NASA CONTRACTOR REPORT

NASA CR-2447

NASA CR-2447

STUDY OF STABILITY OF LARGE MANEUVERS OF AIRPLANES

by Emile K. Haddad

Prepared by
LOCKHEED-GEORGIA COMPANY
Marietta, Ga. 30063
for Langley Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1974

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-2447		

4. Title and Subtitle	5. Report Date
STUDY OF STABILITY OF LARGE MANEUVERS OF AIRPLANES	August 1974
	6. Performing Organization Code
7. Author(s) Emile K. Haddad	8. Performing Organization Report No.
	10. Work Unit No. 501-26-05-06
9. Performing Organization Name and Address Lockheed-Georgia Company Marietta, Georgia	11. Contract or Grant No. NAS1-11667
	13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Final report.

16. Abstract

A predictive method of nonlinear system analysis is used to investigate airplane stability and dynamic response during rolling maneuvers. The maneuver roll-rate is not assumed to be constant, and the airplane motion is represented by a set of coupled nonlinear differential equations.

In the first portion of the study the general rolling maneuver is kinematically specified by its roll-rate variation p(t). A method for relating the airplane dynamic response to p(t) is developed. The method provides analytical expressions for the motion variables in terms of the maneuver descriptor p(t). The accuracy and utilization of the method is illustrated with specific values of airplane parameters. A parameterized family of rolling maneuvers is considered, for which the method is used to predict specific dynamic response information, such as the dependence of the peak angle-of-attack excursion on the maneuver parameters. The accuracy of the method is shown to be substantially better than that of previous linear analytical methods based on approximating the maneuver roll-rate by a constant.

In the second portion of the study the stability and motion of the airplane in response to an arbitrary actuation of aileron input is considered. Analytical expressions relating motion variables to aileron input are obtained. Explicit analytical bounds on the motion variables are derived. A stability criterion which guarantees nondivergence of motion in response to aileron actuation is presented.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
roll coupling nonlinear dynamic stability	Unclassified - Unlimited STAR Category 02

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	74	$3.75

*For sale by the National Technical Information Service, Springfield, Virginia 22151

CONTENTS

CONTENTS (Cont'd)

STUDY OF STABILITY OF LARGE MANEUVERS OF AIRPLANES

By

Emile K. Haddad

LOCKHEED-GEORGIA COMPANY

SUMMARY

A predictive method of nonlinear system analysis is used to investigate airplane stability and dynamic response during rolling maneuvers. The maneuver roll-rate is not assumed to be constant, and the airplane motion is represented by a set of coupled nonlinear differential equations.

In the first portion of the study the general rolling maneuver is kinematically specified by its roll-rate variation p(t). A method for relating the airplane dynamic response to p(t) is developed. The method provides analytical expressions for the motion variables in terms of the maneuver descriptor p(t). The accuracy and utilization of the method is illustrated with specific values of airplane parameters. A parameterized family of rolling maneuvers is considered, for which the method is used to predict specific dynamic response information, such as the dependence of the peak angle-of-attack excursion on the maneuver parameters. The accuracy of the method is shown to be substantially better than that of previous linear analytical methods based on approximating the maneuver roll-rate by a constant.

In the second portion of the study the stability and motion of the airplane in response to an arbitrary actuation of aileron input is considered. Analytical expressions relating motion variables to aileron input are obtained. Explicit analytical bounds on the motion variables are derived. A stability criterion which guarantees nondivergence of motion in response to aileron actuation is presented.

INTRODUCTION

The traditional techniques of flight dynamics analysis have depended on a number of simplifying assumptions and approximations, applied to the complete equations of motion of the aircraft. These equations may be exactly represented by a set of coupled, nonlinear, differential equations. The conventional analytical approaches have generally resorted to various ways of approximation aimed at reducing the problem to the solution of a set of linear differential equations in which the longitudinal and lateral dynamics are uncoupled. The adequacy of such assumptions and approximations leading to decoupling and linearization can be justified only for restricted ranges of dynamic performance: small deviations from steady trimmed straight flight.

The loss of dynamic-response information incurred by the conventional simplified analyses is forcefully exemplified by the well-known rotational coupling problem. Flight tests and simulations have indicated that slender aircraft undergoing rapid rolling maneuvers may exhibit certain dynamic response characteristics, such as divergence in yaw and pitch or excessive excursions in sideslip and angle of attack, which are not predictable from the linearized, decoupled equations. Beginning with the work of W. H. Philips (ref. 1), several studies (for example, ref. 2,3,4) were made to investigate analytically the characteristics of the airplane dynamic response during rolling maneuvers. All of these studies were based on assumptions and approximations sufficient to linearize the equations of motion while retaining some degree of coupling. Key assumptions for linearization were the requirements of constant roll rate and constant forward speed, negligible second-order terms except those involving roll rate, small magnitudes of side and vertical velocities, negligible weight components, and linearized aerodynamics. These approximations, by eliminating all the nonlinear product terms, reduce the degree of coupling among the component equations, and consequently discard some of the dynamic response characteristics attributable to such coupling. Furthermore, the various assumptions made about the dynamic variables are not always representative or realistic. A roll maneuver, for instance, may not be executed at constant roll rate, but it would typically involve a gradual increase in roll rate.

Apart from the analytical studies of the rotational coupling problem, one finds investigations primarily involving extensive simulations on the analog or digital computer (for example, ref. 5). In post-design stages, the computer is quite effective for verifying and checking specific information about the dynamic response of a given aircraft. However, the computer does not lend itself to the formulation of predictive stability and dynamic response criteria in concise mathematical form.

In this study, the airplane dynamic response and stability during rolling maneuvers is investigated using nonlinear system analysis techniques. The investigation is based on a new approach for nonlinear system analysis developed by E. K. Haddad at the Lockheed-Georgia Company (ref. 6, 7, 8). The study develops two analytical methods for predicting dynamic response and stability information from the nonlinear equations of motion representing general rolling maneuvers for which the roll rate or the aileron input may exhibit any time variation. The first method considers the maneuver to be specified by its roll rate time history p(t). In the second method, the maneuver is specified by the aileron time-history input $\delta_a(t)$. This method leads to a stability criterion which defines the range of aileron inputs for which nondivergent motion can be guaranteed.

SYMBOLS

a_q = $|E|\, A_q$

a_r = $|E|\, A_r$

A coefficient matrix in state equations of airplane dynamics $x = Ax + u$

A_o matrix representing time-invariant portion of matrix A

$\tilde{A}$ matrix representing time-dependent portion of matrix A

$A_q(t) = \int_0^t |h_q(t)|\, dt$

$A_r(t) = \int_0^t |h_r(t)|\, dt$

b wing span

$B = \dfrac{\bar{q}Sb}{I_x}\, C_{\ell\beta}$

$\bar{c}$ wing mean aerodynamic chord

$C = \dfrac{\bar{q}Sb^2}{2I_xV}$

C_ℓ rolling moment coefficient, $\dfrac{\text{rolling moment}}{qSb}$

$C_{\ell\beta}$ $\dfrac{\partial C\ell}{\partial\beta}$

$C_{\ell p}$ $\dfrac{\partial C\ell}{\partial(pb/2V)}$

$C_{\ell r}$ $\dfrac{\partial C_\ell}{\partial(rb/2V)}$

$C_{\ell_{\delta_a}}$ $\quad \dfrac{\partial C_\ell}{\partial \delta_a}$

$C_\delta \quad = \quad \dfrac{\bar{q}Sb}{I_x} C_{\ell_{\delta_a}}$

C_m pitching moment coefficient, $\dfrac{\text{pitching moment}}{qS\bar{c}}$

$C_{m\alpha}$ $\quad \dfrac{\partial Cm}{\partial \alpha}$

$C_{m\dot{\alpha}}$ $\quad \dfrac{\partial Cm}{\partial(\dot{\alpha}\bar{c}/2V)}$

C_{m_q} $\quad \dfrac{\partial C_m}{\partial(q\bar{c}/2V)}$

C_L normal-force coefficient, $\dfrac{\text{normal force}}{qS}$

C_{L_α} $\quad \dfrac{\partial C_L}{\partial \alpha}$

C_n Yawing-moment coefficient, $\dfrac{\text{yawing moment}}{qSb}$

C_{n_β} $\quad \dfrac{\partial C_n}{\partial \beta}$

C_{n_p} $\quad \dfrac{\partial C_n}{\partial(pb/2V)}$

C_{n_r} $\quad \dfrac{\partial C_n}{\partial(rb/2V)}$

$C_{n_{\delta_a}}$ $\quad \dfrac{\partial C_n}{\partial \delta_a}$

C_Y Side-force coefficient, $\dfrac{\text{side-force}}{qS}$

C_{Y_β}		$\frac{\partial C_Y}{\partial \beta}$
C_{Y_p}		$\frac{\partial C_Y}{\partial (pb/2V)}$
C_{Y_r}		$\frac{\partial C_Y}{\partial (rb/2V)}$
d	=	$a_q \delta_{ar} - a_r \delta_{aq}$
d_q	=	$\lim_{t \to \infty} \delta_q$
d_r	=	$\lim_{t \to \infty} \delta_r$
D	=	$\frac{\bar{q} S b^2}{2 I_x V} C_{\ell_r}$
E	=	$\frac{I_y - I_z}{I_x}$
g		acceleration due to gravity
h_{mn}		elements of the impulse response matrix H
h_r, h_q		Impulse response functions for system representing roll equation of motion
H		Impulse response matrix for coefficnet matrix A_o, $H(t) = e^{A_o t}$
I_x, I_y, I_z		body axes moments of inertia
m		mass of airplane
M		mapping relating α, β, q, and r to roll rate p
p, q, r		angular rates about body axes
P		average value of p

$\tilde{p}$	$p - P$
$P(t)$	time-history bound on p
$\bar{q}$	dynamic pressure
$Q(t)$	time-history bound on q
$R(t)$	time-history bound on r
$\left\{p(t)\right\}_{P_m}$	parametrized family of rolling maneuvers
P_m	parameter defining members of family of rolling maneuvers.
S	wing area
t	time
u	control input (forcing function) in state equations of motion $\dot{x} = Ax + u$
V	airspeed
x	vector representing motion variables $x = \left[\alpha, \beta, q, r\right]$
α	angle of attack
α_o	angle of attack at initiation of maneuver
$\Delta\alpha$	$\alpha - \alpha_o$
β	angle of side slip
δ_a	aileron deflection
$\hat{\delta}_a =$	$C_{\delta}\delta_a$
*	convolution operator
$\delta_{aq} =$	$\hat{\delta}_a * h_q$
$\delta_{ar} =$	$\hat{\delta}_a * h_r$
$\bar{\delta}_{aq} =$	$\sup\limits_{[0,t]} \lvert \delta_{aq}(x) \rvert$

$\bar{\delta}_{ar} = \sup_{[0,t]} |\delta_{ar}(x)|$

$\delta_q = a_r \bar{\delta}_{aq}$

$\delta_r = a_q \bar{\delta}_{ar}$

ψ, θ, φ angles of yaw, pitch, and roll, respectively

$$\gamma_q = |E| \left[\int_0^\infty |h_q(t)|dt\right]\left[\sup_x \left|\int_0^x h_r(t)dt\right|\right]$$

$$\gamma_r = |E| \left[\int_0^\infty |h_r(t)|dt\right]\left[\sup_x \left|\int_0^x h_q(t)dt\right|\right]$$

AIRPLANE DYNAMIC RESPONSE DURING AN ARBITRARY ROLLING MANEUVER

The first portion of this study was devoted to the analysis of the dynamic response of the airplane during an arbitrarily specified rolling maneuver described by its roll-rate variation p(t). The problem being considered is that of relating the airplane stability and dynamic response characteristics during a rolling maneuver to the kinematic behavior in roll p(t). In the second portion of the study, the airplane dynamic response is directly related to the aileron actuation input $\delta_a(t)$. One rationale for relating the airplane dynamics to the roll-rate p(t) rather than to the aileron input $\delta_a(t)$ is the fact that during the preliminary design effort the specifics of the flight control system may not be known in detail, yet one may be interested in the airplane dynamic performance and capabilities during a rolling maneuver, which would then be appropriately specified by the kinematic descriptor p(t) rather than the control input $\delta_a(t)$.

Analytical Method for Predicting Airplane Motion During a Rolling Maneuver of Arbitrary Roll-rate Variation

A main result of this study was the development of an analytical method for predicting the dynamic response of the aircraft performing a rolling maneuver of arbitrary roll-rate variation p(t).

The maneuver is specified by the roll-rate time-history p(t), and the method provides approximate analytical expressions for the response, in which the roll-rate p(t) appears as an unspecified parameter function p(t).

The method is based on the viewpoint that rolling maneuver may be meaningfully described by specifying the variation of roll-rate p(t). From this standpoint, the first four equations of motion may be regarded as a mapping between p(t) and $x(t) = [\alpha(t), \beta(t), q(t), r(t)]$: $x(t) = M[p(t)]$

The method provides, for any given aircraft, approximate analytical expressions for the nonlinear functional M, in which the response x(t) is expressed explicitly in terms of the argument p(t). The method describes a procedure for obtaining successive expressions representing progressively better approximations to the exact mapping M.

Equations of motion for the "basic case".- The effort in this study has been primarily directed towards consideration of the "basic case" involving a rolling maneuver at constant forward speed. An adequate representation of the dynamics of this maneuver is given by the five-degree-of-freedom equations of motion shown below:

$$\alpha = \alpha_o + \Delta\alpha , \qquad \bar{q}SC_{L\alpha}\alpha_o = mg$$

$$\dot{\alpha} = q - p\beta - \frac{\bar{q}S}{mV}C_{L\alpha}\alpha + \frac{g}{V}\cos\varphi \tag{1}$$

$$= q - p\beta - \frac{\bar{q}S}{mV}C_{L\alpha}\Delta\alpha + \frac{g}{V}(\cos\varphi - 1)$$

$$\dot{\beta} = -r + p\Delta\alpha + \frac{\bar{q}S}{mV}\left(C_{Y_\beta}\beta + \frac{b}{2V}C_{Y_r}r + \frac{b}{2V}C_{Y_p}p\right) + \alpha_o p + \frac{g}{V}\sin\varphi \tag{2}$$

$$\dot{q} = \frac{I_z - I_x}{I_y}pr + \frac{\bar{q}S\bar{c}}{I_y}\left(C_{m\alpha}\Delta\alpha + \frac{\bar{c}}{2V}C_{m\dot{\alpha}}\dot{\alpha} + \frac{\bar{c}}{2V}C_{mq}q\right) \tag{3}$$

$$\dot{r} = \frac{I_x - I_y}{I_z}pq + \frac{\bar{q}Sb}{I_z}\left(C_{n\beta}\beta + \frac{b}{2V}C_{nr}r + \frac{b}{2V}C_{np}p\right) \tag{4}$$

$$\dot{p} = \frac{I_y - I_z}{I_x}qr + \frac{\bar{q}Sb}{I_x}\left(C_{\ell\beta}\beta + \frac{b}{2V}C_{\ell p}p + \frac{b}{2V}C_{\ell r}r + C_{\ell\delta_a}\delta_a\right) \tag{5}$$

$$\dot{\varphi} = p \tag{6}$$

$$\dot{\theta} = q\cos\varphi - r\sin\varphi \tag{7}$$

$$\dot{\psi} = q\sin\varphi + r\cos\varphi \tag{8}$$

The equations are referred to principal inertia axes. Rudder and elevator are assumed fixed during the maneuver. $C_{n\delta_a}$ and $C_{Y_{\delta_a}}$ are assumed to be negligible. The aerodynamic coefficients are linearized. The simplified nonlinear gravity terms in equations (1) and (2), and the kinematic relations (6) through (8), are obtained under the assumption that pitch and yaw andles (θ , ψ) remain small during the rolling maneuver. Consequently, equations (1) - (6) are sufficient to characterize the maneuver. Furthermore, equation (6) can be used to express the bank angle φ as

$$\varphi = \int_0^t p(x)dx.$$

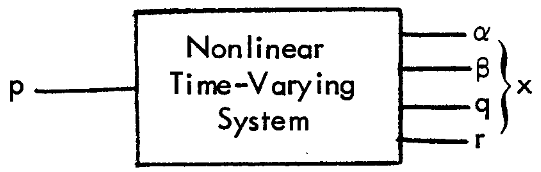
This expression can be used to eliminate the variable φ from the nonlinear gravity terms of equations (1) and (2). The maneuver would then be representable by the five equations (1) - (5) with variables α, β, p, q, and r. During the initial study of the "basic case" the nonlinear weight effects were ignored.

<u>Outline of the method</u>.- Letting $x = [\alpha, \beta, q, r]$, the first four equations of motion may be written as

$$\dot{x} = A(p(t))x + u(p(t)) \tag{9}$$

The system in (9) may be viewed as a nonlinear time-varying mapping from p(t) into x(t):

M



$$x(t) = M[p(t)]$$

Note that in (9) the argument p(t) appears in the forcing vector u and the coefficient matrix A. Thus, doubling the input p(t) does not, in general, result in doubling the output x(t), which makes M a nonlinear system.

Let p(t) be written as

$$p(t) = P + \tilde{p}(t) \tag{10}$$

where P is a constant, which may be conveniently chosen as the average value of p(t). From (9) and (10), one has

$$\dot{x} = A(P + \tilde{p}(t))x + u(p(t)) \tag{11}$$

$$\dot{x} = A_0 x + \tilde{A}(\tilde{p}(t))x + u(p(t)) \tag{12}$$

where the matrix A has been separated into a constant matrix A_0 and a time dependent matrix $\tilde{A}$. A first approximation to x(t) may be obtained by ignoring the second term in (12). Thus, let $x_1(t)$ be the solution to

$$\dot{x}_1 = A_0 x_1 + u(p(t)) \tag{13}$$

The function $x_1(t)$ may be conveniently expressed in terms of the input p(t) by using the impulse response matrix (Green's function matrix) of the system (13):

$$x_1(t) = H_1(p(t)) \tag{14}$$

Note that $x_1(t)$ takes into account the exact time-history in roll p(t) as it appears in u(p(t)), and therefore $x_1(t)$ is expected to furnish a better approximation to the response x(t) than would be obtained from the previous studies based on the approach of replacing p(t) by a constant.

Referring back to the basic equations of motion in (12), and instead of ignoring the term $\tilde{A}(p(t))x$, we now replace x(t) in $\tilde{A}(\tilde{p}(t))x$ by the first approximation $x_1(t)$. Let $x_2(t)$ be the solution to

$$\dot{x}_2 = A_o x_2 + \tilde{\tilde{A}}(p(t))x_1(t) + u(p(t)) \tag{15}$$

$$x_2 = A_o x_2 + \tilde{\tilde{A}}(p(t))\ H_1(p(t)) + u(p(t)) \tag{16}$$

$$\dot{x}_2 = A_o x_2 + u_1(p(t)) \tag{17}$$

where $u_1(p) = \tilde{A}\ H_1(p) + u(p)$. Again, $x_2(t)$ can be expressed in terms of p(t):

$$x_2(t) = H_2(p(t))$$

The function $x_2(t)$ is expected to be a "better" approximation to x(t) than $x_1(t)$. This procedure can be repeated, to obtain successive mappings $x_n(t) = H_n(p(t))$ which represent closer approximations to the mapping x(t) = M(p(t)).

Analytical expressions for the dynamic response.- The method outlined above may be used to obtain explicit analytical expressions for the dynamic response successive approximations $x_n(t)$ in terms of the maneuver roll-rate p(t). For this purpose, let H(t) be the impulse response matrix corresponding to the coefficient matrix A_o:

$$H(t) = e^{Aot} \tag{18}$$

Repeating the steps outlined above, and using H(t) to relate $x_n(t)$ and p(t), one obtains the following general expression for $x_n(t)$:

$$x_n(t) = \sum_{k=1}^{n} \int_0^t u_k(t,z)dz \tag{19}$$

Where the terms $u_k(t,z)$ are defined by the following recurrence relationship

$$u_k(t,z) = \begin{cases} H(t-z)u(z) & k = 1 \\ H(t-z)\widetilde{A}(\widetilde{p}(z)) \int_0^z \underset{k-1}{u}(z,y)dy & k = 2,\ 3,\ \text{---}n \end{cases} \tag{20}$$

Accuracy of the Method and the Analytical Expressions

The usefullness of the above predictive method and the analytical expressions derived from it depends on two considerations:

1. Does the sequence of approximate expressions $\left\{x_n(t)\right\}$ obtained by the method converge to the exact airplane response x(t)?

2. If convergence is guaranteed, how fast is the rate of convergence? What is the magnitude of the error expected in the successively closer approximations $\left\{ x_1(t),\ x_2(t),\ x_3(t), \text{-------}\right\}$.

<u>Convergence of the analytical expressions $\left\{x_n(t)\right\}$ to the exact airplane response x(t).</u> -

The Following is an analytical proof which establishes rigorously the convergence of $\left\{x_n(t)\right\}$ to x(t):

Recall from (9) and (12) that the aircraft response during a rolling maneuver p(t) is given by the solution of the equation

$$\dot{x} = A(t)x + u(t) \tag{21}$$

$$\dot{x} = A_o x + \widetilde{A}(t)x + u(t) \tag{22}$$

and from (19) and (20) the expression for the nth approximate response is

$$x_n = \sum_{k=1}^{n} \int_0^t u_k(t,z)dz \tag{23}$$

where

$$u_k(t,z) = \begin{cases} H(t-z)\, u(z) & k = 1 \qquad (24) \\ H(t-z)\, \widetilde{A}(z) \int_0^z u_{k-1}(z,y)dy & k = 2, 3, \ldots n \qquad (25) \end{cases}$$

$$H(t) = e^{A_o t}$$

We now show that X_∞ is indeed the exact solution to the airplane response equation (22).

$$x_\infty = \sum_{k=1}^{\infty} \int_0^t u_k(t,z)dz \tag{26}$$

$$\dot{x}_\infty = \sum_{k=1}^{\infty} \frac{d}{dt} \int_0^t u_k(t,z)dz$$

Using Leibnitz rule for differentiation, one has

$$\dot{x}_\infty = \sum_{k=1}^{\infty} \left[\int_0^t \frac{\partial}{\partial t} u_k(t,z)dz + u_k(t,t) \right] \tag{27}$$

From the expression for H(t), one obtains

$$\dot{H} = A_o e^{A_o t} = A_o H$$

$$H(0) = I \qquad \text{(Identity Matrix)}$$

From (24) and (25) one has

$$u_1(t,t) = H(0)\; u(t) = u(t) \tag{28}$$

$$u_k(t,t) = \widetilde{A}(t) \int_0^t u_{k-1}(t,y)dy \qquad k=2,3,\ldots \tag{29}$$

$$\begin{aligned} \frac{\partial}{\partial t} u_1(t,z) &= \dot{H}(t-z)\, u(z) = A_o H(t-z)\, u(z) \\ &= A_o u_1(t,z) \end{aligned} \tag{30}$$

$$\frac{\partial}{\partial t} u_k(t,z) = A_o H(t - z) \tilde{A}(z) \int_0^t u_{k-1}(z, y)dy$$

$$= A_o u_k(t,z) \qquad\qquad k = 2,3, \tag{31}$$

Rewriting the expression in (27) as

$$\dot{x}_\infty = \sum_{k=1}^{\infty} \left[\int_0^t \frac{\partial}{\partial t} u_k(t,z)dz\right] + u_1(t,t) + \sum_{k=2}^{\infty} u_k(t,t)$$

and substituting the expressions (28) through (31), one obtains

$$\dot{x}_\infty = \sum_{k=1}^{\infty} \int_0^t A_o u_k(t,z)dz + u(t) + \sum_{k=2}^{\infty} A(t) \int_0^t u_{k-1}(t,z)dz$$

$$\dot{x}_\infty = A_o \sum_{k=1}^{\infty} \int_0^t u_k(t,z)dz + u(t) + \tilde{A}(t) \sum_{k=2}^{\infty} \int_0^t u_{k-1}(t,z)dz$$

$$\dot{x}_\infty = A_o \sum_{k=1}^{\infty} \int_0^t u_k(t,z)dz + u(t) + \tilde{A}(t) \sum_{k=1}^{\infty} \int_0^t u_k(t,z)dz \tag{32}$$

Substituting the expression from (26) into (32),

$$\dot{x}_\infty = A_o x_\infty + \tilde{A}(t)x_\infty + u(t)$$

Thus x_∞ is the exact solution to (22), and therefore represents the required airplane response.

Accuracy of the approximate responses $x_n(t)$ for a parametrized family of rolling maneuvers.- A study was conducted in order to assess the comparative accuracy of the approximations x_1, x_2, and x_3 resulting from different members of a class of rolling maneuvers. A second objective of the study was to compare the accuracy of x_1, x_2, and x_3, as obtained by the present method, to the accuracy of the approximation $x_c(t)$ obtained by replacing p(t) in the equations of motion by a constant (average value of p(t) over the duration of the maneuver). The response $x_c(t)$ represents the type of result that would be obtained if one attempts to apply previous methods (Sternfield, Moul, and Brennan) for the prediction of the airplane dynamic response to a realistic rolling maneuver comprising a gradually increasing roll-rate followed by a gradually decreasing roll-rate. The study was carried out using the numerical values of the mass and aerodynamic parameters of the F-100A. These parameters are shown in Table I. The parametrized class of rolling maneuvers used is given by

$$\left\{p(t)\right\}_{P_m} = P_m(1 - e^{-0.6t}) \qquad o \leq t \leq \frac{2\pi}{P_m}$$

$$= P_m(e^{1.2\pi/P_m} - 1)e^{-0.6t} \qquad t > \frac{2\pi}{P_m}$$

Each member of the family $\left\{p(t)\right\}_{P_m}$, corresponding to a specific choice of the parameter P_m, represents a banking maneuver of 360° executed with a different roll-rate variation. The value of P_m is indicative of the level of aileron applied during the execution of the maneuver. The study was made using the following values of P_m:

$$P_m = -1, -1.5, -2.0, -2.5, -3.0, -3.5, -4.0 \text{ rad/sec.}$$

For these values of P_m, the members of $\left\{p(t)\right\}$ are shown in figure 1.

