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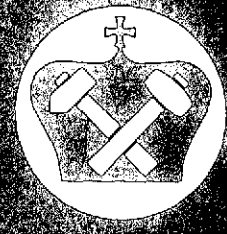

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AN ERROR CRITERION FOR DETERMINING SAMPLING RATES IN CLOSED-LOOP CONTROL SYSTEMS

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ABSTRACT

Discrete-data control involves the sampling of one or more signals in a control system, at a given rate called the sampling rate. Usually the Nyquist-Shannon sampling theorem has been employed to determine the sampling rate. This procedure is proper in dealing with band-limited signals, but it does not allow errors in the performance of the discrete system. Its application to practical cases, which generally do not involve band-limited signals, may demand a faster rate than that necessary for adequate control under practical limitations.

The research reported herein is concerned with the determination of an error criterion which will give a sampling rate for adequate performance of linear, time-invariant, closed-loop discrete-data control systems.

The first part of the research deals with the proper modelling of the closed-loop control system for characterization of the error behavior and the determination of an absolute error definition for performance of the two commonly used holding devices -- the zero-order hold and the first-order polygonal hold.

In the second part, the definition of an adequate relative error criterion as a function of the sampling rate

and the parameters characterizing the system is made, and the determination of sampling rates follows.

The validity of the expressions for the sampling interval has been confirmed by computer simulations. Their application solves the problem of making a first choice in the selection of sampling rates.

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CHAPTER I

INTRODUCTION

1.1. Motivation

In recent years the application of high speed digital computers in the area of control systems has been significantly increased. These applications have included system simulation, signal processing, and use of the computer as a component of the overall control system. For example, some of the many applications are the navigation and guidance systems for aerospace vehicles, some control components in chemical processes and economic models, and the implementation of controllers for general systems [1-10].*

The use of the digital computer in this field, as the controlling element requires a change in the basic concept of control theory. The system has to be controlled at discrete instants of time, because of the nature of the digital computer itself, and not continuously as with analog controllers. The control signal then, is the result of a numerical algorithm, on the observed variables of the plant or

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*The numbers in brackets indicate references, given on pp. 177-181.

process, and it follows a fixed law of variation between computation instants. The observations, or samples, of the behavior of the system are made at a given rate called the sampling rate.

One of the advantages of the use of the digital computer is the possibility of time sharing for controlling many systems, as is being done in process control, where the controlled variables are subject to large time lags and where the variables of the system do not change rapidly with time [8].

It has been observed that digital computer control, or discrete-data control, does not affect the performance of the control system when the sampling rate is much faster than the rate of change of the variables of the system [8].

From the consideration of maximum efficiency in the use of the digital computer, it becomes clear that it is of interest to the designer of discrete-data control systems, to lower the sampling rate without affecting noticeably the desired performance of the system, or keeping the change within acceptable limits.

Usually the Nyquist-Shannon sampling theorem [1], is employed to determine the sampling rate. However, this theorem is properly restricted to systems dealing with band limited signals, but it does not allow errors in the performance of the discrete data system. Its application to many practical cases, which generally do not involve band-

limited signals and where a perfect reproduction of the control signal is not very important from an economic point of view, demands a faster rate than that necessary for adequate control under practical limitations.

The general trend in this field is to use the sampling theorem for a first determination of the sampling rate and then realize computer simulations to verify the validity of this choice.

In view of these problems, it is desirable to find methods for determining economic sampling rates for discrete-data systems. In doing so, Y. K. Kang [11] developed a new method for determining sample rates for open-loop dynamic systems. This research is a continuation of his work and applies to closed-loop dynamic systems. An uniform-rate sampling scheme, slower than that required by the sampling theorem, and appropriate for closed-loop control systems will be determined. The objective will be to keep the performance of the discrete-data control system within acceptable limits with respect to the performance of the continuous system. The result should be a more efficient use of the digital computer.

The interest of obtaining an uniform-rate scheme is based on the convenience of utilizing a sharing system for controlling more than one process with the same digital computer.

1.2. Historical Review

Digital techniques for the solution of numerical problems have been applied since the seventeenth century [12] but their use was quite limited until the appearance of the digital computer.

Prior to 1950, very little attention was given to the subject of analysis and design of discrete-data systems, although early text books in servomechanisms dealt with the problem [13-15].

With the invention and use of digital computers in control systems, in the early 1950's, numerous researchers began to study the problem of discrete-data control and the result was the appearance of many papers and books [1-19]. During this first stage, efforts were made mostly in adapting and making extensions of the existing continuous-data methods for use in discrete-data systems. With the use of the z-transform formulation, the concepts of transfer function, signal flow graphs, stability methods and plots were extended and they resulted in a wide application to analysis and synthesis of discrete-data systems [1-19].

In the decade of the 1960's, the field of discrete-data control has undergone a change in the design techniques. The use of the state variable approach has reformulated the problem of synthesis and analysis bringing new ideas in this field [1]. The theories of optimal control have been applied with remarkable success, leading to the design of

discrete-data systems optimized in some prescribed sense [20].

Another field of research has concerned the application of the digital computer as numerical processor. The analysis of numerical methods and their application has been extensively studied [4, 5, 12, 23].

In the analysis of errors introduced by sampling and the use of the digital computer, much progress has been made. Sampling error, round-off error, truncation error, folding error, discretization error and quantization error have been well defined and studied by many researchers [24-32].

The study of sampling rates, which is the purpose of this research has also been approached from different points of view.

One group studied the problem using the concept of adaptive sampling [33-37]. They select an initial sampling rate and then change it continuously according to the performance of the system.

Another group [39] looked at the problem from the optimal control point of view. They determined the asymptotic behavior of the cost function for an infinite sampling rate (i.e., the continuous system) and a zero sampling rate. They could then select a sampling rate by assuming a smooth variation of the cost function between those two limits. The application of this method is very difficult for high order systems.

Another method for determining sampling rates was developed by Y. K. Kang [11]. He found an upper bound for the discretization error as a function of the sampling rate for open-loop dynamic systems.

Recently a related new field of interest has been the study of digital filters. Researchers have focused their interest on analysis and synthesis problems [4, 5].

1.3. Discrete-Data Control Systems

The term discrete-data control systems has been used to designate systems in which the signal on one or more parts is in the form of either a pulse train or a numerical code. The terms, sampled-data systems and digital systems are often used in control literature as equivalents. However, sampled-data systems refer to systems in which the signals have a pulsed form and digital systems refer to systems in which a digital computer is used for making numerical computations and control.

The term discrete-data system will be used here to include all the possible variations of the above.

A discrete-data system as it was defined demands the existence of pulsed signals. To obtain the data, a sampling operation must be done. The samples are then manipulated and used for controlling a desired process.

To design such a system, the basic components of a closed-loop control system must be recalled. The plant or

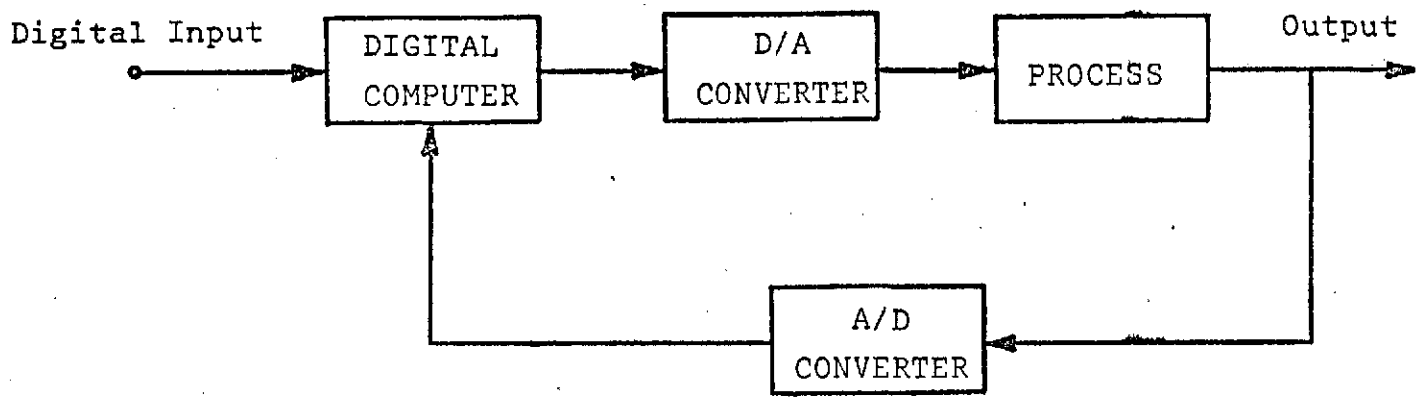

FIGURE 1.1 A DISCRETE-DATA CLOSED-LOOP CONTROL SYSTEM

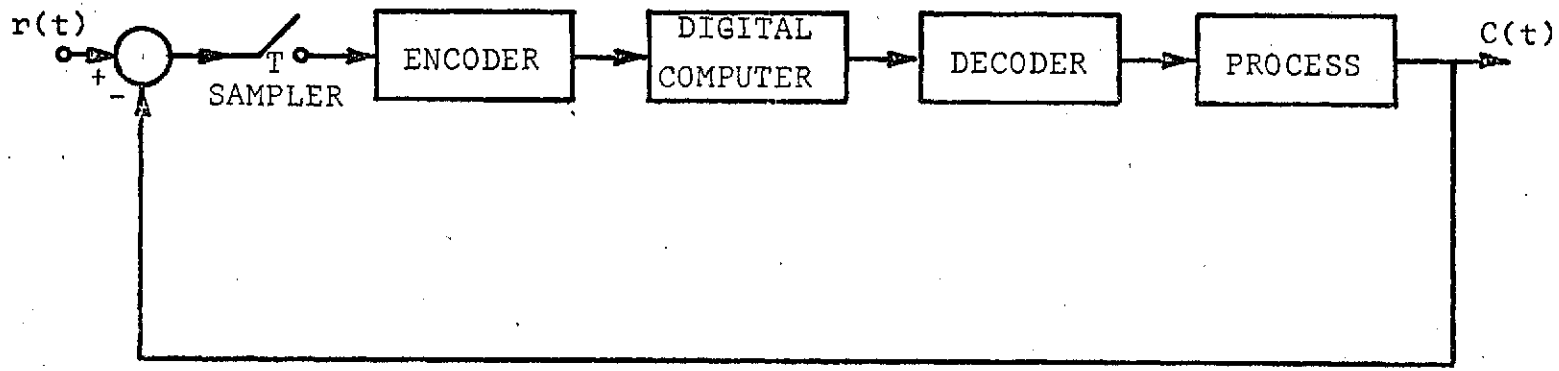

FIGURE 1.2 A DISCRETE-DATA CONTROL SYSTEM WITH A DIGITAL COMPUTER AS CONTROLLER

process to be controlled needs a control signal given by a controller, which acts according to the state of the plant.

A closed-loop control scheme can be realized using continuous control produced by an analog controller or using discrete control generated by sampling and processing in a digital computer.

For operating a digital controller, the nature of the signal emerging from the plant must be changed to digital form by an analog-to-digital converter. The digital signal is processed by the digital computer, and the result is converted again to analog form by a digital-to-analog converter, or an equivalent actuator, such as a stepping motor. Examples of sample-data control systems are shown in the block diagram of Figures 1.1 and 1.2.

The mathematical modeling of this process involves the description of samplers, numerical methods involved with the operation performed by the digital computer, behavior of the converters and the introduction of holding devices for smoothing the discrete-data signal between two successive samples.

The research will deal first with the modelling problem and then with the determination of an expression for finding sampling rates.

1.4. Outline of the Research

The object of this research is to find an economic adequate sampling rate for discrete-data closed-loop control systems. The problem is formulated here in terms of sampling error and expressions are obtained relating the sampling interval T to some parameters of the closed-loop dynamic system.

Chapter II concerns the modeling of discrete-data systems. A review of sampling and reconstruction of signals is made. In the last part, the modelling of a dynamic controller is presented. The model obtained is used later in the research for modelling the behavior of the discrete-data system.

In Chapter III, the closed-loop system is analyzed, and a method for characterizing the behavior of the system is presented. This procedure constitutes the basis for achieving the fundamental results of this research. Expressions for the evolution of the state of the continuous and discrete data systems follow.

Chapter IV deals with the errors present in a discrete-data system. A review of system error, truncation error, round-off error, quantization error and discretization error is made and the importance of each one is analyzed. Finally, the discretization error is studied and an approximation to its behavior is obtained for small sampling intervals.

In Chapter V, two error criteria are introduced. The determination of the sampling rate follows. An example is included and a discussion of sampling rate selection is presented.

Chapter VI states the conclusions of this work and discusses the possibility of further research.

CHAPTER II

MODELLING OF DISCRETE-DATA SYSTEMS

2.1. General Problem

In order to analyze the behavior of discrete-data systems, the modelling of the different basic operations taking place in the system must be reviewed.

The general problem of modelling as applied to this research is to find a mathematical discription of the discrete-data controller for its characterization and digital computer implementation, which will be fast, accurate and stable. Usually these conditions cannot be achieved simultaneously in the same design and the designer must compromise in order to obtain an optimal solution.

Consider now the specific problem of discrete-data control systems. The concepts presented in Chapter I, concerning the digital controller, have to be further studied in order to obtain an adequate modelling of the process.

Assume that it is desired to control some process using a digital computer and that the control laws of the analog system are known. The analog controller can be represented by a block diagram as shown in Figure 2.1. In order to convert this controller into a digital equivalent, the process

described in Chapter I has to be recalled. The analog input signal to the controller is converted into a digital form. This is done by sampling the analog input and converting it using an analog-to-digital converter. Then the data is processed by the digital computer and finally introduced into a digital-to-analog converter for obtaining the analog output signal. These operations are shown in the block diagram of Figure 2.2.

A mathematical model of this process can be obtained by inserting a fictitious sampler at the input of the controller, replacing the digital computer by a holding device followed by the analog controller itself and a fictitious sampler. Another holding device follows for obtaining the analog output signal. The model is shown in block diagram form in Figure 2.3. The fictitious samplers simulate the discrete control of the digital controller and the holding devices convert the discrete-data signal to analog for processing in the analog controller and plant. This method is widely used in modelling digital controllers and it describes perfectly the actual implementation [1].

Therefore the problem of modelling a discrete-data system is related to the study of samplers, holding devices and the formulation of an algorithm for performing the task described by the equations of the analog controller. In this chapter the basic operations just described will be reviewed.

FIGURE 2.1 ANALOG CONTROLLER

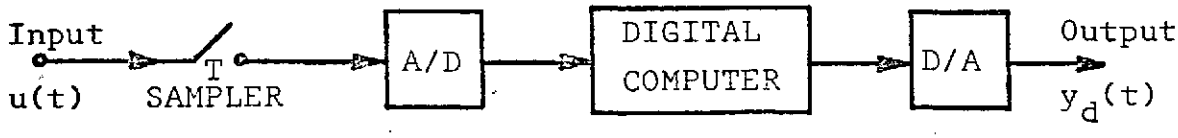

FIGURE 2.2 DISCRETE-DATA CONTROLLER

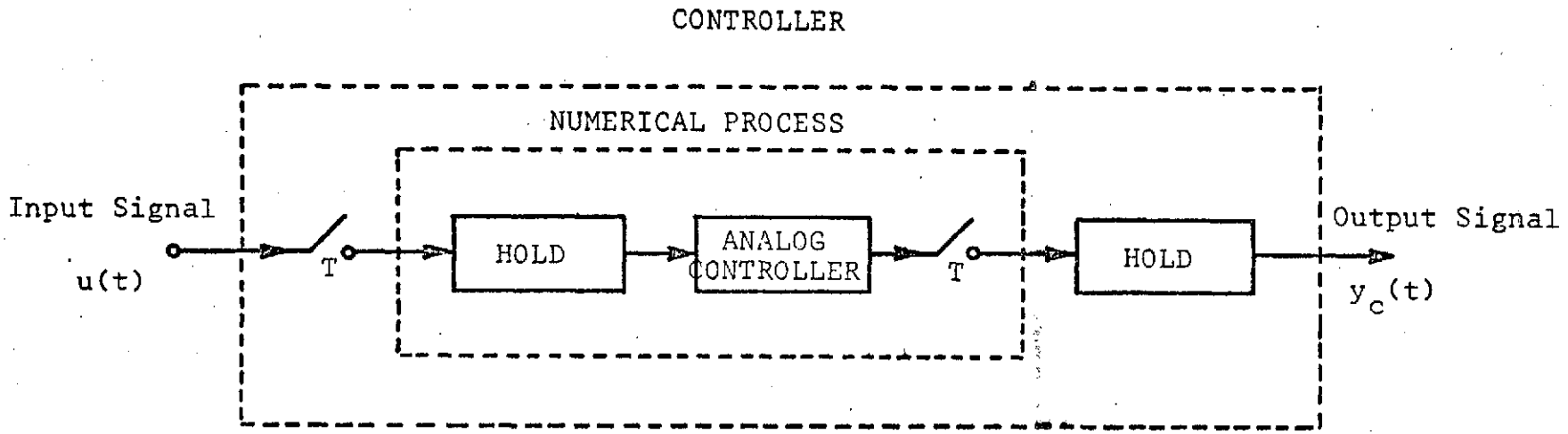

FIGURE 2.3 A DISCRETE-DATA MODEL OF THE CONTINUOUS-DATA CONTROLLER

2.2. Sampling of Continuous Signals

Digital signals can be obtained by sampling a continuous signal using an electronic or mechanical switching device called sampler, which operates at a given rate called sampling rate. It produces a pulse train when a continuous-signal is applied at the input.

As explained in Chapter I, this research deals with uniform-rate sampling. The samplers to be considered are ideal samplers, with the property of having a negligible operation time with respect to the sampling interval.

Throughout this research, the time between two samples is called sampling interval, it will be denoted by T. The instants the samples are made are called sampling times or instants and they will be denoted by kT, with k integer.

It is well known that the sampler behaves as an harmonic generator [1]. The ideal sampler reproduces in its output the spectrum of the continuous input as well as the complimentary components centered at integral multiples of the sampling frequency. The output spectrum is illustrated in Figure 2.4.

If the sampling rate is such that overlapping of the side-bands occurs, it is clear that distortion is present in the system. This problem was studied by Nyquist [21] and later by Shannon [22]; they showed that a signal with the highest frequency f_c, demands a sampling rate no lower than $2f_c$ in order to avoid overlapping. This result has been

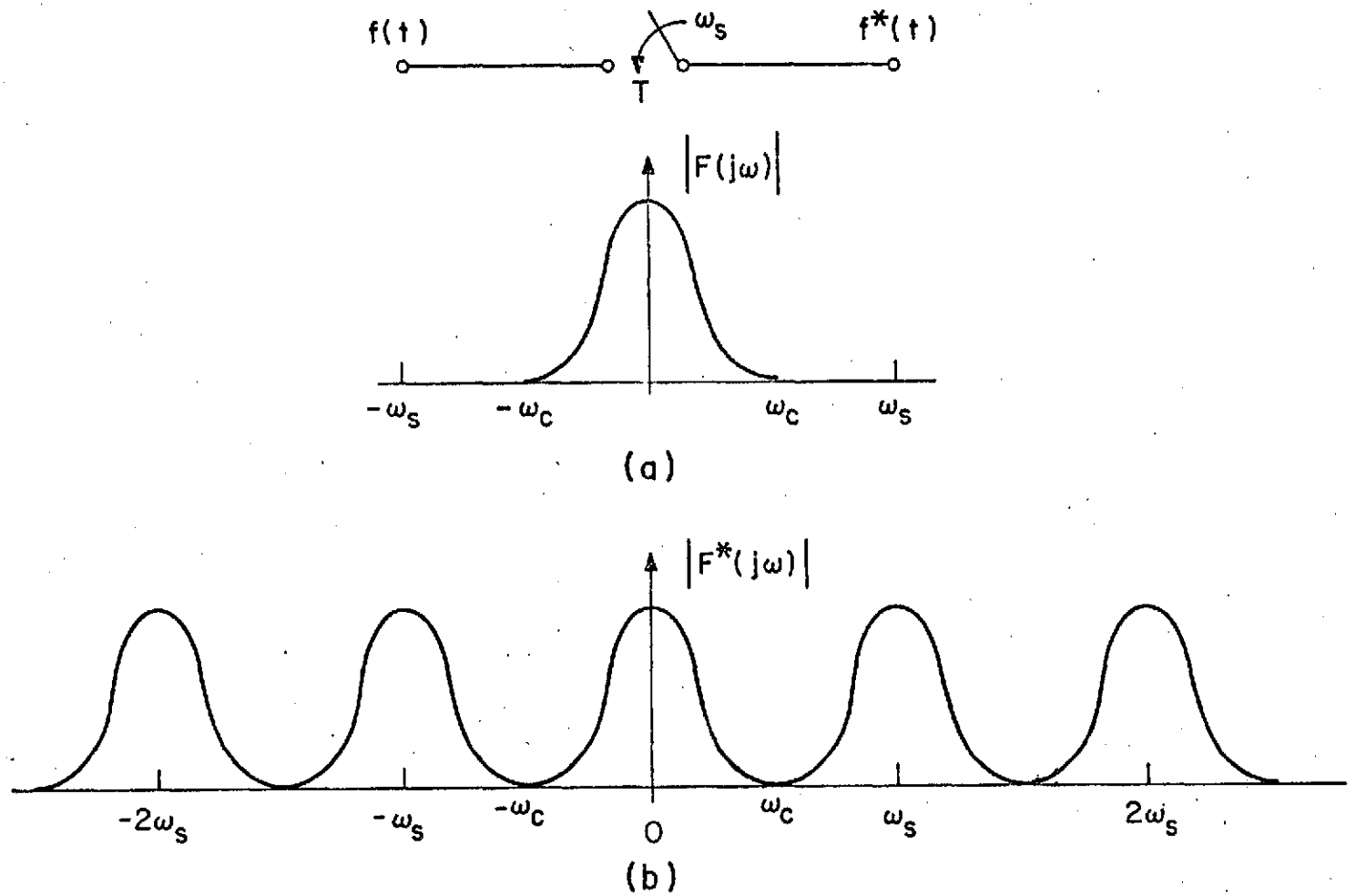

FIG. 2.4
(a) AMPLITUDE SPECTRUM OF CONTINUOUS INPUT f(t)
(b) AMPLITUDE SPECTRUM OF IDEAL SAMPLER OUTPUT ($\omega_s > 2\omega_c$)

stated as a theorem [1].

Theorem 2.1. (Nyquist-Shannon Sampling Theorem). If a signal contains no frequency higher than w_c radians per second, it is completely characterized by the values of the signal measured at instants of time separated by

$$T = \frac{1}{2}\frac{2\pi}{w_c} \text{ seconds.}$$

The interpretation of the theorem implies that it is possible to recover exactly a band-limited signal from its samples, by sampling at a rate such that no overlapping occurs and using an ideal low-pass filter. But a band-limited signal does not exist in practical control systems or communications. Therefore an approximation on the frequency content of the signal must be done, resulting in errors in the performance of the system [11].

This research will determine a sampling scheme such that, the error resulting from sampling at lower rates than those imposed by the theorem, are delimited into acceptable ranges.

2.3. Reconstruction of Sampled Signals

In the model of the discrete-data controller presented in Figure 2.3., a holding device was introduced for reconstructing the output signal of each sampler. It is well known that the holding device has the effect of removing the high frequency components of the sampled signal [1], therefore

its use is proper for reconstruction.

Another justification for using this filtering device is the fact that the signal emerging from it is injected to a continuous system and therefore subject to operations, mostly integrations, if the system has dynamics. As it will be seen, the integrals are evaluated at sampling instants, and their evaluation is simplified by knowing the behavior of the input signals between samples. This behavior is related with the order of the holding device.

The problem is that from a train of impulses with strength $f(kT)$, $k = 0, 1, 2 \ldots$, a continuous signal, $f(t)$, must be reconstructed. The data-reconstruction process may be regarded as an extrapolation process, considering the information available at past sampling instants, or as an interpolation process by considering the data available between two samples and the past data.

Typical holding devices will be reviewed next.

2.3.1. Zero-Order Hold

Consider first the extrapolator type of reconstructor.

A well known method of generating an extrapolation formula is to use the approximation based on the power series expansion of the control signal $f(t)$, in the interval between sampling instants kT and $(k+1)T$ [16]. That is:

$$f(t) = f(kT) + f'(kT)\ (t-kT) + \ldots$$

$$+ f^{(n)}(kT)\ \frac{(t-kT)^n}{n!} + \ldots \qquad (2.1)$$

valid for

$$kT \leq t < kT + T$$

where

$$f^{(n)}(kT) = \left.\frac{d^n f(t)}{dt^n}\right|_{t = kT}$$

To evaluate the coefficients of the series of Eq. (2.1), the derivatives of the function f(t) at sampling instants are usually approximated by backward differences. That is:

$$f^{(n)}(kT) \simeq \frac{1}{T}\ \nabla^n f(kT) \qquad (2.2)$$

where

$$\nabla f(kT) = f(kT) - f(kT-T) \qquad (2.3)$$

and

$$\nabla^n f(kT) = \nabla\ [\nabla^{n-1} f(kT)] \qquad (2.4)$$

By analyzing the nature of the approximation for the derivatives, it can be seen that an n-th order derivative is

a function of the past (n+1) samples of the function [1]. Therefore the higher the order of the approximation, the larger will be the number of past samples required. This fact has a well known adverse effect on the stability of feedback control systems [1]. Also, a high-order extrapolator requires complex circuitry resulting in high costs. For these reasons only the zero-order extrapolator or zero-order hold is used in practical applications [1].

