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DERIVATION OF TRANSFORMATION FORMULAS BETWEEN GEOCENTRIC AND GEODETIC COORDINATES FOR NONZERO ALTITUDES

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SUMMARY

Four formulas, for the nonzero altitude transformation from geodetic coordinates (geodetic latitude and altitude) to geocentric coordinates (geocentric latitude and geocentric distance) and vice versa, are derived. The set of four formulas is expressed in each of three useful forms: series expansion in powers of the earth's flattening; series expansion in powers of the earth's eccentricity; and Fourier series expansion in terms of the geodetic latitude or the geocentric latitude. The error incurred in these series expansions is of the order of one part in 3×10^7.

INTRODUCTION

In astrogeodetic work one frequently needs to transform from geodetic coordinates (geodetic latitude and altitude) to geocentric coordinates (geocentric latitude and geocentric distance). The inverse transformation is also frequently needed.

Before the days of air and space travel, locations of interest remained on the surface of the earth. Because the altitude above mean sea level of locations on the earth's surface is much smaller than the earth's radius, a number of transformation formulas were derived for zero altitude (above mean sea level) and for the geocentric distance equaling the earth's radius. A formula for the geocentric latitude as a function of the geodetic latitude for zero altitude is given as a Fourier series expansion in references 1 to 5. A formula for the geocentric distance as a function of the geodetic latitude for zero altitude is given as a Fourier series expansion in references 3 to 6 and as an analytical expression in references 1 to 3. Also, a formula for the geodetic latitude as a function of the geocentric latitude for the geocentric distance equaling the earth's radius is presented as a Fourier series expansion in references 4 to 6.

After the arrival of the days of air and space travel, locations of interest shifted to above the surface of the earth. A number of transformation formulas have been derived for any altitude and geocentric distance. Analytical expressions for the geocentric distance as a function of the geodetic latitude and the altitude are given in references 7 and 8. A formula for the geodetic latitude as a function of the geocentric latitude and the geocentric distance is given as a Fourier series expansion in reference 9 and as an approximate expression in reference 10. Several analytical expressions (three in ref. 9, one in ref. 10, and one in ref. 11) are presented for the altitude as a function of the geocentric latitude, the geocentric distance, and the geodetic latitude. An approximate expression for the altitude as a function of the geocentric distance and the geodetic latitude is given in reference 6. All six of these formulas, however, possess the same shortcoming; that is, for altitudes close to the earth's surface, the altitude is expressed as the difference of two comparable numbers and, consequently, the accuracy is impaired.

In present-day astrogeodetic work, one needs a complete set of transformation formulas whose accuracy remains consistent for all altitudes. In this paper, formulas for the geodetic latitude and the altitude as a function of the geocentric latitude and the geocentric distance and vice versa are derived for nonzero altitudes. For the reader's convenience, the four formulas are cast as series expansions in powers of the earth's flattening, as series expansions in powers of the earth's eccentricity, and as Fourier series expansions in terms of the geodetic latitude or the geocentric latitude.

Some astronomical applications (such as the laser ranging work in refs. 12 and 13) require the extra precision in coordinate determination that is possible with satellite geodesy techniques. However, for most environmental and space sciences applications (such as the tracking of space vehicles, satellites, and missiles in refs. 6, 10, and 11 and the magnetospheric physics work in refs. 7, 8, and 14), the formulas derived in this paper should be sufficiently accurate and very useful. If improved accuracy is required, additional terms may be derived by use of the procedures described herein.

SYMBOLS

a — semimajor axis of ellipse or equatorial radius of planet, arbitrary units

a_1, a_2, a_1', a_2' — expansion coefficients for ϵ in powers of f

b — semiminor axis of ellipse or polar radius of planet, arbitrary units

b_1, b_2, b_1', b_2' — expansion coefficients for difference between $h + 1$ and ρ in powers of f

C — distance from point on ellipse, along perpendicular to ellipse, to intersection point with minor axis, units of a

e	eccentricity
f	flattening
h	altitude, units of a
S	distance from point on ellipse, along perpendicular to ellipse, to intersection point with major axis, units of a
x,y	coordinates of point on ellipse, units of a
ϵ	difference between geocentric latitude and geodetic latitude
ϕ	geodetic latitude
ϕ'	geocentric latitude
ρ	geocentric distance, units of a

ANALYTICAL FORMULATION

Relations Pertaining to an Ellipse or to a Planet

The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

where a is the semimajor axis (or the equatorial radius of the planet) and b is the semiminor axis (or the polar radius of the planet). The polar radius b is related to the equatorial radius a by

$$b^2 = a^2(1 - e^2) \tag{2}$$

The quantity e is the eccentricity, which is represented by the fraction of the major axis that lies between the two foci of the ellipse.

