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BY:

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THE EFFECT OF SUPPORT FLEXIBILITY AND DAMPING ON THE SYNCHRONOUS RESPONSE OF A SINGLE MASS FLEXIBLE ROTOR

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ABSTRACT

This paper deals with the dynamic unbalance response and transient motion of the single mass Jeffcott rotor in elastic bearings mounted on damped, flexible supports.

A steady state analysis of the shaft and the bearing housing motion was made by assuming synchronous precession of the system. The conditions under which the support system would act as a dynamic vibration absorber at the rotor critical speed were studied and plots of the rotor and support amplitudes, phase angles, and forces transmitted were evaluated by the computer and the performance curves were plotted by an automatic plotter unit. Curves are presented on the optimization of the support housing characteristics to attenuate the rotor synchronous unbalance response.

The complete transient motion including rotor unbalance was examined by integrating the equations of motion numerically using a modified 4th order Runge-Kutta procedure and the resulting whirl orbits were plotted by an automatic plotter unit. The results of the transient analysis are discussed with regards to the design optimization procedure derived from the steady-state analysis.

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INTRODUCTION

The study of rotor dynamics has in recent years, become of increasing importance in the engineering design of power systems. With the increase in performance requirements of high-speed rotating machinery in various fields such as gas turbines, process equipment, auxiliary power machinery and space applications, the engineer is faced with the problem of designing a unit capable of smooth operation under various conditions of speed and load.

In many of these applications the design operating speed is often well beyond the rotor first critical speed, and under these circumstances the problem of insuring that the turbomachine will perform with a stable low-level amplitude of vibration is often difficult to achieve.

At the turn of the century H. H. Jeffcott (1) developed the fundamentals of the dynamic response of the damped single mass unbalanced rotor on a massless elastic shaft mounted on rigid bearing supports. The Jeffcott analysis of the single mass model showed that operating speeds above the first critical speed were possible and that a low level of vibration would be attained once the rotor had exceeded the first critical speed.

As various compressor and turbine manufacturers adapted the flexible rotor design concept in which the rotors were designed to operate above the first critical speed, various units developed severe operating difficulties which could not be explained by the elementary Jeffcott model.

Under certain conditions of high speed operation above the first critical speed, such influences as internal rotor friction (2), hydrodynamic bearing and seal forces, (3) and aerodynamic cross coupling (4) can lead to a destructive nonsynchronous precessive whirl motion being developed in the rotor system.

B. L. Newkirk and Kimball (5), in their early investigations of self-excited instability in compressors due to internal friction, were able to determine experimentally that the introduction of a flexible support system could greatly extend the rotor stability threshold speed. D. M. Smith (6) in 1933 was the first to verify Newkirk's findings theoretically by expanding the Jeffcott model with internal damping to include a massless damped flexible support system. Recent investigators such as Gunter (7), Tondl (8), Dimentberg (9) and others (10) have shown that flexible damped supports may improve the stability characteristics of high speed rotors. The problem of bearing forces transmitted has been examined by various researchers, (11, 12, 13, 14). They have shown that a significant reduction in the forces transmitted can be achieved by the proper design of the bearing support system.

The present analysis was undertaken to determine the influence of flexible supports on the synchronous unbalance response of the single mass Jeffcott rotor, and to optimize the support system characteristics so as to minimize the rotor amplitude and forces transmitted over a given speed range. Den Hartog (15) has shown that the tuned vibration absorber will greatly reduce the response of the forced vibrations of the two-mass system. The following analysis parallels this approach for the case of a single mass rotor excited by an unbalance load.

This paper presents an analytic study of the tuned damper support system similar to that employed by Brock (16) and also presents a generalized study performed on the digital computer to obtain optimum support damping to produce the best response of the rotor over a wide speed range. It is well known that a damper support system can improve the vibration characteristics of a rotating shaft and various investigators have considered the problem either from the standpoint of a continuous elastic system or as a series of lumped masses (17-23).

Although the results presented in this paper apply specifically to the single mass Jeffcott model, the optimization procedure may be readily extended to more complex multi-mass rotor bearing systems by employing a finite element rotor digital computer program similar to the procedure presented by Lund in Ref. 24 or by using the procedure as outlined in the paper presented by Crook and Grantham (25) on the vibration analysis of turbine generators on damped flexible supports.

EQUATIONS OF MOTION

Figure 1*represents the single mass Jeffcott rotor mounted in damped elastic supports. In the Jeffcott model, the shaft is considered as a massless elastic member and the rotor mass is concentrated in a disc mounted at the center of the span. The shaft is supported in linear bearings which are mounted in damped flexible supports.

Neglecting rotor acceleration and the disc gyroscopics, the governing equations of motion for the rotor, bearings, and support system in complex notation reduce to the following (31)

$$M_2\ddot{Z}_2 + C_s\dot{Z}_2 + C_i\dot{Z}_s - iQZ_2 + (K_s - i\omega C_i)Z_s = M_2 e_u \omega^2 e^{i\omega t} \qquad (1)$$

$$C_b\dot{Z}_j - C_i\dot{Z}_s + K_b Z_j - (K_s - i\omega C_i)Z_s = 0 \qquad (2)$$

$$\ddot{Z}_1 + C_1\dot{Z}_1 + K_1 Z_1 - C_i\dot{Z}_s - (K_s - i\omega C_i)Z_s = 0 \qquad (3)$$

where

$$Z_s = Z_2 - Z_j - Z_1 = \text{relative shaft deflection.}$$

If the internal damping C_i and the aerodynamic cross coupling term Q are excluded from the above equations then the system will be stable (26).

After the initial transient motion has damped out, it may be assumed that the system steady-state motion is circular synchronous precession. In this case the displacements are related to the velocity and acceleration vectors as follows:

$$Z_i = A_i e^{i\omega t}$$

$$\dot{Z}_i = i\omega Z_i \qquad (4)$$

$$\ddot{Z}_i = i\omega\dot{Z}_i = -\omega^2 Z_i$$

*Illustrations begin on page 60.

where A_j is in general complex.

The differential equations of motion may be reduced to a set of algebraic equations for the determination of the rotor steady-state motion.

$$(K_s - M_2\omega^2 + iC_s\omega)A_2 - K_sA_j - K_sA_1 = M_2e_u\omega^2 \tag{5}$$

$$-K_sA_2 + (K_b + K_s + i\omega C_b)A_j + K_sA_1 = 0 \tag{6}$$

$$-K_sA_2 + K_sA_j + (K_1 + K_s - M_1\omega^2 + i\omega C_1)A_1 = 0 \tag{7}$$

ROTOR AMPLIFICATION FACTOR

Consider the steady-state orbit of the flexible rotor on rigid supports. The rotor amplitude is a function of both the rotor and bearing stiffness and damping characteristics. Assuming A_1 is zero, the relative journal bearing complex amplitude from Eq. 6 is given by

$$A_j = \frac{K_s(K_s + K_b - i\omega C_b)}{(K_s + K_b)^2 + (\omega C_b)^2} A_2 \tag{8}$$

Solving Eq. 5 for the rotor amplitude yields

$$A_2 = M_2e_u\omega^2 \frac{(K_2 - M_2\omega^2 - i\omega C_2)}{(K_2 - M_2\omega^2)^2 + (\omega C_2)^2} \tag{9}$$

where

$$K_2 = \frac{K_bK_s(K_s + K_b) + K_s(\omega C_b)^2}{(K_s + K_b)^2 + (\omega C_b)^2}$$

$$C_2 = \frac{K_s{}^2C_b}{(K_b + K_s)^2 + (\omega C_b)^2} + C_s$$

The rotor displacement vector Z_2 may be expressed in terms of the absolute displacement R_2 and the phase angle ϕ as follows

$$Z_2 = R_2e^{i(\omega t - \phi)} \tag{10}$$

where

$$R_2 = \frac{M_2 e_u \omega^2}{\sqrt{(K_2 - M_2\omega^2)^2 + (\omega C_2)^2}}$$

$$\phi = \tan^{-1}\left[\frac{\omega C_2}{K_2 - M_2\omega^2}\right]$$

The above results are similar to the rotor amplitude and phase angle results for the single mass flexible rotor on rigid supports as shown by Thomson (27).

The rotor undamped, or natural critical speed is given by

$$\omega_c = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{K_b K_s}{(K_b + K_s)M_2}} \tag{11}$$

For the case of a lightly damped rotor system on rigid supports the maximum rotor amplitude will occur at approximately the rotor critical speed and the dimensionless rotor amplitude or amplification factor at the critical speed is given by

$$A = \frac{R_2}{e_u}\bigg|_{\omega=\omega_c} = \frac{K_2}{\omega_c C_2} \tag{12}$$

Example I

Consider a 97 lb. disc centered on a uniform massless elastic shaft as shown in Fig. (I). Assume that the bearing stiffness $\frac{K_b}{2}$ is 500,000 lb/in and that the effective shaft stiffness K_s at the disc station is 333,000 lb/in. Assuming light damping, the total stiffness K_2 is given by

$$K_2 = \frac{K_s K_b}{K_s + K_b} = \frac{1 \times 0.333 \times 10^{12}}{(1 + 0.333)10^6} = 250{,}000 \text{ lb/in}$$

The rotor critical speed is

$$\omega_c = \sqrt{\frac{K_2}{M_2}} = \sqrt{\frac{250{,}000}{0.25}} = 1{,}000 \text{ rad/sec}$$

or N_c = 9,550 RPM.

If the rotor damping C_s is assumed to be 15 lb - sec/in and the bearing damping coefficient $C_b/2$ is 80 lb - sec/in. then the effective system damping coefficient C_2 is approximately given by

$$C_2 \approx C_s + \frac{K_s{}^2 C_b}{(K_b + K_s)^2} = 15 + \frac{(0.333)^2 \times 10^{12} \times 160}{(1.333)^2 \times 10^{12}} = 25 \text{ lb - sec/in}$$

The amplification factor at the rotor critical speed is given by

$$A_{CR} = A = \frac{K_2}{\omega_c C_2} = \frac{250,000}{1,000 \times 25} = 10.0$$

The amplification factor of 10 represents a very lightly damped rotor system and indicates that the rotor amplitude at the critical speed will be 10 times the rotor unbalance eccentricity e_u.

ROTOR RESPONSE ON DAMPED FLEXIBLE SUPPORTS

Solution of Eq. 6 for the case of synchronous precession for the shaft relative deflection Z_s yields

$$Z_s = (Z_2 - Z_1)\left[\frac{K_b(K_b + K_s) + (\omega C_b)^2 + i\omega C_b K_s}{(K_b + K_s)^2 + (\omega C_b)^2}\right] \tag{13}$$

Hence, in terms of the general coefficients C_2 and K_2

$$Z_s = (Z_2 - Z_1)\left[\frac{K_2 + i\omega(C_2 - C_s)}{K_s}\right] \tag{14}$$

The simultaneous equations for the absolute shaft and support housing motion reduce to the following

$$[K_1 + K_2 - M_1\omega^2 + i\omega(C_2 + C_1 - C_s)]A_1 + [-i\omega(C_2 - C_s) - K_2]A_2 = 0$$

(15)

$$[-K_2 - i\omega(C_2 - C_s)]A_1 + [K_2 - M_2\omega^2 + i\omega C_2]A_2 = M_2 e_u \omega^2 \qquad (16)$$

If the damping terms are neglected, then the natural frequencies of the system may be determined by the expansion of the determinant of coefficients. The resulting frequency equation may be expressed as follows:

$$\omega_{1,2}/\omega_c = \sqrt{\frac{1}{2} + \frac{1 + K}{2M} \pm \sqrt{\left(\frac{1 + K}{2M} + \frac{1}{2}\right)^2 - \frac{K}{M}}} \qquad (17)$$

where

$$\omega_c = \sqrt{\frac{K_2}{M_2}}$$

Figure 2 represents the dimensionless critical speeds vs. the dimensionless support stiffness factor K for various values of support to rotor mass ratios. Note that the incorporation of the flexible support with the rotor bearing system causes two critical speeds to occur; one which is higher and one which is lower than the original rotor critical on rigid supports.

To solve for the complex support and rotor amplitudes A_1 and A_2, Eq. 15 and 16 may be expressed as follows:

$$[a_{ij} + ib_{ij}]A_j = F_i; \quad j = 1, 2; \quad i = 1, 2 \qquad (18)$$

Multiplying Eq. 18 by the complex inverse matrix of coefficients and expanding yields

$$A_1 = \frac{\begin{vmatrix} F_1 & a_{12} + ib_{12} \\ F_2 & a_{22} + ib_{22} \end{vmatrix}}{\Delta}, \qquad (19)$$

Where

$$\Delta = d_r + id_i$$

$$d_r = (K_2 - M_2\omega^2)(K_1 - M_1\omega^2) - K_2M_2\omega^2 - C_1C_2\omega^2 - \omega^2C_s(C_2 - C_s)$$

$$d_i = C_1\omega(K_2 - M_2\omega^2) + C_2\omega(K_1 - M_1\omega^2 - M_2\omega^2) + C_s\omega(K_2 + M_2\omega^2)$$

Expanding Eq. 19

$$A_1 = \frac{F_1 a_{22} - F_2 a_{12} + i(F_1 b_{22} - F_2 b_{12})}{d_r + i d_i} \tag{20}$$

In this case only an external unbalance excitation force F_2 is acting on the shaft and no external exciting force F_1 is assumed to be present on the support system. For example, an excitation force F_1 may be transmitted to the rotor system through the support structure by vibrations of auxiliary or adjacent equipment.

$$A_1 = -\frac{F_2[a_{12}d_r + b_{12}d_i + i(b_{12}d_r - a_{12}d_i)]}{d_r^2 + d_i^2} \tag{21}$$

Assume A_1 is of the form

$$A_1 = A_{1r} - iA_{1i} \tag{22}$$

The complex support amplitude Z_1 after some complex algebraic manipulation is given by

$$Z_1 = A_1 e^{i\omega t} = R_1 e^{i(\omega t - \beta_1)} \tag{23}$$

where

$$R_1 = \sqrt{A_{1r}^2 + A_{1i}^2}\,, \quad \beta_1 = \tan^{-1}\left(\frac{d_i}{d_r}\right)$$

If the shaft damping coefficient C_s is considered small in comparison to the effective damping coefficient C_2 than the system displacements and phase angles are given as follows

$$R_1 = M_2 e_u \omega^2 \sqrt{\frac{K_2^2 + (\omega C_2)^2}{d_r^2 + d_i^2}} \tag{24}$$

and the phase angle of the support motion relative to the rotating unbal- ance is given by

$$\beta_1 = \tan^{-1}\left[\frac{K_2 d_i - \omega C_2 d_r}{K_2 d_r + \omega C_2 d_i}\right] \tag{25}$$

Since the complex rotor support motion Z_1 is given by

$$Z_1 = X_1 + iY_1 = R_1 e^{i(\omega t - \beta_1)}$$

Then for example, the horizontal and vertical components of the support motion are given by

$$\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} = M_2 e_u \omega^2 \sqrt{\frac{K_2{}^2 + (\omega C_2)^2}{d_r{}^2 + d_i{}^2}} \begin{Bmatrix} \cos(\omega t - \beta_1) \\ \sin(\omega t - \beta_1) \end{Bmatrix} \tag{26}$$

In a similar fashion, the complex rotor amplitude Z_2 is given by

$$Z_2 = M_2 e_u \omega^2 \frac{a_{11} + ib_{11}}{d_r + id_i} e^{i\omega t} \tag{27}$$

After some manipulation, Eq. 27 reduces to the following

$$Z_2 = M_2 e_u \omega^2 \sqrt{\frac{(K_1 + K_2 - M_1\omega^2)^2 + ((C_1 + C_2)\omega)^2}{d_r{}^2 + d_i{}^2}} e^{i(\omega t - \beta_2)} \tag{28}$$

where

$$\beta_2 = \tan^{-1}\left(\frac{(K_1 + K_2 - M_1\omega^2)d_i - (C_1 + C_2)\omega d_r}{(K_1 + K_2 - M_1\omega^2)d_r + (C_1 + C_2)\omega d_i}\right) \tag{29}$$

The relative journal displacement is given by

$$Z_j = Z_2 - Z_1 - Z_s \tag{30}$$

Where the relative shaft deflection is

$$Z_s = \frac{(Z_2 - Z_1)}{K_s}\left[K_s - K_2 - i\omega C_2\right] \tag{31}$$

Solving for the journal displacement

$$Z_j = R_j e^{i(\omega t - \beta_j)}$$

where,

$$R_j = M_2 e_u \omega^2 \sqrt{\left[\frac{(K_1 - M_1\omega^2)^2 + (\omega C_1)^2}{d_r^2 + d_i^2}\right] \times \left[\left(\frac{K_s - K_2}{K_s}\right)^2 + \left(\frac{\omega C_2}{K_s}\right)^2\right]} \tag{32}$$

and the phase angle β_j between the journal amplitude and rotating unbalance force is given by

$$\beta_j = \tan^{-1}\left(\frac{(K_1 - M_1\omega^2)d_i - \omega C_1 d_r}{\text{[illegible]} - M_1\omega^2)d_i + \omega C_1 d_i}\right) + \tan^{-1}\left(\frac{\omega C_2}{K_s - K_2}\right) \tag{33}$$

FORCES TRANSMITTED

The magnitude of the resultant forces transmitted through the bearings and the support are of considerable interest to the designer from a standpoint of bearing life and system isolation. It is desirable to minimize the forces transmitted through the supporting structure and foundation so that other machines or piping systems are not excited. The magnitude of the force transmitted through the bearings is given by

$$F_b = R_j \sqrt{K_b^2 + (\omega C_b)^2} \tag{34}$$

and the force transmitted through the support system is given by

$$F_1 = R_j \sqrt{K_1^2 + (\omega C_1)^2} \tag{35}$$

An indication of the effectiveness of the support system in attenuating the forces transmitted to the foundation is the support dynamic transmissibility factor TRD which will be defined as the ratio of the magnitude of the transmitted support force to the rotating unbalance load. If the dynamic transmissibility is less than 1, then the support system possesses good attenuation characteristics. Analysis has shown that if the support housing impedance characteristics, which are determined by the housing mass, stiffness and damping, are mismatched to the rotor-bearing system then under certain speed conditions the dynamic transmissibility may exceed 1.

The dynamic transmissibility for the support is defined as

$$TRD = \frac{F_i}{M_2 e_u \omega^2} = \sqrt{\frac{(K_2{}^2 + (\omega C_2)^2)(K_1{}^2 + (C_1\omega)^2)}{d_r{}^2 + d_i{}^2}} \qquad (36)$$

If it is assumed that the rotor is operating well above any of the system critical speeds then the dynamic transmissibility is approximately given by

$$TRD = \frac{1}{\omega^4}\sqrt{\frac{(K_2{}^2 + (\omega C_2)^2)(K_1{}^2 + (C_1\omega)^2)}{M_1{}^2 M_2{}^2}} \qquad (37)$$

The above expression leads to the well known conclusion that to minimize the forces transmitted through the support for supercritical speed operation in the Jeffcott model, the support damping should be zero and the support stiffness should be as light as possible (28). This is a highly undesirable design practice for several reasons since large rotor amplitudes and forces transmitted may be encountered when passing through the rotor critical speeds, and also the rotor system would be extremely shock sensitive and particularly susceptible to self-excited whirl instability under such conditions.

A compromise support damping coefficient should be selected to either minimize the rotor amplitudes or the forces transmitted over the operating speed range and also be sufficient to insure adequate rotor stability.

ANALYSIS OF SYSTEM UNBALANCE RESPONSE - TUNED SYSTEM

Figure 3 represents a computer generated plot of the dimensionless rotor relative amplitude versus the dimensionless rotor speed for the case of $K = M = 1$. This relative rotor amplitude is equivalent to the motion monitored by a proximity probe mounted in the casing measuring the rotor motion at the center span. This system represents a tuned condition in which the support stiffness ratio K is equal to the support mass ratio M. With no support damping in the system, the tuned support will cause the relative rotor amplitude to be zero at a speed corresponding to the rotor critical speed with rigid supports. The introduction of support mass and flexibility has caused two critical speeds to appear in the system; one above and one below the rigid support rotor critical. Note that when the support damping is relatively low the amplitudes at the two criticals becomes extremely high.

As the dimensionless support damping ratio C increases from 0.01 to 10 the rotor amplitudes at the system critical speeds decrease while the amplitude increases at a speed corresponding to the rigid support critical speed ($\omega/\omega_c = 1$). Note that in this case the damping value of 10 appears to be close to an optimum value for the minimization of the resonance amplitudes. If the support damping is further increased from 10 to 50, Fig. 3 indicates that there will be only one critical speed present in the system which will correspond to the rigid support critical. Although the damping of C = 50 is over 5 times the optimum value, the maximum amplitude is only 1/3 the rigid support value of 10. As the support damping approaches infinity, the rotor amplitude will asymptotically approach 10.

Figure 4 represents the absolute dimensionless rotor motion for various values of support damping ratio and is similar to Fig. 3. It should be noted that the damping coefficient of 10 also appears to be close to the optimum damping for the absolute motion as well as the relative motion.

It is of interest to note that the various damping lines all intersect at a common point P in the plot of absolute as well as relative rotor motion. If the rotor amplification factor A is 100 (implying light rotor damping) then there will be two common points of intersection P and Q on the response plots (see Fig. 10) similar to that shown by Den Hartog for the damped vibration absorber (15). The intersection points P and Q will occur at speeds respectively below and above the rigid support critical speed. The rotor amplitude may be minimized for the case of the absolute rotor motion by selecting the damping such that the slope of the response curve is zero at point P, and zero at point Q to minimize the rotor relative motion.

Figure 5 represents the phase angle between the rotating unbalance vector and the absolute rotor displacement vector for various damping coefficients. The phase angle for the single mass rotor on rigid supports (Jeffcott model) increases with speed from 0 to 90 degrees at the critical speed and asymptotically approaches 180 degrees as the rotor speed greatly exceeds the critical speed. The phase angles of the rotor on damped flexible supports has a considerably different behavior from that of the rigid support rotor. For light values of support damping (C = 0.01), the phase angle increases rapidly to 180° as the system passes through the first critical speed and drops to almost 60° as it passes through the second critical speed. As the speed greatly exceeds the highest critical speed, the phase angle again approaches 180°. The phase angle of 180° indicates that the rotor mass center lies along the rotor spin axis. As the support damping coefficient is increased beyond 5 for the case of the tuned system, the reduction in phase angle above the first critical speed is suppressed. This phenomena of phase angle reversal above the first critical speed has been observed experimentally (30).

Figure 6 represents the support amplitude versus speed for various damping values and indicates that with very light support damping there will be large support resonances. As the damping is increased beyond C = 10 the resonances are suppressed and the amplitude is only slightly greater than 1. For C = 50 there is only a small peak observed in the support system which occurs at a speed corresponding to the rigid support critical speed. The addition of high damping (C > 50) freezes the support and limits its motion drastically.

