations:

$$\frac{\Xi'(\varphi)}{\tau_\Xi} + \frac{Z'(\varphi)}{\tau_z}\left(1-\frac{13}{2}\cos\varphi\right) = -3\varphi\left(\cos\varphi+\frac{1}{2}\cos 2\varphi\right),$$

and
$$\frac{H'(\varphi)}{\tau_\Xi} + 3\varphi\,\frac{Z'(\varphi)}{\tau_z}\left(1+\cos\varphi\right) = \frac{13}{4}\left(1-\cos 2\varphi\right)-\left(1-\cos\varphi\right).$$

(b) Traces for (τ_H, τ_z) only (τ components > 0).

$$\Xi'(\varphi) = \tau_H\left[\frac{11}{4}\left(1-\cos 2\varphi\right)-5\left(1-\cos\varphi\right)-3\varphi\left(\sin\varphi\cos\varphi\right)\right],$$

$$H'(\varphi) = \tau_H\left[\frac{3\varphi}{2}\cos 2\varphi+5\sin\varphi-\frac{11}{2}\sin\varphi\cos\varphi\right],$$

and $Z'(\varphi) = \tau_z \sin\varphi$;

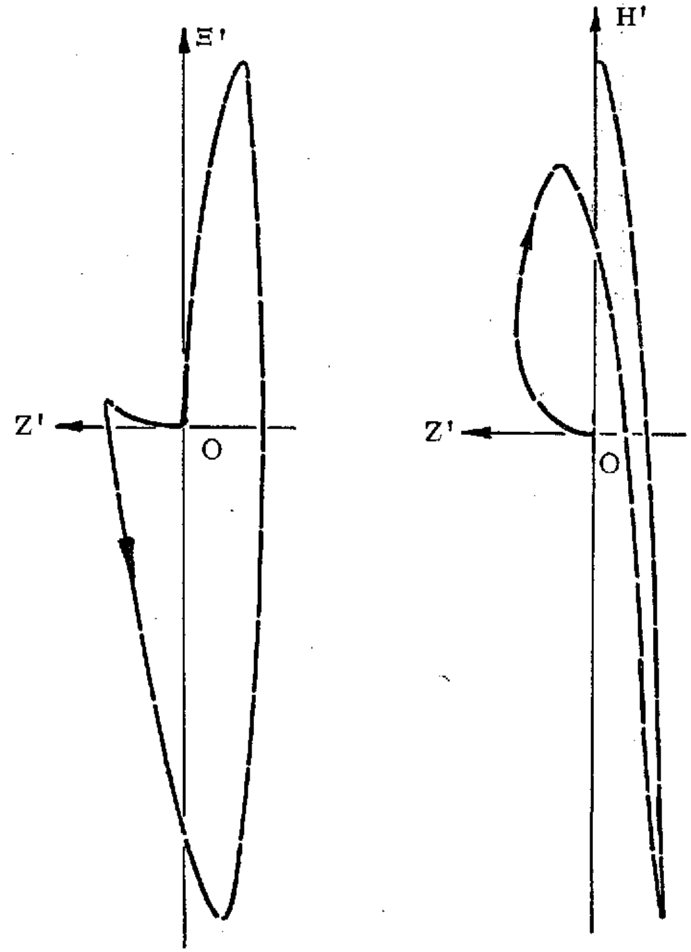
hence, the parametric equations:

$$\frac{\Xi'(\varphi)}{\tau_H} + 3\varphi\,\frac{Z'(\varphi)}{\tau_z}\cos\varphi = \frac{11}{4}\left(1-\cos 2\varphi\right)-5\left(1-\cos\varphi\right),$$

and
$$\frac{H'(\varphi)}{\tau_H} + \frac{Z'(\varphi)}{\tau_z}\left(\frac{11}{2}\cos\varphi-5\right) = \frac{3\varphi}{2}\cos 2\varphi.$$

(c) Hodograph Geometries for $\bar{\tau}_I$.

Note: These traces are defined as the summation of coordinates from (a). and (b). above.



2b. Typical traces illustrating the influence of τ_H, τ_z on the inertially described hodographs.

XI. APPLICATIONS AND SELECTED EXAMPLES

Introduction. This section of the report will be given to the development, description and solution of some selected examples recognized as problems in relative motion; and, to a few application situations. In particular the problem of intercept for a "trusting" particle is described, but for the special case of a fixed magnitude thrust aligned with the inertial triadic reference system. Secondly, some generalizations of the point-to-point transfer problem will be included, and descriptive solutions will be found. A third topic included here is that of defining appropriate "look-angles" for the relatively moving particles. Since these angles may be inferred for both representative coordinate frames, it will be prudent to present a consistent analysis for the two systems.

Basically, the information found in this section is addressed to the problem areas set down above. However, in addition to this there will be found a few illustrative cases noted which point to the utility of the results. Unfortunately time and space limitations do not permit for a more informative and encompassing expose of the analytical (and numerical) capabilities of this work. It will be left to the interested reader to describe and develop solutions, using the current results, which will evolve from his own applications needs.

Look Angles. The concept of "look angles", as the name applies here, is illustrated in the sketch below. On the figure these angles are referred to the local (horizon) frame; however, it must be recognized that the same geometry is applicable in the inertially oriented coordinate system also.

The two angles shown were chosen because: (1) they represent a consistent nomenclature; and, (2) they are relatively simply to describe mathematically. Here the angle, α, is referred to as the "cone" angle, while θ is called the "clock" angle. It should be mentioned that the angular range for these two quantities will be:

$$0 \le \alpha \le \pi,$$

and

$$-\pi \le \theta \le +\pi.$$

In the illustration P is the reference (circular orbiting) particle while Q is the relatively moving orbiter. Hence the angle pair (α, θ) locates Q relative to P -- this is consistent with "location" as defined by the relative position coordinates (x, y, z).

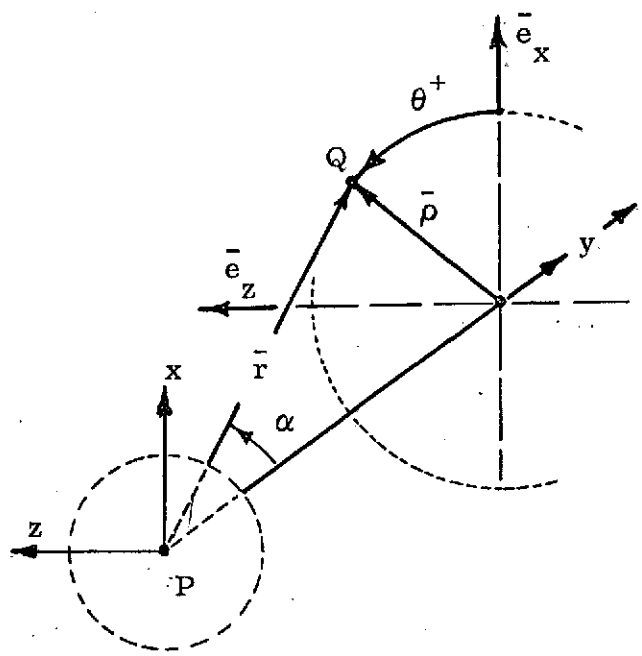


Fig. II.12. Sketch depicting the position angles (α, θ); $\alpha \equiv$ cone angle, $\theta \equiv$ clock angle; both angles are positive, as shown.

(Note: With the two reference frames assumed to be coincident, initially, there is no initial angular separation between them; therefore, the descriptions for α, θ will parallel one another in both reference schemes).

It is schematically apparent that α plays the role of a "cone angle", as its name infers. However, it should be noted that when the coordinate $y < 0$, then $\alpha > \pi/2$.

The "clock angle", θ, will range over a full circle (2π); also, its definition (below) is such that θ is an angle consistent with the algebraic sign of the coordinates (x, z). In addition, one should recognize that θ is measured in a plane whose normal is the unit vector, $\bar{e}_y$.

Taking account of the above information it is seen that when the relative coordinates vanish, identically, these angles are undefined.

Cone and Clock Angles Defined. From the sketch one sees that the relative position vector ($\bar{r}$) can be described as:

$$\bar{r} \equiv \bar{\rho} + y\bar{e}_y, \tag{II.114a}$$

where

$$\bar{\rho} \equiv x\bar{e}_x + z\bar{e}_z, \tag{II.114b}$$

(and similarily for the inertially oriented reference system).