The flattening f, which is the compression of the planet's spheroid from an exact sphere, is defined by

$$f = \frac{a - b}{a}$$

or

$$b = a(1 - f) \tag{3}$$

Comparing equations (2) and (3) yields the following equation relating the eccentricity e to the flattening f:

$$1 - e^2 = (1 - f)^2 \tag{4}$$

Substituting equation (3) into equation (1) results in

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - f)^2} = 1 \tag{5}$$

Differentiating y in equation (5) with respect to x renders the slope dy/dx at the point x,y on an ellipse. This slope is expressed as

$$\frac{dy}{dx} = -(1 - f)^2 \frac{x}{y} \tag{6}$$

Now,

$$\frac{dy}{dx} \tan \phi = -1 \tag{7}$$

where the angle ϕ is the geodetic latitude, shown in figure 1, which is the angle which the perpendicular from the point in space of interest to the earth's spheroid makes with the earth's equatorial plane. The earth's equatorial plane is the plane through the great circle that is halfway between the earth's North and South Poles. Substituting equation (6) into equation (7) furnishes

$$\frac{y}{x} = (1 - f)^2 \tan \phi \tag{8}$$

Substituting equation (8) for y into equation (5) produces

$$\frac{x^2}{a^2}\left[1 + (1 - f)^2 \tan^2\phi\right] = 1$$

or

$$x = \frac{a \cos \phi}{\left[\cos^2\phi + (1 - f)^2 \sin^2\phi\right]^{1/2}} \tag{9}$$

Substituting equation (9) into equation (8) leads to

$$y = \frac{a(1 - f)^2 \sin \phi}{\left[\cos^2\phi + (1 - f)^2 \sin^2\phi\right]^{1/2}} \tag{10}$$

From figure 1 one sees that

$$x = C \cos \phi \tag{11}$$

$$y = S \sin \phi \tag{12}$$

where C and S are the distances from the point x,y along the perpendicular to the ellipse to the intersection points with the minor and major axes, respectively.

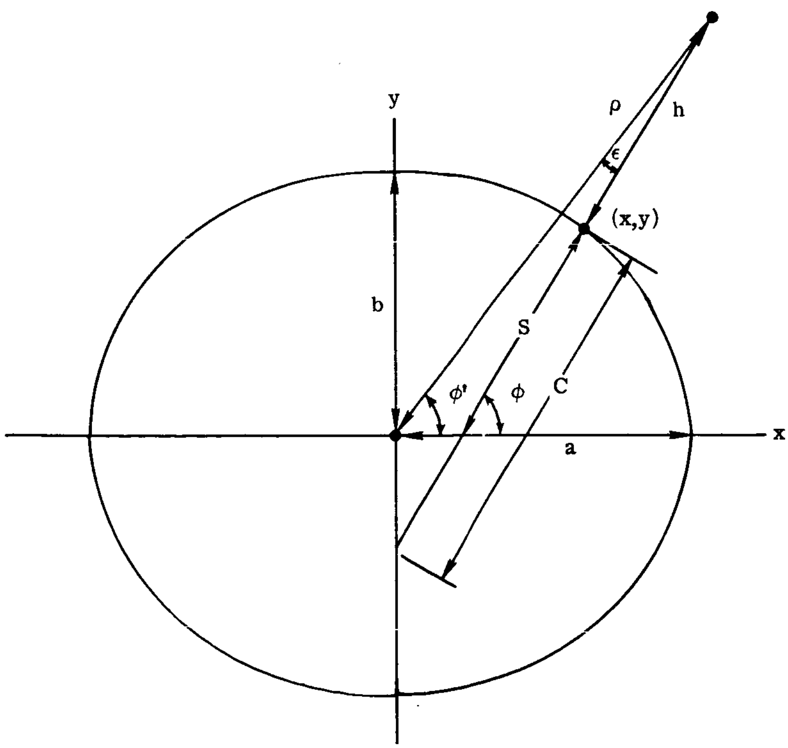

Figure 1.- The geodetic coordinates ϕ and h and the geocentric coordinates ϕ' and ρ.