Figure 7 represents the support phase angles versus speed ratio for various values of support damping. The phase angle for light damping (C = 0.01) is zero at low speeds and goes to 180 degrees as it passes through the first critical and then shifts to 330° upon passing through the second critical speed. If the rotor damping is light (A = 100) the support phase angle will approach 360° after passing through the second critical speed. Note that the various damping lines intersect at three points. The first node point represents the first system critical speed, the second node point represents the rigid support critical speed and the third node point represents the second critical speed on flexible supports. In the discussion of the single mass flexible rotor presented in vibration texts (27) the phase change is only shown from zero to 180 degrees. In more complex systems with flexible supports, the phase change may vary between 0 and 360 degrees. For example in multimass systems the authors have observed phase changes of n times 180 degrees where n represents the number of system critical speeds. The measurement of rotor and support phase angles have been neglected and limited data has been reported in the literature. This is an extremely useful variable which when incorporated with displacement measurements can be used in balancing flexible rotors or impedance calculations of the support system.

Figure 8 represents the dimensionless bearing forces transmitted for the tuned system. The dimensionless force transmitted is obtained by dividing by the transmitted force corresponding to the value at the critical speed of the original rotor on rigid supports. Because of the light shaft damping the force transmitted curves are similar in appearance to the displacement curves. Note that for the support damping coefficient of C = 10 the forces transmitted to the bearings at the rigid support critical are only 10 percent of the value transmitted for the rotor bearing system on rigid supports.

Figure 9 represents the force transmitted through the bearing supports to the foundation or base for various values of supporting damping. With a very lightly damped support system, (C = 0.01) the support amplitude and force transmitted will be particularly high at the first critical speed where the bearing and support motions are in phase. At the second critical speed, the support amplitude is lower than the amplitude attained at the first critical speed. This is because the bearings and support motions are out of phase which enables the bearing damping to help attenuate the support motion. It is of interest to note from Fig. 8, for the tuned rotor system, the bearing force transmitted at $(\omega/\omega_c) = 1$ with an undamped support system is zero. Figure 9 shows that the corresponding force transmitted through the support system at $\omega/\omega_c = 1$ has been reduced to only 10% of the rigid support value.

The force transmitted for an undamped support system at a speed ratio of four is approximately 10% of the rigid support value. This condition would be desireable if it were possible to accelerate through the criticals, thereby avoiding the large steady-state amplitudes and forces developed.

The near optimum damping of 10 increases the support forces transmitted in the supercritical speed region to 30% of the rigid support value and the overdamped support system (C = 50) has increased to nearly 80%. Hence, the support damping introduced to suppress the system resonances will cause the forces transmitted to increase in the supercritical speed region.

If the system is designed to operate over the entire speed range shown, then the near optimum value of damping (i.e., C = 10) for suppressing the rotor absolute amplitude also produces the most desirable attentuation of forces to the system support structure.

OPTIMUM DAMPING FOR TUNED SYSTEM

From the observation of the computer generated displacement and force transmitted plots it is apparent that there exists an optimum damping to either minimize the rotor amplitudes or the forces transmitted over the entire speed range.

For example to minimize the absolute rotor motion as shown in Fig. 4 or the relative rotor motion shown in Fig. 3, the method of (16) may be used in which the damping is chosen so that the slope of the amplitude curve is zero at points P and Q respectively. In the tuned system where K/M = 1 for light rotor damping (A = 100), the rotor amplitudes at points P and Q are independent of the support damping as shown in Fig. 10 and can be shown to be equal to

$$X_2 = x_2/e_u\big|_{P,Q} = \sqrt{1 + 2M} \tag{38}$$

Therefore with the tuned system illustrated with a mass ratio of M = 1, the maximum amplitude at P or Q will be 1.732 times the rotor unbalance eccentricity. The optimum damping may be selected so that the tangent to the amplitude curve at either point P or Q has a zero slope. By selecting the optimum damping in this fashion it is seen that the maximum amplitude in the system will not exceed the value given by Eq. 38. Thus it is readily apparent that to minimize the rotor response over a given speed range, the support mass should be kept as light as possible.

After considerable algebraic manipulation (28) the optimum damping coefficient for both points P and Q is given by the following expression

$$\xi^2 = \frac{4M^3\psi^3 - 3M^2(4M + 3)\psi^2 + M(12M^2 + 13M + 8)\psi - M(1 + 2M)^2}{-12M\psi^2 + 8(1 + 2M)\psi - (1 + 2M)} \qquad (39)$$

where,

$$\xi = C_1/C_c = C_1/C_2 \times 1/2A = C_1 \frac{\omega_c}{2K_2}$$

$\psi = \Omega_1{}^2$ or $\Omega_2{}^2$ depending on whether the value calculated is for point P or Q respectively.

and

$$\Omega_1{}^2 = \frac{\sqrt{1 + 2M}}{1 + \sqrt{1 + 2M}}$$

$$\Omega_2{}^2 = \frac{\sqrt{1 + 2M}}{\sqrt{1 + 2M} - 1}$$

For example, when M = 1 and for the first node, P:

$$\psi = \Omega_1{}^2 = \frac{\sqrt{3}}{1 + \sqrt{3}} = 0.634$$

and

$$\xi^2 = 0.447$$

Hence

$$\left.\frac{C_1}{C_c}\right|_{opt} = 0.688 \qquad \text{for point P}$$

In a similar fashion

$$\left.\frac{C_1}{C_c}\right|_{opt} = 0.559 \qquad \text{for point Q}$$

Example 2

As an example of the application of the tuned support design criteria consider the rotor of Example I mounted in flexibly supported bearing housings which weigh 48.5 lbs and have a stiffness of 125,000 lb/in. The total support weight W_1 and stiffness K_1 is given by

$$W_1 = 2 \times 48.5 = 97 \text{ lb}$$

$$K_1 = 2 \times 125{,}000 = 250{,}000 \text{ lb/in}$$

Hence,

$$M = M_1/M_2 = 1.0$$

$$K = K_1/K_2 = 1.0$$

The critical damping coefficient C_c is given by

$$C_c = \frac{2K_2}{\omega_c} = \frac{500{,}000 \text{ lb/in}}{1{,}000 \text{ rad/sec}} = 500 \text{ lb-sec/in}$$

Thus the support damping coefficients required to make the slope of the rotor amplitude curve zero at points P and Q are respectively given as follows

$$C_1\big|_p = 0.688 \times C_c = 344 \text{ lb-sec/in}$$

$$C_1\big|_q = 0.559 \times C_c = 279.5 \text{ lb-sec/in}$$

These calculations are valid only for the case of zero damping on the rotor and in the bearings (i.e., $A = \infty$) and only for the tuned system (i.e., $K = M$). For a more realistic solution, a value of $A = 10$ was chosen and numerous cases were then programmed on a digital computer to arrive at a value of optimum amplitude and required damping. This approach is discussed in the next section of this paper but the results for the tuned system are very nearly the same as the results arrived at analytically for the case of $A = \infty$ and are presented in Fig. 11.

The results shown in Fig. 11 are approximately correct for systems having moderate to light damping on the rotor (i.e., $10 \leq A < \infty$). Note that the smaller the mass ratio M, the lower will be the peak response and also the lower will be the required support damping. For example, if the mass ratio is 0.1, then the maximum dimensionless amplitude will be only 1.1 and the required damping ratio will be 5 as compared to a value of 13.6 for an M ratio of 1. Figure 12 is a response plot for the tuned system $K = M = 0.1$ which illustrates the validity of the results plotted in Fig. 11. The response curve for a damping ratio of 5 passes almost horizontal through the node point and has the low amplitude ratio as indicated by Fig. 11.

Example 3

Consider a rotor system similar to Example 2 in which the rotor rigid support amplification factor $A = 10$.

For a tuned support system the dimensionless support damping coefficient is obtained from Fig. 11 for $M = 1$ as follows

$$C = C_1/C_2 = 13.6$$

where C_2 is given as 25 lb-sec/in (Example 1).

Therefore,

$$C_1 = 13.6 \times C_2 = 340 \text{ lb-sec/in}$$

Note that this value is approximately the same as the value given in Example 2 for the required damping at point P corresonding to A_{∞}.

This indicates that each support must have 170 lb-sec/in. damping to achieve the optimum response of about 1.7 times the unbalance level of the rotor.

Next consider a tuned support with a mass and stiffness ratio of 0.10 (see Fig. 12). Corresponding support weight and stiffness are given as follows

W_1 = 9.7 lb/in

K_1 = 25,000 lb/in

The required damping is thus found from Fig. 11 to be

$C \doteqdot 5.0$

or

$C_1 = 5 \times 25 = 125$ lb-sec/in

Thus only 62.5 lb-sec/in. damping per support is required to obtain an optimum response of 1.1 times the unbalance level of the rotor. This value of 1.1 is in comparison to a maximum response of 10 times the unbalance level for the rigidly mounted rotor-bearing system.

OPTIMIZATION OF SUPPORT DAMPING FOR OFF-TUNED CONDITIONS

In general it is not possible or necessarily desirable to have a tuned support system. The support to rotor mass ratio is usually dictated by design considerations and can be varied only within certain ranges. Figure 11 shows that for best reduction of rotor amplitude, the support mass should remain as light as possible. However, it will be shown that even with high mass ratio support systems the rotor amplitudes can be attenuated by a factor of 5 by proper selection of the stiffness and damping coefficients.

To evaluate the optimum damping for off-tuned conditions the computer program was run for various support mass and stiffness ratios and each of these for various damping coefficients. For example, Fig. 13 represents the amplitudes at the rotor first and second critical speed for various mass ratios with a dimensionless stiffness ratio of K = 0.01 as the mass ratio and damping are varied. The solid lines represent the amplitude at the second critical speed and the dotted lines represent the amplitude at the first critical speed. With moderate support damping ratios it is observed that as the mass ratio increases the amplitude at the first critical reduces while the amplitude at the second critical increases. The optimum damping was selected as the intersection of the amplitudes at the first and second critical for a particular value of damping. For example in Fig. 14 for K = 1.0, the lowest optimum amplitude point on the plot is given by a damping ratio of 10 and produces an amplitude ratio of about 1.5. Several plots similar to Fig. 13 were produced and the results were then crossplotted to obtain plots of amplitude versus damping ratio such as Fig. 15 for K = 5.0.

Figure 14 represents the maximum rotor amplitude vs. support damping ratio for various values of dimensionless support stiffness for a rotor bearing system with a low support mass ratio of 0.01. Figure 16 shows that for this particular case, the lowest amplitude is achieved by a low support stiffness ratio of K = 0.01 which is of the same order as the mass ratio. With this low support stiffness, there is a wide range of support damping (i.e. C = 1 → 6) that can be used to achieve the low level of rotor response.

Thus, under proper design conditions the support damping may be allowed to vary by a considerable amount without impairing the rotor performance. As the support stiffness ratio increases, the maximum rotor amplitude response also increases and the required support damping must be larger. For example, if the support stiffness ratio increases from 0.01 to 2.0, the optimum damping required increases by a factor of 15 from approximately 2 to 30.

Figures 17 and 18 represent the maximum rotor amplitude for mass ratios of 0.10 and 0.715. Note also that for high stiffness ratio support systems, the permissible range of the support damping coefficient is very narrow, and that either a reduction or an increase of damping beyond the optimum value will result in a rapid gain in rotor response.

It is also of interest to note that if a high support stiffness (K = 2) is used in conjunction with a low value of support damping (C < 2) then the rotor response will be worse than the original rotor response on rigid supports (A = 10).

Figure 19 represents the maximum rotor response vs. support damping for a high support mass ratio system (M = 2). It is obvious from the comparison of Figs. 16 and 19 that the high mass ratio support system is less desirable. The minimum rotor amplitude that can be achieved is $x_2/e_u = 2$ with a tuned support where K = M = 2 and a support damping coefficient of C = 20. (Also see Fig. 11 on the tuned system.) As the support stiffness ratio is reduced, the rotor response curve increases in the optimum damping region.

If it is not possible to incorporate a high value of support damping into the system (C = 20), then the rotor amplitude can still be reduced to 40% of the original rotor response by a low support damping value of C = 1 and a reduced support stiffness ratio of K = 0.7. For low values of support damping, if the support stiffness increases beyond K = 0.7, the rotor response rapidly increases.

A series of plots similar to Figs. 16 - 19 were produced for various mass ratios in order to determine the optimum rotor response for off-tuned support conditions. Figure 20 represents the rotor maximum amplitude vs. the support mass ratio for various values of support stiffness with optimum damping.

For the case of A = 10, Fig. 20 illustrates that the lowest amplitude can be achieved with a low mass ratio support system. With a high mass ratio support system such as M = 5, the rotor amplitude X_2 can be reduced from 10 to 2.8 by means of a tuned support stiffness of K = 5.0 and optimum damping.

Note that as the support stiffness becomes very light, the maximum rotor amplitude increases to 7.5.

At a low value of support mass (M = 0.1), the rotor amplitude increases as the support stiffness increases. The optimum damping required with the tuned support is given by the following approximate relationship

$$c_1 - \frac{1.37 \times K_2}{\omega_c} \times M^{0.437} \tag{40}$$

Figure 21 represents the rotor maximum amplitude vs. stiffness ratio K for various values of mass ratio M and damping values C. This figure illustrates that it is possible to operate with off-tuned conditions and still maintain a low level of vibration. It is seen that the light damping value C = 0.10 will produce the highest amplitude over the range of stiffness plotted for K = 0.1 to 10. For K values less than 1.5, the damping ratio should be less than 10, while for high stiffness supports where K > 2, the damping value C should be > 20 for maximum attenuation. Note that for low stiffness supports (K < 0.2) the value of C = 20 represents an over-damped support system causing the amplitude and transmitted forces to be greater than the optimum value.

Figure 22 represents the rotor maximum amplitude vs. support damping ratio C for various values of support stiffness K with optimum support mass M. Figure 22 shows that the lowest amplitude level can be achieved with a low support stiffness (K = 0.01) and the support damping may vary from C = 0.5 to 10 while maintaining a low level response. The figure also illustrates that as the support stiffness is increased, the amplitude will also increase for a given value of damping. It is also clearly seen that as the stiffness value is increased, a larger support damping value is required to produce a low vibration amplitude with optimum support mass ratio.

Figure 23 represents optimum damping and mass ratios for various values of stiffness ratio. Figure 23 shows that, as the mass ratio increases, the required stiffness ratio increases for a given value of damping. It is of interest to note that for K values between 0.2 and 2.0, for a given M value, there can be two values of optimum damping, a low value of C below 10 and a high damping value > 10. Although high damping values may result in low rotor amplitudes, the bearing forces transmitted through the support will be much higher. Therefore extremely large values of support damping should be avoided.

TRANSIENT ANALYSIS

The previous discussion has been concerned only with the steady-state response of the rotor due to unbalance and has not considered the rotor initial transient motion. As discussed previously, the damped flexible support system is important, not only from the standpoint of reduction of synchronous unbalance response, but also in the control of self excited vibrations such as caused by internal friction, aerodynamic excitation, etc. Therefore to investigate the general rotor motion and also to provide a check on the steady-state analysis, the rotor equations of motion were integrated forward in time on the digital computer using a modified 4th order Runge-Kutta integration procedure. This procedure is of importance particularly if the analysis is extended from a linear bearing or support system to include a nonlinear hydrodynamic damper bearing as presented in Ref. 13.

The dimensionless rotor and support transient orbits were automatically computer plotted with the following dimensionless parameters

$$X = x/e_u, \quad Y = y/e_u$$

Figure 24 represents the initial transient orbit of a 96.6 lb rotor of Example 1 with a highly damped support (C = 43) for the first 12 cycles of shaft motion. The support mass ratio and the support stiffness ratio are both approximately the same (0.10) which represents a tuned system. Because of the excessive support damping, the maximum force transmitted to the support is 2.16 times the unbalance force while the force transmitted to the bearings is reduced by about 40%. The magnifications of the force to the support would be highly undesirable for applications such as aircraft jet engines. For example, various investigators have observed that such a situation occurs with the hydrodynamic squeeze film bearing when operating at excessive eccentricity ratios (29).

Figure 25 represents the bearing absolute and relative motion corresponding to the case as shown in Fig. 24. The solid line represents the bearing absolute motion while the dashed line represents the bearing relative motion. Since the support damping is 10 times the bearing damping, the initial absolute bearing motion is not much larger than the bearing relative motion. Note that the timing marks on the orbit appear in the negative x direction. This indicates that the bearing motion is 180° out of phase with the rotating unbalance load.

Figure 26 illustrates the support housing motion. Because of excessive support damping, the initial support transient motion is quite small and is less than the unbalance eccentricity. The phase angle between the support motion and the rotating unbalance load is approximately 220°.

Figure 27 represents the transient orbit for the same rotor system except that the support damping has been reduced by a factor of 100 from C_1 = 1,000 lb-sec/in. to 10 lb-sec/in. In this case, the maximum force transmitted through the support is less than 9% of the rotating unbalance force and the bearing force transmitted is 16%. This orbit is analogous to a suddenly applied unbalance such as a blade loss in an engine. Although the forces transmitted have been greatly reduced with the low stiffness and damping support system, the rotor has developed a large initial transient motion of over 10 times the unbalance eccentricity and this transient motion is not readily damped out.

Figure 28 represents the absolute and relative bearing motion with the low support damping of C_1 = 10 lb-sec/in. The absolute bearing initial transient motion is extremely large while the relative motion is well behaved. Note that the bearing relative phase angle has shifted from 180° for the highly damped case to 50° for the case with light support housing damping.

Figure 29 represents the support housing motion corresponding to the system with light support damping. A comparison of the absolute support motion and the absolute rotor motion indicates that the two are similar. This implies that the initial transient motion of the rotor is due primarily to the large deflections in the support system.

In Fig. 30, the rotor transient motion is depicted with an optimum damping coefficient of C = 5.5 for minimum rotor response as determined from the steady-state analysis. The transient response is rapidly suppressed after seven cycles of shaft motion to produce a small stable synchronous orbit. The transmitted forces to the bearings and support are nearly balanced to achieve approximately a 75% attenuation of the unbalance load. Figure 30 shows that with the optimum damping as determined by the steady-state analysis, the initial transient rotor motion will be 5 times the rotor unbalance eccentricity.

Figure 31 represents the bearing motion. After approximately 6 cycles, the initial transient motion is damped out. Figure 31 indicates that the absolute bearing motion is equal to the rotor unbalance eccentricity after the transient has died, and is 180° out of phase with the rotating unbalance load. The relative bearing motion is approximately 60% of the unbalanced eccentricity and lags the rotating unbalance by about 120°. Figure 32 represents the support motion and it is also similar to the rotor motion as shown in Fig. 30.

SUMMARY AND CONCLUSIONS

The equations of motion for a single mass rotor-bearing system on damped flexible supports have been derived and studied considering both a steady-state and transient type analysis. Design charts for both tuned and off-tuned support conditions have been presented.

The analysis may be summarized by the following general statements.

1. The critical speed response of the single mass Jeffcott model rotor may be completely eliminated by means of a low mass ratio flexible support with optimum damping. In this case the rotor steady-state amplitude of motion over the entire speed range will only be slightly more than the rotor unbalance eccentricity.

2. The support mass ratio should be kept as light as possible to achieve minimum rotor amplitude.

3. The rotor amplitude may be considerably attenuated even for high mass ratio support systems by tuning the support stiffness such that K = M and incorporating optimum damping for the tuned conditions.

4. With a low mass ratio support system, the required value of optimum damping is not critical and can vary by a factor of 10 without appreciably effecting rotor performance. As the mass ratio increases, the required value of optimum damping increases rapidly and the permissible range of variation of support damping diminishes.

5. The off-tuned support ($K \neq M$) can be designed to produce a considerable improvement in system response in comparison to the rotor on rigid supports. If insufficient damping is incorporated in the support then the resulting rotor steady-state amplitude may be larger than the original rotor response for support stiffness values $K > 1$.

6. If there is excessive support damping ($C > 20$) with a low mass ratio support ($M = 0.1$), then the forces transmitted through the support may exceed the unbalance forces (TRDS > 1.0).

7. Although the steady-state analysis shows that the rotor amplitude will be small for an underdamped ($C < 0.50$) low mass ratio support system, the orbital analysis shows that a large initial transient motion can be

generated due to the suddenly applied unbalance force and that this motion is not readily attenuated.

8. The optimum damping based on minimization of the rotor steady-state amplitude for both tuned and off-tuned conditions produces a satisfactory transient response from the standpoint of rapid reduction of the initial transient motion, improved system stability and reduction of the forces transmitted.

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APPENDIX A

DISCUSSION

A high speed rotor shaft may be considered as a continuous elastic member with variable mass and inertia properties along its length. The rotor shaft usually has attached to it such components as turbine or compressor blades, impeller disks, or spacer assemblies or seals. If the axial dimensions of each rotor component is small in comparison to the overall length of the rotor, then each component may be treated as a concentrated mass with a polar moment of inertia equivalent to that of the original component. If the mass of the components are large in comparison to the shaft mass connecting the components, then the shaft weight can be neglected or considered to be located at the mass stations. If the polar moment of inertia of each section is ignored, then the stations may be considered as concentrated masses, rather than distributed in the plane of the rotor element. However, if the sections whirl in a plane, perpendicular to the spin axis then the gyroscopic moments do not act on the system and hence the equations reduce to the same as if point masses were assumed.

The position vector of the nth mass center is given by

$$\vec{P} = \vec{\delta}_b + \vec{\delta}_j + \vec{\delta}_s + \vec{e}_u$$

where

$\vec{\delta}_b$ = vectoral bearing support deflection

$\vec{\delta}_j$ = vectoral journal deflection

$\vec{\delta}_s$ = vectoral shaft deflection

$\vec{e}_u$ = displacement of mass center from the shaft centerline

The system being analyzed has been reduced to a single mass rotor mounted in idealized linear bearings and the bearings are in turn mounted on damped, elastic supports. By considering only small deflections, the spring rate of the flexible, massless rotor shaft may be considered to be linear.

The rotor disk (see Fig. 1) is considered to whirl in a plane and hence no gyroscopic moments are acting on the system. The orthogonal support and bearing spring rates are assumed symmetric and no cross coupling terms are considered to be acting at the support housings. The aforementioned assumptions allow the equations of motion of the system to be written as total differential equations.