It should be noted that, for the configuration parameters used in this study, the Phillips criterion indicates the airplane is unstable for roll-rates between -1.86 and -2.33 rad/sec. This implies that, for the maneuvers represented by the values $P_m = -2.5, -3.0, -3.5, -4.0$, the airplane goes thru unstable modes of motion during the execution of the maneuver. This is depicted graphically in figure 1, which shows a plot of $\left\{p(t)\right\}_{P_m}$.

For each value of P_m, the computer was used to obtain the response x(t), the approximate responses $x_1(t)$, $x_2(t)$, $x_3(t)$ as provided by the present method, and the approximate response $x_c(t)$ based on the constant roll assumption of previous methods. For each value of P_m, the time histories of α, α_1, α_2, α_3, α_c, β, β_1, β_2, β_3, and β_c were plotted. The results obtained are shown in figures 2 thru 15. Inspection of these results leads to the following conclusions:

1. Although all of the roll maneuvers considered amount to the same bank-angle, namely, 360°, the resulting time-histories of α and β are strongly dependent on the variation of the roll-rate p(t). This fact points to inadequacy of replacing the roll-rate p(t) by a constant value for realistic maneuvers involving variation in roll-rate.

TABLE I

MASS AND AERODYNAMIC PARAMETERS USED IN STUDY

m, slugs	745
I_x, slug-ft^2	10,976
I_y, slug-ft^2	57,100
I_z, slug-ft^2	64,975
C_{Y_β}, per radian	-0.28
C_{Y_r}, per radian	0.34
C_{Y_p}, per radian	0.15
C_{L_α}, per radian	3.85
C_{m_α}, per radian	-0.36
C_{m_q}, per radian	-3.5
$C_{m_{\dot{\alpha}}}$, per radian	-1.25
C_{n_β}, per radian	0.057
C_{n_r}, per radian	-0.095
C_{n_p}, per radian	-0.034
C_{ℓ_p}, per radian	-0.255
C_{ℓ_r}, per radian	0.09
$C_{\ell\beta}$, per radian	0.06
$C_{\ell_{\delta_a}}$, per radian	-0.044
V, ft/sec	691
S, ft^2	377
$\bar{c}$, ft	11.3
b, ft	36.6
$\bar{q}$, lb/ft^2	197

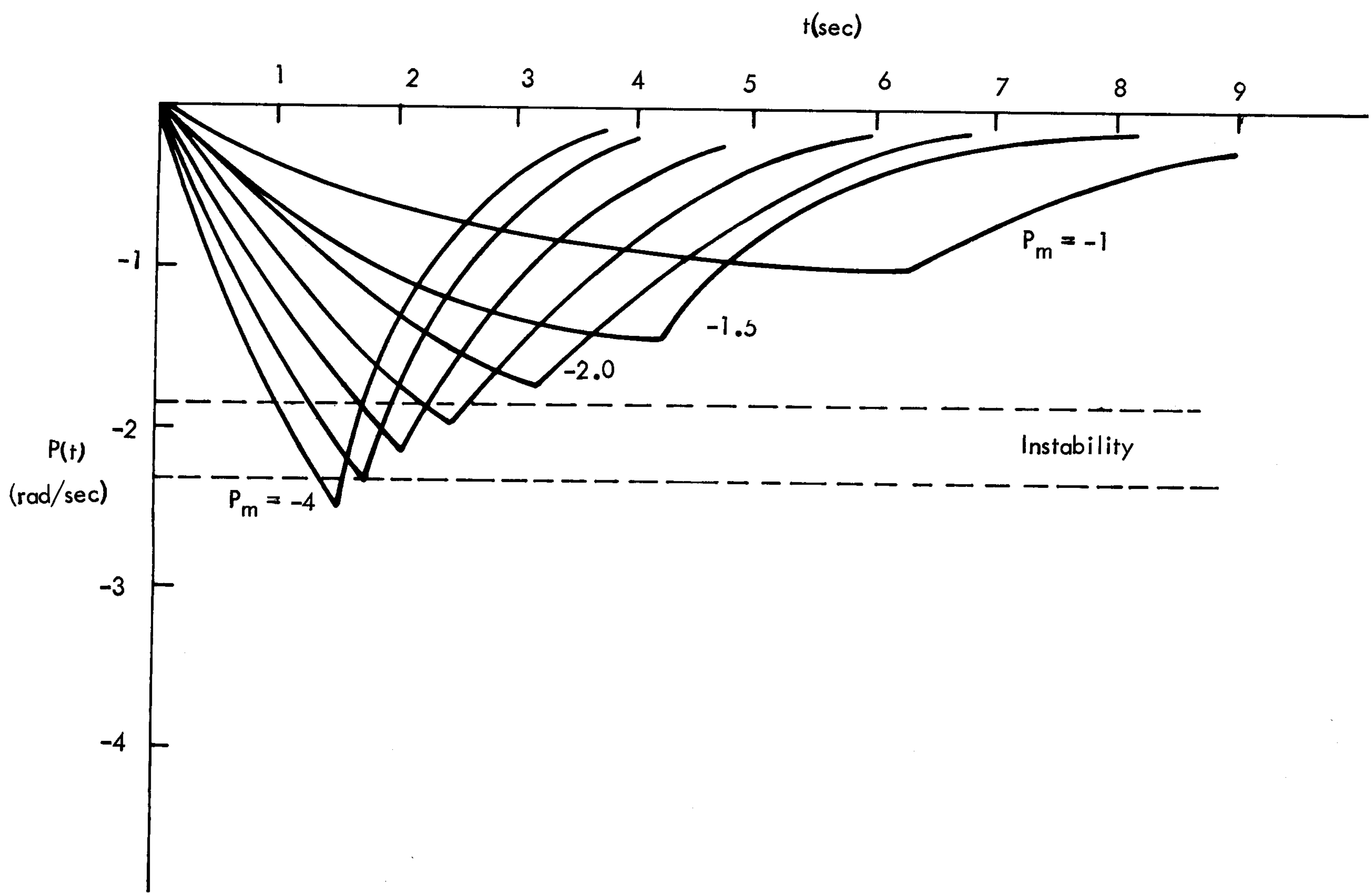


FIGURE 1. FAMILY OF ROLLING MANEUVERS

2. The peaks of α occurred consistently during the termination phase of the maneuver. The peaks of β occurred during the termination phase of the maneuvers represented by P_m = -1, -1.5. Relatively large peak values of β still occurred in the termination phase of the maneuvers $P_m \geq 2$. Again, these facts cannot be predicted from an analysis based on a constant roll-rate approximation.

3. The error plots in figures 16 thru 22 indicate a fairly good rate of convergence of the approximate responses $x_n(t)$ towards the response $x(t)$.

4. The results in figures 16 thru 22 indicate the substantially better results obtained by the present method over the methods based on approximating the maneuver roll-rate by a constant value. For each of the maneuvers considered, the response $x_1(t)$ represented a significantly better approximation to the airplane response $x(t)$ than the approximation $x_c(t)$ obtained by replacing $p(t)$ by its average value.

Comparison with constant roll-rate methods.- In the previous section, comparison was made between the accuracy of the method for obtaining airplane dynamic response as developed in this study, and the accuracy of the methods based on the constant roll-rate assumption. In that comparison, the value of the constant roll-rate used was taken as the average value of the roll-rate variation $p(t)$. In this regard, two questions naturally arise:

1. Is the average value of the roll-rate the best value that can be used in the constant roll-rate methods?

2. How does the accuracy of the method developed in this contract compare with the accuracy of the constant roll-rate method using values other than the average value of roll-rate?

These questions were considered for the case of the family of rolling maneuvers studied previously. For each rolling maneuver, characterized by P_m, the response in α and β was compared with the responses obtained from a number of constant roll-rate maneuvers ranging between 0.8 rad/sec to 3.0 rad/sec, and to responses α_2 and β_2 obtained by the methods used in this study. The results are shown in Table II. Each of the exact response in α (or β) was compared with the approximate responses resulting from constant roll-rates of 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.5, and 3.0 rad/sec, and with the approximate response α_2 (or β_2), obtained by the methods of this study. This is illustrated in figures 23 and 24 for the maneuver represented by $P_m = -2$.

Table II indicates for each maneuver the values of constant roll-rate for which the corresponding response seems to give the least average error and the closest peak estimate. These are shown in rows 3 and 4 of the tabulation. Row 5 is the average of the values in rows 3 and 4, and may be considered as the "optimum" choice of constant roll-rate. Row 6 compares this optimum values of constant roll-rate with the average value P_{av} of the roll-rate for each rolling maneuver. The following conclusions are evident from the tabulation and the figures:

1. The "optimum" value of constant roll-rate is 20 to 30% larger than the average value of the roll-rate for the particular family of maneuvers considered.

2. Even if one uses this optimum value, the accuracy of the method of this work is far better than the accuracy of the constant roll-rate method.

Analytical expressions for the time history of a maneuver starting from arbitrary non-zero initial conditions. - The method was extended to the case of a rolling maneuver starting from non-zero initial conditions in the variables $\Delta\alpha$, β, q, and r. The airplane response $x(t) = [\Delta\alpha(t), \beta(t), q(t), r(t)]$ to an arbitrary rolling maneuver $p(t)$, with initial conditions $x_o = x(0) = [\Delta\alpha_o, \beta_o, q_o, r_o]$, is governed by the equation

$$\begin{aligned} \dot{x} &= A(p(t))x + u(p(t)) \\ &= A(P + \tilde{p}(t))x + u(p(t)) \\ &= A_o x + \tilde{A}(t)x + u(p(t)) \end{aligned}$$

Repeating the steps of the method and taking into consideration the non-zero value of the initial conditions $x(0)$, the successive approximations $x_n(t)$ can be expressed as

$$x_n(t) = \int_0^t \sum_{k=1}^{n} W_k(t,z)dz \tag{33}$$

where the terms $W_k(t,z)$ are defined by the following recurrence relationship:

$$W_k(t,z) = \begin{cases} H(t-z)\,[u(z) + \delta(z)x_o] & k = 1 \\ H(t-z)\,\tilde{A}(z) \int_0^z W_{k-1}(z,y)dy & k = 2,3,\ldots n \end{cases} \tag{34}$$

where $H(t) = e^{A_o t}$ and $\delta(.)$ is the unit-impulse function (delta function).

TABLE II

	α							β						
P_m (Maneuver)	1.0	1.5	2.0	2.5	3.0	3.5	4.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Average Value of Roll-rate P_{av}	.68	.87	.92	.94	.945	.95	.955	.68	.87	.92	.94	.945	.95	.955
Value of Constant Roll-rate for least average error estimate	.8	1.0	1.0	1.0	1.0	1.0	1.0	.8	1.2	1.2	1.2	1.2	1.2	1.2
Value of Constant Roll-rate for best peak estimate	.8	1.2	1.3	1.3	1.3	1.3	1.3	.8	1.0	1.2	1.2	1.2	1.2	1.2
Value of Constant Roll-rate for best "overall" estimate P_c	.8	1.1	1.2	1.2	1.2	1.2	1.2	.8	1.1	1.2	1.2	1.2	1.2	1.2
$\frac{P_c}{P_{av}}$	1.18	1.26	1.30	1.28	1.27	1.26	1.26	1.18	1.26	1.30	1.28	1.27	1.26	1.26

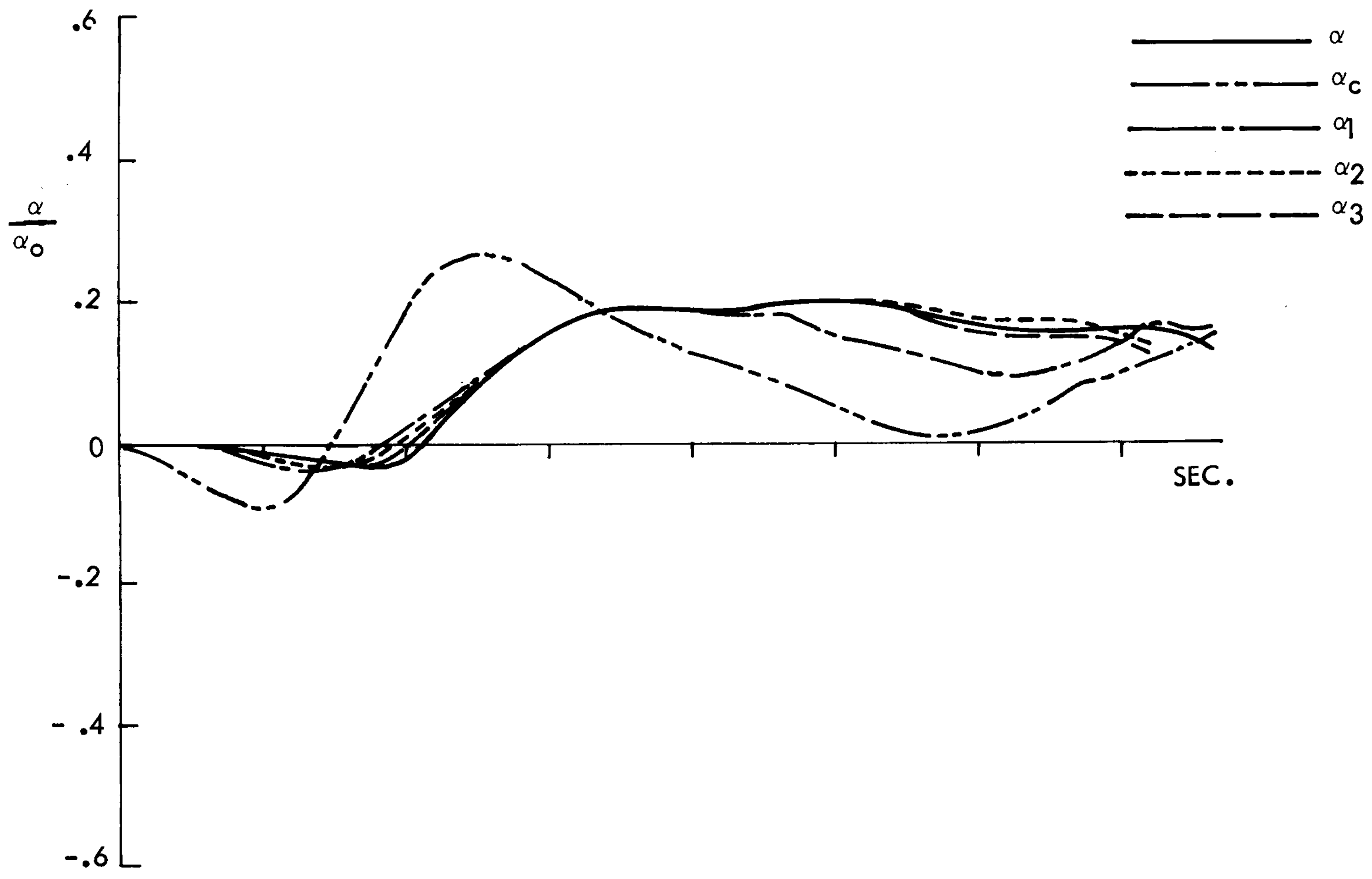


FIGURE 2. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER $P_m = -1$ RAD/SEC

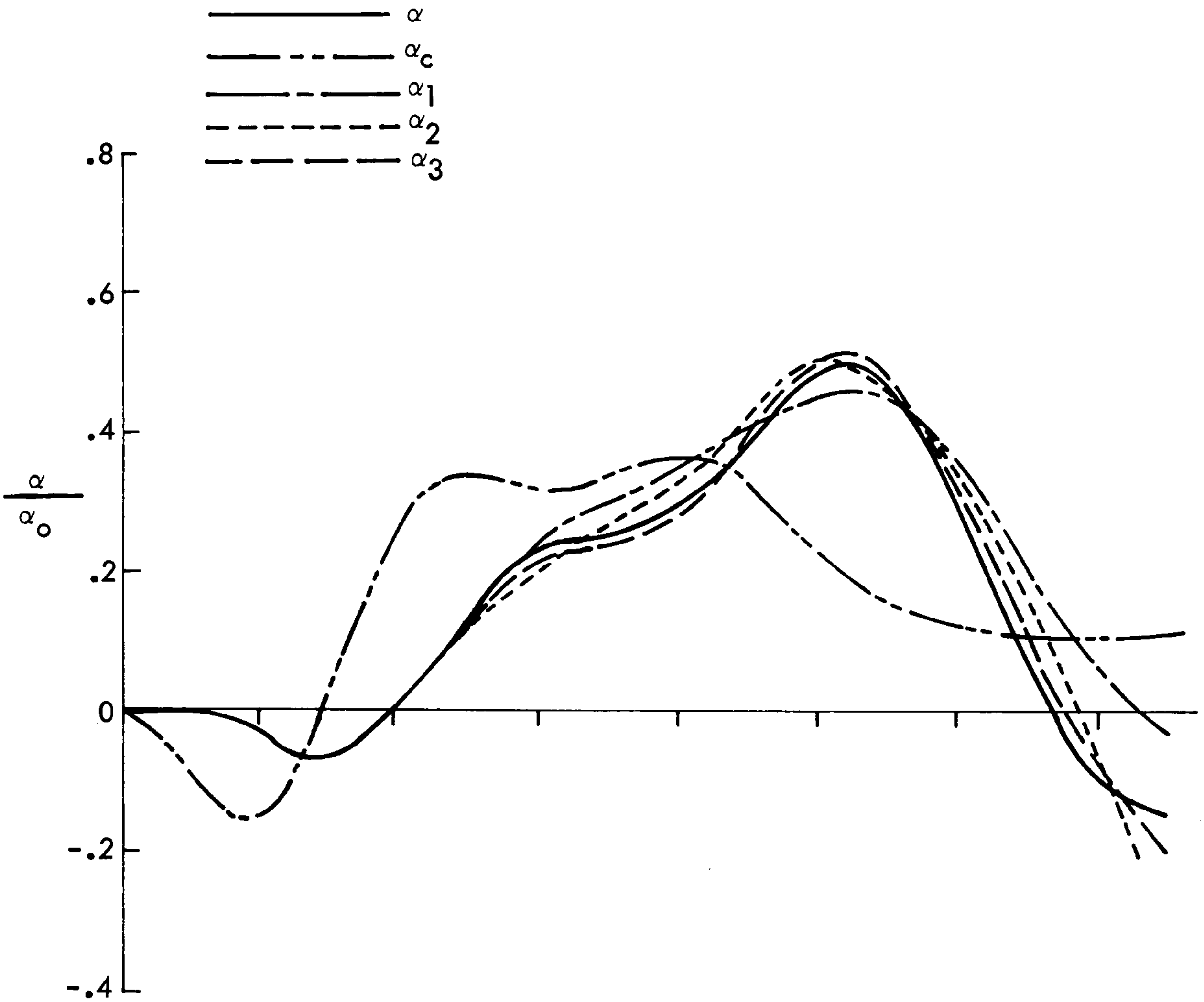


FIGURE 3. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER $P_m = -1.5$ RAD/SEC

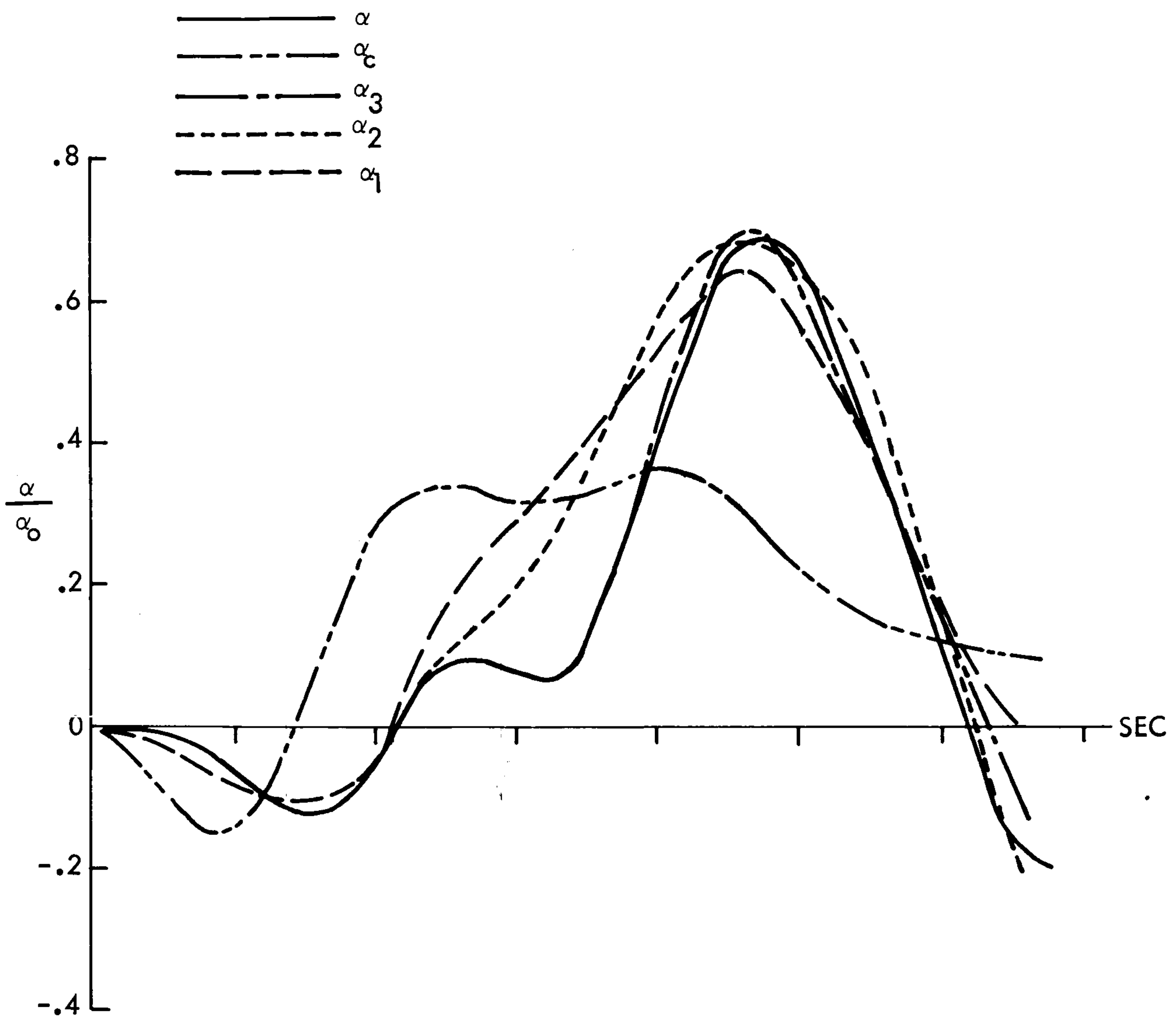


FIGURE 4. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER $P_m = -2.0$ RAD/SEC

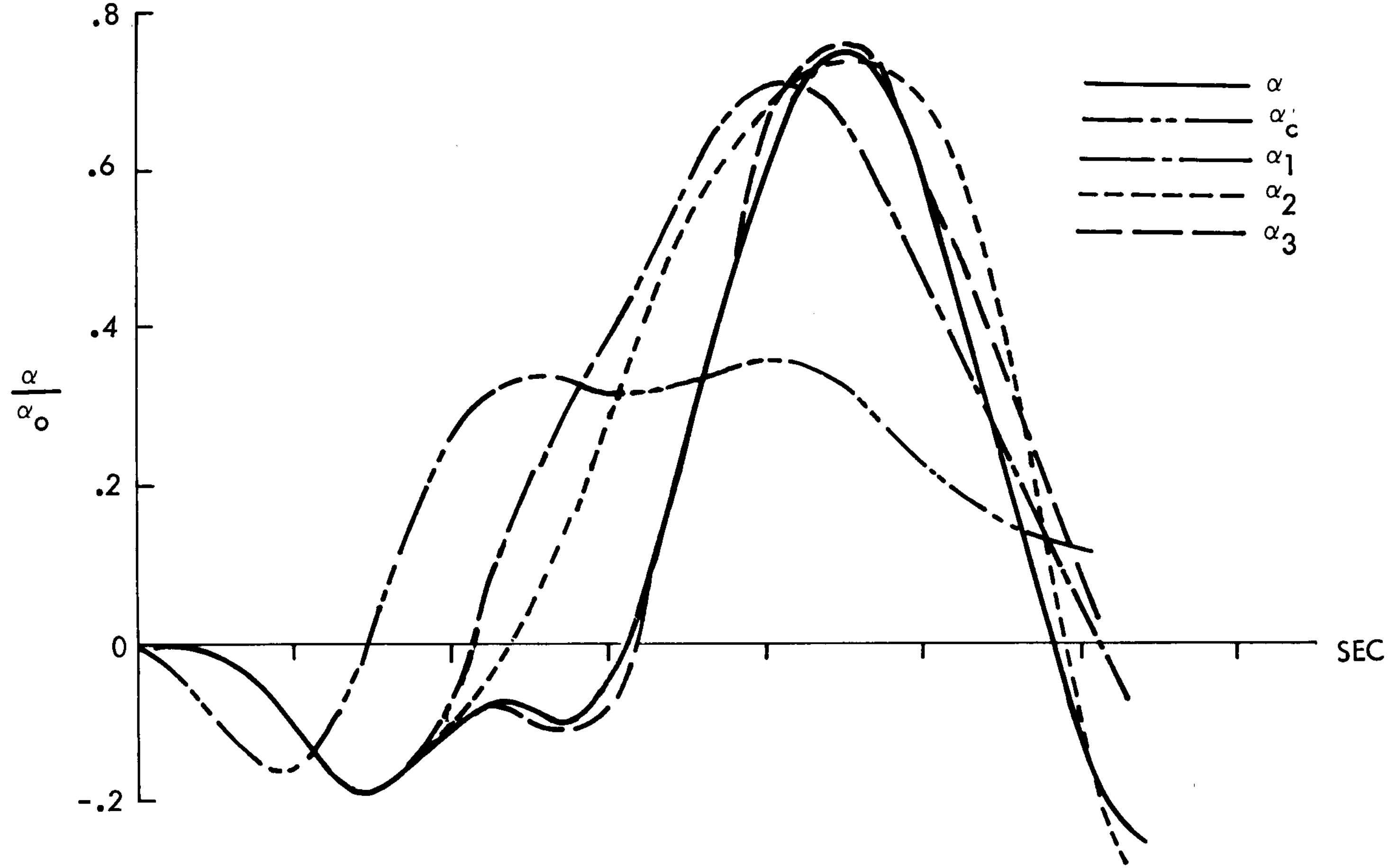


FIGURE 5. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER P_{in} = -2.5 RAD/SEC