The first-order extrapolator can also be used, but its efficiency when used for modelling and the filtering characteristics are inferior in performance to those of the first-order interpolator; therefore the latter is preferred.

The zero-order hold is obtained by considering only the first term of the power series of Eq. (2.1) for approximating the sampled signal. Then Eq. (2.1) becomes

$$f_o(t) = f(kT) \qquad , kT \leq t < kT+T \tag{2.5}$$

Thus, when a sample is made, the reconstructor constantly holds that value until the next sample is obtained, as illustrated in Figure 2.5. From the figure it becomes clear that the accuracy of the zero-order hold depends greatly on the sampling rate. This fact is closely related to the filter behavior of the holding device. The zero-order hold behaves essentially as a low-pass filter, however, when compared with the characteristics of

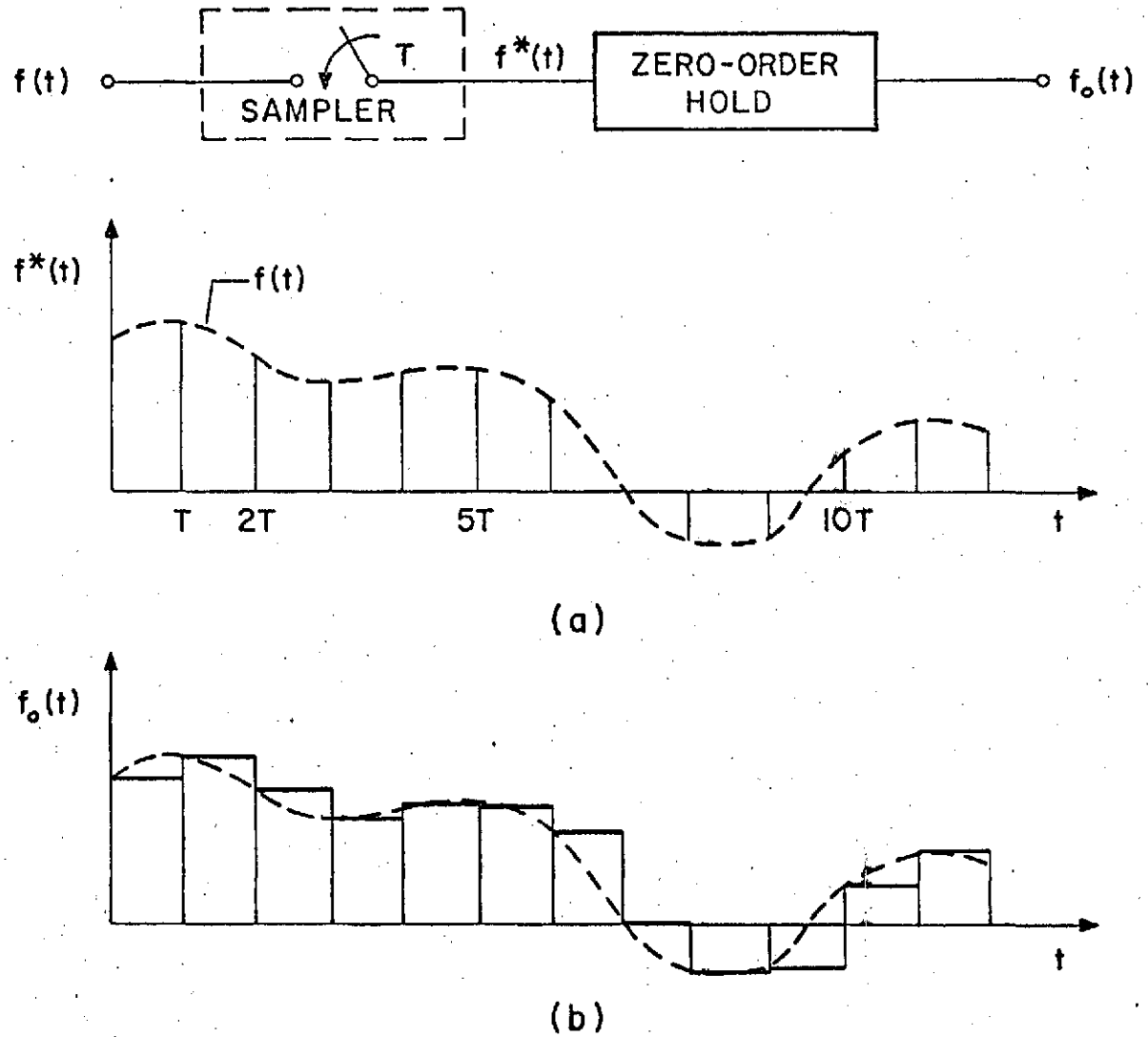

FIG. 2.5 ZERO-HOLD OPERATION

(a) INPUT SIGNAL f(t)
SAMPLED SIGNAL $f^*(t)$

(b) OUTPUT SIGNAL FROM ZERO-ORDER HOLD

an "ideal filter," the amplitude response of the zero-order hold is different from the ideal amplitude response [1].

The expression in Eq. (2.5) for the output of the zero-order hold is in a very convenient form. As will be seen in Section 2.4, the response of a continuous-data dynamic system, excited by a signal of the obtained nature, can be easily evaluated at sampling instants.

2.3.2. First-Order Polygonal Hold

Consider now the interpolator type of reconstructor. The Newton interpolation formula with backward differences describes the behavior of the device [42].

$$f(t) = f(kT+T) + \tau\, \nabla f(kT+T) + \ldots + \frac{\tau\,(\tau+1)\ \ldots\ (\tau+n-1)}{n!}\, \nabla^n f(kT+T) + \ldots \qquad (2.6)$$

valid for

$$kT \leq t < kT+T$$

where

$$\tau = \frac{t - (kT+T)}{T}$$

with the backward differences described by Eqs. (2.3) and (2.4).

By analyzing the nature of the backward differences, it can be seen that an n-th order backward difference requires information of the past (n+1) samples. Then, it becomes clear that the higher the order of the interpolation, the larger will be the number of past samples required. This fact has a well known adverse effect on the stability of closed-loop control systems [1]. Therefore only low order interpolators, up to and including the first-order, are considered in practical applications [1].

Another result of the analysis of the interpolator described by Eq. (2.6) indicates that the device is non-causal. The output, f(t), depends on future values of the input, f(kT+T). But as will be shown in Section 2.4.2, this fact is not a handicap for its usage for modelling dynamic systems if the computational time is negligible with respect to the sampling interval.

The zero-order interpolator is inferior in performance compared with the zero-order hold [1], because its non-causality and filtering characteristics. The first-order interpolator, however, is superior compared with the first-order hold and therefore widely used [1].

When the first two terms of the interpolation formula of Eq. (2.6) are used to approximate the time function between two successive samples, the resulting device is called first-order polygonal hold.

The expression describing the device results

$$f(t) = f(kT) + [f(kT+T) - f(kT)] \frac{t - kT}{T} \qquad (2.7)$$

valid for

$$kT \leq t < kT+T$$

The noncausality of the device is observed by the presence of f(kT+T), but it will be seen in Section 2.4.2 that this fact is not a problem in modelling dynamic systems.

The output of a first-order polygonal hold can be observed in Figure 2.6. From the figure it becomes clear that the accuracy of the device depends greatly on the sampling rate. This fact is closely related with the filter behavior of the first-order polygonal hold. Studies show that the first-order polygonal hold behaves essentially as a low-pass filter with amplitude response closer to the ideal filter amplitude response than the first-order hold [1].

2.4. Numerical Methods of Integration

According to the mathematical model presented in Section 2.1, the analog controller input is the signal from the holding device. This signal is expressed as a function of the values of a continuous signal at sampling

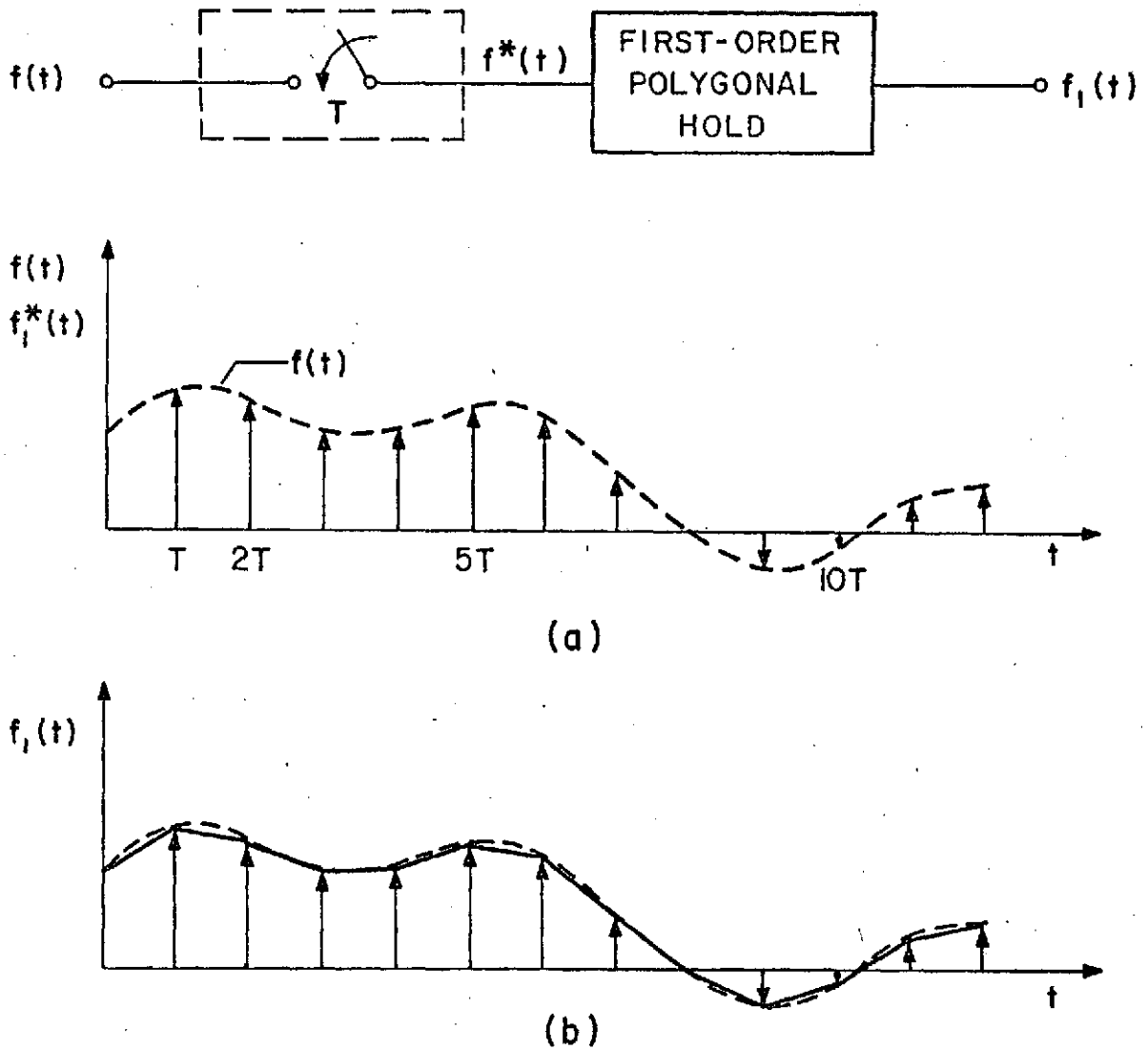

FIG. 2.6 FIRST-ORDER POLYGONAL HOLD

(a) INPUT SIGNAL f(t)
SAMPLED SIGNAL f*(t)

(b) OUTPUT SIGNAL FROM FIRST-ORDER POLYGONAL HOLD

instants as it was seen in Section 2.3. Now it is of interest to analyze the behavior of the analog controller with an input signal of the described nature.

Assume that the analog controller can be described by a set of linear time-invariant differential equations of the form:

$$\dot{q}(t) = F\, q(t) + G\, u(t) \quad ; \; q(t_o) = q_o \tag{2.8}$$

$$y(t) = H\, q(t)$$

where

$q(t)$: m-vector, state

q_o: m-vector, initial state

$u(t)$: 1-vector, input

$y(t)$: r-vector, output

F: m x m matrix, system matrix

G: m x 1 matrix, control matrix

H: r x m matrix, output matrix

The formal solution of the differential equation of Eq. (2.8) from the initial time t_o until actual time t is [43].

$$q(t) = \Phi_F(t-t_o)q(t_o) + \int_{t_o}^{t} \Phi_F(t-\tau)\, G\, u(\tau)\, d\tau \tag{2.9}$$

where

$\Phi_F(t-t_o)$: Transition matrix of F

$$\Phi_F(t-t_o) = e^{F(t-t_o)}$$

with well known properties.

The output y(t) can be obtained from Eq. (2.8).

Reference to the model presented in Section 2.1, as seen in Figure 2.3, the output of the controller is sampled in order to simulate the behavior of the digital computer which operates only at sampling instants. Therefore the behavior of the analog controller at sampling instants must be analyzed.

To obtain the response of the controller at sampling instants, the following change of variables must be made in Eq. (2.9) [1].

$$t_o = kT$$

$$t = kT+T$$

Then Eq. (2.9) becomes

$$q(kT+T) = \Phi_F(T)\ q(kT) + \int_{kT}^{kT+T} \Phi_F(kT+T-\tau)\ G\ u(\tau)\ d\tau \qquad (2.10)$$

and the output is

$$y(kT) = H\ q(kT) \qquad (2.11)$$

The controller input, $u(\tau)$ is the output of the holding device, and as was seen already it has a known variation law for a given hold type. Therefore the integral of Eq.

(2.10) can be evaluated for each of the holding devices presented.

2.4.1. Zero-Order Hold Integrator

Consider first the zero-order hold device. According to Eq. (2.5), the control signal $u(\tau)$ can be expressed as

$$u(\tau) = u(kT), \quad kT \leq \tau < kT+T \tag{2.12}$$

where $u(kT)$ are the sample values of the control signal.

The control vector resulting from using the zero-order hold is constant between two sampling instants; therefore Eq. (2.10) can be written as

$$q(kT+T) = \Phi_F(T)\, q(kT) + \int_{kT}^{kT+T} \Phi_F(kT+T-\tau) G d\tau\, u(kT) \tag{2.13}$$

The integral of the right member of Eq. (2.13) can be evaluated, to yield

$$q(kT+T) = \Phi_F(T)\, q(kT) + F^{-1}[\Phi_F(T) - I]\, G\, u(kT) \tag{2.14}$$

Eq. (2.14) represents a set of first-order difference equations describing the state variables at discrete instants of time. This set is the discrete state equation of the system for the zero-order hold.

The discrete state equation can be solved by means of a simple recursive procedure by setting $k = 0, 1, 2 \ldots$

This fact is adequate for digital computer simulation.

Another observation arising from the analysis of Eqs. (2.13) and (2.14) indicates that the numerical method of integration described is similar to the Euler method of numerical integration. It is known in the field of numerical analysis as the modified Euler method [11, 12].

The analysis of the stability of this scheme is related to the location of the eigenvalues of the difference equation. They must be contained in the unit circle with its center at the origin for stability [12].

2.4.2. First-Order Polygonal Integrator

Consider the control signal emerging from the first-order polygonal hold. According to Eq. (2.7), it can be described as

$$u(\tau) = u(kT) + [u(kT+T) - u(kT)] \frac{\tau - kT}{T} \qquad (2.15)$$

valid for

$$kT \leq \tau < kT+T$$

where u(kT) are the sample values of the control signal.

The control vector u(τ) has a linear variation with respect to time in a sampling interval. Therefore Eq. (2.10) becomes

$$q(kT+T) = \Phi_F(T)\ q(kT)$$

$$+ \int_{kT}^{kT+T} \Phi_F(kT+T-\tau)\ G\ [1 - \frac{\tau - kT}{T}]\ d\tau\ u(kT)$$

$$+ \int_{kT}^{kT+T} \Phi_F(kT+T-\tau) G\ \frac{\tau - kT}{T}\ d\tau\ u(kT+T) \qquad (2.16)$$

By making an appropriate change of the integration variable, the integrals of the right member of Eq. (2.16) can be evaluated, to yield

$$q(kT+T) = \Phi_F(T)\ q(kT)$$

$$+ F^{-1}\{\Phi_F(T) - \frac{1}{T} F^{-1}\ [\Phi_F(T) - I]\}G\ u(kT)$$

$$+ F^{-1}\{\frac{1}{T}F^{-1}[\Phi_F(T) - I]-I\}G\ u(kT+T) \qquad (2.17)$$

This equation, as before, is known as the discrete state equation of the system for the first-order polygonal hold, and it can be solved by using a recursive procedure.

The analysis of Eq. (2.17) shows that the state at the sampling instant kT+T, is a function of the state at kT, and the control signal at instants kT+T and kT. This fact is related to the noncausality discussed in Section 2.3.2. The problem arises because the computation of the present state requires the present input, causing a computational

delay in the evaluation of the system. But considering the speed of the actual digital computers versus the speed of most of control systems, this delay is not a major cause of error unless the system speed is unusually high so that the delay due to computational time must be considered [11].

Another conclusion from the analysis of Eqs. (2.15) and (2.17) is that the first-order polygonal integrator is equivalent to the trapezoidal rule of integration, which approximates the function to be integrated by a linear interpolation between two points. For this reason this method is known as the modified trapezoidal method [11].

As discussed by DiPerna [44], the first-order polygonal integrator belongs to a general class of numerical methods known as the bilinear transformation, which are A-stable. Because of this very desirable property, the modified trapezoidal method is widely used for digital simulation of continuous systems [44].

The discrete-data output from the controller must be reconstructed by using a holding device, as seen in Section 2.1, in order to obtain the analog control signal for the plant. Using one of the hold systems analyzed in Section 2.2, the behavior of the plant can be characterized in the same form as was done with the controller in this section. Next chapter will deal with this problem.

CHAPTER III

CLOSED-LOOP CONTROL SYSTEMS

3.1. Control Systems

A control system is an interconnection of components forming a system configuration to provide a desired performance. An open-loop control system utilizes a controller or control actuator in order to obtain a desired response from a process as shown in Figure 3.1. In contrast to an open-loop control system, a closed-loop control system utilizes in addition a measure of the actual output in order to compare it with the desired output response. A simple closed-loop control system is shown in Figure 3.2. The nature of the controller imposes another classification of control systems. Discrete-data control systems are characterized by the use of a digital controller and continuous-data control systems by an analog controller.

For the purpose of this research, the behavior of a continuous-data and the equivalent discrete-data closed-loop control system must be compared. Therefore

FIGURE 3.1 OPEN-LOOP CONTROL SYSTEM

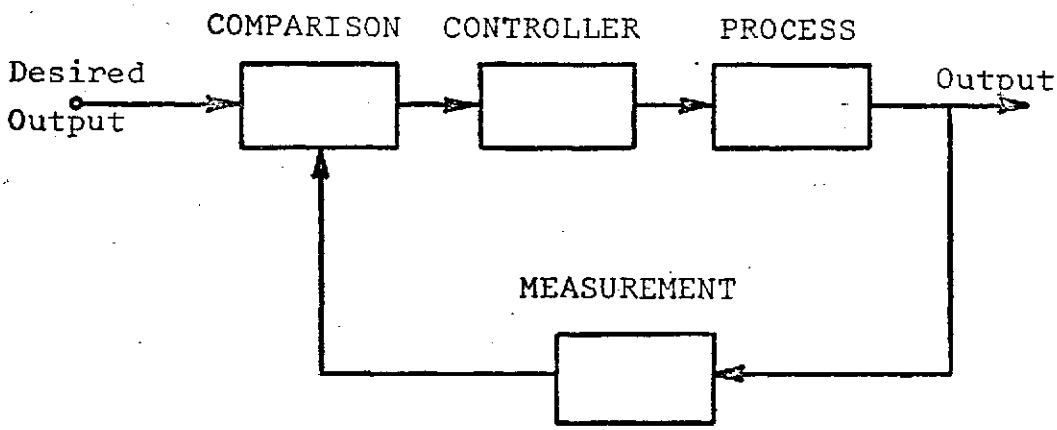

FIGURE 3.2 CLOSED-LOOP CONTROL SYSTEM

the mathematical description of the continuous-data system will be assumed known and the equivalent discrete-data system will be obtained by modelling it according to Section 2.1. A set of linear, time-invariant differential equations in state-variable form will be used to describe each system. Although any physical system, if analyzed in great detail is non linear and time-variant, most of the actual systems can be approximated with sufficient accuracy by linear equations.

3.2. Closed-Loop Continuous-Data System

In order to analyze the closed-loop control system desirable simplifications of the block diagram of Figure 3.2 will be made. The system will be considered with plant, controller, unity feedback loop and null input reference, as shown in Figure 3.3. The mathematical model of each component is known. The reason for null input is to facilitate the mathematical formulation. The case of forced systems will be discussed in Chapter V.

Consider first the description of the plant. Assume that it can be described by a set of linear time-invariant differential equations in the state-variable

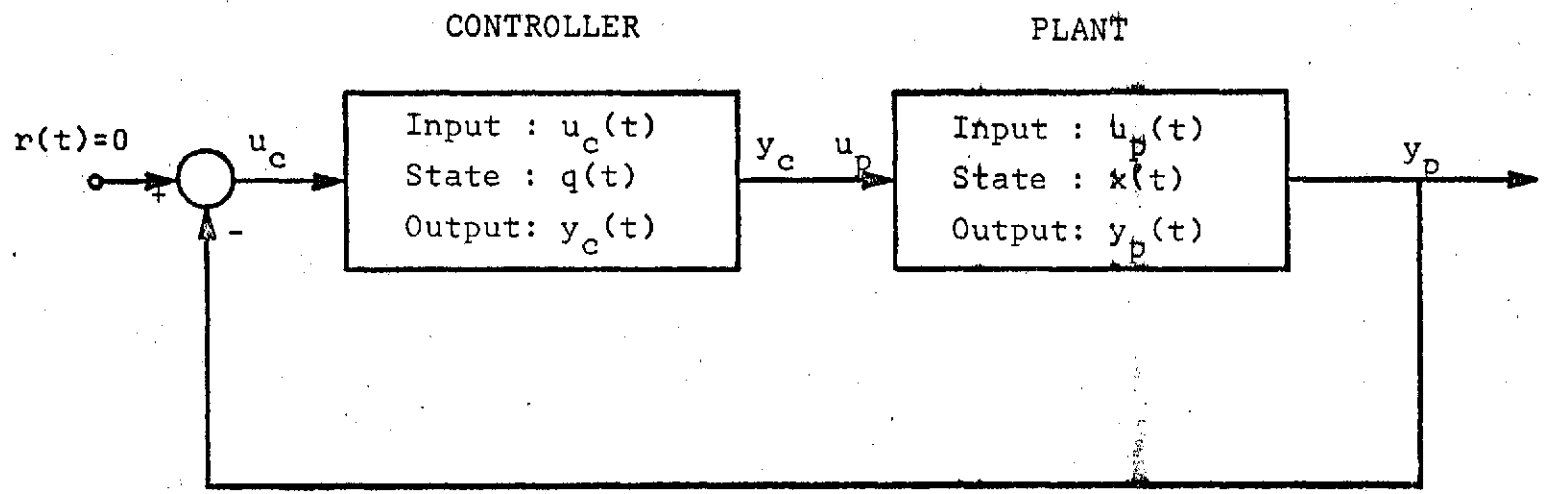

FIGURE 3.3 SIMPLIFIED CLOSED-LOOP CONTROL SYSTEM

form:

$$\dot{x}(t) = A\,x(t) + B\,u_p(t)\ ;\ x(t_o) = x_o \qquad (3.1)$$

$$y_p(t) = C\,x(t)$$

where

$x(t)$: n-vector, state

x_o: n-vector, initial state

$u_p(t)$: r-vector, input

$y_p(t)$: l-vector, output

A: n x n matrix, system matrix

B: n x r matrix, control matrix

C: l x n matrix, output matrix

The first equation gives the plant dynamics and the second specifies the output transformation.

An important observation must be made next. The scope of this research is not to solve a closed-loop control system. The object is to study the behavior of its evolution as a function of some invariant parameters which characterize the system. These parameters are the eigenvalues which can be determined with well known mathematical or computational methods [46, 47]. Therefore, the A-matrix will be assumed simple* and in diagonal form with eigenvalues λ_i. In the case of a non-simple system matrix, a Jordan canonical form will be obtained [35] and the characterization can be made in the same form as for a simple diagonal matrix.

The controller, already presented in Section 2.4, is assumed to be described by a set of linear time-invariant differential equations in the state variable form

$$\dot{q}(t) = F\, q(t) + G\, u_c(t) \; ; \; q(t_o) = q_o \qquad (3.2)$$

$$y_c(t) = H\, q(t)$$

where the matrices and vectors are described in Section

*n x n matrix with n linearly independent eigenvectors.