DERIVATION OF EQUATIONS OF MOTION

A.1 Kinematics

The position vectors to the mass stations are given by

m_1: bearing housing mass

$$\vec{P}^{M_1/O} = X_1\vec{n}_x + Y_1\vec{n}_y \tag{A.1}$$

m_2: rotor mass

$$\vec{P}^{M_2/O} = (X_2 + e_u\cos\theta)\vec{n}_x + (Y_2 + e_u\sin\theta)\vec{n}_y \tag{A.2}$$

The velocities of the mass stations are given by

$$\vec{V}^{M_1/O} = \dot{X}_1\vec{n}_x + \dot{Y}_1\vec{n}_y \tag{A.3}$$

$$\vec{V}^{M_2/O} = (\dot{X}_2 - e_u\dot{\theta}\sin\theta)\vec{n}_x + (\dot{Y}_2 + e_u\dot{\theta}\cos\theta)\vec{n}_y \tag{A.4}$$

A.2 Kinetic Energy

The kinetic energy of the system is given by

$$T = \frac{1}{2} \sum_{i=1}^{2} (M_i \vec{V}_i \bullet \vec{V}_i) + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij} \omega_i \omega_j \tag{A.5}$$

neglecting the gyroscopic couples acting on the disc, the system kinetic energy reduces to

$$T = \frac{1}{2} M_1(\dot{X}_1{}^2 + \dot{Y}_1{}^2) + \frac{1}{2} M_2[(\dot{X}_2 - e_u\dot{\theta}\sin\theta)^2 + (\dot{Y}_2 + e_u\dot{\theta}\cos\theta)^2]$$

$$+ \frac{1}{2} \phi_{zz}\dot{\theta}^2 \tag{A.6}$$

A.3 Potential Energy

The potential energy of the system is composed of the sums of the potential energy of the flexible shaft, the potential energy of the bearings, and the energy of support structure as follows

$$V = \frac{1}{2} [K_s(X_s{}^2 + Y_s{}^2) + K_b(X_j{}^2 + Y_j{}^2) + K_1(X_1{}^2 + Y_1{}^2)] \tag{A.7}$$

where

$$X_s = X_2 - X_1 - X_j$$

A.4 Dissipative Energy

The system dissipative energy consists of the damping functions provided by the bearing support system, the bearings, and the external and internal rotor damping and the aerodynamic rotor cross coupling as described by Alford (4).

$$D = \frac{1}{2} \{C_1(\dot{X}_1{}^2 + \dot{Y}_1{}^2) + C_b(\dot{X}_j{}^2 + \dot{Y}_j{}^2) + C_s(\dot{X}_2{}^2 + \dot{Y}_2{}^2)$$

$$+ C_i[\dot{X}_s{}^2 + \dot{Y}_s{}^2 + 2\omega(Y_s\dot{X}_s - X_s\dot{Y}_s)] + Q(Y_2\dot{X}_2 - X_2\dot{Y}_2)\} \tag{A.8}$$

The internal damping function is dependent upon the rotor precession rate and can cause self excited whirl instability when the rotor is operated above the critical speed (26). Alford has demonstrated that the aerodynamic cross coupling stiffness term can also cause rotor instability when the rotor speed is supercritical. When the system dissipation function is comprised of only the first three terms, the system is inherently stable.

A.5 Lagranges Equations

The governing equations of motion are obtained from Lagranges Equations which state:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_r}\right] - \frac{\partial L}{\partial q_r} + \frac{\partial D}{\partial \dot{q}_r} = F_{q_r} \tag{A.9}$$

where

$$L = T - V$$

The total number of equations of motion obtained will be equal to the number of degrees of freedom of the system which is seven and are given as follows.

<u>Rotor</u>

$$X_2: \quad M_2\ddot{X}_2 + C_s\dot{X}_2 + C_i(\dot{X}_2 - \dot{X}_1 - \dot{X}_j) + K_s(X_2 - X_1 - X_j)$$
$$+ QY_2 + \omega C_i(Y_2 - Y_1 - Y_j) = M_2 e_u(\omega^2\cos(\omega t) + \alpha\sin(\omega t)) \tag{A.10}$$

$$Y_2: \quad M_2\ddot{Y}_2 + C_s\dot{Y}_2 + C_i(\dot{Y}_2 - \dot{Y}_1 - \dot{Y}_j) + K_s(Y_2 - Y_1 - Y_j) - QX_2$$
$$- \omega C_i(X_2 - X_1 - X_j) = M_2 e_u(\omega^2\sin(\omega t) - \alpha\sin(\omega t)) \tag{A.11}$$

Bearings

$$X_j:\quad (C_i + C_b)\dot{X}_j - C_i(\dot{X}_2 - \dot{X}_1) + (K_b + K_s)X_j - K_s(X_2 - X_1) - C_i\omega(Y_2 - Y_1 - Y_j) = 0 \tag{A.12}$$

$$Y_j:\quad (C_i + C_b)\dot{Y}_j - C_i(\dot{Y}_2 - \dot{Y}_1) + (K_b + K_s)Y_j - K_s(Y_2 - Y_1) + C_i\omega(X_2 - X_1 - X_j) = 0 \tag{A.13}$$

Support

$$X_1:\quad M_1\ddot{X}_1 + (C_1 + C_i)\dot{X}_1 - C_i(\dot{X}_2 - \dot{X}_j) + (K_1 + K_s)X_1 - K_s(X_2 - X_j) - C_i\omega(Y_2 - Y_1 - Y_j) = 0 \tag{A.14}$$

$$Y_1:\quad M_1\ddot{Y}_1 + (C_1 + C_i)\dot{Y}_1 - C_i(\dot{Y}_2 - \dot{Y}_j) + (K_1 + K_s)Y_1 - K_s(Y_2 - Y_j) + C_i\omega(X_2 - X_1 - X_j) = 0 \tag{A.15}$$

Angular Acceleration

$$\theta:\quad (\Phi_{zz} + M_2e_u^2)\ddot{\theta} + M_2e_u\left[\ddot{Y}_2\cos\theta - \ddot{X}_2\sin\theta - \dot{\theta}(\dot{Y}_2\sin\theta + \dot{X}_2\cos\theta)\right] = T_z(\dot{\theta}) \tag{A.16}$$

Where

$\dot{\theta} = \omega, \ \ddot{\theta} = \alpha$

The equations A.10 to A.15 may be vectorially combined by representing the displacements in complex notation as follows

$$\left.\begin{aligned} Z_2 &= X_2 + iY_2 \\ Z_j &= X_j + iY_j \\ Z_1 &= X_1 + iY_1 \end{aligned}\right\} \qquad (A.17)$$

$$Z_2:\quad M_2\ddot{Z}_2 + C_s\dot{Z}_2 + C_i(\dot{Z}_2 - \dot{Z}_1 - \dot{Z}_j) + K_s(Z_2 - Z_1 - Z_j) \qquad (A.18)$$

$$- iQZ_2 - i\omega C_i(Z_2 \quad Z_1 - Z_j) = M_2 e_u(\omega^2 - i\alpha)e^{i\omega t}$$

$$Z_j:\quad (C_b + C_i)\dot{Z}_j - C_i(\dot{Z}_2 - \dot{Z}_1) + (K_b + K_s)Z_j \qquad (A.19)$$

$$- K_s(Z_2 - Z_1) + iC_i\omega(Z_2 - Z_1 - Z_j) = 0$$

$$Z_1:\quad M_1\ddot{Z}_1 + (C_1 + C_i)\dot{Z}_1 - C_i(\dot{Z}_2 - \dot{Z}_j) + (K_1 + K_s)Z_1 \qquad (A.20)$$

$$- K_s(Z_2 - Z_j) + iC_i\omega(Z_2 - Z_1 - Z_j) = 0$$

APPENDIX B

TRANSIENT COMPUTER PROGRAM FOR THE THREE MASS SYSTEM

The rotor system used in the transient analysis is of considerably greater complexity than the equations used in the steady-state analysis. In the more general transient analysis, internal and aerodynamic cross coupling may be incorporated in the rotor. The bearing may have 8 stiffness and damping coefficients while the support system may have 4 stiffness and damping coefficients. Because of the more generalized treatment, studies may be conducted on the stability of the rotor due to hydrodynamic bearing forces, internal friction, or aerodynamic cross coupling. The support system can be investigated to show the influence of the support damping in promoting stability.

<u>Equations of Motion</u>

$$X_2:\quad M_2\ddot{X}_2 + C_s\dot{X}_2 + C_i(\dot{X}_2 - \dot{X}_1 - \dot{X}_j)$$

$$+ K_s(X_2 - X_1 - X_j) + QY_2 + \omega C_i(Y_2 - Y_1 - Y_j)$$

$$= M_2 e_u \omega^2 \cos\omega t \tag{B.1}$$

$$Y_2:\quad M_2\ddot{Y}_2 + C_s\dot{Y}_2 + C_i(\dot{Y}_2 - \dot{Y}_1 - \dot{Y}_j)$$

$$+ K_s(Y_2 - Y_1 - Y_j) - QX_2 - \omega C_i(X_2 - X_1 - X_j)$$

$$= M_2 e_u \omega^2 \sin\omega t \tag{B.2}$$

$$X_j:\quad 2M_j(\ddot{X}_1 + \ddot{X}_j) + (C_i + 2C_{xx})\dot{X}_j - C_i(\dot{X}_2 - \dot{X}_1)$$

$$+ (2K_{xx} + K_s)X_j - K_s(X_2 - X_1) - C_i\omega(Y_2 - Y_1 - Y_j)$$

$$+ 2(K_{xy}Y_j + C_{xy}\dot{Y}\ ) = 0 \tag{B.3}$$

$$Y_j:\ 2M_j(\ddot{Y}_1 + \ddot{Y}_j) + (C_i + 2C_{yy})\dot{Y}_j - C_i(\dot{Y}_2 - \dot{Y}_1)$$

$$(2K_{yy} + K_s)Y_j - K_s(Y_2 - Y_1) + C_i\omega(X_2 - X_1 - X_j)$$

$$+ 2(K_{yx}X_j + C_{yx}\dot{X}_j) = 0 \qquad (B.4)$$

$$X_1:\ 2M_1\ddot{X}_1 + 2M_j(\ddot{X}_1 + \ddot{X}_j) + (2C_{1x} + C_i)\dot{X}_1$$

$$- C_i(\dot{X}_2 - \dot{X}_j) + (2K_{1x} + K_s)X_1 - K_s(X_2 - X_j)$$

$$- C_i\omega(Y_2 - Y_1 - Y_j) = 0 \qquad (B.5)$$

$$Y_1:\ 2M_1\ddot{Y}_1 + 2M_j(\ddot{Y}_1 + \ddot{Y}_j) + (2C_{1y} + C_i)\dot{Y}_1$$

$$- C_i(\dot{Y}_2 - \dot{Y}_j) + (2K_{1y} + K_s)Y_1 - K_s(Y_2 - Y_j)$$

$$+ C_i\omega(X_2 - X_1 - X_j) = 0 \qquad (B.6)$$

The support equations can be reduced to the following form by using Eqs. B.3 and B.4.

$$M_1\ddot{X}_1 + C_{1x}\dot{X}_1 + K_{1x}X_1 - C_{xx}\dot{X}_j - C_{xy}\dot{Y}_j$$

$$- K_{xx}X_j - K_{xy}Y_j = 0 \qquad (B.7)$$

$$M_1\ddot{Y}_1 + C_{1y}\dot{Y}_1 + K_{1y}Y_1 - C_{yx}\dot{X}_j - C_{yy}\dot{Y}_j$$

$$- K_{yx}X_j - K_{yy}Y_j = 0 \qquad (B.8)$$

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BEGIN
    COMMENT
    THIS PROGRAM CALCULATES THE TRANSIENT RESPONSE OF A SINGLE-MASS
    FLEXIBLE ROTOR ON FLEXIBLE BEARING SUPPORTS (LINEAR) , THE RESULTS
   ARE PLOTTED AUTOMATICALLY (SEE INPUT CARDS BELOW) AND THE
   INITIAL AND FINAL VALUES ARE PRINTED OUT ON THE LINE PRINTER,
   THEY APPEAR IN THE CORRECT ORDER TO BE TYPED ON DATA CARDS
   TO CONTINUE THE MOTION IF DESIRED (THAT IS CARD 5(4)),

    END OF COMMENT;
%
%
%
%     THE INPUT TO THE PROGRAM IS AS FOLLOWS :
%     (ALL DATA IS IN FREE FIELD )
    COMMENT

    CARD 1  READ(CR,/,TMAX,H,N,CASNO)
    TMAX - NO. OF RADIANS SOL. IS TO BE SOLVED
    H - STEP SIZE USED IN INTEGRATION PROCEDURE     (TMAX/H≤700)
    N - NO. OF EQS. TO BE INTEGRATED
         ( = 12 IF SUPPORT ,  = 8 IF NO SUPPORT)
    CASNO - IDENTIFICATION NUMBER [XXXX.XXXX]
                                 [(MO.)(DAY).(YEAR)(CASE NO.)]

    CARD 2  READ(CR,/,RPM,W,K,DS,CI,QAC,EU,WJ,W1)
    RPM - ROTOR SPEED , REV/MIN
    W - ROTOR WEIGHT , LB.
    K - ROTOR SHAFT STIFFNESS , LB/IN
    DS - ABSOLUTE SHAFT DAMPING , LB-SEC/IN
    CI - INTERNAL FRICTION DAMPING , LB-SEC/IN
    QAC - CROSS-COUPLING, LB/IN
    EU - UNBALANCE ECCENTRICITY OF W IN MILS
                                         ----
    WJ - JOURNAL WEIGHT AT EACH END , LB.
    W1 - SUPPORT WEIGHT AT EACH BEARING , LB.

    CARD 3  READ(CR,/,KXX,KYY,CXX,CYY,KXY,KYX,CXY,CYX)
    BEARING STIFFNESS AND DAMPING OF EACH BEARING

    CARD 4  IF N>8 THEN READ(CR,/,K1X,K1Y,C1X,C1Y)
    SUPPORT STIFFNESS AND DAMPING OF EACH SUPPORT

    CARD 5 (4)  READ(CR,/,FOR I←0 STEP 1 UNTIL N DO[Y[0,I]])
    0 IS FOR INITIAL TIME, RADIANS
    1 - ABSOLUTE ROTOR DISP.  , X-DIR.
    2 - ABSOLUTE ROTOR VELOCITY, X-DIR.
    3 - ABS. ROTOR DISP. , Y-DIR.
    4 - ABS. ROTOR VEL. , Y-DIR.
    5,6,7,8 -  SAME ORDER AS ABOVE FOR JOURNAL RELATIVE MOTION
    9,10,11,12 - SAME AS ABOVE FOR SUPPORT MOTION  (IF REQUIRED)

    CARD 6 (5)READ(CR,/CS,ISCALE,XMIN,DX,XMIN2,DX2,XMIN3,DX3)
    CS - PLOTTER CONTROL   0 - NO PLOT     1 - PLOT
    ISCALE - SCALE CONTROL   0 - PROG. SCALE   1 - USE FOLLOWING INFO.
```

```
     XMIN = NO. TO APPEAR AT ORIGIN      ROTOR DISLPACEMENT
     DX = SCALE INCREMENT PER INCH       ROTOR DISPLACEMENT
     XMIN2,DX2 = SAME  AS ABOVE BUT FOR JOURNAL PLOT
     XMIN3,DX3 = SAME AS ABOVE BUT FOR SUPPORT

     CARD 7 (6)  IF CS≠0 THEN READ(CR,/,RPCS)
     RPCS = NO. OF TIMES INTEG. FOR THAX RADIANS IS TO BE SOLVED

     EXAMPLE DATA INPUT:
12.56,0.05,8,608.7101,
10000,675,280000,0,0,0,0,5,312,50,
351000,606000,739,865,0,0,0,0,
0,-0.5,0,0,-0.5,-0.05,0,0,-0.05,
1,0,0,0,0,0,0,0,
5,

     NOTE ON PLOTTING:
          REQUEST ONE (1) BLOCK PER SET OF DATA
     END OF COMMENT ;
%    ---REPEAT THE SERIES FOR EACH CASE ---
%
     INTEGER  CS ;
     INTEGER I,J,NA,N;
     BOOLEAN  ISCALE ;
     BOOLEAN STABLE ;
     INTEGER RPC,RPCS;
     INTEGER  VV ;
     ALPHA ARRAY ALP1,ALP2[0:8], ALP3,ALP4,ALP5,ALP6,ALP7,ALP8,
     ALP9,ALP10,ALP11,ALP12,ALP13,ALP14,ALP15,ALP16,ALP17[0:3];
     ALPHA ARRAY  ALPHA1, ALPHA2[0:2];
     ALPHA ARRAY ALP18,ALP19[0:4];
     ALPHA ARRAY ALP20[0:8], ALP21[0:8], ALP22[0:3];
     ALPHA ARRAY ALP23,ALP24[0:4];
     ALPHA ARRAY ALP25,ALP26,ALP27,ALP28[0:7];
     ALPHA ARRAY ALP29,ALP30[0:3];
     REAL CASNO;
     REAL K2X,K2Y,C2X,C2Y, WCX,WCY,ACX,ACY;
     REAL K11,K12,K13,K14,FMBH,FMBI,FMSH,FMSI,TIMHB,TIMHS;
     REAL THAXH ;
     REAL FMSHH,FMBHH,TIMHBH,TIMHSH;
     REAL QAC,XMIN2,DX2,XMAX2,XMIN3,DX3,XMAX3, YMIN2,DY2,YMAX2,
     YMIN3,DY3,YMAX3;
     REAL ZZ,ZZ2,Z1,Z2,Z3,Z4,Z5,Z6,Z7,Z8,Z9,Z10   ,Y0,Y1,Y2,Y3,Y4,Y5,
     Y6,Y7,Y8,Y9,Y10;
     REAL Y11,Y12,DS,CI,WJ,KXX,KYY,CXX,CYY,K1X,K1Y,C1X,C1Y;
     REAL MJ,Z22,Z32,Z11,Z12,Z13,Z14,Z15,Z16,Z17,Z18 ,EU ;
     REAL XA,HXA,XI,HXI,YA,HYA,YI,HYI ;
     REAL THAX , H,CC, R, OMEGA, OMEGA2 ;
     REAL NCS ;
     REAL DUM1,DUM2,DUM3,DUM4,DUM22,DUM42,RE1,IM1,M1,W1,C1,K1,KB,CB     ;
     REAL K2,C2 ;
     REAL   RPM,W,K,CD,CQ,DC,RAD,M,DDC,NC1,NCRATIO,DEN1,AA,DEN2,ACR,
     XMAX,YMAX ;
     REAL XMIN, YMIN, DX, DY ;
     REAL KXY,KYX,CXY,CYX;
     REAL Z19,Z20,Z21,Z23;
```

```
      LABEL  ALLDONE , ACARD ;
      REAL ARRAY A,B,C[0:4],Q,KK,Y[0:4,0:12],F[0:12],AY[0:12,0:700] ;
 ARRAY HXR,BYR[0:700];
      LABEL REPEAT;
      FORMAT  OUTDATA (X2,E13.6,X2,F6.2,4(X2,E13.6));


      PROCEDURE MAXMIN (X,Y,N,H);      VALUE N,H;
      REAL H;  INTEGER N;         ARRAY X,Y[0] ;
    BEGIN
      INTEGER I ;
      XI ← XA ← X[1] ;    YI ← YA ← Y[1] ;
      FOR I ← 2 STEP 1 UNTIL N DO
     BEGIN
      IF X[I] >  XA  THEN
     BEGIN
       XA  ← X[I]; HXA ← I ;
      END
      ELSE   BEGIN
      IF X[I] < XI THEN  BEGIN XI ← X[I] ; HXI ← I ; END ;   END ;
      IF Y[I] >  YA  THEN
     BEGIN
       YA  ← Y[I]; HYA ← I ;
     END
      ELSE BEGIN IF Y[I] < YI THEN  BEGIN YI ← Y[I] ; HYI ← I ;END;END;
     END ;
      HYA ← (HYA-1) × H / 6.28 ;
      HYI ← (HYI-1) × H / 6.28 ;
      HXI ← (HXI-1) × H / 6.28 ;
      HXA ← (HXA-1) × H / 6.28 ;
      END OF MAXMIN ;

  PROCEDURE SAMESCALE(X,Y,N,XMIN,XMAX,DX,YMIN,YMAX,DY);   VALUE N;
      ARRAY X,Y[0] ;  INTEGER N ;
      REAL XMIN,XMAX,DX,YMIN,YMAX,DY;
     BEGIN
      REAL ANS, ANSX  ;
      INTEGER  I;
      ANS ← ABS (Y[1]); ANSX ← ABS (X[1]);
      FOR I ← 2 STEP 1 UNTIL N DO
     BEGIN
      IF ABS (Y[I]) > ANS THEN ANS ← ABS (Y[I]);
      IF ABS (X[I]) > ANSX THEN ANSX ← ABS (X[I]);
     END;
      IF ANSX > ANS THEN ANS ← ANSX;
      IF ANS < 1 THEN
      BEGIN
      XMIN ← -1;  DX ← 0.3333333333;
      END
      ELSE
      IF ANS < 3 THEN
     BEGIN
      XMIN ← -3 ;  DX ← 1 ;
     END
      ELSE
      IF ANS < 6 THEN
```

```
    BEGIN
      XMIN ← -6;
      DX ← 2;
    END
      ELSE IF ANS <12 THEN
    BEGIN
      XMIN ←-12;
      DX ←4;
    END
      ELSE IF ANS < 18 THEN
     BEGIN
      XMIN ← -18 ;  DX ← 6 ;
     END
      ELSE
    BEGIN
      XMIN ← - 24;
DX ←8;
    END;
      DY ← DX; YMIN ← XMIN;
      XMAX ← 3×DX; YMAX ← 3×DY;
    END OF SAMESCALE ;

      PROCEDURE PLOTCHEK(X,Y,N,XMIN,XMAX,YMIN,YMAX);  VALUE N;
      ARRAY X,Y [0]; INTEGER N;
      REAL XMIN,XMAX,YMIN,YMAX;
    BEGIN
      INTEGER I;
      FOR I ← 1 STEP 1 UNTIL N DO
    BEGIN
      IF X [I] > XMAX THEN X [I] ← XMAX
      ELSE
      IF X[I] < XMIN THEN X[I] ← XMIN;
      IF Y [I] > YMAX THEN Y[I] ← YMAX
      ELSE
      IF Y[I] < YMIN THEN Y[I] ← YMIN;
    END ;
  END OF PLOTCHEK;


PROCEDURE ORBITTOP (I);
VALUE I; INTEGER I ;
BEGIN
IF I = 1 THEN
      SYMBOL(1.50,9.00,.21,ALP1,0,24)
ELSE IF I = 2 THEN
      SYMBOL(2.00,9.00,.21,ALP2,0,18)
ELSE
      SYMBOL(2.00,9.00,.21,ALP3,0,18);
      SYMBOL(5.00,9.50,0.10,ALP30,0,3);
      NUMBER(5.20,9.50,0.10,CASNO,0,4);
      SYMBOL(1.55,8.75,.14,ALP25,0,24);
      NUMBER(2.03,8.75,.14,AY[0,1]/6.28,0,2);
      K ←K/1000; KXX←KXX/1000; KYY←KYY/1000; K1X←K1X/1000;K1Y←K1Y/1000;
      SYMBOL(0.75,8.50,.14,ALP4 ,0,18);NUMBER(1.35,8.50,.14,RPM    ,0,0);
      SYMBOL(3.75,8.50,.14,ALP6 ,0,19);NUMBER(4.23,8.50,.14,EU     ,0,3);
```

