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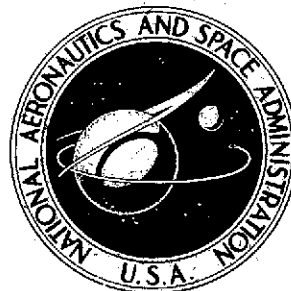
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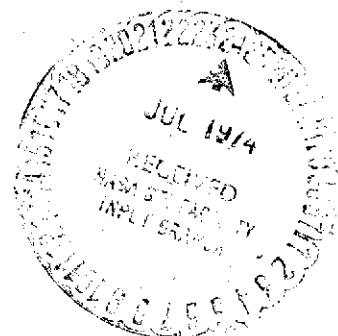
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TIME-FIXED RENDEZVOUS BY IMPULSE FACTORING WITH AN INTERMEDIATE TIMING CONSTRAINT

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16. Abstract

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SUMMARY

A method is presented for factoring a two-impulse orbital transfer into a three- or four-impulse transfer which solves the rendezvous problem and satisfies an intermediate timing constraint. Both the time of rendezvous and the intermediate time of alinement are formulated as any element of a finite sequence of times. These times are integer multiples of a constant plus an additive constant. The rendezvous condition is an equality constraint, whereas the intermediate alinement is an inequality constraint. The two timing constraints are satisfied by factoring the impulses into collinear parts that vectorially sum to the original impulses and by varying the resultant period differences and the number of revolutions in each orbit. Five different types of solutions arise by considering factoring either or both of the two impulses into two or three parts with a limit of four total impulses. The impulse-factoring technique may be applied to any two-impulse transfer which has distinct orbital periods.

INTRODUCTION

A major component of space exploration is the mission involving orbital operations. The maneuvers in orbit often require rendezvous. In the present study, rendezvous is defined as a trajectory which starts at a terminal and ends at a time-related terminal (ref. 1); and a terminal is a specified Cartesian position and velocity vector, or equivalently, the set of six Keplerian orbital elements. A requirement of the orbital rendezvous may be that the maneuver be completed at specified times (time-fixed rendezvous). The time-fixed rendezvous is contrasted to the trajectory which has no limitation on the rendezvous time (time-open rendezvous). Another requirement of the orbital rendezvous may be that the transfer trajectory satisfy an intermediate timing constraint. For example, to reach a particular true anomaly within a prespecified timing error is an intermediate inequality constraint on the transfer trajectory. The problem addressed in this analysis can thus be described as a time-fixed rendezvous with an intermediate timing constraint.

An orbital transfer between two general orbits can be achieved in two impulses. However, the time limitation for fixed-time rendezvous severely hinders the various rendezvous schemes which require waiting for an initial orientation before making the two-impulse transfer. Several authors have investigated alternative methods which require more than two impulses to complete the rendezvous. Van Gelder and associates (ref. 2) accomplished a three-impulse time-open rendezvous between coplanar orbits by splitting one Hohmann impulse into two equivalent impulses to obtain rendezvous with the same ΔV as the Hohmann transfer. Straly (ref. 3) investigated three-impulse time-open rendezvous with a circular target orbit by making a phasing impulse after arriving in the plane of the target orbit. Roth (ref. 4) used a bi-elliptic transfer between circular coplanar orbits to achieve time-open rendezvous. Bender (ref. 5) investigated three-impulse time-open rendezvous by splitting one of the impulses of the nonrendezvous optimum two-impulse transfer. Eckel (ref. 6) found a minimum-time rendezvous by a technique similar to Bender's. Finally, Doll (ref. 7) determined optimal multiple impulse time-fixed rendezvous trajectories, but his method requires elaborate estimates of the transfer and does not always yield a solution. In no instance do the rendezvous schemes consider the possibility of an intermediate timing constraint, and most do not consider time-fixed transfers between general terminals.

The purpose of the analysis presented herein is to present a solution to the time-fixed rendezvous between general orbits with an intermediate timing constraint. The approach taken is first to determine a two-impulse orbital transfer which satisfies the geometry (position and velocity) of the rendezvous condition and then to adjust the timing in the geometry solution to satisfy the time constraints. The two-impulse geometry solution can be any set of maneuvers which transfers from a completely specified initial orbit to a completely specified final orbit or rendezvous orbit. One example is the two-impulse transfer of McCue (ref. 8). An example of a two-impulse transfer tailored to a particular mission is presented in appendix A. Next, the timing in the geometry solution is adjusted to satisfy the intermediate timing constraint and the time of rendezvous. A coarse adjustment of the timing can be achieved by allowing the spacecraft to wait in orbit for a number of revolutions before proceeding to the next orbit of the two-impulse transfer. Varying the number of revolutions in each orbit will permit the timing to be adjusted without altering the geometry solution. Vernier adjustments of the timing can be achieved by applying only part of the impulse, waiting a number of revolutions, and then applying the remainder of the impulse. If the two parts of the impulse are both applied at the same point and vectorially summed to the original impulse, then the timing in orbit is adjusted while the resulting geometry is unchanged. This approach to solving the rendezvous problem is termed the impulse-factoring technique. It has also been called splitting, looping, phasing, and epoch changing by various authors.

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The factoring of an impulsive velocity maneuver to adjust the timing is quite simple and is depicted in figure 1. The impulsive velocity maneuver is ΔV and is applied to the initial orbit to establish the transfer orbit. If the impulse is factored by the velocity factor α_V and only part of the impulse is applied $(\alpha_V \Delta V)$, then an inter-mediate or factored orbit is established. This orbit has a period which is generally different from either the initial or the

Figure 1.- Impulse-velocity factoring.

transfer orbit and causes a change in the time of periapsis passage. Adjustments in timing can be made by the proper choice of the factor α_V. To complete the maneuver from the initial orbit to the transfer orbit requires that the remaining part of the impulse, $(1 - \alpha_V)\Delta V$, be applied. Thus, an impulsive velocity maneuver can be factored to alter the timing while the orbital geometry is preserved.
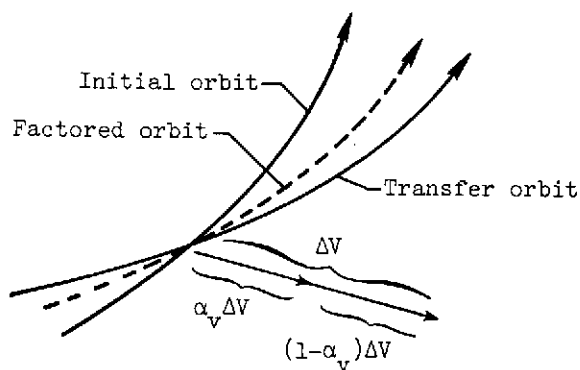
The full potential of impulse factoring has previously been ignored. Some authors have applied one factor and restricted it to lie in the range 0 to 1. This restriction results in no increase in the cost of the ΔV maneuver. However, in some cases the restriction does not allow a time-fixed rendezvous. In the present investigation, the range of rendezvous is extended by allowing factors outside the range 0 to 1. Even though the sum of the two resulting impulses may be larger than the original impulse, this technique yields a time-fixed rendezvous solution that is often acceptable. In addition, this investigation considers using two factors to satisfy two timing constraints. One factor may be applied to each impulse of the two-impulse geometry solution, or both factors may be applied to the same impulse, whereby the impulse may be factored into three parts. There are three such schemes to factor a two-impulse geometry solution into a four-impulse rendezvous solution. In addition, there are two three-impulse rendezvous solutions which result from applying only one factor to either impulse. In all, five distinct cases arise and are discussed subsequently in detail in the section "Analysis."

An example problem using the impulse-factoring technique is presented in the section "Application." A further application is presented in reference 9.

SYMBOLS

a   semimajor axis, kilometers

$\left.\begin{array}{l} c_1, c_2, c_3, \\ c_4, c_5, c_6 \end{array}\right\}$ constants

e	eccentricity
f	true anomaly, degrees
H	component of $\hat{H}$
$\hat{H}$	unit vector along angular momentum vector
i	inclination, degrees
I,J,K,L	number of revolutions relative to f^o in initial, α-, transfer, and β-orbit, respectively
I',J',K',L'	number of revolutions relative to f^a in initial, α-, transfer, and β-orbit, respectively, prior to alinement
I_a	number of revolutions relative to f^a prior to alinement
I_s	upper bound on total number of revolutions
m	integer denoting time of alinement (see eq. (2))
M	number of revolutions relative to f^a in final orbit prior to alinement
n	integer denoting time of rendezvous (see eq. (1))
P	orbital period, hours
$\hat{P},\hat{Q},\hat{W}$	orbital plane coordinate system, where $\hat{P}$ is directed toward periapsis, $\hat{Q}$ is in orbit plane advanced from $\hat{P}$ by a right angle in direction of increasing true anomaly, and $\hat{W}$ completes right-handed system
r	radius from center of planet, kilometers
$\hat{r}$	unit radius vector
R	rotation matrix from the $\hat{P},\hat{Q},\hat{W}$ to the $\hat{X},\hat{Y},\hat{Z}$ coordinate system
t	time, hours

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t_a	time of alinement, hours
t_r	time of rendezvous, hours
V	orbital velocity, kilometers/second
$\hat{X}, \hat{Y}, \hat{Z}$	rectangular Cartesian coordinate system
$\hat{X}', \hat{Y}', \hat{Z}'$	rectangular Cartesian coordinate system defined in appendix A (sketch B)
α	period factor to establish α-orbit
α_V	velocity impulse factor to establish α-orbit
β	period factor to establish β-orbit
β_V	velocity impulse factor to establish β-orbit
γ	flight-path angle, degrees
δ	latitude constraint on spacecraft, degrees
δ_P	latitude of periapsis, degrees
ΔV_T	sum of velocity impulses, kilometers/second
ϵ	timing error at alinement, hours
ζ	time between alinement opportunities, hours
η	time between rendezvous opportunities, hours
θ	declination of spacecraft, degrees
λ	discretization variable for ϵ
Λ	right ascension, degrees
μ	gravitational constant, kilometers^3/second^2

ξ	angle between two orbital planes, degrees (see appendix A, sketch C)
τ_a	time of first alinement opportunity, hours
τ_r	time of first rendezvous opportunity, hours
ϕ	partial sum of orbital revolutions
ψ	true anomaly constraint on spacecraft, degrees
ω	argument of periapsis, degrees
Ω	longitude of ascending node, degrees

Subscripts:

b	upper bound
f	final orbit
min	minimum
o	initial orbit
r,h,n	radial, heading, normal
p,q,w	$\hat{P},\hat{Q},\hat{W}$ coordinate system
t	transfer orbit
x,y,z	$\hat{X},\hat{Y},\hat{Z}$ coordinate system
x',y',z'	$\hat{X}',\hat{Y}',\hat{Z}'$ coordinate system
α	alpha orbit
β	beta orbit
1, 2, . . ., 6	first, second, . . ., sixth

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Superscripts:

a   alinement

i   in

o   out

Notation convention:

$ci(x)$ closest integer to x

Δx increment of x

$|x|$ absolute value of x

$\min_{x}\left\{f(x)\right\}$ minimum value of $f(x)$ over the range of x

$\rightarrow$ implies

$\bar{x}$ vector x
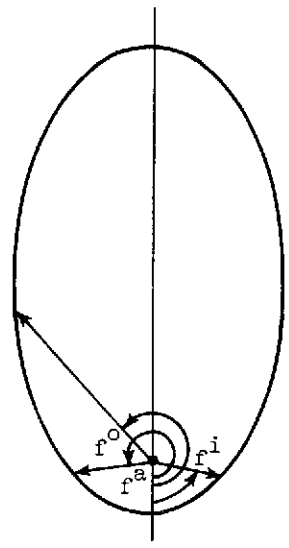
$sgn(x)$ sign of x

ANALYSIS

The method of impulse factoring is applied to a two-impulse orbital transfer so that both an intermediate timing constraint and a final rendezvous constraint are satisfied. The intermediate timing constraint is defined as an alinement of the spacecraft with some object of interest. For example, alinement could correspond to a close approach with another spacecraft or satellite or it could correspond to the reconnaissance of a surface feature. Essentially, it requires that the spacecraft reach a given true anomaly within a specified interval of time. In addition, the rendezvous condition requires that the spacecraft achieve a specified position and velocity at a specified time.

Prior to the application of the impulse-factoring method, a two-impulse orbital transfer which satisfies the orbital geometry of the rendezvous condition is determined. This "geometry" solution establishes the required position and velocity at rendezvous but does not establish the proper timing. The geometry solution can be determined by various means and is dependent upon the specific problem. Since the impulse-factoring method

is independent of the characteristics of the geometry solution, except that the orbital periods are distinct, the determination of the geometry solution is not discussed herein; however, an example of a typical two-impulse solution is presented in the section "Application."

The geometry solution specifies the orbital elements of the initial, transfer, and final orbits, which remain fixed. Quantities defined on these three orbits are denoted by subscripts o, t, and f, respectively. Also specified (fig. 2) are the true anomalies where the spacecraft goes "into" the orbits f^i and where it goes "out" of the orbits f^o. In addition the true anomaly of the alinement f^a can be determined for the specific problem of interest. All true anomalies represent a point in orbit and are defined between $0°$ and $360°$. The time increment Δt between two true anomalies is denoted by its superscripts. For example, Δt_t^{io} is interpreted as the time interval on the transfer orbit from f_t^i to f_t^o and is a positive quantity less than one orbital period. The actual time in an orbit is Δt^{io} plus an integral number of orbital periods. The magnitudes of the two-impulse maneuvers are denoted by ΔV_1 and ΔV_2. These two velocity maneuvers are factored to satisfy the timing constraints.

Figure 2.- True anomalies.