Making use of the appropriate vector and scalar products one sees that the circular functions are ascertainable from the following operations:

$$\bar{r} \times \bar{e}_y \sim \sin\alpha,$$

$$\bar{\rho} \times \bar{e}_x \sim \sin\theta,$$

$$\bar{\rho} \cdot \bar{e}_x \sim \cos\theta,$$

and

$$\bar{\rho} \cdot \bar{e}_y \sim \cos\alpha\,;$$

therefore, a description of these quantities can be given by:

$$\theta \equiv \tan^{-1}(z/x), \tag{II.115a}$$

and

$$\alpha \equiv \tan^{-1}(|\bar{\rho}|/y); \tag{II.115b}$$

where $|\bar{\rho}|$ is the magnitude of the vector in Eq. (II.114b).

Expressed in non-dimensional form, and for corresponding frame(s) of reference, these angle definitions may be recast as:

(a) for the local rotating (horizon) frame:

$$\tan\theta = \zeta/\xi, \tag{II.116a}$$

and

$$\tan\alpha = \sqrt{\xi^2 + \zeta^2}/\eta\,; \tag{II.116b}$$

(b) for the inertially oriented frame:

$$\tan \theta = Z/\Xi , \qquad \text{(II.116c)}$$

and

$$\tan \alpha = \sqrt{\Xi^2 + Z^2}/H. \qquad \text{(II.116d)}$$

The definitions stated above indicate where an observer in "P" should look -- relative to his own frame of reference -- to "locate" Q. Logically, one would (as a first guess) expect that for the converse situation the angles should be incremented by π radians. Unfortunately, this is not quite the case; a "correction" should be added (or accounted for) before a true definition can be had. For instance, true imagery of these angles will exist only when the inertial reference frame is used for observational information; and, only then if the frames in "P" and "Q" are aligned. When these conditions exist, the corresponding angles, from the two particles, are phased at π radians.

A Correction to Look Angles. For the "local" rotating frame of reference the clock angles are phased as noted above; however, the cone angles cannot be so simply related due to an angular rotation needed in locating one local triad relative to another (see Fig. II.13, below).

Assuming that only the in-plane angularity needs the correction, then it is easily shown that the reference triads are connected by the transformation:

$$\begin{bmatrix} \bar{e}_x \\ \bar{e}_y \\ \bar{e}_z \end{bmatrix}_Q = \begin{bmatrix} \cos \Delta\varphi & \sin \Delta\varphi & 0 \\ -\sin \Delta\varphi & \cos \Delta\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{e}_x \\ \bar{e}_y \\ \bar{e}_z \end{bmatrix}_P . \qquad \text{(II.117)}$$

In this expression the subscripts "Q" and "P" signify an appropriate origin for the triads. Recognizing that the relative position vectors satisfy the relation, $\bar{r}_1 = -\bar{r}$, then the positional coordinates, for P relative to Q are given by:

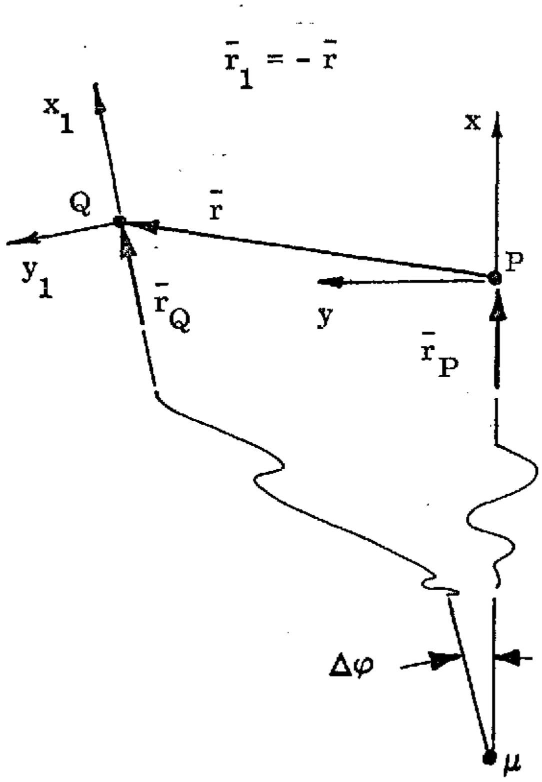


Fig. II.13. Sketch, illustrating the in-plane angular separation between two local triads of reference. Note that the relative position vector, locating "P" from "Q" would satisfy the relationship: $\bar{r}_1 = -\bar{r}$.

if

$$\{\bar{r}_1\} \equiv [x_1, y_1, z_1]^T,$$

then

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = - \begin{bmatrix} \cos\Delta\varphi & \sin\Delta\varphi & 0 \\ -\sin\Delta\varphi & \cos\Delta\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad \text{(II.118)}$$

(An evaluation for $\Delta\varphi$ can be approximated by the relationship:

$$\Delta\varphi \cong \tan^{-1}(\eta).$$

For most application purposes, consistent with the "smallness" of the relative displacements, this is a valid approximation).

Making use of the above transformation, and denoting the new position angles by (α_1, θ_1), then it is evident that:

$$\theta_1 \equiv \tan^{-1}(\zeta_1/\xi_1), \quad \text{(II.119a)}$$

and

$$\alpha_1 \equiv \tan^{-1}\left(\sqrt{\xi_1^2+\zeta_1^2}\,/\eta_1\right). \quad \text{(II.119b)}$$

Here the dimensionless coordinates are described in a manner consistent with the definitions for (ξ, η, ζ).

One statement of caution should be made before leaving this section. That is, the expressions immediately above apply only to the relative positioning of P as measured from Q; and, then only for the local rotating (or horizon oriented) triad reference system. Normally, the premise would be that the position coordinates are measured from P to Q; hence, the two position angles are appropriately defined in Eqs. (II.116).

A Powered ($\bar{\tau}_I$) Intercept Problem. Earlier, in Reference [1], the problem of a two-particle intercept was introduced, solved and illustrated. There, both the impulsive action and a fixed magnitude thrust intercept case were documented. However, with the introduction of a second frame of reference (the inertially oriented one), in this present work, there is reason to go back and reexamine the intercept problem. The purpose in doing so is to resolve it for the case of a fixed thrust aligned with the inertial reference frame's triad.

For clarity and continuity in notation the previously solved thrusting-intercept result will be restated (or alluded to) here. The reason for doing so is to introduce the notational scheme employed now; but, to do so by means of a known resultant. Hopefully this will alleviate any difficulties which could arise in notational understanding.

The meaning of intercept, as inferred here, was described in Article III.6, Reference [1]; and, the powered intercept problem was solved in Appendix D (see Section D.6.2). Briefly, there one finds that Eq. (D.12b) represents (symbolically) the linear solution for a relative motion with thrusting ($\bar{T}$). Introducing the concept of intercept, then that solution was represented by Eq. (D.13); and, the dimensionless resultant, corresponding to this, was written out in Eq. (D.15).

Now, for the present case (an intercept to occur by an inertially directed dimensionless thrust, $\bar{\tau}_I$) the solution equations to be manipulated are noted as Eqs. (II.30), herein. Symbolically, the general solution can be noted as:

$$\bar{\varkappa} = A\bar{\varkappa}_o + B\bar{\varkappa}'_o + C\bar{\tau}_I . \qquad \text{(II.120a)}$$

(Here $\bar{\varkappa}$ is a representation for position; $\bar{\varkappa}'$ represents velocity; and A, B, C are the appropriate coefficient matrices. For reference purposes, the corresponding velocity expression, Eq. (II.31), is expressed as:

$$\bar{\varkappa}' = A'\bar{\varkappa}_o + B'\bar{\varkappa}'_o + C'\bar{\tau}_I . \qquad \text{(II.120b)}$$

The primes, here, infer differentiation, as before. The "zero" subscript is used to signify an initial valued quantity).

For a solution, the concept of intercept in applied ($\bar{\varkappa} \equiv \bar{0}$), and the value of $\bar{\tau}_I$ is determined accordingly. For consistency in notation, the intercept is assumed to occur in t^* seconds (or in a transfer of φ^* radians); thus, this solution for $\bar{\tau}_I$ is denoted as $\bar{\tau}^*_I$. Consequently, the symbolic description for the intercept is written as:

$$\bar{\tau}^*_I = - C^{-1}(A\,\bar{\varkappa}_o + B\,\bar{\varkappa}'_o), \tag{II.121a}$$

wherein C^{-1} is the inverse matrix corresponding to C. An expanded representation for this resultant is written as:

$$\begin{bmatrix} \tau^*_{\Xi} \\ \tau^*_H \\ \tau^*_Z \end{bmatrix} = -(C^{-1}A) \begin{bmatrix} \xi_o \\ \eta_o \\ \zeta_o \end{bmatrix} -(C^{-1}B) \begin{bmatrix} \xi'_o \\ \eta'_o \\ \zeta'_o \end{bmatrix} . \tag{II.121b}$$

A more useful expression, for present purposes, is set down into the following form:

$$\bar{\tau}^*_I \Delta \equiv A^*\bar{\varkappa}_o + B^*\bar{\varkappa}'_o \equiv \begin{bmatrix} a^*_{11} & a^*_{12} & a^*_{13} \\ a^*_{21} & a^*_{22} & a^*_{23} \\ a^*_{31} & a^*_{32} & a^*_{33} \end{bmatrix} \left\{\bar{\varkappa}_o\right\} + \left[b^*_{ij}\right]\left\{\bar{\varkappa}'_o\right\}, \tag{II.122a}$$

wherein Δ is related to the determinant of C^{-1}, and A^*, B^* are the results of matrix multiplication noted in Eq. (II.121b).