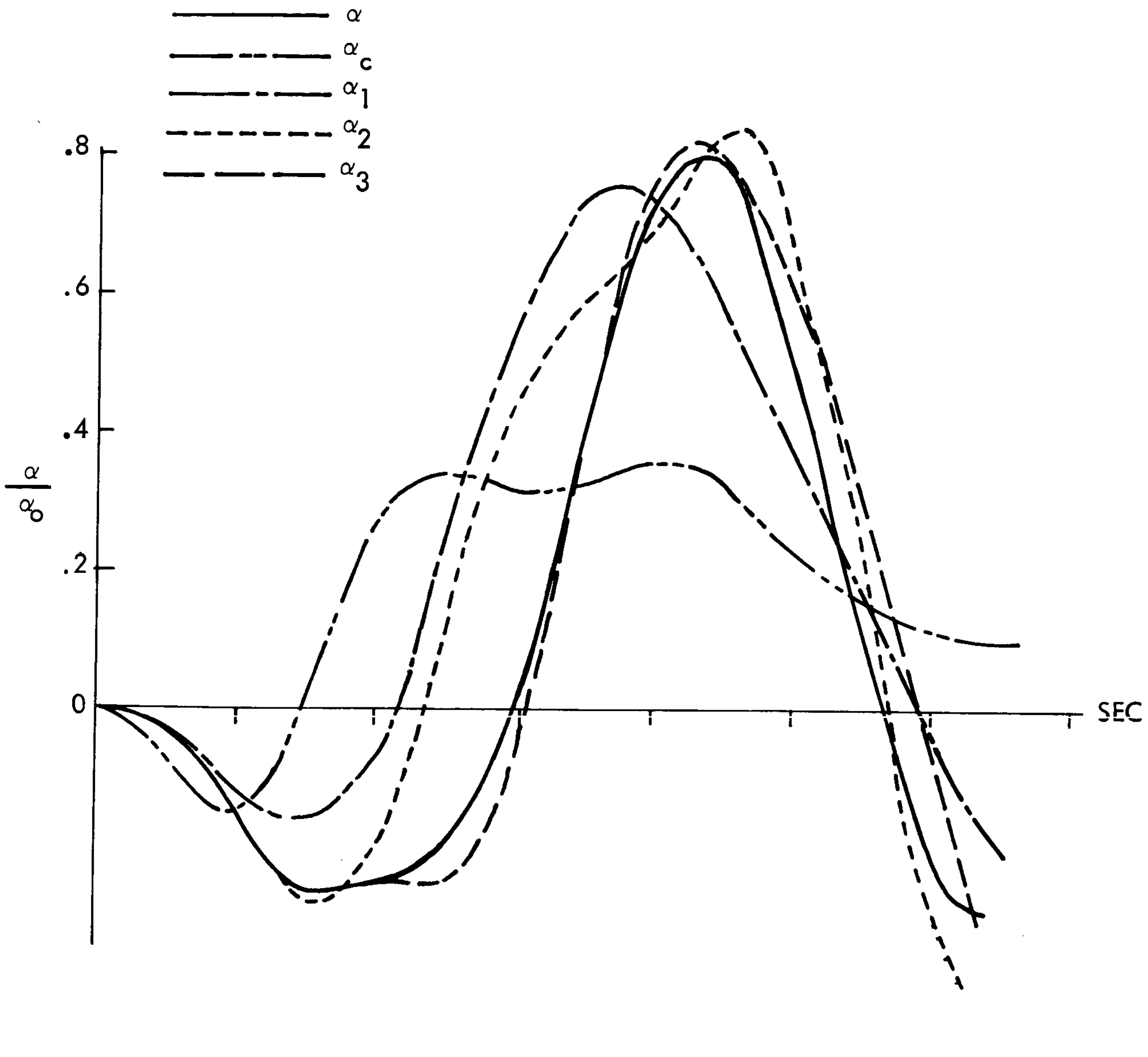


FIGURE 6. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER P_m = -3.0 RAD/SEC

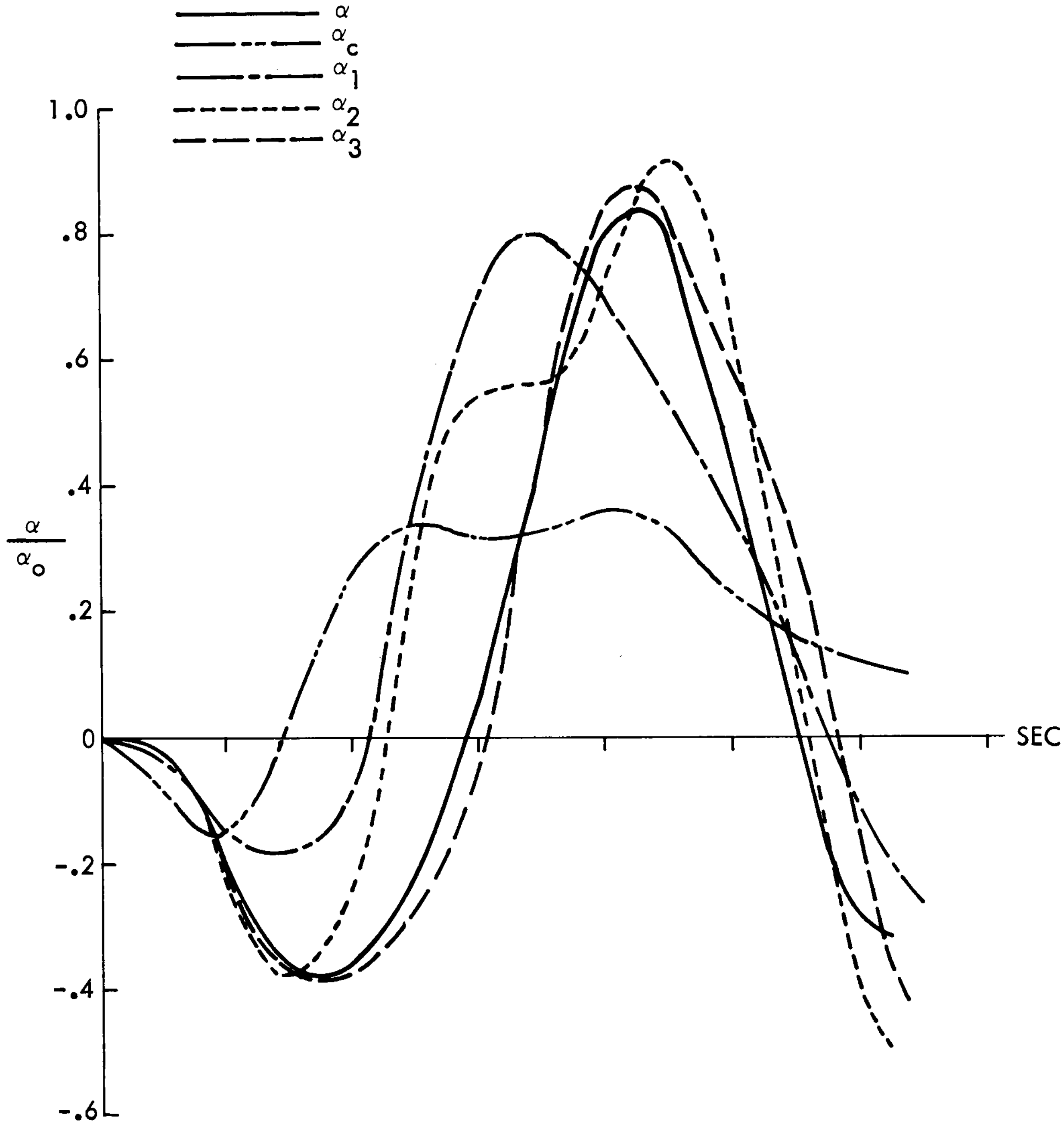


FIGURE 7. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER P_m = -3.5 RAD/SEC

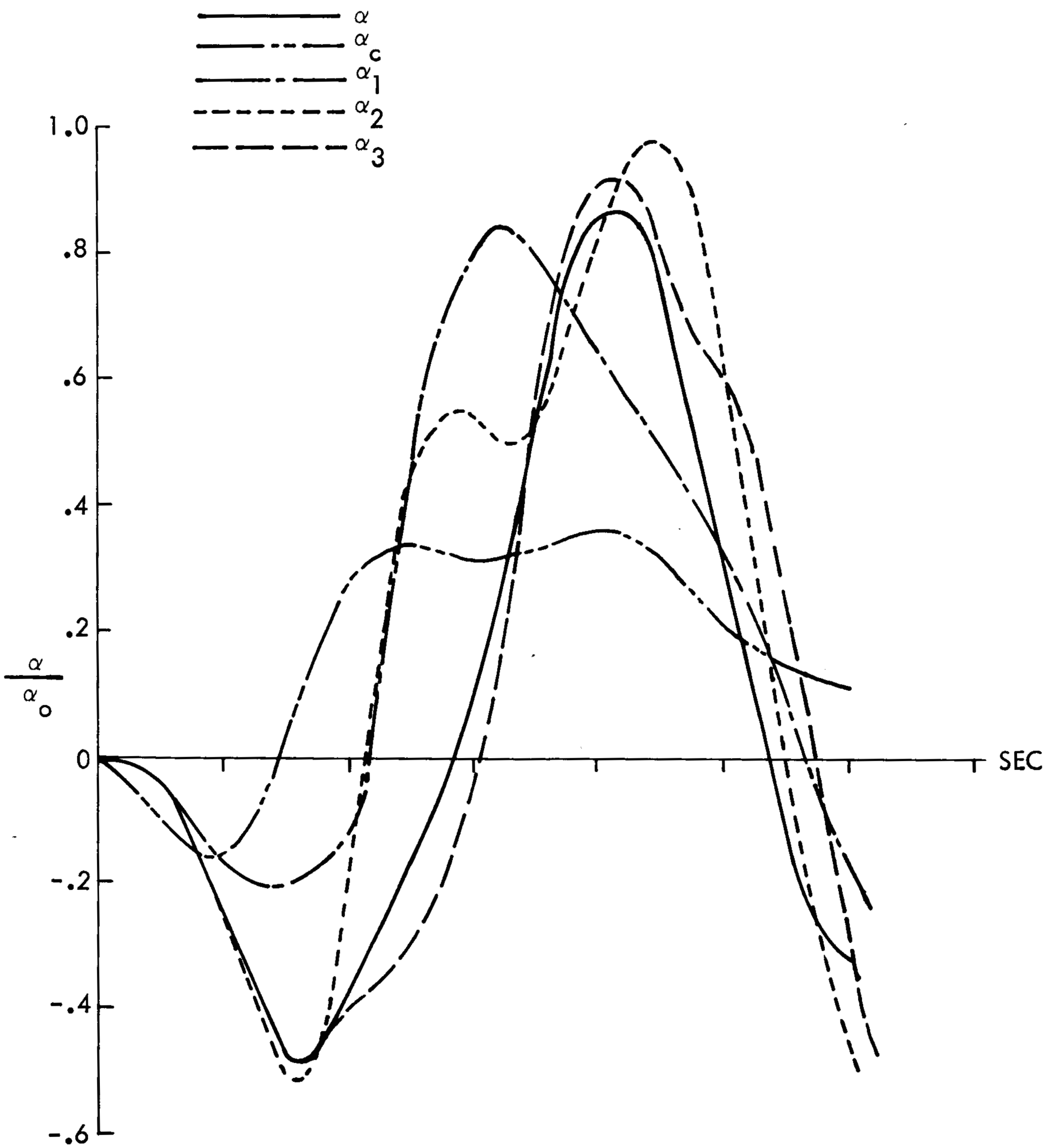


FIGURE 8. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN α FOR ROLLING MANEUVER $P_m = -4.0$ RAD/SEC

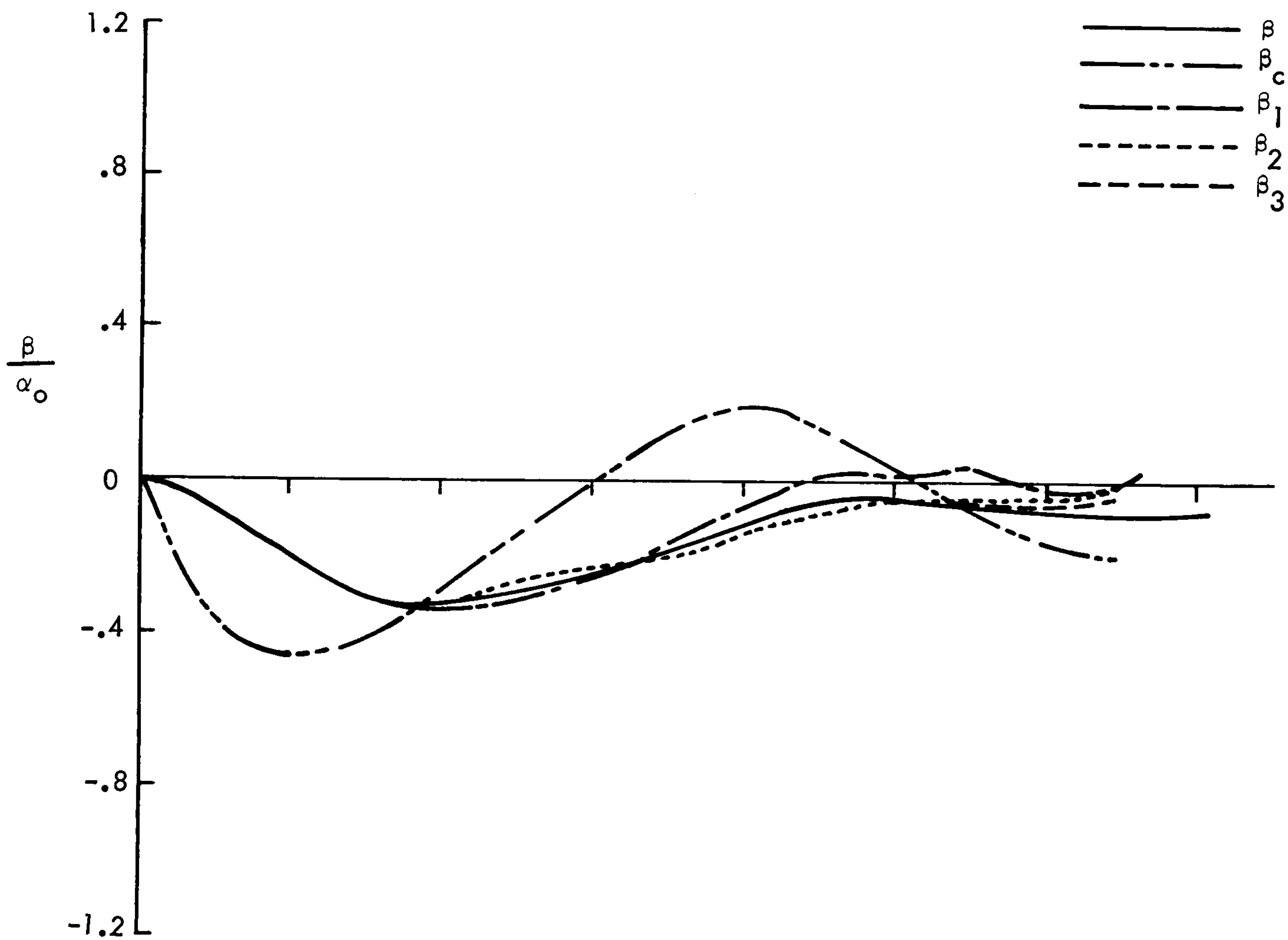


FIGURE 9. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER $P_m = -1.0$ RAD/SEC

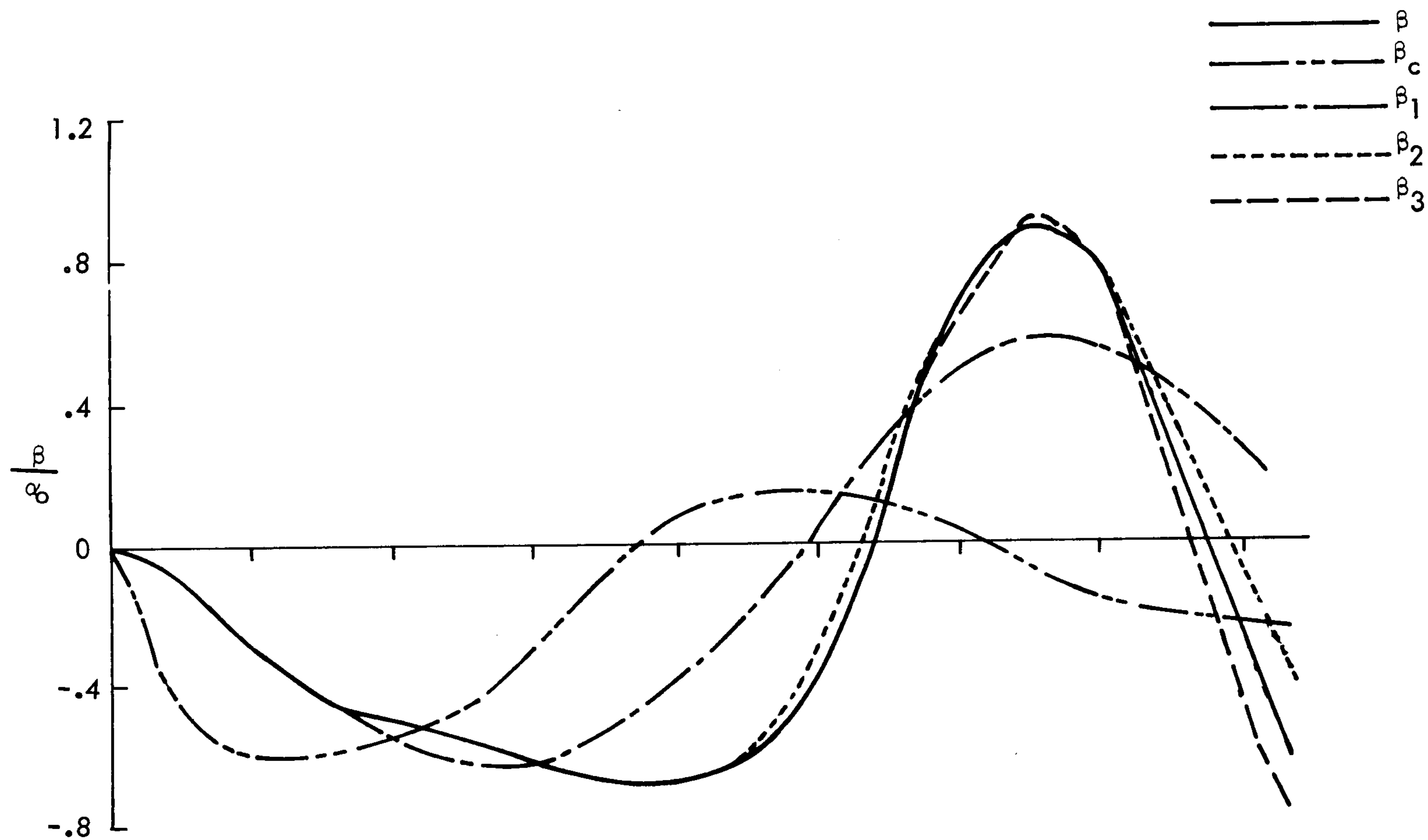


FIGURE 10. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER $P_m = -1.5$ RAD/SEC

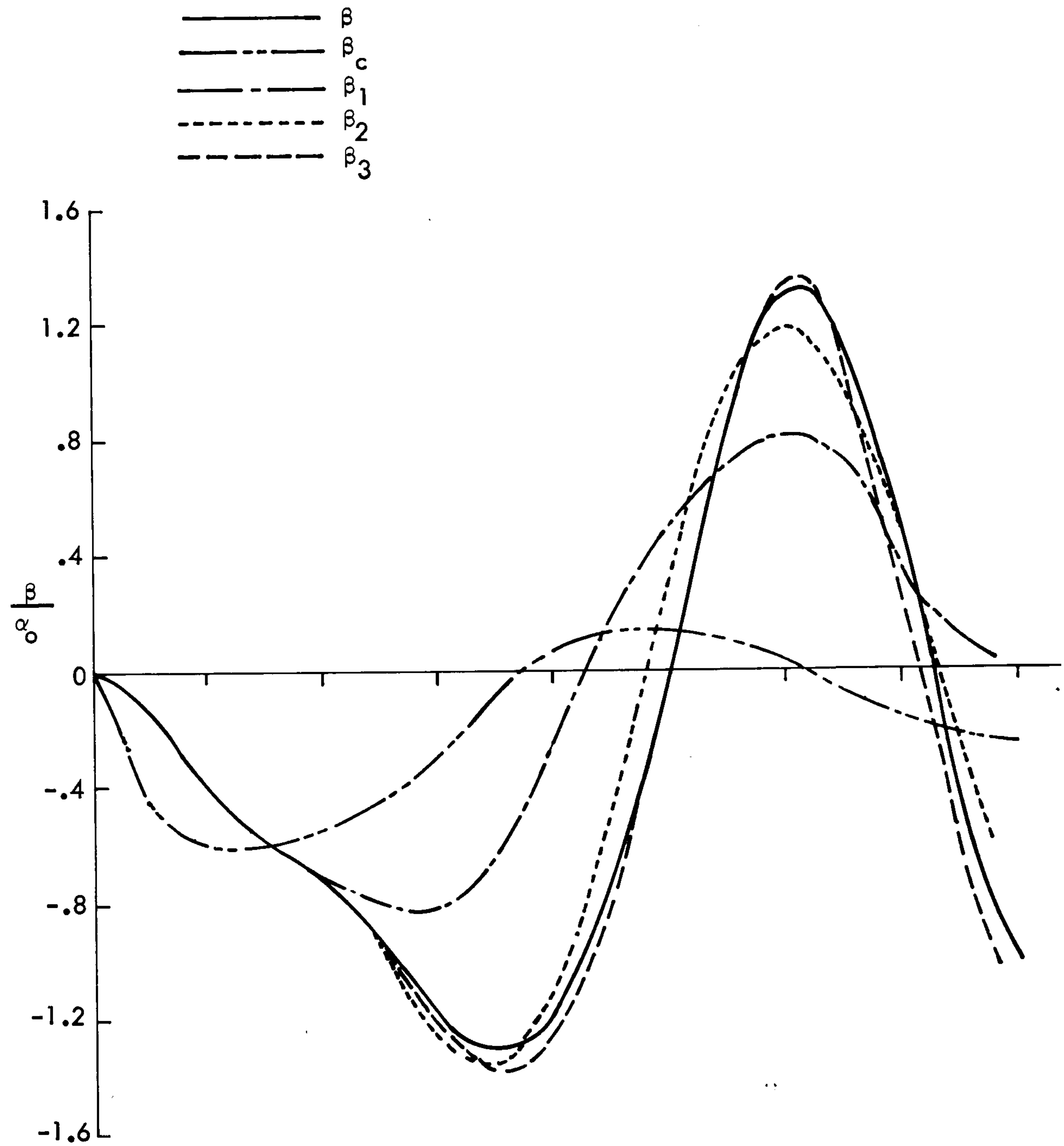


FIGURE 11. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER $P_m = -2$ RAD/SEC

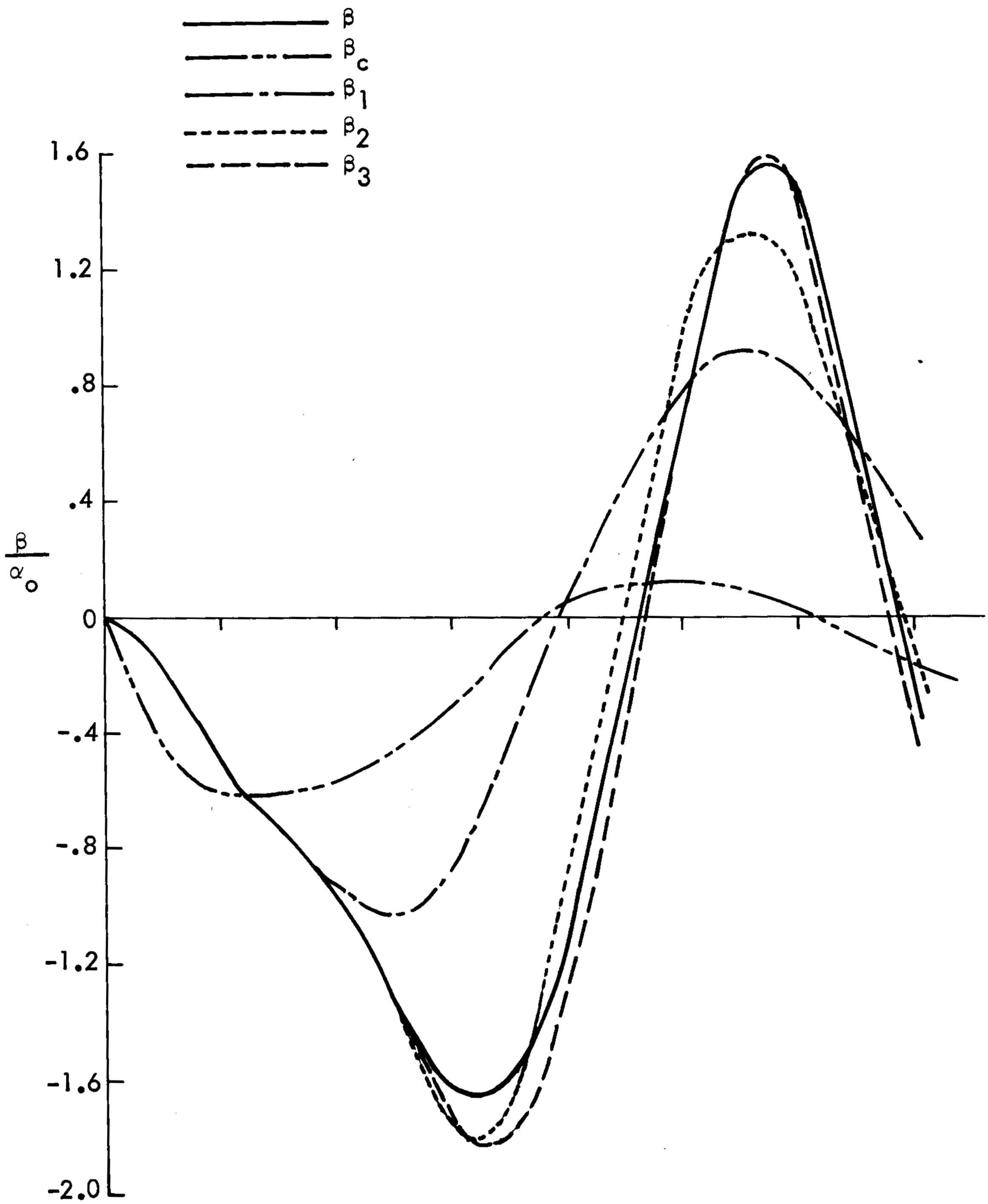


FIGURE 12. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER P_m = -2.5 RAD/SEC