```
SYMBOL(0,75,8,25,,14,ALP5 ,0,17);NUMBER(1,23,8,25,,14,N      ,0,2));
SYMBOL(3,75,8,25,,14,ALP8 ,0,17);NUMBER(4,23,8,25,,14,NJ     ,0,2));
SYMBOL(0,75,8,00,,14,ALP7 ,0,21);NUMBER(1,23,8,00,,14,K      ,0,3));
SYMBOL(3,75,8,00,,14,ALP10,0,22);NUMBER(4,35,8,00,,14,KXX    ,0,3));
SYMBOL(0,75,7,75,,14,ALP9 ,0,23);NUMBER(1,23,7,75,,14,DS     ,0,2));
SYMBOL(3,75,7,75,,14,ALP12,0,22);NUMBER(4,35,7,75,,14,KYY    ,0,3));
SYMBOL(0,75,7,50,,14,ALP11,0,23);NUMBER(1,23,7,50,,14,CI     ,0,2));
SYMBOL(3,75,7,50,,14,ALP14,0,24);NUMBER(4,35,7,50,,14,CXX    ,0,2));
SYMBOL(0,75,7,25,,14,ALP13,0,18);NUMBER(1,12,7,25,,14,QAC    ,0,2));
SYMBOL(3,75,7,25,,14,ALP16,0,24);NUMBER(4,35,7,25,,14,CYY    ,0,2));
SYMBOL(0,75,7,00,,14,ALP15,0,19);NUMBER(1,47,7,00,,14,WCX    ,0,2));
SYMBOL(3,75,7,00,,14,ALP18,0,6 );NUMBER(4,23,7,00,,14,ACX    ,0,2));
SYMBOL(0,75,6,75,,14,ALP17,0,19);NUMBER(1,47,6,75,,14,WCY    ,0,2));
SYMBOL(3,75,6,75,,14,ALP20,0,6 );NUMBER(4,23,6,75,,14,ACY    ,0,2));
IF N > 8 THEN  BEGIN
SYMBOL(0,75,6,50,,14,ALP19,0,16);NUMBER(1,23,6,50,,14,W1     ,0,2));
SYMBOL(0,75,6,27,,14,ALP21,0,21);NUMBER(1,23,6,27,,14,K1X    ,0,3));
SYMBOL(3,75,6,27,,14,ALP22,0,24);NUMBER(4,35,6,27,,14,C1X    ,0,2));
SYMBOL(0,75,6,05,,14,ALP23,0,21);NUMBER(1,23,6,05,,14,K1Y    ,0,3));
SYMBOL(3,75,6,05,,14,ALP24,0,24);NUMBER(4,35,6,05,,14,C1Y    ,0,2));
END;
K ←K×1000; KXX←KXX×1000; KYY←KYY×1000; K1X←K1X×1000;K1Y←K1Y×1000;
END OF ORBITTOP ;
PROCEDURE AGRID (XMIN,DX,YMIN,DY);
REAL XMIN,DX,YMIN,DY;
BEGIN
AXIS(0,0,0,ALPHA1,14,6,90,YMIN,DY);
AXIS(0,0,ALPHA2,-14,6,0,XMIN,DX);
 AXIS(6,0,ALPHA1,0,6,90,YMIN,DY);
AXIS(6,6,ALPHA2,0,6,180,XMIN,DX);
END;

PROCEDURE FUNCTION (J,Y);
VALUE J;  INTEGER J;  ARRAY Y[0,0];
BEGIN
Y0←Y[J,0];
Y1←Y[J,1];
Y2←Y[J,2];
Y3←Y[J,3];
Y4←Y[J,4];
Y5←Y[J,5];
Y6←Y[J,6];
Y7←Y[J,7];
Y8←Y[J,8];
IF N > 8 THEN BEGIN
Y9←Y[J,9];
Y10 ← Y[J,10];
Y11 ← Y[J,11];
Y12 ← Y[J,12];
END;
F[0] ← 1;
F[1] ← Y2;
 F[2] ← EU×COS(Y0)-Z1×Y2-Z2×(Y2-Y10-Y6)
-Z3×(Y1-Y9-Y5)-Z4×Y3-Z2×(Y3-Y11-Y7);
F[3] ← Y4;
F[4] ← EU×SIN(Y0)-Z1×Y4-Z2×(Y4-Y12-Y8)
```

```
-Z3×(Y3-Y11-Y7)+Z4×Y1+Z2×(Y1-Y9 -Y5);
IF N > 8 THEN BEGIN
F[9] ← Y10;
F[10] ← -Z10×(Z11×Y9+Z12×Y10-Z13×Y5-Z14×Y6 -Z19×Y7-Z20×Y8);
F[11] ← Y12;
F[12] ← -Z10×(Z15×Y11+Z16×Y12-Z17×Y7-Z18×Y8-Z21×Y5-Z23×Y6);
END ELSE F[9]←F[10]←F[11]←F[12]←0.0;
F[5] ← Y6;
F[6] ← -F[10]-Z5×(Z6×Y6-Z22×(Y2-Y10) +Z7×Y5
-Z32×(Y1-Y9)-Z22×(Y3-Y11-Y7)+Z19×Y7+Z20×Y8);
F[7] ← Y8;
F[8] ← -F[12] -Z5×(Z8×Y8-Z22×(Y4-Y12) + Z9×Y7
-Z32×(Y3-Y11) + Z22×(Y1-Y9-Y5)+Z21×Y5+Z23×Y6);

END OF FUNCTION ;


PROCEDURE  RKG (K,Y); INTEGER K ;
ARRAY Y[0,0] ;
BEGIN
REAL P ; INTEGER I,J ;
FOR J ← 1 STEP 1 UNTIL 4 DO
BEGIN
FUNCTION (J-1,Y) ;
FOR I ← 0 STEP 1 UNTIL N DO
KK[J,I] ← F[I] ;
FOR I ← 0 STEP 1 UNTIL N DO
BEGIN
P ← A[J] × (KK[J,I] - B[J] × Q[J-1,I]) ;
Y[J,I] ← Y[J-1,I] + H × P ;
Q[J,I] ← Q[J-1,I] + 3 × P - C[J] × KK[J,I] ;
END;
END;
FOR I←0 STEP 1 UNTIL N DO
BEGIN
Y[0,I]←Y[4,I];
Q[0,I]←Q[4,I] ;
AY[I,K]←Y[4,I];
END;
FMBI ←(Z13×Y[0,5] + Z14×Y[0,6] + Z19×Y[0,7] + Z20×Y[0,8])*2
    + (Z21×Y[0,5] + Z23×Y[0,6]    +  Z17×Y[0,7] + Z18×Y[0,8])*2 ;
FMSI ← (Z11×Y[0,9]+Z12×Y[0,10])*2+(Z15×Y[0,11]+Z16×Y[0,12])*2   ;
IF FMBI>FMBH THEN BEGIN
FMBH ← FMBI; TIMHB ← Y[0,0]; END;
IF FMSI > FMSH THEN BEGIN
FMSH ← FMSI;  TIMHS ← Y[0,0];  END;
END OF RKG;
PROCEDURE TIMESTEP (TMAX, H, N, AY,  NA,Y);
VALUE  TMAX,H,N;
REAL TMAX,H;  INTEGER N, NA ;
REAL ARRAY  AY[0,0] ;
ARRAY Y[0,0] ;
BEGIN
INTEGER  I,J,K;    LABEL  REPEAT ;
FOR I←0 STEP 1 UNTIL N DO
BEGIN
```

```
        Q[0,I]←0;
        AY[I,1]←Y[0,I];
    END;

        FMBH ←(Z13×Y[0,5] + Z14×Y[0,6] + Z19×Y[0,7] + Z20×Y[0,8])*2
            + (Z21×Y[0,5] + Z23×Y[0,6]    +  Z17×Y[0,7] + Z18×Y[0,8])*2 ;
        FMSH ← (Z11×Y[0,9]+Z12×Y[0,10])*2+(Z15×Y[0,11]+Z16×Y[0,12])*2    ;
        TIMHB ← Y[0,0];  TIMHS ← Y[0,0];

        K←1;
    REPEAT:
        K←K+1;
        RKG (K,Y);
        IF Y[0,0]<TMAX THEN GO TO REPEAT ;
        NA←K;
    END OF TIMESTEP;

        A[1] ← C[1] ← C[4] ← 0.5 ;
        A[2] ← C[2] ← 1-SQRT(0.5) ;
        A[3] ← C[3] ← 1+SQRT(0.5) ;
        A[4] ← 1/6 ;
        B[1] ← B[4] ← 2 ;
        B[2] ← B[3] ← 1;
        WRITE(LP,< / /, X10,"******* SMFROLS ******",//,
      " THIS PROGRAM CALCULATES THE TRANSIENT RESPONSE OF A SINGLE-MASS",/,
      "FLEXIBLE ROTOR ON FLEXIBLE BEARING SUPPORTS (LINEAR) . THE RESULTS",
     /,"ARE PLOTTED AUTOMATICALLY (SEE INPUT CARDS BELOW) AND THE         ",/,
      "INITIAL AND FINAL VALUES ARE PRINTED OUT ON THE LINE PRINTER.    ",/,
      "THEY APPEAR IN THE CORRECT ORDER TO BE TYPED ON DATA CARDS       ",/,
      "TO CONTINUE THE MOTION IF DESIRED (THAT IS CARD 5(4)).          ",/,
    ///," THE INPUT TO THE PROGRAM IS AS FOLLOWS :              ",/,
      "  (ALL DATA IS IN FREE FIELD )              ",///,
      " CARD 1  READ(CR,/,TMAX,H,N,CASNO)             ",/,
      " TMAX - NO. OF RADIANS SOL. IS TO BE SOLVED              ",/,
      " H - STEP SIZE USED IN INTEGRATION PROCEDURE    (TMAX/H≤700)  ",/,
      " N - NO. OF EQS. TO BE INTEGRATED            ",/,
      "      ( = 12 IF SUPPORT ,  = 8 IF NO SUPPORT)          ",/,
      " CASNO - IDENTIFICATION NUMBER [XXXX.XXXX] ",/,
      "                               [(MO.)(DAY).(YEAR)(CASE NO.)]  ",/,
    //," CARD 2  READ(CR,/,RPM,W,K,DS,CI,QAC,EU,WJ,W1)           ",/,
      " RPM - ROTOR SPEED , REV/MIN             ",/,
      " W - ROTOR WEIGHT , LB.               ",/,
      " K - ROTOR SHAFT STIFFNESS , LB/IN           ",/,
      " DS - ABSOLUTE SHAFT DAMPING , LB-SEC/IN           ",/,
      " CI - INTERNAL FRICTION DAMPING , LB-SEC/IN            ",/,
      " QAC - CROSS-COUPLING, LB/IN            ",/,
      " EU - UNBALANCE ECCENTRICITY OF W IN MILS          ",/,
      "                                  ----          ",/,
      " WJ - JOURNAL WEIGHT AT EACH END , LB.          ",/,
      " W1 - SUPPORT WEIGHT AT EACH BEARING , LB.           ",/,
    //," CARD 3  READ(CR,/,KXX,KYY,CXX,CYY,KXY,KYX,CXY,CYX)        ",/,
      " BEARING STIFFNESS AND DAMPING OF EACH BEARING            ",/,
    //," CARD 4  IF N>8 THEN READ(CR,/,K1X,K1Y,C1X,C1Y)        ",/,
      " SUPPORT STIFFNESS AND DAMPING OF EACH SUPPORT             ",/,
    //," CARD 5 (4)  READ(CR,/,FOR I←0 STEP 1 UNTIL N DO[Y[0,I]])      ",/,
      " 0 IS FOR INITIAL TIME, RADIANS           ",/,
```

```
" 1 - ABSOLUTE ROTOR DISP.  , X-DIR.                  ",/,
" 2 - ABSOLUTE ROTOR VELOCITY, X-DIR.                  ",/,
" 3 - ABS. ROTOR DISP. , Y-DIR.                  ",/,
" 4 - ABS. ROTOR VEL. , Y-DIR.                  ",/,
" 5,6,7,8 -  SAME ORDER AS ABOVE FOR JOURNAL RELATIVE MOTION   ",/,
" 9,10,11,12 - SAME AS ABOVE FOR SUPPORT MOTION  (IF REQUIRED) ",/,
//," CARD 6 (5)READ(CR,/CS,ISCALE,XMIN,DX,XMIN2,DX2,XMIN3,DX3)     ",/,
" CS - PLOTTER CONTROL   0 - NO PLOT      1 - PLOT          ",/,
" ISCALE - SCALE CONTROL   0 - PROG. SCALE    1 - USE FOLLOWING INFO."
,/," XMIN - NO. TO APPEAR AT ORIGIN      ROTOR DISLPACEMENT         ",/,
" DX - SCALE INCREMENT PER INCH      ROTOR DISPLACEMENT          ",/,
" XMIN2,DX2 - SAME  AS ABOVE BUT FOR JOURNAL PLOT            ",/,
" XMIN3,DX3 - SAME AS ABOVE BUT FOR SUPPORT                  ",/,
//," CARD 7 (6)  IF CS≠0 THEN READ(CR,/,RPCS)                 ",/,
" RPCS - NO. OF TIMES INTEG. FOR TMAX RADIANS IS TO BE SOLVED   ",/,
//," EXAMPLE DATA INPUT:                ",/,
"12.56,0.05,8,608,7101,    ",/,
"10000,675,280000,0,0,0,0,5,312,50, ",/,
"351000,606000,739,865,  ",/,
"0,0,0,0,0,0,0,0,0,     ",/,
"1,0,0,0,0,0,0,0,   ",/,
"5, ",/,
//," NOTE ON PLOTTING:                ",/,
"       REQUEST ONE (1) BLOCK PER SET OF DATA               ",/,
" ---REPEAT THE SERIES FOR EACH CASE ---                "));
  WRITE(LP[PAGE]);
  WRITE(LP,<"     PLOTTER OUTPUT INFORMATION AND SUGGESTIONS",///,
"MAKE TMAX A MULTIPLE OF 6.28 (BUT LESS THAN 75 IF H=0.05)    ",/,
"A REASONABLE VALUE OF H IS 0.05 WHICH GIVES 125 STEPS PER ",/,
"CYCLE OF RUNNING SPEED FOR THE INTEGRATION.",///,
"TWO (2) MINUTES PROCESSOR TIME IS REQUIRED FOR 10 CYCLES OF ",/,
"SOLUTION FOR N = 8. ",/,
"THREE (3) MINUTES PROCESSOR TIME IS REQUIRED FOR 10 CYCLES OF ",/,
"SOLUTION FOR N = 12. ",/,
/,"A SMALL CIRCLE APPEARS ON THE ORBIT EVERY 6.28 RADIANS OF ",/,
"SOLUTION AND IS EQUIVALENT TO A KEY PHASOR MARK ON A CRO TRACE",/,
"THIS IS TRUE ONLY WHEN TMAX IS A MULTIPLE OF 6.28.",/,
"A PLUS SIGN APPEARS AT THE POINT THE SOLUTION IS INITIALLY STARTED
",/,"OR CONTINUED WITH RPCS > 1. ",///,
"WHEN N = 12, THE ABSOLUTE JOURNAL MOTION APPEARS AS A DASHED",/,
"LINE AND THE RELATIVE MOTION APPEARS AS A SOLID LINE. ",///,
"THE CROSS COUPLING TERMS FOR THE BEARINGS ARE NOT PRINTED OUT",/,
"ON THE PLOTTER OUTPUT BUT THEY DO APPEAR ON THE LP OUTPUT. ",/,
"AS A SUGGESTION YOU COULD PUT A NEGATIVE CASE NUMBER WHEN AND ",/,
"IF THE CROSS COUPLING TERMS FOR THE BEARINGS ARE NOT ZERO .",/,
"THIS WOULD INDICATE TO LOOK AT THE LP I=0 FOR THE VALUES">);
  ACARD :
  FMSHH ← 0.0;  FMBHH ← 0.0;
  VV ← 1 ;
  RPC  ← 1;
  READ (CR,/,TMAX,H,N,CASNO)[ALLDONE];
  READ(CR,/,RPM,W,K,DS,CI,QAC,EU,WJ,W1);
  READ(CR,/,KXX,KYY,CXX,CYY,KXY,KYX,CXY,CYX);
  IF N > 8 THEN READ(CR,/,K1X,K1Y,C1X,C1Y);
   READ(CR,/,FOR I←0 STEP 1 UNTIL N DO [Y[0,I]]);
  WRITE(LP[PAGE]);
```

```
WRITE(LP,<"INITIAL CONDITIONS">);
WRITE(LP,<F11.6>,FOR I ← 0 STEP 1 UNTIL N  DO[Y[0,I]]);
READ (CR,/,CS,ISCALE,XMIN,DX,XMIN2,DX2,XMIN3,DX3);
IF CS ≠ 0 THEN
READ(CR,/,RPCS);
RAD ← RPM × 0.1047 ;
M ← W / 386 ;
M1 ← W1/386 ;
MJ ← WJ/386.0;
ZZ ← M× RAD;
ZZ2 ← ZZ×RAD;
Z1 ← DS/ZZ; Z2←CI/ZZ; Z3←K/ZZ2; Z4←QAC/ZZ2;
Z5← M/MJ; Z6←(CI+2×CXX)/(2×ZZ); Z22←Z2/2;
Z7←(2×KXX+K)/(2×ZZ2); Z32←Z3/2;
Z8←(CI+2×CYY)/(2×ZZ); Z9←(2×KYY+K)/(2×ZZ2);
IF N > 8 THEN
Z10← M/M1; Z11←K1X/ZZ2; Z12←C1X/ZZ; Z13←KXX/ZZ2;
Z14←CXX/ZZ; Z15←K1Y/ZZ2; Z16←C1Y/ZZ;
Z17←KYY/ZZ2; Z18←CYY/ZZ;
Z19←KXY/ZZ2; Z20←CXY/ZZ; Z21←KYX/ZZ2; Z23←CYX/ZZ;
K2X ← (2×KXX×K×(K+2×KXX)+4×K×(RAD×CXX)*2)/((K+2×KXX)*2 +
4×(RAD×CXX)*2);
K2Y ← (2×KYY×K×(K+2×KYY) + 4×K×(RAD×CYY)*2)/ ((K+2×KYY)*2 +
4×(RAD×CYY)*2);
C2X ← 2×K×K×CXX/((2×KXX+K)*2 + 4×(RAD×CXX)*2) + DS ;
C2Y ← 2×K×K×CYY/((2×KYY+K)*2 + 4×(RAD×CYY)*2) + DS ;
WCX ← SQRT(2×K×KXX/((2×KXX+K)×M))    ;
WCY ← SQRT(2×K×KYY/((2×KYY+K)×M)) ;
ACX ← K2X/(WCX×C2X) ;
ACY ← K2Y/(WCY×C2Y) ;
WCX ← WCX/0.1047;    WCY ← WCY/0.1047;
TMAXH ← TMAX ;
TMAX ← TMAX + Y[0,0];
REPEAT: RPC ← RPC + 1;
TIMESTEP (TMAX, H, N, AY,NA,Y);
WRITE (LP[PAGE]);
 WRITE (LP,<X35,"ORBITAL MOTION OF THE SINGLE MASS UNBALANCED ",
"ROTOR",///>);
WRITE(LP,<X2,"CASE NO.",F11.4,//,X2,"ROTOR WEIGHT =",
F9.3," LB.",X12,"ROTOR SPEED =",F10.2," RPM",/,X2,
"ROTOR STIFFNESS =",F11.4," LB/MIL.",X3,
"UNBALANCE =",F8.3," MILS",/,X2,
"SHAFT DAMPING =",F8.3," LB-SEC/IN", X6,
"INTERNAL DAMPING =",F9.3," LB-SEC/IN",/,X2,
"CROSS COUPLING =",F9.2, " LB/IN",//,X2,
"BEARING STIFFNESS",X11,"BEARING DAMPING",/,X3,
"KXX =",F10.3," LB/MIL",X6,"CXX =",F10.3," LB-SEC/IN",/,X3,
"KYY =",F10.3," LB/MIL",X6,"CYY =",F10.3," LB-SEC/IN">,
CASNO,W,RPM,K/1000.0,EU,DS,CI,QAC,KXX/1000.0,CXX,KYY/1000.0,CYY);
WRITE(LP,<X3,"KXY =",F10.3," LB/MIL",X6,"CXY =",F10.3," LB-SEC/IN",
/,X3,        "KYX =",F10.3," LB/MIL",X6,"CYX =",F10.3," LB-SEC/IN">
,KXY/1000.0,CXY,KYX/1000.0,CYX);
WRITE(LP,<X3,"WEIGHT OF EACH BEARING =",F9.3," LB.",//>,WJ);
IF N > 8 THEN
WRITE(LP,<X2,"SUPPORT STIFFNESS",X11,"SUPPORT DAMPING",/,X3,
"K1X =",F10.3," LB/MIL",X6,"C1X =",F10.3," LB-SEC/IN",/,X3,
```

```
    "K1Y =",F10.3," LB/MIL",X6,"C1Y =",F10.3," LB-SEC/IN",//>,
    K1X/1000.0,C1X,K1Y/1000.0,C1Y);
    IF N > 8 THEN  WRITE(LP,<X3,"WEIGHT OF EACH SUPPORT =",F9.3,
    " LB.",//>,W1);
    WRITE(LP,<X4,"THAXH=",F8.2,/,X7,"N =",I3,/,X7,"H =",F8.3,/,X4,
    "RPCS =",I4>,THAXH,N,H,RPCS);

    IF CS ≠ 0 THEN
BEGIN
    IF VV = 1 THEN   BEGIN
    PLOT (2,0,-4);  PLOT (2,0,-5);
    FILL ALPHA1[*] WITH "Y-DIR,","  [MIL","S]    ";
    FILL ALPHA2[*] WITH "X-DIR,","  [MIL","S]    ";
    FILL ALP1 [*] WITH "ABSOLU","TE ROT","OR MOT","ION    ";
    FILL ALP2 [*] WITH "JOURNA","L MOTI","ON    ";
    FILL ALP3 [*] WITH "SUPPOR","T MOTI","ON    ";
    FILL ALP4 [*] WITH "N =   ","      ","RPM   ";
    FILL ALP5 [*] WITH "W2 =  ","      ","  LB. ";
    FILL ALP6 [*] WITH "EU =  ","      ","   MIL","S     ";
    FILL ALP7 [*] WITH "KS =  ","      ","   LB/","MIL    ",
    FILL ALP8 [*] WITH "WJ =  ","      ","  LB. ";
    FILL ALP9 [*] WITH "CS =  ","      ","  LB-S","EC/IN ";
    FILL ALP10[*] WITH "KXX = ","      ","    LB","/MIL  ";
    FILL ALP11[*] WITH "CI =  ","      ","  LB-S","EC/IN ";
    FILL ALP12[*] WITH "KYY = ","      ","    LB","/MIL  ";
    FILL ALP13[*] WITH "Q =   ","      "," LB/IN";
    FILL ALP14[*] WITH "CXX = ","      ","   LB-","SEC/IN";
    FILL ALP15[*] WITH "WCX = ","      ","    RP","M     ";
    FILL ALP16[*] WITH "CYY = ","      ","   LB-","SEC/IN";
    FILL ALP17[*] WITH "WCY = ","      ","    RP","M     ";
    FILL ALP18[*] WITH "ACX = ";
    FILL ALP20[*] WITH "ACY = ";
    IF N > 8 THEN BEGIN
    FILL ALP19[*] WITH "W1 =  ","      ","  LB  ";
    FILL ALP21[*] WITH "K1X = ","      ","   LB/","MIL   ";
    FILL ALP22[*] WITH "C1X = ","      ","   LB-","SEC/IN";
    FILL ALP23[*] WITH "K1Y = ","      ","   LB/","MIL   ";
    FILL ALP24[*] WITH "C1Y = ","      ","   LB-","SEC/IN";
    END;
    FILL ALP25[*] WITH "CYCLES","      ","     T","HROUGH";
    FILL ALP26[*] WITH "FU =  ","      ","   LB.","      ";
    FILL ALP27[*] WITH "TRDB =","            ","    ( ","      )";
    FILL ALP28[*] WITH "TRDS =","            ","    ( ","      )";
    FILL ALP29[*] WITH "FB =  ","      ","   LB.";
    FILL ALP30[*] WITH "NO,   ";
    END ;
    VV ← VV + 1 ;
    BEGIN
    REAL KIRKRG ;
    FOR I ← 0 STEP 1 UNTIL N  DO
    Y[0,I] ← AY[I,NA] ;
    WRITE(LP,<X2,"END CONDITIONS FOR RPC =",I3>,RPC-1);
    WRITE(LP,<F11.6>,FOR I ← 0 STEP 1 UNTIL N  DO[Y[0,I]]);
    IF N > 8 THEN BEGIN
    FOR I ← 1 STEP 1 UNTIL NA DO  BEGIN
    BXR[I] ← AY[5,I]+AY[9,I];
```