The details of the required matrix operations (above) are not included here; however, the scalars for the two matrices (A^*, B^*), and the value of Δ is noted immediately below. That is:

$$a^*_{11} = 9\varphi\,(\varphi \cos\varphi - \sin\varphi) + (1 - \cos\varphi)[6(1 - \cos\varphi) + 23] - 12\varphi \sin\varphi,$$

$$a^*_{21} = 3[3\varphi^2 \sin\varphi + 2(\varphi\cos\varphi - \sin\varphi) - (3\varphi + 2\sin\varphi)(1 - \cos\varphi)],$$

$$a^*_{31} = 0,$$

$$a^*_{12} = -\frac{3}{2}(\varphi\cos\varphi - \sin\varphi),$$

$$a^*_{22} = -\frac{3\varphi}{2}\sin\varphi + 2(1 - \cos\varphi),$$

$$a^*_{32} = 0 = a^*_{13} = a^*_{23},$$

$$a^*_{33} = -\Delta\cos\varphi/(1 - \cos\varphi);$$

$$b^*_{11} = -(1 - \cos\varphi)(3\varphi - 2\sin\varphi),$$

$$b^*_{21} = 4[\frac{3}{2}\sin\varphi(\varphi - \sin\varphi) - (1 - \cos\varphi)^2],$$

$$b^*_{31} = 0,$$

$$b^*_{12} = 4(4 - \cos\varphi)(1 - \cos\varphi) - \frac{3\varphi}{2}(7\sin\varphi - 3\varphi\cos\varphi),$$

$$b^*_{22} = 2[\frac{9\varphi^2}{4}\sin\varphi - (3\varphi + 2\sin\varphi)(1 - \cos\varphi)],$$

$$b^*_{32} = 0 = b^*_{13} = b^*_{23},$$

$$b^*_{33} = -\Delta/\tan\frac{\varphi}{2};$$

and

$$\Delta \equiv 9(\varphi - \sin\varphi)^2 + \frac{9\varphi}{2}(1 - \cos\varphi) + 10(1 - \cos\varphi)^2. \quad \text{(II.122b)}$$

Equations (II.122) describe the magnitude of "thrust" needed to produce an intercept from an initial state ($\bar{\varkappa}_o$, $\bar{\varkappa}'_o$). As before, the "time" required for the operation is presumed known (a priori). Once the resultant $\bar{\tau}^*_I$ is acquired a "time" history of the displacement ($\bar{\varkappa}$) and the velocity ($\bar{\varkappa}'$) is obtained from Eqs. (II.120)*. In Eqs. (II.120) the proper value for $\bar{\tau}_I$ is (obviously) that solved for above; namely, $\bar{\tau}^*_I$.

*These results, being analytical, infer a linear solution to this problem.

This intercept solution describes the state of motion as it would be "seen" in the local rotating frame of reference, even though the "thrust" $(\bar{T}_I)$ is aligned with an inertially oriented triad. There is a corresponding, or companion, solution for this situation -- one which would refer the state to an inertial frame of reference, per se. In order to acquire this resultant one could proceed in the same manner as is indicated by the generalization alluded to in Eqs. (II.33b), and the procedure used to obtain Eq. (II.36). These particular operations are not to be carried out here; the interested reader can obtain the desired results, readily, but at the expense of some needed algebraic gyrations.

A General, Point-to-Point, Transfer. The general point-to-point transfer involves the application of an impulse, and/or a specified thrusting action, which will drive an orbiting particle from some initial state $(\bar{\kappa}_o, \bar{\kappa}'_o)$* to some final state $(\bar{\kappa}_f, \bar{\kappa}'_f)$. Normally, the final state is described in terms of either position $(\bar{\kappa}_f)$ or velocity $(\bar{\kappa}'_f)$, as a priori information, but not both. This is the consequence of preselecting the "time" (or transfer angle) during which the operation is to occur.

In effect the solution(s) here follow, identically, the pattern set down in the foregoing section. For this reason the procedures, here, will be abbreviated; however, the cognizant results will be set down in sufficient detail to evaluate solutions, if desired. Following the established pattern the present solutions evolve, generally, from the state equations written in rotating frame coordinates; nevertheless the companion solution (in inertial relative coordinates) can be acquired as above. There is an exception to this procedure. This is for the case of an impulsive transfer which is referred to the inertially oriented frame of reference.

(a) The Impulsive Transfer. The impulsive point-to-point transfer is developed from the results in Eqs. (II.30) and (II.31). For instance, denoting the final state as $(\bar{\kappa}_f, \bar{\kappa}'_f)$, then from these solutions, one can write:

*The quantities $(\bar{\kappa}_i, \bar{\kappa}'_i)$ are employed as general relative motion state vectors.

$$\bar{\varkappa}_f = A\bar{\varkappa}_o + B\bar{\varkappa}'_o, \tag{II.123a}$$

and

$$\bar{\varkappa}'_f = A'\bar{\varkappa}_o + B'\bar{\varkappa}'_o. \tag{II.123b}$$

Here it must be assumed that A, B, A', B' are known functions of the "time required" to complete the intended maneuver.

In order to ascertain the impulse ($\bar{\varkappa}'_o$) needed to reach $\bar{\varkappa}_f$ from ($\bar{\varkappa}_o$, $\bar{\varkappa}'_o$) one should solve Eq. (II.123a). The form of that solution is:

$$\bar{\varkappa}'_o = B^{-1}\bar{\varkappa}_f - (B^{-1}A)\,\bar{\varkappa}_o, \tag{II.124}$$

wherein B^{-1} is the inverse of matrix B (see Eq. (II.123a)). Remembering that the matrices are known functions of the transfer time (and/or angle), then the resultant is acquired as follows:

Since $\bar{\varkappa}_i \equiv \bar{\varkappa}\,(\xi, \eta, \zeta)_i$; $\bar{\varkappa}'_o \equiv \bar{\varkappa}'_o\,(\xi'_o, \eta'_o, \zeta'_o)$, the solution is expressed as:

$$\begin{bmatrix} \xi'_o \\ \eta'_o \\ \zeta'_o \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} b_{ij} \end{bmatrix}\begin{bmatrix} \xi_f \\ \eta_f \\ \zeta_f \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} a_{ij} \end{bmatrix}\begin{bmatrix} \xi_o \\ \eta_o \\ \zeta_o \end{bmatrix}, \tag{II.125*}$$

for (i, j) ≡ (1, 2, 3).

Herein the various scalar components of the matrices are:

$$b_{ij} \equiv \begin{bmatrix} 4\cos\varphi & -2(1-\cos\varphi) & 0 \\ 2(1-\cos\varphi) & \sin\varphi & 0 \\ 0 & 0 & \dfrac{\Delta}{\sin\varphi} \end{bmatrix} \tag{II.126a}$$

and

*Subscripts $(\sim)_o$ and $(\sim)_f$ denote initial and final values, respectively.

$$a_{ij} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} , \text{ wherein} \qquad \text{(II.126b)}$$

$$a_{11} = (3\varphi - 4 \cos \varphi) + 3(1 - \cos \varphi)[(3\varphi - 4 \cos \varphi) - 4(\varphi - \sin \varphi)] ,$$

$$a_{21} = 2[3\varphi \sin \varphi - 7(1 - \cos \varphi)] , \quad a_{13} = 0,$$

$$a_{21} = 2(1 - \cos \varphi), \quad a_{22} = - \sin \varphi, \quad a_{32} = 0, \qquad \text{(II.126c)}$$

$$a_{13} = a_{23} = 0, \quad a_{33} = - \Delta/\tan \varphi , \text{ and}$$

$$\Delta \equiv 4[\sin \varphi (\cos \varphi - \frac{3\varphi}{4}) + (1 - \cos \varphi)^2] .$$

The consequence here is that the general point-to-point transfer is now known; and that the sought for impulse is described by Eq. (II.125).