The rendezvous condition requires that the spacecraft achieve a specified position and velocity at a specified time. The proper position and velocity are satisfied by the choice of the geometry solution and the proper timing is achieved by factoring the impulse maneuvers. If the point of rendezvous is defined as the entry point into the final orbit f_f^i the time of rendezvous t_r is the sum of the time spent in each orbit from the initial point f_o^i to the final point f_f^i. The initial point f_o^i corresponds to zero time. Thus, the timing constraint for rendezvous is given by

$$t_r = \tau_r + n\eta \tag{1}$$

where $n = 0, 1, 2, \ldots, n_b$ and τ_r and η are constants. The parameter τ_r is the time of the first rendezvous opportunity and η is the time between rendezvous opportunities. The value of the integer variable n is bounded by n_b which limits the time of rendezvous. The different rendezvous opportunities correspond to the times at which a

rendezvous is possible. For example, if the problem were to dock the spacecraft with a satellite for which the orbit coincided with the final orbit, then a rendezvous opportunity would occur each time the satellite passed through the rendezvous point f_f^i. In this case τ_r would be the length of time for the satellite initially to achieve the true anomaly f_f^i and η would be the period of the final orbit. It might be that other mission constraints only permit rendezvous opportunities on alternate passages of the rendezvous point. In this case η would be twice the orbit period.

The alinement constraint requires that the spacecraft reach a given true anomaly f^a within a specified interval of time. If the time of alinement t_a is defined as the time from the initial point f_0^i to the alinement point f^a, then the inequality timing constraint for alinement is given by

$$-\epsilon_b \leqq t_a - \left(\tau_a + m\zeta\right) \leqq \epsilon_b \tag{2}$$

where $m = 0, 1, 2, \ldots, m_b$. The parameter τ_a depends upon the orbit on which alinement occurs; ζ is a constant denoting the time between alinement opportunities. The bounds on the timing error for alinement are $\pm\epsilon_b$ and the bound on the integer variable m is m_b.

The orbit on which alinement occurs is not specified. However, it is required that alinement occur on the I_a revolution. That is, f^a is passed I_a times before alinement occurs. It should be emphasized that I_a is a specified constant and imposes a constraint on the solution. If $I_a = 0$, then the alinement must occur the first time the spacecraft reaches f^a. Although I_a defines the number of revolutions prior to alinement, it does not define the alinement orbit until the number of revolutions in each orbit has been determined. Once this is accomplished, the alinement orbit can be determined by the logic of table 1.

The timing constraints on rendezvous and alinement are satisfied by the method of impulse factoring. Applying only part of the ΔV at f^o, waiting a number of revolutions on the intermediate orbit, and then applying the remainder of the ΔV at the same place allows the timing in orbit to be altered while the orbital geometry is preserved. Thus, varying the percentage of the ΔV applied and the number of revolutions in the various orbits satisfies the timing constraints. The factored intermediate orbits are designated the α- and β-orbits since they result from applying the factors α_v and β_v to either or both ΔV_1 and ΔV_2. Corresponding to each velocity factor is a period factor which denotes the accompanying change in period. For example, if ΔV_1 is factored by α_v so that the α-orbit is intermediate to the initial and transfer orbits, then the resulting period of the α-orbit is given by

$$P_\alpha = P_0 + \alpha\left(P_t - P_0\right) = P_0 + \alpha\ \Delta P_{to} \qquad (3)$$

where α is the period factor corresponding to α_v. Since the constraints are expressions of time, the period factors α and β are more tractable than the velocity factors α_v and β_v. Once the period factors are determined, the corresponding velocity factors are easily determined (appendix B). A factor between zero and one results in two impulses which have the same total ΔV as the single impulsive maneuver. A factor outside this range results in two impulses which sum to a higher total ΔV than the original single impulse. If the factor is between zero and one, the transfer is said to be "free" (without penalty).

The impulse-factoring method also determines the number of revolutions in each orbit. A revolution is defined as the number of times the spacecraft passes the maneuver point f^o without performing the maneuver. The number of revolutions in each of the four orbits (initial, α, transfer, and β) are denoted by the integer variables I, J, K, and L, respectively. Frequently, it is advantageous to limit the total number of revolutions prior to rendezvous. That is,

$$I + J + K + L \leqq I_s \qquad (4)$$

The two-impulse geometry solution may be factored into three or four impulses by considering all combinations of factoring either or both impulses into two or three parts. For convenience, define "bisect" to denote factoring into two collinear parts that vectorially sum to the original impulse, and "trisect" to denote factoring into three collinear parts that vectorially sum to the original impulse. If the impulse is not factored, it is said to be applied in "full." Limiting the total number of impulses to four or less allows five different types of solution: (1) bisect-full (BF), (2) full-bisect (FB), (3) bisect-bisect (BB), (4) trisect-full (TF), and (5) full-trisect (FT). The first two types are three-impulse solutions and the remainder are four-impulse solutions. It has been assumed that the full-full or geometry solution does not satisfy the time constraints.

The rendezvous problem with an intermediate time constraint as proposed herein does not have a unique solution. In fact, it is possible for all five types of solutions to satisfy the time constraints. The desired solution is the one which minimizes the sum of the impulses. For example, in a solution of the bisect-bisect type, the sum of the impulses is given by

$$\Delta V_T = \left|\alpha_v\right| \Delta V_1 + \left|1 - \alpha_v\right| \Delta V_1 + \left|\beta_v\right| \Delta V_2 + \left|1 - \beta_v\right| \Delta V_2 \qquad (5)$$

Thus, the problem consists of determining the set of integer variables I, J, K, L, m, n and the factors α and β which satisfy the rendezvous and alinement constraints at

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the least cost for the given geometry solution. To determine this solution, the variables I, J, K, and L are systematically varied within the following limits:

$$I \geqq I_{min} \geqq 0$$

$$J \geqq J_{min} > 0$$

$$K \geqq K_{min} \geqq 0$$

$$L \geqq L_{min} > 0$$

$$I + J + K + L \leqq I_S$$

where the minimum values of I, J, K, and L are imposed on the solution. The minimum values of J and L must be greater than zero to avoid eliminating their respective orbits. Next, for each set of I, J, K, and L, the best values of α, β, m, and n are determined for each of the five different types of solutions. In this manner, the type of solution and the values of I, J, K, L, m, n, α, and β which minimize the sum of the impulses are determined.

In the remainder of this section the necessary equations for each of the five types of solution are developed.

Bisect-Full Solution

The bisect-full type of solution is a three-impulse solution where ΔV_1 is factored by α_v to establish the α-orbit, which follows the initial orbit and precedes the transfer orbit. The ΔV_2 impulse is unaltered and applied in "full"; that is, the spacecraft traverses first the initial orbit, then the α-orbit, the transfer orbit, and the final orbit. The rendezvous time is satisfied by the α-factor which also dictates the alinement time. If the alinement time is within bounds, then the solution is acceptable, which means that the solution satisfies the two time constraints. This result is not intended to infer, however, that the solution is complete since other acceptable solutions may yield a lower total ΔV. If the alinement time is not within bounds, then the bisect-full solution is unacceptable.

For the bisect-full type of solution, the spacecraft proceeds from f_o^i on the initial orbit to the true anomaly of exit f_o^o. It then makes I complete revolutions in the initial orbit relative to f_o^o before transferring to the α-orbit. Since the transfer in and out of the α-orbit occurs at the same point $\left(f_\alpha^i = f_\alpha^o\right)$, the spacecraft makes J complete revolutions in the α-orbit before transferring into the transfer orbit. It then traverses to f_t^o, makes K complete revolutions relative to f_t^o, and transfers to the final orbit at f_f^i

which is the rendezvous point. Thus, equating the rendezvous time (eq. (1)) and the time in orbit until the spacecraft arrives at f_f^i yields

$$\Delta t_o^{io} + IP_o + JP_\alpha + \Delta t_t^{io} + KP_t = \tau_r + n\eta \tag{6}$$

where $n = 0, 1, 2, \ldots, n_b$. Substituting equation (3) for P_α into equation (6) yields

$$\Delta t_o^{io} + IP_o + J\left(P_o + \alpha \, \Delta P_{to}\right) + \Delta t_t^{io} + KP_t = \tau_r + n\eta$$

For a given value of I, J, and K,

$$C_1\alpha + C_2 = n\eta \tag{7}$$

where

$$C_1 = J \, \Delta P_{to}$$

$$C_2 = (I + J)P_o + KP_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$$

and the C's are constants. Thus, the value of α is given by

$$\alpha = \frac{n\eta - C_2}{C_1} \tag{8}$$

and the total cost is

$$\Delta V_T = \left|\alpha_v\right| \Delta V_1 + \left|1 - \alpha_v\right| \Delta V_1 + \Delta V_2 \tag{9}$$

where α_v is determined from α (appendix B).

Next, the defining equation for the alinement is written. Since this equation takes a different form for alinement on different orbits, each possibility is considered in turn. For a given value of I, J, and K, the alinement orbit is determined by table 1.

(a) Initial orbit.- Assume alinement on the initial orbit. Equating the time of aline-ment to the time of arrival of the spacecraft at the alinement point yields

$$\Delta t_o^{ia} + I'P_o + \epsilon = \tau_{a,o} + m\zeta \tag{10}$$

where $m = 0, 1, 2, \ldots, m_b$ and Δt_o^{ia} is the time increment on the initial orbit from f_o^i to f_o^a. The time error in the spacecraft arrival at the alinement point is denoted by ϵ. Solving equation (10) for ϵ and minimizing over m gives

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$$\epsilon = \min_{m}\left\{\left|m\zeta + C_3\right|\right\} \qquad\qquad \left(m = 0, 1, 2, \ldots, m_b\right) \qquad\qquad (11)$$

where

$$C_3 = \tau_{a,o} - \Delta t_o^{ia} - I'P_o$$

The parameter I' is the number of revolutions referenced to f_o^a before the spacecraft reaches the alinement point. Summing the angular traverse to the alinement point and equating it to the required traverse yields

$$\Delta f_o^{ia} + 360^{\circ}\, I_a = \Delta f_o^{ia} + 360^{\circ}\, I'$$

or

$$I' = I_a$$

Now, if $\epsilon \leqq \epsilon_b$, the solution is acceptable and ΔV_T can be minimized over n, where $n = 0, 1, 2, \ldots, n_b$.

(b) α-orbit.- Assume alinement on the α-orbit. The defining equation for alinement is

$$\Delta t_o^{io} + IP_o + \Delta t_\alpha^{ia} + J'P_\alpha + \epsilon = \tau_{a,\alpha} + m\zeta \qquad\qquad (12)$$

The values of Δt_α^{ia} and $\tau_{a,\alpha}$ require a knowledge of the α-orbit which is yet to be determined. To overcome this difficulty, assume a linear transition of these quantities with respect to α between the initial and transfer orbits. Thus, the following approximations are made:

$$\Delta t_\alpha^{ia} \approx \Delta t_o^{oa} + \alpha\left(\Delta t_t^{ia} - \Delta t_o^{oa}\right) \qquad\qquad (13)$$

$$\tau_{a,\alpha} \approx \tau_{a,o} + \alpha\left(\tau_{a,t} - \tau_{a,o}\right) \qquad\qquad (14)$$

Substituting equations (3), (13), and (14) into equation (12) yields

$$\Delta t_o^{io} + IP_o + \Delta t_o^{oa} + \alpha\left(\Delta t_t^{ia} - \Delta t_o^{oa}\right) + J'\left(P_o + \alpha\, \Delta P_{to}\right) + \epsilon = \tau_{a,o} + \alpha\left(\tau_{a,t} - \tau_{a,o}\right) + m\zeta$$

or

$$C_3\alpha + C_4 + \epsilon = m\zeta \qquad\qquad (15)$$

where

$$C_3 = J' \Delta P_{to} + \Delta t_t^{ia} - \Delta t_0^{oa} - \tau_{a,t} + \tau_{a,o}$$

$$C_4 = (I + J')P_0 + \Delta t_0^{io} + \Delta t_0^{oa} - \tau_{a,o}$$

If the position of alinement is assumed to be relatively insensitive to changes in orbit so that $\Delta f_0^{ia} + 360^{\circ} I_a$ is a good approximation to the angular traverse prior to alinement, then

$$\Delta f_0^{ia} + 360^{\circ} I_a \approx \Delta f_0^{io} + 360^{\circ} I + \Delta f_{\alpha}^{ia} + 360^{\circ} J'$$

Assuming that $\Delta f_{\alpha}^{ia} \approx \Delta f_0^{oa}$, which is consistent with the previous assumption, and solving for J' yields

$$J' = ci\left[I_a - I - \left(\frac{\Delta f_0^{io} + \Delta f_0^{oa} - \Delta f_0^{ia}}{360^{\circ}} \right) \right]$$

where the operator "ci" denotes closest integer. Solving equation (15) for ϵ and eliminating α with equation (8) gives

$$\epsilon = \min_{m,n} \left\{ \left| m\zeta - \left(\frac{n\eta - C_2}{C_1} \right) C_3 - C_4 \right| \right\} \tag{16}$$

where $n = 0, 1, 2, \ldots, n_b$ and $m = 0, 1, 2, \ldots, m_b$. If $\epsilon \leqq \epsilon_b$, then the solution is acceptable. Usually only one set of m and n will satisfy $\epsilon \leqq \epsilon_b$. However, if more than one acceptable solution exists, one can determine the acceptable values of m and n which yield the least cost according to equation (9).

(c) Transfer orbit.- Assume alinement on the transfer orbit. The defining equation for alinement is

$$\Delta t_0^{io} + IP_0 + JP_{\alpha} + \Delta t_t^{ia} + K'P_t + \epsilon = \tau_{a,t} + m\zeta$$

Substituting for P_{α} and collecting terms yields an equation of the form

$$C_3 \alpha + C_4 + \epsilon = m\zeta \tag{17}$$

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where

$$C_3 = J \, \Delta P_{to}$$

$$C_4 = (I + J)P_o + K'P_t + \Delta t_o^{io} + \Delta t_t^{ia} - \tau_{a,t}$$

$$K' = ci\left[I_a - I - J - \left(\frac{\Delta f_o^{io} + \Delta f_t^{ia} - \Delta f_o^{ia}}{360^o}\right)\right]$$

The simultaneous solution of equations (8) and (17) is given by equation (16).