```
BYR[I] ← AY[7,I]+AY[11,I];
END;
END;
IF ISCALE THEN   BEGIN
IF VV ≤ 2 THEN BEGIN
XMAX ← YMAX ← 3 × DX ;
YMIN ← XMIN ;        DY ← DX ;
XMAX2 ← YMAX2 ← 3×DX2;
XMAX3 ← YMAX3 ← 3×DX3;
YMIN2 ← XMIN2 ; DY2 ← DX2;
YMIN3 ← XMIN3; DY3 ← DX3;
END;
PLOTCHEK(AY[1,*],AY[3,*],NA,XMIN,XMAX,YMIN,YMAX);
SCALES (AY[3,*],NA,YMIN,DY,CS);
SCALES (AY[1,*],NA,XMIN,DX,CS);
PLOTCHEK(AY[5,*],AY[7,*],NA,XMIN2,XMAX2,YMIN2,YMAX2);
SCALES(AY[5,*],NA,XMIN2,DX2,CS);
SCALES(AY[7,*],NA,YMIN2,DY2,CS);
IF N > 8 THEN BEGIN
I ← 11;
PLOTCHEK(AY[9,*],AY[I,*],NA,XMIN3,XMAX3,YMIN3,YMAX3);
SCALES(AY[I,*],NA,YMIN3,DY3,CS);
SCALES(AY[9,*],NA,XMIN3,DX3,CS);
PLOTCHEK(BXR,BYR,NA,XMIN2,XMAX2,YMIN2,YMAX2);
SCALES(BYR,NA,YMIN2,DY2,CS);
SCALES(BXR,NA,XMIN2,DX2,CS);
END;
END
ELSE
BEGIN
SAMESCALE(AY[1,*],AY[3,*],NA,XMIN,XMAX,DX,YMIN,YMAX,DY);
PLOTCHEK(AY[1,*],AY[3,*],NA,XMIN,XMAX,YMIN,YMAX);
SAMESCALE(AY[5,*],AY[7,*],NA,XMIN2,XMAX2,DX2,YMIN2,YMAX2,DY2);
PLOTCHEK(AY[5,*],AY[7,*],NA,XMIN2,XMAX2,YMIN2,YMAX2);
SCALES (AY[3,*],NA,YMIN,DY,CS);
SCALES (AY[1,*],NA,XMIN,DX,CS);
SCALES(AY[7,*],NA,YMIN2,DY2,CS);
SCALES(AY[5,*],NA,XMIN2,DX2,CS);
IF N > 8 THEN BEGIN
PLOTCHEK(BXR,BYR,NA,XMIN2,XMAX2,YMIN2,YMAX2);
SCALES(BXR,NA,XMIN2,DX2,CS);
SCALES(BYR,NA,YMIN2,DY2,CS);
I ← 11;
SAMESCALE(AY[9,*],AY[I,*],NA,XMIN3,XMAX3,DX3,YMIN3,YMAX3,DY3);
PLOTCHEK(AY[9,*],AY[I,*],NA,XMIN3,XMAX3,YMIN3,YMAX3);
SCALES(AY[I,*],NA,YMIN3,DY3,CS);
SCALES(AY[9,*],NA,XMIN3,DX3,CS);
END;
END ;
IF VV ≤ 2 THEN
BEGIN
ORBITTOP(1);
AGRID(XMIN,DX,YMIN,DY);
END;
SYMBOL(AY[1,1],AY[3,1],.14,ALPHA1,0,-13);
XA ← 6.28/H;
```

```
FOR I ← XA STEP XA UNTIL NA DO
SYMBOL(AY[1,I],AY[3,I],0.07,ALPHA1,0,-5);
LYNE(AY[1,*],AY[3,*],NA,CS);
PLOT(12,0,-5);
IF VV ≤ 2 THEN
BEGIN
ORBITTOP(2);
AGRID(XMIN2,DX2,YMIN2,DY2);
END;
SYMBOL(AY[5,1],AY[7,1],0.14,ALPHA1,0,-13);
FOR I ← XA STEP XA UNTIL NA DO
SYMBOL(AY[5,I],AY[7,I],0.07,ALPHA1,0,-5);
LYNE(AY[5,*],AY[7,*],NA,CS);
IF N > 8 THEN BEGIN
SYMBOL(BXR[1],BYR[1],0.14,ALPHA1,0,-13);
FOR I ← XA STEP XA UNTIL NA DO
SYMBOL(BXR[I],BYR[I],0.07,ALPHA1,0,-5);
DASHLINE(BXR,BYR,NA,CS);
END;
PLOT(12,0,-5);
IF N > 8 THEN BEGIN
IF VV ≤ 2 THEN
BEGIN
ORBITTOP(3);
AGRID(XMIN3,DX3,YMIN3,DY3);
END;
J ← 11;
SYMBOL(AY[9,1],AY[J,1],0.14,ALPHA1,0,-13);
FOR I ← XA STEP XA UNTIL NA DO
SYMBOL(AY[9,I],AY[J,I],0.07,ALPHA1,0,-5);
LYNE(AY[9,*],AY[J,*],NA,CS);
END;

END;
WRITE(LP[DBL]);
WRITE(LP,<"TRDB-MAX = ",F11.4," AND OCCURS AT ", F7.2,
" CYCLES">,IF EU=0.0 THEN SQRT(FMBH)×2.0 ELSE SQRT(FMBH)×2.0/EU,
TIMHB/6.28);
IF N>8 THEN
WRITE(LP,<"TRDS-MAX = ",F11.4, "AND OCCURS AT ",F7.2,
" CYCLES">,IF EU=0.0 THEN SQRT(FMSH)×2.0 ELSE SQRT(FMSH)×2.0/EU,
TIMHS/6.28);
WRITE(LP,<"UNBALANCE STIFFNESS = ",F11.2, " LB/MIL",X2,
"AND UNBALANCE = ",F8.3," MILS">,ZZ2/1000.0,EU);
IF FMBH > FMBHH THEN BEGIN
FMBHH ← FMBH;  TIMHBH ← TIMHB;
END;
IF FMSH > FMSHH THEN BEGIN
FMSHH ← FMSH;  TIMHSH ← TIMHS;
END;
IF RPC ≤ RPCS THEN
BEGIN
PLOT(-24,0,-5);
TMAX ← TMAX + TMAXH;
ISCALE ← TRUE;
GO TO REPEAT;
```

```
        END;
        IF EU = 0.0 THEN BEGIN
        FMBHH ← SQRT(FMBHH)×2.0;
        FMSHH ← SQRT(FMSHH)×2.0;
        END ELSE
        BEGIN
        FMBHH ← SQRT(FMBHH)×2.0/EU;
        FMSHH ← SQRT(FMSHH)×2.0/EU;
        END;
        TIMHBH ← TIMHBH/6.28;
        TIMHSH ← TIMHSH/6.28;
        PLOT(-24,0,-5);
        IF N=8 THEN BEGIN
        SYMBOL(0.75,6.50,0.14,ALP29,0,18);
   NUMBER(1.23,6.50,0.14,FMBHH×ZZ2×EU/1000,0,0,3);
        END;
        NUMBER(4.31,8.75,.14,AY[0,NA]/6.28,0,2);
        SYMBOL(3.75,6.50,.14,ALP26,0,21);
        NUMBER(4.23,6.50,.14,ZZ2/1000.0×EU,0,3);
        PLOT(12,0,-5);
        IF N=8 THEN BEGIN
        SYMBOL(0.75,6.50,0.14,ALP29,0,18);
   NUMBER(1.23,6.50,0.14,FMBHH×ZZ2×EU/1000,0,0,3);
        END;
        NUMBER(4.31,8.75,.14,AY[0,NA]/6.28,0,2);
        SYMBOL(3.75,6.50,.14,ALP27,0,24);NUMBER(4.35,6.50,.14,FMBHH,0,3);
        NUMBER(5.43,6.50,.14,TIMHBH,0,2);
        PLOT(12,0,-5);
        IF N>8 THEN BEGIN.
        NUMBER(4.31,8.75,.14,AY[0,NA]/6.28,0,2);
        SYMBOL(3.75,6.50,.14,ALP28,0,24);NUMBER(4.35,6.50,.14,FMSHH,0,3);
        NUMBER(5.43,6.50,.14,TIMHSH,0,2);
        END;
        PLOT(12,0,-5);
        GO TO ACARD ;
        ALLDONE :
        PLOT(1,0 -3);
      END;
        WRITE (LP[PAGE]);
        WRITE (LP,<"TOTAL PROCESSOR TIME = ", F6.2,X1,"MINUTES">,
        TIME(2) / 3600 ) ;
        WRITE (LP[PAGE ] , <"TOTAL I-O TIME = ", F6.2, X1, "MINUTES" > ,
        TIME(3) / 3600 ) ;
  END.
ARCTAN IS SEGMENT NUMBER 0080,PRT ADDRESS IS 0117
COS IS SEGMENT NUMBER 0081,PRT ADDRESS IS 0075
EXP IS SEGMENT NUMBER 0082,PRT ADDRESS IS 0072
LN IS SEGMENT NUMBER 0083,PRT ADDRESS IS 0071
SIN IS SEGMENT NUMBER 0084,PRT ADDRESS IS 0076
SQRT IS SEGMENT NUMBER 0085,PRT ADDRESS IS 0436
OUTPUT(W) IS SEGMENT NUMBER 0086,PRT ADDRESS IS 0440
BLOCK CONTROL IS SEGMENT NUMBER 0087,PRT ADDRESS IS 0005
INPUT(W) IS SEGMENT NUMBER 0088,PRT ADDRESS IS 0444
X TO THE I IS SEGMENT NUMBER 0089,PRT ADDRESS IS 0073
GO TO SOLVER  IS SEGMENT NUMBER 0090,PRT ADDRESS IS 0065
ALGOL WRITE   IS SEGMENT NUMBER 0091,PRT ADDRESS IS 0014
```

```
     ALGOL READ    IS SEGMENT NUMBER 0092,PRT ADDRESS IS 0015
     ALGOL SELECT  IS SEGMENT NUMBER 0093,PRT ADDRESS IS 0016
     FILE ATTRBUTS IS SEGMENT NUMBER 0094,PRT ADDRESS IS 0033
COMPILATION TIME(SECONDS):    PR =   57    I/O =  162
NUMBER OF ERRORS DETECTED = 000, LAST ERROR ON CARD #
NUMBER OF SEQUENCE ERRORS COUNTED =    0,
NUMBER OF SLOW WARNINGS =   0,
PRT SIZE= 334;    TOTAL SEGMENT SIZE= 3952 WORDS,
DISK STORAGE REQ,=  360 SEGS,;    NO, SEGS,=  95,
ESTIMATED CORE STORAGE REQUIREMENT = 15355 WORDS,
```

PLOTTER OUTPUT INFORMATION AND SUGGESTIONS

MAKE TMAX A MULTIPLE OF 6.28 (BUT LESS THAN 35 IF H=0.05)
A REASONABLE VALUE OF H IS 0.05 WHICH GIVES 125 STEPS PER
CYCLE OF RUNNING SPEED FOR THE INTEGRATION.

TWO (2) MINUTES PROCESSOR TIME IS REQUIRED FOR 10 CYCLES OF
SOLUTION FOR N = 8.
THREE (3) MINUTES PROCESSOR TIME IS REQUIRED FOR 10 CYCLES OF
SOLUTION FOR N = 12.

A SMALL CIRCLE APPEARS ON THE ORBIT EVERY 6.28 RADIANS OF
SOLUTION AND IS EQUIVALENT TO A KEY PHASOR MARK ON A CRO TRACE
THIS IS TRUE ONLY WHEN TMAX IS A MULTIPLE OF 6.28.
A PLUS SIGN APPEARS AT THE POINT THE SOLUTION IS INITIALLY STARTED
OR CONTINUED WITH RPCS > 1.

WHEN N = 12, THE ABSOLUTE JOURNAL MOTION APPEARS AS A DASHED
LINE AND THE RELATIVE MOTION APPEARS AS A SOLID LINE.

THE CROSS COUPLING TERMS FOR THE BEARINGS ARE NOT PRINTED OUT
ON THE PLOTTER OUTPUT BUT THEY DO APPEAR ON THE LP OUTPUT.
AS A SUGGESTION YOU COULD PUT A NEGATIVE CASE NUMBER WHEN AND
IF THE CROSS COUPLING TERMS FOR THE BEARINGS ARE NOT ZERO .
THIS WOULD INDICATE TO LOOK AT THE LP I=O FOR THE VALUES

ORBITAL MOTION OF THE SINGLE MASS UNBALANCED ROTOR

```
CASE NO.   707.7100

ROTOR WEIGHT =  675.000 LB.              ROTOR SPEED =  10000.00 RPM
ROTOR STIFFNESS =   280.0000 LB/MIL.     UNBALANCE =   0.500 MILS
SHAFT DAMPING =   0.000 LB-SEC/IN        INTERNAL DAMPING =  100.000 LB-SEC/IN
CROSS COUPLING =     0.00 LB/IN

BEARING STIFFNESS                 BEARING DAMPING
 KXX =   351.000 LB/MIL            CXX =   739.000 LB-SEC/IN
 KYY =   606.000 LB/MIL            CYY =   865.000 LB-SEC/IN
 KXY =     0.000 LB/MIL            CXY =     0.000 LB-SEC/IN
 KYX =     0.000 LB/MIL            CYX =     0.000 LB-SEC/IN
 WEIGHT OF EACH BEARING =  312.000 LB.

  TMAXH=   12.56
      N =  8
      H =   0.050
   RPCS =   5
END CONDITIONS FOR RPC =  1
75.400000
-0.758303
-0.497459
 1.748020
-0.[illegible]37636
 0.368998
-0.208504
 0.435189
 0.060081

TROB-MAX =       0.7590 AND OCCURS AT    11.82 CYCLES
UNBALANCE STIFFNESS =      1916.95 LB/MIL  AND UNBALANCE =    0.500 MILS