Should the desired solution have been one to determine an impulse ($\bar{\chi}'_o$) to reach a final velocity ($\bar{\chi}'_f$); then, that the result would have appeared as:

$$\bar{\chi}'_o = (B')^{-1} \bar{\chi}'_f - [(B')^{-1} A'] \bar{\chi}_o . \qquad \text{(II.127)}$$

(Recall that primes in this expression denote differentiation with respect to the transfer angle).

The results above account, in general, for an impulsive transfer from an initial state to a prescribed final state. However, the specific case set down here was for a problem referred to the local rotating frame of reference.

The corresponding results, cast in inertially described relative motion coordinates, could be acquired by means of the relations noted in Eqs. (II.32a); or, the more direct approach could be taken using the solution expressions, Eqs. (II.39), (II.40). Making use of these latter formulae, and applying to them the concept noted in Eq. (II.124) -- but introducting the inertially described coordinates -- then:

$$\text{with } \bar{x}_i \equiv x(\Xi, H, Z)_i; \quad \bar{x}'_o \equiv x'_o(\Xi'_o, H'_o, Z'_o)$$

the solution would be expressed as:

$$\begin{bmatrix} \Xi'_o \\ H'_o \\ Z'_o \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} b_{ij} \end{bmatrix} \begin{bmatrix} \Xi_f \\ H_f \\ Z_f \end{bmatrix} + \frac{1}{\Delta} \begin{bmatrix} a_{jk} \end{bmatrix} \begin{bmatrix} \Xi_o \\ H_o \\ Z_o \end{bmatrix}, \quad \text{for } (j,k) \equiv (1, 2, 3). \tag{II.128}$$

Herein the two coefficient matrices are:

$$\begin{bmatrix} b_{jk} \end{bmatrix} \equiv \begin{bmatrix} (2\sin\varphi + \sin 2\varphi - 3\varphi\cos\varphi) & -[2\cos\varphi + (\cos 2\varphi - 3) + 3\varphi\sin\varphi] & 0 \\ 2\cos\varphi - \frac{1}{2}(3 + \cos 2\varphi) & 2\sin\varphi - \frac{1}{2}\sin 2\varphi & 0 \\ 0 & 0 & \Delta/\sin\varphi \end{bmatrix}, \tag{II.129a}$$

and

$$\begin{bmatrix} a_{jk} \end{bmatrix} \equiv \begin{bmatrix} 3\varphi\cos\varphi - 4\sin\varphi & 3[\varphi\sin\varphi - 2(1 - \cos\varphi)] & 0 \\ 3[\varphi\sin\varphi - 2(1 - \cos\varphi)] & -\sin\varphi & 0 \\ 0 & 0 & -\Delta/\tan\varphi \end{bmatrix}; \tag{II.129b}$$

with the quantity Δ being:

$$\Delta \equiv 8(1 - \cos\varphi) - 3\varphi\sin\varphi. \tag{II.129c}$$

This last result describes the same transfer situation as does Eq. (II.125); the exception is that here the coordinates are inertially oriented -- the previous case was for motions referred to the rotating frame of reference.

These two solutions describe (in a linearized sense) the general impulsive transfer from one prescribed relative position to another. It must be recognized that the primary piece of information acquired here is the required initial impulse; the final impulse, and a time history for the state variables -

during these transfer maneuvers - must be obtained from the general expressions (see Eqs. (II.30), (II.31), (II.40)).

The next problem area to be examined is also concerned with a point-to-point transfer, but there the primary consideration will be a determination of the thrust required to accomplish the maneuver. In this next sub-section the results will be reported only for cases where the coordinates are described in a local rotating frame. To determine the transfer in relative inertial coordinates the reader should apply the non-familiar transformations for coordinates and speeds.

(b) The Thrusting Transfer. The outline which follows will show results obtained for a point-to-point transfer due to thrusting actions. Here, as mentioned earlier, the two cases of interest will consider the thrust aligned with one or the other of the reference triads. However, both solutions will be for the state variables referred to the local rotating (horizon oriented) triad only.

Since the basic expressions to be manipulated stem from the formal solution results, these will be symbolically represented, at first; the specific equations will be noted when final statement formats are set down.

In this regard a general form of the state equations, referred to either frame of reference, can be written as:

$$\bar{\kappa} = A\,\bar{\kappa}_o + B\,\bar{\kappa}'_o + C\,\bar{\tau} \qquad \text{(II.130a)}$$

and

$$\bar{\kappa}' = A'\bar{\kappa}_o + B'\bar{\kappa}'_o + C'\bar{\tau}\,, \qquad \text{(II.130b)}$$

where the $\bar{\kappa}_i$ are state vectors, $\bar{\tau}$ is the dimensionless specific "thrust", primes denote derivative forms and A, B, C, etc. are coefficient matrices.

Quite simply, the solution, that of defining a thrust to accomplish the maneuver, can be represented by:

$$\bar{\tau} = C^{-1}\bar{\kappa}_f - C^{-1}[A\,\bar{\kappa}_o + B\,\bar{\kappa}'_o]\,. \qquad \text{(II.131)}$$

This result suggests that (now) a more general solution type is acquired since the complete beginning state ($\bar{\varkappa}_o$, $\bar{\varkappa}'_o$) is present. The fact that only the terminal position ($\bar{\varkappa}_f$) is preset is a necessary consequence of the general solution. These being fixed thrusts ($\bar{\tau}$), then the terminal velocity ($\bar{\varkappa}'_f$) will be acquired from a time-history description of the motion state, per se.

To make Eq. (II.131) operationally useful it is necessary, next, to set down the results in an appropriate algebraic/matrix format. This is carried out below.

(1) <u>The solution for $\bar{\tau} \equiv \bar{\tau}(\tau_\xi, \tau_\eta, \tau_\zeta)$.</u> On the assumption that $\bar{\tau}$ has the designation indicated, then following the solution described in Eq. (II.131), it can be shown that the various coefficient matrices are obtained as indicated below:

First, the solution is cast into the format;

$$\begin{bmatrix} \tau_\xi \\ \tau_\eta \\ \tau_\zeta \end{bmatrix} = \frac{1}{\Delta}\left\{ \begin{bmatrix} c_{ij} \end{bmatrix} \begin{bmatrix} \xi_f \\ \eta_f \\ \zeta_f \end{bmatrix} + \begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} \xi_o \\ \eta_o \\ \zeta_o \end{bmatrix} + \begin{bmatrix} b_{ij} \end{bmatrix} \begin{bmatrix} \xi'_o \\ \eta'_o \\ \zeta'_o \end{bmatrix} \right\}, \qquad \text{(II.132)}$$

for (i, j) = (1, 2, 3).

Here,

$$c_{11} = 4(1-\cos\varphi) - \frac{3\varphi^2}{2}, \quad c_{12} = -2(\varphi - \sin\varphi), \quad c_{13} = 0,$$

$$c_{21} = 2(\varphi - \sin\varphi), \quad c_{22} = (1-\cos\varphi), \quad c_{23} = 0, \qquad \text{(II.133a)}$$

$$c_{31} = 0, \quad c_{32} = 0, \quad c_{33} = \Delta/(1-\cos\varphi);$$

and

$$a_{11} = 4[6\varphi\sin\varphi - 7(1-\cos\varphi)], \quad a_{12} = 2(\varphi - \sin\varphi), \quad a_{13} = 0,$$

$$a_{21} = -2(\varphi - \sin\varphi), \quad a_{22} = -(1-\cos\varphi), \quad a_{23} = 0,$$

$$a_{31} = 0, \quad a_{32} = 0, \quad a_{33} = -\Delta \cos\varphi/(1 - \cos\varphi). \qquad \text{(II.133b)}$$

Next,

$$b_{11} = \frac{3\varphi^2}{2} \sin\varphi - 4\varphi(1 - \cos\varphi),$$

$$b_{21} = 2[2(1 - \cos\varphi) - \varphi \sin\varphi],$$

$$b_{31} = 0,$$

$$b_{12} = 2[7\varphi \sin\varphi - 8(1 - \cos\varphi)] - \frac{3\varphi^2}{2}(1 + \cos\varphi),$$

$$b_{22} = -\varphi(1 - \cos\varphi),$$

$$b_{32} = 0,$$

$$b_{31} = b_{32} = 0,$$

and

$$b_{33} = -\Delta \sin\varphi/(1 - \cos\varphi). \qquad \text{(II.133c)}$$

For the above, the quantity Δ is defined by:

$$\Delta \equiv 8(1 - \cos\varphi - \varphi \sin\varphi) + \frac{5\varphi^2}{2}\left(1 + \frac{3}{5}\cos\varphi\right). \qquad \text{(II.133d)}$$

(2) <u>The solution for $\bar{\bar{\tau}} \equiv \bar{\bar{\tau}}_I (\tau_\Xi, \tau_H, \tau_Z)$.</u> In this solution example the motion traces and the transfer arise from the application of a non-dimensional, specific, fixed thrusting action; but one where the components are <u>aligned</u> with the inertially oriented triad of reference.