(d) Final orbit.- Assume alinement on the final orbit. To this point it has been accepted that the rendezvous condition is satisfied at the entry point into the final orbit. However, once the rendezvous condition is satisfied, it is continuously satisfied unless the orbit is altered. Since the final orbit remains unaltered, alinement on the final orbit is allowed to occur on any successive passage of f_f^a provided it is on the I_ath revolution and the time of alinement does not exceed $t_a = \tau_{a,f} + m_b\zeta + \epsilon_b$. If the number of revolutions relative to f^a in the final orbit prior to alinement is denoted by M, the defining equation for alinement is then

$$\Delta t_o^{io} + IP_o + JP_\alpha + \Delta t_t^{io} + KP_t + \Delta t_f^{ia} + MP_f + \epsilon = \tau_{a,f} + m\zeta$$

Substituting for P_α and collecting terms yields an equation of the form

$$C_3\alpha + C_4 + \epsilon = m\zeta \tag{18}$$

where

$$C_3 = J \, \Delta P_{to}$$

$$C_4 = (I + J)P_o + KP_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$$

$$M = ci\left[I_a - I - J - K - \left(\frac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_f^{ia} - \Delta f_o^{ia}}{360^o}\right)\right]$$

The simultaneous solution of equations (8) and (18) is given by equation (16).

For the bisect-full type of solution the alinement can occur on four different orbits. Each of these possibilities has been fully investigated and results are summarized in table 2.

Full-Bisect Solution

The full-bisect type of solution is a three-impulse solution where ΔV_2 is factored by β_v so that the β-orbit follows the transfer orbit and precedes the final orbit. The ΔV_1 impulse is unaltered. This type of solution is similar to the bisect-full type with the difference being the factoring of ΔV_2 as opposed to the factoring of ΔV_1. Since the derivations of the governing equations are so similar, they are not developed here but are summarized in table 3.

Bisect-Bisect Solution

The bisect-bisect type of solution is a four-impulse solution where ΔV_1 is factored by α_v and ΔV_2 is factored by β_v in such manner that the order of the orbits is as follows: initial orbit, α-orbit, transfer orbit, β-orbit, final orbit. In this type of solution the two factors (α and β) are so determined that the two time constraints are satisfied.

The solution is determined by the simultaneous solution of the equations governing the alinement and rendezvous conditions. The rendezvous condition is

$$\Delta t_o^{io} + IP_o + JP_\alpha + \Delta t_t^{io} + KP_t + LP_\beta = \tau_r + n\eta$$

An expression for the period of the β-orbit is

$$P_\beta = P_t + \beta\,\Delta P_{ft} \tag{19}$$

which is similar to the expression for P_α (eq. (3)). Substituting these expressions for the periods in the rendezvous equation yields

$$C_1\alpha + C_2\beta + C_3 = n\eta \tag{20}$$

where

$$C_1 = J\,\Delta P_{to}$$

$$C_2 = L\,\Delta P_{ft}$$

$$C_3 = (I + J)P_o + (K + L)P_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$$

The cost of the solution is given by

$$\Delta V_T = \left|\alpha_v\right|\Delta V_1 + \left|1 - \alpha_v\right|\Delta V_1 + \left|\beta_v\right|\Delta V_2 + \left|1 - \beta_v\right|\Delta V_2 \tag{21}$$

Next, the governing equations for alinement are derived.

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(a) Initial orbit.- Assume alinement on the initial orbit. The alinement condition is given by equation (11) as

$$\epsilon = \min_{m}\left\{\left|m\zeta + C_4\right|\right\} \qquad \left(m = 0, 1, 2, \ldots, m_b\right)$$

where

$$C_4 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$$

If $\epsilon \leqq \epsilon_b$, then the solution is acceptable and ΔV_T is minimized subject to equation (20). From equation (20),

$$\alpha = \frac{n\eta - C_2\beta - C_3}{C_1} \qquad \qquad ($$

and substituting into equation (21) with the assumption that $\alpha_V \approx \alpha$ and $\beta_V \approx \beta$ gives

$$\Delta V_T \approx \left|\frac{n\eta - C_2\beta - C_3}{C_1}\right|\Delta V_1 + \left|\frac{C_1 - n\eta + C_2\beta + C_3}{C_1}\right|\Delta V_1 + \left|\beta\right|\Delta V_2 + \left|1 - \beta\right|\Delta V_2$$

It is desired to minimize ΔV_T over n and β. For a given value of n, ΔV_T is a piecewise linear function of β with corners (that is, discontinuous first derivatives) at the points where each of the four terms equal zero. Since ΔV_T is a piecewise linear function, its minimum must occur at one of the four corners created by the absolute-value operator. Another possibility exists if there is a range of β in the closed interval 0 to 1 so that α is also in the same range. In this case, the factoring is "free" and $\Delta V_T = \Delta V_1 + \Delta V_2$ over the closed interval 0 to 1. Thus, there exists a family of solutions which minimize ΔV_T. It can be shown that one of these solutions corresponds to one of the solutions obtained by setting each term of ΔV_T equal to zero. It follows that for either case ΔV_T is minimized by either $\alpha = 0$, $\alpha = 1$, $\beta = 0$, or $\beta = 1$ and the four-impulse bisect-bisect solution degenerates to one of the types of three-impulse solutions. All four cases must be investigated by using equation (20) to determine the other factor; and ΔV_T is minimized over $n = 0, 1, 2, \ldots, n_b$. Since $\alpha_V = \alpha$ and $\beta_V = \beta$ at the two points zero and one, the approximation is exact for the four degenerate cases.

(b) α-orbit.- Assume alinement on the α-orbit. The alinement condition is given by equation (15) as

$$C_4\alpha + C_6 + \epsilon = m\zeta \tag{22}$$

where

$$C_4 = J' \, \Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$$

$$C_6 = (I + J')P_o + \Delta t_o^{io} + \Delta t_o^{oa} - \tau_{a,o}$$

$$J' = ci\left[I_a - I - \left(\frac{\Delta f_o^{io} + \Delta f_o^{oa} - \Delta f_o^{ia}}{360^o}\right)\right]$$

The solution is obtained by minimizing ΔV_T subject to equations (20) and (22) where $-\epsilon_b \leqq \epsilon \leqq \epsilon_b$, $n = 0, 1, 2, \ldots, n_b$, and $m = 0, 1, 2, \ldots, m_b$. The solution is simplified by discretizing ϵ as $\epsilon = \lambda \epsilon_b$ where $\lambda = -1, 0, 1$. Thus, solving equation (22) for α and equation (20) for β yields

$$\Delta V_T = \min_{n,m,\lambda}\left\{\left|\alpha_v\right| \Delta V_1 + \left|1 - \alpha_v\right| \Delta V_1 + \left|\beta_v\right| \Delta V_2 + \left|1 - \beta_v\right| \Delta V_2\right\} \tag{23}$$

where

$$\alpha = \frac{m\zeta - \lambda\epsilon_b - C_6}{C_4} \qquad \left(\alpha \rightarrow \alpha_v\right)$$

$$\beta = \frac{n\eta - C_1\alpha - C_3}{C_2} \qquad \left(\beta \rightarrow \beta_v\right)$$

$$n = 0, 1, 2, \ldots, n_b$$

$$m = 0, 1, 2, \ldots, m_b$$

$$\lambda = -1, 0, 1$$

and α_v and β_v are derived in appendix B.

(c) Transfer orbit.- Assume alinement on the transfer orbit. The alinement is given by equation (17) as

$$C_4\alpha + C_6 + \epsilon = m\zeta \tag{24}$$

where

$$C_4 = J \, \Delta P_{to}$$

$$C_6 = (I + J)P_o + K'P_t + \Delta t_o^{io} + \Delta t_t^{ia} - \tau_{a,t}$$

$$K' = ci \left[I_a - I - J - \left(\frac{\Delta f_o^{io} + \Delta f_t^{ia} - \Delta f_o^{ia}}{360^o} \right) \right]$$

The solution to equations (20) and (24) is given by equation (23).

(d) β-orbit.- Assume alinement on the β-orbit. The defining equation for alinement
is

$$\Delta t_o^{io} + IP_o + JP_\alpha + \Delta t_t^{io} + KP_t + \Delta t_\beta^{ia} + L'P_\beta + \epsilon = \tau_{a,\beta} + m\zeta$$

where

$$\Delta t_\beta^{ia} \approx \Delta t_t^{oa} + \beta \left(\Delta t_f^{ia} - \Delta t_t^{oa} \right)$$

$$\tau_{a,\beta} \approx \tau_{a,t} + \beta \left(\tau_{a,f} - \tau_{a,t} \right)$$

Making these substitutions along with equations (3) and (19) yields

$$C_4\alpha + C_5\beta + C_6 + \epsilon = m\zeta \tag{25}$$

where

$$C_4 = J \, \Delta P_{to}$$

$$C_5 = \Delta t_f^{ia} - \Delta t_t^{oa} + L' \, \Delta P_{ft} - \tau_{a,f} + \tau_{a,t}$$

$$C_6 = (I + J)P_o + (K + L')P_t + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_t^{oa} - \tau_{a,t}$$

$$L' = ci \left[I_a - I - J - K - \left(\frac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_t^{oa} - \Delta f_o^{ia}}{360^o} \right) \right]$$

In general, equations (20) and (25) can be solved simultaneously for α and β. Thus, the solution is

$$\Delta V_T = \min_{n,m,\lambda} \left\{ \left| \alpha_v \right| \Delta V_1 + \left| 1 - \alpha_v \right| \Delta V_1 + \left| \beta_v \right| \Delta V_2 + \left| 1 - \beta_v \right| \Delta V_2 \right\} \tag{26}$$

where

$$\alpha = \frac{-\left(n\eta - C_3 \right) C_5 + \left(m\zeta - C_6 - \lambda\epsilon_b \right) C_2}{C_1 \left(C_2 - C_5 \right)} \qquad \left(\alpha \rightarrow \alpha_v \right) \tag{27}$$

$$\beta = \frac{\left(n\eta - C_3 \right) - \left(m\zeta - C_6 - \lambda\epsilon_b \right)}{C_2 - C_5} \qquad \left(\beta \rightarrow \beta_v \right) \tag{28}$$

$$n = 0, 1, 2, \ldots, n_b$$

$$m = 0, 1, 2, \ldots, m_b$$

$$\lambda = -1, 0, 1$$

and the relation $C_4 = C_1$ has been incorporated.

This solution is well behaved unless $C_1 = 0$ or $C_2 = C_5$. The constant $C_1 = J \Delta P_{to}$ must be nonzero since $J > 0$ and $\Delta P_{to} \neq 0$ because the geometry solution is restricted to orbits with distinct periods. If $C_2 = C_5$, then from equations (20) and (25) the minimum value of ϵ is

$$\epsilon = \min_{m,n} \left\{ \left| \left(n\eta - C_3 \right) - \left(m\zeta - C_6 \right) \right| \right\} \tag{29}$$

where $n = 0, 1, 2, \ldots, n_b$ and $m = 0, 1, 2, \ldots, m_b$, and

$$\alpha = \frac{\left(n\eta - C_3 \right) - C_2\beta}{C_1} \qquad \left(\alpha \rightarrow \alpha_v \right) \tag{30}$$

If $\epsilon \leq \epsilon_b$, then the solution is acceptable and ΔV_T is minimized. From equation (30) ΔV_T can be expressed in terms of four continuous linear functions of either α or β. Since the minimum of ΔV_T must occur at one of the corners created by the absolute operator, either $\alpha = 0$, $\alpha = 1$, $\beta = 0$, or $\beta = 1$. The minimum of ΔV_T is chosen from these four possibilities.

(e) Final orbit.- Assume alinement on the final orbit. The defining condition for alinement is given by

$$\Delta t_o^{io} + IP_o + JP_\alpha + \Delta t_t^{io} + KP_t + LP_\beta + \Delta t_f^{ia} + MP_f + \epsilon = \tau_{a,f} + m\zeta$$

Substituting for P_α and P_β and collecting terms yields

$$C_1\alpha + C_2\beta + C_4 + \epsilon = m\zeta \tag{31}$$

where

$$C_4 = (I + J)P_o + (K + L)P_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$$

$$M = ci\left[I_a - I - J - K - L - \left(\frac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_f^{ia} - \Delta f_o^{ia}}{360^0}\right)\right]$$

From equations (20) and (31) the minimum value of ϵ is

$$\epsilon = \min_{m,n}\left\{\left|\left(n\eta - C_3\right) - \left(m\zeta - C_4\right)\right|\right\}$$

where $n = 0, 1, 2, . . ., n_b$ and $m = 0, 1, 2, . . ., m_b$. If $\epsilon \leqq \epsilon_b$, the solution is acceptable and ΔV_T is minimized subject to equation (20); that is,

$$\alpha = \frac{n\eta - C_2\beta - C_3}{C_1} \qquad (\alpha \rightarrow \alpha_v) \tag{32}$$

From equations (32) and (26), ΔV_T can be expressed as the sum of four continuous linear functions of either α or β. Since the minimum of ΔV_T must occur at one of the corners created by the absolute operator, either $\alpha = 0$, $\alpha = 1$, $\beta = 0$, or $\beta = 1$. The minimum of ΔV_T is chosen from these four possibilities.

For the bisect-bisect type solution the alinement can occur on any of the five orbits. Each case has been investigated and is summarized in table 4.