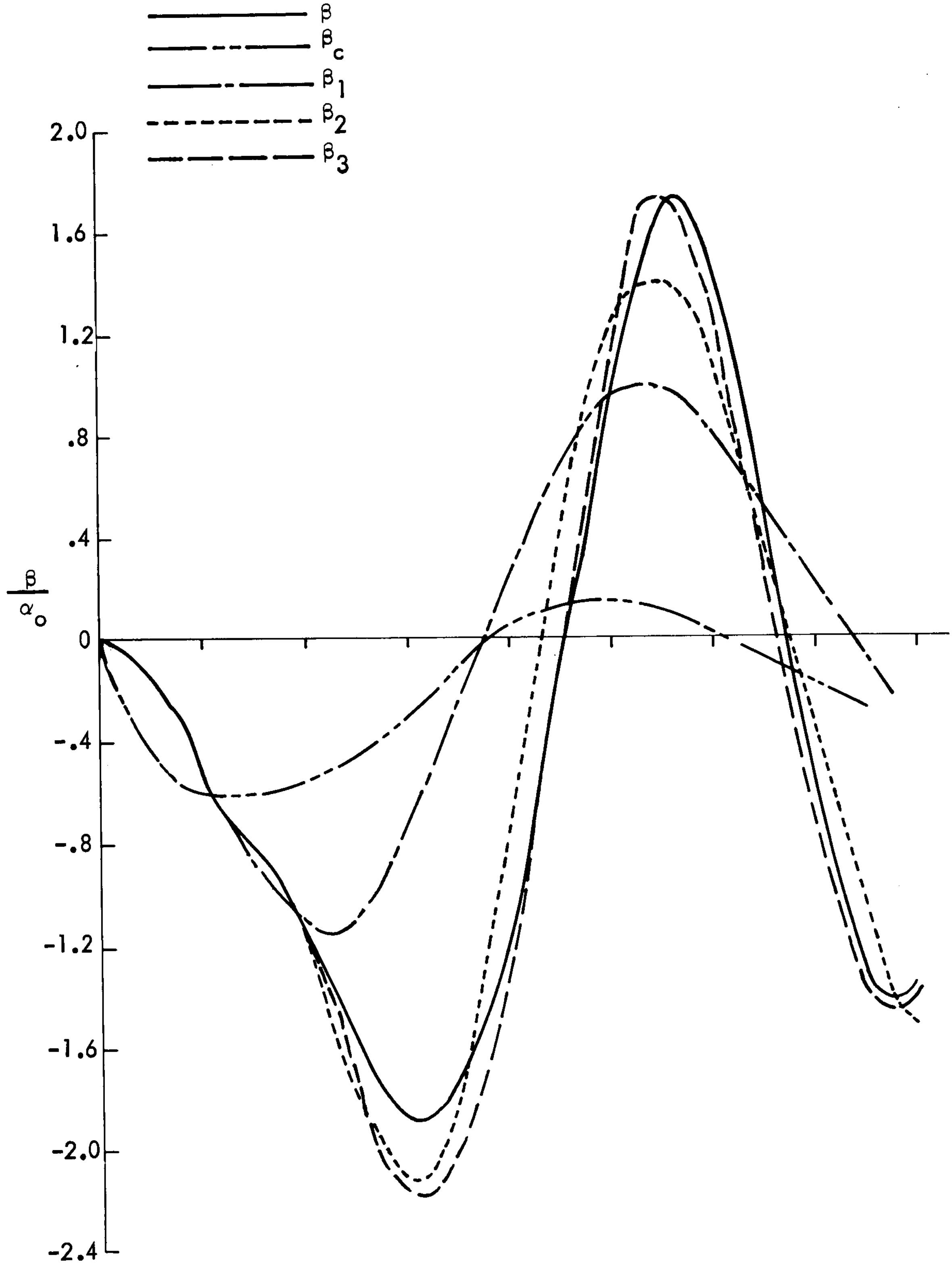


FIGURE 13. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER P_m = -3.0 RAD/SEC

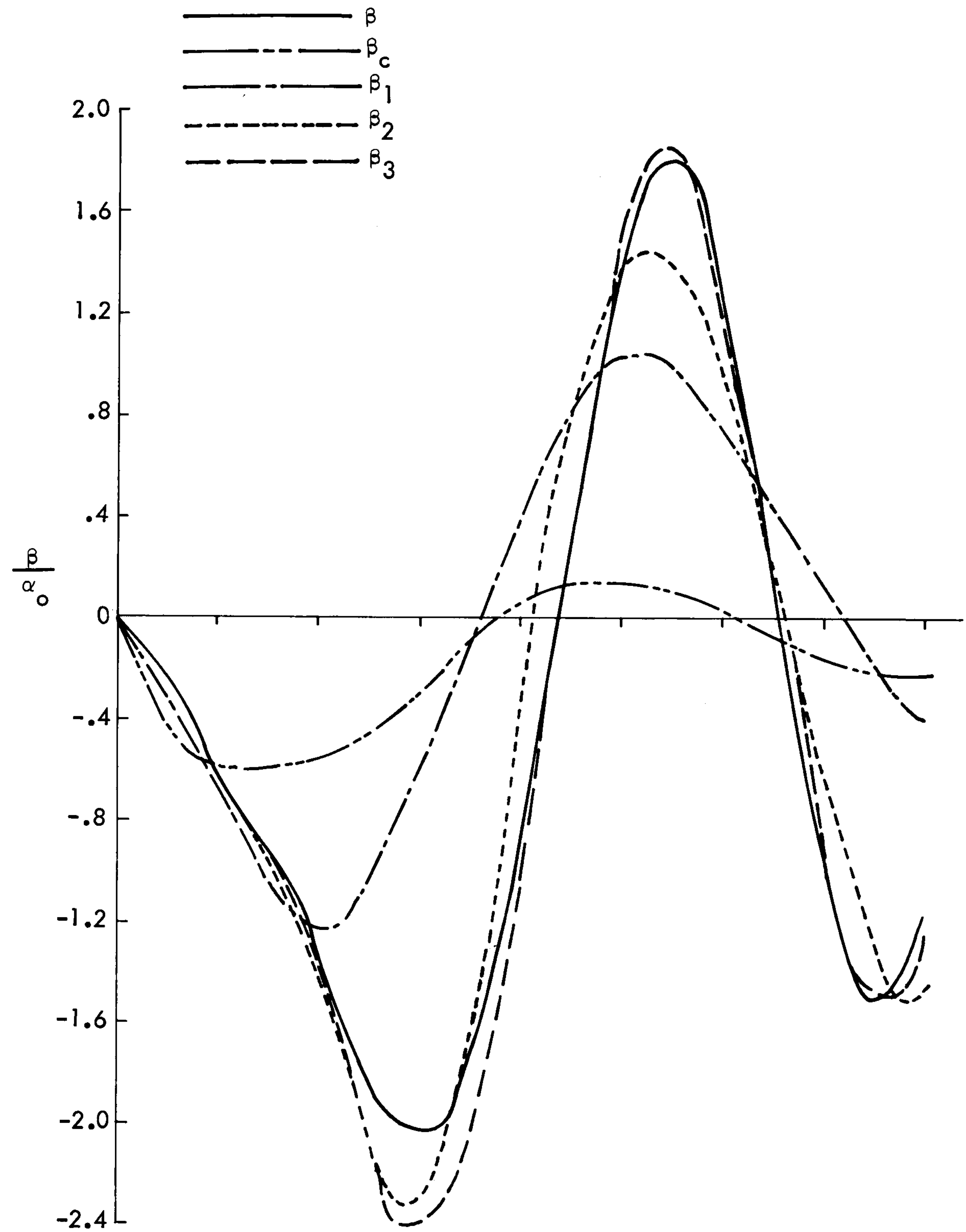


FIGURE 14. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER $P_m = -3.5$ RAD/SEC

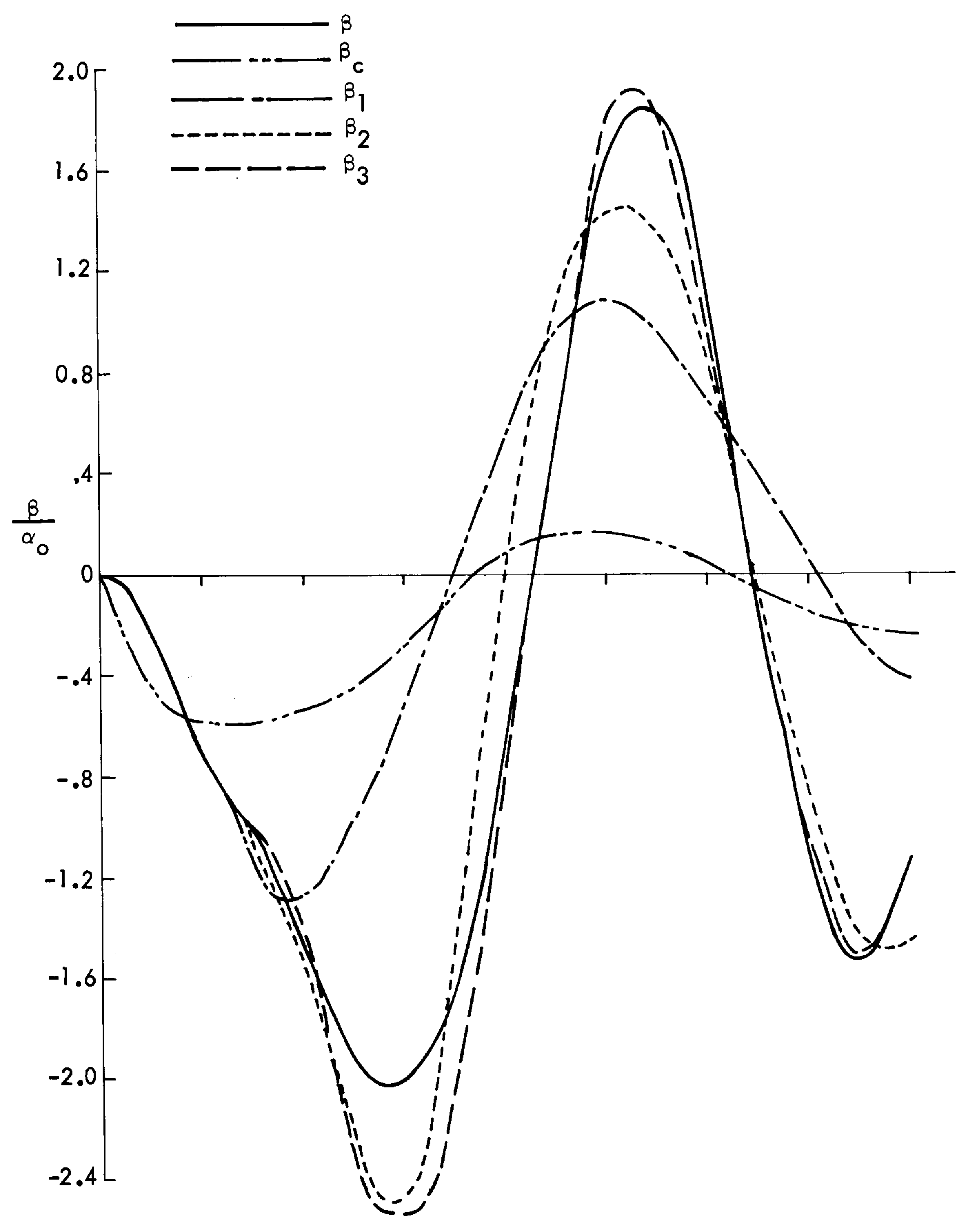


FIGURE 15. COMPARISON OF EXACT AND APPROXIMATE RESPONSES IN β FOR ROLLING MANEUVER $P_m = -4.0$ RAD/SEC

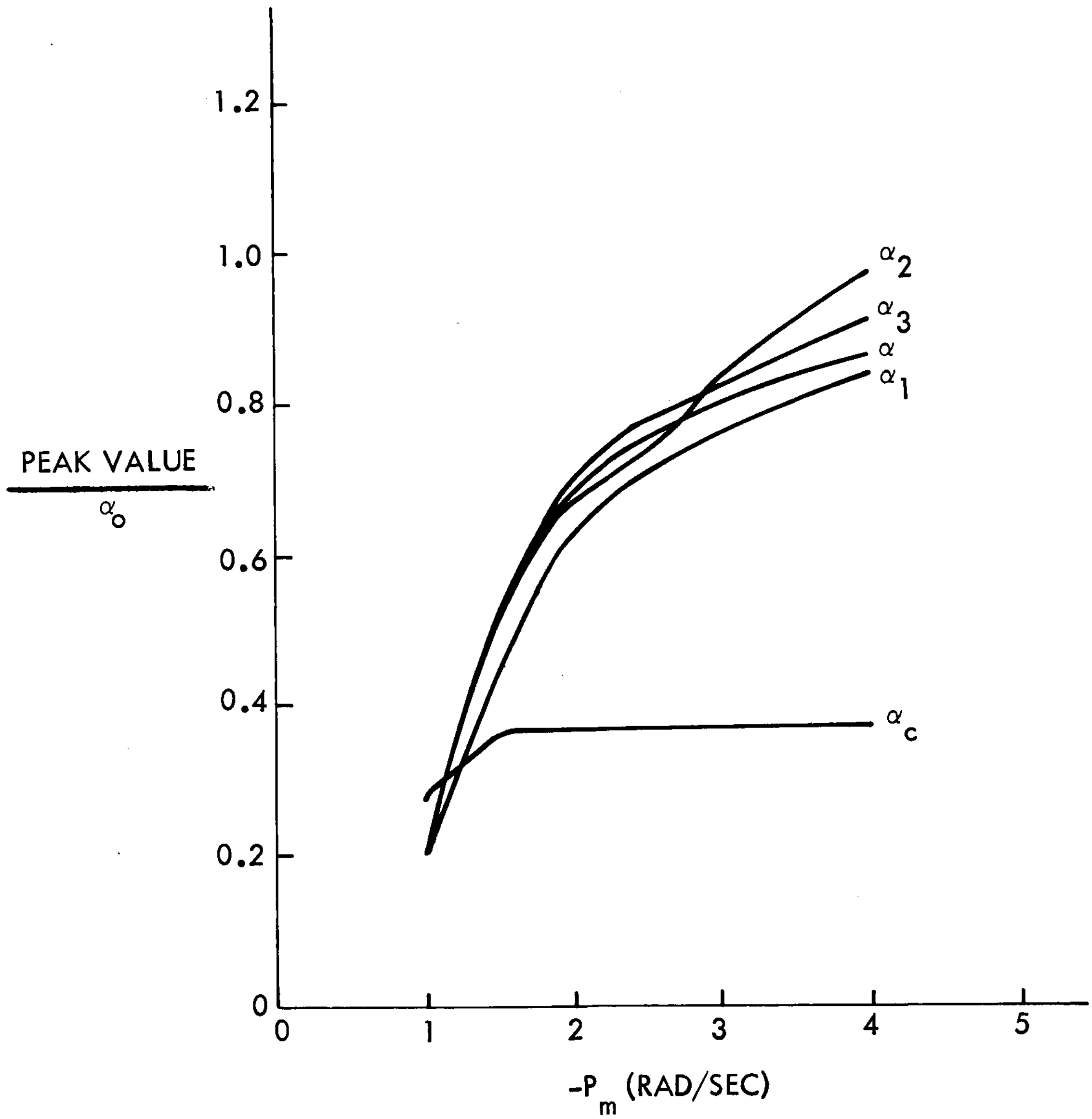


FIGURE 16. COMPARISON OF EXACT AND APPROXIMATE PEAK VALUES OF α RESPONSE FOR FAMILY OF ROLLING MANEUVERS $\{P(t)\}_{P_m}$