   INITIAL CONDITIONS
     62.800000
     -1.890000
      0.163980
     -0.801630
     -1.133000
     -0.353260
     -0.167480
      0.128290
     -0.107260
```

NOMENCLATURE

A_{cr}	Amplification factor at rigid support critical $= \frac{K_2}{\omega_c C_2}$ (DIM)
A_j	Complex bearing amplitude, in
A_1	Complex support amplitude, in
A_2	Complex rotor amplitude, in
C	Damping ratio $= C_1/C_2$ (DIM)
C_b	Bearing damping, lb-sec/in
C_c	Critical damping coefficient, lb-sec/in
C_i	Rotor internal damping, lb-sec/in
C_s	Absolute shaft damping, lb-sec/in
C_1	Support damping, lb-sec/in
C_2	Effective rotor-bearing damping, lb-sec/in
e_u	Rotor mass eccentricity, in
F_1	Force transmitted to foundation, lb
F_b	Force transmitted to bearing housing, lb
K	Stiffness ratio, K_1/K_2
K_b	Bearing stiffness, lb/in
K_s	Rotor-shaft stiffness, lb/in
K_1	Support stiffness, lb/in
K_2	Effective rotor-bearing stiffness, lb/in
M	Mass ratio, $= M_1/M_2$ (DIM)
M_1	Support mass, $lb\text{-}sec^2/in$
M_2	Rotor mass, $lb\text{-}sec^2/in$
N_c	Rotor critical speed, [RPM]
P	1st node point on response plot
Q	Rotor cross-coupling stiffness, lb/in
Q	2nd node on response plots

R_2	Rotor absolute displacement amplitude
T	Kinetic energy
TRD	Transmissibility = $F_1/(M_2 e_u \omega^2)$
V	Potential Energy
$\bar{V}$	Velocity, in/sec
W_1	Support weight, lb
$\bar{X}$	Defined as X^2
X_s	Shaft relative displacement in x - direction
X_1	Support displacement in x - direction
X_2	Rotor absolute displacement in x - direction
X_j	Journal relative displacement in x - direction
Y_s	Shaft relative displacement in y - direction
Y_1	Support displacement in y - direction
Y_2	Rotor absolute displacement in y - direction
Y_j	Journal relative displacement in y - direction
Z_s	Complex shaft relative amplitude, in.
Z_1	Complex support amplitude, in.
Z_2	Complex rotor amplitude, in.
Z_j	Complex journal amplitude, in.
α	Rotor angular acceleration, rad/sec^2
β_1	Phase angle of support motion relative to rotor unbalance, DEG
β_2	Phase angle of rotor motion relative to rotor unbalance, DEG
β_b	Phase angle of bearing motion relative to rotor unbalance, DEG