2.4. Here it is also assumed that the F-matrix is simple and diagonal with eigenvalues μ_i.

Consider now the description of the closed-loop system. It is clear from Figure 3.3 that

$$y_c(t) = u_p(t) \; ; \quad r\text{-vectors} \tag{3.3}$$

$$y_p(t) = -u_c(t) \; ; \quad l\text{-vectors} \tag{3.4}$$

Then substituting Eqs. (3.3) and (3.4) into Eqs. (3.1) and (3.2), the closed-loop control system can be described in vector-matrix form by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A & BH \\ -GC & F \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} \; ; \; \begin{bmatrix} x_o \\ q_o \end{bmatrix} \tag{3.5}$$

In order to simplify the notation call

$$z(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} \tag{3.6.a}$$

(n + m) - state vector.

$$\mathcal{A} = \begin{bmatrix} A & BH \\ -GC & F \end{bmatrix} \tag{3.6.b}$$

$(n + m) \times (n + m)$ - system matrix.

Therefore Eq. (3.5) can be expressed as

$$\dot{z}(t) = \mathcal{A}\, z(t) \;,\; z(t_o) = z_o \tag{3.7}$$

which is a homogeneous vector-matrix differential equation with known initial conditions.

As it can be seen, $z(t)$ represents the state of the interconnected system, $\mathcal{A}$ is the system matrix, it reflects the dependence and influence between states and describes completely the behavior of the closed-loop system. It is assumed that the closed-loop control system is asymptoticaly stable; therefore, the $\mathcal{A}$-matrix has eigenvalues with negative real part [38].

Again, it is assumed that $\mathcal{A}$ is a simple matrix and that there exists a non-singular similarity transformation P which converts $\mathcal{A}$ into a diagonal form, with eigenvalues δ_i. Therefore, setting

$$z(t) = P\, w(t) \tag{3.8}$$

and substituting Eq. (3.8) into Eq. (3.7), yields

$$\dot{w}(t) = \Delta \; w(t) \tag{3.9}$$

$$w(t_o) = P^{-1} z_o \tag{3.10}$$

where

$$\Delta = P^{-1} \mathcal{A} P \tag{3.11}$$

diagonal matrix with elements δ_i.

The closed-loop system is described in a simple differential equation form. In the case of a non-simple $\mathcal{A}$-matrix, a Jordan canonical form appears and the problem can be solved following a similar procedure.

In order to know the performance of the closed-loop control system, equations (3.5) and (3.9) must be solved. The well known solution of linear differential equations applies in this case [35].

$$x(t) = \Phi_\lambda(t-t_o)x(t_o) + \int_{t_o}^{t} \Phi_\lambda(t-\tau)BH \; q(\tau) \; d\tau \tag{3.12}$$

$$q(t) = \Phi_\mu(t-t_o)q(t_o) - \int_{t_o}^{t} \Phi_\mu(t-\tau)\ GC\ x(\tau)\ d\tau \qquad (3.13)$$

$$w(t) = \Phi_\delta(t-t_o)\ w(t_o) \qquad (3.14)$$

where

$$\Phi_\lambda(t) = e^{A\,t} = \text{Diag}\ [e^{\lambda_i t}]$$

$$\Phi_\mu(t) = e^{F\,t} = \text{Diag}\ [e^{\mu_i t}]$$

$$\Phi_\delta(t) = e^{\Delta\,t} = \text{Diag}\ [e^{\delta_i t}]$$

are the fundamental matrices.

Consider now Eq. (3.8); the states x(t) and q(t) can be expressed as

$$z(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} = Pw(t) = \begin{bmatrix} P_n \\ P_m \end{bmatrix} w(t) \qquad (3.15)$$

where

P_n: First n-rows of P, affecting x(t).

P_m: Last m-rows of P, affecting q(t).

then, according to Eq(3.14) and (3.15) the states are

$$x(t) = P_n \Phi_\delta(t-t_o)\ w(t_o) \tag{3.16}$$

$$q(t) = P_m \Phi_\delta(t-t_o)\ w(t_o) \tag{3.17}$$

Substituting Eq. (3.16) into Eq. (3.13) and Eq. (3.17) into Eq. (3.12) results in the evolution of the state of the plant and the controller.

$$x(t) = \Phi_\lambda(t-t_o)\ x(t_o) + \int_{t_o}^{t} \Phi_\lambda(t-\tau)\ BHP_m\ \Phi_\delta(\tau)\ d\tau\ w_1 \tag{3.18}$$

$$q(t) = \Phi_\mu(t-t_o)\ q(t_o) - \int_{t_o}^{t} \Phi_\mu(t-\tau)\ GCP_n\ \Phi_\delta(\tau)\ d\tau\ w_1 \tag{3.19}$$

where

$$w_1 = \Phi_\delta(-t_o)w(t_o) \tag{3.20}$$

is a constant vector.

As stated in Section 3.1, the evolution of the continuous and discrete-data systems has to be compared. It becomes clear from Section 2.4 that the sampling instants are adequate for the comparison. Therefore, the evolution of the continuous system has to be determined at those points. Replace t_o by kT and t by kT+T in Eqs. (3.18) and (3.19), furthermore make the change of variable v = t - kT. The evolution of the state of the plant and controller at sampling times becomes

$$x(kT+T) = \Phi_\lambda(T)\, x(kT) + \Theta_{\lambda\delta}(T)\, \Phi_\delta(kT)\, w_1 \qquad (3.21)$$

$$q(kT+T) = \Phi_\mu(T)\, q(kT) - \Theta_{\mu\delta}(T)\, \Phi_\delta(kT)\, w_1 \qquad (3.22)$$

where

$$\Theta_{\lambda\delta}(T) = \int_o^T \Phi_\lambda(T-v)\, R\, \Phi_\delta(v)\, dv \qquad (3.23)$$

n x (n + m) matrix

$$\Theta_{\mu\delta}(T) = \int_o^T \Phi_\mu\,(T-v)\, S\, \Phi_\delta(v)\, dv \qquad (3.24)$$

m x (n + m) matrix

$$R = B\ H\ P_m \ ;\ n \times (n + m) \text{ matrix} \tag{3.25}$$

$$S = G\ C\ P_n \ ;\ m \times (n + m) \text{ matrix} \tag{3.26}$$

These equations express the state of the plant and the controller of the continuous system at time t = kT+T as an exact function of its value at time t = kT.

The integrals of Eqs. (3.23) and (3.24) can be evaluated explicitly as matrices with elements

$$\theta_{\lambda\delta}(T)\Big|_{ij} = \frac{e^{\lambda_i T} - e^{\delta_j T}}{\lambda_i - \delta_j}\, r_{ij} \tag{3.27}$$

with

$$1 \le i \le n, \quad 1 \le j \le n + m$$

and

$$\theta_{\mu\delta}(T)\Big|_{ij} = \frac{e^{\mu_i T} - e^{\delta_i T}}{\mu_i - \delta_j}\, s_{ij} \tag{3.28}$$

with

$$1 \le i \le m, \quad 1 \le j \le n + m$$

where

λ_i : Eigenvalues of the plant

μ_i : Eigenvalues of the controller

δ_j : Eigenvalues of the closed-loop system

r_{ij} : Elements of the matrix R

s_{ij} : Elements of the matrix S

The case in which a closed-loop eigenvalue is identical with a plant or controller eigenvalue may be treated by taking an appropriate limit, i.e.,

$$\lim_{\delta_j \to \lambda_i} \Theta_{\lambda\delta}(T) \Big|_{ij} = T\, e^{\delta_j T}$$

finite for stable systems. A more interesting case arises when T = 0, it is

$$\lim_{T \to 0} \Theta_{\lambda\delta}(T) \Big|_{ij} = 0$$

This result, which is not surprising, indicates a zero sampling interval or no sampling. It therefore results in no change in the state, as can be seen in Eqs. (3.21) and (3.22) by letting T become zero.

The analysis of Eqs. (3.27) and (3.28) shows that the matrices $\Theta_{\lambda\delta}(T)$ and $\Theta_{\mu\delta}(T)$ can be expressed as the result of a transformation of the fundamental matrices. That is, it is written as

$$\Theta_{\lambda\delta}(T) = \Phi_{\lambda}(T)\ Q_x - Q_x\ \Phi_{\delta}(T) \tag{3.29}$$

$$\Theta_{\mu\delta}(T) = \Phi_{\mu}(T)\ Q_q - Q_q\ \Phi_{\delta}(T) \tag{3.30}$$

with elements

$$\theta_{\lambda\delta}(T)\Big|_{ij} = e^{\lambda_i T}\ Q_x\Big|_{ij} - Q_x\Big|_{ij}\ e^{\delta_j T} \tag{3.29.a}$$

and

$$\theta_{\mu\delta}(T)\Big|_{ij} = e^{\mu_i T}\ Q_q\Big|_{ij} - Q_q\Big|_{ij}\ e^{\delta_j T} \tag{3.30.a}$$

Equating the elements of the matrices of Eqs. (3.29) and (3.30) with the given by Eqs. (3.27) and (3.28), the elements of the Q matrices become

$$Q_x\Big|_{ij} = \frac{r_{ij}}{\lambda_i - \delta_j} \tag{3.31}$$

with

$$1 \leq i \leq n, \quad 1 \leq j \leq n + m$$

and

$$Q_q\Big|_{ij} = \frac{s_{ij}}{\mu_i - \delta_j} \tag{3.32}$$

with

$$1 \leq i \leq m, \quad 1 \leq j \leq n + m$$

Here the elements of the Q matrices seem to be undefined for a zero in the denominator, but the fact that the Θ matrices are finite implies that this pole is cancelled by a zero in the final expression.

Equations (3.29) and (3.30) can be substituted into Eqs. (3.21) and (3.22). It follows that

$$x(kT+T) = \Phi_\lambda(T)\, x(kT) + [\Phi_\lambda(T)\, Q_x - Q_x\, \Phi_\delta(T)] \cdot \Phi_\delta(kT)\, w_1 \tag{3.33}$$

$$q(kT+T) = \Phi_\mu(T)\, q(kT) - [\Phi_\mu(T)\, Q_q - Q_q\, \Phi_\delta(T)] \cdot \Phi_\delta(kT)\, w_1 \tag{3.34}$$

These equations give the evolution of the state of the plant and the controller of the continuous system as a function of the fundamental matrices of the plant, controller and closed-loop system, the initial conditions and the transformation matrices Q. The equations will be used in Chapter IV for comparing the evolutions of the continuous and discrete-data systems.

In the next section, the discrete-data system will be analyzed using the same approach as used here.

3.3. Closed-Loop Discrete-Data System

Consider now the closed-loop discrete-data system. The digital controller was presented and modelled in Chapter II and shown in Figure 2.3. The plant is the same as the one analyzed in Section 3.2 because the discrete-data system has only a change in nature of the controller. Therefore the closed-loop discrete-data system can be assumed to be of the form shown in Figure 3.4.

For the purpose of this research it is necessary to analyze the behavior of the discrete-data control system for different complexity of holding devices and compare them with the behavior of the continuous system. In this section, the discrete-data control system using the two different holding devices presented in Chapter II will be analyzed.

3.3.1. Zero-Order Discrete-Data System

Consider first that the zero-order hold is used as the holding device. The resulting system is called zero-order discrete-data system. The input signal to the controller and the plant is characterized by its

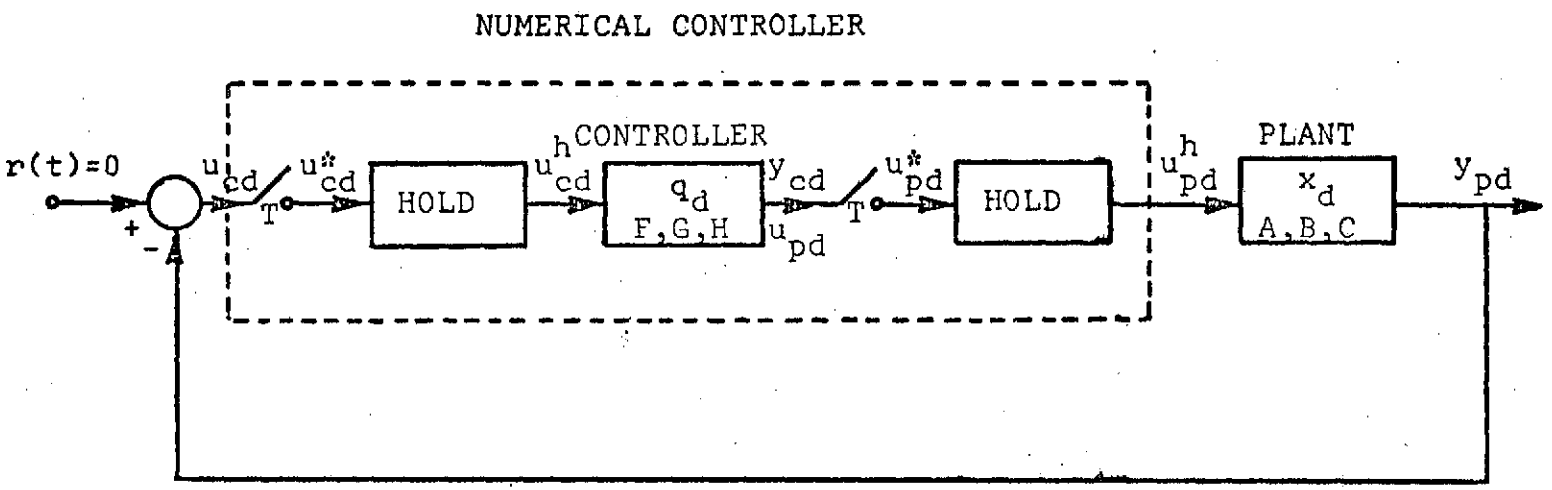

FIGURE 3.4 DISCRETE-DATA CLOSED-LOOP CONTROL SYSTEM

values at sampling times and held constant in between them, as seen in Section 2.3. Also, it is of piecewise-continuous nature and differs from the continuous signal as shown in Figure 2.5. Because of this behavior, the states of the plant and the controller differ from those of the continuous system. They will be denoted by $x_d(t)$ and $q_d(t)$ respectively.

Assume that the plant is as described in Section 3.2. Using the notation introduced in Figure 3.4, its behavior can be described by

$$\dot{x}_d(t) = A\, x_d(t) + B\, u^o_{pd}(t),\ x_d(t_o) = x_o \tag{3.35}$$

$$y_{pd}(t) = C\, x_d(t)$$

where the A, B and C matrices have been presented in Eq. (3.1) and $u^o_{pd}(t)$ is the piecewise-continuous control output of the zero-order hold.

The controller has been presented in Section 2.4. According to Eq. (3.2) and Figure 3.4, it can be described by

$$\dot{q}_d(t) = F\, q_d(t) + G\, u^o_{cd}(t),\ q_d(t_o) = q_o \tag{3.36}$$

$$y_{cd}(t) = H\, q_d(t)$$

where the F, G and H matrices have been presented in Eq. (3.1) and $u^o_{cd}(t)$ is the piecewise-continuous output of the holding device.

In order to study the evolution of the state of the plant and the controller of the discrete-data system for its comparison with the continuous system, Eqs. (3.35) and (3.36) must be solved. As shown earlier,

$$x_d(t) = \Phi_\lambda(t-t_o)\, x_d(t_o) + \int_{t_o}^{t} \Phi_\lambda(t-\tau)\, B\, u^o_{pd}(\tau)\, d\tau \tag{3.37}$$

$$q_d(t) = \Phi_\mu(t-t_o)\, q_d(t_o) + \int_{t_o}^{t} \Phi_\mu(t-\tau)\, G\, u^o_{cd}(\tau)\, d\tau \tag{3.38}$$

As may be seen in Figure 3.5, the controls $u^h_{pd}(\tau)$ and $u^h_{cd}(\tau)$ are the output of the zero-order hold device. According to Eq. (2.5), they are described by

$$u^o_{p/cd}(\tau) = u_{p/cd}(kT) \qquad (3.39)^*$$

a constant vector for

$$kT \leq t < kT + T$$

From Figure 3.5 it is evident that

$$u_{pd}(kT) = y_{cd}(kT)$$

and

$$u_{cd}(kT) = -y_{pd}(kT)$$

so that, according to Eqs. (3.35), (3.36) and (3.39)

---

*The notation p/c indicates either the plant (p) or controller (c) input or parameter.

$$u^{o}_{pd}(\tau) = H\, q_d(kT) \tag{3.40}$$

and

$$u^{o}_{cd}(\tau) = C\, x_d(kT) \tag{3.41}$$

Consider now the piecewise nature of the control vectors and their value given by Eqs. (3.40) and (3.41). The Eqs. (3.37) and (3.38) become

$$x_d(kT+T) = \Phi_\lambda(T)\, x_d(kT) + \int_{kT}^{kT+T} \Phi_\lambda(kT+T-\tau)\, B\, H\, d\tau\, q_d(kT) \tag{3.42}$$

$$q_d(kT+T) = \Phi_\mu(T)\, q_d(kT) - \int_{kT}^{kT+T} \Phi_\mu(kT+T-\tau)\, G\, C\, d\tau\, x_d(kT) \tag{3.43}$$

These equations are valid only for one sampling interval since the input vector $u^{o}_{p/cd}(kT)$ is constant

only for that duration. It is possible to evaluate the integrals of the right member of Eq. (3.42) and (3.43); they have the same form as Eq. (2.13); thus

$$x_d(kT+T) = \Phi_\lambda(T)\ x_d(kT) + A^{-1}\ [\Phi_\lambda(T) - I]\ \ BH\ q_d\ (kT) \tag{3.44}$$

$$q_d(kT+T) = \Phi_\mu(T)\ q_d(kT) - F^{-1}\ [\Phi_\mu(T) - I]\ G\ C\ x_d\ (kT) \tag{3.45}$$

Equations (3.44) and (3.45) represent a set of linear difference equations in vector-matrix form. They are the time-discrete state equations of the digital system.

In order to facilitate the formulation for the comparison of the discrete and continuous systems it is desirable to express Eqs. (3.44) and (3.45) in the same format as Eqs. (3.33) and (3.34). To do so, a definition of state digitalization error must be presented.

Definition 3.1. The difference between the state variables of a continuous system and the discretized version of it is called state digitalization error.

It is given as

$$e_d^x(kT) = x(kT) - x_d(kT) \tag{3.46}$$

$$e_d^q(kT) = q(kT) - q_d(kT) \tag{3.47}$$

The error, which represents the difference between performance of the continuous and discrete systems, will be discussed in the following chapters. The use of this definition will permit the characterization of the discrete-data system in the desired format and this object is followed in this chapter. In Chapter IV, an important component of the state digitalization error will be analyzed, the error introduced by sampling.

Substituting Eqs. (3.46) and (3.47) into Eqs. (3.44) and (3.45) yields

$$x_d(kT+T) = \Phi_\lambda(T)\, x_d(kT) + A^{-1}\,[\Phi_\lambda(T) - I]\, B\, H \cdot [q(kT) - e_d^q(kT)] \tag{3.48}$$

$$q_d(kT+T) = \Phi_\mu(T)\, q_d(kT) - F^{-1}\,[\Phi_\mu(T) - I]\, G\, C \cdot [x(kT) - e_d^x(kT)] \tag{3.49}$$

Consider now Eqs. (3.16), (3.17), (3.20) for t = kT, that is,

$$x(kT) = P_n \Phi_\delta (kT) w_1 \tag{3.50}$$

$$q(kT) = P_m \Phi_\delta (kT) w_1 \tag{3.51}$$

and substitute these values into Eqs. (3.48) and (3.49) to yield

$$x_d(kT+T) = \Phi_\lambda(T) x_d(kT) - A^{-1}[\Phi_\lambda(T) - I] B H e_d^q(kT) + \Theta^o_{\lambda\delta}(T) \Phi_\delta(kT) w_1 \tag{3.52}$$

$$q_d(kT+T) = \Phi_\mu(T) q_d(kT) + F^{-1}[\Phi_\mu(T) - I] G C e_d^x(kT) - \Theta^o_{\mu\delta}(T) \Phi_\delta(kT) w_1 \tag{3.53}$$

where

$$\theta^{o}_{\lambda\delta}(T) = A^{-1}\,[\Phi_{\lambda}(T) - I]R \tag{3.54}$$

$$\vartheta^{o}_{\mu\delta}(T) = F^{-1}[\Phi_{\mu}(T) - I]S \tag{3.55}$$

with R and S described by Eqs. (3.25) and (3.26) respectively.

In order to be consistent with Section 3.2, it is desirable to express the Eqs. (3.54) and (3.55) in the same form as Eqs. (3.29) and (3.30). Consider first Eqs. (3.54) and (3.55), the elements of those matrices are

$$\theta^{o}_{\lambda\delta}(T)\Big|_{ij} = \lambda_i^{-1}(e^{\lambda_i T} - 1)\, r_{ij} \tag{3.56}$$

for

$$1 \leq i \leq n, \quad 1 \leq j \leq n + m$$

and

$$\theta^{o}_{\mu\delta}(T)\Big|_{ij} = \mu_i^{-1}(e^{\mu_i T} - 1)s_{ij} \qquad (3.57)$$

for

$$1 \leq i \leq m, \quad 1 \leq j \leq n + m$$

The Eqs. (3.29) and (3.30), as applied to the case of the zero-order hold modelling, take the form

$$\theta^{o}_{\lambda\delta}(T) = \Phi_\lambda(T)Q^{o}_x(T) - Q^{o}_x(T)\Phi_\delta(T) \qquad (3.58)$$

$$\theta^{o}_{\mu\delta}(T) = \Phi_\mu(T)Q^{o}_q(T) - Q^{o}_q(T)\Phi_\delta(T) \qquad (3.59)$$

with elements of the form of Eqs. (3.29.a) and (3.30.a).

Since the elements of the θ^{o} matrices given by Eqs. (3.54), (3.55), (3.58) and (3.59) must be identical, the elements of the Q^{o} matrices become

$$Q^{o}_x(T)\Big|_{ij} = \frac{e^{\lambda_i T} - 1}{\lambda_i(e^{\lambda_i T} - e^{\delta_j T})}\, r_{ij} \qquad (3.60)$$

for

$$1 \leq i \leq n, \quad 1 \leq j \leq n + m$$

and

$$Q_q^o(T)\Big|_{ij} = \frac{e^{\mu_i T} - 1}{\mu_i (e^{\mu_i T} - e^{\delta_j T})} s_{ij} \tag{3.61}$$

for

$$1 \leq i \leq m, \quad 1 \leq j \leq n + m$$

The cases in which a closed-loop and an open-loop eigenvalue are identical and when the sampling interval become zero can be treated in a manner similar to that of Section 3.2.

By substituting Eqs. (3.58) and (3.59) into (3.52) and (3.53), the discrete-state equations of the system for the zero-order hold are obtained. They are:

$$x_d(kT+T) = \Phi_\lambda(T)x_d(kT) - A^{-1}[\Phi_\lambda(T) - I]\ B\ H\ e_d^q(kT)$$

$$+ [\Phi_\lambda(T)Q_x^o(T) - Q_x^o(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \quad (3.62)$$

$$q_d(kT+T) = \Phi_\mu(T)x_d(kT) + F^{-1}[\Phi_\mu(T) - I]\ G\ C\ e_d^x(kT)$$

$$- [\Phi_\mu(T)Q_q^o(T) - Q_q^o(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \quad (3.63)$$

These equations are similar in format to the equations describing the evolution of the continuous system at sampling instants. They will be used in the next chapter for comparison between continuous and discrete-data systems.

3.3.2. First-Order Discrete-Data System

Consider now that the first-order polygonal hold is used as the holding device. The resulting system is called first-order discrete-data system. As shown in Section 2.3.2, the input signal to the controller and the plant is characterized by its behavior at sampling instants and has a linear variation in between them.

Its effect on the system is a change on the value of the state because of the difference with the continuous signal. As before a change on the state occurs and $x_d(t)$ and $q_d(t)$ will be the modified states of the plant and the controller respectively.

Consider the system as previously described by Eqs. (3.35) and (3.36). The evolution of the states of the plant and the controller are given by Eqs. (3.37) and (3.38) and are presented here for the first-order polygonal hold.

$$x_d(t) = \Phi_\lambda(t-t_o)x_d(t_o) + \int_{t_o}^{t} \Phi_\lambda(t-\tau)B\, u^1_{pd}(\tau)\, d\tau \tag{3.64}$$

$$q_d(t) = \Phi_\mu(t-t_o)q_d(t_o) + \int_{t_o}^{t} \Phi_\mu(t-\tau)G\, u^1_{cd}(\tau)\, d\tau \tag{3.65}$$

where $u^1_{p/cd}(\tau)$ is the piecewise-continuous control, output of the first-order polygonal hold. According Eq. (2.7), it can be described as

$$u^1_{p/cd}(\tau) = u^1_{p/cd}(kT) + [u^1_{p/cd}(kT+T) - u_{p/cd}(kT)]\frac{\tau-kT}{T} \quad (3.66)$$

for

$$kT \leq \tau < kT + T$$

and with $u^1_{p/cd}(kT)$ and $u^1_{p/cd}(kT+T)$ constant vectors in the interval.