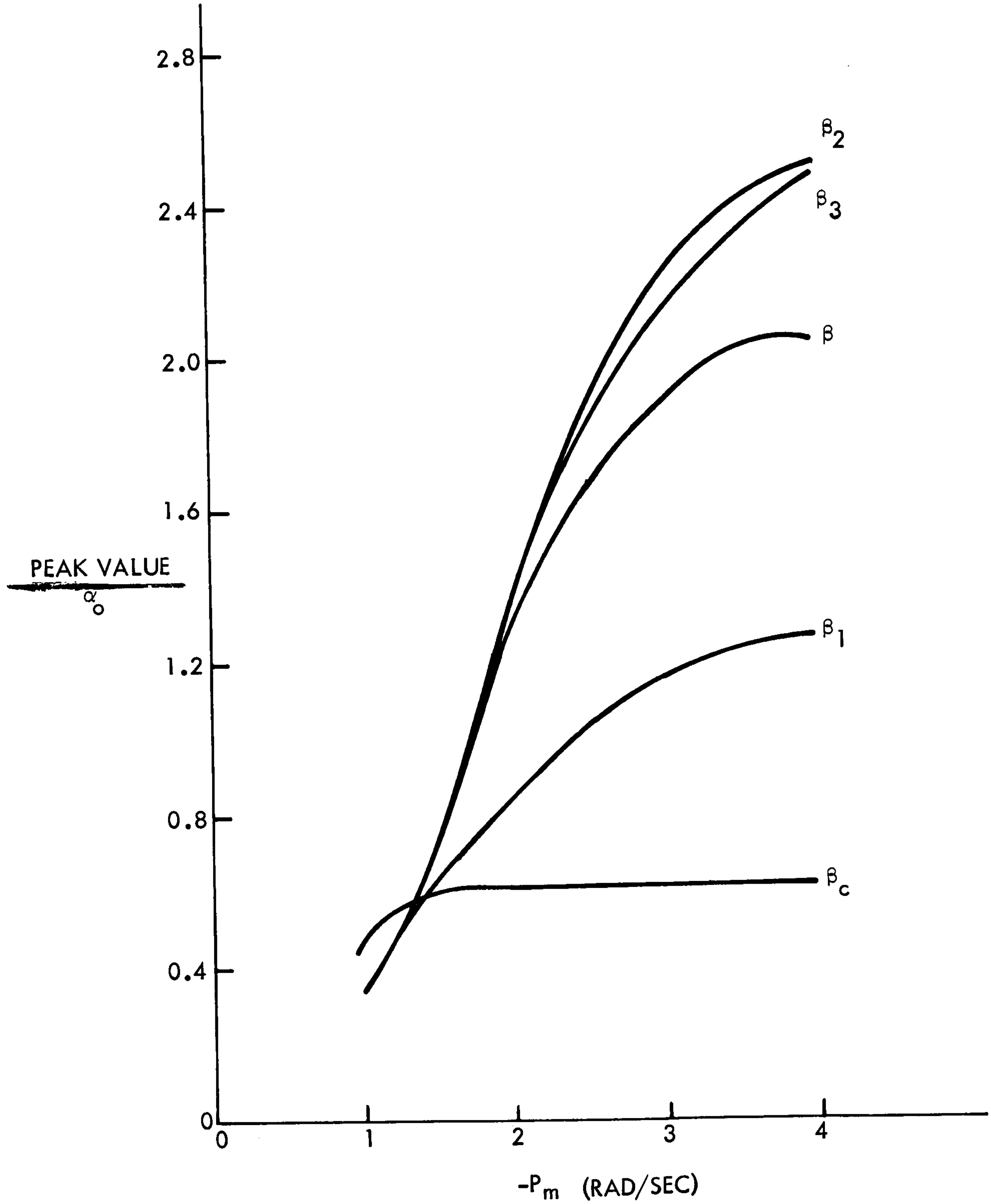


FIGURE 17. COMPARISON OF EXACT AND APPROXIMATE FIRST PEAK VALUES OF β RESPONSE FOR FAMILY OF ROLLING MANEUVERS

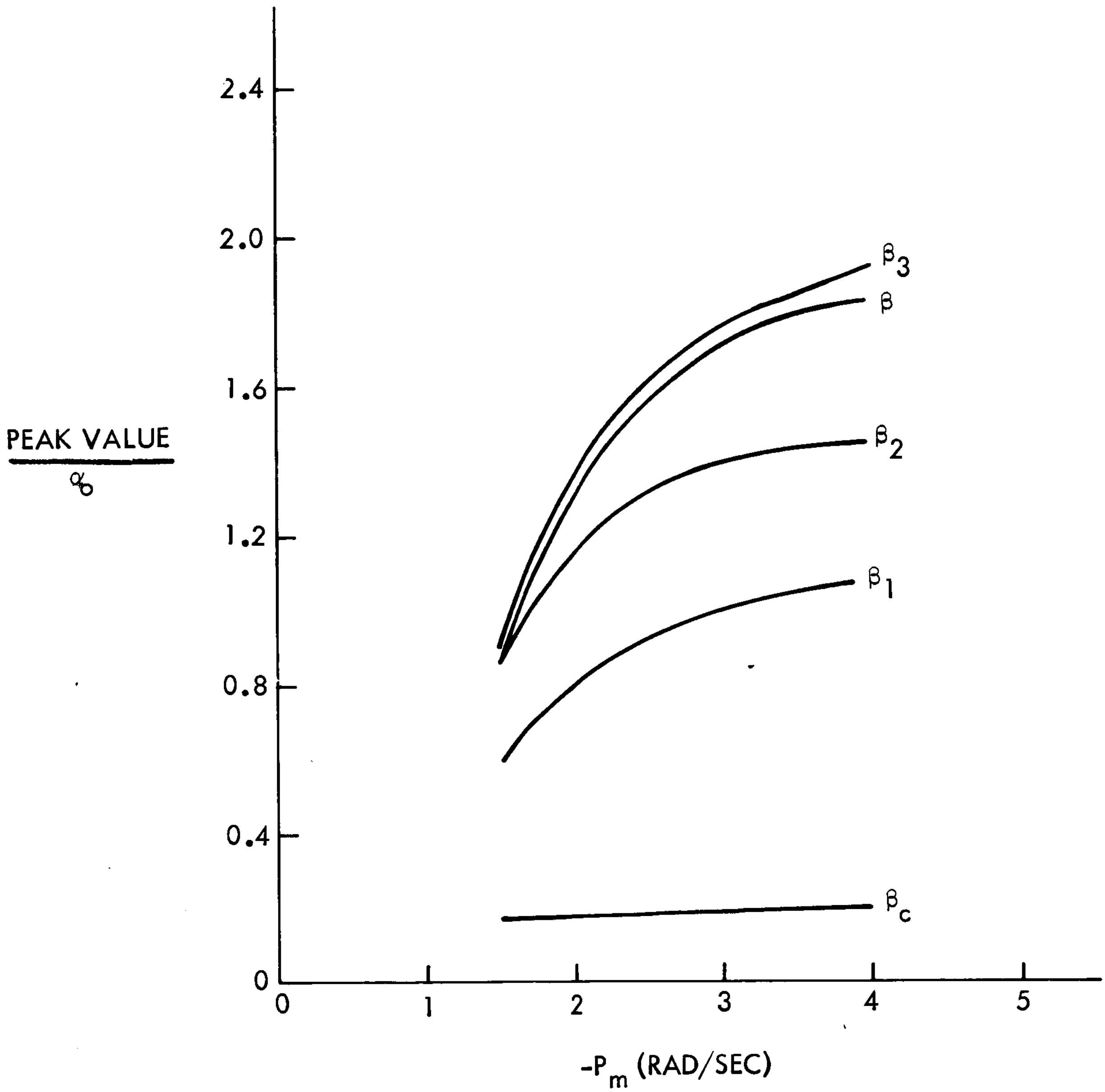


FIGURE 18. COMPARISON OF EXACT AND APPROXIMATE SECOND PEAK VALUES OF β RESPONSE FOR FAMILY OF ROLLING MANEUVERS

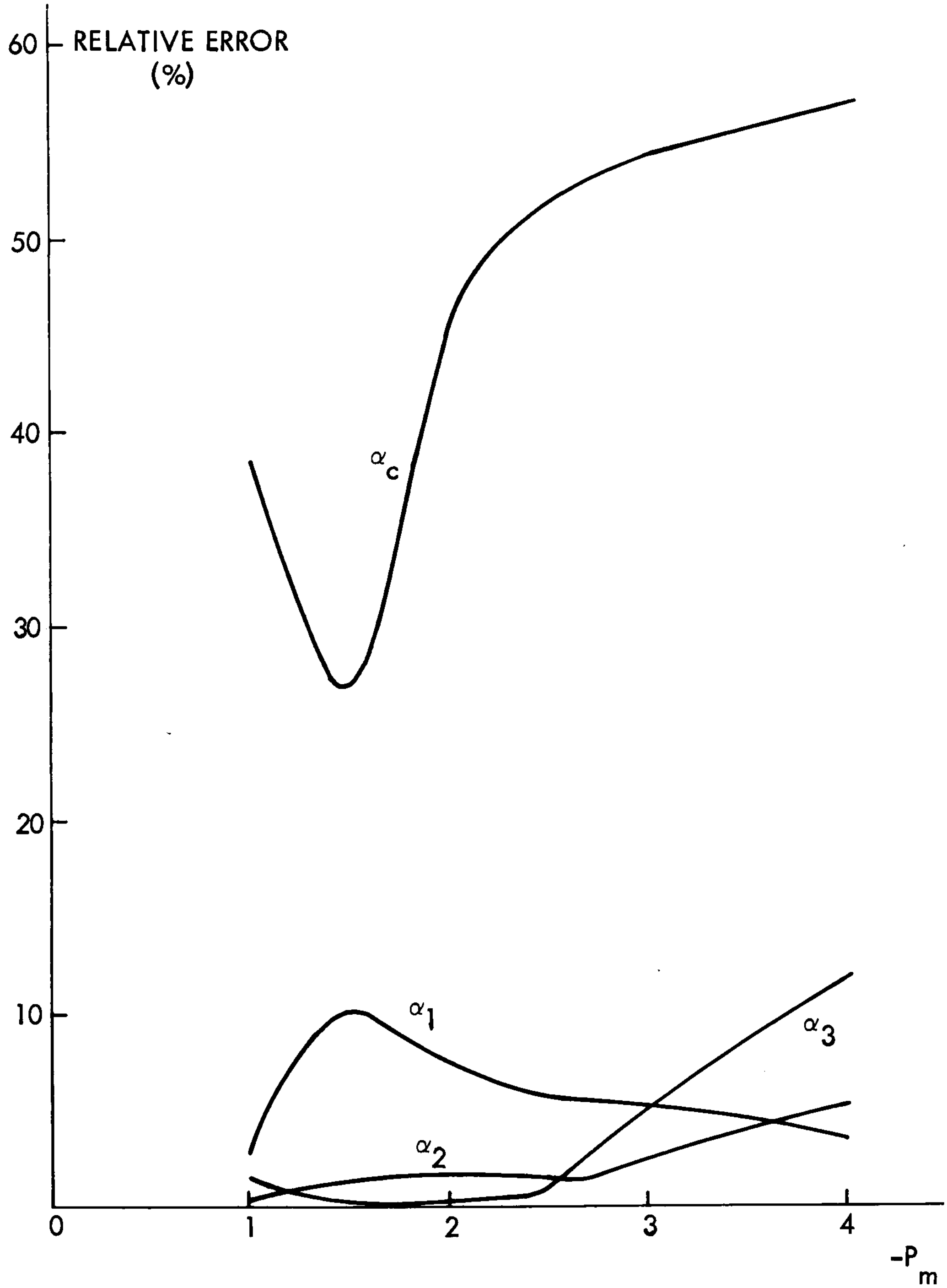


FIGURE 19. COMPARISON OF ERRORS IN PEAK VALUE FOR THE APPROXIMATE RESPONSES IN α

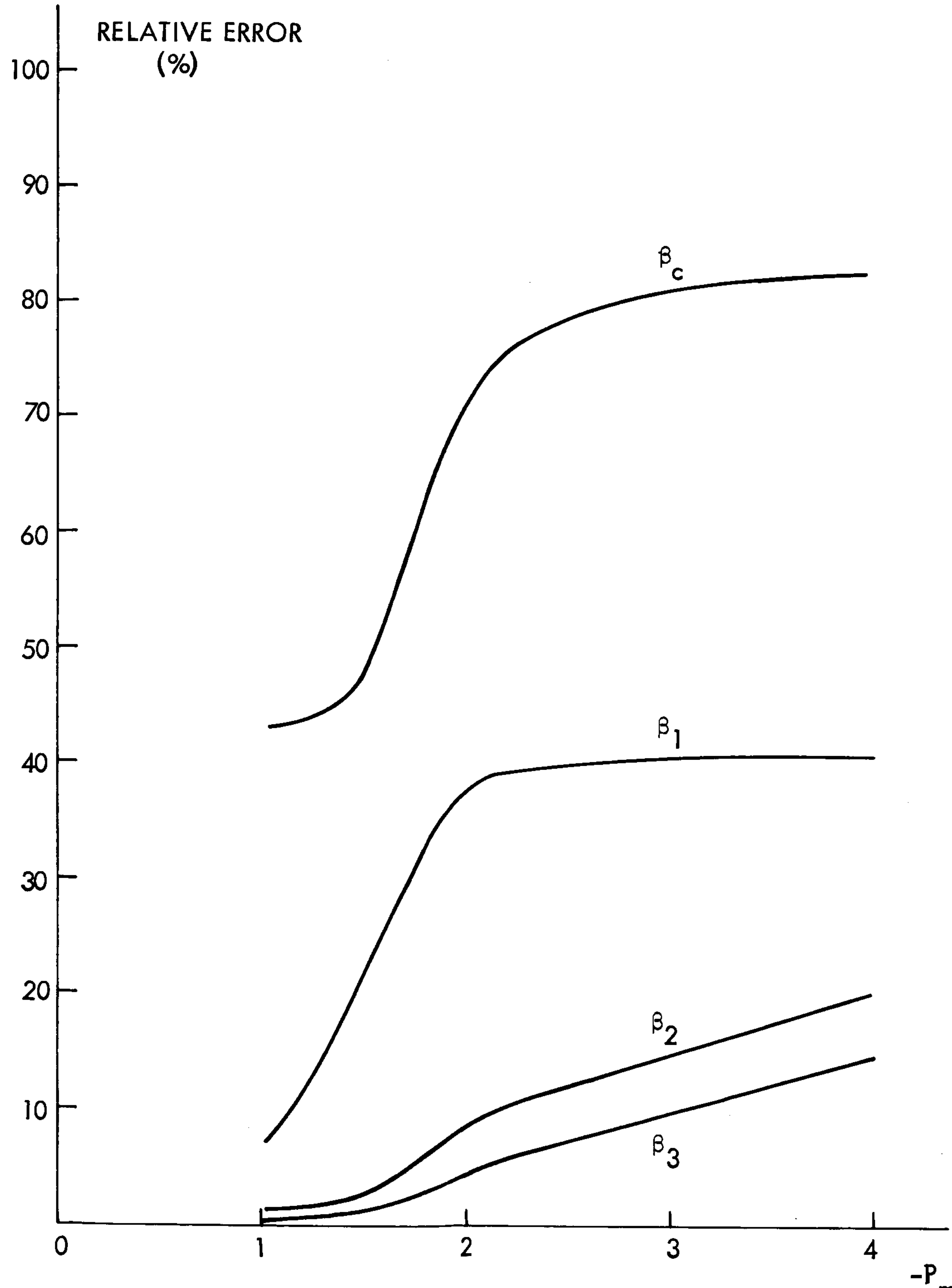


FIGURE 20. COMPARISON OF ERRORS IN PEAK VALUE FOR THE APPROXIMATE RESPONSES IN β

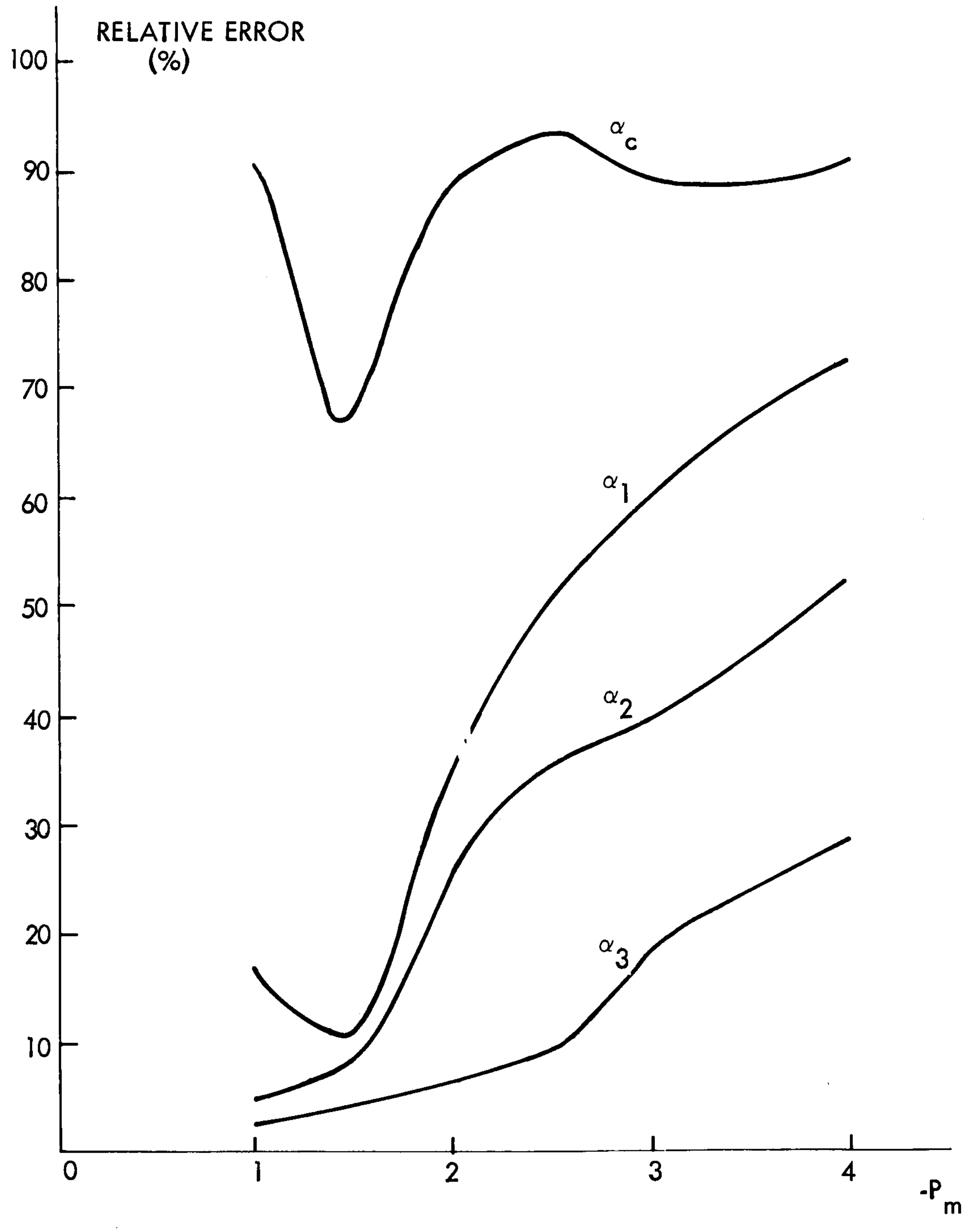


FIGURE 21. COMPARISON OF AVERAGE ERRORS OF APPROXIMATE RESPONSES IN α

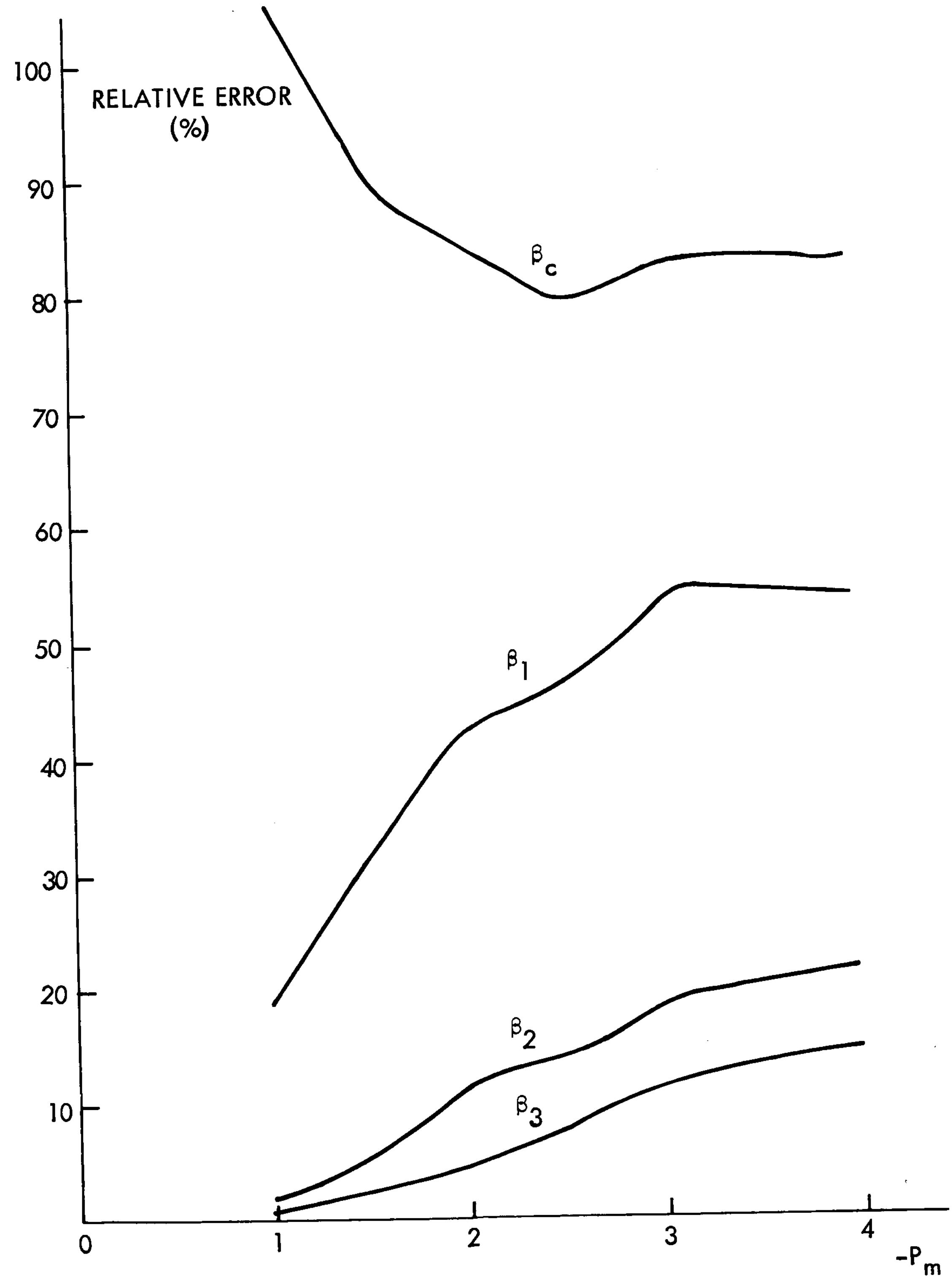


FIGURE 22. COMPARISON OF AVERAGE ERROR OF APPROXIMATE RESPONSES IN β

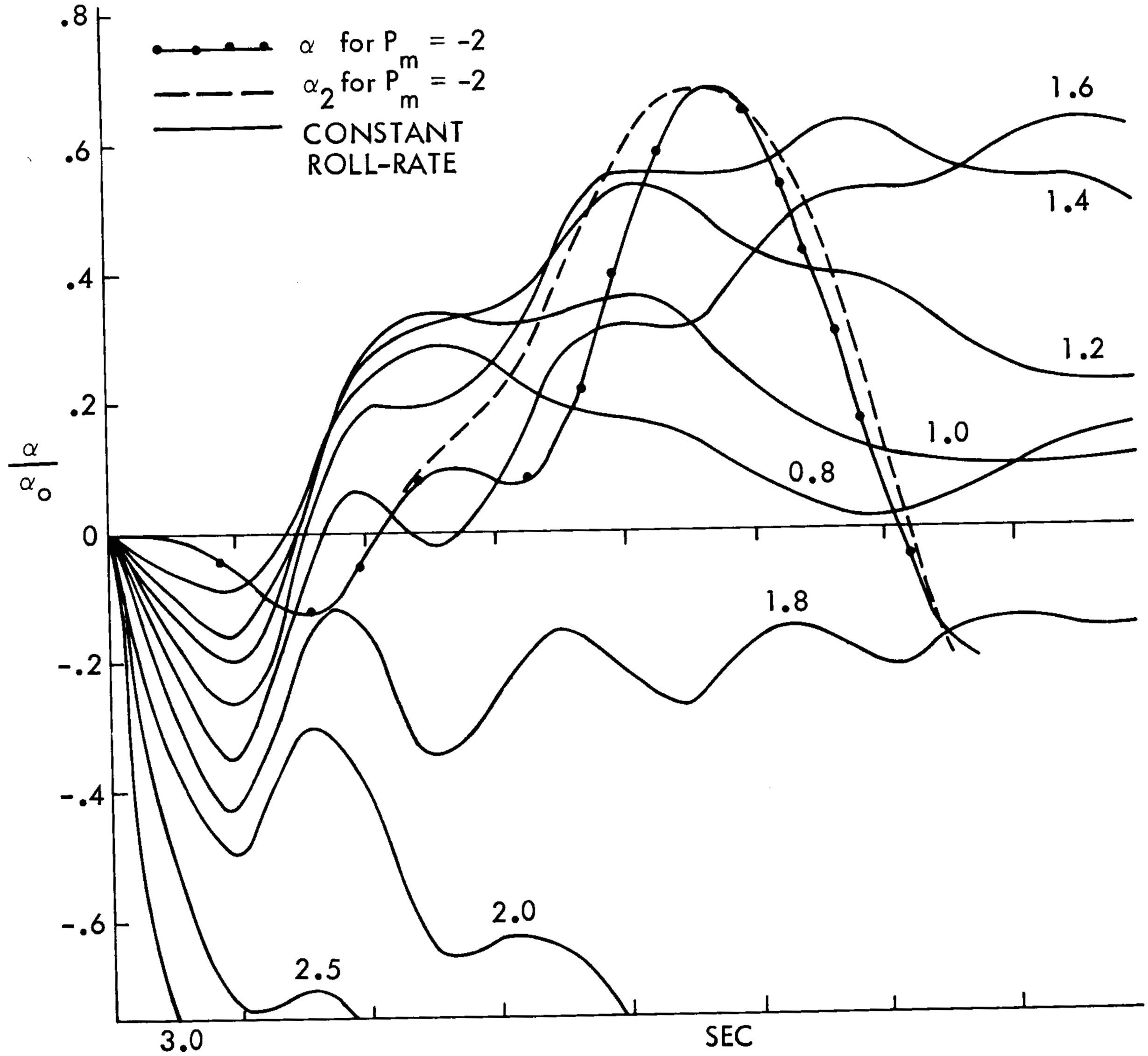


FIGURE 23. COMPARISON OF α AND α_2 RESPONSES WITH RESPONSES RESULTING FROM VARIOUS VALUES OF CONSTANT ROLL-RATE (0.8 - 3.0 RAD/SEC)

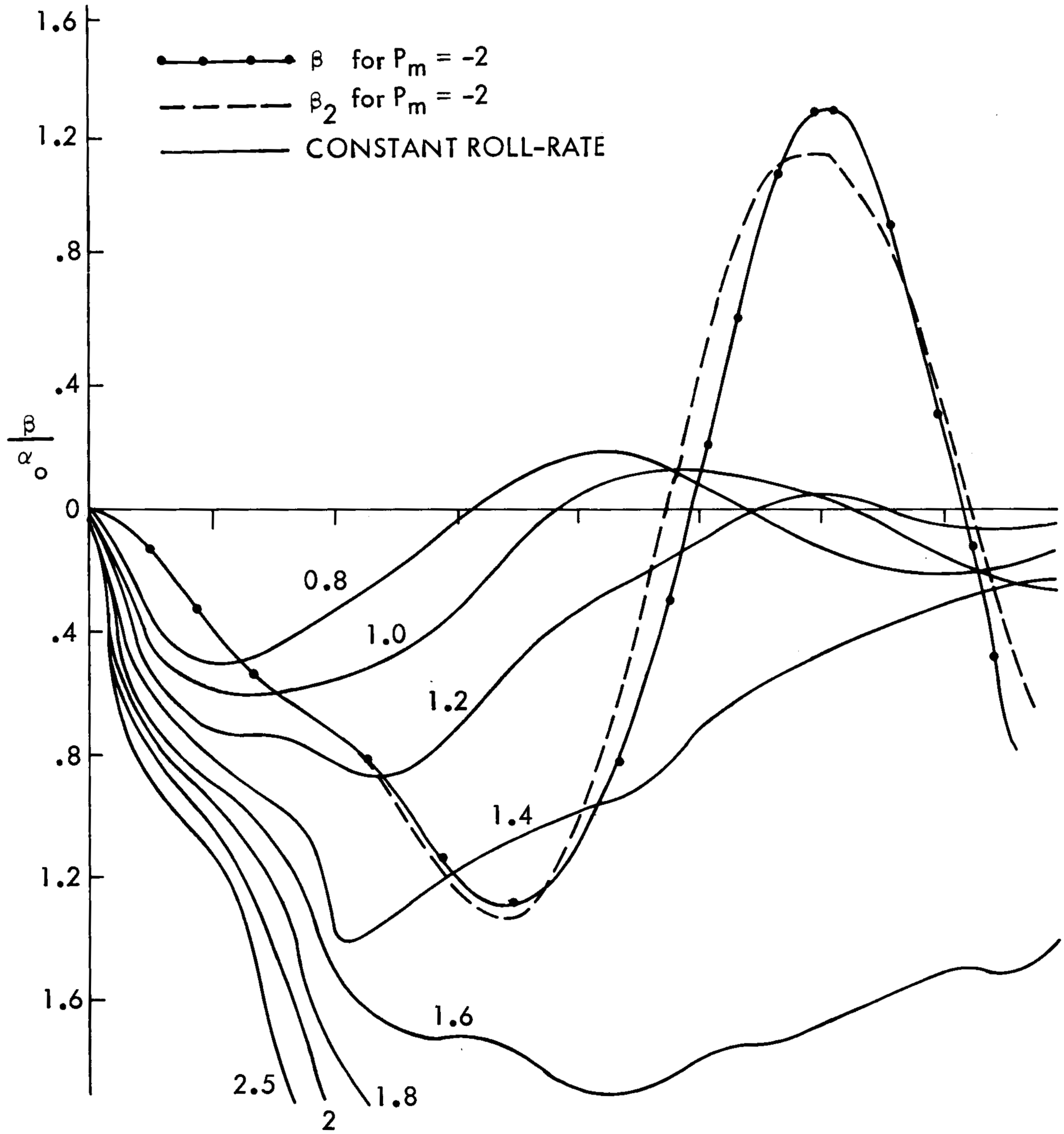


FIGURE 24. COMPARISON OF β AND β_2 RESPONSES WITH RESPONSES FROM VARIOUS VALUES OF CONSTANT ROLL-RATE (0.8 - 2.5 RAD/SEC)

Accuracy of the method for maneuvers involving prolonged durations of unstable roll-rate. - The accuracy of the method was tested for rolling maneuvers that assume unstable values of roll-rates for a relatively long period of time. The responses $x_1(t)$, $x_2(t)$ and $x_3(t)$ were obtained for the maneuver p(t) given by

$$p(t) = -2.5(1 - e^{-1.2t}) \qquad 0 \leq t \leq 2.5 \tag{35}$$

$$p(t) = -2.34\,e^{-1.2t} \qquad t \geq 2.5 \tag{36}$$

Compared to the family of maneuvers studied in a previous section, this maneuver involves a relatively lengthy interval of unstable roll-rates, as shown in figure 25. Figures 26 and 27 give the time histories for β and α respectively. As might be expected, these plots indicate that the approximate responses α_n and β_n exhibit a slower rate of convergence at the values of t for which p(t) is within, or close to, the unstable range of roll-rates. Nontheless, the agreement of the first approximations α_1 and β_1 with exact responses is substantially better than the agreement provided by α_c and β_c which are computed on the basis of the constant roll-rate approximation of previous studies. Table III gives the peak values of the various approximate responses:

TABLE III

$\frac{\alpha}{\alpha_o}$	$\frac{\alpha_c}{\alpha_o}$	$\frac{\alpha_1}{\alpha_o}$	$\frac{\alpha_2}{\alpha_o}$	$\frac{\alpha_3}{\alpha_o}$
1.02	0.38	0.95	1.04	1.12
$\frac{\beta}{\beta_o}$	$\frac{\beta_c}{\beta_o}$	$\frac{\beta_1}{\beta_o}$	$\frac{\beta_2}{\beta_o}$	$\frac{\beta_3}{\beta_o}$
2.46	0.16	1.35	1.92	2.80

Improving accuracy of method by piece-wise consideration of maneuver.- The convergence and accuracy of the responses $x_n(t)$ can be ameliorated by considering a given maneuver p(t) as a sequence of two maneuvers of shorter duration, with the terminal values of the first maneuver acting as the initial values of the second maneuver. This is feasible since we can obtain expressions for the time history of a maneuver starting from any initial values in the variables $\Delta\alpha$, β, q, and r, as given by (33) and (34). This procedure was applied to the maneuver considered in the previous section as given by (35) and (36). The responses $x_n(t)$ were first obtained for the initiation phase of the maneuver given by (35), and the values $x_n(2.5)$ were used as initial values for the termination phase given by (36). Figures (28) and (29) show the time histories thus obtained for β and α, which indicate a substantial enhancement in the accuracy (convergence) of the approximate responses over the corresponding responses given in figures (26) and (27), respectively.