Once more the solution format is that described in Eq. (II.131); however, the coefficient matrices are not identical to those set down above, for obvious reasons.

For identification, let the present solution be given in the form:

$$\begin{bmatrix} \tau_\Xi \\ \tau_H \\ \tau_Z \end{bmatrix} = \frac{1}{\Delta}\left\{ \begin{bmatrix} c_{jk} \end{bmatrix} \begin{bmatrix} \xi_f \\ \eta_f \\ \zeta_f \end{bmatrix} + \begin{bmatrix} a_{jk} \end{bmatrix} \begin{bmatrix} \xi_o \\ \eta_o \\ \zeta_o \end{bmatrix} + \begin{bmatrix} b_{jk} \end{bmatrix} \begin{bmatrix} \xi_o' \\ \eta_o' \\ \zeta_o' \end{bmatrix} \right\} , \qquad \text{(II.134)}$$

for (j, k) = (1, 2, 3).

The various scalar components of the coefficient matrices are:

$$c_{11} = 3\varphi\sin\varphi - 5(1-\cos\varphi), \quad c_{12} = \frac{3}{2}(\varphi\cos\varphi - \sin\varphi), \quad c_{13} = 0,$$

$$c_{21} = 3[2\sin\varphi - \varphi(1+\cos\varphi)], \quad c_{22} = \frac{3\varphi}{2}\sin\varphi - 2(1-\cos\varphi), \quad c_{23} = 0,$$

$$c_{31} = 0, \quad c_{32} = 0, \quad c_{33} = \Delta/(1-\cos\varphi). \qquad \text{(II.135a)}$$

Next,

$$a_{11} = 9\varphi(\varphi\cos\varphi - \sin\varphi) + 23(1-\cos\varphi) + 6(1-\cos\varphi)^2 - 12\varphi\sin\varphi,$$

$$a_{21} = 3[2(\varphi\cos\varphi - \sin\varphi) + 3\varphi^2\sin\varphi - (1-\cos\varphi)(3\varphi + 2\sin\varphi)],$$

$$a_{31} = 0,$$

$$a_{12} = \frac{3}{2}(\sin\varphi - \varphi\cos\varphi),$$

$$a_{22} = 2(1-\cos\varphi) - \frac{3\varphi}{2}\sin\varphi,$$

$$a_{32} = 0,$$

$$a_{13} = a_{23} = 0, \quad \text{and} \quad a_{33} = -\Delta\cos\varphi/(1-\cos\varphi). \qquad \text{(II.135b)}$$

Lastly,

$$b_{11} = (1-\cos\varphi)(2\sin\varphi - 3\varphi),$$

$$b_{21} = 4[\frac{3}{2}\sin\varphi(\varphi - \sin\varphi) - (1-\cos\varphi)^2],$$

$$b_{31} = 0,$$

$$b_{12} = 4(1-\cos\varphi)(4-\cos\varphi) + \frac{3\varphi}{2}(3\varphi\cos\varphi - 7\sin\varphi),$$

$$b_{22} = \frac{9\varphi^2}{2}\sin\varphi - 2(1-\cos\varphi)(2\sin\varphi + 3\varphi),$$

$$b_{32} = 0,$$

$$b_{13} = b_{23} = 0, \quad \text{and} \quad b_{33} = -\Delta/\tan\frac{\varphi}{2}\ . \tag{II.135c}$$

In the above expressions,

$$\Delta \equiv 9(\varphi - \sin\varphi)^2 + \frac{9\varphi^2}{2}(1-\cos\varphi) + 10(1-\cos\varphi)^2. \tag{II.135d}$$

Equations (II.132) and (II.134) describe a "thrust" vector needed to accomodate some preselected point-to-point transfer -- one which is to occur in a known time (or, over a given transfer angle). To describe the "time-history" of such a maneuver, it would be necessary to insert the $\bar{\tau}$ vectors into the proper state equations (set down) and to evaluate those accordingly. Once again, it must be remembered that these are linear resultants and, correspondingly, they have a limited applicability for prediction purposes.

Also, as mentioned earlier, these results are applicable for local horizon state coordinate descriptions only; if the reader desires to have a similar result in the inertially defined relative motion coordinates, it will be necessary to transform these results over to that frame of reference.

In the next sub-section an operational type example will be described. The purpose there will be to illustrate how the information developed here can be put to use -- a mathematical solution is not set down.

Example: A Stationkeeping Mode. The idea of "stationkeeping" as it is inferred here, has to do with maintaining some position* which is in close proximity to (say) an orbiting vehicle. Previously, in Reference [1], ideas were mentioned which fit this concept. There the elimination of secular influences allowed for

*position, here, suggests a region rather than a point.

adjacent relative orbits, which fit this idea. The "messenger" transfer, as depicted on Fig. III.7 (Ref. [1]), could also be classed as a candidate operation - even though that maneuver was time restricted in its stationkeeping mode.

To be more realistic and to present a rather general scheme of operation, the maneuvering concept set forth in Reference [22] will be utilized here. This will allow the reader to make ample use of the motion traces, described for a general relative motion, and to ascertain how a stationkeeping mode could be established.

For present purposes the reader is referred to (say) Fig. III.3 in Ref. [1]; and, more specifically, to trace (4) on page 73, and to the hodograph (trace (4) on page 81) of this report. These two sketches are reproduced below; they will be utilized in describing an in-plane stationkeeping mode - one which will "hold" a relative motion in the near vicinity of an orbiting spacecraft.

On Fig. II.14a, the origin (P) is assumed to be the locus for the orbiting spacecraft while Q is an instantaneous position for the relatively moving orbiter. Fig. II.14b is the (relative) hodograph corresponding to the trace on the first sketch.

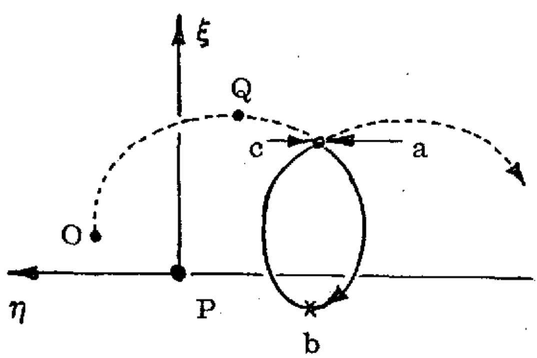


Fig. II.14a. Sketch of a general, in-plane relative motion trace. P is the reference (spacecraft) particle: Q is the local "orbiter".

In keeping with the geometry shown here, the stationkeeping mode will consist of the trace curve (a, b, c); the idea is to repeat this trace geometry, in a continuous fashion, thereby retaining a closed local "orbit" for Q, near to P. Obviously, if some corrective action is not introduced then Q will continue to fall further and further behind P, during its free motion.

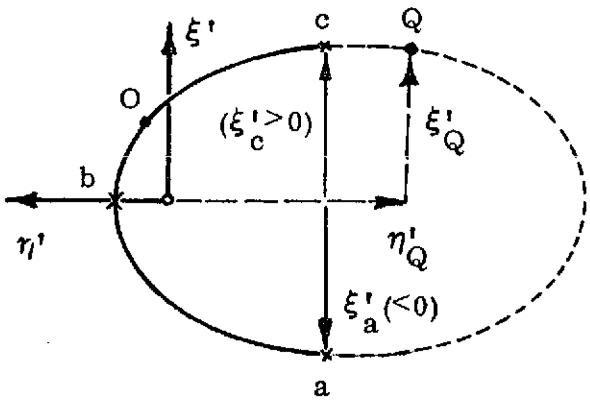


Fig. II.14b. Hodograph for the relative motion above. Note that the velocity for "Q" is shown, and that the speed at a, b, c are described. Also, recognize that $|\xi_a'| = |\xi_o'|$.

Focusing attention, now, on Fig. II.14b, one notes that the hodograph (for the trace (a, b, c)) is also noted by a, b, c (here). That is, at position "a", the hodograph locus shows particle Q to have a relative velocity composed of $(-\xi_a', -\eta_a')$ speeds. At position "b" the hodograph suggests speed components $(0, \eta_b)$; while at position "c" the hodograph is <u>similar</u> to that at "a" except that now the speeds are $(+\xi_c', -\eta_c')$.