As before, from Figure 3.5

$$u^1_{pd}(kT) = H\, q_d(kT) \quad (3.67)$$

$$u^1_{cd}(kT) = -C\, x_d(kT) \quad (3.68)$$

Then evaluating Eq. (3.66) for the controls given by Eqs. (3.67) and (3.68) and substituting the result into Eqs. (3.64) and (3.65) yields

$$x_d(kT+T) = \Phi_\lambda(T)\, x_d(kT) + \int_{kT}^{kT+T} \Phi_\lambda(kT+T-\tau)\, BH\, \frac{kT+T-\tau}{T}\, d\tau\, q_d(kT) + \int_{kT}^{kT+T} \Phi_\lambda(kT+T-\tau)\, BH\, \frac{\tau-kT}{T}\, d\tau\, q_d(kT+T) \tag{3.69}$$

A similar expression is obtained for $q_d(kT+T)$ but with the $x_d(kT)$ and $q_d(kT)$ interchanged, the λ replaced by μ, and BH replaced by -GC.

These equations as before are valid for one sampling interval since $u^1_{p/cd}(\tau)$ is continuous only for that duration. The integrals of the right member can be evaluated; they are in the same form as Eq. (2.16); thus

$$x_d(kT+T) = \Phi_\lambda(T)x_d(kT) + A^{-1}\{\Phi_\lambda(T) - \frac{A^{-1}}{T}[\Phi_\lambda(T)-I]\}\, BH\, q_d(kT) + A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\}\, BH\, q_d(kT+T) \tag{3.70}$$

and

$$q_d(kT+T) = \Phi_\mu(T)\, q_d(kT)$$

$$-F^{-1}\left\{\Phi_\mu(T) - \frac{F^{-1}}{T}\left[\Phi_\mu(T)-I\right]\right\} GC\, x_d(kT)$$

$$-F^{-1}\left\{\frac{F^{-1}}{T}\left[\Phi_\mu(T)-I\right] -I\right\} GC\, x_d(kT+T) \qquad (3.71)$$

These discrete state equations represent a set of linear difference equation in vector-matrix form, describing the evolution of the digital system when modelled with the first-order polygonal hold.

In order to facilitate the formulation for comparison recall the digitalization error defined in Section 3.3.1. Substituting Eqs. (3.46) and (3.47) into (3.70) and (3.71) and considering the continuous states given by Eqs. (3.50) and (3.51), the evolution of the states can be expressed as

$$x_d(kT+T) = \Phi_\lambda(T)\,x_d(kT)$$

$$- A^{-1}\left\{\Phi_\lambda(T) - \frac{A^{-1}}{T}\left[\Phi_\lambda(T)-I\right]\right\} BH\; e_d^q\,(kT)$$

$$- A^{-1}\left\{\frac{A^{-1}}{T}\left[\Phi_\lambda(T)-I\right] -I\right\} BH\; e_d^q\,(kT+T)$$

$$+ \Theta^1_{\lambda\delta}(T)\;\Phi_\delta(kT)\;w_1 \qquad (3.72)$$

and

$$q_d(kT+T) = \Phi_\mu(T)\,q_d(kT)$$

$$+ F^{-1}\left\{\Phi_\mu(T) - \frac{F^{-1}}{T}\left[\Phi_\mu(T)-I\right]\right\} GC\; e_d^x\,(kT)$$

$$+ F^{-1}\left\{\frac{F^{-1}}{T}\left[\Phi_\mu(T)-I\right] -I\right\} GC\; e_d^q\,(kT+T)$$

$$- \Theta^1_{\mu\delta}(T)\,\Phi_\delta(kT)\,w_1 \qquad (3.73)$$

where

$$\theta^{1}_{\lambda\delta}(T) = A^{-1}\left\{\Phi_{\lambda}(T) - \frac{A^{-1}}{T}\left[\Phi_{\lambda}(T) - I\right]\right\} R$$

$$+ A^{-1}\left\{\frac{A^{-1}}{T}\left[\Phi_{\lambda}(T) - I\right] - I\right\} R\, \Phi_{\delta}(T) \qquad (3.74)$$

and

$$\theta^{1}_{\mu\delta}(T) = F^{-1}\left\{\Phi_{\mu}(T) - \frac{F^{-1}}{T}\left[\Phi_{\mu}(T) - I\right]\right\} S$$

$$+ F^{-1}\left\{\frac{F^{-1}}{T}\left[\Phi_{\mu}(T) - I\right] - I\right\} S\, \Phi_{\delta}(T) \qquad (3.75)$$

Here, for simplicity, the digitalization error is written with the same notation as in Section 3.3.1, but assumes different values.

In order to express Eqs. (3.72) and (3.73) in the same form as Eqs. (3.29) and (3.30), they must be modified accordingly. Start first with Eqs. (3.74) and (3.75), the elements of those matrices have elements of the form

$$\theta^1_{\lambda\delta}(T)\Big|_{ij} = \frac{1}{\lambda_i}\left\{\left[e^{\lambda_i T} - \frac{1}{\lambda_i T}\left(e^{\lambda_i T} - 1\right)\right]\right.$$

$$\left. + \frac{1}{\lambda_i T}\left[\left(e^{\lambda_i T} - 1\right) - 1\right] e^{\delta_j T}\right\} r_{ij} \tag{3.76}$$

for

$$1 \le i \le n, \quad 1 \le j \le n + m$$

and

$$\theta^1_{\lambda\delta}(T)\Big|_{ij} = \frac{1}{\mu_i}\left\{\left[e^{\mu_i T} - \frac{1}{\mu_i T}\left(e^{\mu_i T} - 1\right)\right]\right.$$

$$\left. + \frac{1}{\mu_i T}\left[\left(e^{\mu_i T} - 1\right) - 1\right] e^{\delta_j T}\right\} s_{ij} \tag{3.77}$$

for

$$1 \le i \le m, \quad 1 \le j \le n + m$$

It can be shown that in the limit for T, λ_i and μ_i approaching zero, the elements are finite.

Finally, Eqs. (3.74) and (3.75) must be expressed in the form of Eqs. (3.29) and (3.30), thus

$$\Theta^{1}_{\lambda\delta} = \Phi_{\lambda}(T)Q^{1}_{x}(T) - Q^{1}_{x}(T)\Phi_{\delta}(T) \tag{3.78}$$

$$\Theta^{1}_{\mu\delta} = \Phi_{\mu}(T)Q^{1}_{q}(T) - Q^{1}_{q}(T)\Phi_{\delta}(T) \tag{3.79}$$

These matrices have elements of the form of Eqs. (3.29.a) and (3.29.b) and they must be identical with the elements given by Eqs. (3.76) and (3.77). Equating the identities the elements of the transformation matrix become

$$Q^{1}_{x}(T)\Big|_{ij} = \frac{(1-e^{\lambda_i T})(1-e^{\delta_j T}) + \lambda_i T(e^{\lambda_i T}-e^{\delta_j T})}{\lambda_i^2 T\,(e^{\lambda_i T}-e^{\delta_j T})}\, r_{ij} \tag{3.80}$$

for

$$1 \leq i \leq n, \quad 1 \leq j \leq n + m$$

and

$$Q_q^1(T)\Big|_{ij} = \frac{(1-e^{\mu_i T})(1-e^{\delta_j T}) + \mu_i T(e^{\mu_i T}-e^{\delta_j T})}{\mu_i^2 T(e^{\mu_i T}-e^{\delta_j T})} s_{ij} \tag{3.81}$$

for

$$1 \leq i \leq m, \quad 1 \leq j \leq n + m$$

The behavior of these equations in the limit can be analyzed as in Section 3.2.

By substituting Eqs. (3.78) and (3.79) into Eqs. (3.72) and (3.73), the discrete-state equations of the system for the first-order polygonal hold are obtained. They are:

$$x_d(kT+T) = \Phi_\lambda(T)x_d(kT)$$

$$- A^{-1}\{\Phi_\lambda(T) - \frac{A^{-1}}{T}[\Phi_\lambda(T) - I]\} BH\, e_d^q(kT)$$

$$- A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T) - I] - I\} BH\, e_d^q(kT+T)$$

$$+ [\Phi_\lambda(T)Q_x^1(T) - Q_x^1(T)\Phi_\delta(T)]\,\Phi_\delta(kT)w_1 \qquad (3.82)$$

and

$$q_d(kT+T) = \Phi_\mu(T)q_d(kT)$$

$$+ F^{-1}\{\Phi_\mu(T) - \frac{F^{-1}}{T}[\Phi_\mu(T) - I]\} GC\, e_d^x(kT)$$

$$+ F^{-1}\{\frac{F^{-1}}{T}[\Phi_\mu(T) - I] - I\} GC\, e_d^x(kT+T)$$

$$- [\Phi_\mu(T)Q_q^1(T) - Q_q^1(T)\Phi_\delta(T)]\,\Phi_\delta(kT)w_1 \qquad (3.83)$$

These equations are similar in format to the equations describing the continuous data system at sampling instants.

A close analysis of the obtained equations will show that they are similar to those obtained in [11] for open-loop systems; therefore comparable techniques can be used for approaching the determination of sampling rates in open and closed-loop systems. In the next chapter a study of the elements appearing in the equations describing the continuous and discrete-data systems will be made and an expression measuring the difference between continuous and discrete-data states will be obtained.

CHAPTER IV

ERROR ANALYSIS

4.1. Introduction

The response of a continuous and computer controlled closed-loop system was introduced in the last chapter. The problem of relating them in order to compare their performance is constrained by a proper determination of error or differences between systems. In this section a brief review of the different errors present in a discrete-data system will be made in order to define a proper error criterion.

An early work in this area [24] defines "system error" as the result of the imperfect response of the discrete-data system to an applied input. The "system error" is composed of two components. One referred to as "organic error," is introduced by lags or leads of the continuous part of the system [24]. The second component, called "ripple," is the error introduced by the sampler. This error, in the steady state, contains

only those frequency components and its harmonics. In practical systems, this "ripple" is generally suppressed by the forward transmission function of the system, by filtering. In the referred paper [24], the authors obtained a mathematical description of the "ripple" by using the z-transform and Laplace transform techniques. Also they make clear that during the transient the ripple component is insignificant. The "system error" approach is of interest to this research, because the problem is similar. The concept will be applied to the state variables of the plant instead of the output error dealt within the referred paper.

Another interpretation of the system error is given by researchers in the field of numerical analysis and is referred to as truncation error [8, 42, 48, 55]. The error is defined as that resulting from the manner in which the differential equation describing a dynamic system is approximated. This definition applies more directly to this research because it involves a differential equation, that is, the mathematical description of the continuous data control system, and an approximation which is the result of discretizing the system.

Another source of error present in the system is due to the imperfect analog-to-digital and digital-to-

analog conversion. The practical converters have a finite number of conversion bits, therefore error appears and is known as quantization error. In general the error is neglected in the design of digital control systems, but the effect of quantization should be considered. It can serve as a practical guide in the determination of the height of quantization levels and the size of the registers of the digital computer used for control.

The effects of quantization of continuous-time signals have been extensively investigated from the probabilistic viewpoint [49-52]. Deterministic studies on the effect of quantization errors in linear systems have also been studied [30, 31, 32, 53]. They give a mathematical formulation for the problem and determine an upper bound for the quantization error in the output. Lately the problem has been approached from the optimum control viewpoint [54], and a performance criterion for the minimization of the worst effect of the error measure has been defined.

In previous research [11], it has been shown that the variance of the state quantization error due to input quantization depends linearly on the sampling interval T, when T is small, and varies as the square of the quantization level.

From these considerations it is clear that further investigation is required of quantization errors in closed-loop systems, and how to select a converter for a given sampling interval and digital computer. This subject is presently under study in other research [56]. The error due to sampling only is analyzed in this chapter.

4.2. Discretization Error

Consider now the error introduced in the system due to sampling and hold and called in the research preceding this discretization error. It reflects the difference between the performance of the continuous and discrete systems and is defined by

Definition 4.1. The difference between the state variables of the continuous system and its discrete-data version in the absence of quantization noise is called state discretization error, defined as

$$e_x(kT) = x(kT) - x_d(kT) \tag{4.1}$$

$$e_q(kT) = q(kT) - q_d(kT) \tag{4.2}$$

This definition in essence is the same as Definition 3.1 (state digitalization error) but it holds in the case of absence of quantization error. The state variables were already defined in Eqs. (3.1), (3.2), (3.35) and (3.36).

In a completely similar manner the output discretization error can be formulated.

Definition 4.2. The difference between outputs of the continuous system and its discrete-data version, in absence of quantization noise, is called output discretization error, defined as

$$e_y(kT) = y_p(kT) - y_{pd}(kT) \tag{4.2}$$

According to Eqs. (3.1) and (3.35)

$$y_p(kT) = C\ x\ (kT) \tag{4.3}$$

and

$$y_{pd}(kT) = C\ x_d\ (kT) \tag{4.4}$$

therefore, the output discretization error can be

expressed as

$$e_y(kT) = C\, e_x\ (kT) \tag{4.5}$$

These error definitions will be used in the remaining of this chapter for characterizing the behavior of the discrete-data system with respect to its continuous equivalent.

4.3. Discretization Error for Zero-Order Discrete-Data System

For the application of the definition of state discretization error considered in the last section, recall the expressions for the evolution of the states of the continuous and discrete system for the zero-order hold reconstructor, developed in Chapter III and repeated here.

For the continuous system,

$$x(kT+T) = \Phi_\lambda(T)\ x(kT)$$

$$+ [\Phi_\lambda(T)\ Q_x - Q_x\Phi_\delta(T)]\ \Phi_\delta(kT)\ w_1 \tag{4.6}$$

$$q(kT+T) = \Phi_{\mu}(T)\, q(kT)$$

$$- [\Phi_{\mu}(T) Q_q - Q_q \Phi_{\delta}(T)]\, \Phi_{\delta}(kT)\, w_1 \qquad (4.7)$$

where

$$Q_x \Big|_{ij} = \frac{r_{ij}}{\lambda_i - \delta_j} \; ; \quad [n \times (n + m)] \qquad (4.8)$$

$$Q_q \Big|_{ij} = \frac{s_{ij}}{\mu_i - \delta_j} \; ; \quad [m \times (n + m)] \qquad (4.9)$$

are all known elements presented in Chapter III.

For the discrete-data system, in which the quantization error is assumed to be zero or negligible, the digitalization error is equal to the discretization error; therefore the states become

$$x_d(kT+T) = \Phi_\lambda(T)x_d(kT) - A^{-1}[\Phi_\lambda(T) - I]\, BHe_q(kT) + [\Phi_\lambda(T)Q_x^o(T) - Q_x^o(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \tag{4.10}$$

$$q_d(kT+T) = \Phi_\mu(T)q_d(kT) + F^{-1}[\Phi_\mu(T) - I]\, GCe_x(kT) - [\Phi_\mu(T)Q_q^o(t) - Q_q^o(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \tag{4.11}$$

where

$$Q_x^o(T)\Big|_{ij} = \frac{e^{\lambda_i T} - 1}{\lambda_i(e^{\lambda i T} - e^{\delta j T})}\, r_{ij} \; ; \quad [n \times (n+m)] \tag{4.12}$$

$$Q_q^o(T)\Big|_{ij} = \frac{e^{\mu_i T} - 1}{\mu_i(e^{\mu i T} - e^{\delta j T})}\, s_{ij} \; ; \quad [m \times (n+m)] \tag{4.13}$$

Then by direct application of Definition 4.1, the state discretization error is

$$e_x(kT+T) = \Phi_\lambda(T)\ e_x(kT) + A^{-1}[\Phi_\lambda(T)-I]BH\ e_q(kT) + [\Phi_\lambda(T)\ M_x^o(T) - M_x^o(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \tag{4.14}$$

and

$$e_q(kT+T) = \Phi_\mu(T)\ e_q(kT) - F^{-1}[\Phi_\mu(T)-I]GC\ e_x(kT) - [\Phi_\mu(T)\ M_q^o(T) - M_q^o(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \tag{4.15}$$

where

$$M_x^o(T) = Q_x - Q_x^o(T)\ ;\quad [n \times (n+m)] \tag{4.16}$$

$$M_q^o(T) = Q_q - Q_q^o(T) \; ; \; [m \times (n + m)] \tag{4.17}$$

Since the initial conditions of the continuous and discrete systems are the same, the initial errors are

$$e_x(t_o) = x(t_o) - x_d(t_o) = 0 \tag{4.18}$$

$$e_q(t_o) = q(t_o) - q_d(t_o) = 0 \tag{4.19}$$

In order to facilitate the notation, Eqs. (4.15) and (4.16) may be expressed in vector-matrix form

$$\begin{bmatrix} e_x(kT+T) \\ e_q(kT+T) \end{bmatrix} = \begin{bmatrix} \Phi_\lambda(T) & A^{-1}[\Phi_\lambda(T)-I]BH \\ -F^{-1}[\Phi_\mu(T)-I]GC & \Phi_\mu(T) \end{bmatrix} \begin{bmatrix} e_x(kT) \\ e_q(kT) \end{bmatrix}$$

$$+ \begin{bmatrix} \Phi_\lambda(T)M_x^o(T) - M_x^o(T)\Phi_\delta(T) \\ -[\Phi_\mu(T)M_q^o(T) - M_q^o(T)\Phi_\delta(T)] \end{bmatrix} \Phi_\delta(kT)w_1 \qquad (4.20)$$

with the initial condition

$$e(o) = 0$$

Or

$$e(kT+T) = \Lambda^o(T)\ e(kT) + \Theta^o(T)\ \Phi_\delta(kT)w_1 \qquad (4.21)$$

where

$$e(kT) = \begin{bmatrix} e_x(kT) \\ e_q(kT) \end{bmatrix} \qquad (4.22)$$

$$\Lambda^{o}(T) = \begin{bmatrix} \Phi_{\lambda}(T) & A^{-1}[\Phi_{\lambda}(T)-I]BH \\ -F^{-1}[\Phi_{\mu}(T)-I]GC & \Phi_{\mu}(T) \end{bmatrix} \tag{4.23}$$

$$\Theta^{o}(T) = \begin{bmatrix} \Phi_{\lambda}(T)M_{x}^{o}(T) - M_{x}^{o}(T)\Phi_{\delta}(T) \\ -[\Phi_{\mu}(T)M_{q}^{o}(T) - M_{q}^{o}(T)\Phi_{\delta}(T)] \end{bmatrix} \tag{4.24}$$

Equation (4.21) represents a vector-matrix difference equation which gives the discretization error at sampling instants for the zero-order discrete-data system.

To study the behavior of Eq. (4.21), the matrix $\Theta^{o}(T)$ must be investigated further. Take its components $M_{x}^{o}(T)$ and $M_{q}^{o}(T)$.

In view of Eqs. (4.16), (4.17), (4.8), (4.9), (4.12) and (4.13), the elements of the matrices $M_{x}^{o}(T)$ and $M_{q}^{o}(T)$ are

$$m^{o}_{xij}(T) = \frac{r_{ij}}{\lambda_i - \delta_j} \left[1 - \frac{(\lambda_i - \delta_j)\ (e^{\lambda_i T} - 1)}{\lambda_i (e^{\lambda_i T} - e^{\delta_j T})}\right] \qquad (4.25)$$

and

$$m^{o}_{qij}(T) = \frac{s_{ij}}{\mu_i - \delta_j} \left[1 - \frac{(\mu_i - \delta_j)(e^{\mu_i T} - 1)}{\mu_i (e^{\mu_i T} - e^{\delta_j T})}\right] \qquad (4.26)$$

Therefore the matrix $\Theta^{o}(T)$ has elements of the form

$$\theta^{o}_{xij}(T) = \frac{r_{ij}}{\lambda_i - \delta_j} \left[1 - \frac{(\lambda_i - \delta_j)(e^{\lambda_i T} - 1)}{\lambda_i (e^{\lambda_i T} - e^{\delta_j T})}\right] \quad (e^{\lambda_i T} - e^{\delta_j T}) \qquad (4.27)$$

for

$$1 \leq i \leq n, \quad 1 \leq j \leq n + m$$

and

$$\theta^{o}_{qij}(T) = \frac{s_{ij}}{\mu_i - \delta_j}\left[1 - \frac{(\mu_i - \delta_j)(e^{\mu_i T} - 1)}{\mu_i (e^{\mu_i T} - e^{\delta_j T})}\right](e^{\mu_i T} - e^{\delta_j T}) \tag{4.28}$$

for

$$n + 1 \leq i \leq n + m, \quad 1 \leq j \leq n + m$$

These elements have the following property:

By application of the L'Hospital rule,

$$\lim_{T \to 0} \theta^{o}_{ij}(T) = 0$$

$$\lim_{\substack{\lambda_i/\mu_i \to 0 \\ \lambda_i/\mu_i \to \delta_i}} \theta_{ij}(T) = k(T), \text{ finite for } T \text{ finite.}$$

This property will be applied next to study the behavior of Eq. (4.21).

The first question concerning Eq. (4.21) refers to its stability. It is clear that it represents a forced, fixed, linear discrete time system at rest at the starting time k = o, but in which an excitation is present thereafter. According to [43], the stability of such a system is determined by the eigenvalues of the system matrix and by the forcing function. Therefore, for stability the eigenvalues $Z_i^o(T)$ of $\Lambda^o(T)$ must be:

$$|Z_i^o(T)| < 1$$

for

$$1 \leq i \leq n + m$$

Since these eigenvalues are functions of T, a root-locus analysis may be done in order to find the least value of T which will make the system unstable. For the present purposes it will be seen in Chapter V that the T chosen will be less than that maximum and therefore the scheme will be stable.

The nature of the forcing function will be studied next. As seen in Eq. (4.21), the forcing function is

$$w(kT) = \Phi_{\delta}(kT)w_1 \tag{4.29}$$

where $\Phi_{\delta}(kT)$ is a stable fundamental matrix, which for k approaching infinity, approaches zero. Therefore the forcing function is bounded, Eq. (4.21) is stable, and the error e(kT) approaches zero as k approaches infinity. The matrix $\Theta^{o}(T)$ does not affect the analysis of stability because as shown, it has the property of being stable for all possible eigenvalues.

A second observation concerns the behavior of Eq. (4.21) when T approaches zero. It can be seen that

$$\lim_{T \to 0} \Lambda^{o}(T) = I \text{ , unit matrix}$$

$$\lim_{T \to 0} \Theta^{o}(T) = 0, \text{ zero matrix}$$

Thus, with kT = t and T approaching zero, Eq. (4.21) becomes

$$\lim_{T \to 0} e(t+T) = e(t)$$

That is, for T = o, there is no sampling, and therefore the discrete-data system becomes the continuous system according to the model discussed in Chapter II. This is reflected in the fact that the error does not change and has the value of its initial condition which as seen in Eqs. (4.18) and (4.19) is zero.

In order to characterize the error in terms of parameters of the system, Eq. (4.21) must be solved. The solution of this equation is for $k \geq 1$ [43]

$$e(kT) = \sum_{n=o}^{k-1} [\Lambda^{o}(T)]^{(k-n-1)} \Theta^{o}(T)\ \Phi_{\delta}(nT) w_1 \qquad (4.30)$$

In order to see the meaning of each matrix, the original factors must be substituted in Eq. (4.30); then

$$e(kT) = \sum_{n=o}^{k-1} \begin{bmatrix} \Phi_{\lambda}(T) & A^{-1}[\Phi_{\lambda}(T)-I]BH \\ -F^{-1}[\Phi_{\mu}(T)-I]GC & \Phi_{\mu}(T) \end{bmatrix}^{(k-1-n)} \cdot \begin{bmatrix} \Phi_{\lambda}(T)M_x^{o}(T) - M_x^{o}(T)\Phi_{\delta}(T) \\ -[\Phi_{\mu}(T)M_q^{o}(T) - M_q^{o}(T)\Phi_{\delta}(T)] \end{bmatrix} \Phi_{\delta}(nT) w_1 \qquad (4.31)$$

Considering that a zero-order hold device has been used for modelling the discrete-data system, the $\Lambda^{o}(T)$ matrix appearing in Eq. (4.30) can be considered as a first-order approximation of its series expansion, because the method is first-order [48]. Assume then, that T is such that the expressions of the fundamental matrices can be evaluated by

$$\Phi_{\lambda/\mu}(T) = e^{(A/F)T} \simeq I + (A/F)\ T \qquad (4.32)$$

In this connection, it is worth noting that an exponential $e^{\lambda_i T}$ can be approximated within 1% by $(1 + \lambda_i T)$ if $\lambda_i T$ is less than 0.15; this bound is very liberal on $\lambda_i T$ for practical purposes. It will be seen in Chapter V that the selected T will be much less than this bound. Therefore Eq. (4.31) will be examined using the approximation in Eq. (4.32).

Consider first

$$\Lambda^{o}(T) = \begin{bmatrix} \Phi_{\lambda}(T) & A^{-1}[\Phi_{\lambda}(T)-I]BH \\ -F^{-1}[\Phi_{\mu}(T)-I]GC & \Phi_{\mu}(T) \end{bmatrix}$$

$$\simeq \begin{bmatrix} I + AT & A^{-1}ATBH \\ -F^{-1}FTGC & I + FT \end{bmatrix}$$

$$\simeq I + \begin{bmatrix} A & BH \\ -GC & F \end{bmatrix} T \qquad (4.33)$$

Comparing Eq. (4.33) with Eq. (3.5), it can be seen that

$$\Lambda^{o}(T) \simeq I + \mathcal{A}T \simeq \Phi_{\mathcal{A}}(T) \qquad (4.34)$$

which is the fundamental matrix of the closed-loop continuous system.