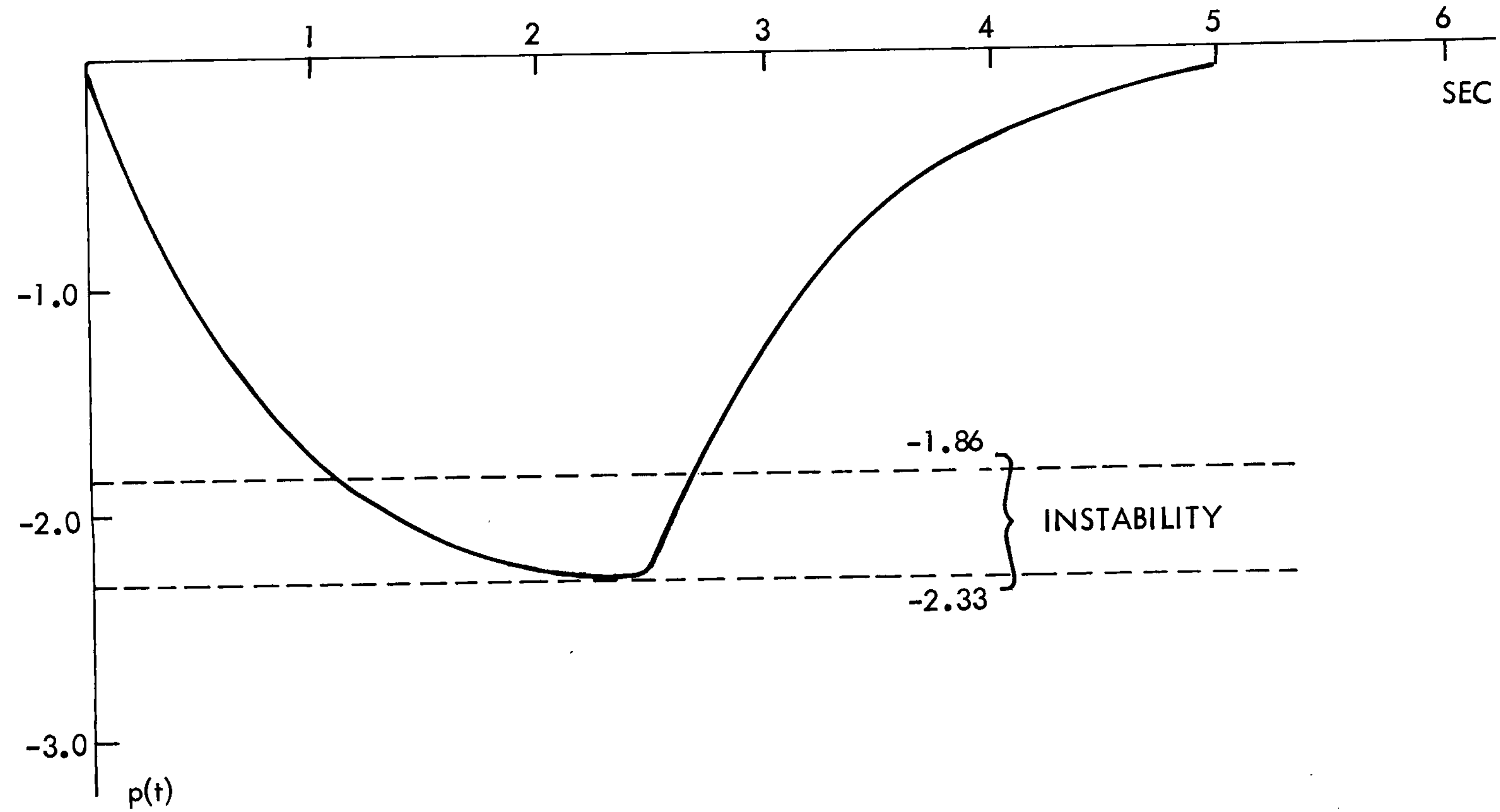


FIGURE 25. ROLLING MANEUVER INVOLVING PROLONGED DURATION OF UNSTABLE ROLL - RATES

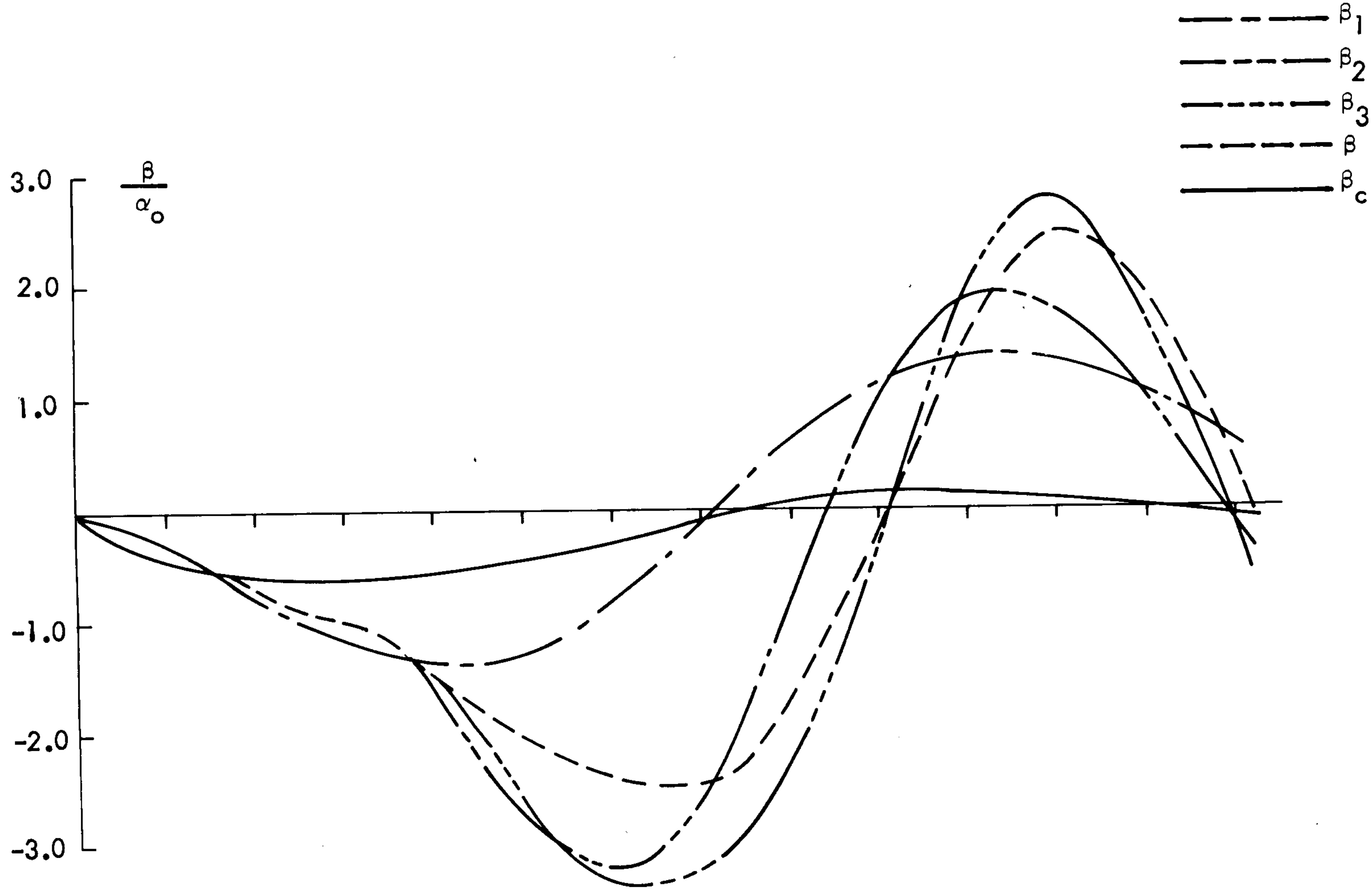


FIGURE 26. β RESPONSES FOR MANEUVER SHOWN IN FIGURE 25

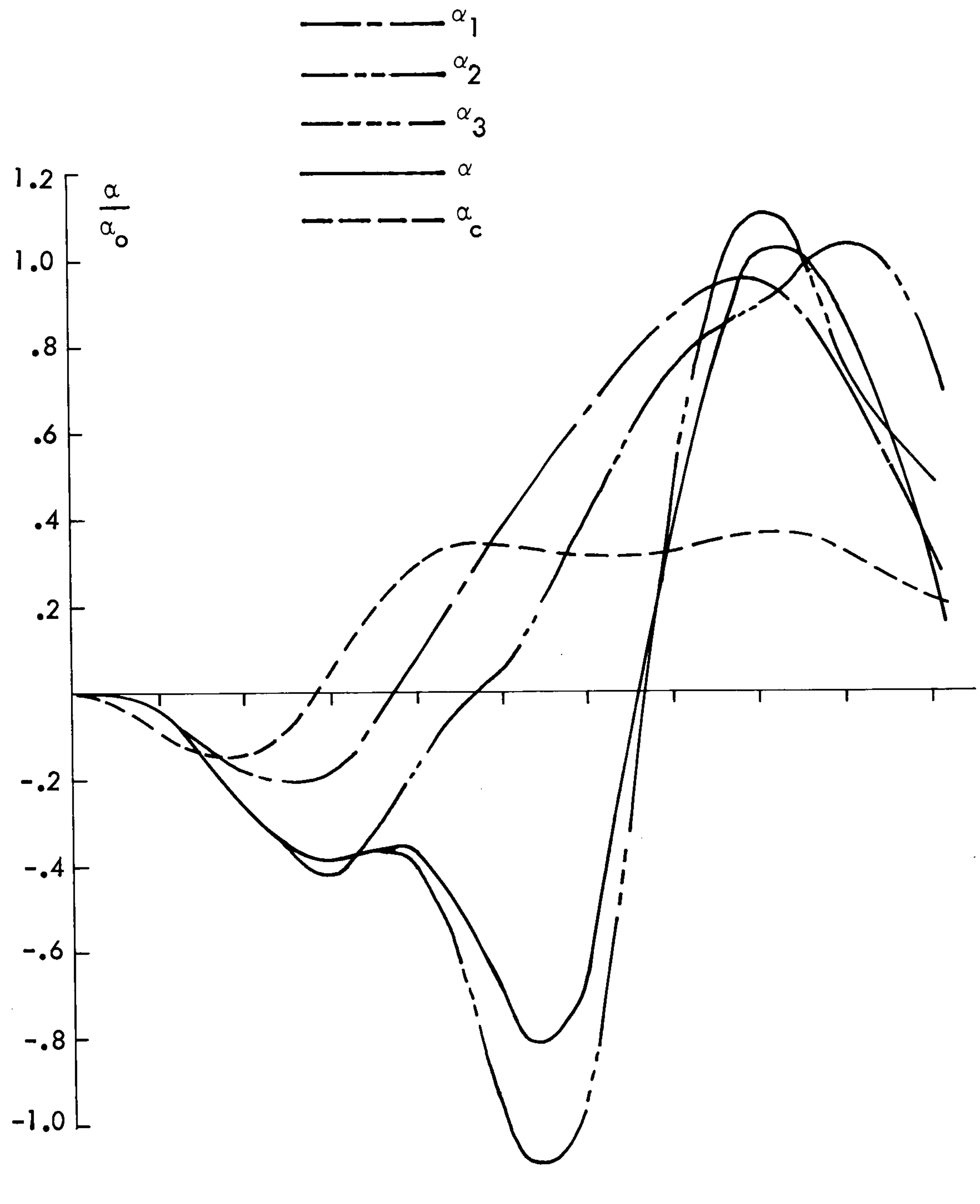


FIGURE 27. α RESPONSES FOR MANEUVER SHOWN IN FIGURE 25

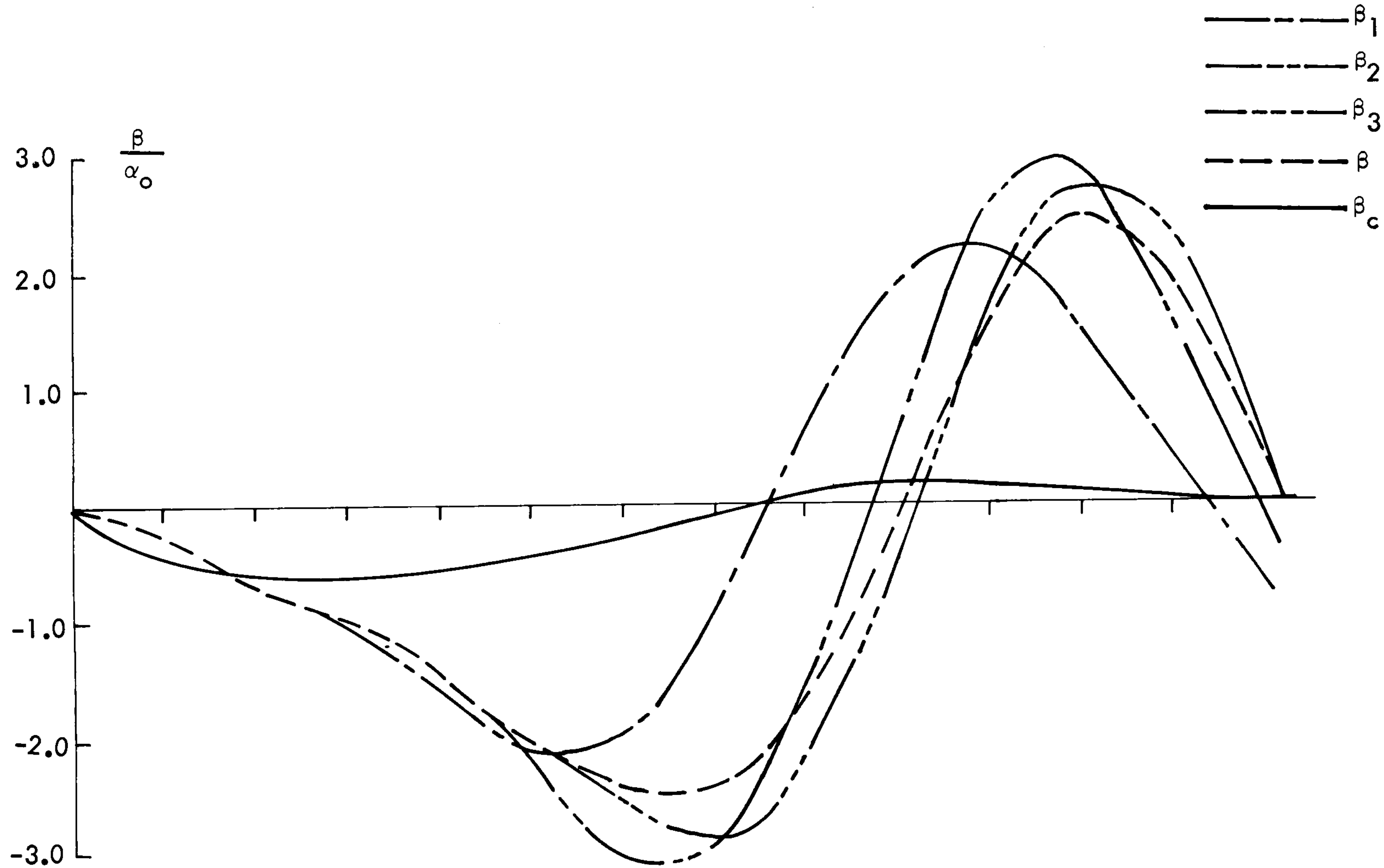


FIGURE 28. IMPROVED ACCURACY OF β RESPONSES RESULTING FROM PIECE-WISE CONSIDERATION OF MANEUVER IN FIGURE 25

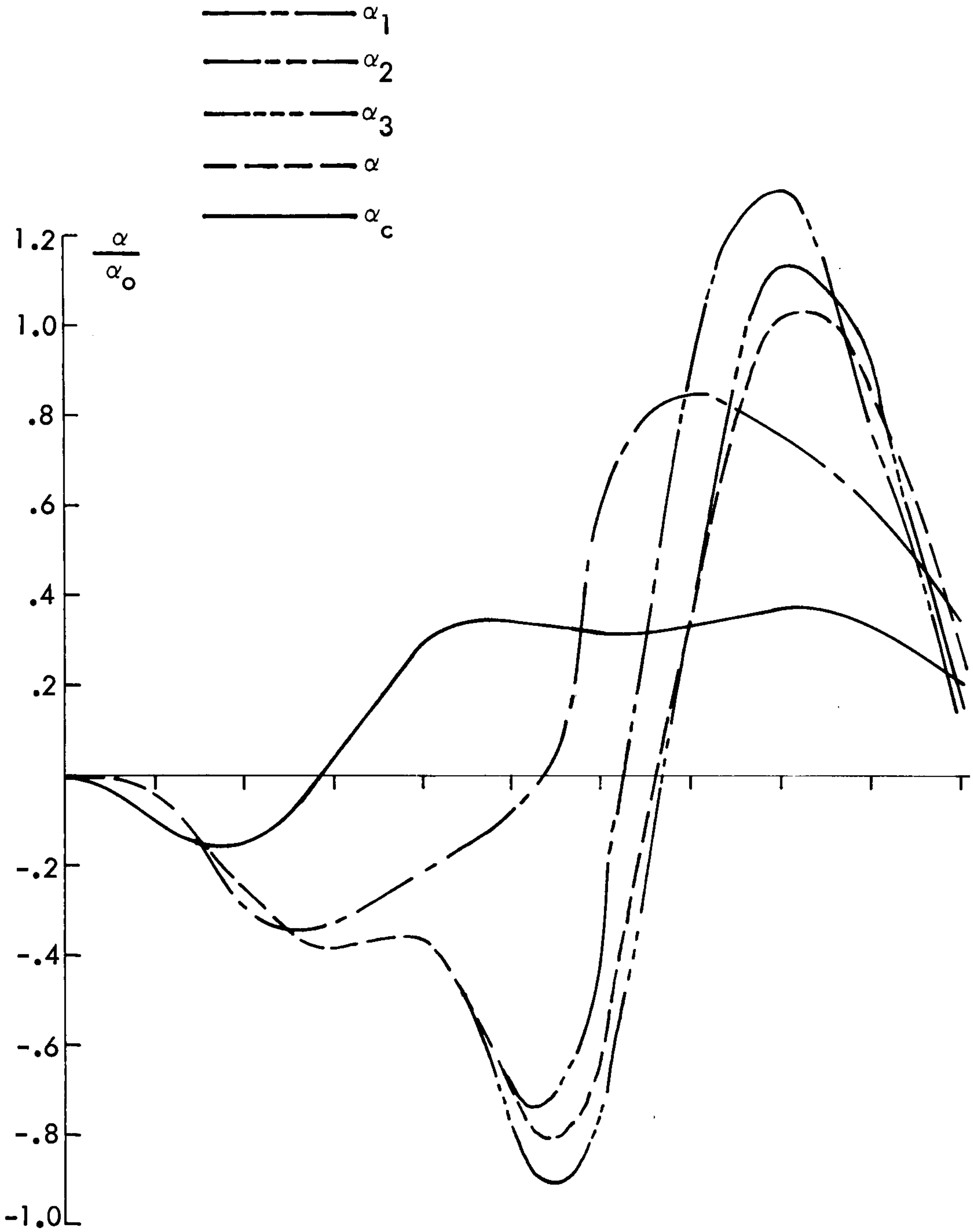


FIGURE 29. IMPROVED ACCURACY OF α RESPONSES RESULTING FROM PIECE-WISE CONSIDERATION OF MANEUVER IN FIGURE 25

Utilization of the Method - Usefulness of the Analytical Expressions

In this section we illustrate the utilization of the method and demonstrate the usefulness of the analytical expression in the predictive study of the airplane dynamic response. Recall that the method provides analytical expressions for the mapping $x(t) = M[p(t)]$, in which the response $x(t)$ is expressed explicitly in terms of the maneuver roll-rate $p(t)$. The analytical expressions for M may be useful in different ways. For any specific maneuver, one may substitute the given expression of $p(t)$ into M to obtain the response $x(t)$, from which the peak excursions of the various variables may be obtained. The analytical expressions for the mapping M may also be used to study the dependence (and sensitivity) of the response $x(t)$ on the details of the maneuver $p(t)$, such as level, rise time, slope, duration, etc. Simple ways for parametrizing the maneuver $p(t)$ may be employed, and the mapping $x = M(p)$ may then be used to obtain expressions for the response $x(t)$ in terms of the selected parameters. Furthermore, in view of the simpler analytical nature of the mapping $x = M(p)$ in comparison to the original equations of motion, one may find it advantageous to use the expressions of the mapping $x = M(p)$ to obtain the response $x(t)$ on the digital computer with considerable savings in programming effort and computation time.

To illustrate these points, a demonstration case-study was performed using the specific parameters of the F-100A. Explicit analytical expressions were obtained for the impulse response matrix $H(t)$, which was then used to generate the approximate analytical expressions for $\alpha(t)$ and $\beta(t)$ in response to the family of parameterized rolling-maneuvers $\{p(t)\}_{P_m}$ described earlier. The analytical expressions were then employed to obtain information about peak excursions. Analytical expressions were obtained relating the magnitude of the response peak excursion to the value of the maneuvers index P_m. All of the predictions based on the analytical expressions were in agreement with the information previously obtained via numerical integration.

The impulse-response matrix $H(t)$.- The matrix $H(t) = e^{A_o t}$ appearing in the analytical expressions as given by (24) and (25) may be determined by any of the standard methods for obtaining the state-transition matrix of a linear system (ref. 9). The matrix $H(t)$ was determined for the parameters of the F-100A as:

$$H(t) = 10^{-2} V(t) \begin{bmatrix} (71,11,29,8) & (3,47,3,-36) & (4,32,-4,11) & (27,2,-27,-2) \\ (-100,84,.5,-66) & (29,10,71,-8) & (24,2,-24,-3) & (68,-27,-68,-13) \\ (-23,-161,23,-81) & (105,8,-105,-8) & (67,-8,33,11) & (.5,-17,-.5,-50) \\ (87,5,-87,-1.6) & (-15,50,15,99) & (-4,44,4,-55) & (25,9,75,-12) \end{bmatrix}$$

where

$$V(t) = (e^{-.355t}\cos 2.84t,\ e^{-.355t}\sin 2.84t,\ e^{-.206t}\cos .96t, e^{-.206t}\sin .96t)$$

and where the multiplication $V(a,b,c,d)$ is an inner product (dot product), e.g.

$$V(71,11,29,8) = e^{-.355t}(71 \cos 2.84t + 11 \sin 2.84t) + e^{-.206t}(29 \cos .96t + 8 \sin .96t)$$

Analytical expressions for the approximate responses.- The matrix H(t) can be used to generate analytical expressions for the approximate time histories in response to a rolling maneuver p(t). Ignoring the effects of the weight terms and the elevator and rudder inputs (basic case), the input vector becomes

$$u(t) = [0, p(t), 0, 0] \tag{37}$$

Substituting for u(z) and H(t - z) in the expressions (23) - (25), the following expression for $\alpha_1(t)$ was obtained:

$$\alpha_1(t) = 10^{-2} \int_0^t p(t - z)\ [(-3, 47, 3, -36)\ \bar{V}(z)]dz$$

Ignoring the smaller terms, one has

$$\alpha_1(t) \simeq 10^{-2} \int_0^t p(t - z)[47e^{-.355z} \sin 2.84z - 36e^{-.206z} \sin .96z]dz \tag{38}$$

Substituting the expression for the parametrized family of maneuvers described previously, namely,

$$\frac{p(t)}{P_m} = P_m(1 - e^{-0.6t}) \quad , \quad 0 \leq t \leq \frac{2\pi}{|P_m|} \text{ (Initiation)}$$

$$= P_m\left(\frac{1.2\pi}{e|P_m|} - 1\right) e^{-0.6t} \quad , \quad t \geq \frac{2\pi}{|P_m|} \text{ (Recovery)}$$

the following approximate expression for the α response was obtained in terms of the maneuver parameter P_m:

Initiation Phase

$$\alpha_1(t) \simeq P_m[.226e^{-.206t} \sin .96t + .2(e^{-.6t} - 1)] \quad , 0 \leq t \leq \frac{2\pi}{P_m}$$

Recovery Phase ($t \geq 2\pi/P_m$)

$$\alpha_1(t) \simeq P_m\left\{ .226e^{-.206t}\left[\sin .96t - e^{1.3/|P_m|} \sin .96\left(t - \frac{2\pi}{|P_m|}\right)\right] + .2(1 - e^{3.8/|P_m|})e^{-.6t}\right\}$$

Prediction of dynamic response information from the analytical expressions.- Considering these expressions for the range of values of P_m of the family of maneuvers studied earlier, namely $-4 \leq P_m \leq -1$, the following information about the response in α was predicted in a straightforward fashion:

1. The motion in α is a damped oscillation with a period of approximately 6 sec.

2. The first relative peak value of α_1 occurs during the initiation phase of the maneuver at $t_1 = 1.4$ sec, and the value of the peak is

$$\alpha_1(t_1) = \alpha_1(1.4) = 0.053P_m$$

3. The second relative peak value occurs at approximately $t_2 = \frac{2\pi}{|P_m|} + \frac{\pi}{(2)(.96)} = \frac{2\pi}{|P_m|} + 1.65$, i.e. 1.65 seconds after the removal of aileron. The magnitude of this peak is

$$\alpha_1(t_2) = P_m \left[.16e^{-1.3/|P_m|} \cos\frac{6}{P_m} + .07e^{-3.8/|P_m|} - .24\right]$$

For the range of P_m considered, $|\alpha_1(t_2)| > |\alpha_1(t_1)|$, therefore, the expression for $\alpha_1(t_2)$ represents the variation of the absolute peak value as a function of P_m. Hence, this analytical expression corresponds to the graph α_1 shown in figure (30) which was obtained previously (see figure 16) from several time-history runs on the digital computer.

It should be noted that the above information predicted by inspection of the analytical expression is in good agreement with the time-history runs in figures (2) thru (15) obtained by numerical integration of the equations of motion on the digital computer.