θ	Angular displacement, rad
γ	Defined as K/M (DIM)
ξ	Damping ration= c_1/c_c (DIM)
ϕ	Rotor absolute amplitude phase angle, deg.

Φ	Moment of inertia
χ	Optimum amplitude for tuned system
Ψ	Defined as Ω_1^2 or Ω_2^2 when calculating required damping at point P or Q respectively
ω	Rotor angular velocity, rad/sec
$\omega_{1,2}$	Rotor system critical speeds, rad/sec
ω_c	Rigid support critical speed, rad/sec
Ω_1, Ω_2	Speeds at which the node point P and Q occur on response plots

(AUTOMATIC PLOTTER NOMENCLATURE)

A	Amplification factor at rigid support critical (DIM)
CB	Bearing damping, lb-sec/in
CD	Shaft damping coefficient, lb-sec/in
DC	Internal damping, lb-sec/in
E	Rotor mass eccentricity, in
FTR@WC	Force transmitted at rigid support critical
N	Rotor speed, RPM
QAC	Aerodynamic cross-coupling coef., lb/in.
TRDB	Maximum bearing force transmitted (DIM)
TRDS	Maximum support force transmitted (DIM)
FU*	Rotating unbalance load per mil unbalance eccentricity, lb
W	Rotor speed, rad/sec
WC	Rigid support critical speed, rad/sec.

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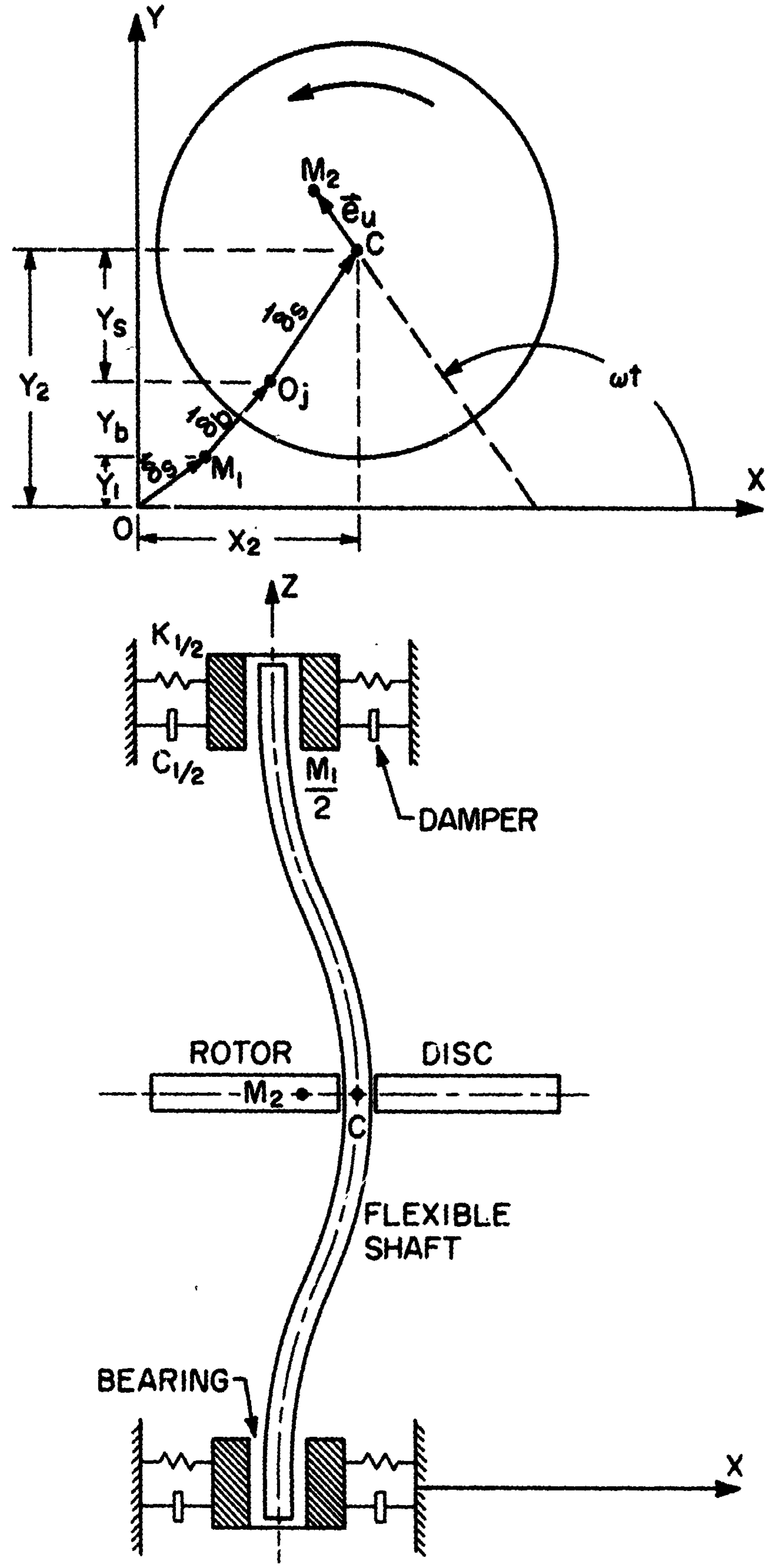

Figure 1 Schumatic Diagram of a Single Mass Rotor on Damped Elastic Supports

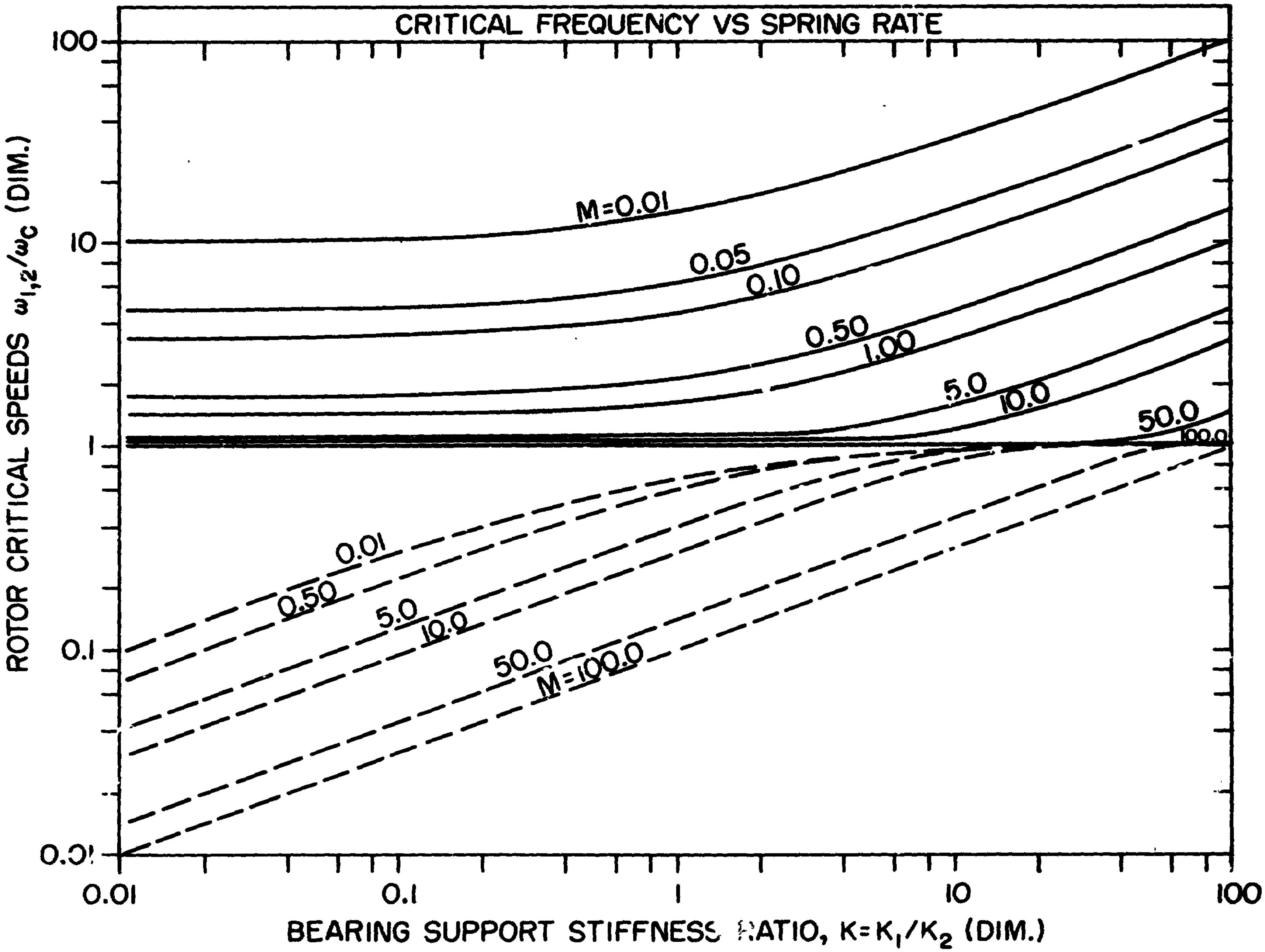

Figure 2 Dimensionless Critical Speeds vs. Support Stiffness Ratio for Various Support Housing Mass Ratios

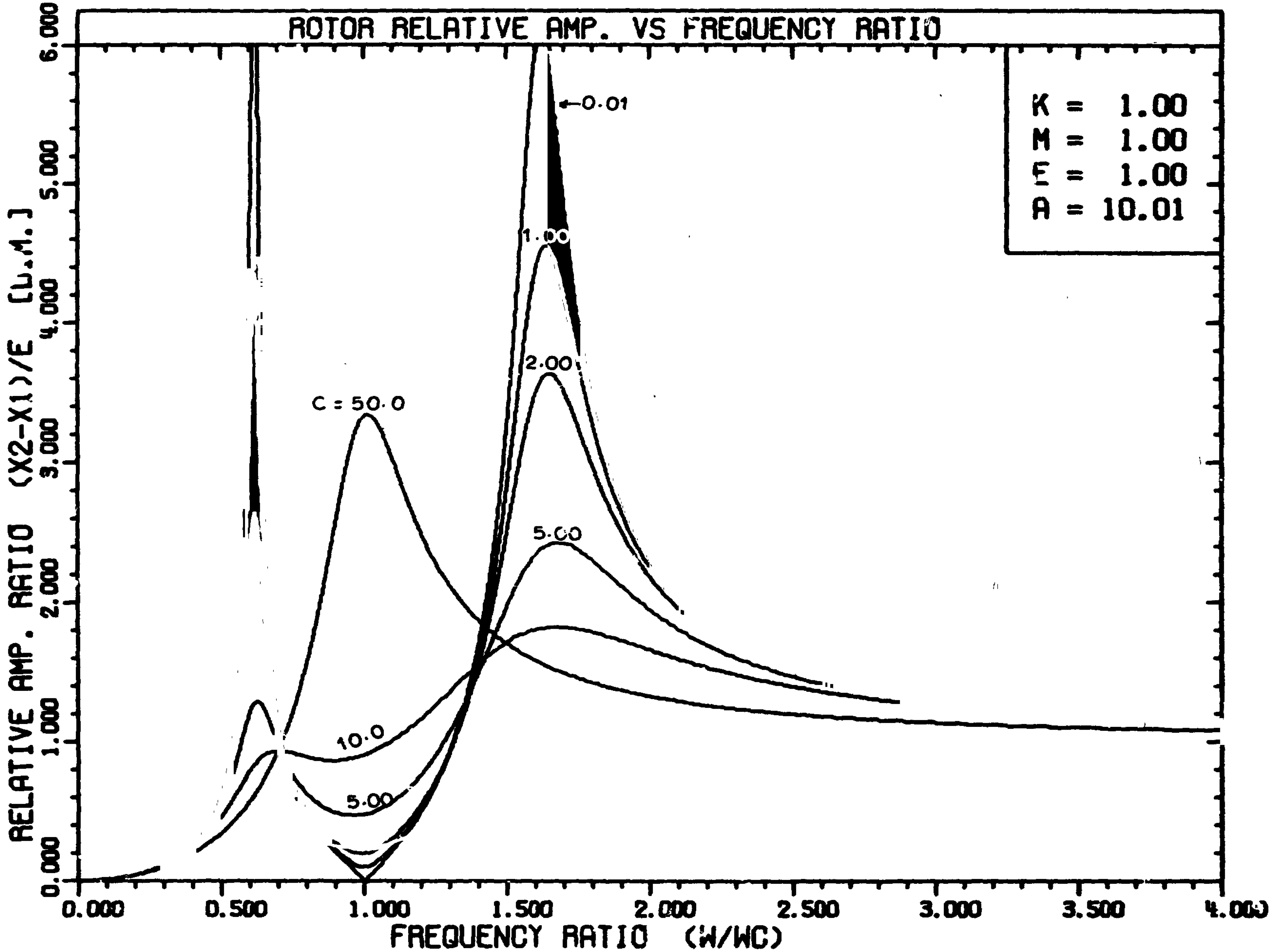

Figure 3. Dimensionless Relative Rotor Amplitude Vs Speed Ratio for Various Values of Support Damping for a Tuned Support System, K = M = 1

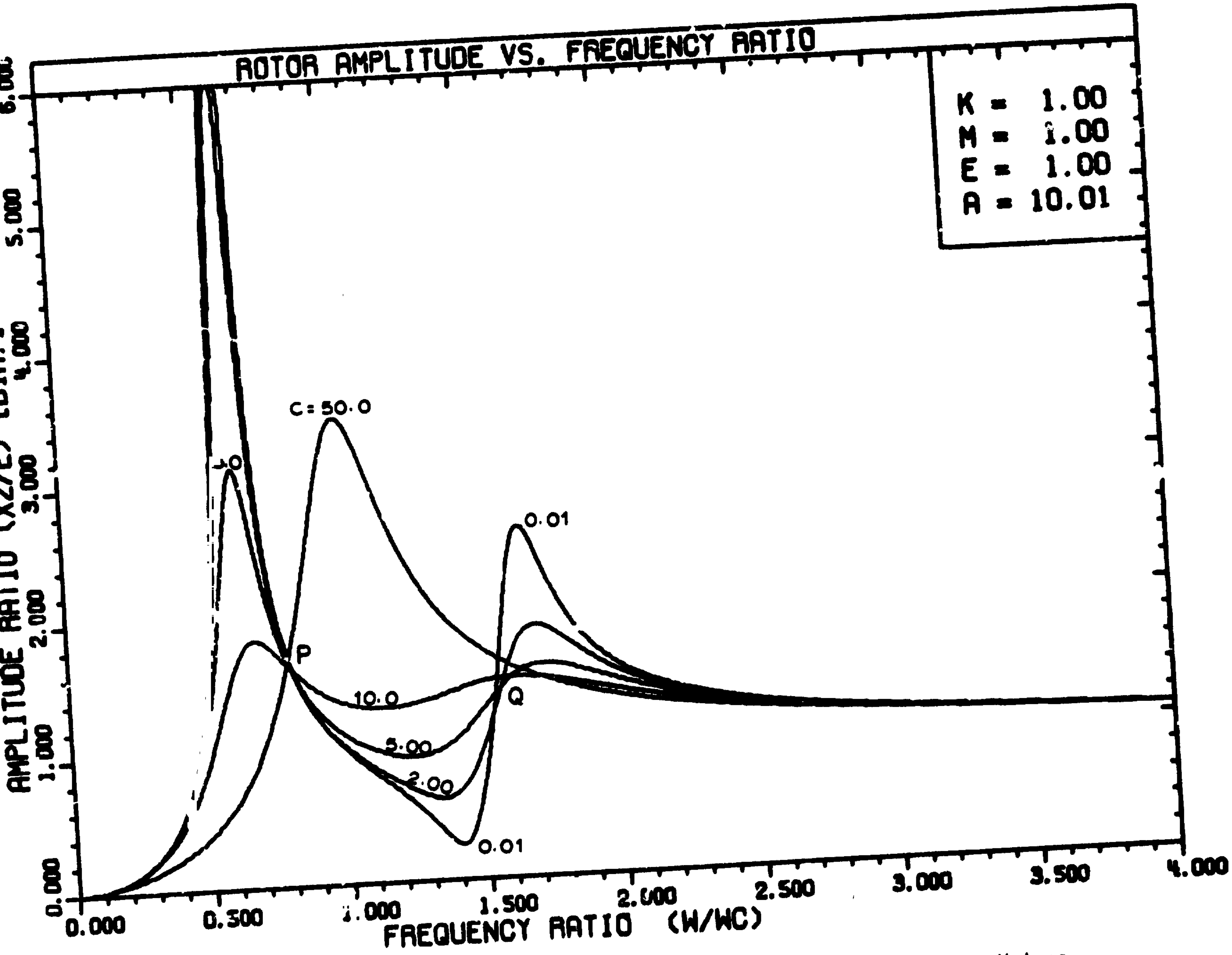

Figure 4. Absolute Rotor Motion with a Tuned Support System for Various Values of Support Damping, K = M = 1, A = 10

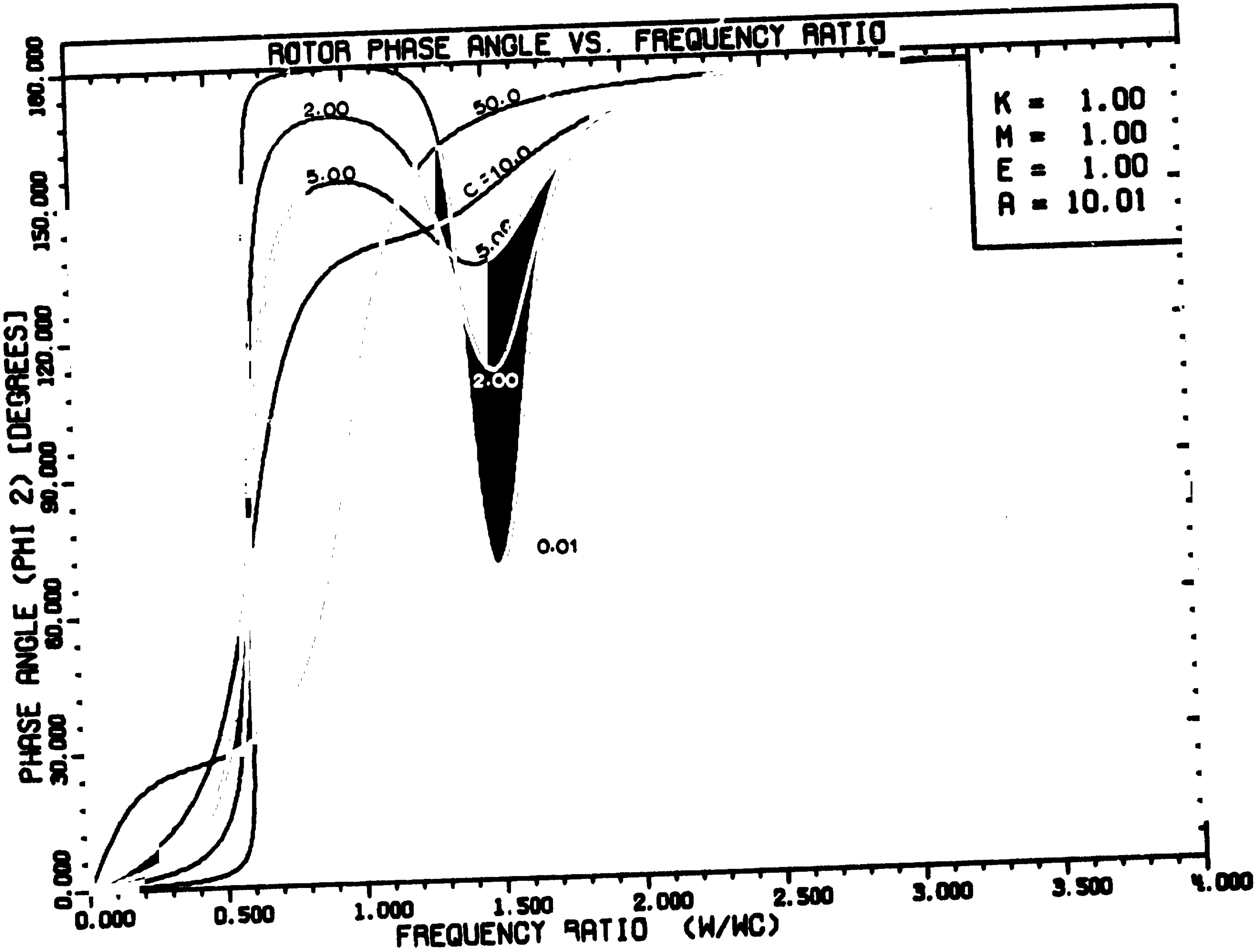

Figure 5. Phase Angle of Absolute Rotor Motion Relative to Unbalance for Various Values of Support Damping

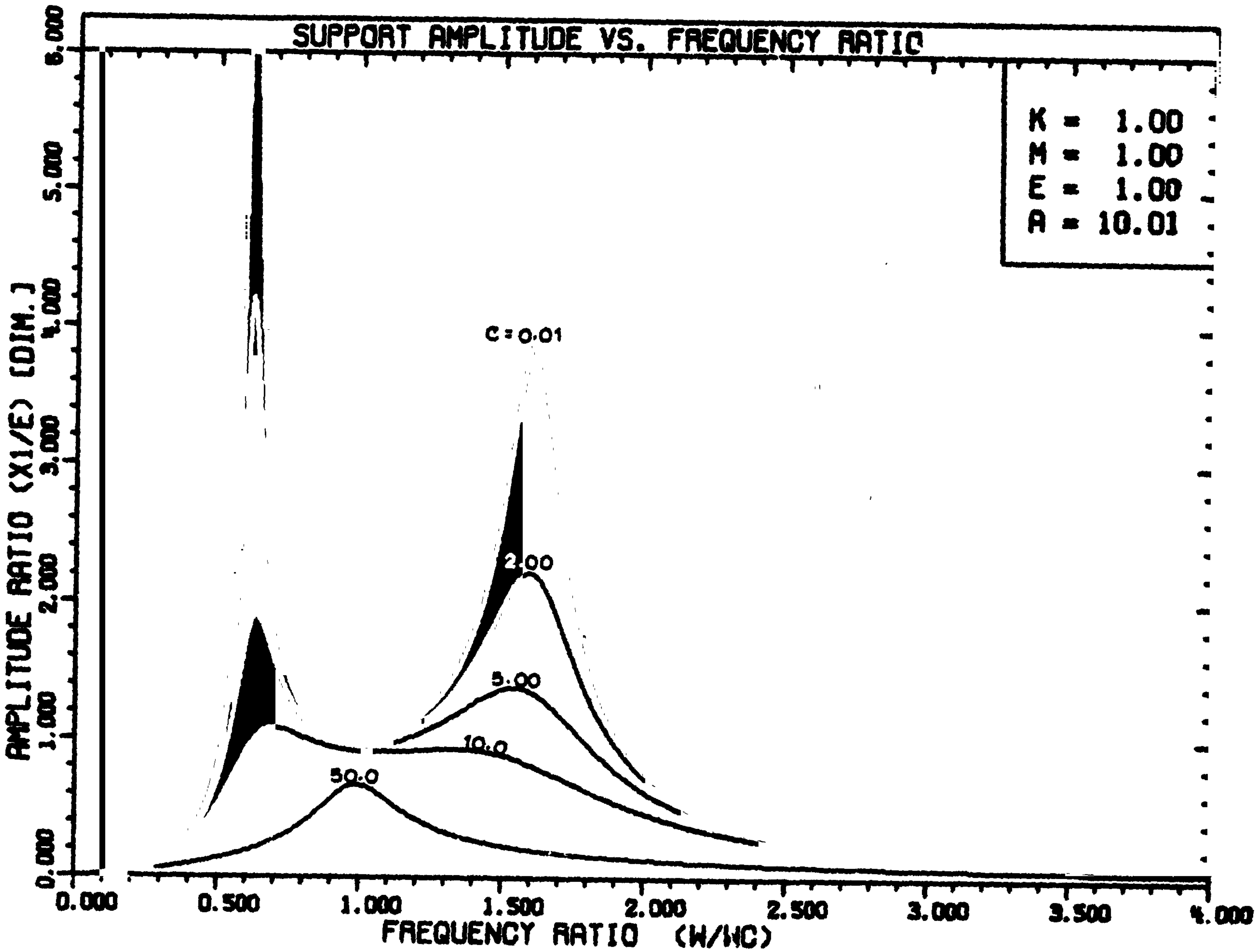

Figure 6. Support Amplitude Vs Speed for Various Values of Support Damping

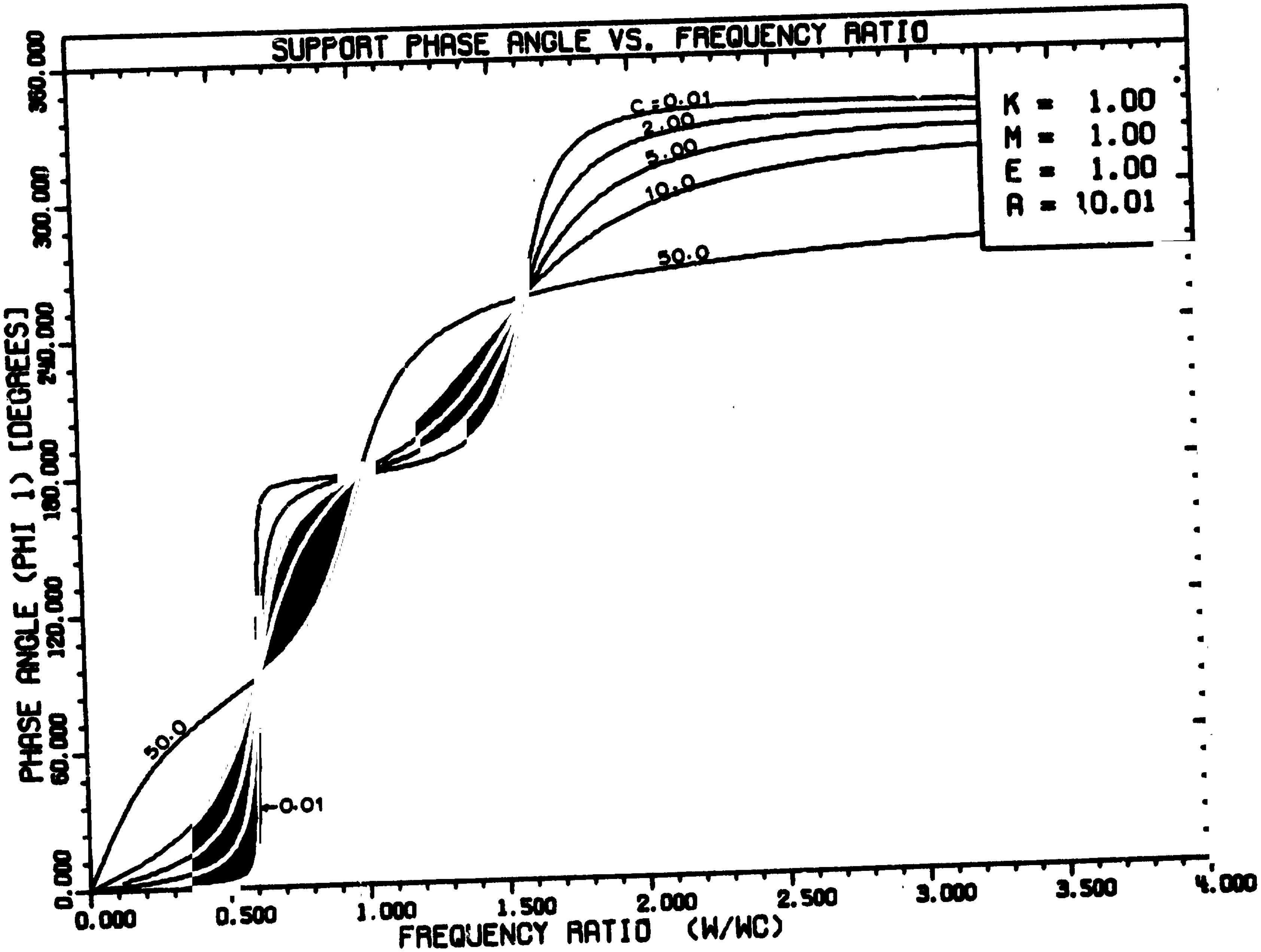

Figure 7. Phase Angle of Support Motion Relative to Rotor Unbalance for Various Values of Support Damping

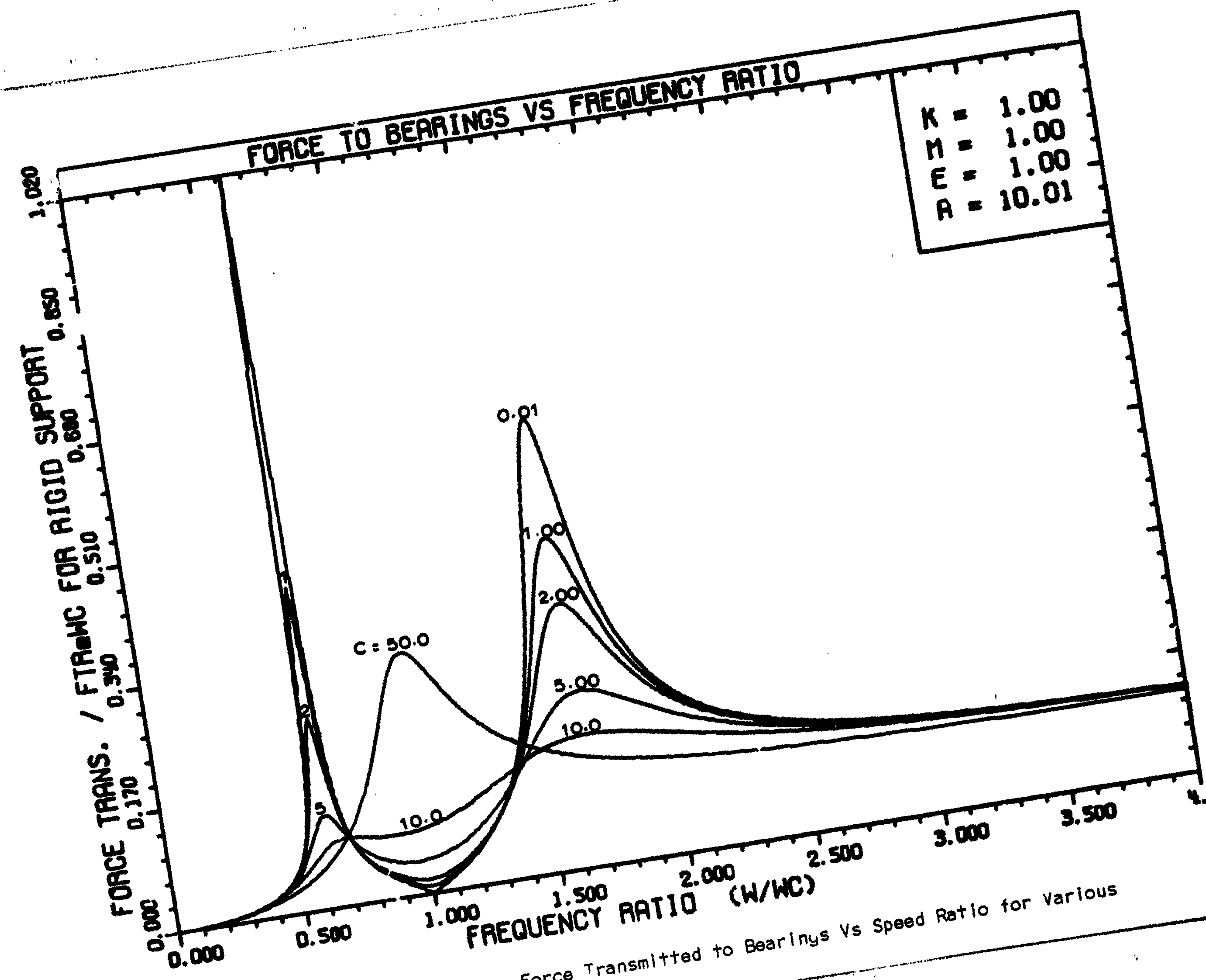