In order to express Eq. (4.34) in a more manageable form, the use of the similarity transformation of Eq. (3.11) is required. The fundamental matrix is

$$\Phi_{\mathcal{A}}(T) = P\Phi_{\delta}(T)\ P^{-1} \tag{4.35}$$

Then the state discretization error for the zero-order discrete-data system can be expressed, by considering Eqs. (4.34) and (4.35) as

$$e(kT) = P \sum_{n=o}^{k-1} \Phi_{\delta}[(k-1-n)T]P^{-1}\theta^{o}(T)\Phi_{\delta}(nT)w_1 \tag{4.36}$$

Of primary interest is the error in the plant, because it is the process to be controlled; thus the state discretization error becomes

$$e(kT) = P_n \sum_{n=o}^{k-1} \Phi_{\delta}[(k-1-n)T]P^{-1}\theta^{o}(T)\Phi_{\delta}(nT)w_1 \tag{4.37}$$

where P_n has the first n rows of the matrix P and defined in Eq. (3.15).

Before continuing a check on the stability of Eq. (4.37) can be made. It has been seen in this section that the stability depends on the eigenvalues of $\Lambda^o(T)$. From Eq. (4.34) it is clear that they are of the form $e^{\delta_j T}$ for small T, and due to the stability of the closed-loop system $|e^{\delta_i T}| < 1$. Therefore the scheme is stable for T small.

Consider now the factor $\Theta^o(T)$ of Eq. (4.37), which is given by Eq. (4.24), and study its behavior for small T. Take first the matrices $M_x^o(T)$ and $M_q^o(T)$ given by Eqs. (4.16) and (4.17) under consideration. As will be seen they play an important role in the determination of the sampling interval. Each will be called <u>error coefficient matrix</u> with elements

$$m_{xij}^o(T) = \frac{r_{ij}}{\lambda_i - \delta_j} \left[1 - \frac{(\lambda_i - \delta_j)(e^{\lambda_i T} - 1)}{\lambda_i (e^{\lambda_i T} - e^{\delta_j T})} \right] \qquad (4.38)$$

$$m_{qij}^o(T) = \frac{s_{ij}}{\mu_i - \delta_j} \left[1 - \frac{(\mu_i - \delta_j)(e^{\mu_i T} - 1)}{\mu_i (e^{\mu_i T} - e^{\delta_j T})} \right] \qquad (4.39)$$

As a first observation, it is seen that the factor $r_{ij}/(\lambda_i-\delta_j)$ describes how the control mode δ_j affects the plant mode λ_i. In order to have a better understanding of these expressions they may be expanded in Taylor's series about the point T = 0. Then, it is obtained that

$$m^{o}_{xij}(T) = \frac{r_{ij}}{\lambda_i-\delta_j}\left[\delta_j\frac{T}{2} - \delta_j(\delta_j+\lambda_i)\frac{T^2}{12}+\delta_j^2\lambda_i\frac{T^3}{24} + \ldots\right]$$

$$= \frac{r_{ij}}{\lambda_i-\delta_j} D^{o}_{xij}(T) \qquad (4.40)$$

$$m^{o}_{qij}(T) = \frac{s_{ij}}{\mu_i-\delta_j}\left[\delta_j\frac{T}{2} - \delta_j(\delta_j+\mu_i)\frac{T^2}{12}+\delta_j^2\mu_i\frac{T^3}{24} + \ldots\right]$$

$$= \frac{s_{ij}}{\mu_i-\delta_j} D^{o}_{qij}(T) \qquad (4.41)$$

From these expressions it can be seen

i) The magnitude of the discretization error is dependent on the magnitude of these elements and

$$\lim_{T \to 0} \; m^{o}_{x/qij}(T) = 0$$

meaning that it is zero for no sampling.

ii) The elements have a leading term in the first power of T meaning that the model of the discrete system is equivalent to a first-order method of numerical integration [48].

iii) The eigenvalues of the closed-loop system affects the expression in the first term.

iv) In order to keep the magnitude of the discretization error smaller than the effect of the control (i.e., the term $r_{ij}/\lambda_i - \delta_j$), it is reasonable to have a T such that

$$|D^{o}_{ij}(T)| \ll 1$$

Then taking the magnitudes of Eqs. (4.40) and (4.41)

and approximating them by the second order term, yields

$$|D^{o}_{xij}(T)| \leq \frac{T}{2} |\delta_j| + \frac{T^2}{12} |\delta_j| \ (|\delta_j|+|\lambda_i|) \tag{4.42}$$

$$|D^{o}_{qij}(T)| \leq \frac{T}{2} |\delta_j| + \frac{T^2}{12} |\delta_j| \ (|\delta_j|+|\mu_i|) \tag{4.43}$$

This approximation is very conservative for complex eigenvalues. It permits the definition of relative discretization error coefficient to be used in Chapter V.

Definition 4.3. The relative discretization error coefficient for the zero-order discrete-data system is

$$\gamma^{o}_{x}(T) = |\delta| \ \frac{T}{2} \ [1 + \frac{T}{6} \ (|\delta|+|\lambda|)] \tag{4.44}$$

for the plant, and

$$\gamma^{o}_{q}(T) = |\delta| \ \frac{T}{2} \ [1 + \frac{T}{6} \ (|\delta|+|\mu|)] \tag{4.45}$$

for the controller.

Where

$$|\delta| = \max_j \{|\delta_j|\} \tag{4.46}$$

$$|\lambda| = \max_i \{|\lambda_i|\} \tag{4.47}$$

$$|\mu| = \max_i \{|\mu_i|\} \tag{4.48}$$

Consider now the $\Theta(T)$ matrix. It will be called the error matrix. Its elements are given by Eqs. (4.27) and (4.28). Under the same assumption as for the matrix $M^o(T)$, the elements may be expanded in Taylor's series about $T = 0$, it is obtained

$$\theta^o_{xij}(T) = r_{ij}\ \delta_j \frac{T^2}{2}\ [1+(\delta_j+\lambda_i)\frac{T}{2} - (\delta_j+\lambda_i)^2\ \frac{T^2}{12} + \ldots] \tag{4.49}$$

and

$$\theta^{o}_{qij}(T) = s_{ij}\,\delta_j\,\frac{T^2}{2}\left[1 + (\delta_j+\mu_i)\frac{T}{2} - (\delta_j+\mu_i)^2\,\frac{T^2}{12} + \ldots\right] \tag{4.50}$$

Similar conclusions as the ones obtained for the matrix $M^{o}(T)$ can be observed by analyzing Eqs. (4.49) and (4.50). These results will be used in the next chapter for determination of criteria for choosing an acceptable sampling rate.

4.4. Discretization Error For First-Order Discrete-Data System

Consider now the definition of state discretization error, the expressions of the evolution of the continuous system, given by Eqs. (4.6) and (4.7), and the evolution of the discrete-data system, for the first-order polygonal hold reconstructor and repeated here

$$x_d(kT+T) = \Phi_\lambda(T)x(kT) - A^{-1}\{\Phi_\lambda(T) - \frac{A^{-1}}{T}[\Phi_\lambda(T)-I]\}BH\ e_q(kT)$$

$$- A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\}BH\ e_q(kT+T)$$

$$+ [\Phi_\lambda(T)Q_x^1(T) - Q_x^1(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \qquad (4.51)$$

$$q_d(kT+T) = \Phi_\mu(T)q(kT) + F^{-1}\{\Phi_\mu(T) - \frac{F^{-1}}{T}[\Phi_\mu(T)-I]\}GC\ e_x(kT)$$

$$+ F^{-1}\{\frac{F^{-1}}{T}[\Phi_\mu(T)-I]-I\}GC\ e_x(kT+T)$$

$$- [\Phi_\mu(T)Q_q^1(T) - Q_q^1(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \qquad (4.52)$$

where $e_x(kT)$ and $e_q(kT)$ are the state discretization errors and

$$Q_x^1(T)\Big|_{ij} = \frac{(1-e^{\lambda_i T})(1-e^{\delta_j T}) + \lambda_i T(e^{\lambda_i T}-e^{\delta_j T})}{\lambda_i^2 T(e^{\lambda_i T}-e^{\delta_j T})}\, r_{ij}$$

(4.23.a)

for $1 \leq i \leq n, \quad 1 \leq j \leq n + m$

$$Q_q^1(T)\Big|_{ij} = \frac{(1-e^{\mu_i T})(1-e^{\delta_j T}) + \mu_i T(e^{\mu_i T}-e^{\delta_j T})}{\mu_i^2 T(e^{\mu_i T}-e^{\delta_j T})} \; s_{ij} \tag{4.53}$$

for $1 \leq i \leq m, \quad 1 \leq j \leq n + m$

Direct application of the definition of state discretization error yields

$$\begin{aligned} e_x(kT+T) &= \Phi_\lambda(T)\, e_x(kT) \\ &+ A^{-1}\left\{\Phi_\lambda(T) - \frac{A^{-1}}{T}\left[\Phi_\lambda(T) - I\right]\right\} BH\, e_q(kT) \\ &+ A^{-1}\left\{\frac{A^{-1}}{T}\left[\Phi_\lambda(T) - I\right] - I\right\} BH\, e_q(kT+T) \\ &+ \left[\Phi_\lambda(T) M_x^1(T) - M_x^1(T)\Phi_\delta(T)\right]\, \Phi_\delta(kT) w_1 \end{aligned} \tag{4.54}$$

and

$$e_q(kT+T) = \Phi_\mu(T)e_q(kT)$$

$$- F^{-1}\{\Phi_\mu(T) - \frac{F^{-1}}{T}[\Phi_\mu(T) - I]\}GCe_x(kT)$$

$$- F^{-1}\{\frac{F^{-1}}{T}[\Phi_\mu(T) - I] - I\}GCe_x(kT+T)$$

$$- [\Phi_\mu(T)M_q^1(T) - M_q^1(T)\Phi_\delta(T)]\Phi_\delta(kT)w_1 \qquad (4.55)$$

where

$$M_x^1(T) = Q_x - Q_x^1(T); \quad [n \times (n+m)] \qquad (4.56)$$

$$M_q^1(T) = Q_q - Q_q^1(T); \quad [m \times (n+m)] \qquad (4.57)$$

and the initial errors are

$$e_x(t_o) = x(t_o) - x_d(t_o) = 0 \tag{4.58}$$

$$e_q(t_o) = q(t_o) - q_d(t_o) = 0 \tag{4.59}$$

In order to facilitate the notation Eqs. (4.54) and (4.55) may be expressed in vector-matrix form

$$\begin{bmatrix} e_x(kT+T) \\ e_q(kT+T) \end{bmatrix} = \begin{bmatrix} I & -A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\ BH \\ F^{-1}\{\frac{F^{-1}}{T}[\Phi_\mu(T)-I]-I\ GC & I \end{bmatrix}^{-1}$$

$$\cdot\{\begin{bmatrix} \Phi_\lambda(T) & A^{-1}\{\Phi_\lambda(T)-\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]\ BH \\ -F^{-1}\{\Phi_\mu(T)-\frac{F^{-1}}{T}[\Phi_\mu(T)-I]\}GC & \Phi_\mu(T) \end{bmatrix}\begin{bmatrix} e_x(kT) \\ e_q(kT) \end{bmatrix}$$

$$+ \begin{bmatrix} \Phi_\lambda(T)M_x^1(T) - M_x^1(T)\Phi_\delta(T) \\ -[\Phi_\mu(T)M_q^1(T) - M_q^1(T)\Phi_\delta(T)] \end{bmatrix} \Phi_\delta(kT)w_1\ \} \tag{4.60}$$

with the initial condition $e(0) = 0$.

Or

$$e(kT+T) = U(T)\Lambda^{1}(T)e(kT) + U(T)\Theta^{1}(T)\Phi_{\delta}(kT)w_{1} \tag{4.61}$$

where

$$U(T) = \begin{bmatrix} I & -A^{-1}\{\frac{A^{-1}}{T}[\Phi_{\lambda}(T)-I]-I\}BH \\ F^{-1}\{\frac{F^{-1}}{T}[\Phi_{\mu}(T)-I]-I\}GC & I \end{bmatrix}^{-1} \tag{4.62}$$

$$\Lambda^{1}(T) = \begin{bmatrix} \Phi_{\lambda}(T) & A^{-1}\{\Phi_{\lambda}(T)-\frac{A^{-1}}{T}[\Phi_{\lambda}(T)-I]\}BH \\ -F^{-1}\{\Phi_{\mu}(T)-\frac{F^{-1}}{T}[\Phi_{\mu}(T)-I]\}GC & \Phi_{\mu}(T) \end{bmatrix} \tag{4.63}$$

$$\Theta^{1}(T) = \begin{bmatrix} \Phi_{\lambda}(T)M_{x}^{1}(T) - M_{x}^{1}(T)\Phi_{\delta}(T) \\ -[\Phi_{\mu}(T)M_{q}^{1}(T) - M_{q}^{1}(T)\Phi_{\delta}(T)] \end{bmatrix} \tag{4.64}$$

Equation (4.61) represents a vector-matrix difference equation giving the discretization error at sampling times for the first-order discrete-data system.

To examine the behavior of Eq. (4.61), take the error matrix $\Theta^1(T)$. Its components $M_x^1(T)$ and $M_q^1(T)$ have elements

$$m_{xij}^1(T) = \frac{r_{ij}}{\lambda_i - \delta_j} \left\{1 - \frac{(\lambda_i-\delta_j)\left[(1-e^{\lambda_i T})(1-e^{\delta_j T})+\lambda_i T(e^{\lambda_i T}-e^{\delta_j T})\right]}{\lambda_i^2 T(e^{\lambda_i T}-e^{\delta_j T})}\right\} \tag{4.65}$$

and

$$m_{qij}^1(T) = \frac{s_{ij}}{\mu_i - \delta_j} \left\{1 - \frac{(\mu_i-\delta_j)\left[(1-e^{\mu_i T})(1-e^{\delta_j T})+\mu_i T(e^{\mu_i T}-e^{\delta_j T})\right]}{\mu_i^2 T(e^{\mu_i T}-e^{\delta_i T})}\right\} \tag{4.66}$$

Therefore the error matrix $\Theta^1(T)$ has elements of the form

$$\theta^1_{ij}(T) = m^1_{xij}(T)\ (e^{\lambda_i T} - e^{\delta_j T}) \tag{4.67}$$

for

$$1 \leq i \leq n, \quad 1 \leq j \leq n + m$$

and

$$\theta^1_{ij}(T) = m^1_{qij}(T)\ (e^{\mu_i T} - e^{\delta_j T}) \tag{4.68}$$

for

$$n + 1 \leq i \leq n + m, \ 1 \leq j \leq n + m$$

The analysis of the expressions obtained for the elements of $\Theta^1(T)$ leads to the same conclusions as

those obtained for the elements of the error matrix of the zero-order discrete-data system.

Concerning the stability of Eq. (4.61), the same procedure as in Section 4.3 can be applied. Consider first the system matrix $U(T)\Lambda^1(T)$. Its eigenvalues $z_i^1(T)$ determine the stability of the unforced scheme. Then the condition

$$| z_i^1(T) | < 1$$

for

$$1 \leq i \leq n + m$$

is necessary and sufficient for stability [43]. A root-locus analysis might be done in order to find the last value of T which will make the system unstable. As will be seen in Chapter V, the sampling interval T will be chosen below that maximum and therefore stability will be preserved.

The other term determining the stability is the forcing term. Comparing Eqs. (4.61) and (4.21) and considering the nature of the matrix $U(T)\Theta^{1}(T)$ it can be seen that the forcing function in both equations is the same. Therefore the same discussion as in Section 4.3 can be applied in this case and will yield in a stable scheme.

The behavior of Eq. (4.61) when T approaches zero will be studied next. It can be seen from Eqs. (4.62) and (4.63) that

$$\lim_{T \to 0} U(T)\Lambda^{1}(T) = I$$

and from the property of the $\Theta^{1}(T)$ matrix that

$$\lim_{T \to 0} \Theta^{1}(T) = 0$$

The meaning of this behavior, as in Section 4.3, is that when no sampling is made the system does not

introduce discretization error, or that according to Chapter II, the discrete-data system behaves as a continuous one.

The solution of Eq. (4.61) may be found directly by recursion [43]. It is for $k \geq 1$

$$e(kT) = \sum_{n=o}^{k-1} [U(T)\Lambda^{1}(T)]^{(k-n-1)} U(T)\Theta^{1}(T)\Phi_{\delta}(nT)w_{1} \tag{4.69}$$

In order to understand better Eq. (4.69) the original factors have to be substituted. Then

$$e(kT) = \sum_{n=o}^{k-1} \{ \begin{bmatrix} I & -A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\}BH \\ F^{-1}\{\frac{F^{-1}}{T}[\Phi_\mu(T)-I]-I\}GC & I \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} \Phi_\lambda(T) & A^{-1}\{\Phi_\lambda(T)-\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\}BH \\ -F^{-1}\{\Phi_\mu(T)-\frac{F^{-1}}{T}[\Phi_\mu(T)-I]-I\}GC & \Phi_\mu(T) \end{bmatrix}^{(k-n-1)} \}$$

$$\cdot \begin{bmatrix} I & -A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\}BH \\ F^{-1}\{\frac{F^{-1}}{T}[\Phi_\mu(T)-I]-I\}GC & I \end{bmatrix}^{-1}$$

$$\cdot \begin{bmatrix} \Phi_\lambda(T)\, M_q^1(T) - M_q^1(T)\Phi_\delta(T) \\ -[\Phi_\mu(T) M_q^1(T) - M_q^1(T)\Phi_\delta(T)] \end{bmatrix} \Phi_\delta(nT)\, w_1 \qquad (4.70)$$

Considering now that a first-order hold device has been used for modelling the discrete-data system, and recalling that this is equivalent to a second order numerical approximation of the continuous system [48], therefore

it is reasonable to evaluate for small sampling interval the fundamental matrix by

$$\Phi_{\lambda/\mu}(T) \simeq I + (A/F)T + (A/F)^2 \frac{T^2}{2} \tag{4.71}$$

and examine Eq. (4.70) under this assumption.

Consider first

$$U(T) = \begin{bmatrix} I & -A^{-1}\{\frac{A^{-1}}{T}[\Phi_\lambda(T)-I]-I\}BH \\ F^{-1}\{\frac{F^{-1}}{T}[\Phi_\lambda(T)-I]-I\}GC & I \end{bmatrix}^{-1}$$

$$\simeq \begin{bmatrix} I & -A^{-1}\{\frac{A^{-1}}{T}[AT+A^2\frac{T^2}{2}]-I\}BH \\ F^{-1}\{\frac{F^{-1}}{T}[FT+F^2\frac{T^2}{2}]-I\}GC & I \end{bmatrix}^{-1}$$

$$\simeq \{I + \frac{T}{2}\begin{bmatrix} o & -BH \\ GC & o \end{bmatrix}\}^{-1} \tag{4.72}$$

But for small T, the resulting matrix can be considered as an infinitesimal transformation matrix and be approximated by [46]

$$U(T) \simeq I - \frac{T}{2}\begin{bmatrix} o & -BH \\ GC & o \end{bmatrix} + \frac{T^2}{4}\begin{bmatrix} o & -BH \\ GC & o \end{bmatrix}^2 \tag{4.73}$$

Then

$$U(T) \simeq \begin{bmatrix} I - \frac{T^2}{4}BHGC & \frac{T}{2}BH \\ -\frac{T}{2}GC & I - \frac{T^2}{4}GCBH \end{bmatrix} \tag{4.74}$$

The same approximation can be applied to $\Lambda^1(T)$.

$$\Lambda^1(T) = \begin{bmatrix} \Phi_\lambda(T) & A^{-1}\{\Phi_\lambda(T) - \frac{A^{-1}}{T}[\Phi_\lambda(T)-I]\}BH \\ -F^{-1}\{\Phi_\mu(T) - \frac{F^{-1}}{T}[\Phi_\mu(T)-I]\}GC & \Phi_\mu(T) \end{bmatrix}$$

$$\simeq \begin{bmatrix} I + AT + A^2\frac{T^2}{2} & \frac{T}{2}BH + \frac{T^2}{4}ABH \\ -\frac{T}{2}GC - \frac{T^2}{4}FGC & I + FT + F^2\frac{T^2}{2} \end{bmatrix} \qquad (4.75)$$

In order to obtain the approximation for Eq. (4.69), Eqs. (4.74) and (4.75) must be multiplied, yielding

$$U(T)\Lambda^1(T) \simeq \begin{bmatrix} I+AT+(A^2-BHGC)\frac{T^2}{2} & BHT+(ABH+BHF)\frac{T^2}{2} \\ -GCT-(FGC+GCA)\frac{T^2}{2} & I+FT+(F^2-GCBH)\frac{T^2}{2} \end{bmatrix}$$

$$\simeq I + T\begin{bmatrix} A & BH \\ -GC & F \end{bmatrix} + \frac{T^2}{2}\begin{bmatrix} A^2-BHGC & ABH+BHF \\ -FGC-GCA & F^2-GCBH \end{bmatrix} \qquad (4.76)$$

Comparing Eq. (4.76) with (3.6.b) it can be seen that the obtained equation is essentially the fundamental matrix of the closed-loop continuous system for small T. Recalling the similarity transformation of Eq. (3.11) the system matrix of Eq. (4.61) can be approximated by

$$U(T)\Lambda^{1}(T) \simeq \Phi_{a}(T) = P\Phi_{\delta}(T)P^{-1} \tag{4.77}$$

Therefore, the state discretization error for the first-order discrete-data system can be expressed as

$$e(kT) = P\sum_{n=o}^{k-1} \Phi_{\delta}[(k-n-1)T]P^{-1}U(T)\Theta^{1}(T)\Phi_{\delta}(nT)w_{1} \tag{4.78}$$

Of primary interest is the error in the plant, because it is the controlled process; thus according to Eq. (3.15), Eq. (4.78) becomes

$$e(kT) = P_{n}\sum_{n=o}^{k-1} \Phi_{\delta}[(k-n-1)T]P^{-1}U(T)\Theta^{1}(T)\Phi_{\delta}(nT)w_{1} \tag{4.79}$$

where P_n are the first n rows of P.

The stability of Eq. (4.79) as it was seen in this section depends on the eigenvalues of $U(T)\Lambda^1(T)$. According to Eq. (4.77) for small T, they are given by $e^{\delta_j T}$. Due to the stability of the closed-loop system, it is $|e^{\delta_j T}| < 1$, therefore the scheme is stable.

Consider now the error matrix $\Theta^1(T)$ of Eq. (4.64), and study its behavior for small T. Take first the error coefficient matrices $M_x^1(T)$ and $M_q^1(T)$ given by Eqs. (4.56) and (4.57) with elements given by Eqs. (4.65) and (4.66), and analyze them by expanding in Taylor's series their elements about the point T = o. Then, it is obtained that

$$m_{xij}^1(T) = \frac{r_{ij}}{\lambda_i - \delta_j}$$

$$\cdot \left[-\delta_j^2 \frac{T^2}{12} + \delta_j^2(\delta_j^2 - 4\lambda_i\delta_j + \lambda_i^2)\frac{T^4}{720} + \ldots \right]$$

$$= \frac{r_{ij}}{\lambda_i - \delta_j} \cdot D_{xij}^1(T) \qquad (4.80)$$

and

$$m^1_{qij}(T) = \frac{s_{ij}}{\mu_i - \delta_j}$$

$$\cdot \left[- \delta_j^2 T^2 + \delta_j^2 (\delta_j^2 - 4\mu_i \delta_j + \mu_i^2) \frac{T^4}{720} + \cdots \right]$$

$$= \frac{s_{ij}}{\mu_i - \delta_j} \; D^1_{qij}(T) \qquad (4.81)$$

From these expressions it can be seen that

i) The magnitude of the discretization error depends on these elements and is zero for no sampling due to

$$\lim_{T \to o} m^1_{x/qij}(T) = o$$

ii) The elements have a leading term in the second power of T, this means that the model of the discrete system is equivalent to a second order method of numerical integration [48].

iii) The eigenvalues of the closed-loop system appear in the first term of the expansion.

iv) The relative discretization coefficient, under the same assumptions as in Section 4.3, is

$$\gamma^1(T) = |\delta|^2 \frac{T^2}{12} \tag{4.82}$$

where $|\delta|$ is given by Eq. (4.46).

Consider now the elements of the matrix $\Theta^1(T)$ given by Eqs. (4.67) and (4.68). The Taylor's expansion about T = o is

$$\theta^1_{xij}(T) = r_{ij}\ \delta_j^2 \frac{T^3}{12} \left[1 + (\delta_j + \lambda_i) \frac{T}{2} - (\delta_j^2 - 4\lambda_i\delta_j + \lambda_i^2) \frac{T^2}{60} + \ldots\right] \tag{4.83}$$

and

$$\theta^1_{qij}(T) = s_{ij}\ \delta_j^2 \frac{T^3}{12} \left[1 - (\delta_j + \mu_i) \frac{T}{2} - (\delta_j^2 - 4\mu_i\delta_j + \mu_i^2) \frac{T^2}{60} + \ldots\right] \tag{4.84}$$

These equations describe the behavior of the discretization error. They can be compared with the equivalent results for the zero-order discrete system and it can be observed that they have the same format with a difference in the power of T in the leading term of the series. The reason for this behavior is that they are equivalent to two different methods of numerical integration, one first order and the other second. These results will be used in the next chapter for determination of a criterion for choosing an acceptable sampling rate.