Comparing equations (9) and (11) establishes that

$$C = \frac{a}{\left[\cos^2\phi + (1 - f)^2 \sin^2\phi\right]^{1/2}} \tag{13}$$

Comparing equations (10) and (12) gives

$$S = \frac{a(1 - f)^2}{\left[\cos^2\phi + (1 - f)^2 \sin^2\phi\right]^{1/2}}$$

Comparing this equation with equation (13) yields

$$S = (1 - f)^2 C \tag{14}$$

Expanding equation (13) in a binomial series expansion, setting a equal to unity, and omitting the terms of order f^3 and higher result in

$$C = 1 + \frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2} (f + f^2)\cos 2\phi + \frac{3}{16} f^2 \cos 4\phi \tag{15}$$

Note that, because of the small value of the earth's flattening f, the error incurred because of the omission of the terms of order f^3 and higher is of the order of one part in 3×10^7.

The use of equation (14) for S renders

$$C - S = C - (1 - f)^2 C$$

$$= (2f - f^2)C$$

Substituting equation (15) into this equation and omitting the terms of order f^3 and higher yield

$$C - S = 2f - f^2 \cos 2\phi \tag{16}$$

From figure 1 one observes that for an arbitrary point in space

$$\rho \cos \phi' = (C + h)\cos \phi \tag{17}$$

$$\rho \sin \phi' = (S + h)\sin \phi \tag{18}$$

The distance h is the altitude, which is the perpendicular distance from the point in space of interest to the earth's spheroid. The distance ρ is the geocentric distance, which is the distance from the point in space of interest to the earth's center; and the angle ϕ' is the geocentric latitude, which is the angle that this radial line from the earth's center makes with the earth's equatorial plane.

The difference between the geocentric latitude ϕ' and the geodetic latitude ϕ is denoted by ϵ and is shown in figure 1. This difference is expressed as

$$\epsilon = \phi - \phi' \tag{19}$$

Since the geodetic latitude differs by a small amount from the geocentric latitude because of the earth's flattening f, which is a small value, then

$$\sin \epsilon = \epsilon \tag{20}$$

The quantity ϵ is of the same order of magnitude as the earth's flattening f. The following Taylor series expansions where the terms of order ϵ^3 and higher have been omitted will be used in the subsequent derivations of the transformation formulas:

$$\cos(\phi' + \epsilon) = \cos \phi' - \epsilon \sin \phi' - \frac{\epsilon^2}{2} \cos \phi' \tag{21}$$

$$\cos(\phi - \epsilon) = \cos \phi + \epsilon \sin \phi - \frac{\epsilon^2}{2} \cos \phi \tag{22}$$

$$\sin(\phi' + \epsilon) = \sin \phi' + \epsilon \cos \phi' - \frac{\epsilon^2}{2} \sin \phi' \tag{23}$$

$$\cos(2\phi' + 2\epsilon) = \cos 2\phi' - 2\epsilon \sin 2\phi' - 2\epsilon^2 \cos 2\phi' \tag{24}$$

$$\sin(2\phi' + 2\epsilon) = \sin 2\phi' + 2\epsilon \cos 2\phi' - 2\epsilon^2 \sin 2\phi' \tag{25}$$

$$\cos(4\phi' + 4\epsilon) = \cos 4\phi' - 4\epsilon \sin 4\phi' - 8\epsilon^2 \cos 4\phi' \tag{26}$$

Transformation Formulas in Forms of Series Expansions in Powers of Earth's Flattening

Geodetic latitude.- Multiplying equation (17) by $\sin \phi$ and equation (18) by $\cos \phi$ and then subtracting the second of these two resulting equations from the first produce

$$\rho \sin(\phi - \phi') = \frac{1}{2}(C - S)\sin 2\phi \tag{27}$$

The ϕ-dependence in equation (27) can be eliminated through the use of equation (19) to obtain

$$\rho \sin \epsilon = \frac{1}{2}(C - S)\sin(2\phi' + 2\epsilon) \tag{28}$$

Substituting equation (20) on the left-hand side and substituting equation (25) on the right-hand side of equation (28) lead to

$$\rho\epsilon = \frac{1}{2}(C - S)(\sin 2\phi' + 2\epsilon \cos 2\phi' - 2\epsilon^2 \sin 2\phi') \tag{29}$$