Trisect-Full Solution

The trisect-full type of solution is a four-impulse solution in which ΔV_1 is factored into three parts by α and β and ΔV_2 is applied in full. The order of the orbits are the initial orbit, α-orbit, β-orbit, transfer orbit, and final orbit.

The period of the α-orbit is the same as previously defined; that is,

$$P_\alpha = P_0 + \alpha\left(P_t - P_0\right) = P_0 + \alpha\,\Delta P_{to} \tag{33}$$

and the period of the β-orbit is

$$P_\beta = P_\alpha + \beta\left(P_t - P_\alpha\right)$$

or

$$P_\beta = P_0 + (\alpha + \beta - \alpha\beta)\,\Delta P_{to} \tag{34}$$

where P_α was eliminated by use of equation (33). Thus, the α-factor bisects the period difference between the initial and transfer orbit, and the β-factor bisects the period difference between the α- and transfer orbits. The result is to trisect the ΔV_1 impulse. The defining equation for rendezvous is

$$\Delta t_0^{io} + IP_0 + JP_\alpha + LP_\beta + \Delta t_t^{io} + KP_t = \tau_r + n\eta$$

Substituting for P_α and P_β (eqs. (33) and (34)) and collecting terms yields

$$C_1\alpha + C_2\beta(1 - \alpha) + C_3 = n\eta \tag{35}$$

where

$$C_1 = (J + L)\,\Delta P_{to}$$

$$C_2 = L\,\Delta P_{to}$$

$$C_3 = (I + J + L)P_0 + KP_t + \Delta t_0^{io} + \Delta t_t^{io} - \tau_r$$

Equation (35) can be solved for either α or β in terms of the other factor; that is,

$$\alpha = \frac{n\eta - C_2\beta - C_3}{C_1 - C_2\beta} \qquad \left(\alpha \to \alpha_v\right) \tag{36}$$

or

$$\beta = \frac{n\eta - C_1\alpha - C_3}{C_2(1 - \alpha)} \qquad \left(\beta \to \beta_v\right) \tag{37}$$

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From equation (36) note that a value of $\beta = \dfrac{C_1}{C_2} = \dfrac{J + L}{L}$ cannot be tolerated. The reason for this restriction becomes clear by examination of the time spent in the α- and β-orbits; that is,

$$JP_\alpha + LP_\beta = JP_\alpha + L\left[P_\alpha + \beta\left(P_t - P_\alpha\right)\right] = (J + L)P_t$$

Thus, the time spent in the α- and β-orbits is independent of α if $\beta = \dfrac{J + L}{L}$ and the rendezvous condition cannot be satisfied. Essentially, the problem is that the α-orbit causes a phase shift and the β-orbit causes an equal and opposite shift relative to the period of the transfer orbit. From equation (37) note that $\alpha = 1$ cannot be tolerated. Physically, a value of $\alpha = 1$ results in the α-orbit coinciding with the transfer orbit and there is no period difference for β to factor. For the acceptable solutions the cost is given by

$$\Delta V_T = \left|\alpha_v\right| \Delta V_1 + \left|\beta_v\right|\left|1 - \alpha_v\right| \Delta V_1 + \left|1 - \beta_v\right|\left|1 - \alpha_v\right| \Delta V_1 + \Delta V_2 \tag{38}$$

Next, the governing equations for alinement are derived.

(a) Initial orbit.- Assume alinement on the initial orbit. The alinement condition is given by equation (11) as

$$\epsilon = \min_m \left\{ \left| m\zeta + C_4 \right| \right\} \qquad \left(m = 0,\ 1,\ 2,\ \ldots,\ m_b\right)$$

where

$$C_4 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$$

If $\epsilon \leqq \epsilon_b$, the solution is acceptable and ΔV_T is minimized subject to equation (35). As before ΔV_T can be expressed as the sum of three continuous functions of either α or β. Since the minimum of ΔV_T must occur at one of the corners created by the absolute operator, either $\alpha = 0$, $\beta = 0$, or $\beta = 1$. The minimum of ΔV_T is chosen from these three possibilities.

(b) α-orbit.- Assume alinement on the α-orbit. The alinement condition is given by equation (22) as

$$C_4 \alpha + C_5 + \epsilon = m\zeta \tag{39}$$

where

$$C_4 = J' \Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$$

$$C_5 = (I + J')P_0 + \Delta t_0^{io} + \Delta t_0^{oa} - \tau_{a,o}$$

$$J' = ci\left[I_a - I - \left(\frac{\Delta f_0^{io} + \Delta f_0^{oa} - \Delta f_0^{ia}}{360^0}\right)\right]$$

Solving equation (39) for α gives

$$\alpha = \frac{m\zeta - \lambda \epsilon_b - C_5}{C_4} \qquad (\alpha \to \alpha_v) \tag{40}$$

Thus, the solution is obtained by minimizing ΔV_T where α is given by equation (40); β is given by equation (37); and $n = 0, 1, 2, \ldots, n_b$; $m = 0, 1, 2, \ldots, m_b$; $\lambda = -1, 0, 1$.

(c) β-orbit. - Assume alinement on the β-orbit. The alinement equation is given by

$$\Delta t_0^{io} + IP_0 + JP_\alpha + \Delta t_\beta^{ia} + L'P_\beta + \epsilon = \tau_{a,\beta} + m\zeta \tag{41}$$

The values of Δt_β^{ia} and $\tau_{a,\beta}$ require a knowledge of the α- and β-orbits which are yet to be determined. This difficulty can be overcome by making the following approximations:

$$\Delta t_\beta^{ia} \approx \Delta t_0^{oa} + (\alpha + \beta - \alpha\beta)\left(\Delta t_t^{ia} - \Delta t_0^{oa}\right)$$

$$\tau_{a,\beta} \approx \tau_{a,o} + (\alpha + \beta - \alpha\beta)\left(\tau_{a,t} - \tau_{a,o}\right)$$

Substituting these approximations, along with equations (33) and (34), into equation (41) yields

$$C_4\alpha + C_5\beta(1 - \alpha) + C_6 + \epsilon = m\zeta \tag{42}$$

where

$$C_4 = (J + L')\Delta P_{to} + \Delta t_t^{ia} - \Delta t_0^{oa} - \tau_{a,t} + \tau_{a,o}$$

$$C_5 = L'\Delta P_{to} + \Delta t_t^{ia} - \Delta t_0^{oa} - \tau_{a,t} + \tau_{a,o}$$

$$C_6 = (I + J + L')P_0 + \Delta t_0^{io} + \Delta t_0^{oa} - \tau_{a,o}$$

$$L' = ci\left[I_a - I - J - \left(\frac{\Delta f_0^{io} + \Delta f_0^{oa} - \Delta f_0^{ia}}{360^0}\right)\right]$$

Solving equations (35) and (42) for α yields

$$\alpha = \frac{(n\eta - C_3)C_5 - (m\zeta - C_6)C_2 + C_2\lambda\epsilon_b}{C_1C_5 - C_2C_4} \qquad (\alpha \rightarrow \alpha_v) \qquad (43)$$

Thus, the solution is obtained by minimizing ΔV_T where α is given by equation (43); β is given by equation (37); and $n = 0, 1, 2, \ldots, n_b$; $m = 0, 1, 2, \ldots, m_b$; $\lambda = -1, 0, 1$.

(d) Transfer orbit.- Assume alinement on the transfer orbit. The alinement equation is given by

$$\Delta t_o^{io} + IP_o + JP_\alpha + LP_\beta + \Delta t_t^{ia} + K'P_t + \epsilon = \tau_{a,t} + m\zeta$$

Substituting for P_α and P_β and collecting terms yields

$$C_1\alpha + C_2\beta(1 - \alpha) + C_4 + \epsilon = m\zeta \qquad (44)$$

where

$$C_4 = (I + J + L)P_o + K'P_t + \Delta t_o^{io} + \Delta t_t^{ia} - \tau_{a,t}$$

$$K' = ci\left[I_a - I - J - L - \left(\frac{\Delta t_o^{io} + \Delta t_t^{ia} - \Delta t_o^{ia}}{360^o}\right)\right]$$

and C_1 and C_2 are given by equation (35). It is readily seen that equations (35) and (44) cannot be solved simultaneously for α and β. Thus, solving these equations for ϵ and minimizing over m and n gives

$$\epsilon = \min_{m,n}\left\{\left|-(n\eta - C_3) + (m\zeta - C_4)\right|\right\}$$

where $n = 0, 1, 2, \ldots, n_b$ and $m = 0, 1, 2, \ldots, m_b$. If $\epsilon \leq \epsilon_b$, the solution is acceptable and ΔV_T is minimized over the factors subject to equation (35). Here, again, the minimization of ΔV_T can be reduced to three possibilities, $\alpha = 0$, $\beta = 0$, and $\beta = 1$.

(e) Final orbit.- Assume alinement on the final orbit. This solution is very similar to the solution for alinement on the transfer orbit. Therefore, it is not repeated but is summarized in table 5 along with the five cases of the trisect-full type of solution.

Full-Trisect Solution

The full-trisect type of solution is a four-impulse solution where ΔV_1 is applied in full and ΔV_2 is factored into three parts by α and β. The order of the orbits is

the initial, transfer, α-, β-, and final orbit. This type of solution is similar to the trisect-full solution with the difference being the trisecting of ΔV_2 as compared with trisecting ΔV_1. Since the derivation of the governing equations are so similar, they are not developed here but are summarized in table 6.

APPLICATION

As an example of the impulse-factoring technique, consider a satellite mission about the planet Mars. Assume the satellite is initially in a known orbit. A sequence of impulsive maneuvers is desired that will cause the spacecraft to fly over a selected surface feature for reconnaissance and then establish a synchronous orbit in such a way that the surface feature is directly beneath the spacecraft at each periapsis passage. This type of mission might involve surface mapping or it might involve establishing a proper orbit from which to launch a surface probe.

The geometry of the mission is presented in figure 3. The Keplerian orbital elements of the initial orbit and the right ascension of the surface feature are known at some epoch. The first task is to define a two-impulse geometry solution that establishes the desired final orbit. The conditions imposed on the final orbit are that it have a synchronous period and that periapsis be directly over the latitude of the surface feature. In
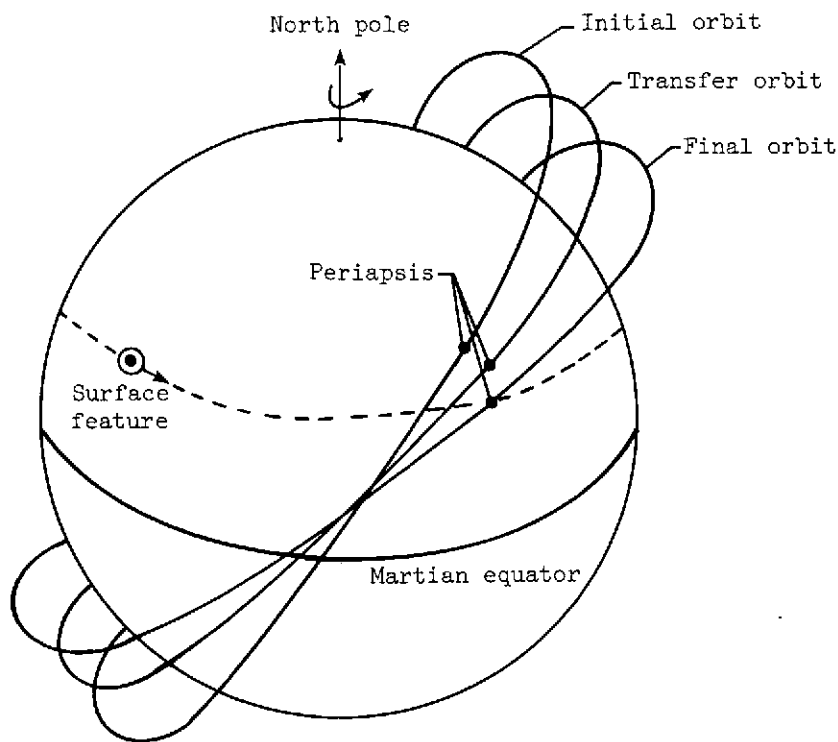

Figure 3.- Mission geometry.

addition, the altitude of periapsis is fixed. Thus, the synchronous period and the periapsis altitude define the shape of the orbit, that is, the semimajor axis a and the eccentricity e. The condition of periapsis over the correct latitude leads to a relationship between the orientation angles i, ω, Ω. Since the final orbit is not fully defined, the geometry solution is not unique and various two-impulse solutions are possible. A specific geometry solution can be defined, however, by choosing the magnitude and direction of the first impulse, the position in the initial orbit at which it is applied, and the true anomaly of entry into the final orbit. These five independent quantities along with the quantities already specified uniquely define the geometry solution. (See appendix A.) Since the choice of the five independent quantities is arbitrary, these quantities can be varied to minimize the sum of the impulses for the geometry solution. (See ref. 10.) This minimization gives the following solution:

	Initial orbit	Transfer orbit	Final orbit
a	*20762	21242	*20428
e	*0.77524	0.76285	*0.76045
i . . . . . i	*33.20	33.86	33.24
ω	*34.38	30.32	28.17
Ω	*104.85	107.39	107.93
P	*25.23	26.11	*24.62
f^i	*0.00	213.51	125.87
f^o	211.58	124.18	-------
f^a	*-6.17	-2.63	*0.00
Δf^{io}	211.58	270.67	-------
Δf^{ia}	*353.83	143.86	234.13
Δf^{oa}	142.25	233.19	-------
Δt^{io}	21.04	6.30	-------
Δt^{ia}	*25.20	4.22	22.55
Δt^{oa}	4.16	24.03	-------
τ_a	*7.86	7.99	8.06
δ_p	*18.01	16.33	*15.00

*Fixed quantity.