To retrace the position plot (a, b, c) it is only necessary to re-establish the hodograph locus at "a". This can be simply done by adding an impulse at "c" (of size $-2\xi_a'$) -- then the initial hodograph is reformed, and the trace (a, b, c) is followed again.

Consequently, each time particle Q reaches position "c", the impulse is applied again and path (a, b, c) is restored. This obviously establishes (in a hueristic sense) the stationkeeping mode for this displacement configuration. Some thought on this operation will convince the reader that a like stationkeeping path (for Q, relative to P) could be determined for the entire (in-plane) space adjacent to P. Of course, in a more restricted sense a stationkeeping mode <u>could</u> be developed, adjacent to P, which would eliminate the repeative application of impulses. Such a scheme would be reminiscent of the relative motions with secular terms removed. This stationkeeping operation would be repetitive and <u>without</u> impulse input, but it would be cyclic in the spacecraft (P) period. The operation described above does not utilize the entire hodograph, hence that operation is repetitive at <u>less</u> than orbit period.

Needless to say, there are a variety of operational schemes which can be developed from these ideas -- the inventive reader could easily envision a number of such.

Here, a word description has been used to develop a stationkeeping mode. The mathematics of such a scheme would readily follow from study and implementation of the accompanying sketches, coupled with several of the formulae provided herein. It is hoped that by now the reader has acquired an appreciation and use for the relative motion traces and hodographs.

Example: A Minimum Time, Relative Transfer. The problem to be solved, next, has to do with a minimum time transfer. The transfer is to begin at an orbiting spacecraft (P), and proceed to a lower altitude orbit; but, to a position which is close by to P (close enough so that the linearized relative motion equations will be applicable).

Such an operation, as that described here, could depict the "rescue" of a stranded astronaut from a position below the parent vehicle (see Fig. II.15). If time would be of the essence, here, then the information gleaned (below) could be of great value.

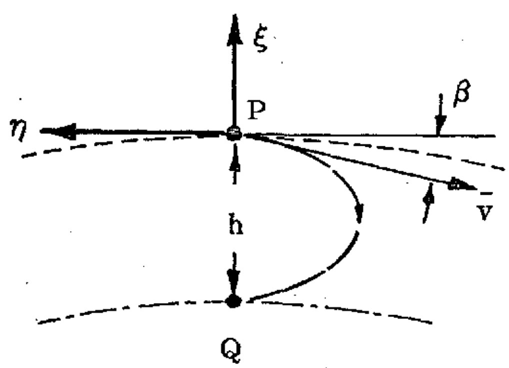


Fig. II.15. Sketch of the minimum time transfer operation. P is the (circular) orbiter; Q is the final position to be achieved, $\Delta\bar{v}$ is the impulse, and β its direction.

For mathematical manipulations this problem will be cast into the format set down in Eqs. (II.30). That is, all state quantities are non-dimensionalized, and the minimum time requirement becomes a minimum transfer ($\Delta\varphi$) requirement.

In the parlance of present nomenclature, the following quantities are employed:

$$\xi = \frac{-h}{r_p} \equiv -\boldsymbol{\mathit{h}} \;;\; \eta = \zeta = 0;$$

$$\xi_o = \eta_o = \zeta_o = 0;$$

$$\xi_o' = \frac{-v}{V_P}\sin\beta \equiv \nu \sin\beta,$$

$$\eta_o' = -\nu\cos\beta;\ \zeta_o' = 0. \qquad \text{(II.136)}$$

From Eqs. (II.30) this relative motion problem will be expressed (in general) as:

$$\bar{x}(\varphi) = A\,\bar{x}_o + B\,\bar{x}_o' + c\,\bar{\tau}; \qquad \text{(II.137a)}$$

and, in particular, for the in-plane format, write:

$$\begin{bmatrix} -h \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\varphi & 2(1-\cos\varphi) \\ -2(1-\cos\varphi) & 4\sin\varphi - 3\varphi \end{bmatrix} \begin{bmatrix} -\nu\sin\beta \\ -\nu\cos\beta \end{bmatrix} . \qquad \text{(II.137b)*}$$

From this expression it is found that a general solution statement is:

$$\frac{h}{\nu} - 2\cos\beta = \sin\varphi\sin\beta - 2\cos\varphi\cos\beta . \qquad \text{(II.138)}$$

Examining this expression for an extremal (presuming that h and ν are not variables) then the condition for an extremal is found to be:

$$\tan\frac{\varphi}{2} = \frac{1}{2}\,\text{ctn}\ \beta, \qquad \text{(II.139)}$$

describing a requirement between the "aiming" angle and the transfer angle.

When this requirement is incorporated into Eq. (II.138), it is found (after some manipulation) that the minimal maneuver must satisfy the constraint condition:

$$\frac{h}{\nu} = \frac{4\cos\beta}{4 - 3\cos^2\beta} \quad . \qquad \text{(II.140}$$

*No $\bar{\tau}$-terms are apparent since "thrusting" is not assumed.

(Note that, according to this result, $\frac{k}{\nu} \leq 4$, in order to satisfy the minimal condition).

With this requirement, above, one sees that:

(1) If k and ν are known, a priori, then Eq. (II.140) shows a constraint on β for the minimum transfer (time).

(2) If k and β are known, a priori, the equation defines a constraint on ν for the minimum transfer (time).

This example concludes the work in this section. Here, the attempt was to show how practical results could be extracted from the general solutions; and, to illustrate, by example, a few operational situations which could be simulated and studied as by-products of the more formal developments. With this insight into relative motion, as might be gained herein, it is obvious that there will be many opportunities and situations which can be examined and studied -- each making ample use of the solutions which have been found and set down in this report.

In the following section a few remarks, as concluding statements, will be offered regarding this investigation.

XII. CONCLUDING REMARKS

The work which has been reported in this document represents the results gleaned from an analytical study on a class of relative motion problems. The general case here considers two particles orbiting about a central attracting mass. There is no mutual attraction between particles assumed, and, one particle is designed to travel on a circular orbit; this serves as a reference (or datum) basis for the system.

The major portion of effort in this part of the investigation was directed to obtaining general solutions for the relative motions. These evolved as a consequence of an initial state ($\bar{r}_o$, $\bar{V}_o$) and in response to a system of prescribed forces. Of course the results must be considered as reliable only in the vicinity of the reference particle; this is a condition which is imposed in addition to the constraint of a simple central attracting center. Nevertheless, the equations obtained here are invaluable to understanding the types of motions (variations in state, relatively speaking) which do occur. And, in view of the fact that these results describe variations in the two primary frames of reference, it is apparent that what has been determined here represents a most complete and comprehensive data base.

Aside from acquiring the equations, they have been put to use in describing traces of the motion on the principal coordinate planes. This graphical representation adds a new dimension to the results, providing the reader with a pictorial as well as analytic description of the influence played by initial values and by the applied disturbing forces. It is the investigator's belief that the data summaries, which are made up of these sets of information, will materially aid in the study and understanding of relative motions.

Even though the results which have been acquired here are complete, as far as they go, they have considered only one particular class of disturbing forces. Consequently, it is not unreasonable to expect that the tasks commenced here should be continued; at least to the point of examining other input systems to ascertain their influence of the motion. Hopefully this latter task will be undertaken and completed.

In addition, the work described here, and that which is expected to come later can be applied to real world problems -- or simulations -- so that many more realistic situations may be examined. It would be a mistake to presume that the equations, developed from this investigation, have "been used"; and, that they can be discarded now. Instead, they should be put to use for explicit and for implicit data acquisition. Here, in the body of this report, and in the references cited, an interested reader can find other suggestions on how the information can be utilized.

Once more the reader is reminded that this investigation has given equal emphasis to the relative motion velocity, as compared to the displacements. This was done because it is felt that the full state of a motion is essential to a complete understanding of these problems and for their manipulation. Generally, orbital motions cannot be altered without a change in the total state; and many "changes in state" are due to either a direct or implied change in the velocity.

Finally, it is remarked that the work described here is complete insofar as it goes. Obviously there are other areas and problems which should be examined; it is hoped that some of these, at least, can and will be carried forward to successful conclusions.

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APPENDICES

APPENDIX A

Definition of a Rotation. Consider an initially oriented Cartesian frame of reference, $(\sim)^{o}$, and a second reference frame, $(\sim)'$, which is located by an angle θ_1 with respect to $(\sim)^{o}$. Let this angular displacement be described as a "right-hand rule" rotation; and, let the rotation take place about the z-axis. As a consequence of these conditions only the (x, y) plane will undergo a reorientation during the displacement.