Using equation (19) to eliminate the ϕ-dependence from equation (16) and then using equation (24) establish that

$$C - S = 2f - f^2(\cos 2\phi' - 2\epsilon \sin 2\phi' - 2\epsilon^2 \cos 2\phi')$$

Recalling that ϵ is of the same order of magnitude as f and then omitting the terms of order f^3 and higher render

$$C - S = 2f - f^2 \cos 2\phi' \tag{30}$$

Substituting equation (30) into equation (29) gives

$$\rho\epsilon = \frac{1}{2}(2f - f^2 \cos 2\phi')(\sin 2\phi' + 2\epsilon \cos 2\phi' - 2\epsilon^2 \sin 2\phi') \tag{31}$$

Since the quantity ϵ is of the same order of magnitude as the earth's flattening f, one can assume that ϵ can be expanded in powers of f. This power series expansion can be written as

$$\epsilon = a_1 f + a_2 f^2 \tag{32}$$

where the terms of order f^3 and higher have been omitted and the coefficients a_1 and a_2 are to be determined.

Substituting equation (32) into equation (31) and then equating like powers of f yield the coefficients a_1 and a_2 as follows:

$$a_1 = \frac{\sin 2\phi'}{\rho} \tag{33}$$

$$a_2 = \left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\phi' \tag{34}$$

Substituting equations (33) and (34) into equation (32) yields

$$\epsilon = \left(\frac{\sin 2\phi'}{\rho}\right)f + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\phi'\right]f^2 \tag{35}$$

Substituting equation (19) into the left-hand side of equation (35) proves that

$$\phi = \phi' + \left(\frac{\sin 2\phi'}{\rho}\right)f + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\phi'\right]f^2 \tag{36}$$

Therefore, equation (36) is the transformation formula, in the form of a series expansion in powers of the earth's flattening, for the geodetic latitude as a function of the geocentric latitude and the geocentric distance.

Altitude.- Multiplying equation (17) by $\sin \phi'$ and equation (18) by $\cos \phi'$ and then subtracting the second of these two resulting equations from the first result in

$$0 = C \cos \phi \sin \phi' - S \sin \phi \cos \phi' - h \sin(\phi - \phi')$$

or

$$(h + 1)\sin(\phi - \phi') = (C - S)\sin \phi \cos \phi' - (C - 1)\sin(\phi - \phi') \tag{37}$$

Subtracting equation (37) from equation (27) renders

$$\left[\rho - (h + 1)\right]\sin(\phi - \phi') = (C - S)\sin \phi(\cos \phi - \cos \phi') + (C - 1)\sin(\phi - \phi') \tag{38}$$

The ϕ-dependence in equation (38) can be eliminated through the use of equation (19) to obtain

$$\left[\rho - (h + 1)\right]\sin \epsilon = (C - S)\sin(\phi' + \epsilon)\left[\cos(\phi' + \epsilon) - \cos \phi'\right] + (C - 1)\sin \epsilon \qquad (39)$$

Substituting equations (20), (23), and (21) into equation (39) furnishes

$$\left[\rho - (h + 1)\right]\epsilon = (C - S)\left(\sin \phi' + \epsilon \cos \phi' - \frac{\epsilon^2}{2} \sin \phi'\right)\left(-\epsilon \sin \phi' - \frac{\epsilon^2}{2} \cos \phi'\right) + (C - 1)\epsilon$$

or

$$\rho - (h + 1) = (C - S)\left(\sin \phi' + \epsilon \cos \phi' - \frac{\epsilon^2}{2} \sin \phi'\right)\left(- \sin \phi' - \frac{\epsilon}{2} \cos \phi'\right) + (C - 1)$$

Omitting the ϵ^3 term in this equation gives

$$\rho - (h + 1) = -(C - S)\left[\sin^2\phi' + \frac{3\epsilon}{4} \sin 2\phi' + \frac{\epsilon^2}{2}\left(\cos^2\phi' - \sin^2\phi'\right)\right] + (C - 1)$$