The true anomaly of alinement f^a depends on the orbit. For this example it is defined as the true anomaly directly above the latitude of the surface feature which is 15°. Likewise, the time constant for alinement τ_a depends on the orbit and is defined as the time required for the surface feature to rotate beneath the orbit, that is, to rotate to the point of alinement. Since the object of alinement is a surface feature, the time between alinement opportunities ζ is the rotational period of Mars (24.62 hours) which is also the

time between rendezvous opportunities η since the final orbit is synchronous. The time constant for rendezvous τ_r causes the surface feature to be directly beneath the spacecraft at periapsis passage on the final orbit. Numerically, it is equal to the time required for the surface point to rotate beneath the final orbit minus the time required for the spacecraft to pass from f_f^i to periapsis $\left(\text{recall that rendezvous occurs at } f_f^i\right)$, that is

$$\tau_r = \tau_{a,f} - \Delta t_f^{ia}$$

$$\tau_r = 8.06 - 22.55$$

$$\tau_r = -14.49 \text{ or } 10.13$$

Since rendezvous cannot occur at a negative time, and since the time of rendezvous is $t_r = \tau_r + n\eta$, where $n = 0, 1, 2, \ldots, n_b$, the time constant is modulo η or $\tau_r = 10.13$. As a matter of interest, the latitude of periapsis δ_P on the three orbits is shown. The only other quantities needed for the impulse-factoring technique are:

$\Delta V_1 = 0.03467$ $\qquad\qquad$ $L_{min} = 2$

$\Delta V_2 = 0.02688$ $\qquad\qquad$ $I_a = 4$

$\epsilon_b = 0.5$ $\qquad\qquad$ $I_s = 8$

$I_{min} = 1$ $\qquad\qquad$ $m_b = 12$

$J_{min} = 2$ $\qquad\qquad$ $n_b = 11$

$K_{min} = 1$

The sum of velocity impulses is minimized for each type of solution and the results are presented below:

Type of solution	I	J	K	L	m	n	α	β	ΔV_T, km/sec
Bisect-full	1	2	1	---	5	5	2.336	-----	0.14369
Full-bisect	1	---	1	2	5	5	----	-0.790	.09995
Bisect-bisect	1	3	1	3	5	9	1.676	1.206	.11635
Trisect-full	1	3	1	3	5	9	1.623	4.188	.24715
Full-trisect	1	2	1	4	5	9	-.586	.936	.09036

None of the five types of solutions are free since all have factors outside the range 0 to 1. The least expensive solution, however, is the "full-trisect" type of solution where alinement occurs on the α-orbit (note f^a is passed twice on the transfer orbit) after approximately 5 Martian days and rendezvous on the ninth day. In order to satisfy the alinement constraint, α is required to be negative and results in additional cost relative to the cost of the geometry solution. However, this is the least expensive way to satisfy the rendezvous condition with an intermediate timing constraint with the given geometry solution. The flow diagram for this example is given in figure 4(a).

In the preceding numerical example the cost of the geometry solution $(\Delta V_1 + \Delta V_2)$ was minimized by varying the five independent quantities associated with the solution and then the impulse-factoring technique was applied. If the resulting factors are outside the range 0 to 1, then the total cost ΔV_T will exceed the cost of the geometry solution. For this situation the total cost may be reduced at the expense of additional calculations by minimizing ΔV_T over the five independent quantities. This approach involves applying the impulse-factoring technique at each step during the minimization process instead of only once as shown in figure 4(b). The result is to change the geometry solution in such a way that the factoring technique is less expensive. The geometry solution resulting from minimizing ΔV_T is given in the following tabulation:

	Initial orbit	Transfer orbit	Final orbit
a	*20762	21563	*20428
e	*0.77524	0.76868	*0.76045
i	*33.20	33.34	33.04
ω	*34.38	30.33	28.34
Ω	*104.85	106.54	106.98
P	*25.23	26.70	*24.62
f^i	*0.00	230.14	113.58
f^o	227.51	111.95	-------
f^a	*-6.17	-2.24	*0.00
Δf^{io}	227.51	241.81	-------
Δf^{ia}	*353.83	127.62	246.42
Δf^{oa}	126.32	245.81	-------
Δt^{io}	22.81	3.89	-------
Δt^{ia}	*25.20	2.42	23.15
Δt^{oa}	2.38	25.23	-------
τ_a	*7.86	7.96	8.01
δp	*18.01	16.12	*15.00

*Fixed quantity.
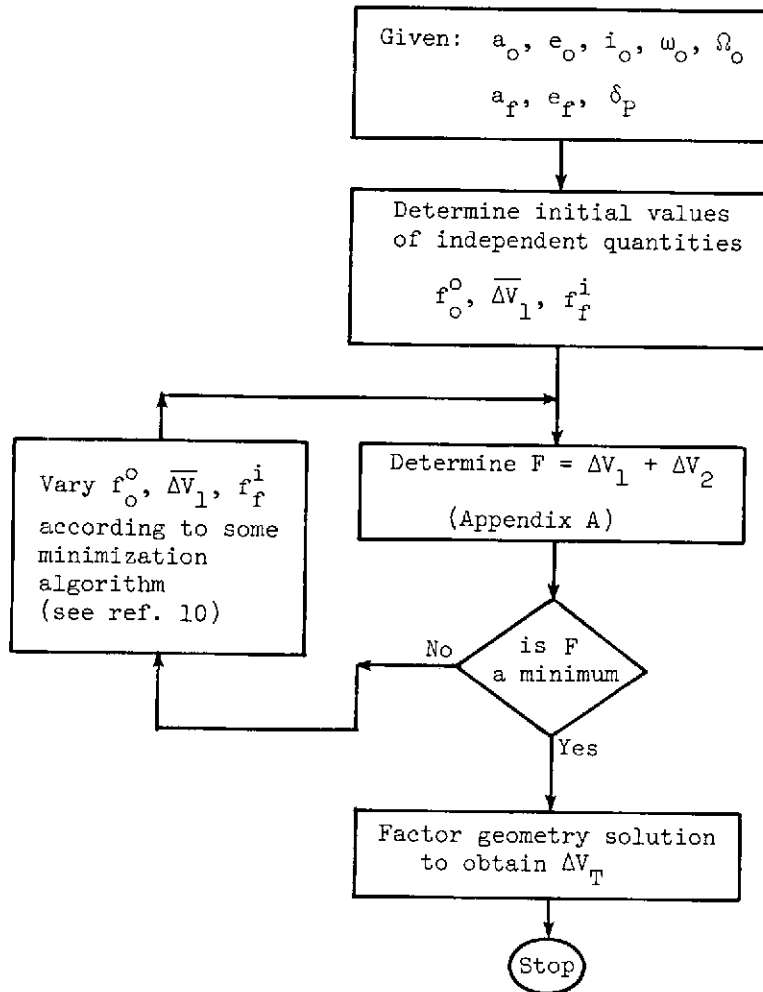
(a) Minimum geometry cost solution.
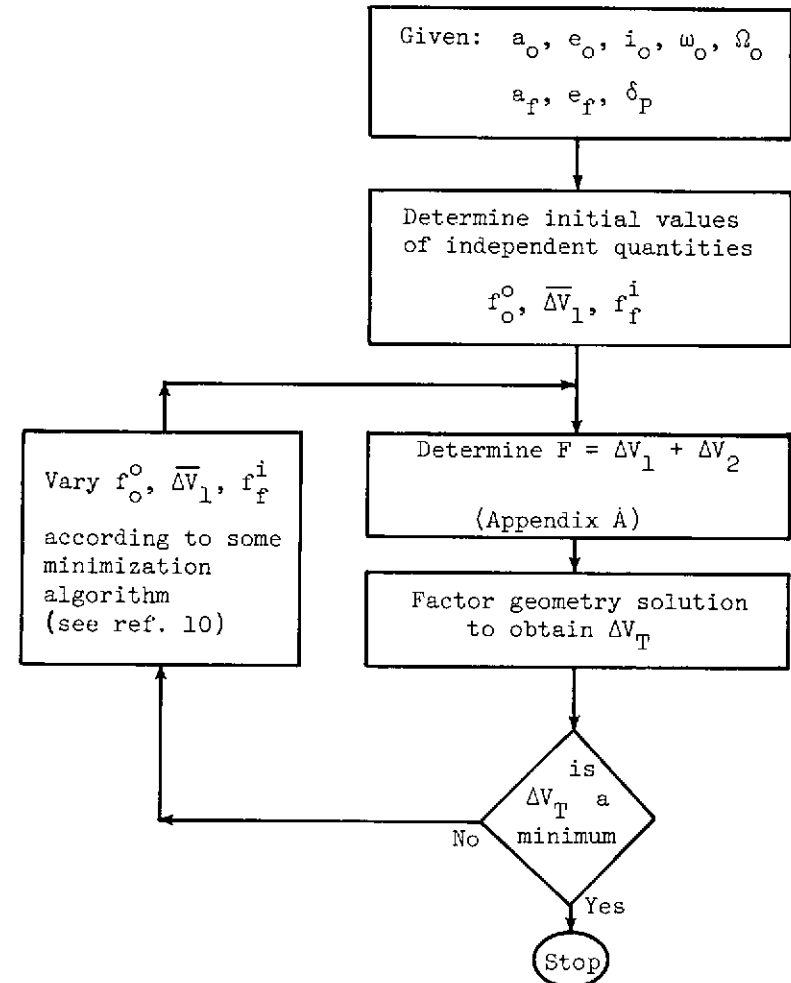
(b) Minimum total cost solution.

Figure 4.- Flow diagrams of impulse-factoring solution.

The cost of the geometry solution is $\Delta V_1 = 0.03906$ and $\Delta V_2 = 0.02719$ for a total cost of 0.06625 km/sec. This result is slightly greater than that of the first example which cost 0.06155 km/sec. However, in the second example the geometry solution has been changed to reflect the factoring process. The results of the five types of solutions are presented below:

Type of solution	I	J	K	L	m	n	α	β	ΔV_T, km/sec
Bisect-full	1	2	1	---	5	5	1.186	------	0.07969
Full-bisect	1	---	3	2	5	7	----	0.868	.06625
Bisect-bisect	1	2	1	2	5	7	1.000	.868	.06625
Trisect-full	1	3	1	3	5	9	.973	-63.013	.20688
Full-trisect	1	2	1	2	5	7	.000	.868	.06625

It is seen that the "full-bisect," "bisect-bisect," and the "full-trisect" solutions are identical and that they are free. Thus, selecting a geometry solution which minimizes ΔV_T instead of $\left(\Delta V_1 + \Delta V_2\right)$ reduces the total cost from 0.09036 to 0.06625 km/sec. The reduction in cost is not always this appreciable; however, when the cost of the factored solution greatly exceeds the cost of the geometry solution, the potential for a cost reduction exists by minimizing ΔV_T.

CONCLUDING REMARKS

A method has been presented for factoring a two-impulse orbital transfer into a three- or four-impulse transfer which solves the rendezvous problem and satisfies an intermediate alinement constraint. Five types of solutions exist and depend upon the factoring scheme employed. The equations governing each solution have been derived.

The impulse-factoring technique has many advantages which make it desirable. The rendezvous problem is simplified by first solving the orbital geometry transfer and then satisfying the time constraints by factoring the velocity-impulse maneuvers. The method generates a number of different solutions which satisfy the rendezvous condition and the alinement constraint. This set of solutions is finite for a given geometry transfer, and the minimum velocity solution can be chosen from the set. The method is easily programed and circumvents many of the mathematical and computational problems associated with more classical approaches.

An application of the impulse-factoring technique has been presented. An example problem was solved by minimizing the sum of the two impulsive velocity maneuvers of

the geometry transfer and then factoring the impulses to satisfy the time constraints. This rendezvous solution required more velocity increment ΔV than the geometry transfer. Therefore, the example problem was reworked to minimize the ΔV of the rendezvous solutions by varying the geometry transfer, resulting in a reduction of ΔV. It is concluded that if the rendezvous solution costs more than the geometry transfer, it is better to determine the geometry transfer which reflects the factoring process than merely to factor the geometry transfer of least ΔV.

Langley Research Center,
    National Aeronautics and Space Administration,
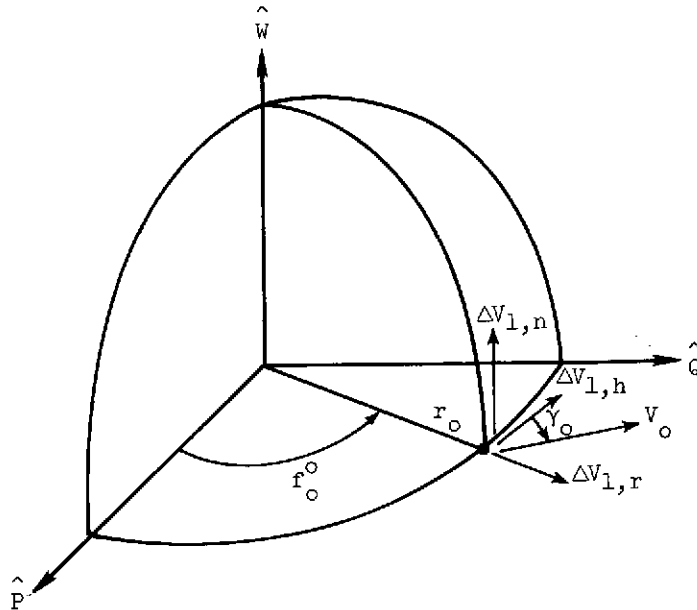   Hampton, Va., April 4, 1974.