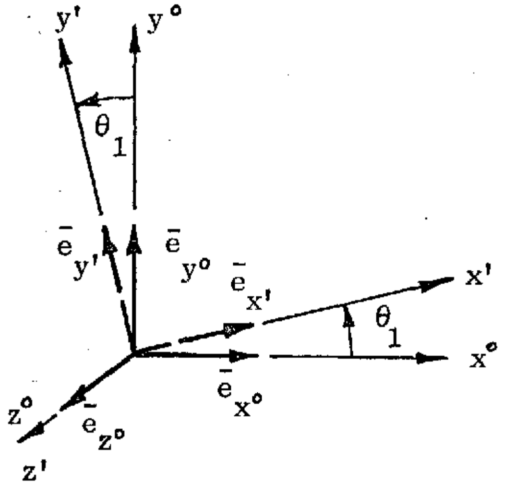

Fig. A.1. Sketch of a displacement.

For convenience, in the description and representation of this situation, assume that these Cartesian frames are assigned a set of orthogonal unit vectors each. To identify them, a superscript is added to the rotated triad. Therefore, after a displacement, of (say) θ_1, the unit vector $\bar{e}_{x'}$ is related to the inertial $(\bar{e}_i)$ vectors by:

$$\bar{e}_{x'} \equiv (\bar{e}_X \cdot \bar{e}_{x'})\bar{e}_X + (\bar{e}_Y \cdot \bar{e}_{x'})\bar{e}_Y + (\bar{e}_Z \cdot \bar{e}_{x'})\bar{e}_Z$$

$$= \cos\theta_1 \bar{e}_X + \sin\theta_1 \bar{e}_Y + 0\,\bar{e}_Z . \qquad \text{(A.1)}$$

In a corresponding manner the other unit vectors may be described by:

$$\bar{e}_{y'} = -\sin\theta_1 \bar{e}_X + \cos\theta_1 \bar{e}_Y + 0\,\bar{e}_Z ,$$

and

$$\bar{e}_{z'} = \bar{e}_Z . \qquad \text{(A.2)}$$

Treated as a matrix operation the expressions above can be written as:

$$\begin{bmatrix} \bar{e}_{x'} \\ \bar{e}_{y'} \\ \bar{e}_{z'} \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{e}_X \\ \bar{e}_Y \\ \bar{e}_Z \end{bmatrix} \qquad \text{(A.3a)}$$

or

$$\bar{e}_{i'} = T(\theta^+)\,(\bar{e}_i)^o, \qquad (i = x,\ y,\ z) \qquad \text{(A.3b)}$$

The notation in Eq. (A.3b) implies that a +-rotation is needed to locate the $(\sim)'$-triad from the $(\sim)^o$-triad. In order to reverse this situation - that is, to provide for a reverse rotation - or, to locate the $(\sim)^o$-triad relative to the $(\sim)'$-triad - one could write (symbolically),

$$(\bar{e}_i)^o \equiv T(\theta^+)^{-1}\ \bar{e}_{i'}, \qquad (i = x,\ y,\ z). \qquad \text{(A.4a)}$$

Here $T(\theta^+)^{-1}$ is the inverse matrix (operator) obtained from $T(\theta^+)$! (Note: since $T(\theta^+)$ is an orthonormal matrix, then its inverse is identical to its transpose. Consequently,

$$T(\theta^+)^T \equiv \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv T(\theta^-); \qquad \text{(A.4b)}$$

that is, the transpose, $T(\theta^+)^T$, is identical to the "transform matrix" developed for an angular displacement, $\theta = -\theta_1$).

There are some interesting and useful relations which can be developed for these transforms. Several of these are shown below:

(a). Demonstrate that $T(\theta^-)$ is indeed the inverse of $T(\theta^+)$.

Proof: If the statement is true then; $T(\theta^-) * T(\theta^+) \equiv I_3$;

that is,

$$I = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv I_3 \qquad \text{(A.5)*}$$

wherein $(\sim)_3$ infers the full (3 by 3) unit matrix (Q.E.D.).

(b). Evaluate the time derivative of the transform operator, $T(\theta^-)$. In this regard:

$$\dot{T}(\theta^-) \equiv \frac{d}{dt} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\theta} \begin{bmatrix} -s\theta & -c\theta & 0 \\ c\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \text{(A.6a)}$$

Also, note that:

$$\dot{T}(\theta^+) \equiv \frac{d}{dt} \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\theta} \begin{bmatrix} -s\theta & c\theta & 0 \\ -c\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad \text{(A.6b)}$$

Now, having the derivative(s), define a matrix, B, such that $\dot{T}(\theta^-) \equiv BT(\theta^-)\,\dot{\theta}$.

Using the assumed relationship it is apparent that,

$$\dot{\theta}BT(\theta^-)\,T(\theta^-)^{-1} \equiv B\dot{\theta} = \dot{T}(\theta^-)\,T(\theta^-)^{-1},$$

at least symbolically. However, $T(\theta^-)^{-1} = T(\theta^+)$, therefore;

$$\dot{\theta}B = \dot{T}(\theta^-)T(\theta^+) = \dot{\theta} \begin{bmatrix} -s\theta & -c\theta & 0 \\ +c\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

which leads directly to,

---

*For these operations a shorthand notation is adopted for the circular functions; here, $s\theta \equiv \sin\,\theta$ and $c\theta \equiv \cos\,\theta$.

$$B \equiv \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \equiv \left[\begin{array}{c|c} B_2 & 0 \\ \hline 0 & 0 \end{array}\right]. \qquad \text{(A.6c)*}$$

For reference purposes, it should be remembered that

$$\dot{T}(\theta^-) = \dot{\theta}\, BT(\theta^-). \qquad \text{(A.6d)}$$

For the converse operation define $\dot{T}(\theta^+)$ in a similar manner, and relate this operator to the transform $T(\theta^+)$: that is, let

$$\dot{T}(\theta^+) \equiv \dot{\theta}\, B^+ T(\theta^+);$$

then

$$\dot{\theta} B^+ = \dot{T}(\theta^+) T(\theta^-) = \dot{\theta} \begin{bmatrix} -s\theta & c\theta & 0 \\ -c\theta & -s\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

or,

$$\dot{\theta} B^+ = \dot{\theta} \times \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\theta}(-B);$$

therefore,

$$\dot{T}(\theta^+) = -\dot{\theta} BT(\theta^+). \qquad \text{(A.6e)}$$

(c). Develop some properties for B:

First note that the product of B with itself is:

---

*In view of this definition one can recognize that $\dot{\theta}B$ is a specialization of the matrix operator used to replace the vector operator "$\bar{\omega} \times$ ", where $\bar{\omega}$ is an instantaneous "rotation vector". For the present case $\dot{\theta}B$ represents an $\bar{\omega} \equiv \bar{\omega}(\theta)$; thus the <u>two elements</u> in this skew-symmetric operator (B).

$$BB \equiv B^2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;$$

therefore,

$$B^2 = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} -I_2 & 0 \\ \hline 0 & 0 \end{array}\right] = - I_2, \qquad \text{(A.7a)}$$

where I_2 is the two-element idem matrix!

Making use of the definition for B (Eq. (A.6c)), then Eqs. (A.3b) and (A.4b) may be rewritten as:

$$T(\theta^-) = I_2 \cos\theta + B \sin\theta + (I_3 - I_2) \equiv (T_-) , \qquad \text{(A.7b)}$$

and

$$T(\theta^+) = I_2 \cos\theta - B \sin\theta + (I_3 - I_2) \equiv (T_+) . \qquad \text{(A.7c)}$$