Using equation (35) in this equation and omitting the terms of order f^3 and higher produce

$$\rho - (h + 1) = -(C - S)\left\{\sin^2\phi' + \frac{3}{4}\left\{\left(\frac{\sin 2\phi'}{\rho}\right)f + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\phi'\right]f^2\right\}\sin 2\phi'\right.$$
$$\left. + \frac{1}{2}\left[\left(\frac{\sin 2\phi'}{\rho}\right)^2 f^2\right]\left(\cos^2\phi' - \sin^2\phi'\right)\right\} + (C - 1)$$

$$= -(C - S)\left\{\frac{(1 - \cos 2\phi')}{2} + 3f \frac{(1 - \cos 4\phi')}{8\rho}\right.$$
$$\left. + f^2\left[\frac{3}{4}\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\phi' \sin 2\phi' + \frac{1}{2}\left(\frac{\sin 2\phi'}{\rho}\right)^2\left(\cos^2\phi' - \sin^2\phi'\right)\right]\right\}$$
$$+ (C - 1) \qquad (40)$$

The ϕ-dependence in equation (15) can be eliminated through the use of equation (19) to obtain

$$C - 1 = \frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2}\left(f + f^2\right)\cos(2\phi' + 2\epsilon) + \frac{3}{16} f^2 \cos(4\phi' + 4\epsilon)$$

Using equations (24) and (26) in this equation leads to

$$C - 1 = \frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2}\left(f + f^2\right)(\cos 2\phi' - 2\epsilon \sin 2\phi' - 2\epsilon^2 \cos 2\phi')$$

$$+ \frac{3}{16} f^2(\cos 4\phi' - 4\epsilon \sin 4\phi' - 8\epsilon^2 \cos 4\phi')$$

Using equation (35) in this equation and omitting terms of order f^3 and higher establish that

$$C - 1 = \frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2}\left(f + f^2\right)\left[\cos 2\phi' - 2f\left(\frac{\sin 2\phi'}{\rho}\right)\sin 2\phi'\right] + \frac{3}{16} f^2 \cos 4\phi'$$

$$= \frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2}\left(f + f^2\right)\left[\cos 2\phi' - f \frac{(1 - \cos 4\phi')}{\rho}\right] + \frac{3}{16} f^2 \cos 4\phi' \qquad (41)$$

Substituting equations (30) and (41) into equation (40) and omitting terms of order f^3 and higher give

$$\rho - (h + 1) = -\left(2f - f^2 \cos 2\phi'\right)\left[\frac{(1 - \cos 2\phi')}{2} + 3f \frac{(1 - \cos 4\phi')}{8\rho}\right]$$

$$+ \frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2}\left(f + f^2\right)\left[\cos 2\phi' - f \frac{(1 - \cos 4\phi')}{\rho}\right] + \frac{3}{16} f^2 \cos 4\phi' \qquad (42)$$

Just as one assumed that the difference ϵ between the geocentric latitude ϕ' and the geodetic latitude ϕ could be expanded in powers of the earth's flattening f, one can assume that the difference between the quantity $h + 1$ and the geocentric distance ρ can also be so expanded. (Recall that, in this paper, the altitude h is in units of the equatorial radius a and a has been set equal to unity; hence, the quantity $h + 1$ actually represents the quantity $h + a$.) The power series expansion for the difference $\rho - (h + 1)$ can be written as

$$\rho - (h + 1) = b_1 f + b_2 f^2 \qquad (43)$$

where the terms of order f^3 and higher have been omitted and the coefficients b_1 and b_2 are to be determined.

Substituting equation (43) into the left-hand side of equation (42) and then equating like powers of f yield the coefficients b_1 and b_2 as follows:

$$b_1 = \frac{\cos 2\phi' - 1}{2} \tag{44}$$

$$b_2 = \left(\frac{1}{4\rho} - \frac{1}{16}\right)(\cos 4\phi' - 1) \tag{45}$$

Substituting equations (44) and (45) into equation (43) proves that

$$h = (\rho - 1) + \left(\frac{1 - \cos 2\phi'}{2}\right)f + \left[\left(\frac{1}{4\rho} - \frac{1}{16}\right)(1 - \cos 4\phi')\right]f^2 \tag{46}$$

Therefore, equation (46) is the transformation formula, in the form of a series expansion in powers of the earth's flattening, for the altitude as a function of the geocentric latitude and the geocentric distance.

Geocentric latitude.- The ϕ'-dependence in equation (37) can be eliminated through the use of equation (19) to obtain

$$(h + 1)\sin \epsilon = (C - S)\sin \phi \cos(\phi - \epsilon) - (C - 1)\sin \epsilon \tag{47}$$

(Note that, just as the quantity $h + 1$ actually represents the quantity $h + a$, the quantity $C - 1$ actually represents the quantity $C - a$.)