TWO-IMPULSE ORBITAL TRANSFER TO ESTABLISH a, e, δ

The necessary equations for defining a two-impulse transfer from a known Keplerian initial orbit to a partially specified final orbit are derived in this appendix. The initial orbit is defined by a_o, e_o, i_o, ω_o, Ω_o; and the final orbit is partially specified by a_f, e_f, and δ where δ is the latitude of the spacecraft when the true anomaly has the value ψ. The two velocity maneuvers are defined as (see sketch A):

$$\overline{\Delta V_1} \equiv \left(\Delta V_{1,r}, \ \Delta V_{1,h}, \ \Delta V_{1,n} \right)$$

$$\overline{\Delta V_2} \equiv \left(\Delta V_{2,r}, \ \Delta V_{2,h}, \ \Delta V_{2,n} \right)$$

Sketch A

The problem to be solved can be stated as follows:

Given: a_o, e_o, i_o, ω_o, Ω_o, a_f, e_f, δ, ψ, μ, f_o^o, $\Delta V_{1,r}$, $\Delta V_{1,h}$, $\Delta V_{1,n}$, f_f^i

Find: $\overline{\Delta V_2}$ and the complete description of the transfer and final orbits.

The parameter f_o^o is the true anomaly in the initial orbit where $\overline{\Delta V_1}$ is applied and f_f^i is the true anomaly in the final orbit corresponding to the $\overline{\Delta V_2}$ maneuver.

Although the parameters f_o^o, $\Delta V_{1,r}$, $\Delta V_{1,h}$, $\Delta V_{1,n}$, f_f^i could be considered independent variables and $\Delta V_1 + \Delta V_2$ could be minimized, this problem is not addressed here.

The first transfer is straightforward. Consider the initial orbit in the $\hat{P}, \hat{Q}, \hat{W}$ coordinate system (sketch A). The magnitude of the first maneuver is simply

$$\Delta V_1 = \left(\Delta V_{1,r}^2 + \Delta V_{1,h}^2 + \Delta V_{1,n}^2\right)^{1/2}$$

and the elements of the resulting transfer orbit can be established by computing its components of position $(\vec{r})$ and velocity $(\vec{V})$ in the $\hat{P}, \hat{Q}, \hat{W}$ system, rotating these components to the original $\hat{X}, \hat{Y}, \hat{Z}$ system, and finally converting them to orbital elements. These calculations are as follows:

$$r_o = \frac{a_o\left(1 - e_o^2\right)}{1 + e_o \cos f_o^o}$$

$$V_o = \left[\mu\left(\frac{2}{r_o} - \frac{1}{a_o}\right)\right]^{1/2}$$

$$\sin \gamma_o = \frac{e_o \sin f_o^o}{\left(1 + 2e_o \cos f_o^o + e_o^2\right)^{1/2}}$$

$$\cos \gamma_o = \left(1 - \sin^2 \gamma_o\right)^{1/2}$$

$$r_p = r_o \cos f_o^o$$

$$r_q = r_o \sin f_o^o$$

$$r_w = 0$$

$$V_p = \left(V_o \sin \gamma_o + \Delta V_{1,r}\right) \cos f_o^o - \left(V_o \cos \gamma_o + \Delta V_{1,h}\right) \sin f_o^o$$

$$V_q = \left(V_o \sin \gamma_o + \Delta V_{1,r}\right) \sin f_o^o + \left(V_o \cos \gamma_o + \Delta V_{1,h}\right) \cos f_o^o$$

$$V_w = \Delta V_{1,n}$$

$$R_{11} = \cos \omega_o \cos \Omega_o - \sin \omega_o \sin \Omega_o \cos i_o$$

$$R_{12} = -\sin \omega_O \cos \Omega_O - \cos \omega_O \sin \Omega_O \cos i_O$$

$$R_{13} = \sin \Omega_O \sin i_O$$

$$R_{21} = \cos \omega_O \sin \Omega_O + \sin \omega_O \cos \Omega_O \cos i_O$$

$$R_{22} = -\sin \omega_O \sin \Omega_O + \cos \omega_O \cos \Omega_O \cos i_O$$

$$R_{23} = -\cos \Omega_O \sin i_O$$

$$R_{31} = \sin \omega_O \sin i_O$$

$$R_{32} = \cos \omega_O \sin i_O$$

$$R_{33} = \cos i_O$$

$$r_x = R_{11} r_p + R_{12} r_q + R_{13} r_w$$

$$r_y = R_{21} r_p + R_{22} r_q + R_{23} r_w$$

$$r_z = R_{31} r_p + R_{32} r_q + R_{33} r_w$$

$$V_x = R_{11} V_p + R_{12} V_q + R_{13} V_w$$

$$V_y = R_{21} V_p + R_{22} V_q + R_{23} V_w$$

$$V_z = R_{31} V_p + R_{32} V_q + R_{33} V_w$$

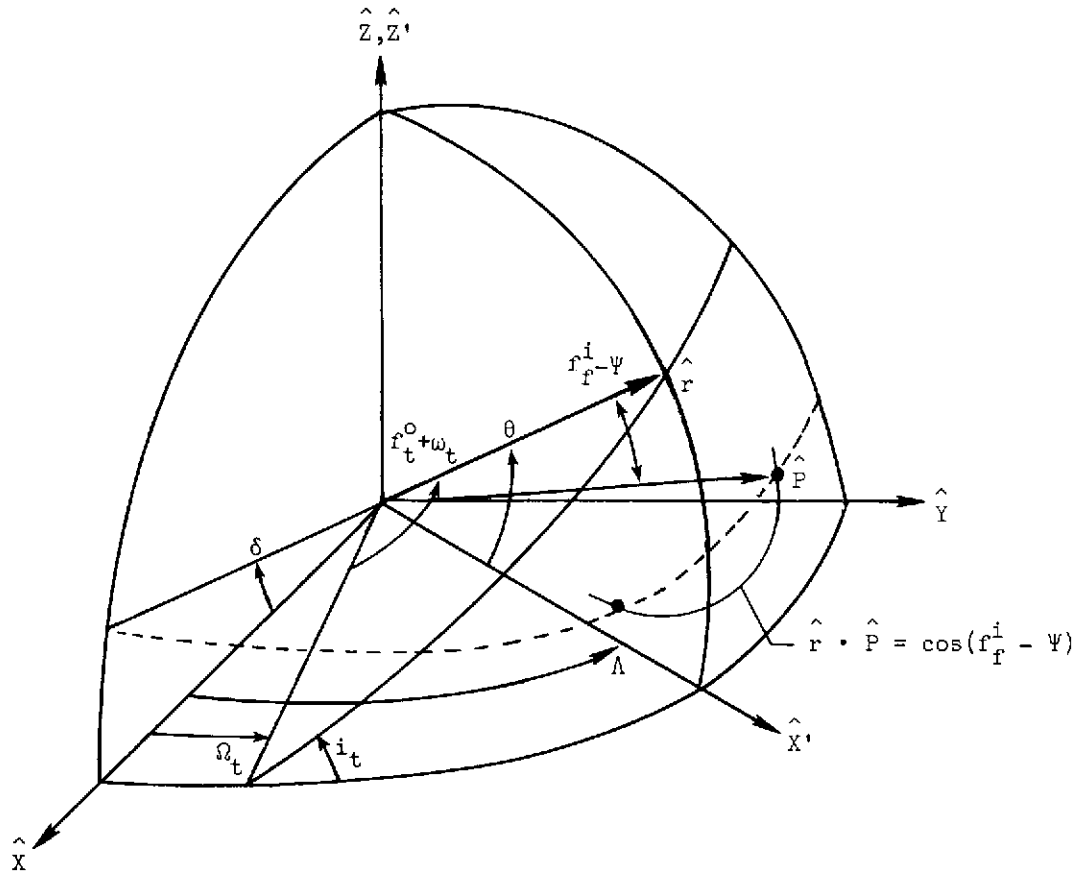$$r_x, r_y, r_z, V_x, V_y, V_z, \mu \xrightarrow{\ T\ } a_t, e_t, i_t, \omega_t, \Omega_t$$

where the transformation T from Cartesian position and velocity to Keplerian orbital elements is by reference 11.

The geometry of the second maneuver is defined in sketch B. Knowing f_f^i, one can calculate the radius at the maneuver as

$$r_f = \frac{a_f \left(1 - e_f^2\right)}{1 + e_f \cos f_f^i}$$

Sketch B

which defines the $\cos f_t^o$ as

$$r_f = \frac{a_t\left(1 - e_t^2\right)}{1 + e_t \cos f_t^o}$$

or

$$\cos f_t^o = \frac{a_t\left(1 - e_t^2\right) - r_f}{e_t r_f}$$

$$\sin f_t^o = \text{sgn}\left(\sin f_f^i\right)\left(1 - \cos^2 f_t^o\right)^{1/2}$$

If $\left|\dfrac{a_t\left(1 - e_t^2\right) - r_f}{e_t r_f}\right| > 1$, then the transfer orbit does not contain a radius r_f as computed

on the final orbit and there is no solution to the transfer problem for the set of param-

eters being considered. If this is not the case, then the assumption is made that the $\text{sgn}\left(\sin f_t^O\right) = \text{sgn}\left(\sin f_f^i\right)$. This assumption is valid except for small values of $\left|\sin f_t^O\right|$. It is recommended that both signs of $\sin f_t^O$ be examined for the smallest ΔV_2 if this is the case. The unit vector $\hat{r}$ to the maneuver point can now be computed by

$$r_x = \cos\left(\omega_t + f_t^O\right)\cos \Omega_t - \cos i_t \sin\left(\omega_t + f_t^O\right)\sin \Omega_t$$

$$r_y = \cos\left(\omega_t + f_t^O\right)\sin \Omega_t + \cos i_t \sin\left(\omega_t + f_t^O\right)\cos \Omega_t$$

$$r_z = \sin\left(\omega_t + f_t^O\right)\sin i_t$$

To simplify the further development, rotate to the $\hat{X}', \hat{Y}', \hat{Z}'$ coordinate system (sketch B) and compute

$$\sin \theta = r_z$$

$$\cos \theta = +\left(1 - \sin^2 \theta\right)^{1/2}$$

$$\sin \Lambda = \frac{r_y}{\cos \theta}$$

$$\cos \Lambda = \frac{r_x}{\cos \theta}$$

If $\cos \theta = 0$, there is no solution as will be shown later. Thus, one can write $\hat{r}$ as

$$r_{x'} = \cos \theta$$

$$r_{y'} = 0$$

$$r_{z'} = \sin \theta$$

Now impose the constraint that the latitude of ψ equals δ. To this end define a unit vector $\hat{P}$ in the final orbit which is at a true anomaly of ψ and is at a latitude of δ. Since the true anomaly of $\hat{r}$ is f_f^i and the true anomaly of $\hat{P}$ is ψ, the angle between $\hat{r}$ and $\hat{P}$ is $\left(f_f^i - \psi\right)$. Thus, the minor circle $\hat{r} \cdot \hat{P} = \cos\left(f_f^i - \psi\right)$ can intersect the δ latitude line at two points (see sketch B) and yield two solutions to the transfer problem. The transfer with the lowest ΔV is desired. The vector $\hat{P}$ can be determined from the following system of equations:

$$\hat{r} \cdot \hat{P} = \cos\left(f_f^i - \psi\right)$$

$$\hat{P} \cdot \hat{z} = \cos\left(90^O - \delta\right) = \sin \delta$$

$$\hat{P} \cdot \hat{P} = 1$$

From the second equation obtain

$$P_{z'} = \sin \delta$$

and from the first equation

$$P_{x'} \cos \theta + \sin \theta \sin \delta = \cos\left(f_f^i - \psi\right)$$

or

$$P_{x'} = \frac{\cos\left(f_f^i - \psi\right) - \sin \theta \sin \delta}{\cos \theta}$$

If $\cos \theta = 0$, then $\theta = \pm 90^O$ and $\hat{r}$ is along $\hat{z}$. In this case the minor circle defined by $\hat{r} \cdot \hat{P} = \cos\left(f_f^i - \psi\right)$ is a latitude line and a solution is possible only if $\cos\left(f_f^i - \psi\right) = \sin \delta$. Since this is very unlikely, this case is not considered and no solution is assumed if $\cos \theta = 0$. The third equation yields

$$P_{y'}^2 = 1 - P_{x'}^2 - P_{z'}^2$$

If $P_{y'}^2 < 0$, then the minor circle $\hat{r} \cdot \hat{P} = \cos\left(f_f^i - \psi\right)$ does not intersect the δ latitude line and there is no solution. If $P_{y'}^2 > 0$, then two solutions exist; namely,

$$P_{y'} = \pm\left(1 - P_{x'}^2 - P_{z'}^2\right)^{1/2}$$

Both of these solutions must be examined since they both satisfy all the constraints placed on the transfer. The solution which has a plane of motion lying closest to the transfer orbit costs less and is chosen. To determine this orbit, rotate $\hat{P}$ in the prime system to the original $\hat{X}, \hat{Y}, \hat{Z}$ system; that is

$$P_X = P_{x'} \cos \Lambda - P_{y'} \sin \Lambda$$

$$P_y = P_{x'} \sin \Lambda + P_{y'} \cos \Lambda$$

$$P_Z = \sin \delta$$

where $P_{y'} > 0$ has been arbitrarily chosen. The unit vector $\hat{H}_f$ normal to the plane of the final orbit and in the direction of angular momentum can be found from

$$\hat{H}_f = \frac{\hat{P} \times \hat{r}}{\left|\hat{P} \times \hat{r}\right|} = \frac{\hat{P} \times \hat{r}}{\sin\left(f_f^i - \psi\right)}$$

or

$$\begin{bmatrix} H_{f,x} \\ H_{f,y} \\ H_{f,z} \end{bmatrix} = \frac{1}{\sin\left(f_f^i - \psi\right)} \begin{bmatrix} P_y r_z - P_z r_y \\ P_z r_x - P_x r_z \\ P_x r_y - P_y r_x \end{bmatrix}$$