CHAPTER V

DETERMINATION OF SAMPLING RATES

5.1. Introduction

The determination of an acceptable sampling rate is related to a proper definition of a relative error. In Chapter IV, an absolute measure of the error introduced by discretizing a system was obtained but its value is not weighted with respect to the performance of the system. A relative error criterion is appropriate for comparing the behavior of the system with respect to a parameter characterizing it, but the determination of the proper parameter is very difficult and it is subject to interpretation.

In this chapter two relative error criteria will be presented. One relates a measure of the discretization error to the initial state of the system. The second is obtained by extending the relative error criterion presented in [11].

5.2. Relative Local State Discretization Error

In numerical analysis a basic measure of the accuracy of a method is the order of magnitude of the error introduced in each step of calculation [48]. In order to apply this concept to the present research, suppose that the exact solution of the state equation of a system at sampling instants (kT-T) and kT is given by x(kT-T) and x(kT) respectively and that the states of the discrete system are given by $x_d(kT-T)$ and $x_d(kT)$. Assume that at the sampling instant kT-T, the states of the two systems are the same; since the states at the next sampling instant will generally not be equal the following definition is appropriate.

Definition 5.1. The local state discretization error is defined as

$$LDE(kT) = \| x(kT) - x_d(kT) \|$$

if the states at the previous sampling instants are equal, i.e.

$$x(kT-T) = x_d(kT-T)$$

where $\|v\|$ stands for the norm of the vector.

For the purpose of this research, the definition of the norm is very important. The norm of a vector may be defined as in [42]

$$\text{(i)} \qquad M(v) = \max_{1 \le i \le n} |v_i|$$

$$\text{(ii)} \qquad S(v) = \sum_{i=1}^{n} |v_i|$$

$$\text{(iii)} \qquad E(v) = \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

with the corresponding norms for matrices given by:

$$\text{(i)} \qquad M(A) = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\text{(ii)} \qquad S(A) = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$\text{(iii)} \qquad E(A) = \sqrt{\text{Maximum eigenvalue of } A^*A}$$

From the three definitions above given, the norm (i) will be used. It simplifies the mathematical formulation and is not as conservative as the other two.

The definition of local state discretization error can be modified by assigning a proper weighting factor to each component of state discretization error. In this manner the effect of some states errors could be magnified and/or penalized during a desired time interval.

Another error definition will be introduced next. It will give a relative measure of the total discretization error introduced along the trajectory of the system.

Definition 5.2. The accumulated local state discretization error is defined as

$$ALDE = \sum_{k=o}^{\infty} LDE(kT)$$

This definition gives an idea of the system behavior. It is more restrictive than the local discretization error because it looks at the system during its evolution from the initial state until it reaches the final state. It gives a measure of the norm of the error matrix as will be seen in Sections 5.2.1 and 5.2.2.

The relative local state discretization error will be defined next. It is classical in control systems to relate the system behavior to the input control signal and not to the output. For the regulator problem this concept can be extended to a measure of the initial input, because as the states are evolving the input approaches zero. Consider the system under analysis. The plant error is of interest and its initial input is obtained from Eq. (3.5) as

$$u_p(t_o) = HP_m w_o \tag{5.1}$$

thus the following definition is appropriate.

Definition 5.3. The relative accumulated local state discretization error is defined as

$$\varepsilon_1 = \frac{ALDE}{\|HP_m w_o\|}$$

This definition gives a measure of how the discretization error is affected by the initial value of the input to the plant. It will be applied next to the zero and first-order discrete-data systems.

5.2.1. Zero-Order Discrete-Data System

Consider Definition 5.1 and Eq. (4.21). The local state discretization error for the zero-order discrete data system is

$$LDE^{o}(kT) = \| \Theta_{x}^{o}(T) \Phi_{\delta}(kT-T) w_1 \| \tag{5.2}$$

where $\Theta^{o}(T)$ is the error matrix given by Eq. (4.24).

The application of the Schwarz inequality to Eq. (5.2) yields

$$LDE^{o}(kT) \leq \| \Theta_{x}^{o}(T) \| \cdot \| \Phi_{\delta}(kT) \| \cdot \| w_1 \| \tag{5.3}$$

Consider now the norm of each factor of Eq. (5.3). Take first the norm of the error matrix. Using the selected definition of norm and the elements of $\Theta^{o}(T)$ given by Eq. (4.49), it is

$$\left\| \theta_x^o(T) \right\| = \max_{1 \le i \le n} \sum_{j=1}^{n+m} \left| r_{ij}\, \delta_j \frac{T^2}{2} \left[1 + (\delta_j + \lambda_i)\frac{T}{3} - (\delta_j + \lambda_i)^2 \frac{T^2}{12} + \ldots \right] \right| \tag{5.4}$$

By using the Schwarz inequality and approximating the series by the first two terms as justified in Chapter IV, Eq. (5.4) can be reduced to

$$\left\| \theta_x^o(T) \right\| \le \frac{T^2}{2} |\delta| \left[1 + \frac{T}{3}(|\delta| + |\lambda|)\right] \cdot \max_{1 \le i \le n} \sum_{j=1}^{n} |r_{ij}| \tag{5.5}$$

where $|\delta|$ and $|\lambda|$ are the maximum eigenvalue magnitudes given by Eq. (4.46) and (4.47). Consider the matrix R given by Eq. (3.25) and the proper definition of norm. Eq. (5.5) can then be expressed as

$$\left\| \theta_x^o(T) \right\| \le \frac{T^2}{2} |\delta| \left[1 + \frac{T}{3}(|\delta| + |\lambda|)\right] \left\| BHP_m \right\| \tag{5.6}$$

Consider now the norm of $\Phi_\delta(kT-T)$. The use of the proper definition of norm of a matrix yields

$$\| \Phi_\delta(kT-T) \| = e^{-d(kT-T)} \tag{5.7}$$

where

$$d = -\min_{1\leq j\leq n+m} [Re(\delta_j)] \tag{5.8}$$

Therefore considering Eqs. (5.6) and (5.8) the local state discretization error is

$$LDE^{o}(kT) \leq \frac{T^2}{2} |\delta| \left[1 + \frac{T}{3}(|\delta|+|\lambda|)\right] \|BHP_m\| \| w_1 \| e^{-d(kT-T)} \tag{5.9}$$

Consider now the accumulated local state discretization error. The application of Definition 5.2 to Eq. (5.9) yields

$$\mathrm{ALDE}^{o} \leq \frac{T^2}{2} |\delta| \left[1 + \frac{T}{3} (|\delta| + |\lambda|)\right] \cdot \| \mathrm{BHP}_m \| \cdot \| w_1 \| \sum_{i=1}^{\infty} e^{-d(kT-T)} \tag{5.10}$$

The infinite sum in the right side of inequality (5.10) can be evaluated as,

$$\sum_{k=1}^{\infty} e^{-d(kT-T)} = \frac{1}{1-e^{-dT}} \simeq \frac{1}{dT} (1-d\tfrac{T}{2}) \tag{5.11}$$

for $d\frac{T}{2} << 1$. This approximation is proper as will be seen for the values of T to be chosen. Substituting Eq. (5.11) into (5.10) the accumulated local state discretization error for the zero-order discrete system is

$$\text{ALDE}^{o} \leq \frac{T}{2} \frac{|\delta|}{d} \left[1 + \frac{T}{3} \left(|\delta| + |\lambda| - \frac{3}{2} d\right)\right]$$

$$\cdot \| \text{BHP}_m \| \cdot \| w_1 \| \qquad (5.12)$$

Finally, consider the relative accumulated local state discretization error given by Definition 5.3 and apply it to Eq. (5.12). It is

$$\varepsilon_1^{o} \leq \frac{T}{2} \frac{|\delta|}{d} \left[1 + \frac{T}{3} \left(|\delta| + |\lambda| - \frac{3}{2} d\right)\right] N \qquad (5.13)$$

where

$$N = \frac{\| \text{BHP}_m \| \quad \| w_1 \|}{\| \text{HP}_m w_o \|} \qquad (5.13a)$$

The parameter N depends on the topology of the system and is available to the designer from its equations.

The relative error expression obtained above will be used next to determine sampling rates, but first a similar expression for the first-order discrete system will be obtained.

5.2.2. First-Order Discrete-Data System

Consider the Definition 5.1 and Eq. (4.61). The local state discretization error for the first-order discrete-data system is

$$LDE^1(kT) = \| U(T)\Theta_x^1(T)\Phi_\delta(kT-T)w_1 \| \tag{5.14}$$

The application of the Schwarz inequality yields

$$LDE^1(kT) \leq \| U(T) \| \cdot \| \Theta_x^1(T) \| \cdot \| \Phi_\delta(kT-T) \| \cdot \| w_1 \| \tag{5.15}$$

Consider now the norm of each factor. Take first U(T). According to Eq. (4.73), the norm of U(T) is

$$\| U(T) \| = 1 + \frac{T}{2} m_1 + \frac{T^2}{4} m_2 \tag{5.16}$$

where

$$m_1 = M \left\{ \begin{bmatrix} o & -BH \\ GC & o \end{bmatrix} \right\} \tag{5.17}$$

$$m_2 = M \left\{ \begin{bmatrix} o & -BH \\ GC & o \end{bmatrix}^2 \right\} \tag{5.18}$$

are functions known to the designer.

Consider next the norm of the error matrix $\Theta_x^1(T)$ with elements given by Eq. (4.83).

$$\| \Theta_x^1(T) \| = \max_{1 \le i \le n} \sum_{j=1}^{n+m} | r_{ij} \, \delta_j^2 \, \frac{T^3}{12} \left[1 + (\delta_j + \lambda_i) \, \frac{T}{2} \right.$$

$$\left. - (\delta_j^2 - 4\lambda_i \delta_j + \lambda_i^2) \, \frac{T^2}{60} + \ldots \right] | \tag{5.19}$$

By using the Schwarz inequality and approximating the series by the first two terms as justified in Chapter IV, Eq. (5.19) is reduced to

$$\| \Theta_x^1(T) \| \leq \frac{T^3}{12} |\delta|^2 \left[1 + \frac{T}{2} (|\delta| + |\lambda|)\right] \| BHP_m \| \tag{5.20}$$

where $\| BHP_m \|$ is obtained from Eq. (3.25).

The norm of $\Phi_\delta(kT-T)$ has been obtained in Eq. (5.7). Therefore considering Eqs. (5.16), (5.20) and (5.7), the local state discretization error is

$$LDE^1(kT) \leq \frac{T^3}{12} |\delta|^2 \left[1 + \frac{T}{2} (|\delta| + |\lambda| + m_1)\right] \| BHP_m \| \cdot \| w_1 \| e^{-d(kT-T)} \tag{5.21}$$

Consider now the accumulated local state discretization error. The application of Definition 5.2 to Eq. (5.21) yields

$$ALDE^1 \leq \frac{T^3}{12} |\delta|^2 \left[1 + \frac{T}{2} (|\delta|+|\lambda|+m_1)\right] \|BHP_m\|$$

$$\cdot \|w_1\| \sum_{k=1}^{\infty} e^{-d(kT-T)} \qquad (5.22)$$

With the same considerations as in Section 5.2.1, the sum of the right member of Eq. (5.22) can be evaluated. Thus, the accumulated local state discretization error for the first-order discrete system is

$$ALDE^1 \leq \frac{T^2}{12} \frac{|\delta|^2}{d} \left[1 + \frac{T}{2} (|\delta|+|\lambda|+m_1-d)\right]$$

$$\cdot \|BHP_m\| \cdot \|w_1\| \qquad (5.23)$$

Finally, consider the relative accumulated local state discretization error given by Definition 5.3 and apply it to Eq. (5.23).

$$\varepsilon_1^1 \leq \frac{T^2}{12} \frac{|\delta|^2}{d} \left[1 + \frac{T}{2} (|\delta|+|\lambda|+m_1-d)\right] N \qquad (5.24)$$

where N is given by Eq. (5.13a).

Equations (5.12) and (5.24) will be used in Section 5.4 for determining an appropriate sampling rate.

5.3. Normalized Sum of Squared Output Discretization Error

Another manner of defining a relative measure of the effect of the discretization error in the evolution of the plant can be obtained by using the procedure of the previous research [11]. Kang defines an error criterion called normalized sum of squared output discretization error defined as

$$\varepsilon_2 = \sqrt{\frac{\sum_{k=o}^{\infty} |y(kT) - y_d(kT)|^2}{\sum_{k=o}^{\infty} |y(kT)|^2}} \qquad (5.25)$$

where $y(kT)$ and $y_d(kT)$ are the output of the continuous and discrete systems respectively.

For absolutely stable closed-loop control systems, the infinite sums in the numerator and denominator can be evaluated by means of the complex convolution theorem and the Cauchy residue theorem [57]. However, as it was shown in [11], it is of interest to approximate the value of ε_2 for relatively small values of the sampling interval. Therefore the same procedure as used in [11] will be followed here.

Consider first the output of the continuous system. According to Eq. (3.1) and (3.16) it can be expressed as

$$y(kT) = CP_n \Phi_\delta(kT) w_1 \tag{5.26}$$

then

$$\sum_{k=o}^{\infty} y^*(kT)y(kT) = \sum_{k=o}^{\infty} \sum_{i=1}^{l} \left| \sum_{j=1}^{n+m} d_{ij}\, e^{\delta_j kT} w_{1j} \right|^2 \tag{5.27}$$

where * stands for transpose-conjugate and d_{ij} are the elements of the CP_n matrix.

The output discretization error according to Definition 4.2 is

$$e_y(kT) = C\, e_x(kT) \tag{5.28}$$

where $e_x(kT)$ is given by Eqs. (4.37) and (4.78). Thus,

$$e_y(kT) = CP_n \sum_{n=o}^{k-1} \Phi_\delta[(k-n-1)T]P^{-1}\theta(T)\Phi_\delta(nT)w_1 \tag{5.29}$$

and

$$\sum_{k=o}^{\infty} e_y^*(kT)e_y(kT) = \sum_{k=o}^{\infty} \sum_{i=1}^{l} \left| \sum_{j=1}^{n+m} \sum_{p=1}^{n+m} d_{ip} \sum_{n=o}^{k-1} \Phi_\delta[(k-n-1)T]P^{-1} \theta(T)\ \Phi_\delta(nT)w_1 \right|^2 \tag{5.30}$$

Equations (5.27) and (5.30) are similar to the expressions obtained in [11], and therefore for small

values of the sampling interval, the normalized sum of squared output discretization error can be approximated by the relative discretization error coefficient defined by Eqs. (4.44) and (4.82). That is

$$\varepsilon_2 \simeq \gamma^i (T) \tag{5.31}$$

where i is zero and one for the zero and first-order discrete system respectively.

The normalized sum of squared output discretization error can be modified by assigning a proper weight to each component of the output discretization error as suggested in Section 5.2.

Considering the two obtained relative measurements of the discretization error given by Eqs. (5.12), (5.24) and (5.31), the determination of sampling intervals can be obtained, as it will be seen next.

5.4. Determination of Sampling Rates. Error Criterion I.

Consider the expressions for the relative accumulated local state discretization error given by Eqs. (5.12) and (5.24). Take first the zero-order discrete-data system. The relative error is

$$\varepsilon_1^o = \frac{T}{2}\frac{|\delta|}{d}\left[1 + \frac{T}{3}\left(|\delta|+|\lambda| - \frac{3}{2}d\right)\right] N$$

The sampling interval T can be selected by making the relative accumulated local state discretization error less than an allowed error ε. Then

$$\frac{T}{2}\frac{|\delta|}{d}\left[1 + \frac{T}{3}\left(|\delta| + |\lambda| - \frac{3}{2}d\right)\right] N \leq \varepsilon \qquad (5.32)$$

A quadratic equation in T has been obtained. The solution is

$$T \leq \frac{-\frac{3}{2} + \frac{3}{2}\sqrt{1 + \frac{8}{3} d \left(1 + \frac{|\lambda|}{N|\delta|} - \frac{3}{2}\frac{d}{N|\delta|}\right)\varepsilon}}{N|\delta| \left(1 + \frac{|\lambda|}{N|\delta|} - \frac{3}{2}\frac{d}{N|\delta|}\right)} \tag{5.33}$$

This expression gives an upper bound for T for an accepted error ε. The radical with the minus sign is not considered because it gives a negative T.

In most of the cases an approximation to Eq. (5.33) can be made. If

$$\frac{8}{3} d \left(1 + \frac{|\lambda|}{N|\delta|} - \frac{3}{2} \frac{d}{N|\delta|}\right) \varepsilon << 1$$

the approximation

$$\sqrt{1 + a} \simeq 1 + \frac{1}{2} a, \text{ for } a << 1$$

can be used, and Eq. (5.33) becomes

$$T \leq \frac{2\,d\,\varepsilon}{|\delta|\,N} \tag{5.34}$$

An upper bound for T has thus been obtained. As can be seen it is a function of the eigenvalues, the topology of the system and an allowed performance error.

Consider now the first-order discrete-data system. The relative accumulated local state discretization error is

$$\varepsilon_1^1 = \frac{T^2}{12}\frac{|\delta|^2}{d}\left[1 + \frac{T}{2}\left(|\delta|+|\lambda|+m_1-d\right)\right]N$$

The sampling interval can be selected as before by making the relative error less than an allowed error ε. Then

$$\frac{T^2}{12}\frac{|\delta|^2}{d}\left[1 + \frac{T}{2}\left(|\delta|+|\lambda|+m_1-d\right)\right]N \leq \varepsilon \tag{5.35}$$

This expression is a cubic equation in T with a negative constant term, implying the existence of a real solution for positive T. The solution can be obtained by any computer algorithm or a graphical method. The sampling interval T obtained is an upper bound for an accepted error ε and is a function of the eigenvalues of the system.

5.5. Determination of Sampling Rates. Error Criterion II.

Consider now the expressions of the normalized sum of squared output discretization error given by Eq. (5.31) and the relative discretization error coefficient defined by Eqs. (4.44) and (4.82). Take first the zero-order discrete-data system. The relative measure of the error is

$$\varepsilon_2^o = |\delta| \frac{T}{2} \left[1 + \frac{T}{6} (|\delta| + |\lambda|)\right] \tag{5.36}$$

The sampling interval can be selected by making this error less than an allowed overall performance error ε, that is

$$|\delta| \frac{T}{2} \left[1 + \frac{T}{6} (|\delta| + |\lambda|)\right] \leq \varepsilon$$

A quadratic equation in T has been obtained. The solution is

$$T \leq \frac{-3+3\sqrt{1 + \frac{4}{3}\left(1 + \frac{|\lambda|}{|\delta|}\right) \varepsilon}}{|\delta| \left(1 + \frac{|\lambda|}{|\delta|}\right)} \qquad (5.37)$$

This expression gives an upper bound for T for an accepted error ε.

In most of the cases an approximation to Eq. (5.37) can be made. Consider the case

$$\frac{4}{3}\left(1 + \frac{|\lambda|}{|\delta|}\right) \varepsilon \ll 1$$

The approximation used in Section 5.4 can be used and the sampling interval is approximated by

$$T \leq \frac{2\varepsilon}{|\delta|} \tag{5.38}$$

An upper bound for T has thus been obtained. As seen, it depends on the eigenvalues of the closed-loop system and the overall performance error.

Consider now the first-order discrete-data system. The normalized sum of squared output discretization error is according to Eqs. (5.31) and (4.82)

$$\varepsilon_2^1 = |\delta|^2 \frac{T^2}{12} \tag{5.39}$$

The sampling interval can be determined by making this error less than an allowed overall performance error ε. Then

$$|\delta|^2 \frac{T^2}{12} \leq \varepsilon \tag{5.40}$$

or

$$T \leq \frac{\sqrt{12\varepsilon}}{|\delta|} \tag{5.41}$$

An upper bound for T has thus been obtained. As before, it is related to the eigenvalues of the closed-loop system and the overall performance error.

Two very simple formulas have been obtained for determining sampling rates. They show that the sampling interval depends primarily on the eigenvalues of the closed-loop system and they give an explicit relation between the error, the sampling interval and the parameters of the system. Their application to practical cases solve the problem of determining a sampling interval for systems which do not have band-limited characteristics. Therefore, the engineering criterion for limiting the bandwidth of an actual system, for applying the Nyquist-Shannon sampling theorem, is not required.

5.6. Examples

Consider now the expressions derived for the sampling interval T, and apply them to some examples.

Example 5.1: A fifth order closed-loop control system has been chosen for this purpose and it is shown in Fig. 5.1.

The system is characterized by:

(i) Plant

$$\dot{x}(t) = Ax(t) + B\, u_p\,(t)$$

$$y_p(t) = C\, x\,(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -25 & -15 & -7 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 4 \\ -33 \end{bmatrix} \; ; \qquad C = [\,1 \quad 0 \quad 0\,]$$

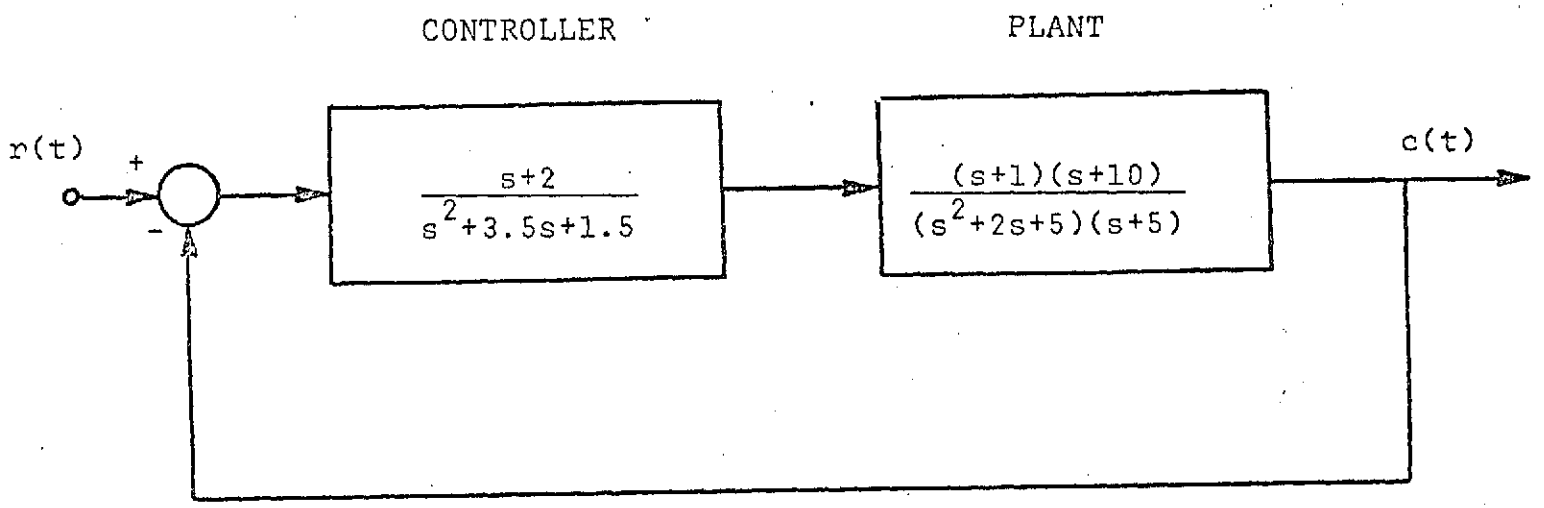

FIGURE 5.1 A FIFTH-ORDER CLOSED-LOOP CONTROL SYSTEM (Example 5.1)

$$x_o = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Eigenvalues

$$\lambda_1 = -5$$

$$\lambda_{2/3} = -1 \pm j2$$

(ii) Controller

$$\dot{q}(t) = F\, q(t) + G\, u_c(t)$$

$$y_c(t) = H\, q(t)$$

where

$$F = \begin{bmatrix} o & 1 \\ -1.5 & -3.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} \quad ; \qquad C = [\,1 \quad 0\,]$$

$$q_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Eigenvalues

$$\mu_1 = -3$$

$$\mu_2 = -0.5$$

(iii) Closed-loop system

$$\dot{z}(t) = \mathcal{A}\, z(t)$$

$$z(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}$$

$$a = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ -25 & -15 & -7 & -33 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 1.5 & 0 & 0 & -1.5 & -3.5 \end{bmatrix}$$

$$z_o = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Eigenvalues

$$\delta_1 = -5.273717$$

$$\delta_{2/3} = -0.936565 \pm j2.365428$$

$$\delta_4 = -2.737927$$

$$\delta_5 = -0.615265$$

(iv) Selection of the sampling rate.

Since the output is to be observed, error criterion II is selected. The maximum eigenvalues magnitudes needed are

$$|\delta| = 5.273717$$

$$|\lambda| = 5$$

and let the acceptable error be five percent, that is

$$\varepsilon = 0.05$$

a) Zero-Order Discrete-Data System

For the given data the approximate expression for T can be used, then by Eq. (5.49),

$$T \leq \frac{2\varepsilon}{|\delta|} = \frac{0.1}{5.273717} \simeq 0.019$$

Take

$$T = 0.02$$

b) First-Order Discrete-Data System

The sampling interval is according to Eq. (5.51)

$$T \leq \frac{\sqrt{12\varepsilon}}{|\delta|} = \frac{\sqrt{0.6}}{5.273717} \approx 0.143$$

Take

$$T = 0.15$$

(v) Simulation

The evolution of the states was calculated from Eqs. (3.44) and (3.45) for the zero-order discrete system and Eqs. (3.70) and (3.71) for the first-order discrete system. Both evolutions are compared against the exact solution.