Substituting equations (20) and (22) into equation (47) yields

$$(h + 1)\epsilon = (C - S)\sin \phi\left(\cos \phi + \epsilon \sin \phi - \frac{\epsilon^2}{2} \cos \phi\right) - (C - 1)\epsilon$$

$$= (C - S)\left[\frac{\sin 2\phi}{2} + \epsilon \frac{(1 - \cos 2\phi)}{2} - \frac{\epsilon^2}{2} \cos \phi\right] - (C - 1)\epsilon$$

Using equations (16) and (15) in this equation results in

$$(h + 1)\epsilon = (2f - f^2 \cos 2\phi)\left[\frac{\sin 2\phi}{2} + \epsilon \frac{(1 - \cos 2\phi)}{2} - \frac{\epsilon^2}{2} \cos \phi\right]$$

$$- \left[\frac{1}{2} f + \frac{5}{16} f^2 - \frac{1}{2}\left(f + f^2\right)\cos 2\phi + \frac{3}{16} f^2 \cos 4\phi\right]\epsilon \tag{48}$$

Again, one can assume that the difference ϵ between the geocentric latitude ϕ' and the geodetic latitude ϕ can be expanded in powers of the earth's flattening f. This power series expansion can be written as

$$\epsilon = a_1' f + a_2' f^2 \tag{49}$$

where the terms of order f^3 and higher have been omitted and the coefficients a_1' and a_2' are to be determined.

Substituting equation (49) into equation (48) and then equating like powers of f yield the coefficients a_1' and a_2' as follows:

$$a_1' = \frac{\sin 2\phi}{h + 1} \tag{50}$$

$$a_2' = \frac{\sin 2\phi}{2(h + 1)^2} - \left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]\sin 4\phi \tag{51}$$

Substituting equations (50) and (51) into equation (49) renders

$$\epsilon = \left(\frac{\sin 2\phi}{h + 1}\right)f + \left\{\frac{\sin 2\phi}{2(h + 1)^2} - \left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]\sin 4\phi\right\}f^2 \tag{52}$$

Substituting equation (19) into the left-hand side of equation (52) proves that

$$\phi' = \phi + \left(\frac{-\sin 2\phi}{h + 1}\right)f + \left\{\frac{-\sin 2\phi}{2(h + 1)^2} + \left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]\sin 4\phi\right\}f^2 \tag{53}$$

Therefore, equation (53) is the transformation formula, in the form of a series expansion in powers of the earth's flattening, for the geocentric latitude as a function of the geodetic latitude and the altitude.

Geocentric distance.- The ϕ'-dependence in equation (38) can be eliminated through the use of equation (19) to obtain

$$\left[\rho - (h + 1)\right]\sin \epsilon = (C - S)\sin \phi\left[\cos \phi - \cos(\phi - \epsilon)\right] + (C - 1)\sin \epsilon \tag{54}$$

Substituting equations (20) and (22) into equation (54) furnishes

$$\left[\rho - (h + 1)\right]\epsilon = (C - S)\sin\phi\left(-\epsilon\sin\phi + \frac{\epsilon^2}{2}\cos\phi\right) + (C - 1)\epsilon$$

or

$$\rho - (h + 1) = (C - S)\sin\phi\left(-\sin\phi + \frac{\epsilon}{2}\cos\phi\right) + (C - 1)$$

$$= -(C - S)\left(\sin^2\phi - \epsilon\,\frac{\sin 2\phi}{4}\right) + (C - 1)$$

The use of equation (52) in this equation produces

$$\rho - (h + 1) = -(C - S)\left\{\sin^2\phi - \left(\left(\frac{\sin 2\phi}{h + 1}\right)f\right.\right.$$

$$\left.\left. + \left\{\frac{\sin 2\phi}{2(h + 1)^2} - \left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]\sin 4\phi\right\} f^2\right)\frac{\sin 2\phi}{4}\right\} + (C - 1)$$

$$= -(C - S)\left(\frac{(1 - \cos 2\phi)}{2} - f\,\frac{(1 - \cos 4\phi)}{8(h + 1)}\right.$$

$$\left. + f^2\left\{\frac{\sin 2\phi}{2(h + 1)^2} - \left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]\sin 4\phi\right\}\frac{\sin 2\phi}{4}\right) + (C - 1)$$