The corresponding vector for the transfer orbit is given by

$$H_{t,x} = \sin \Omega_t \sin i_t$$

$$H_{t,y} = -\cos \Omega_t \sin i_t$$

$$H_{t,z} = \cos i_t$$

From sketch C it can be seen that the angle ξ which is the angle between the two planes of motion at their line of intersection is given by $\cos \xi = \hat{H}_t \cdot \hat{H}_f$, or

$$\cos \xi = H_{t,x} H_{f,x} + H_{t,y} H_{f,y} + H_{t,z} H_{f,z}$$

At this point the other solution of $\hat{P}$ is examined, that is $P_{y'} < 0$, and the value of $\cos \xi$ corresponding to this solution is calculated. The desired $\hat{P}$ is that one which has the largest value of $\cos \xi$, or that solution which has a plane of motion lying closest to the transfer orbit. Once this value is obtained, the sine of ξ is calculated; that is,

$$\sin \xi = \left(\hat{H}_t \times \hat{H}_f\right) \cdot \hat{r}$$

or

$$\sin \xi = \left(H_{t,y} H_{f,z} - H_{t,z} H_{f,y}\right) r_x + \left(H_{t,z} H_{f,x} - H_{t,x} H_{f,z}\right) r_y + \left(H_{t,x} H_{f,y} - H_{t,y} H_{f,x}\right) r_z$$
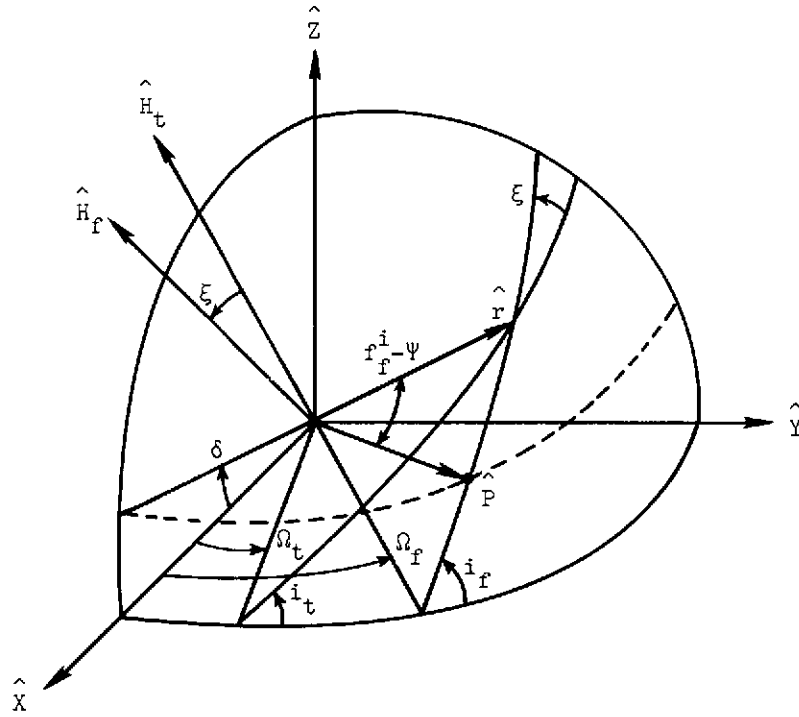
Sketch C

Sketch D

The difference in velocities ΔV_2 between the transfer orbit and the final orbit at $\hat{r}$ remains to be calculated. The geometry of the velocity vectors is shown in sketch D, where the following calculations lead to ΔV_2:
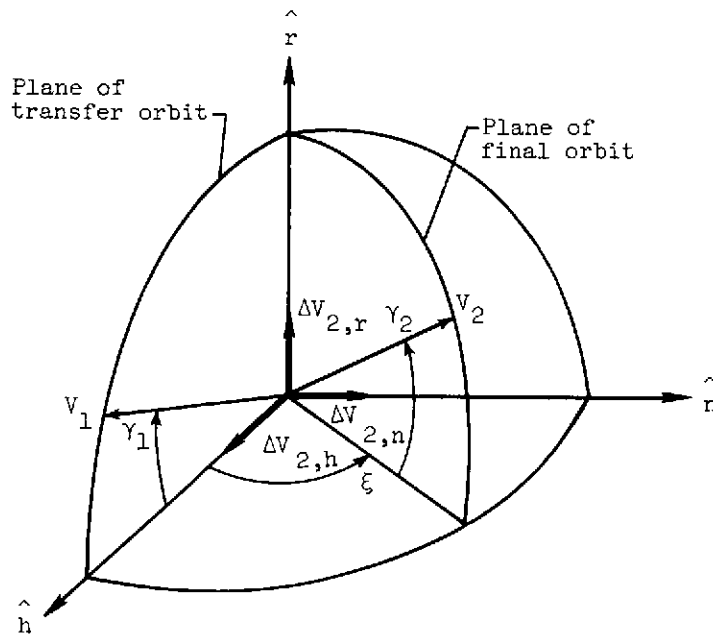
$$V_1 = \left[\mu\left(\frac{2}{r} - \frac{1}{a_t}\right)\right]^{1/2}$$

$$\sin \gamma_1 = \frac{e_t \sin f_t^o}{\left(1 + 2e_t \cos f_t^o + e_t^2\right)^{1/2}}$$

$$\cos \gamma_1 = +\left(1 - \sin^2 \gamma_1\right)^{1/2}$$

$$V_2 = \left[\mu\left(\frac{2}{r} - \frac{1}{a_f}\right)\right]^{1/2}$$

$$\sin \gamma_2 = \frac{e_f \sin f_f^i}{\left(1 + 2e_f \cos f_f^i + e_f^2\right)^{1/2}}$$

$$\cos \gamma_2 = +\left(1 - \sin^2 \gamma_2\right)^{1/2}$$

$$\Delta V_{2,r} = V_2 \sin \gamma_2 - V_1 \sin \gamma_1$$

$$\Delta V_{2,h} = V_2 \cos \gamma_2 \cos \xi - V_1 \cos \gamma_1$$

$$\Delta V_{2,n} = V_2 \cos \gamma_2 \sin \xi$$

$$\Delta V_2 = \left(\Delta V_{2,r}^2 + \Delta V_{2,h}^2 + \Delta V_{2,n}^2\right)^{1/2}$$

APPENDIX B

RELATION BETWEEN PERIOD FACTOR AND VELOCITY FACTOR

Assume that the elements of two intersecting elliptical orbits are known. The impulsive velocity maneuver between them is $\overline{\Delta V} = \overline{V_2} - \overline{V_1}$. Further assume that the period factor α required to obtain an intermediate orbit with period P_α is known; that is,

$$P_\alpha = P_1 + \alpha\left(P_2 - P_1\right)$$

It is desired to obtain a velocity factor α_v which when applied to $\overline{\Delta V}$ yields an intermediate orbit with period P_α. The semimajor axis of the α-orbit is

$$a_\alpha = \left[\frac{P_\alpha^2 \mu}{(2\pi)^2}\right]^{1/3}$$

The maneuver occurs at the true anomaly f_1^O and a radius of

$$r = \frac{a_1\left(1 - e_1^2\right)}{1 + e_1 \cos f_1^O}$$

Thus, the magnitude of the velocity in the α-orbit is

$$V_\alpha = \left[\mu\left(\frac{2}{r} - \frac{1}{a_\alpha}\right)\right]^{1/2}$$

From the definition of the velocity factor,

$$\overline{V_\alpha} = \overline{V_1} + \alpha_v \overline{\Delta V}$$

Squaring both sides yields

$$\overline{V_\alpha} \cdot \overline{V_\alpha} = \left(\overline{V_1} + \alpha_v \overline{\Delta V}\right) \cdot \left(\overline{V_1} + \alpha_v \overline{\Delta V}\right)$$

or

$$\Delta V^2 \alpha_v^2 + 2\overline{V_1} \cdot \overline{\Delta V} \alpha_v + \left(V_1^2 - V_\alpha^2\right) = 0$$

and the velocity factor is given by the quadratic formula as

$$\alpha_V = \frac{-\overline{V_1} \cdot \overline{\Delta V} \pm \left[\left(\overline{V_1} \cdot \overline{\Delta V}\right)^2 - \Delta V^2\left(V_1^2 - V_\alpha^2\right)\right]^{1/2}}{\Delta V^2}$$

where the sign of the radical is so chosen that $\left\{\left|\alpha_V\right| + \left|1 - \alpha_V\right|\right\}$ is a minimum.

A similar expression may be obtained for β_V by replacing the α related parameters with similar β related parameters.

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TABLE 1.- DETERMINATION OF ALINEMENT ORBIT

Given: I, J, K, L, I_a
TYPE SOLUTION (BF, FB, BB, TF, FT)

(o) Alinement on initial orbit
(α) Alinement on α-orbit
(T) Alinement on transfer orbit
(β) Alinement on β-orbit
(F) Alinement on final orbit

$\Delta f_o^{io} \leqq \Delta f_o^{ia}$ NO $\phi = I + 1$

YES

$\phi = I$

$I_a < \phi$ YES (o)

NO   FB / FT   BF / BB / TF $\phi = \phi + J$

$\Delta f_t^{io} \leqq \Delta f_t^{ia}$ NO $\phi = \phi + K + 1$

$I_a < \phi$ YES (α)

YES

$\phi = \phi + K$ NO   TF   BF / BB

$I_a < \phi$ YES (T)

$\phi = \phi + L$

NO   FT   FB

$\Delta f_t^{io} \leqq \Delta f_t^{ia}$ $\phi = \phi + K + 1$

$\phi = \phi + J$ $\phi = \phi + L$ $I_a < \phi$ YES (β)   YES

$I_a < \phi$ (α)   YES

NO   NO $\Delta f_t^{io} \leqq \Delta f_t^{ia}$ $\phi = \phi + K$

$\phi = \phi + L$ $I_a < \phi$ YES $\phi = \phi + K + 1$ YES $I_a < \phi$ YES (T)

NO   (F) $\phi = \phi + K$ NO   BB   BF   (F)

$I_a < \phi$ YES (β) $I_a < \phi$ YES (T) $\phi = \phi + L$

NO   NO

(F)   (F) $I_a < \phi$ YES (β)

NO

(F)

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TABLE 2.- BISECT-FULL TYPE SOLUTION

Rendezvous condition	$\alpha = \dfrac{n\eta - C_2}{C_1}$; $\alpha \to \alpha_v$ $\Delta V_T = \|\alpha_v\|\Delta V_1 + \|1 - \alpha_v\|\Delta V_1 + \Delta V_2$ $C_1 = J\,\Delta P_{to}$ $C_2 = (I + J)P_o + KP_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$ $m = 0, 1, 2, \cdots, m_b$; $\quad n = 0, 1, 2, \cdots, n_b$
Alinement on initial orbit	$\epsilon = \min\limits_m \left\{ \|m\zeta + C_3\| \right\}$ $C_3 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$ If $\epsilon \leqq \epsilon_b$, minimize ΔV_T over n.
Alinement on α-orbit	$\epsilon = \min\limits_{m,n} \left\{ \left\| m\zeta - \left(\dfrac{n\eta - C_2}{C_1}\right) C_3 - C_4 \right\| \right\}$ $C_3 = J'\Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$ $C_4 = (I + J')P_o + \Delta t_o^{io} + \Delta t_o^{oa} - \tau_{a,o}$ $J' = ci\left[I_a - I - \left(\dfrac{\Delta r_o^{io} + \Delta r_o^{oa} - \Delta r_o^{ia}}{360°}\right) \right]$ If $\epsilon \leqq \epsilon_b$, solution is acceptable.
Alinement on transfer orbit	$\epsilon = \min\limits_{m,n} \left\{ \left\| m\zeta - \left(\dfrac{n\eta - C_2}{C_1}\right) C_3 - C_4 \right\| \right\}$ $C_3 = J\,\Delta P_{to}$ $C_4 = (I + J)P_o + K'P_t + \Delta t_o^{io} + \Delta t_t^{ia} - \tau_{a,t}$ $K' = ci\left[I_a - I - J - \left(\dfrac{\Delta r_o^{io} + \Delta r_t^{ia} - \Delta r_o^{ia}}{360°}\right) \right]$ If $\epsilon \leqq \epsilon_b$, solution is acceptable.
Alinement on final orbit	$\epsilon = \min\limits_{m,n} \left\{ \left\| m\zeta - \left(\dfrac{n\eta - C_2}{C_1}\right) C_3 - C_4 \right\| \right\}$ $C_3 = J\,\Delta P_{to}$ $C_4 = (I + J)P_o + KP_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$ $M = ci\left[I_a - I - J - K - \left(\dfrac{\Delta r_o^{io} + \Delta r_t^{io} + \Delta r_f^{ia} - \Delta r_o^{ia}}{360°}\right) \right]$ If $\epsilon \leqq \epsilon_b$, solution is acceptable.