The exact and approximate value of the normalized sum of squared output discretization errors are obtained by simulation over a time span of t=o to $t=t_f$ such that $y_p(t_f)<10^{-70}$. The results are shown in Figures 5.4 and 5.5.

(vi) Results

As can be seen in Figures 5.2 and 5.3, a very close approximation to the exact solution has been obtained. A very crude measurement might be obtained by considering the relative error of the maximum output discretization error with respect to the output at that instant of time. The relative errors are 3.05% and 1.25% for the zero and first-order discrete data systems respectively.

The normalized sum of squared output discretization error, is within acceptable value, it resulted in

$$\varepsilon_2^o = 5.23\%$$

for the zero-order discrete data system, and

$$\varepsilon_2^1 = 4.88\%$$

for the first-order discrete data system. The results can be considered acceptable for practical applications.

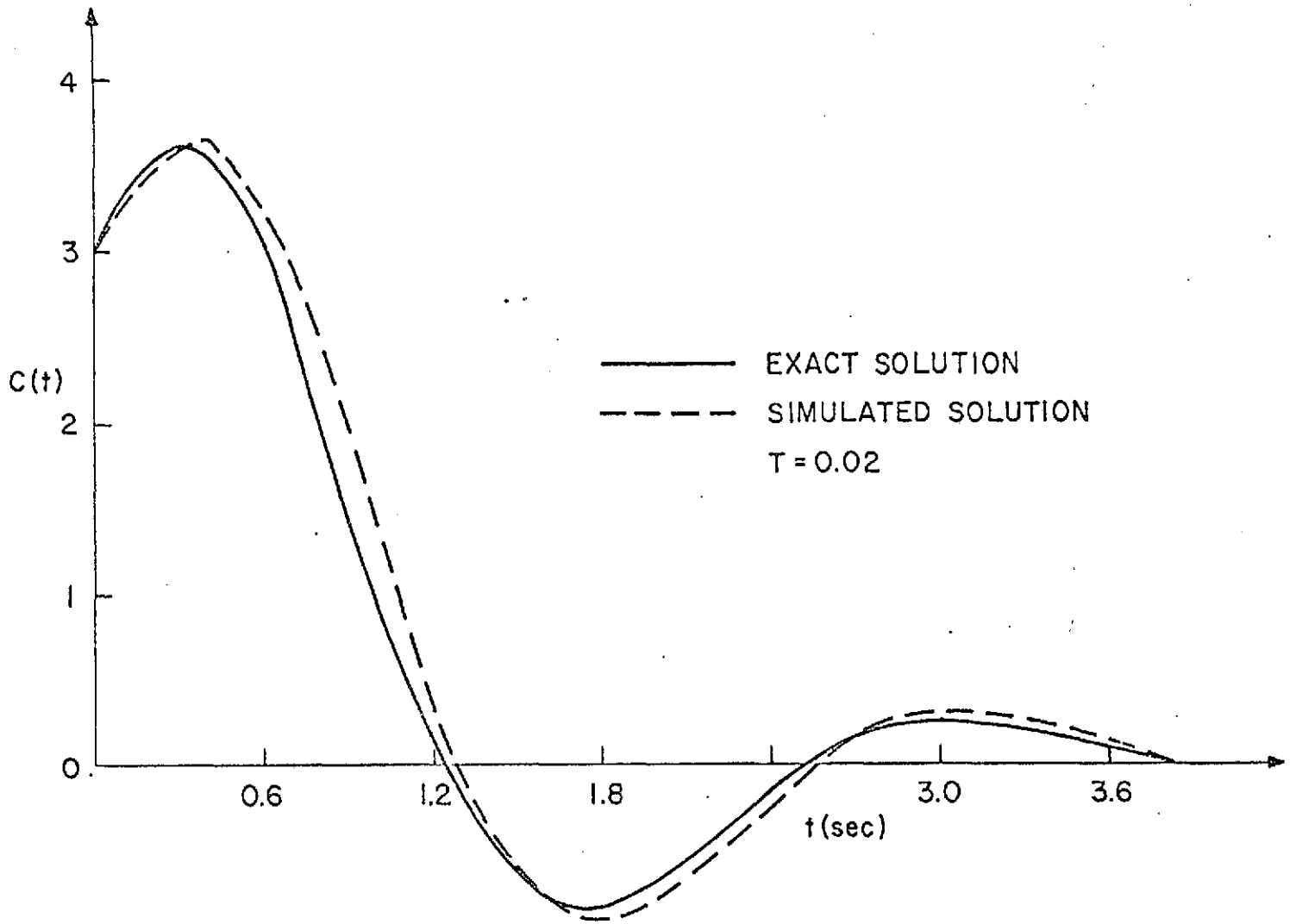

FIG. 5.2 ZERO-ORDER DISCRETE-DATA SYSTEM
(Example 5.1)

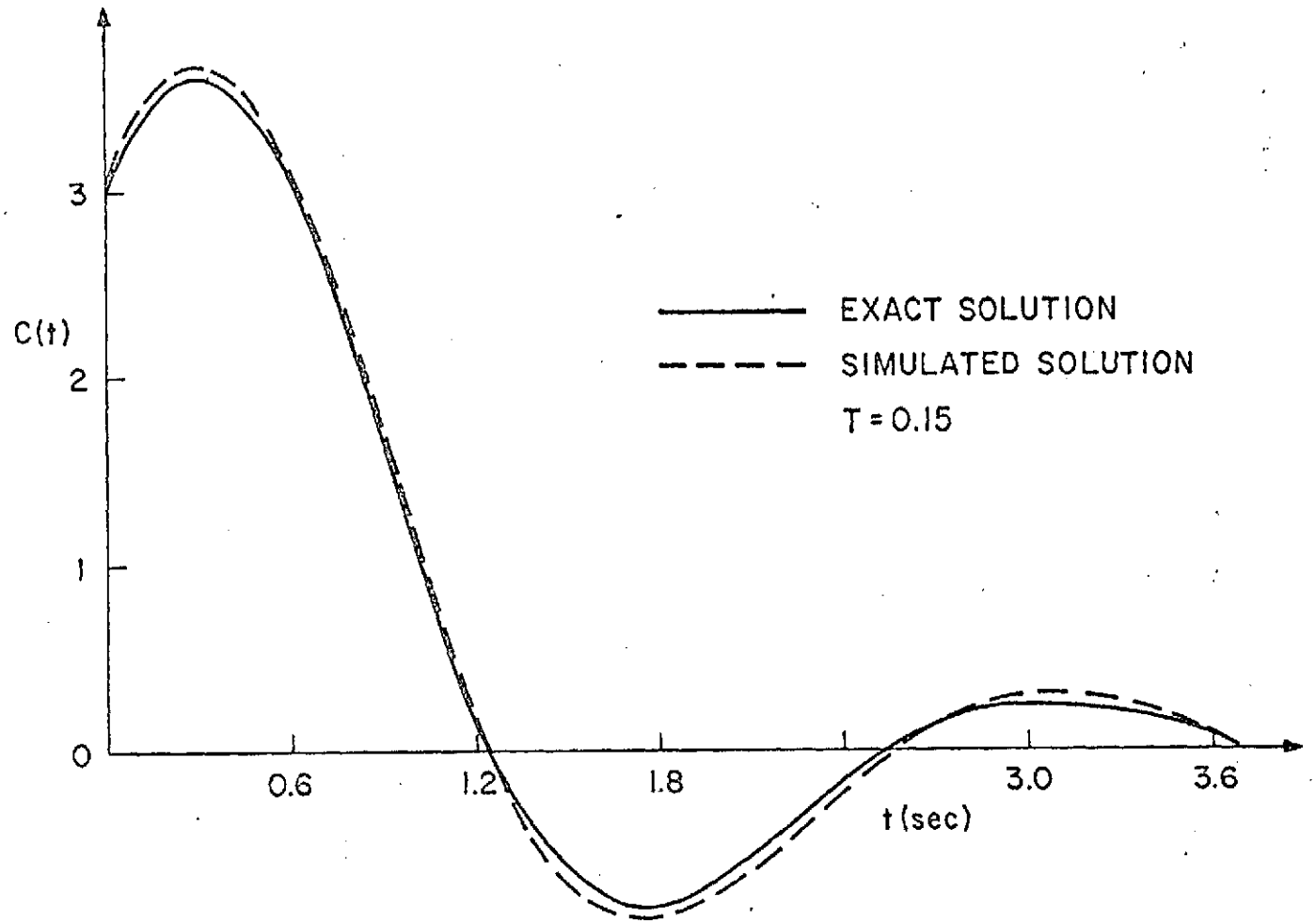

FIG. 5.3 FIRST-ORDER DISCRETE-DATA SYSTEM
(Example 5.1)

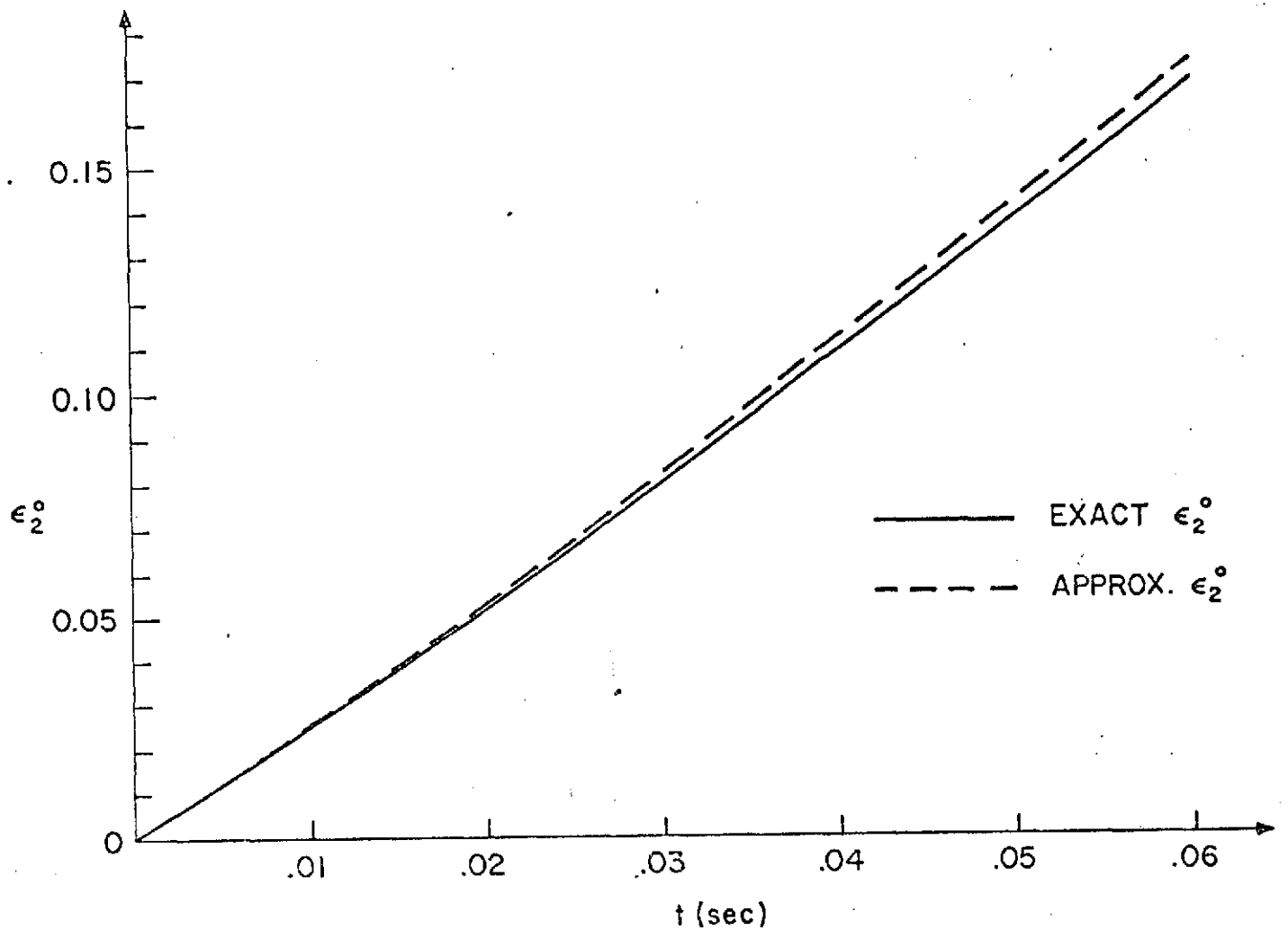

FIG. 5.4 EXACT AND APPROXIMATE NORMALIZED SUM OF SQUARED OUTPUT DISCRETIZATION ERROR IN THE ZERO-ORDER DISCRETE-DATA SYSTEM (Example 5.1)

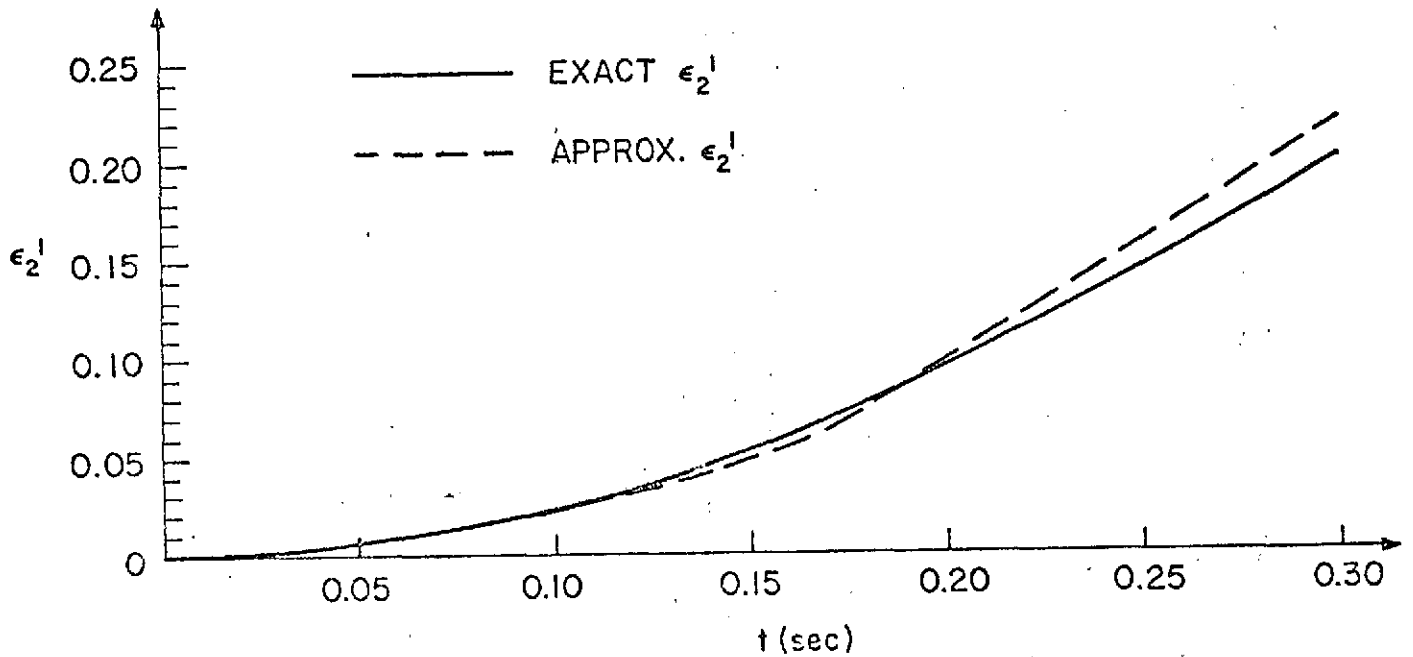

FIG. 5.5 EXACT AND APPROXIMATE NORMALIZED SUM OF SQUARED OUTPUT DISCRETIZATION ERROR IN THE FIRST-ORDER DISCRETE-DATA SYSTEM (Example 5.1)

Example 5.2: A follow-up (position) control system is shown in Figure 5.6. A correcting controller was introduced in the forward path to improve the dynamic response of the control system. The object of the designer is to digitalize the analog controller by using the models presented in this research.

The closed-loop control system is characterized by:

(i) Plant

Dynamics

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u_p(t)$$

$$y_p(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Initial Conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

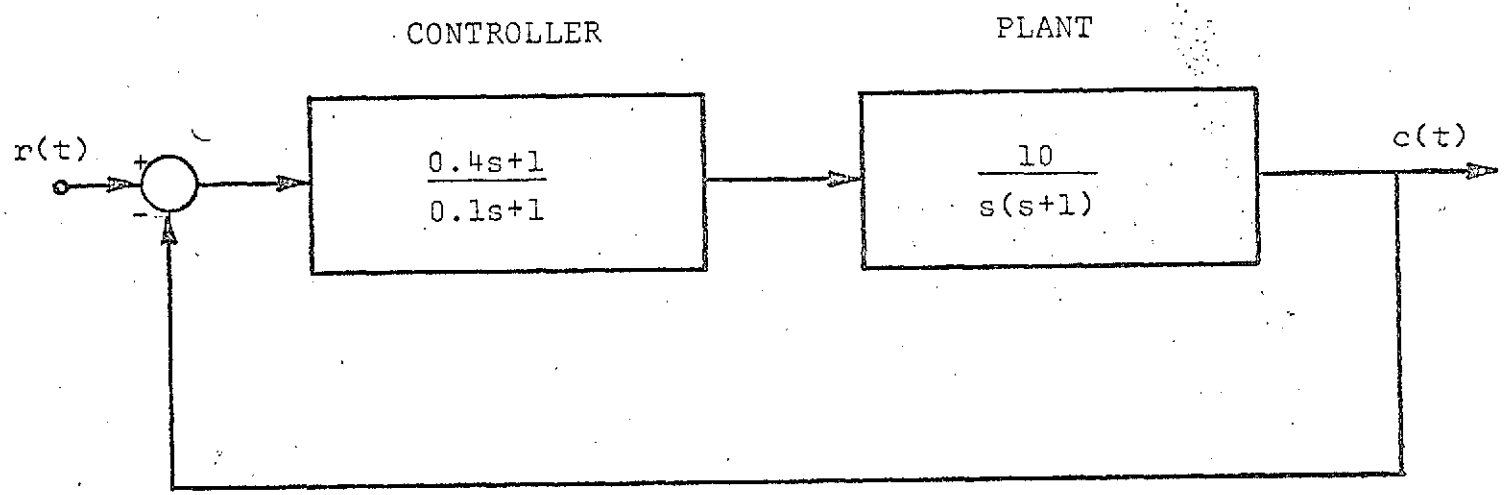

FIGURE 5.6 POSITION SERVO (Example 5.2)

Eigenvalues

$$\lambda_1 = 0$$

$$\lambda_2 = -1$$

(ii) Controller

Dynamics

$$\dot{q}(t) = -10\, q(t) - 30\, u_c(t)$$

$$y_c(t) = q(t) + 4\, u_c(t)$$

Initial Condition

$$q(0) = -1$$

Eigenvalue

$$\mu = -10$$

(iii) Closed-Loop System

$$\dot{z}(t) = \mathcal{A}\, z(t)$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ -40 & -1 & 10 \\ 30 & 0 & -10 \end{bmatrix}$$

Initial Conditions

$$z(0) = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

Eigenvalues

$$\delta_1 = -5$$
$$\delta_2 = -3 + j\sqrt{11}$$
$$\delta_3 = -3 - j\sqrt{11}$$

(iv) Selection of the Sampling Rate.

Since the output is to be observed, select error criterion II. The maximum eigenvalue magnitudes are

$$|\delta| = 5$$

$$|\lambda| = 1$$

and let the acceptable error be five per cent as before, that is

$$\varepsilon = 0.05$$

For the zero-order discrete data system,

$$T \leq 0.02$$

and for the first-order discrete data system,

$$T \leq 0.155.$$

(v) Simulation

The evolution of the states was obtained by numerical simulation. The outputs are plotted in Figure 5.7 for the zero-order system and Figure 5.8 for the first-order discrete-data system. The normalized sum of squared output discretization error is for the zero-order system.

$$\varepsilon_2^0 = 4.795\%$$

and

$$\varepsilon_2^1 = 4.93\%$$

for the first-order discrete-data system.

Again here, the application of the obtained formulas for selecting the sampling interval T, leads to an acceptable result.

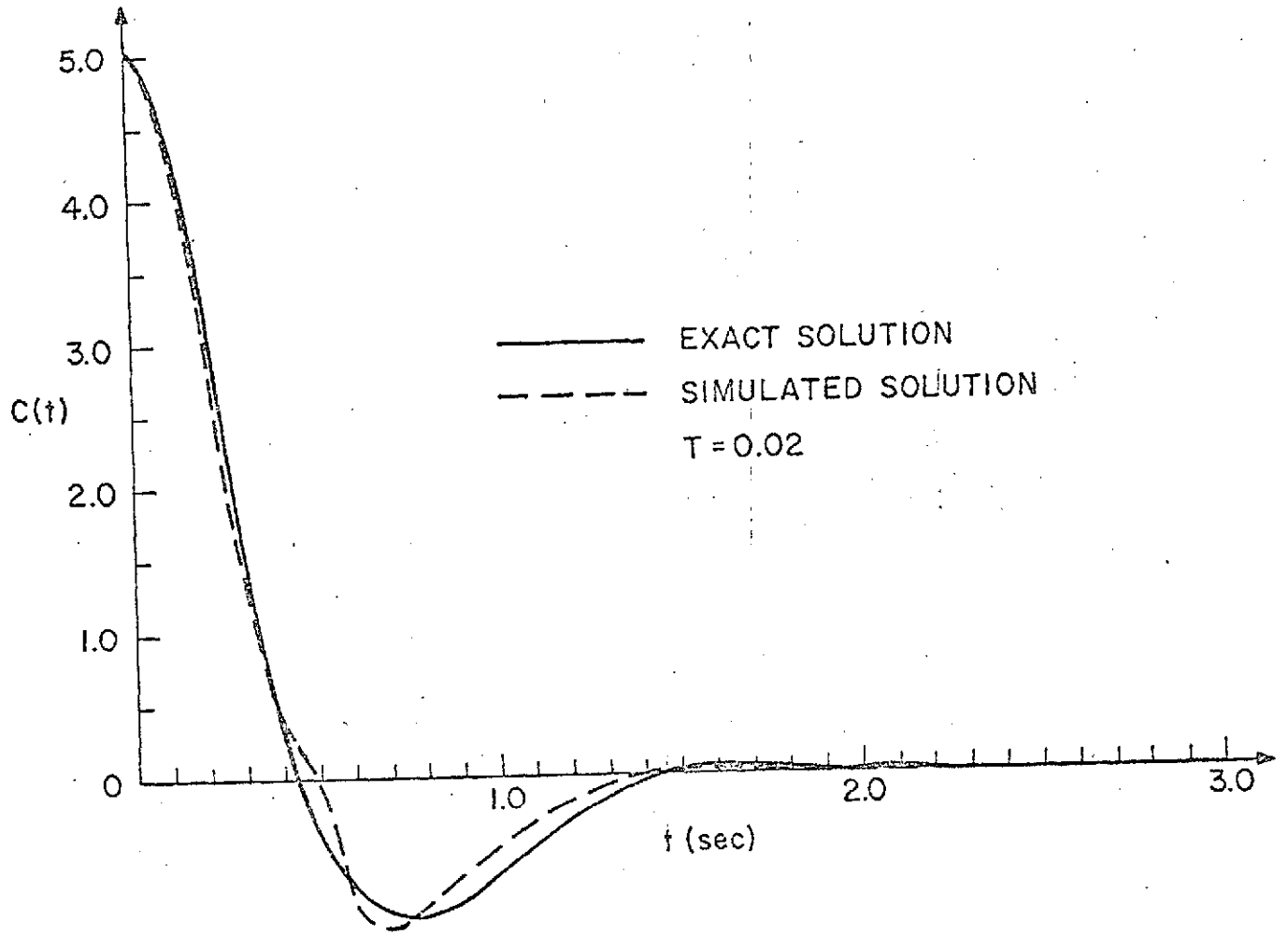

FIG. 5.7 ZERO-ORDER DISCRETE-DATA SYSTEM
(Example 5.2)

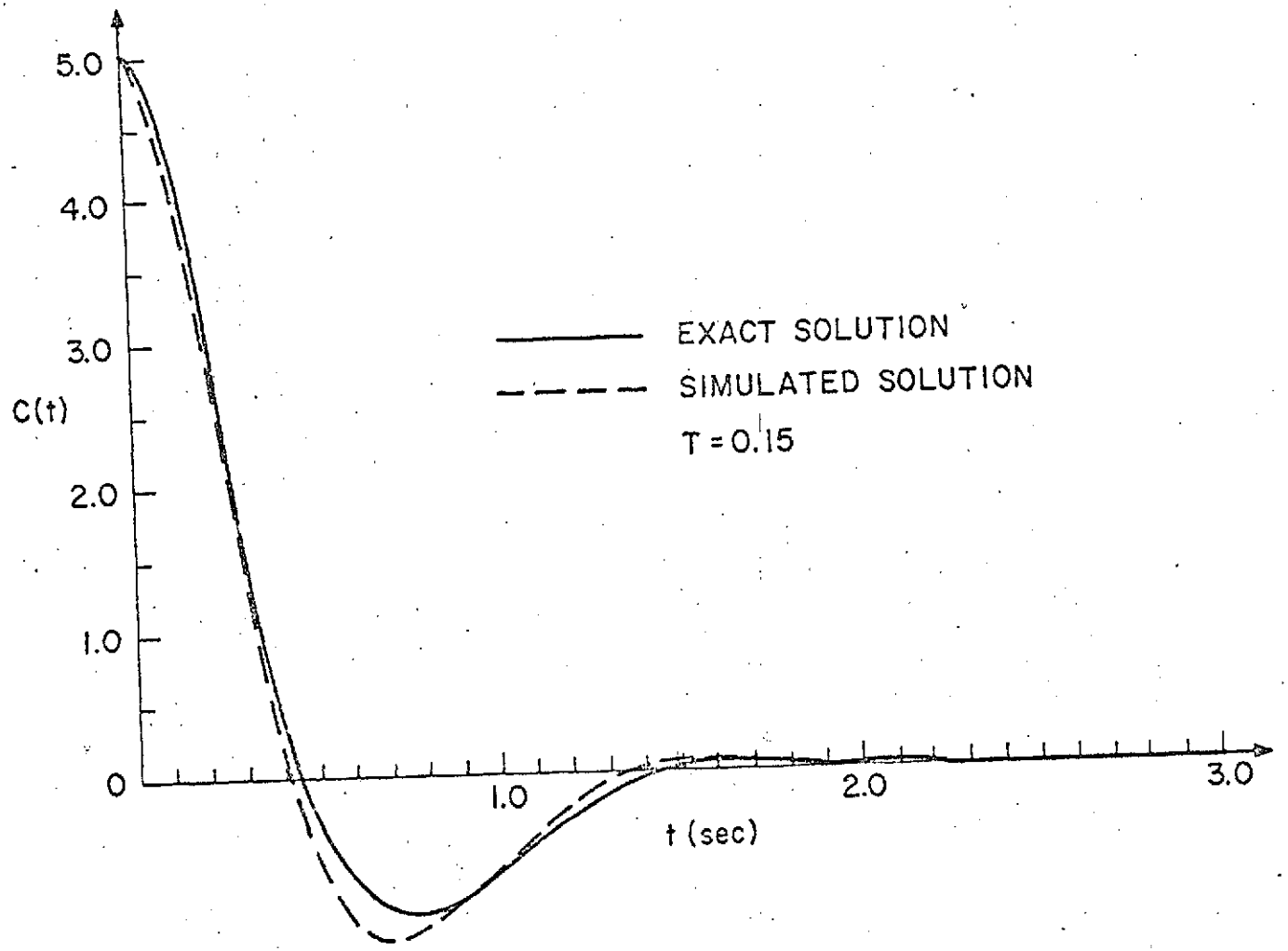

FIG. 5.8 FIRST-ORDER DISCRETE-DATA SYSTEM
(Example 5.2)

5.7. Limitations in the Choice of T

Some further comments must be made concerning the choice of the sampling interval.