Using equations (16) and (15) in this equation and omitting terms of order f^3 and higher lead to

$$\rho - (h + 1) = -\left(2f - f^2\cos 2\phi\right)\left[\frac{(1 - \cos 2\phi)}{2} - f\,\frac{(1 - \cos 4\phi)}{8(h + 1)}\right] + \frac{1}{2}f + \frac{5}{16}f^2$$

$$- \frac{1}{2}\left(f + f^2\right)\cos 2\phi + \frac{3}{16}f^2\cos 4\phi \qquad (55)$$

Again, one can assume that the difference between the quantity $h + 1$ and the geocentric distance ρ can be expanded in powers of the earth's flattening f. This power series expansion can be written as

$$\rho - (h + 1) = b_1' f + b_2' f^2 \tag{56}$$

where the terms of order f^3 and higher have been omitted and the coefficients b_1' and b_2' are to be determined.

Substituting equation (56) into the left-hand side of equation (55) and then equating like powers of f yield the coefficients b_1' and b_2' as follows:

$$b_1' = \frac{\cos 2\phi - 1}{2} \tag{57}$$

$$b_2' = \left[\frac{1}{4(h + 1)} + \frac{1}{16}\right](1 - \cos 4\phi) \tag{58}$$

Substituting equations (57) and (58) into equation (56) proves that

$$\rho = (h + 1) + \left(\frac{\cos 2\phi - 1}{2}\right)f + \left\{\left[\frac{1}{4(h + 1)} + \frac{1}{16}\right](1 - \cos 4\phi)\right\} f^2 \tag{59}$$

Therefore, equation (59) is the transformation formula, in the form of a series expansion in powers of the earth's flattening, for the geocentric distance as a function of the geodetic latitude and the altitude.

Summation of formulas.- The four transformation formulas for the geodetic latitude and the altitude as a function of the geocentric latitude and the geocentric distance for nonzero altitudes and vice versa, in the forms of series expansions in powers of the earth's flattening, are given by equations (36), (46), (53), and (59), respectively. For convenience, these formulas are listed as follows:

$$\phi = \phi' + \left(\frac{\sin 2\phi'}{\rho}\right)f + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)\sin 4\phi'\right]f^2$$

$$h = (\rho - 1) + \left(\frac{1 - \cos 2\phi'}{2}\right)f + \left[\left(\frac{1}{4\rho} - \frac{1}{16}\right)(1 - \cos 4\phi')\right]f^2$$

$$\phi' = \phi + \left(\frac{-\sin 2\phi}{h + 1}\right)f + \left\{\frac{-\sin 2\phi}{2(h + 1)^2} + \left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]\sin 4\phi\right\}f^2$$

$$\rho = (h + 1) + \left(\frac{\cos 2\phi - 1}{2}\right)f + \left\{\left[\frac{1}{4(h + 1)} + \frac{1}{16}\right](1 - \cos 4\phi)\right\}f^2$$

These power series expansions in f are useful forms for two reasons: (1) the power series expansions allow one to readily discern the accuracy of the formulas, and (2) the earth's flattening f is an important physical parameter.

Transformation Formulas in Forms of Series Expansions in Powers of Earth's Eccentricity

The earth's eccentricity is also a useful and important physical parameter. Equation (4) relates the eccentricity e to the flattening f. From equation (4)

$$f = 1 - (1 - e^2)^{1/2}$$

Expanding this equation in a binomial series expansion and omitting the terms of order e^6 and higher establish that

$$f = \frac{1}{2}e^2 + \frac{1}{8}e^4 \tag{60}$$

Substituting equation (60) into equations (36), (46), (53), and (59), respectively, and omitting the terms of order e^6 and higher give

$$\phi = \phi' + \left(\frac{\sin 2\phi'}{2\rho}\right)e^2 + \left[\frac{\sin 2\phi'}{8\rho} + \left(\frac{1}{4\rho^2} - \frac{1}{16\rho}\right)\sin 4\phi'\right]e^4 \tag{61}$$

$$h = (\rho - 1) + \left(\frac{1 - \cos 2\phi'}{4}\right)e^2 + \left[\frac{(1 - \cos 2\phi')}{16} + \left(\frac{1}{16\rho} - \frac{1}{64}\right)(1 - \cos 4\phi')\right]e^4 \tag{62}$$

$$\phi' = \phi + \left[\frac{-\sin 2\phi}{2(h + 1)}\right]e^2 + \left\{-\left[\frac{1}{8(h + 1)^2} + \frac{1}{8(h + 1)}\right]\sin 2\phi + \left[\frac{1}{16(h + 1)^2} + \frac{1}{16(h + 1)}\right]\sin 4\phi\right\}e^4 \tag{63}$$

$$\rho = (h + 1) + \left(\frac{\cos 2\phi - 1}{4}\right)e^2 + \left\{\frac{(\cos 2\phi - 1)}{16} - \left[\frac{1}{16(h + 1)} + \frac{1}{64}\right](\cos 4\phi - 1)\right\} e^4 \tag{64}$$