Next, evaluate the operational quantity, $T(\theta^-) * B * T(\theta^+)$:

$$(T_-) B (T_+) = \left[I_2 (\cos\theta - 1) + B \sin\theta + I_3\right] * B * \left[I_2 (\cos\theta - 1) - B \sin\theta + I_3\right]$$

$$= \left[I_2 (\cos\theta - 1) + B \sin\theta + I_3\right] * \left[B_2 (\cos\theta - 1) + I_2 \sin\theta + B\right]$$

$$= \left[I_2 (\cos\theta - 1) + B \sin\theta + I_3\right] * \left[B_2 (\cos\theta - 1 + 1) + I_2 \sin\theta\right]$$

$$= B_2 (\cos^2\theta - \cos\theta) - I_2 \sin\theta \cos\theta + B \cos\theta + I_2 (\cos\theta \sin\theta$$

$$- \sin\theta) + B_2 \sin^2\theta + I_2 \sin\theta = B_2 ;$$

therefore,

$$(T_-)\, B_2 (T_+) = B_2. \tag{A.7d}$$

Note that past and pre-multiplication of this result leads to the following: That is, since $(T_+) = (T_-)^{-1}$, and vice versa;

$$(T_+)\left[(T_-)\, B_2 (T_+)\right](T_-) = (T_+)\, B_2\, (T_-)$$

or

$$B_2 \quad = (T_+)\, B_2\, (T_-). \tag{A.7d2}$$

(d). The <u>composition for two subsequent rotations, about a common axis</u> can be expressed by

$$T(\theta_1^+) * T(\theta_2^+) = T(\theta_1^+ + \theta_2^+). \tag{A.7e}$$

This is an a priori statement. The proof of the statement follows directly from Eq. (A.7c) wherein;

$$T(\theta_1^+) * T(\theta_2^+) = \left[I_2 (\cos\theta_1 - 1) - B \sin\theta_1 + I_3\right] * \left[I_2 (\cos\theta_2 - 1) - B \sin\theta_2 + I_3\right];$$

which, after manipulation, is noted to be,

$$= I_2\left[\cos(\theta_1 + \theta_2) - 1\right] - B_2 \sin(\theta_1 + \theta_2) + I_3;$$

or,

$$T(\theta_1^+) * T(\theta_2^+) = T(\theta_1^+ + \theta_2^+) \qquad \text{(Q.E.D.)}$$

(Hueristically, it is apparent that this technique may be used for a sequence of displacements: $\sum_i \theta_i$).

(e). <u>An application of the above result, applied to the $T(\theta^+)$ transform</u> matrix.

According to Eq. (A.3a) and/or (A.7c), $T(\theta^+)$ when $\theta = 0$, gives $T(0^+)\left[= T(0^-)\right] = I_3$. Correspondingly, for $|\theta| = \pi/2$ and π, respectively;

$T(\pi/2^+) = I_3 - I_2 - B$, while $T(\pi/2^-) = I_3 - I_2 + B$;

and

$T(\pi^+) = I_3 - 2I_2$ and $T(\pi^-) = I_3 - 2I_2$.

Now, for instance, $T(\pi^+) \equiv T(\pi/2^+) * T(\pi/2^+) = (I_3 - I_2 - B) * (I_3 - I_2 - B) = = I_3 - 2I_2$.

Correspondingly, the reversed rotations would lead to a like resultant.

<u>The Consequence of a Rotation.</u> The idea to be illustrated here is that the relation of a position-vector, projected back onto a reference set of axes, can be defined by means of the transformation operators, $T(\theta^\pm)$. For the example (here) let the $(\sim)^o$-triad be an inertial triad (of reference) as before. Also suppose that the $(\sim)'$-triad is a rotating frame of reference; one following a particle "P". (A graphical description of this situation is been below).

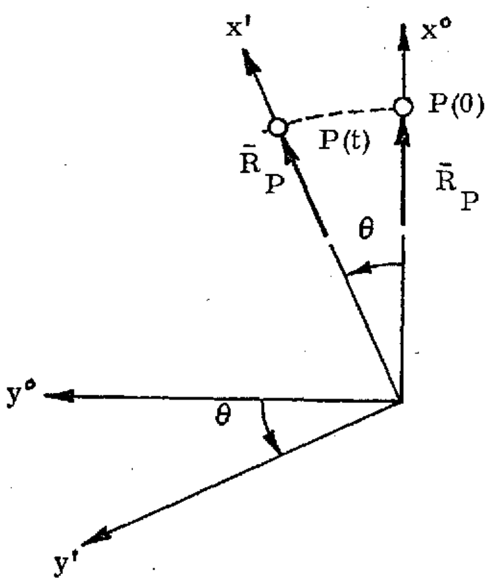

Fig. A.2.

The condition which will be illustrated considers a simple rotation of the position vector $\bar{R}_P$; a displacement through the angle, θ. To be described is a representation of this displacement in the $(\sim)^o$-frame of reference.

Since the vector $\bar{R}_P(t)$ can be defined by

$$\bar{R}_P(t) \equiv R_P \bar{e}_{x'}$$

in the moving (or displaced) frame, then its projection onto the $(\sim)^o$-triad is (see Eq. (A.1))

$$\bar{R}_P(t) = R_P (\cos\theta \bar{e}_X + \sin\theta \bar{e}_Y).$$

This is equivalent to transforming from the $(\sim)'$-triad back to the $(\sim)^o$-triad; or

$$(\bar{R}_P)^o \equiv T(\theta^-)(\bar{R}_P)'. \qquad (A.8)$$

This is a description of how the transform operators may be employed to relate a rotated vector back to an inertial frame of reference.

Obviously, the operator $T(\theta^+)$ should be employed whenever it is desired to transform a vector from the inertial-frame to the displaced-frame of reference. That is,

$$(\bar{R}_P)' \equiv T(\theta^+)(\bar{R}_P)^o. \qquad (A.9)$$

APPENDIX B

Some Special Matrices. In this appendix some special matrices, and some elementary operations with them, are described. Specifically these matrices are introduced here in order to maintain continuity of the main text.

Ordered Unit Matrices. The general nth order unit matrix is defined as an (n x n) unit diagonal matrix. That is

$$I_n \equiv \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & 1 \end{bmatrix}. \qquad (B.1)$$

In this regard, 3-dimensional problems should lead to a unit matrix, I_3, defined as:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (B.2)$$

Now, special variations of I_3 lead to the matrices I_2 and I_1, which are defined as:

$$I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } I_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{respectively.} \qquad (B.3)$$

Next, for convenience, three single unit matrices are defined. These are:

$$J_1 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = I_1; \qquad J_2 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (I_2 - I_1);$$

and

$$J_3 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (I_3 - I_2). \tag{B.4}$$

Operations with the Matrix, B_2. Making use of the special matrix, B_2, (or B), from Appendix A, the following operations are performed and identified. Since

$$B \equiv \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \equiv \left[\begin{array}{c|c} B_2 & 0 \\ \hline 0 & 0 \end{array}\right]$$

then it can be shown that

$$BJ_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = J_2B, \text{ and } J_1B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = BJ_2. \tag{B.5}*$$

As a consequence, it is recognized that

$$BJ_1 = BI_1 = (I_2 - I_1)B = J_2B$$

and

$$J_1B = I_1B = B(I_2 - I_1) = BJ_2$$

then, with

*This result is obvious since $B_2 \equiv BI_2 \equiv B(J_1 + J_2) = BJ_1 + BJ_2$ and $B_2 = BJ_1 + J_1B$, etc.

$$B_2 = BJ_1 + J_1B \; [= J_2B + BJ_2] \equiv BJ_1 + BJ_2 = B(J_1 + J_2) \equiv BI_2. \tag{B.6}$$

Making an observation of the matrices described in Eq. (B.4), it is apparent that

$$J_1J_1 = J_1^2 = J_1, \; J_2J_2 = J_2, \; J_3J_3 = J_3. \tag{B.7}$$

while the mixed product of these special matrices vanish.

Operations with a Combination of the Special Matrices. During the formulation and solution to a three dimensional problem, one is likely to have need for the inverse of a matrix formed by the special matrices defined above. To illustrate the nature of such, suppose there is a matrix combination like

$$A \equiv aJ_1 + bJ_2 + cJ_3 + eB \tag{B.8a}$$

wherein (a, b, c, e) are coefficients describing the problem. Now, from the definition of these special matrices, it can be shown that

$$A = \begin{bmatrix} a & -e & 0 \\ e & b & 0 \\ 0 & 0 & c \end{bmatrix}, \tag{B.8b}$$

by the addition of the matrices, as implied. In forming the inverse one needs the determinant of A (Det A = $c(ab + e^2)$), and the adjoint transpose of A ($\equiv A^T$) -- which can be shown to be:

$$A_*^T \equiv \begin{bmatrix} bc & ec & 0 \\ -ec & ac & 0 \\ 0 & 0 & (ab+e^2) \end{bmatrix}. \tag{B.8c}$$

Now, since $A^{-1} = A_*^T/\text{Det } A$, then it is immediately apparent that

$$A^{-1} \equiv \frac{A_*^T}{c(ab+e^2)} = \frac{1}{c(ab+e^2)} \times \begin{bmatrix} bc & ec & 0 \\ -ec & ac & 0 \\ 0 & 0 & (ab+e^2) \end{bmatrix}, \qquad \text{(B.8d)}$$

which can be written as:

$$A^{-1} = \frac{bJ_1 + aJ_2 - eB_2}{(ab+e^2)} + \frac{J_3}{c} . \qquad \text{(B.8e)}$$

In the above operations

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \equiv \left[\begin{array}{c|c} B_2 & 0 \\ \hline 0 & 1 \end{array}\right] + \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = B_2 + I_3 .$$