<u>Relating the maneuver roll-rate time history p(t) to aileron input $\delta_a(t)$.</u>- It has been argued so far that the analysis relating the airplane dynamic response to the kinematic description of the maneuver p(t), rather than to the aileron control input $\delta_a(t)$, is useful in its own right, particularily in a preliminary design effort where the characteristics of the flight control system are not fully specified. In the second portion of this study the airplane dynamic response to an arbitrary aileron input $\delta_a(t)$ was investigated. However, it should be noted at this point that for any rolling maneuver specified spatially by the kinematic descriptor p(t), one may use the roll equation of motion (5) to obtain the corresponding aileron deflection $\delta_a(t)$. In other words, equation (5) provides a means of deducing the aileron input $\delta_a(t)$ that would be required to realize a specified roll-rate time-history p(t). Equation (5) can be written as

$$\delta_a(t) = \frac{1}{C_{\ell\delta_a}}\left[\frac{I_x}{\bar{q}Sb}\dot{p} - \left(\frac{I_y - I_z}{\bar{q}S\,b}\right) qr - C_{\ell\beta}\beta - \frac{b}{2V}\left(C_{\ell r} r + C_{\ell p} p\right)\right]$$

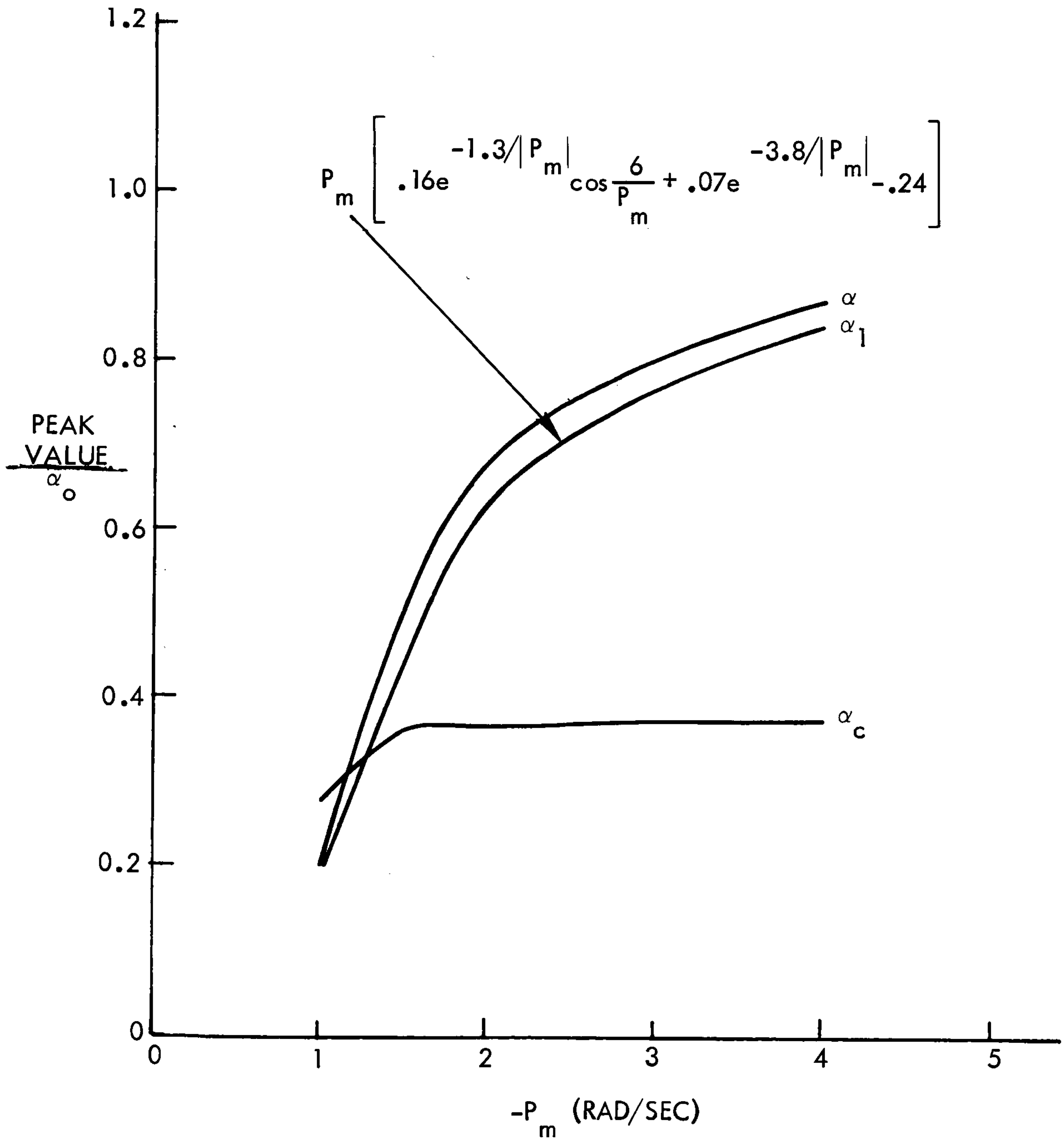


FIGURE 30. ANALYTICAL EVALUATION OF PEAK α AS A FUNCTION OF THE MANEUVER PARAMETER P_m

Substituting the expressions for β, q, and r in terms of $p(t)$, namely,

$$x(t) = [\alpha, \beta, q, r] = M[p] = [M_\alpha(p), M_\beta(p), M_q(p), M_r(p)]$$

one obtains an expression for $\delta_a(t)$ in terms of $p(t)$:

$$\delta_a(t) = \frac{1}{C_{\ell\delta_a}}\left[\frac{I_x}{\bar{q}Sb}\dot{p} - \left(\frac{I_y - I_z}{\bar{q}Sb}\right)M_q(p)\,M_r(p) - C_{\ell\beta}M_\beta(p) - \frac{b}{2V}\left(C_{\ell r}M_r(p) + C_{\ell p}\,p\right)\right] \quad (39)$$

Recall that the above relationships are valid under the assumption $C_{n_{\delta_a}} \simeq C_{Y_{\delta_a}} \simeq 0$.

Another consideration, which has a bearing on the problem of relating $p(t)$ and $\delta_a(t)$, is the question of how to specify the shape of maneuver roll-rate time-history $p(t)$, and the related question of how to parameterized $p(t)$ in meaningful and representative way for any given airplane. The results to be presented the remaining portion of this study on the airplane response to specified aileron input $\delta_a(t)$ are helpful in answering these questions. A simple way to answer these questions for any particular airplane configuration would be to obtain a number of time-histories of the $p(t)$ response to what might be considered as representative aileron inputs $\delta_a(t)$ applied during typical rolling maneuvers of the specific airplane under consideration.

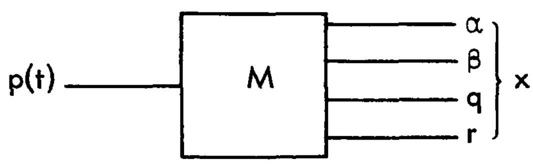
AIRPLANE MOTION IN RESPONSE TO AN ARBITRARY AILERON ACTUATION

In this portion of the study, the dynamic response of the airplane resulting from a general actuation of the aileron input $\delta_a(t)$ is considered. Two types of results are obtained. The first type pertains to the derivation of analytical expressions of the airplane motion in terms of the aileron input $\delta_a(t)$, analogous to the expressions in terms of p(t) obtained in the first portion of the study. The second category of results pertains to the stability of motion under a general aileron input $\delta_a(t)$. Bounds on the motion variables are derived. A stability criterion is also presented.

Analytical Expressions for Airplane Motion in Terms of Aileron Input $\delta_a(t)$

The method presented in the first portion of this study was extended to relate the motion variables during a rolling maneuver to the aileron input $\delta_a(t)$. The extended method generates analytical expressions for α, β, q, r, p in terms of an arbitrary given $\delta_a(t)$, analogous to the expressions of α, β, q, r in terms of p(t) obtained previously.

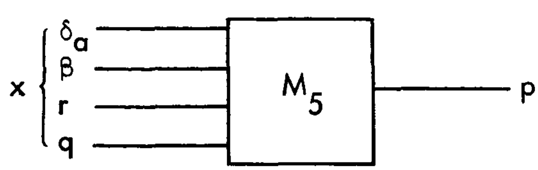
Outline of method.- Let the expressions for the motion variables in terms of p(t) be denoted by the mapping x = M(p):



The fifth equation of motion (roll equation), namely,

$$\dot{p} - Cp = C_\delta \delta_a + B\beta + Dr + Eqr$$

may be regarded as a mapping from (δ_a, x) into p:



$$p = M_5(\delta_a, x)$$

The iterative procedure for obtaining successively more accurate expressions for x and p in terms of δ_a is described as follows: For the given δ_a obtain from M_5 a first estimate on p denoted as p_1. Use p_1 into M to obtain an estimate $x_1 = M(p_1)$, then use x_1 in M_5 to obtain a second estimate p_2, and so on. The accuracy and rate of convergence of this iterative procedure is dependent on the accuracy of the first estimate p_1. Two methods may be used to obtain the first estimate p_1 from the roll equation in a straightforward manner:

1. Consider p_1 as resulting only from the aileron input $\delta_a(t)$:

$$\dot{p}_1 - Cp_1 = C_\delta \, \delta_a(t)$$

2. Ignore the term Eqr in the roll equation and replace β and r by their expressions in terms of p(t) as obtained previously. The resulting integro-differential equation is a mapping from $\delta_a(t)$ into p(t):

$$\dot{p} - Cp - B\beta(p) - Dr(p) = C_\delta \delta_a$$

from which an expression for p_1 in terms of δ_a can be obtained.

Derivation of motion analytical expressions in terms of $\delta_a(t)$. - Recall the expressions for the airplane response $x_n(t) = [\alpha_n(t), \beta_n(t), q_n(t), r_n(t)]$ in terms of the roll-rate variation p(t). Denote the dependence of $x_n(t)$ on p(t) by the functional M_n,

$$x_n = [\alpha_n, \beta_n, q_n, r_n] = [M_{\alpha n}(p), M_{\beta n}(p), M_{qn}(p), M_{rn}(p)] = M_n(p) \tag{40}$$

where $M_{\alpha n}(p)$, $M_{\beta n}(p)$, $M_{qn}(p)$, $M_{rn}(p)$ are the expressions for α_n, β_n, q_r, and r_n in terms of p(t). We shall use these expressions together with the roll equation, namely,

$$\dot{p} - Cp - B\beta - Dr - Eqr = C_\delta \delta_a \tag{41}$$

to derive analytical expressions for α, β, p, q, r in terms of an arbitrary given $\delta_a(t)$. Rewriting (41) as

$$\dot{p} - Cp = C_\delta \delta_a + B\beta + Dr + Eqr = C_\delta \delta_a + f \tag{42}$$

where

$$f = B\beta + Dr + Eqr \tag{43}$$

The function f will be treated as a forcing function in equation (42). For the given δ_a, one can use equation (42) to obtain a first estimate $p_1(t)$ for p(t), as outlined in the previous section. Let α_{n1}, β_{n1}, q_{n1}, r_{n1} be the expressions for α_n, β_n, q_n, r_n corresponding to p_1 as given by (40):

$$[\alpha_{n1}, \beta_{n1}, q_{n1}, r_{n1}] = [M_{\alpha n}(p_1), M_{\beta n}(p_1), M_{qn}(p_1), M_{rn}(p_1)]$$

Let f_{n1} be the first approximation to f obtained from (43) by replacing β, q, and r by β_{n1}, q_{n1}, and r_{n1}, respectively:

$$f_{n1}(p_1) = B\beta_{n1} + Dr_{n1} + Eq_{n1}r_{n1} = BM_{\beta n}(p_1) + DM_{rn}(p_1) + EM_{qn}(p_1)\, M_{rn}(p_1)$$

$$f_{n1}(p_1) = L_n(p_1)$$

where

$$L_n(\cdot) \equiv BM_{\beta n}(\cdot) + DM_{rn}(\cdot) + EM_{qn}(\cdot)\, M_{rn}(\cdot)$$

Using f_{n1} as an approximation to f, one obtains from (42) the second approximation p_2 in a straightforward fashion:

$$\dot{p}_2 - Cp_2 = C_\delta\, \delta_a + f_{n1}(p_1) = C_\delta\, \delta_a + L_n(p_1)$$

$$p_2 = [C_\delta\, \delta_a + L_n(p_1)] * e^{Ct} = \int_0^t [C_\delta\, \delta_a(x) + L_n(p_1(x))]e^{C(t-x)}\, dx$$

Repeating this procedure for p_2 to obtain p_3, and so on, one obtains the following recurrence relationships for p_m:

$$p_m = [C_\delta\, \delta_a + L_n(p_{m-1})] * e^{Ct} \qquad m = 2,3,4,\ldots \tag{44}$$

and the corresponding expressions for α, β, q, and r

$$\alpha_{nm} = M_{\alpha n}(p_m) \quad , \quad \beta_{nm} = M_{\beta n}(p_m) \quad , \quad q_{nm} = M_{qn}(p_m) \quad , \quad M_{rn}(p_m) \tag{45}$$

The relationships in (44) and (45), together with the initial expression for p_1, are the required analytical relationships representing the approximate responses in α, β, p, q, and r for an arbitrary given $\delta_a(t)$. The index m represents the number of iterations and, therefore, the degree of accuracy, in relating p(t) to the given $\delta(t)$, while the index n represents the degree of accuracy in relating α, β, q, and r to p(t), as described in the results of the previous sections which indicate that fairly accurate results should be obtained by taking n = 1 or n = 2.

First approximation of p(t) in terms of $\delta_a(t)$.- The initial approximation p_1 can be related to $\delta_a(t)$ via equation (42) by either of the following methods:

1. Consider p_1 as resulting only from the aileron input $\delta_a(t)$:

$$\dot{p}_1 - Cp_1 = C_\delta\, \delta_a(t)$$

$$p_1(t) = C_\delta\, \delta_a(t) * e^{Ct} = \int_0^t C_\delta\, \delta_a(x)\, e^{C(t-x)}\, dx$$

2. Ignore the term Eqr in equation (42) and replace β and r by their first approximate expressions β_1 and r_1 in terms of p as obtained previously:

$$\dot{p}_1 - Cp_1 - B\beta_1(p_1) - Dr_1(p_1) = C_\delta \delta_a$$

$$\dot{p}_1 - Cp_1 - Bp_1 * h_{22} - Dp_1 * h_{42} = C_\delta \delta_a \tag{46}$$

where h_{22} and h_{42} are entries from the impulse response matrix $H(t) = [h_{ij}]$. The linear equation in (46) can be solved for p_1 in terms of δ_a. Using the Laplace transform method one obtains an impulse response function $h_{p\delta}$ that relates p_1 to δ_a in (46):

$$p_1(t) = C_\delta \delta_a * h_{p\delta} = C_\delta \int_0^t \delta_a(x)\, h_{p\delta}(t-x)dx$$

where $h_{p\delta}$ is the inverse Laplace transform of

$$H_{p\delta}(S) = \frac{C_\delta}{S - C - BH_{22}(S) - DH_{42}(S)}$$

Method 2 provides a better approximation of δ_a. However, method 1 involves a simpler computation.

Stability of Airplane Motion Resulting from an Arbitrary Aileron Input $\delta_a(t)$

In the last part of this study, a method for analyzing the stability of motion in response to a general aileron actuation $\delta_a(t)$ is presented. The predictive method provides bounds on motion variables. The method is also used to derive a stability criterion for boundedness in the form of an algebraic inequality in terms of the aileron input and the airplane parameters.

Bounds on the motion variables.- Consider the roll equation

$$\dot{p} - Cp - B\beta - Dr - Eqr = C_\delta \delta_a \equiv \hat{\delta}_a$$

Replacing β, r, and q by their expressions in terms of p, one has

$$\dot{p} - Cp - Bp * h_{22} - Dp * h_{42} - E(p * h_{32})(p * h_{42}) = \hat{\delta}_a$$

where h_{22}, h_{32}, h_{42} are the appropriate entries from the impulse response matrix $H(t) = [h_{ij}]$. This nonlinear integro-differential equation relating p to δ_a is modeled into a feedback system with a single product nonlinearity as shown in Figure 31. The linear system G with a single input and two outputs is simplified in Figure 32, and characterized by the two impulse-response functions h_r and h_q:

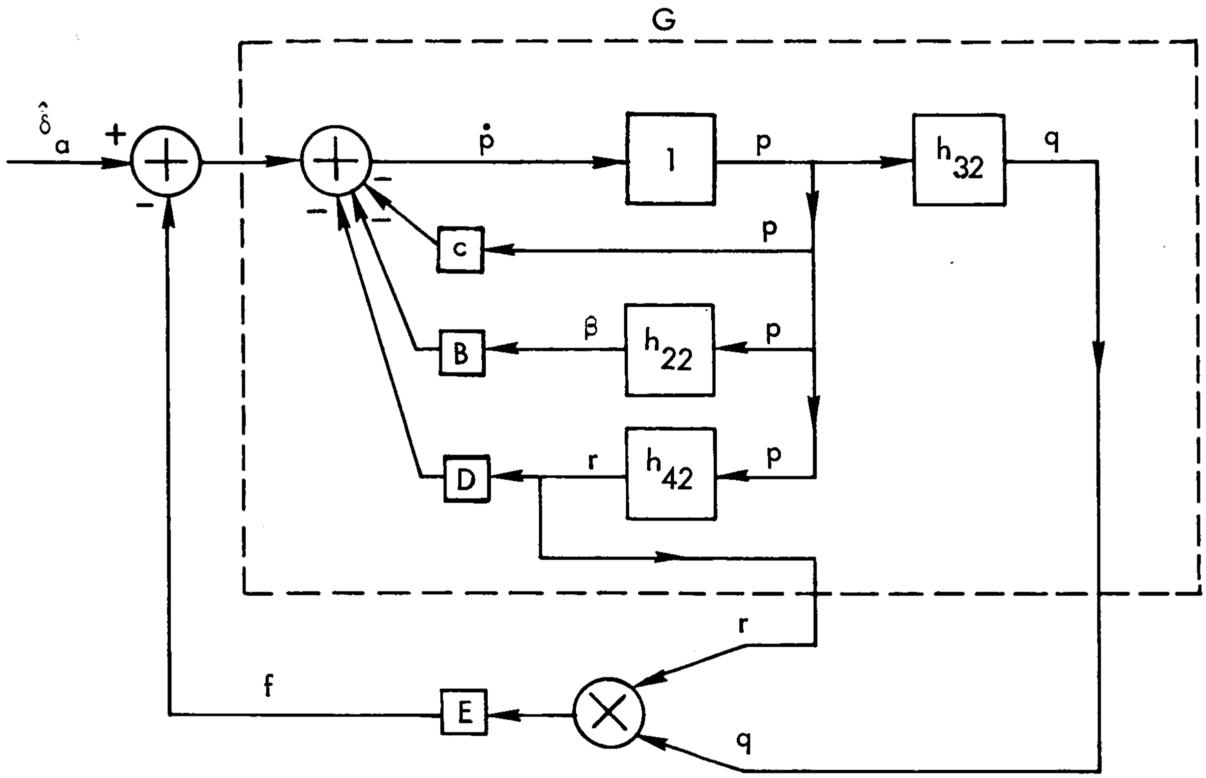


FIGURE 31 SYSTEM MODELLING OF EQUATIONS OF MOTION

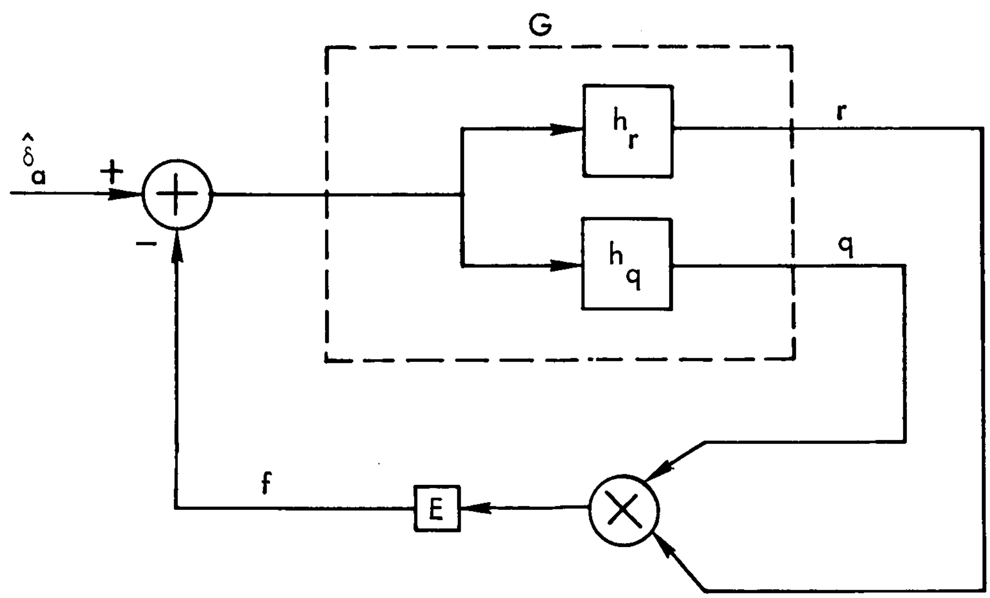


FIGURE 32 SIMPLIFICATION OF SYSTEM IN FIGURE (31)

$$r = (\hat{\delta}_a - f) * h_r = \hat{\delta}_a * h_r - f * h_r \tag{47}$$

$$q = (\hat{\delta}_a - f) * h_q = \hat{\delta}_a * h_q - f * h_q \tag{48}$$

$$f = Eqr$$

The functions $\hat{\delta}_a * h_r$ and $\hat{\delta}_a * h_q$, which represent outputs of the linear system G in response to the aileron input $\hat{\delta}_a(t)$, will be denoted by δ_{ar} and δ_{aq}, respectively:

$$\delta_{ar} \equiv \hat{\delta}_a * h_r \quad , \quad \delta_{aq} \equiv \hat{\delta}_a * h_q \tag{49}$$

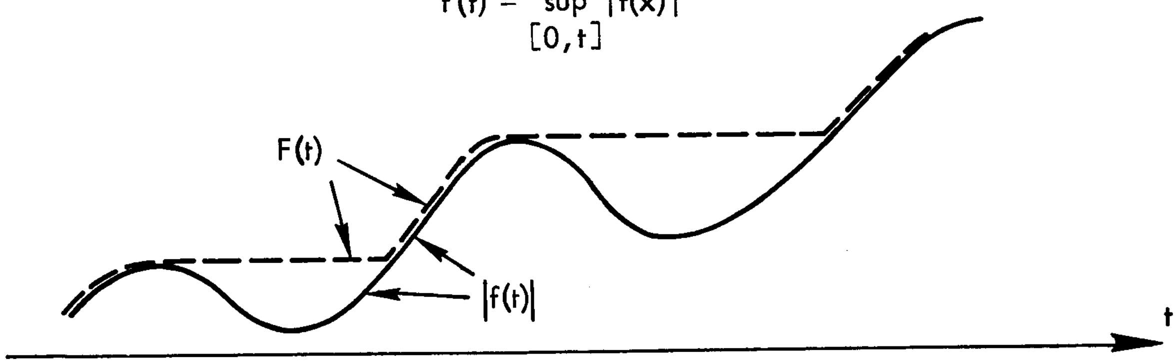
From equation (47) and (48), one has

$$|r| \leq |\delta_{ar}| + |f * h_r| = |\delta_{ar}| + \left| \int_0^t f(x) h_r(t-x)dx \right| \tag{50}$$

$$|q| \leq |\delta_{aq}| + |f * h_q| = |\delta_{aq}| + \left| \int_0^t f(x) h_q(t-x)dx \right| \tag{51}$$

Consider the function $\overline{f}(t)$ defined as follows:

$$\overline{f}(t) = \sup_{[0,t]} |f(x)|$$



Note that $\overline{f}(t)$ is a non-decreasing function, and that its increasing portions are identical to $|f(t)|$. Similarly, define the functions $\overline{r}(t)$, $\overline{q}(t)$, $\overline{\delta}_{ar}(t)$, $\overline{\delta}_{aq}(t)$ as

$$\overline{r}(t) = \sup_{[0,t]} |r(t)|$$

$$\overline{q}(t) = \sup_{[0,t]} |q(t)|$$

$$\bar{\delta}_{ar}(t) = \sup_{[0,t]} |\delta_{ar}(t)|$$

$$\bar{\delta}_{aq}(t) = \sup_{[0,t]} |\delta_{aq}(t)|$$

From (50) and (52) one obtains

$$|r(t)| \le \bar{\delta}_{ar}(t) + \bar{f}(t)\, A_r(t) \tag{52}$$

$$|q(t)| \le \bar{\delta}_{aq}(t) + \bar{f}(t)\, A_q(t) \tag{53}$$

where

$$A_r(t) \equiv \int_0^t |h_r(y)|dy = \int_0^t |h_r(t - x)|dx$$

$$A_q(t) \equiv \int_0^t |h_q(y)|dy = \int_0^t |h_q(t - x)|dx$$

Note that $A_r(t)$ and $A_q(t)$ are non-decreasing functions of t. Since $\bar{\delta}_{ar}(t)$, $\bar{\delta}_{aq}(t)$, $\bar{f}(t)$ are also non-decreasing functions of t, it follows that the right-hand-sides of inequality (52) and of (53) are non-decreasing functions of t. Consequently it follows from (52) and (53) that

$$\sup_{[0,t]} |r(t)| \le \bar{\delta}_{ar}(t) + \bar{f}(t)\, A_r(t)$$

$$\sup_{[0,t]} |q(t)| \le \bar{\delta}_{aq}(t) + \bar{f}(t)\, A_q(t)$$

or

$$\bar{r}(t) \le \bar{\delta}_{ar}(t) + A_r(t)\, \bar{f}(t)$$

$$\bar{q}(t) \le \bar{\delta}_{aq}(t) + A_q(t)\, \bar{f}(t)$$

And since $f(t) = Eq(t)r(t)$, one has

$$\bar{f}(t) \le |E|\, \bar{q}(t)\, \bar{r}(t)$$

hence,

$$\bar{r}(t) \le \bar{\delta}_{ar}(t) + |E|\, \bar{r}(t)\, \bar{q}(t)\, A_r(t)$$

$$\bar{q}(t) \le \bar{\delta}_{aq}(t) + |E|\, \bar{r}(t)\, \bar{q}(t)\, A_q(t)$$

Denoting $|E|\, A_r(t)$ by $a_r(t)$ and $|E|\, A_q(t)$ by $a_q(t)$,

$$\bar{r}(t) \le \bar{\delta}_{ar}(t) + a_r(t)\, \bar{r}(t)\, \bar{q}(t) \tag{54}$$

$$\bar{q}(t) \le \bar{\delta}_{aq}(t) + a_q(t)\, \bar{r}(t)\, \bar{q}(t) \tag{55}$$

$$\bar{r}(t) \ge 0 \quad , \quad \bar{q}(t) \ge 0 \tag{56}$$

From these inequality conditions we shall now deduce conditions that guarantee the boundedness of r(t) and q(t), and derive explicit bounds R(t) and Q(t) on r(t) and q(t), respectively. Combining the inequalities (54), (55), and (56), keeping in mind that all terms are positive, one obtains

$$\bar{r}(t)\bar{q}(t) \le [\bar{\delta}_{ar}(t) + a_r(t)\, \bar{r}(t)\, \bar{q}(t)][\bar{\delta}_{aq}(t) + a_q(t)\, \bar{r}(t)\, \bar{q}(t)]$$

$$\bar{r}(t)\bar{q}(t) \ge 0$$

Letting $x(t) \equiv \bar{r}(t)\bar{q}(t)$ one has

$$a_r(t)\, a_q(t)\, x^2(t) + [a_r(t)\, \bar{\delta}_{aq}(t) + a_q(t)\, \bar{\delta}_{ar}(t) - 1]x(t) + \bar{\delta}_{ar}(t)\, \bar{\delta}_{aq}(t) \ge 0$$

$$x(t) \ge 0$$

Letting

$$a(t) \equiv a_r(t)\, a_q(t) \tag{57}$$

$$b(t) \equiv a_r(t)\, \bar{\delta}_{aq}(t) + a_q(t)\, \bar{\delta}_{ar}(t) - 1 \tag{58}$$

$$c(t) \equiv \bar{\delta}_{ar}(t)\, \bar{\delta}_{aq}(t) \tag{59}$$

the conditions may be expressed as

$$a(t)\, x^2(t) + b(t)\, x(t) + c(t) \ge 0 \tag{60}$$

$$x(t) \ge 0 \tag{61}$$

These two inequalities must be satisfied for all $t \ge 0$. We can now determine conditions on a(t), b(t), and c(t) which would ensure that x(t) satisfying (60) and (61) be a bounded function. We shall assume that $x(0) = \bar{r}(0)\, \bar{q}(0) = 0$, which would be true for a maneuver starting from zero initial conditions. The analysis can be extended to the general case $x(0) \neq 0$. We shall also assume that $x(t)$ is a continuous function. This is justified by the fact that the physical quantities $r(t)$ and $q(t)$ cannot be discontinuous variables.