Therefore, equations (61), (62), (63), and (64) are the four transformation formulas, in the forms of series expansions in powers of the earth's eccentricity, for the geodetic latitude and the altitude as a function of the geocentric latitude and the geocentric distance for nonzero altitudes and vice versa. Note that equation (61) for the geodetic latitude, if rearranged, is the same as the Fourier series expansion for the geodetic latitude found in reference 9.

Transformation Formulas in Forms of Fourier Series Expansions

Fourier series expansions are also useful forms. Equations (36), (46), (53), and (59) can be rearranged to represent Fourier series expansions in terms of the geodetic latitude or the geocentric latitude. Therefore, the four transformation formulas for the geodetic latitude and the altitude as a function of the geocentric latitude and the geocentric distance for nonzero altitudes and vice versa, in the forms of Fourier series expansions, are given by the following equations:

$$\phi = \phi' + \left(\frac{f}{\rho}\right)\sin 2\phi' + \left[\left(\frac{1}{\rho^2} - \frac{1}{4\rho}\right)f^2\right]\sin 4\phi' \tag{65}$$

$$h = (\rho - 1) + \frac{1}{2}f + \left(\frac{1}{4\rho} - \frac{1}{16}\right)f^2 + \left(\frac{-f}{2}\right)\cos 2\phi' + \left[-\left(\frac{1}{4\rho} - \frac{1}{16}\right)f^2\right]\cos 4\phi' \tag{66}$$

$$\phi' = \phi + \left[\frac{-f}{h + 1} - \frac{f^2}{2(h + 1)^2}\right]\sin 2\phi + \left\{\left[\frac{1}{4(h + 1)^2} + \frac{1}{4(h + 1)}\right]f^2\right\}\sin 4\phi \tag{67}$$

$$\rho = (h + 1) - \frac{1}{2}f + \left[\frac{1}{4(h + 1)} + \frac{1}{16}\right]f^2 + \left(\frac{f}{2}\right)\cos 2\phi + \left\{-\left[\frac{1}{4(h + 1)} + \frac{1}{16}\right]f^2\right\}\cos 4\phi \tag{68}$$

Note that equation (67) for the geocentric latitude, if the altitude h is set equal to zero, reduces to the Fourier series expansion for the geocentric latitude found in references 3 to 5. Also, note that equation (68) for the geocentric distance, if the altitude h is set equal to zero, reduces to the Fourier series expansion for the geocentric distance found in references 3 to 5. Therefore, in this paper, the Fourier series expansions for the geocentric latitude and the geocentric distance as a function of the geodetic latitude have been generalized to include nonzero altitudes.

CONCLUDING REMARKS

Four formulas, for the nonzero altitude transformation from geodetic coordinates (geodetic latitude and altitude) to geocentric coordinates (geocentric latitude and geocentric distance) and vice versa, are derived. The set of four formulas is expressed in each of three useful forms: series expansion in powers of the earth's flattening; series expansion in powers of the earth's eccentricity; and Fourier series expansion in terms of the geodetic latitude or the geocentric latitude. The error incurred due to omitting the third order and higher terms in the series expansions is of the order of one part in 3×10^7, which for most environmental and space sciences applications is inconsequential. This paper has three significant contributions: (1) the four transformation formulas, which are accurate for any altitude, have been derived as series expansions; (2) the formulas are listed in a complete and consistent set; and (3) the set of formulas has been cast into three different and useful forms.

Langley Research Center,
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Hampton, Va., February 6, 1974.

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16. Abstract

Four formulas, for the nonzero altitude transformation from geodetic coordinates (geodetic latitude and altitude) to geocentric coordinates (geocentric latitude and geocentric distance) and vice versa, are derived. The set of four formulas is expressed in each of three useful forms: series expansion in powers of the earth's flattening; series expansion in powers of the earth's eccentricity; and Fourier series expansion in terms of the geodetic latitude or the geocentric latitude. The error incurred in these series expansions is of the order of one part in 3×10^7.

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