First the definition of error must be recalled. It must be remembered that in this research only the case of discretization error was considered, leaving the quantization error and the round-off error for further research. Considering only the discretization error, one may arrive at a conclusion (not always true in practical applications) that the smaller the sampling interval T, the better a discrete-time model results. But, taking the other errors, the round-off error has a contrary effect on the choice of T. It increases the total error as T decreases. Fortunately, in practical discrete-data control-systems the situation of having important weight from the round-off error is rarely present.

A second observation to be made concerns the loss of controllability of the system due to a particular selection of the sampling interval [58]. It is a well known fact that sometimes sampling can destroy controllability. The problem is analyzed in the above referred paper and a theorem stated. They proved that

a time invariant discrete data system (derived from a controllable continuous system) is completely controllable if

$$\text{Im}\{\delta_i(A) - \delta_j(A)\} \neq \frac{2\pi}{T} n$$

whenever

$$\text{Re}\{\delta_i(A)\} = \text{Re}\{\delta_j(A)\}$$

and

$$n = \pm 1, \pm 2, \ldots$$

$$\delta(A) = \text{Eigenvalues of } A$$

and if the control is scalar, then the condition is necessary as well. Then an extra constraint in the selection of T must be considered. According to the expressions for T obtained in this chapter, and because the presence of ε in them, the situation of loss of controllability is very unlikely. Nevertheless, this case must be checked. Physically, this fact means that the periodicity inherent in sampling is not

allowed to interact with the natural frequencies of the system to be controlled.

A third consideration is the case of discretization error in uncontrollable and unobservable systems. Consider the model introduced in Chapter III for describing the system under consideration. Since the A and F matrices were assumed to be diagonal then the loss of controllability and observability is related to zero elements in the matrices B and G, and C and H respectively. This fact affects the discretization error by causing the R and S matrices to have a zero element. Therefore there will be no contribution to the discretization error from the uncontrollable mode. The same effect can be observed on the output due to an unobservable mode.

Finally consider the case of forced systems. The analysis of the expressions obtained in this research for determining sampling rates shows that the sampling interval depends on the fastest natural mode present in the closed-loop system. A similar conclusion was reported in [11] where the system was open-loop and the forcing function was modelled as the output of a companion system which has an impulse response identical to the forcing signal. The sampling interval was determined by the fastest natural

mode of the companion system. In view of these similar results and the fact that the closed-loop control system from an input/output point of view can be regarded as a block which can be either closed or open-loop, the following approach can be suggested. The sampling interval of the forced closed-loop system can be determined by the fastest natural frequency of the augmented system, that is the original control system and the companion system. This generalization of the results reported herein and in [11] should be investigated further to show the validity of this assumption. This engineering approach to the problem is sustained by an example.

Example 5.3: Consider the position servo presented in Example 5.2. Assume that two different forcing functions are applied at the input. One a "slow" step function and the other a "fast" damped sinusoidal function.

For the impulse signal

$$r_1(t) = 1 \quad , t \geq 0$$

$$= 0 \quad , t < 0$$

the maximum eigenvalue magnitude is

$$|\delta_1^s| = 0$$

For the damped sinusoidal signal.

$$r_2(t) = 6e^{-4t} \cos \sqrt{39}\, t \quad , t \geq 0$$

$$= 0 \quad , t < 0$$

the maximum eigenvalue magnitude is

$$|\delta_2^s| = 7.416$$

The closed-loop system has

$$|\delta| = 5 .$$

Consider the system relaxed. The application of the above reasoning for the determination of T for an acceptable error of five per cent yields

(a) Zero-order discrete-data system

$$T = 0.02$$

for the step input, and

$$T = 0.0135$$

for the damped sinusoidal.

(b) First-order discrete-data system

$$T = 0.155$$

for the step input, and

$$T = 0.1045$$

for the damped sinusoidal.

The evolution of the states was obtained by numerical simulation. The outputs are plotted in Figures 5.9 and 5.10, for the zero-order discrete system and Figures 5.11 and 5.12 for the first-order discrete system. As can be seen, a good approximation to the exact solution has been obtained. The normalized sum of squared output discretization error for each case is within acceptable limits.

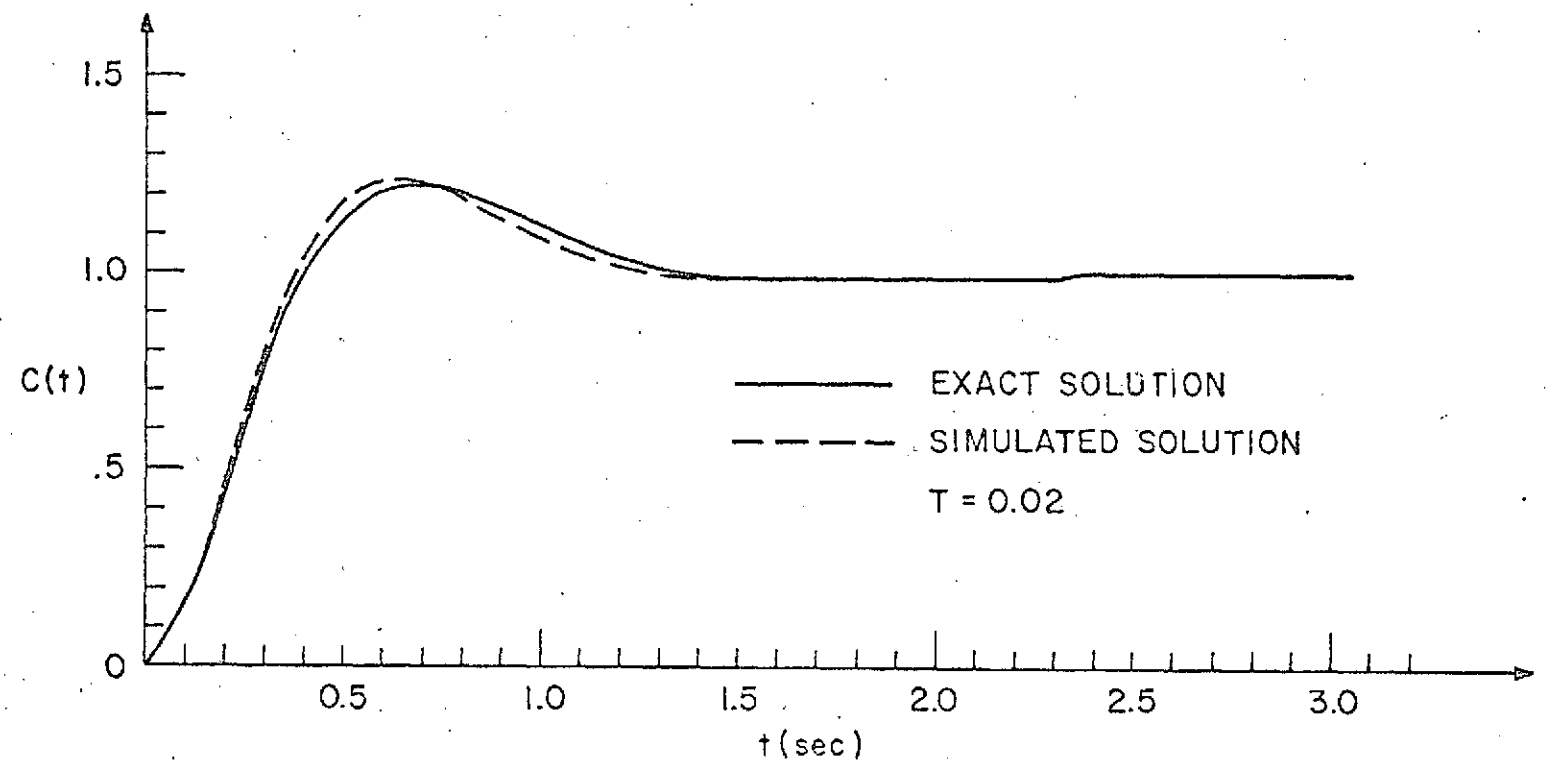

FIG. 5.9 ZERO-ORDER DISCRETE-DATA SYSTEM
(Example 5.3)

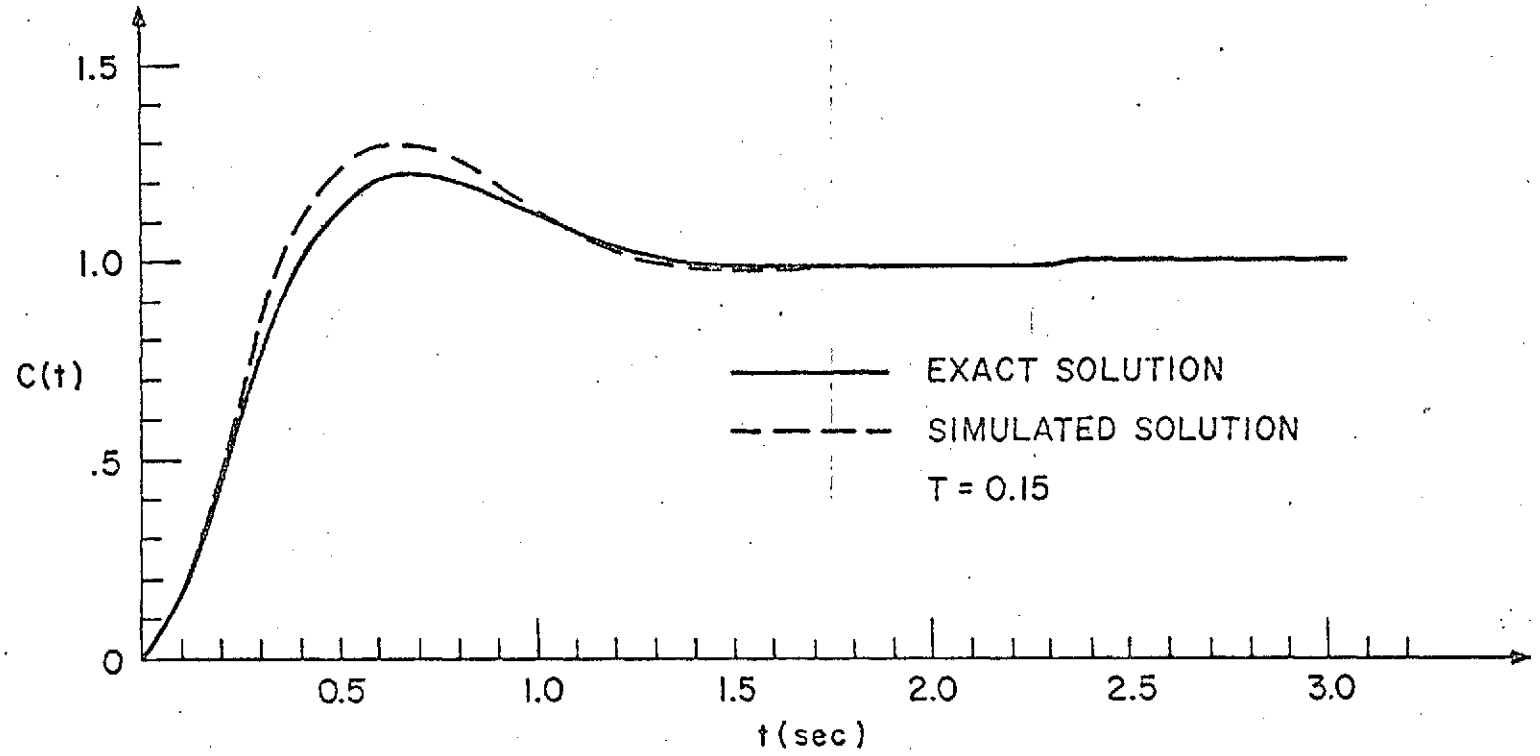

FIG. 5.10 FIRST-ORDER DISCRETE-DATA SYSTEM
(Example 5.3)

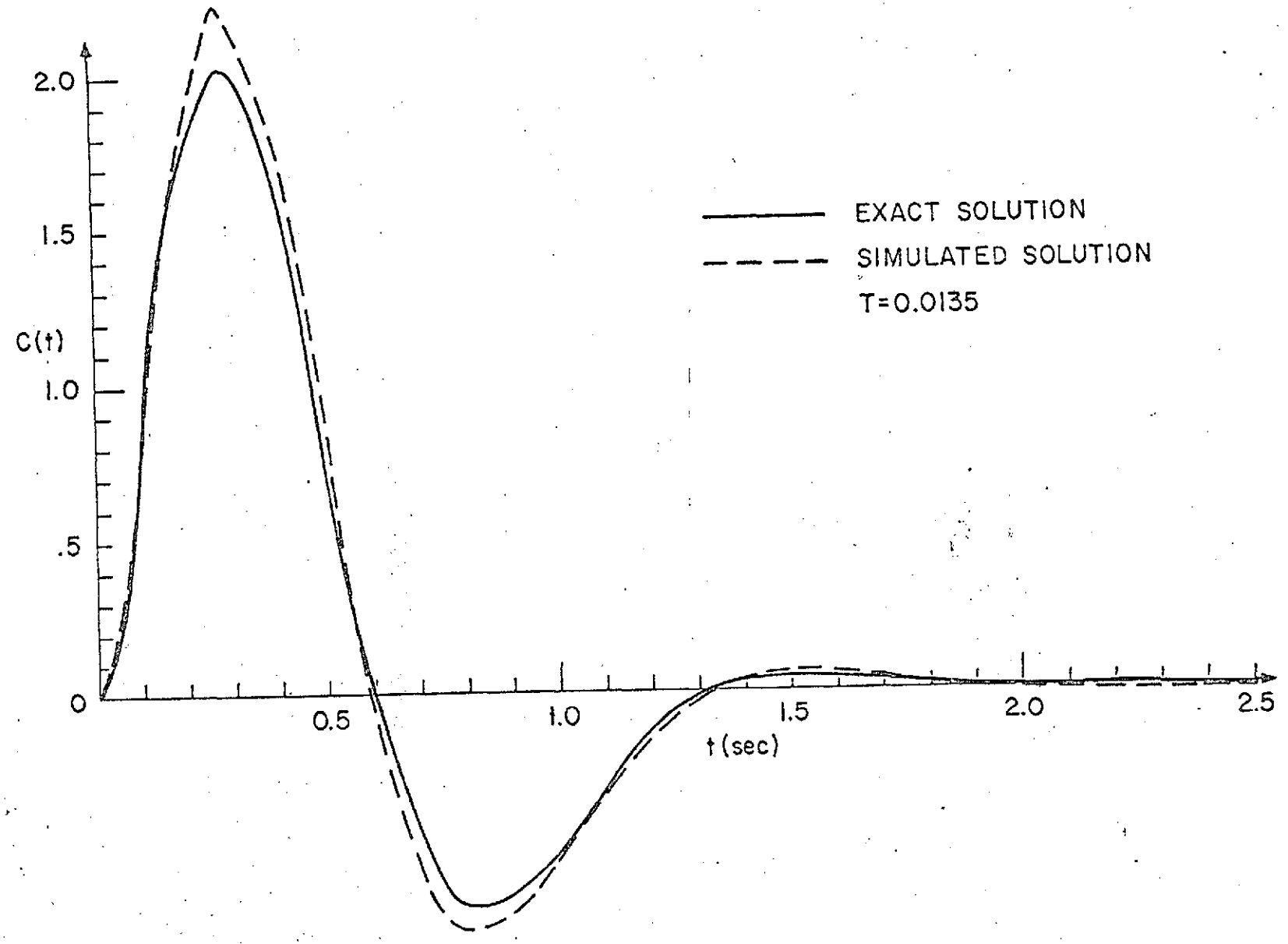

FIG. 5.11 ZERO-ORDER DISCRETE-DATA SYSTEM
(Example 5.3)

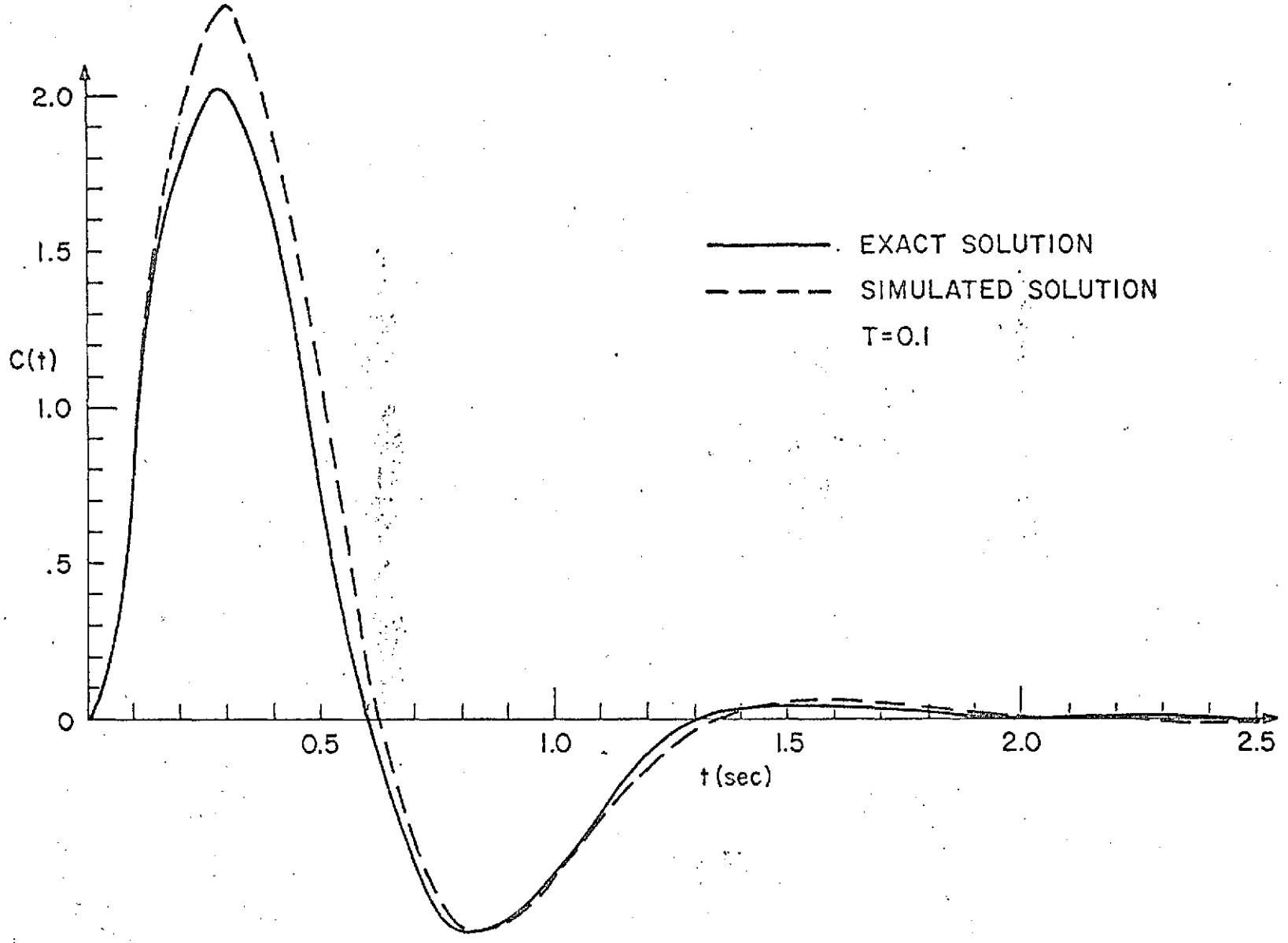

FIG. 5.12 FIRST-ORDER DISCRETE-DATA SYSTEM
(Example 5.3)

CHAPTER VI

CONCLUSIONS

A closed-loop discrete-data control system is a dynamic process where the control signal is the result of a numerical algorithm performed by a digital computer at given instants of time called sampling instants. This investigation was concerned with the determination of sampling rates for linear, time-invariant, closed-loop discrete-data control systems. The research may be regarded as an attempt to establish a formula to be applied to practical systems for making a first choice in the selection of sampling rates. The motivation for such an attempt is that the use of the Nyquist-Shannon sampling theorem for this purpose in practical applications, may lead to a faster rate than that necessary for adequate control under actual limitations.

The basic idea in the research is to compare the state evolution of a discrete-data closed-loop control system with the state evolution of the continuous-data

version of the system. The model chosen to represent the continuous closed-loop control system is a set of n linear differential equations in state-variable form for the plant, and a set of m linear differential equations in the same form for the controller. The reference input was assumed zero (i.e., the regulator problem is treated). The model chosen for the discrete-data controller is the typical configuration of a sampler, followed by a holding device, the analog model of the controller, another sampler (real or fictitious) and another holding device. The closed-loop discrete-data system was also modelled using the state variable approach. The study was made for the two most commonly used holding devices--the zero-order hold and the first-order polygonal hold. The comparison between the evolution of the states was done at sampling instants.

A particularly important step in the achievement of this research is the observation that the closed-loop control system should be characterized and not solved. The solution of the differential equation describing the system is assumed known. The characterization is made by the eigenvalues of the plant, controller and closed-loop system; therefore the system matrices can be

considered diagonal and simple. This fact permits the integration of the state equations and leads to the state characterization at sampling instants as a function of the sampling interval, eigenvalues, matrices and initial conditions of the system.

Of considerable importance to the results obtained here is the determination of a relative error criterion for the evaluation of the discrete-data control system performance with respect to the continuous-data equivalent system. This determination comes from a careful analysis of the different errors present in the system and interpretations of the meaning of a relative error definition. The relative error criterion leads to the objective of this research, the determination of the sampling interval as a function of the maximum allowed error and the eigenvalues of the system, for the two holding devices considered.

Thus, a formula has been obtained for determining the sampling interval or rate for closed-loop discrete-data systems, for two different realizations. It shows that the sampling interval depends primarily on the eigenvalues of the closed-loop continuous-data control system for an allowed performance error. Its application

solves the practical problem of making a first choice in the selection of sampling rates. Further research may be directed toward the extension of this work to the determination of sampling rates for open-loop and closed-loop non-linear systems, systems with random input and related topics for simulation of continuous-data systems.

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