The inequalities (60) and (61), which must be satisfied for all t, impose a restriction on the values that $x(t)$ may take. For any given t, let X_t be the set of all values of $x(t)$ that satisfy (60) and (61):

$$X_t = \left\{ x(t) \middle| \begin{array}{l} ax^2 + bx + c \geq 0 \\ x \geq 0 \end{array} \right\}$$

Figure 33 shows the set X_t for various conditions on the functions $a(t)$, $b(t)$, and $c(t)$. Recall from (57), (58), and (59) that $a(t) \geq 0$, $c(t) \geq 0$, but $b(t)$ can take negative values. From Figure 33a, it is easily seen that if $b^2 - 4ac \leq 0$, then $X_t = [0,\infty]$. Figure 33b represents the condition $b^2 - 4ac > 0$, $b > 0$, for which also $X_t = [0,\infty]$. In Figure 33c, the condition $b^2 - 4ac > 0$, $b < 0$, forces the set X_t into two disjoint sets on the positive axis

$$X_t = [0, r_1(t)] \cup [r_2(t), \infty] \quad , \quad r_2 > r_1$$

where $r_1(t)$ and $r_2(t)$ are the roots of the equation

$$a(t)x^2 + b(t)x + c(t) = 0$$

Recall that $x(t) \in X_t$ for all t. Recall also that $x(0) = 0$ and $x(t)$ is continuous. This implies that $x(t)$ starts in the set $[0, r_1(t)]$ and remains in $[0, r_1(t)]$ for all t. Thus, under the conditions

$$b^2(t) - 4a(t)\,c(t) > 0 \quad , \quad b(t) < 0 \quad , \quad \text{for all } t \tag{62}$$

the time history $x(t)$ is bounded by the function $r_1(t)$

$$x(t) \leq r_1(t) = \frac{-b(t) - \sqrt{b^2(t) - 4a(t)\,c(t)}}{2a(t)}$$

Recalling that $x(t) = \bar{r}(t)\,\bar{q}(t)$, and substituting the expressions for $a(t)$, $b(t)$, $c(t)$ from (57) through (59) one has

$$\bar{r}\,\bar{q} \leq \frac{1 - a_r\bar{\delta}_{aq} - a_q\bar{\delta}_{ar} - \sqrt{[1 - a_r\bar{\delta}_{aq} - a_q\bar{\delta}_{ar}]^2 - 4a_r\bar{\delta}_{aq}a_q\bar{\delta}_{ar}}}{2a_r a_q}$$

Substituting into (54) and (55), and letting

$$\delta_r(t) \equiv a_q(t)\,\bar{\delta}_{ar}(t) \quad , \quad \delta_q(t) \equiv a_r(t)\,\bar{\delta}_{aq}(t) \tag{63}$$

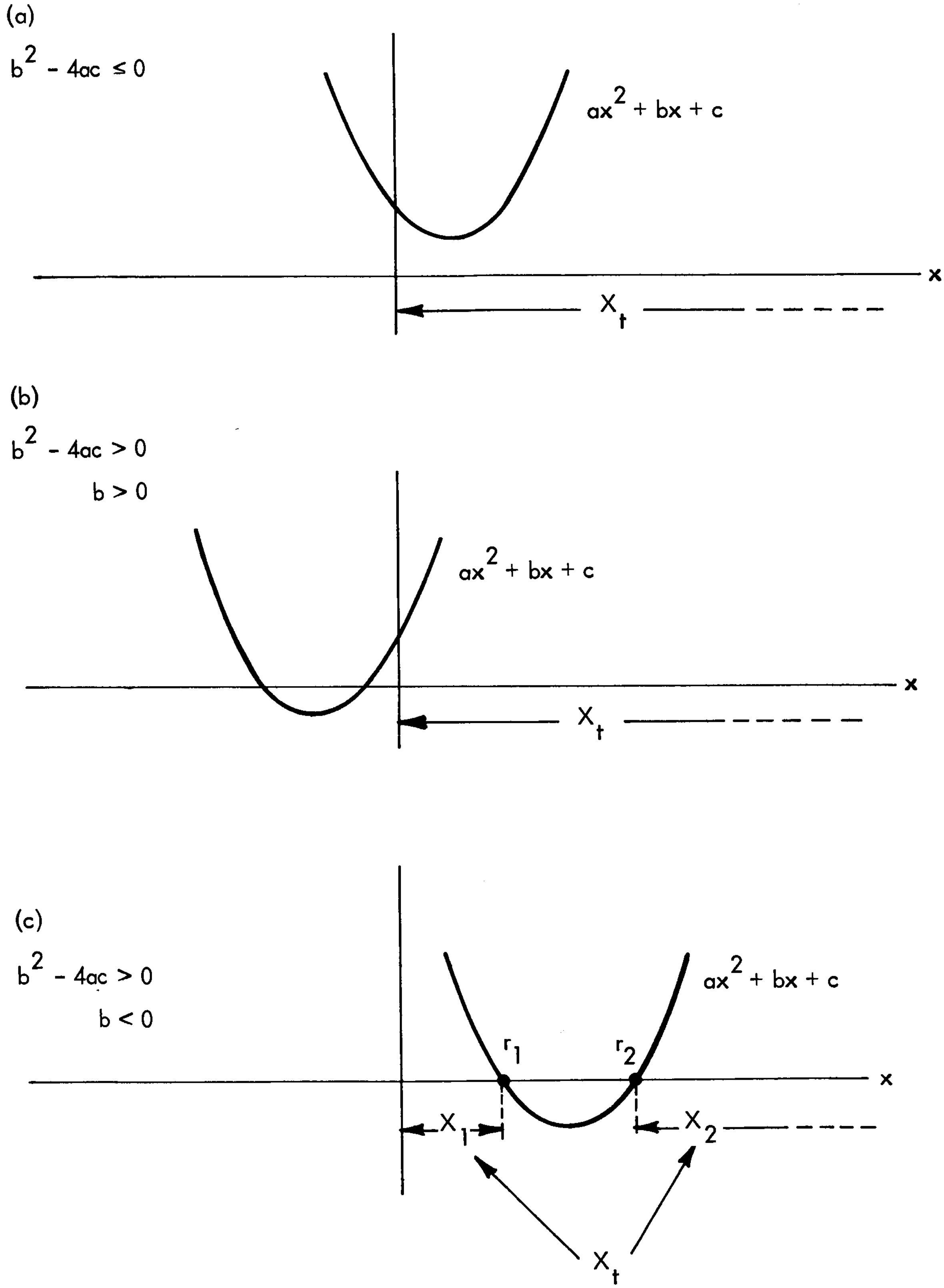


FIGURE 33. GRAPHICAL REPRESENTATION OF BOUNDS

one obtains

$$|r(t)| \le \bar{r}(t) \le R(t) \equiv \frac{1 - \delta_q(t) + \delta_r(t) - \sqrt{[1 - \delta_q(t) - \delta_r(t)]^2 - 4\delta_q(t)\delta_r(t)}}{2a_q(t)}$$

$$|q(t)| \le \bar{q}(t) \le Q(t) \equiv \frac{1 - \delta_r(t) + \delta_q(t) - \sqrt{[1 - \delta_q(t) - \delta_r(t)]^2 - 4\delta_q(t)\delta_r(t)}}{2a_r(t)} \qquad (64)$$

These expressions are the required bounds R(t) and Q(t) on r(t) and q(t), respectively. For any given aileron input $\delta_a(t)$, one determines $\delta_r(t)$ and $\delta_q(t)$ from (63) and (49) and substitutes in the above expressions to determine the time-history bounds R(t) and Q(t). Recall that the expressions for the bounds in (64) are valid only if the condition in (62) is satisfied for all t:

$$[1 - \delta_q(t) - \delta_r(t)]^2 - 4\delta_q(t)\,\delta_r(t) > 0$$

$$\delta_q(t) + \delta_r(t) - 1 < 0 \qquad (65)$$

These conditions will be used in a following section to derive a stability criterion which defines the range of aileron inputs $\delta_a(t)$ for which boundedness (non-divergence) of the airplane motion can be guaranteed.

Bounds on $\alpha(t)$, $\beta(t)$, and p(t).- Having determined the bounds R(t) and Q(t) on r(t) and q(t), corresponding bounds on p(t), $\alpha(t)$, and $\beta(t)$ can be determined by relating these time histories to r(t) and q(t). From figure 31 it is easily seen that p(t) is the output of a linear system whose input is $\delta_a(t)$ - Eq(t) r(t). This is shown explicitly in figure 34, where the appropriate linear system is denoted by L_p, and its impulse response by $h_p(t)$. The time history bound P(t) on p(t) may now be expressed in terms of $\delta_a(t)$, Q(t), R(t), and $h_p(t)$ as follows:

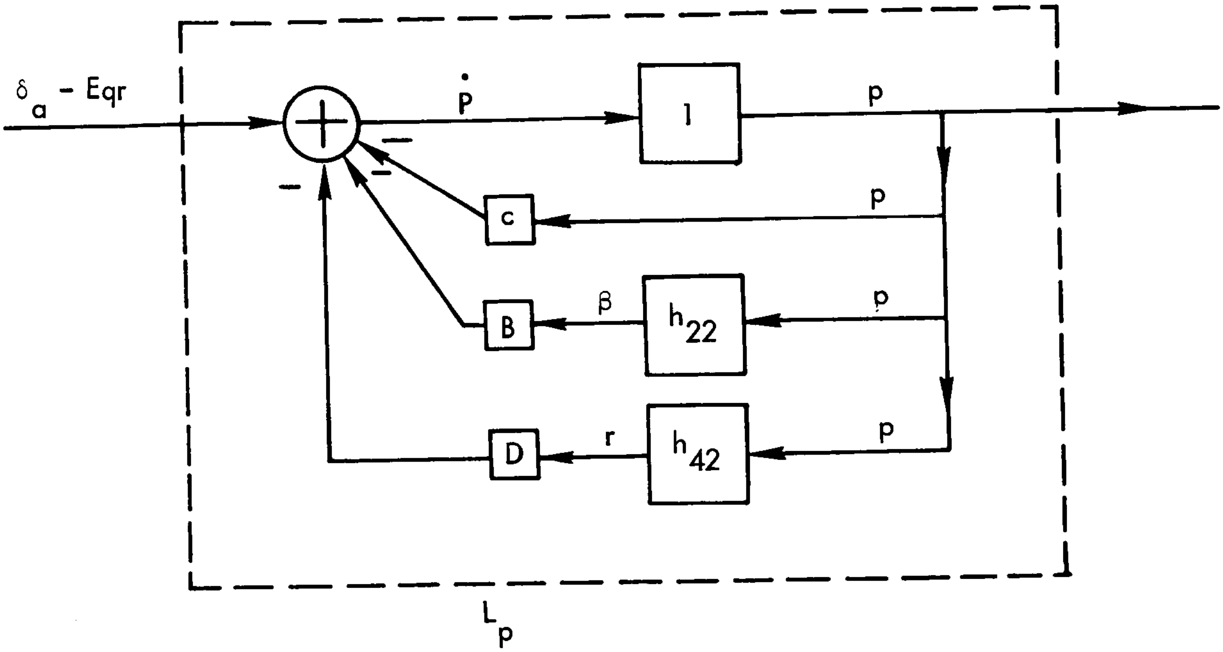
$$p(t) = (\delta_a - Eqr) * h_p \qquad (66)$$

$$|p(t)| \le |\delta_a(t) - Eqr| * |h_p|$$

$$|p(t)| \le [|\delta_a(t)| + |E|\ Q(t)\ R(t)] * |h_p(t)| \equiv P(t)$$

The function P(t) is a time-history bound on |p(t)|. Alternatively, one may determine upper and lower bounds on p(t) which we shall denote as $P_1(t)$ and $P_2(t)$, respectively.

$$P_2(t) \le p(t) \le P_1(t)$$



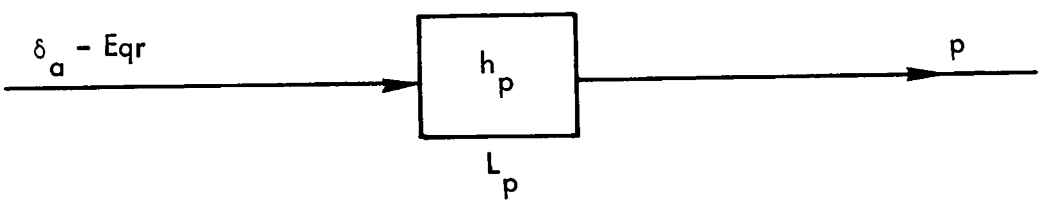


FIGURE 34 LINEAR SYSTEM L_p RELATING p(t) TO δ_a, q, AND r

From (66), one has

$$p(t) = \delta_a * h_p - (Eqr) * h_p$$

$$|p - \delta_a * h_p| = |E|\,|(qr) * h_p| \le |E|\,(|q|\,|r|) * |h_p|$$

$$|p(t) - \delta_a(t) * h_p(t)| \le |E|[Q(t)R(t)] * |h_p(t)|$$

$$\delta_a(t) * h_p(t) - |E|[Q(t)R(t)] * |h_p(t)| \le p(t) \le \delta_a(t) * h_p(t) + |E|[Q(t)R(t)] * |h_p(t)|$$

Thus, the upper and lower bounds on p(t) are:

$$P_1(t) = \delta_a(t) * h_p(t) + |E|\,[Q(t)\,R(t)] * |h_p(t)|$$

$$P_2(t) = \delta_a(t) * h_p(t) - |E|\,[Q(t)\,R(t)] * |h_p(t)|$$

Having determined bounds on p(t), the corresponding bounds on $\alpha(t)$ and $\beta(t)$ can now be obtained from the expressions of $\alpha(t)$ and $\beta(t)$ in terms of p(t):

$$\alpha(t) = p(t) * h_{12}(t) \quad , \quad \beta(t) = p(t) * h_{22}(t) \tag{67}$$

$$|\alpha(t)| \le P(t) * |h_{12}(t)| \equiv \Omega(t)$$

$$|\beta(t)| \le P(t) * |h_{22}(t)| \equiv \Phi(t)$$

Alternatively, one may use the upper and lower bounds on p(t), namely $P_1(t)$ and $P_2(t)$, to determine upper and lower bounds $\Omega_1(t)$ and $\Omega_2(t)$ on $\alpha(t)$, $\Phi_1(t)$ and $\Phi_2(t)$ on $\beta(t)$. For this purpose, we separate $h_{12}(t)$ into two functions $\bar{h}_{12}(t)$ and $\underline{h}_{12}(t)$, representing its positive and negative values, respectively, i.e.

$$\bar{h}_{12}(t) \equiv h_{12}(t) \qquad \text{whenever } h_{12}(t) \ge 0$$

$$\underline{h}_{12}(t) \equiv h_{12}(t) \qquad \text{whenever } h_{12}(t) \le 0$$

Note that $h_{12}(t) = \bar{h}_{12}(t) + \underline{h}_{12}(t)$ for all t. From (67), one has

$$\alpha(t) = p(t) * [\bar{h}_{12}(t) + \underline{h}_{12}(t)] = p(t) * \bar{h}_{12}(t) + p(t) * \underline{h}_{12}(t)$$

and since $P_2(t) \le p(t) \le P_1(t)$, one obtains

$$P_2(t)\,\bar{h}_{12}(t) + P_1(t)\,\underline{h}_{12}(t) \le \alpha(t) \le P_1(t) * \bar{h}_{12}(t) + P_2(t) * \underline{h}_{12}(t)$$

Thus

$$\Omega_1(t) = P_1(t) * \overline{h}_{12}(t) + P_2(t) * \underline{h}_{12}(t)$$

$$\Omega_2(t) = P_2(t) * \overline{h}_{12}(t) + P_1(t) * \underline{h}_{12}(t)$$

Similarly for $\beta(t)$:

$$\Phi_2(t) \le \beta(t) \le \Phi_1(t)$$

where

$$\Phi_1(t) \equiv P_1(t) * \overline{h}_{22}(t) + P_2(t) * \underline{h}_{22}(t)$$

$$\Phi_2(t) \equiv P_2(t) * \overline{h}_{22}(t) + P_1(t) * \underline{h}_{22}(t)$$

Stability criterion.- In the two previous sections we derived expressions for the bounds on the time-histories of $r(t)$, $q(t)$, $p(t)$, $\alpha(t)$, and $\beta(t)$. Recall that the existence of these bounds and the boundedness of the airplane motion was guaranteed if condition (65) is satisfied for all t:

$$\begin{gathered}[1 - \delta_q(t) - \delta_r(t)]^2 - 4\delta_q(t)\,\delta_r(t) > 0 \\ 1 - \delta_q(t) - \delta_r(t) > 0 \qquad \text{for all } t \end{gathered} \tag{68}$$

It can be easily verified from the definitions that $\delta_q(t)$ and $\delta_r(t)$ are non-decreasing functions of time. This implies that the left-hand sides of the inequalities in (68) are non-increasing functions of t. Define

$$d_q = \lim_{t\to\infty} \delta_q(t) \quad , \quad d_r = \lim_{t\to\infty} \delta_r(t) \tag{69}$$

It follows that the conditions in (68) are satisfied iff

$$\begin{gathered}[1 - d_q - d_r]^2 - 4d_q d_r > 0 \\ 1 - d_q - d_r > 0 \end{gathered} \tag{70}$$

or

$$\begin{gathered}(d_q - 1)^2 + (d_r - 1)^2 - 2d_q d_r > 1 \\ d_q + d_r < 1 \end{gathered} \tag{71}$$

Thus, if for a given aileron input $\delta_a(t)$, the parameters d_q and d_r satisfy the conditions in (71), the response of the airplane is nondivergent. The set of values (d_q, d_r) which satisfy (71) is obtained graphically in figure 35 by plotting the boundary curves whose equations are

$$(d_q - 1)^2 + (d_r - 1)^2 - 2d_q d_r = 1$$

$$d_q + d_r = 1 \tag{72}$$

The first boundary curve is in fact a parabola inclined at 45° and tangent to the axis at (1,0) and (0,1). The shaded region in figure 35 represents all the points (d_q, d_r) which satisfy (71) and for which boundedness is guaranteed. By combining the various definitions that led to the definitions of d_q and d_r, one can express d_q and d_r in terms of the aileron time-history input $\delta_a(t)$ and the system functions $h_q(t)$ and $h_r(t)$ of figure 32:

$$d_r = |E| \left[\int_0^{\infty} |h_q(t)| dt \right] \sup_x \left| \int_0^x \delta_a(t)\, h_r(x - t) dt \right| \tag{73}$$

$$d_q = |E| \left[\int_0^{\infty} |h_r(t)| dt \right] \sup_x \left| \int_0^x \delta_a(t)\, h_q(x - t) dt \right| \tag{74}$$

Stability of motion resulting from a step aileron.- This criterion for boundedness, presented in the previous section takes a simple form for the special case where $\delta_a(t)$ is a step input. Let

$$\delta_a(t) = \text{constant} = \delta_a$$

Then from (73) and (74) we obtain

$$d_r = |E|\, A_q\, m_r\, |\delta_a| \equiv \gamma_q\, |\delta_a| \tag{75}$$

$$d_q = |E|\, A_r\, m_q\, |\delta_a| \equiv \gamma_r\, |\delta_a| \tag{76}$$

where

$$A_q \equiv \int_0^{\infty} |h_q(t)|\, dt \quad , \quad A_r \equiv \int_0^{\infty} |h_r(t)|\, dt$$

$$m_q \equiv \sup_x \left| \int_0^x h_q(t) dt \right| \quad , \quad m_r \equiv \sup_x \left| \int_0^x h_r(t) dt \right|$$

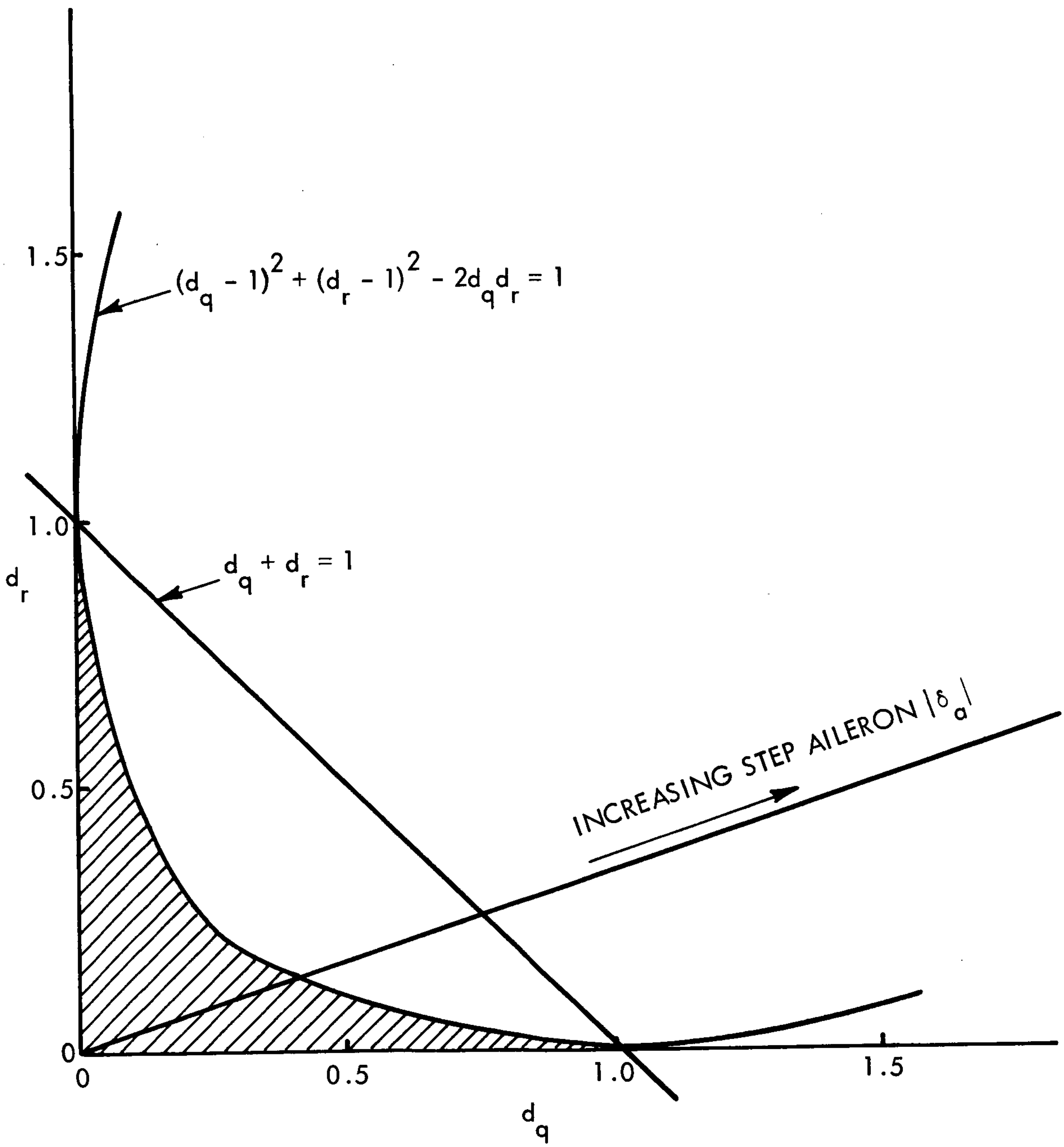


FIGURE 35. STABILITY CRITERION

In the (d_q, d_r) plane of figure 35, equations (75) and (76) are the parametric equations of a straight line through the origin of slope γ_r/γ_q. The intersection of this line with the boundary of the shaded region gives the largest value of $|\delta_a|$ for which stability is guaranteed. This value is obtained from (72), (75), and (76) as:

$$|\delta_a|_{max} = \left(\frac{\sqrt{\gamma_q} - \sqrt{\gamma_r}}{\gamma_q - \gamma_r}\right)^2 \quad , \quad \delta_q \neq \delta_r \tag{77}$$

If $\gamma_q = \gamma_r = \gamma$, then $|\delta_a|_{max} = 1/4\gamma$, which is also the limit of the expression in (77) as γ_q tends to γ_r.

REFERENCES

1. W. H. Phillips, "Effect of Steady Rolling on Longitudinal and Directional Stability," NASA TN 1627, 1948.

2. L. Sternfield, "A Simplified Method for Approximating the Transient Motion in Angles of Attack and Sideslip During a Constant Rolling Maneuver," NACA Report 1344, 1958.

3. M. T. Moul and T. R. Brennan, "Approximate Method for Calculating Motions in Angles of Attack and Sideslip Due to Step Pitching and Yawing Moment Inputs During Steady Roll," NACA TN 4346, Sept. 1958.

4. W. J. G. Pinsker, "Preliminary Note on the Effect of Inertia Cross Coupling on Aircraft Response in Rolling Manoeuvres," R.A.E. Tech. Note Aero 2419, November 1955.

5. W. J. Pinsker, "Charts of Peak Amplitudes in Incidence and Sideslip in Rolling Maneuvers Due to Inertia Cross Coupling," RAE, AERO 2604, April 1958.

6. E. K. Haddad, "On the Lagrange Stability of Nonlinear Systems," Proceedings of the Southeastern Symposium on System Theory, Georgia Institute of Technology, April 1971.

7. E. K. Haddad, "New Criteria for Bounded-Input-Bounded-Output and Asymptotic Stability of Nonlinear Systems," Proceedings of the Fifth International Federation of Automatic Control Congress, Paris, June 1972.

8. E. K. Haddad, "A Criterion for the Bounded Input, Bounded-Output Stability of Time-Varying Nonlinear Systems," SIAM Journal on Control, Vol. 11, No. 2, May 1973.

9. L. A. Zadeh and C. A. Desoer, Linear System Theory, McGraw-Hill, 1963.

NASA-Langley, 1974 CR-2447