TABLE 5.- FULL-BISECT TYPE SOLUTION

Rendezvous condition	$\beta = \dfrac{n\eta - C_2}{C_1}$; $\beta \to \beta_v$ $\Delta V_T = \Delta V_1 + |\beta_v|\Delta V_2 + |1 - \beta_v|\Delta V_2$ $C_1 = L\,\Delta P_{ft}$ $C_2 = IP_o + (K + L)P_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$ $m = 0, 1, 2, \cdots, m_b$; $n = 0, 1, 2, \cdots, n_b$
Alinement on initial orbit	$\epsilon = \min_m \left\{ \left| m\zeta + C_3 \right| \right\}$ $C_3 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$ If $\epsilon \leqq \epsilon_b$, minimize ΔV_T over n.
Alinement on transfer orbit	$\epsilon = \min_m \left\{ \left| m\zeta + C_3 \right| \right\}$ $C_3 = \tau_{a,t} - \Delta t_o^{io} - IP_o - \Delta t_t^{ia} - K'P_t$ $K' = ci\left[I_a - I - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{ia} - \Delta f_o^{ia}}{360°} \right) \right]$ If $\epsilon \leqq \epsilon_b$, minimize ΔV_T over n.
Alinement on β-orbit	$\epsilon = \min_{m,n} \left\{ \left| m\zeta - \left(\dfrac{n\eta - C_2}{C_1} \right) C_3 - C_4 \right| \right\}$ $C_3 = \Delta t_f^{ia} - \Delta t_t^{oa} + L'\Delta P_{ft} - \tau_{a,f} + \tau_{a,t}$ $C_4 = IP_o + (K + L')P_t + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_t^{oa} - \tau_{a,t}$ $L' = ci\left[I_a - I - K - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_t^{oa} - \Delta f_o^{ia}}{360°} \right) \right]$ If $\epsilon \leqq \epsilon_b$, solution is acceptable.
Alinement on final orbit	$\epsilon = \min_{m,n} \left\{ \left| m\zeta - \left(\dfrac{n\eta - C_2}{C_1} \right) C_3 - C_4 \right| \right\}$ $C_3 = L\,\Delta P_{ft}$ $C_4 = IP_o + (K + L)P_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$ $M = ci\left[I_a - I - K - L - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_f^{ia} - \Delta f_o^{ia}}{360°} \right) \right]$ If $\epsilon \leqq \epsilon_b$, solution is acceptable.

TABLE 4.- BISECT-BISECT TYPE SOLUTION

Rendezvous condition	$\alpha = \dfrac{-(n\eta - C_3)C_5 + (m\zeta - C_6 - \lambda\varepsilon_b)C_2}{C_2 C_4 - C_1 C_5}; \ \alpha \to \alpha_v$ $\beta = \dfrac{(n\eta - C_3)C_4 - (m\zeta - C_6 - \lambda\varepsilon_b)C_1}{C_2 C_4 - C_1 C_5}; \ \beta \to \beta_v$ $\Delta V_T = |\alpha_v|\Delta v_1 + |1 - \alpha_v|\Delta v_1 + |\beta_v|\Delta v_2 + |1 - \beta_v|\Delta v_2$ $C_1 = J \ \Delta P_{to}$ $C_2 = L \ \Delta P_{ft}$ $C_3 = (I + J)P_o + (K + L)P_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$ $m = 0, 1, 2, \cdots, m_b; \ n = 0, 1, 2, \cdots, n_b; \ \lambda = -1, 0, 1$
Alinement on initial orbit	$\varepsilon = \min_m \left\{ |m\zeta + C_4| \right\}$ $C_4 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$ If $\varepsilon \leq \varepsilon_b$, minimize ΔV_T over n, for the following four cases: 1. $\alpha_v = 0$ and $\beta = \dfrac{n\eta - C_3}{C_2}; \ \beta \to \beta_v$ 2. $\alpha_v = 1$ and $\beta = \dfrac{n\eta - C_1 - C_3}{C_2}; \ \beta \to \beta_v$ 3. $\beta_v = 0$ and $\alpha = \dfrac{n\eta - C_3}{C_1}; \ \alpha \to \alpha_v$ 4. $\beta_v = 1$ and $\alpha = \dfrac{n\eta - C_2 - C_3}{C_1}; \ \alpha \to \alpha_v$
Alinement on α-orbit	$C_4 = J'\Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$ $C_5 = 0$ $C_6 = (I + J')P_o + \Delta t_o^{io} + \Delta t_o^{oa} - \tau_{a,o}$ $J' = ci\left[I_a - I - \left(\dfrac{\Delta r_o^{io} + \Delta r_o^{oa} - \Delta r_o^{ia}}{360°} \right) \right]$ Minimize ΔV_T over m, n, λ.
Alinement on transfer orbit	$C_4 = C_1$ $C_5 = 0$ $C_6 = (1 + J)P_o + K'P_t + \Delta t_o^{io} + \Delta t_t^{ia} - \tau_{a,t}$ $K' = ci\left[I_a - I - J - \left(\dfrac{\Delta r_o^{io} + \Delta r_t^{ia} - \Delta r_o^{ia}}{360°} \right) \right]$ Minimize ΔV_T over m, n, λ.

TABLE 4.- BISECT-BISECT TYPE SOLUTION - Concluded

Alinement on β-orbit	$C_4 = C_1$ $C_5 = \Delta t_f^{ia} - \Delta t_t^{oa} + L'\Delta P_{ft} - \tau_{a,f} + \tau_{a,t}$ $C_6 = (I + J)P_o + (K + L')P_t + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_t^{oa} - \tau_{a,t}$ $L' = ci\left[I_a - I - J - K - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_t^{oa} - \Delta f_o^{ia}}{360°}\right)\right]$ Minimize ΔV_T over m, n, λ.
Alinement on final orbit	$\varepsilon = \min_{m,n}\left\{\left\|(n\eta - C_3) - (m\zeta - C_4)\right\|\right\}$ $C_4 = (I + J)P_o + (K + L)P_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$ $M = ci\left[I_a - I - J - K - L - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_f^{ia} - \Delta f_o^{ia}}{360°}\right)\right]$ If $\varepsilon \leq \varepsilon_b$, minimize ΔV_T over the four cases given for alinement on initial orbit.

TABLE 5.- TRISECT-FULL TYPE SOLUTION

Rendezvous condition	$\Delta V_T = |\alpha_v|\Delta V_1 + |\beta_v||1 - \alpha_v|\Delta V_1 + |1 - \beta_v||1 - \alpha_v|\Delta V_1 + \Delta V_2$ $\beta = \dfrac{n\eta - C_1\alpha - C_3}{C_2(1-\alpha)}$; $\alpha \neq 1$; $\beta \neq \dfrac{J+L}{L}$; $\beta \to \beta_v$ $C_1 = (J + L)\Delta P_{to}$ $C_2 = L\,\Delta P_{to}$ $C_3 = (I + J + L)P_o + KP_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$ $m = 0, 1, 2, \cdots, m_b$; $n = 0, 1, 2, \cdots, n_b$; $\lambda = -1, 0, 1$
Alinement on initial orbit	$\epsilon = \min_m \left\{ \left| m\zeta + C_4 \right| \right\}$ $C_4 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$ If $\epsilon \leqq \epsilon_b$, minimize ΔV_T over n for the following 3 cases: 1. $\alpha_v = 0$ and $\beta = \dfrac{n\eta - C_3}{C_2}$; $\beta \to \beta_v$ 2. $\beta_v = 0$ and $\alpha = \dfrac{n\eta - C_3}{C_1}$; $\alpha \to \alpha_v$ 3. $\beta_v = 1$ and $\alpha = \dfrac{n\eta - C_2 - C_3}{C_1 - C_2}$; $\alpha \to \alpha_v$
Alinement on α-orbit	$\alpha = \dfrac{m\zeta - \lambda\epsilon_b - C_5}{C_4}$; $\alpha \to \alpha_v$ $C_4 = J'\Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$ $C_5 = (I + J')P_o + \Delta t_o^{io} + \Delta t_o^{oa} - \tau_{a,o}$ $J' = ci\left[I_a - I - \left(\dfrac{\Delta f_o^{io} + \Delta f_o^{oa} - \Delta f_o^{ia}}{360°} \right) \right]$ Minimize ΔV_T over m, n, λ.
Alinement on β-orbit	$\alpha = \dfrac{(n\eta - C_3)C_5 - (m\zeta - C_6)C_2 + C_2\lambda\epsilon_b}{C_1 C_5 - C_2 C_4}$; $\alpha \to \alpha_v$ $C_4 = (J + L')\Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$ $C_5 = L'\Delta P_{to} + \Delta t_t^{ia} - \Delta t_o^{oa} - \tau_{a,t} + \tau_{a,o}$ $C_6 = (I + J + L')P_o + \Delta t_o^{io} + \Delta t_o^{oa} - \tau_{a,o}$ $L' = ci\left[I_a - I - J - \left(\dfrac{\Delta f_o^{io} + \Delta f_o^{oa} - \Delta f_o^{ia}}{360°} \right) \right]$ Minimize ΔV_T over m, n, λ.

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TABLE 5.- TRISECT-FULL TYPE SOLUTION - Concluded

Alinement on transfer orbit	$\varepsilon = \min\limits_{m,n} \left\{ \left\| (n\eta - C_3) - (m\zeta - C_4) \right\| \right\}$ $C_4 = (I + J + L)P_o + K'P_t + \Delta t_o^{io} + \Delta t_t^{ja} - \tau_{a,t}$ $K' = ci\left[I_a - I - J - L - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{ia} - \Delta f_o^{ia}}{360^\circ} \right) \right]$ If $\varepsilon \leqq \varepsilon_b$, minimize ΔV_T over the 3 cases given for alinement on initial orbit.
Alinement on final orbit	$\varepsilon = \min\limits_{m,n} \left\{ \left\| (n\eta - C_3) - (m\zeta - C_4) \right\| \right\}$ $C_4 = (I + J + L)P_o + KP_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$ $M = ci\left[I_a - I - J - K - L - \left(\dfrac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_f^{ia} - \Delta f_o^{ia}}{360^\circ} \right) \right]$ If $\varepsilon \leqq \varepsilon_b$, minimize ΔV_T over the 3 cases given for alinement on initial orbit.

TABLE 6.- FULL-TRISECT TYPE SOLUTION

Rendezvous condition	$\Delta V_T = \Delta V_1 + \|\alpha_v\|\Delta V_2 + \|\beta_v\|\|1 - \alpha_v\|\Delta V_2 + \|1 - \beta_v\|\|1 - \alpha_v\|\Delta V_2$ $\beta = \dfrac{n\eta - C_1\alpha - C_3}{C_2(1 - \alpha)}$; $\alpha \neq 1$; $\beta \neq \dfrac{J+L}{L}$; $\beta \rightarrow \beta_v$ $C_1 = (J + L)\Delta P_{ft}$ $C_2 = L \Delta P_{ft}$ $C_3 = IP_o + (J + K + L)P_t + \Delta t_o^{io} + \Delta t_t^{io} - \tau_r$ $m = 0, 1, 2, \cdots, m_b$; $n = 0, 1, 2, \cdots, n_b$; $\lambda = -1, 0, 1$
Alinement on initial orbit	$\varepsilon = \min\limits_{m} \left\{ \|m\zeta + C_4\| \right\}$ $C_4 = \tau_{a,o} - \Delta t_o^{ia} - I_a P_o$ If $\varepsilon \leqq \varepsilon_b$, minimize ΔV_T over n for the following 3 cases: 1. $\alpha_v = 0$ and $\beta = \dfrac{n\eta - C_3}{C_2}$; $\beta \rightarrow \beta_v$ 2. $\beta_v = 0$ and $\alpha = \dfrac{n\eta - C_3}{C_1}$; $\alpha \rightarrow \alpha_v$ 3. $\beta_v = 1$ and $\alpha = \dfrac{n\eta - C_2 - C_3}{C_1 - C_2}$; $\alpha \rightarrow \alpha_v$
Alinement on transfer orbit	$\varepsilon = \min\limits_{m} \left\{ \|m\zeta + C_4\| \right\}$ $C_4 = \tau_{a,t} - \Delta t_o^{io} - IP_o - \Delta t_t^{ia} - K'P_t$ $K' = ci\left[I_a - I - \left(\dfrac{\Delta r_o^{io} + \Delta r_t^{ia} - \Delta r_o^{ia}}{360^o} \right) \right]$ If $\varepsilon \leqq \varepsilon_b$, minimize ΔV_T over n for the cases given for alinement on initial orbit.
Alinement on α-orbit	$\alpha = \dfrac{m\zeta - \lambda\varepsilon_b - C_5}{C_4}$; $\alpha \rightarrow \alpha_v$ $C_4 = J'\Delta P_{ft} + \Delta t_f^{ia} - \Delta t_t^{oa} - \tau_{a,f} + \tau_{a,t}$ $C_5 = IP_o + (J' + K)P_t + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_t^{oa} - r_{a,t}$ $J' = ci\left[I_a - I - K - \left(\dfrac{\Delta r_o^{io} + \Delta r_t^{io} + \Delta r_t^{oa} - \Delta r_o^{ia}}{360^o} \right) \right]$ Minimize ΔV_T over m, n, λ.

TABLE 6.- FULL-TRISECT TYPE SOLUTION - Concluded

Alinement on β-orbit	$$\alpha = \frac{(n\eta - C_3)C_5 - (m\zeta - C_6)C_2 + C_2\lambda\varepsilon_b}{C_1 C_5 - C_2 C_4} \; ; \; \alpha \to \alpha_v$$ $$C_4 = (J + L')\Delta P_{ft} + \Delta t_f^{ia} - \Delta t_t^{oa} - \tau_{a,f} + \tau_{a,t}$$ $$C_5 = L'\Delta P_{ft} + \Delta t_f^{ia} - \Delta t_t^{oa} - \tau_{a,f} + \tau_{a,t}$$ $$C_6 = IP_o + (J + K + L')P_t + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_t^{oa} - \tau_{a,t}$$ $$L' = ci\left[I_a - I - J - K - \left(\frac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_t^{oa} - \Delta f_o^{ia}}{360°}\right)\right]$$ Minimize ΔV_T over m, n, λ.
Alinement on final orbit	$$\varepsilon = \min_{m,n} \left\{ \left| (n\eta - C_3) - (m\zeta - C_4) \right| \right\}$$ $$C_4 = IP_o + (J + K + L)P_t + MP_f + \Delta t_o^{io} + \Delta t_t^{io} + \Delta t_f^{ia} - \tau_{a,f}$$ $$M = ci\left[I_a - I - J - K - L - \left(\frac{\Delta f_o^{io} + \Delta f_t^{io} + \Delta f_f^{ia} - \Delta f_o^{ia}}{360°}\right)\right]$$ If $\varepsilon \leqq \varepsilon_b$, minimize ΔV_T over the 3 cases given for alinement on initial orbit.