Figure 8. Dimensionless Force Transmitted to Bearings Vs Speed Ratio for Various Values of Support Damping

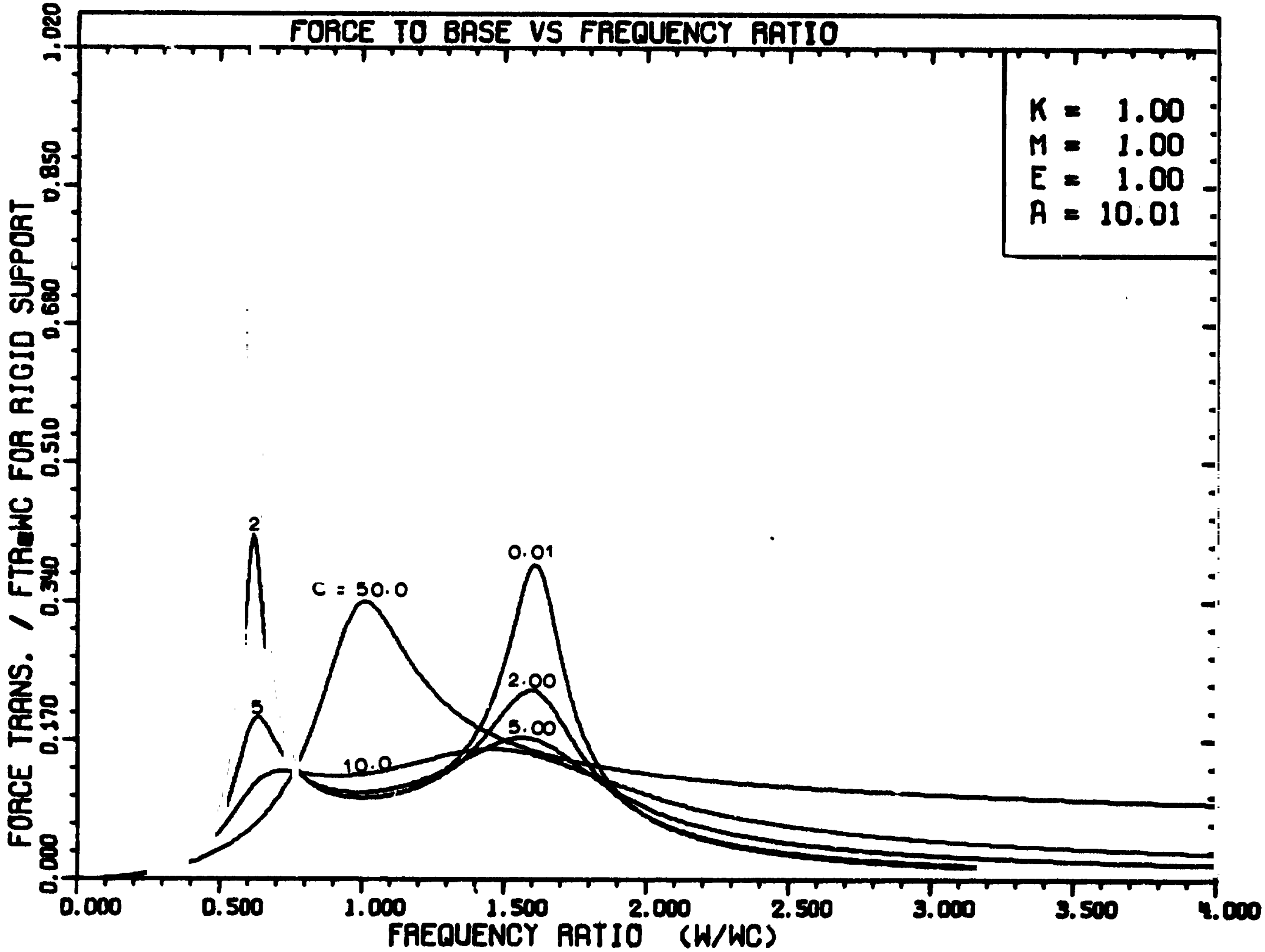

Figure 9. Dimensionless Force Transmitted to Foundation Vs Speed Ratio for Various Values of Support Damping

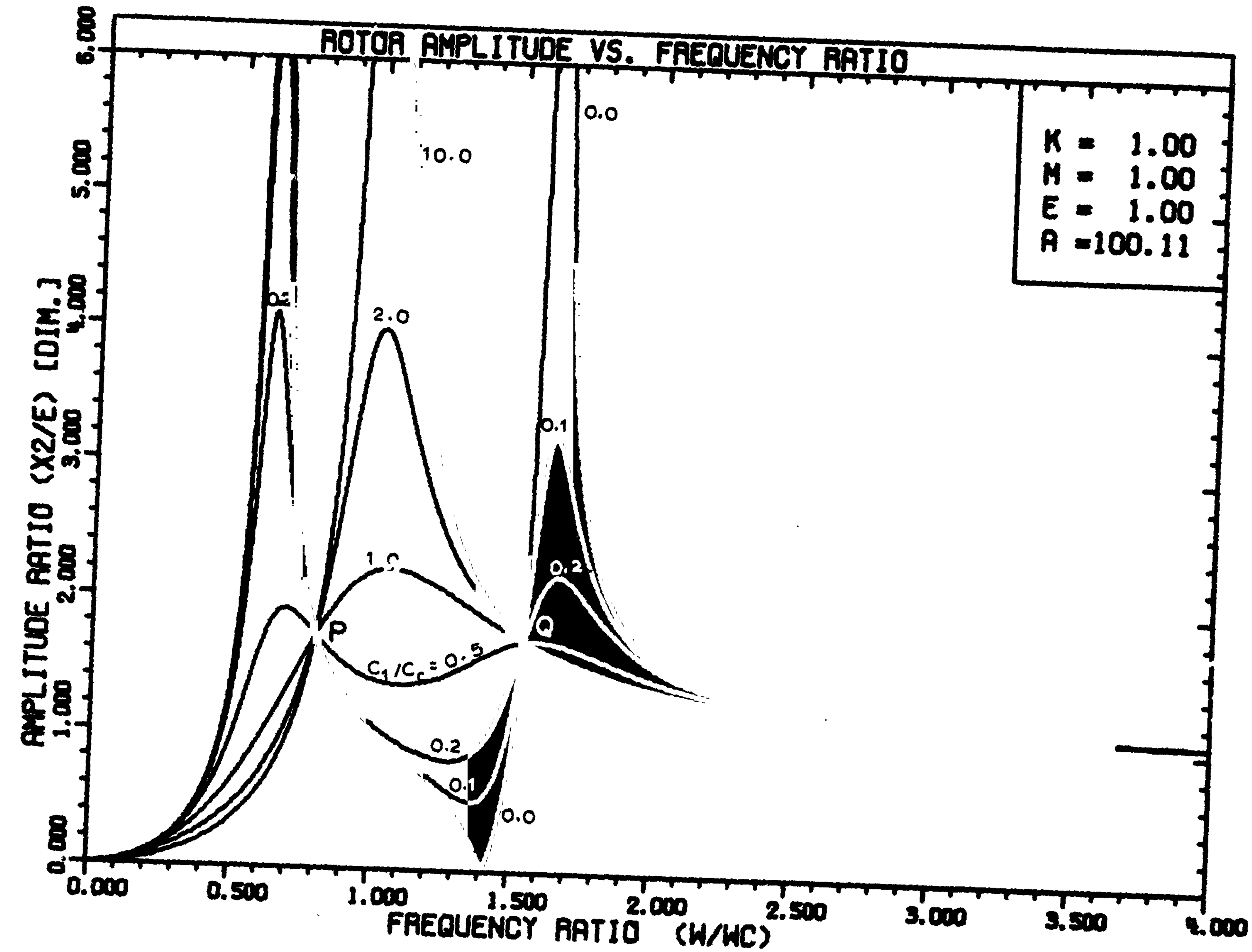

Figure 10. Amplitude of Motion Vs Speed with Light Rotor Damping [A = 100] for Various Values of Support Damping, K = M = 1

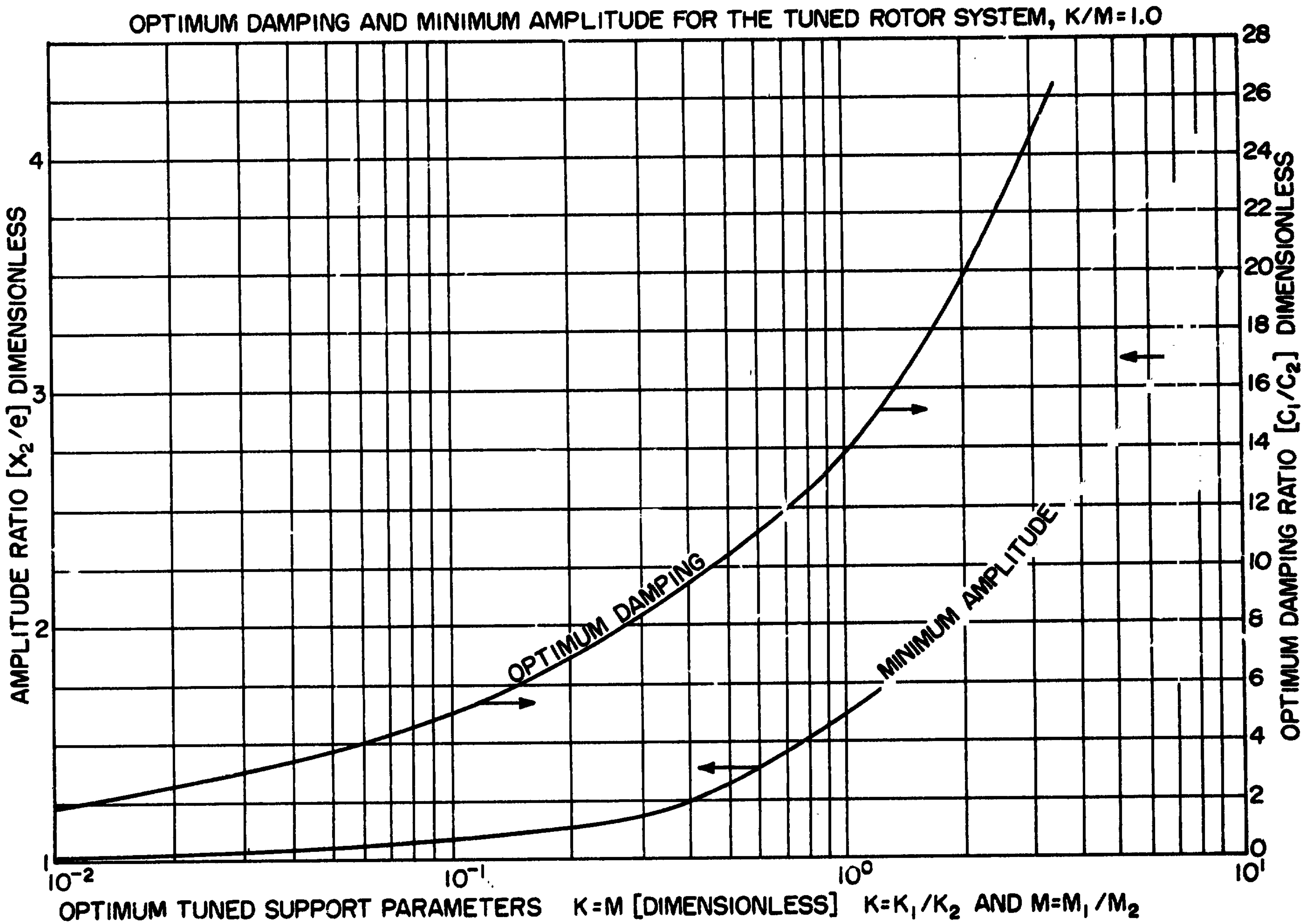

Figure 11. Optimum Support Damping and Maximum Rotor Amplitude Vs Mass Ratio for A = 10

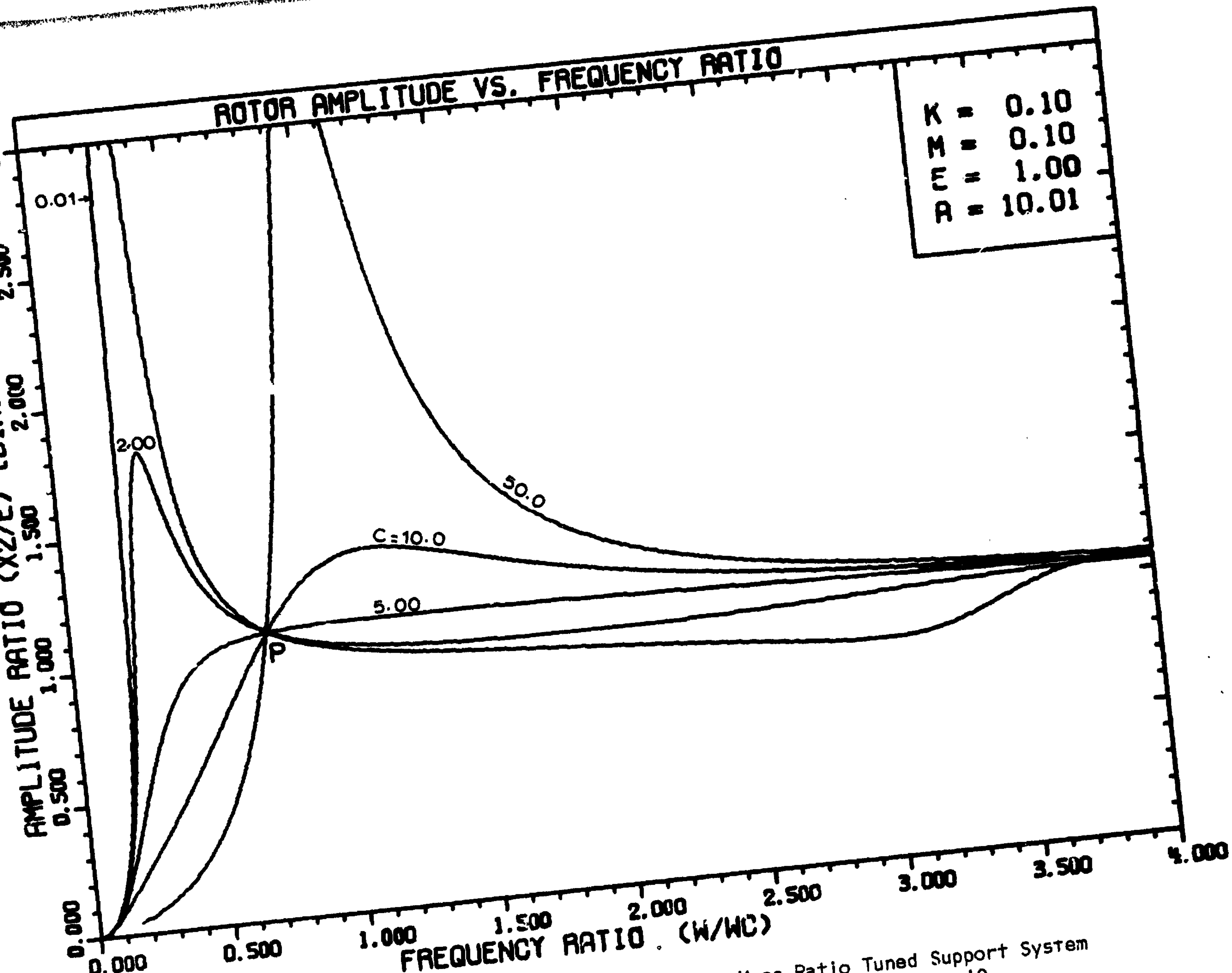

Figure 12. Rotor Amplitude Vs Speed for a Low Mass Ratio Tuned Support System for Various Values of Support Damping K = M = 0.1, A = 10

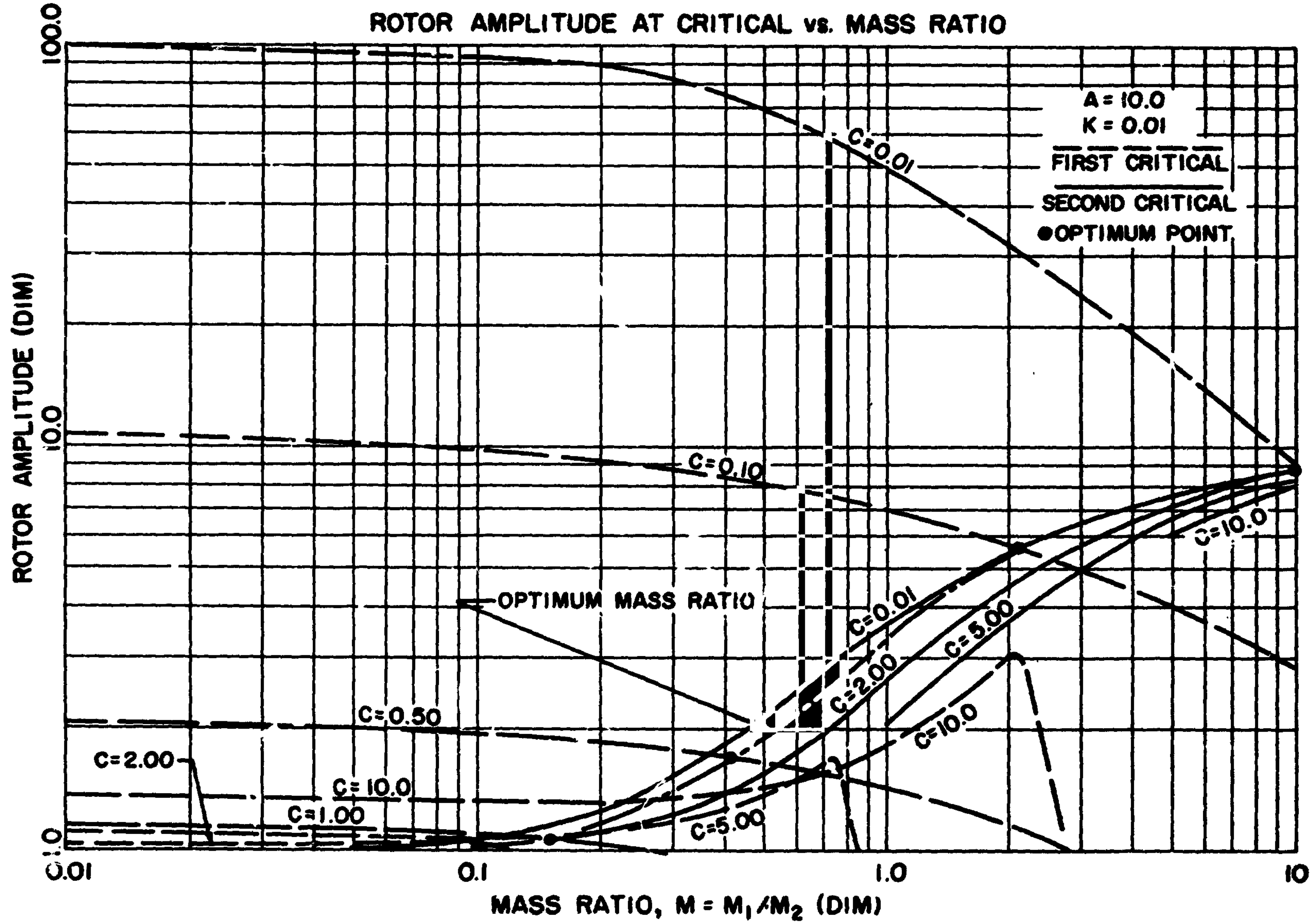

Figure 13 Rotor Amplitude at Critical Speeds vs. Mass Ratio Various Values for Various Damping Ratios, A = 10, K = .01

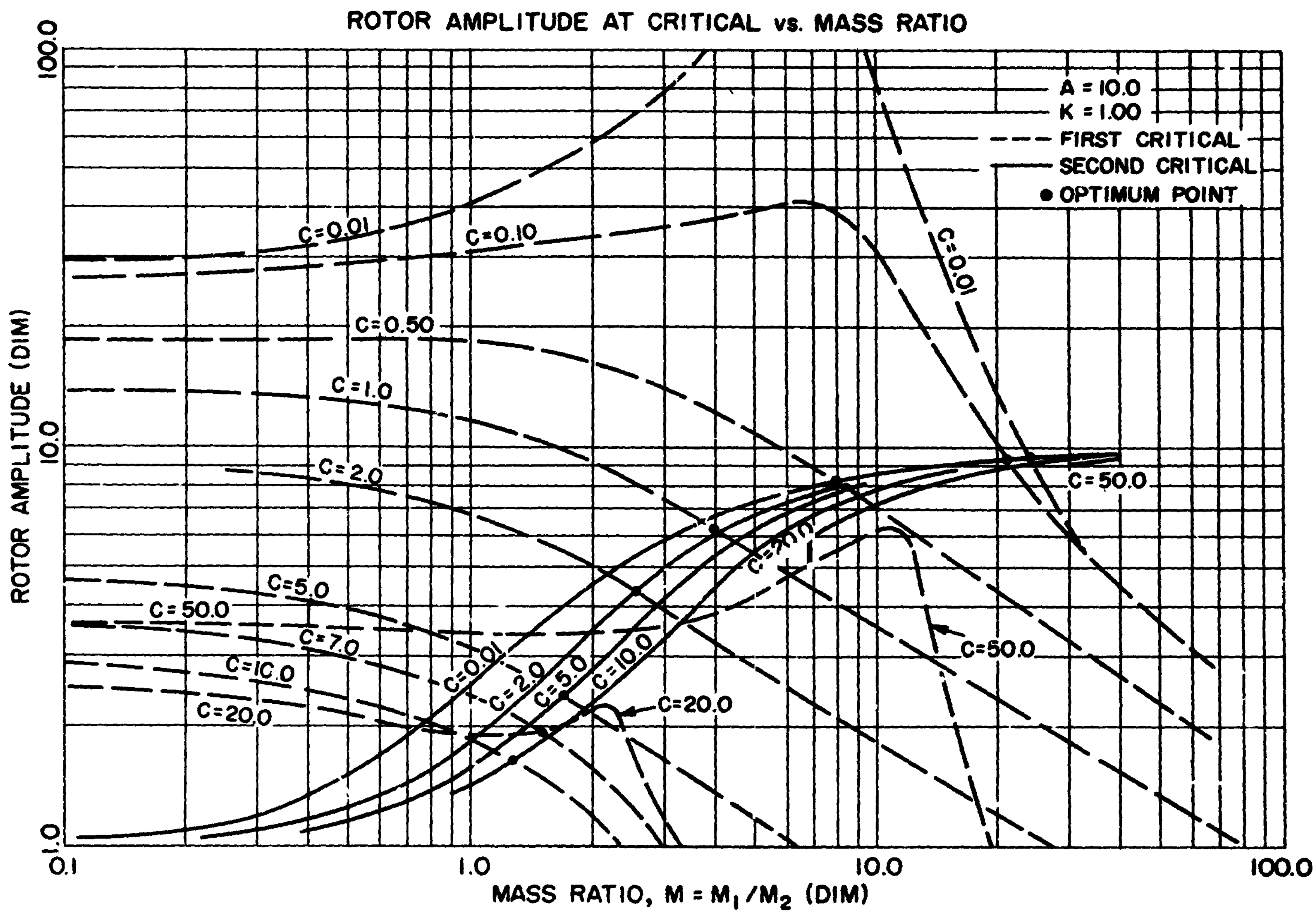

Figure 14   Rotor Amplitude at Critical Speeds Vs Mass Ratio for Various Damping Ratios, A = 10.0, K = 1.0

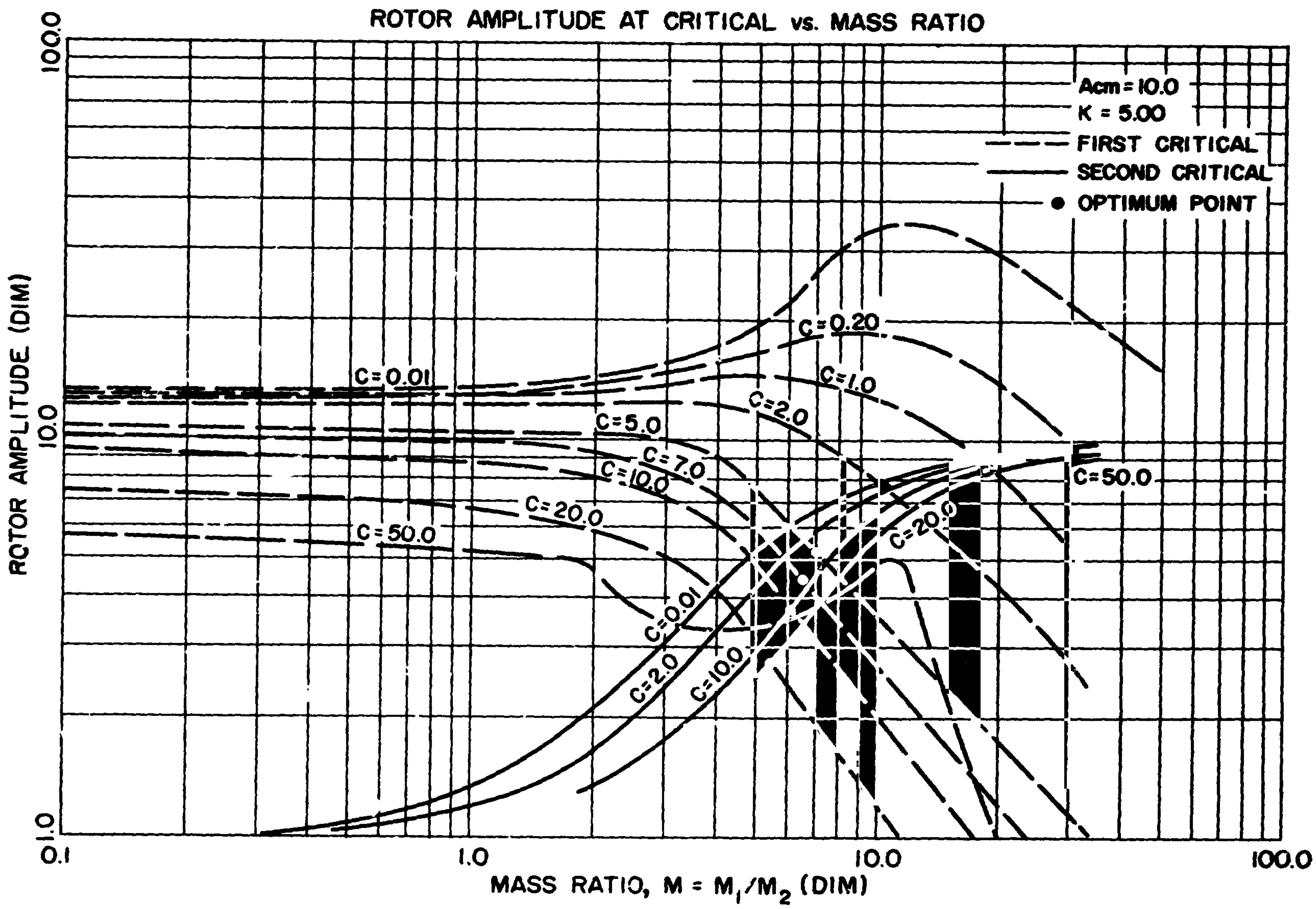

Figure 15 Rotor Amplitude at Critical Speeds vs. Mass Ratio for Various Damping Ratios, A = 10, K = 5.0

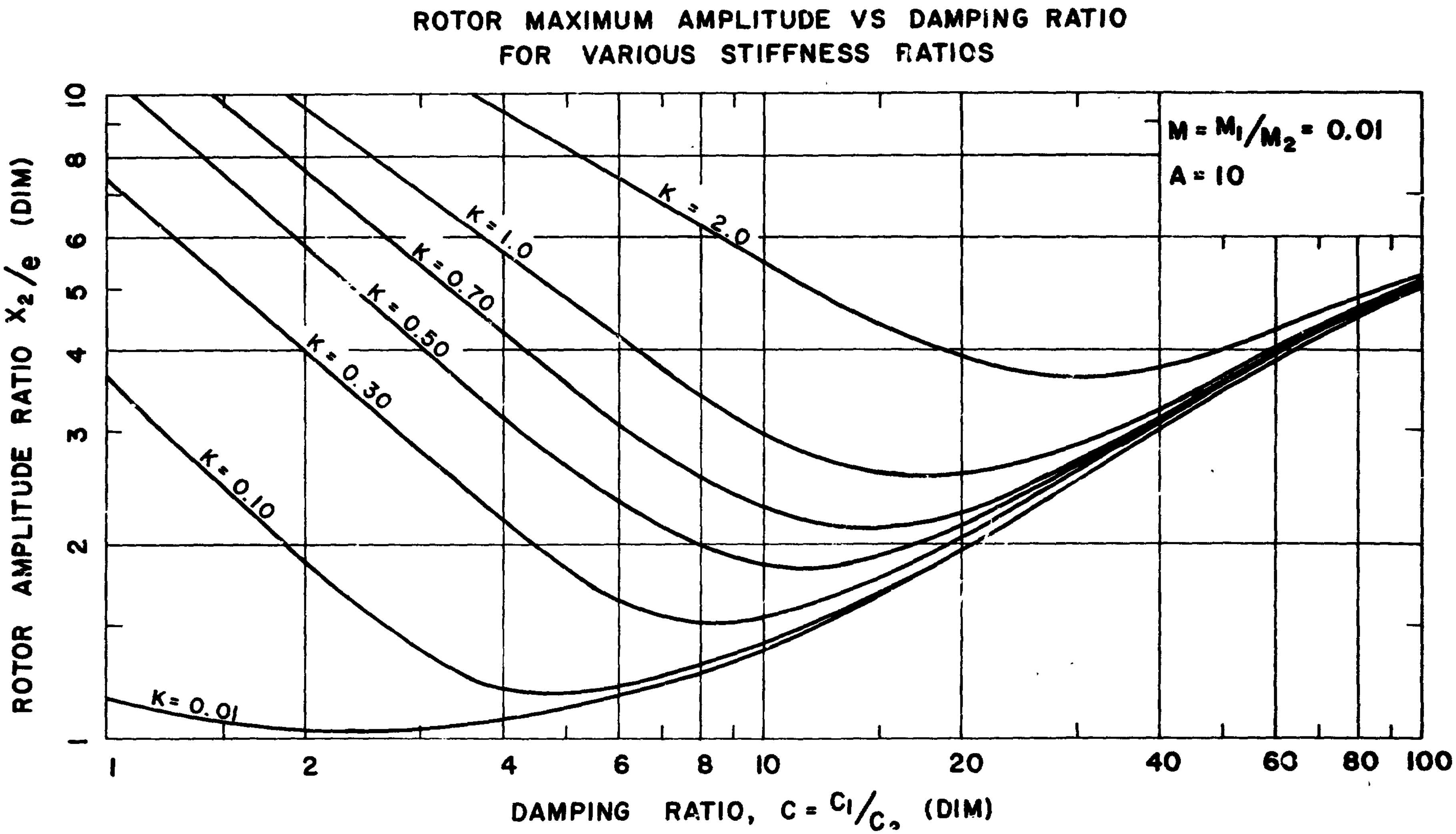

Figure 16 Rotor Maximum Amplitude vs. Damping Ratio for Various Values of Stiffness Ratios for a Low Mass Ratio Support, M = 0.01, A = 10

ROTOR MAXIMUM AMPLITUDE VS DAMPING RATIO
FOR VARIOUS STIFFNESS RATIOS, K

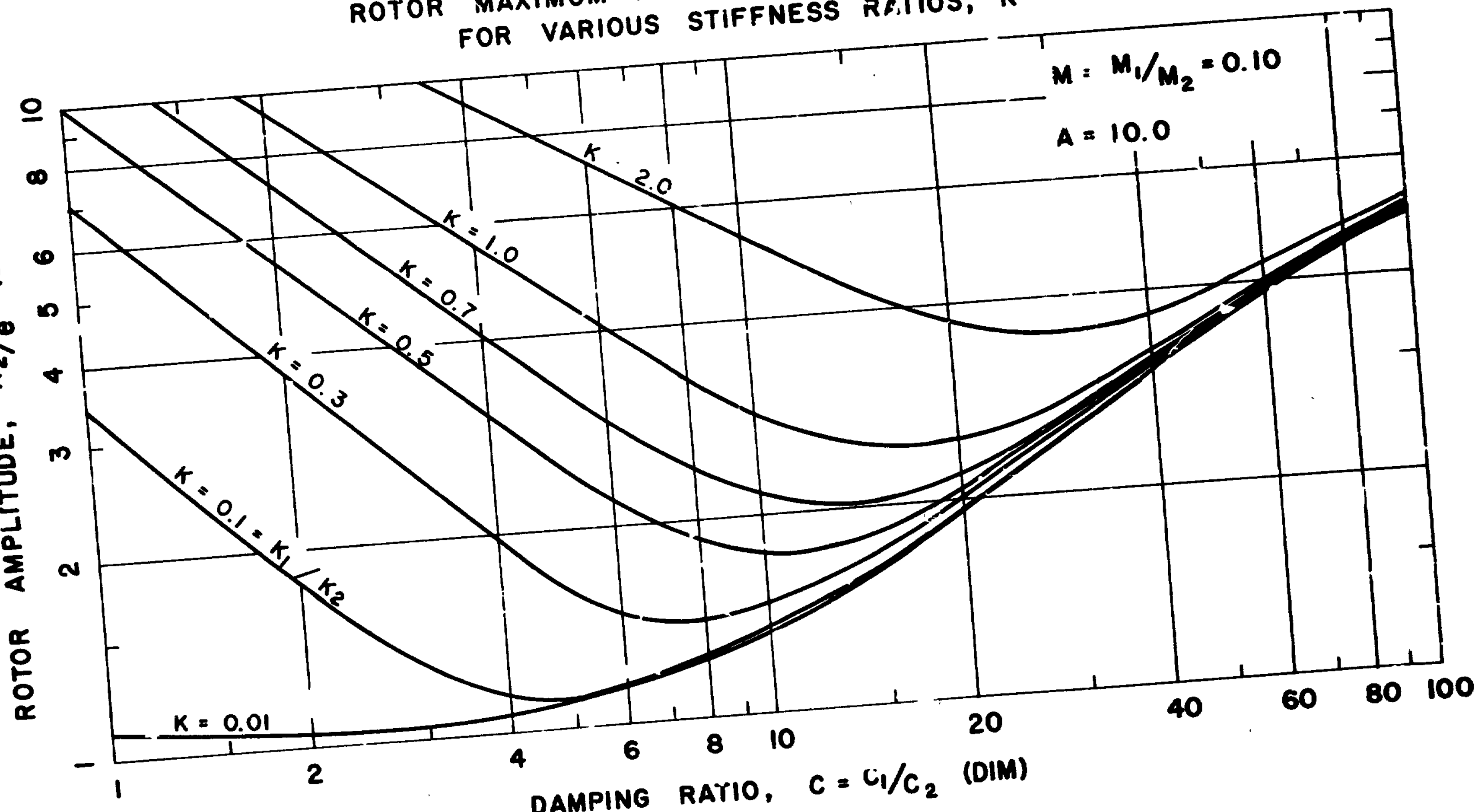

Figure 17 Rotor Maximum Amplitude vs. Damping Ratio for Various Values of Stiffness Ratios, $M = 0.10$, $A = 10$

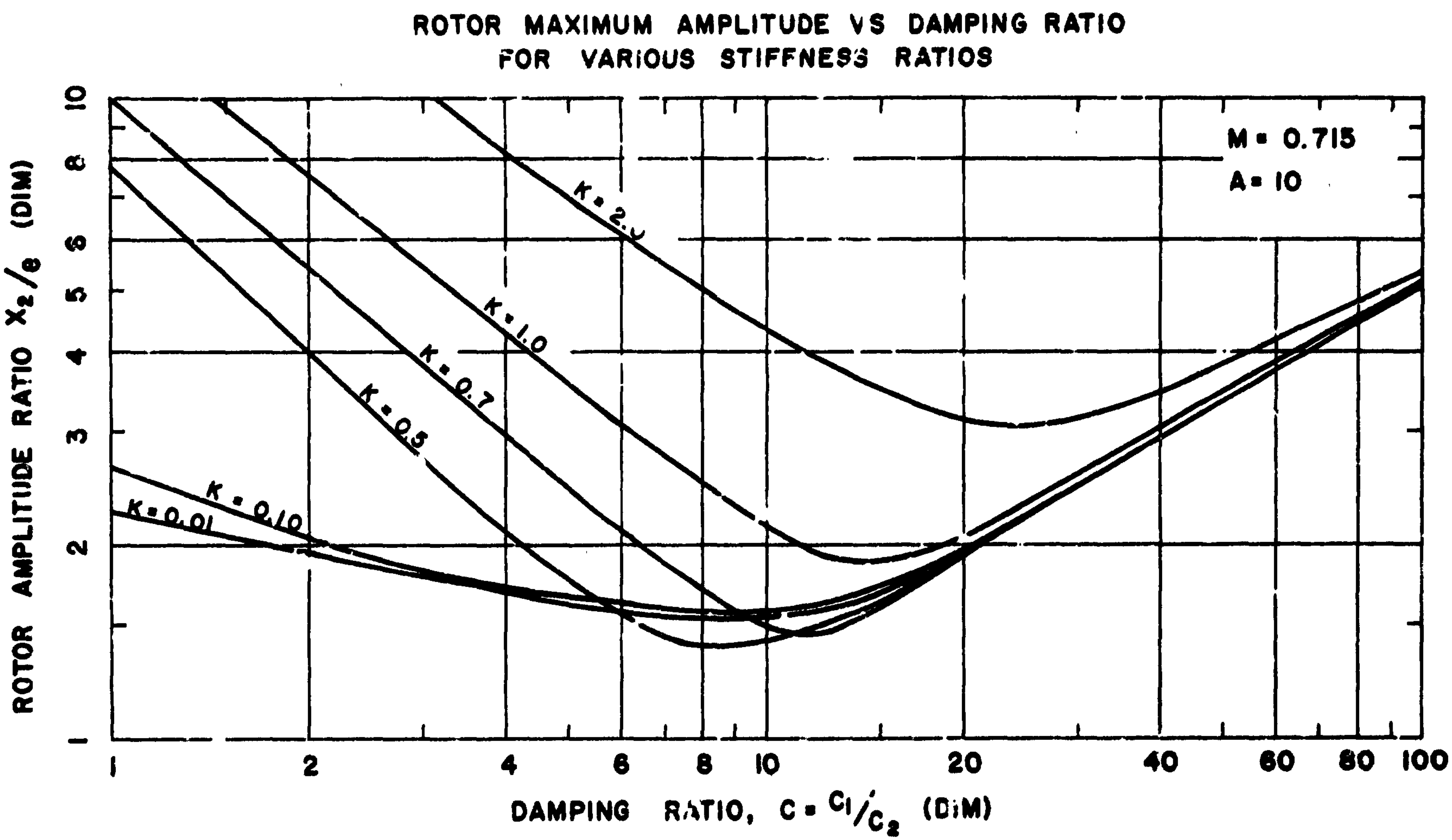

Figure 18 Rotor Maximum Amplitude vs. Damping Ratio for Various Values of Stiffness Ratios, M = 0.715, A = 10

ROTOR MAXIMUM AMPLITUDE VS DAMPING RATIO FOR VARIOUS STIFFNESS RATIOS, K

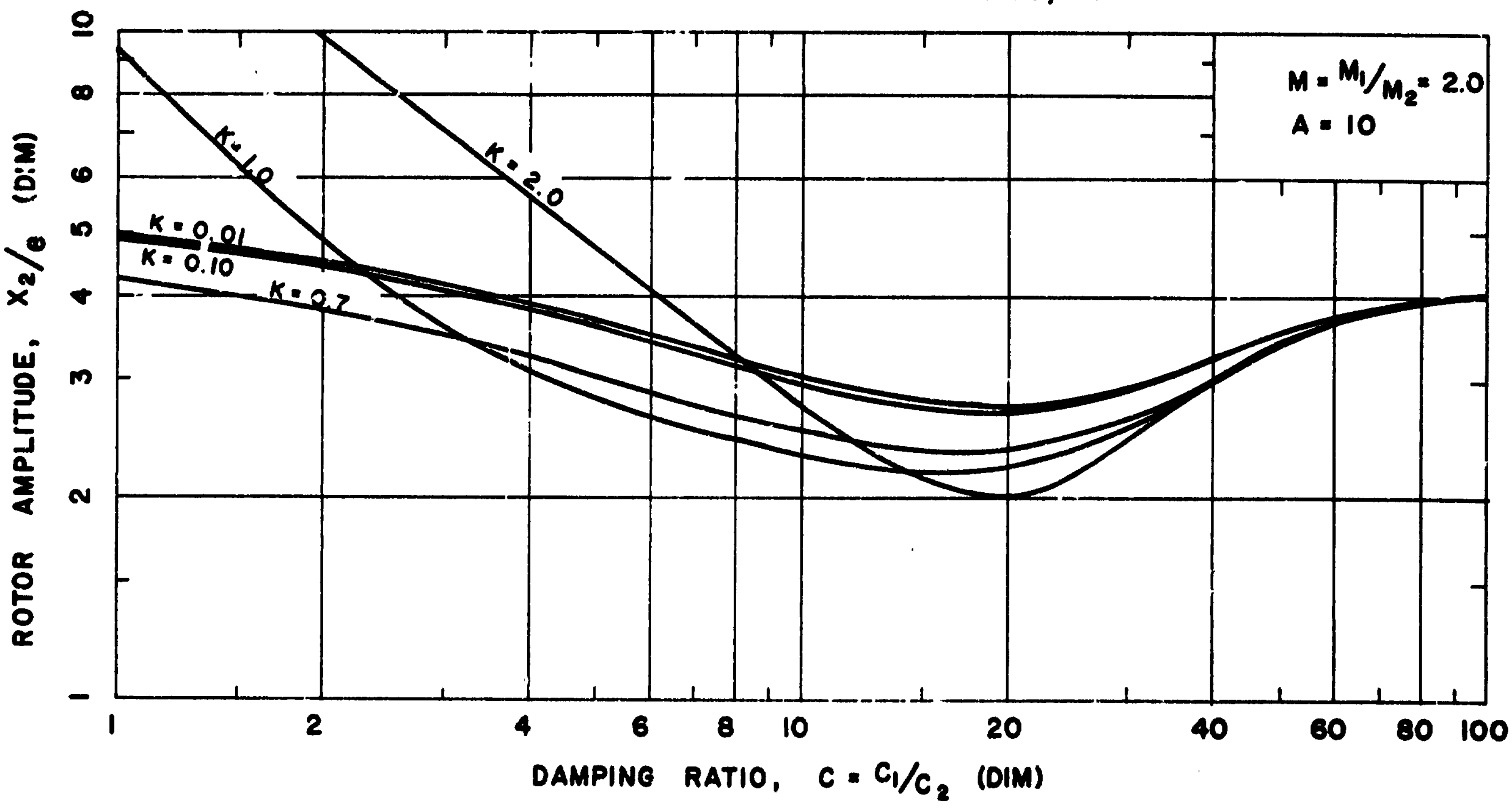

Figure 19 Rotor Maximum Amplitude Vs Damping Ratio for Various Values of Stiffness Ratios for a High Mass Ratio Support, M = 2, A = 10

ROTOR MAXIMUM AMPLITUDE FOR VARIOUS VALUES OF STIFFNESS AND MASS RATIO WITH OPTIMUM SUPPORT DAMPING

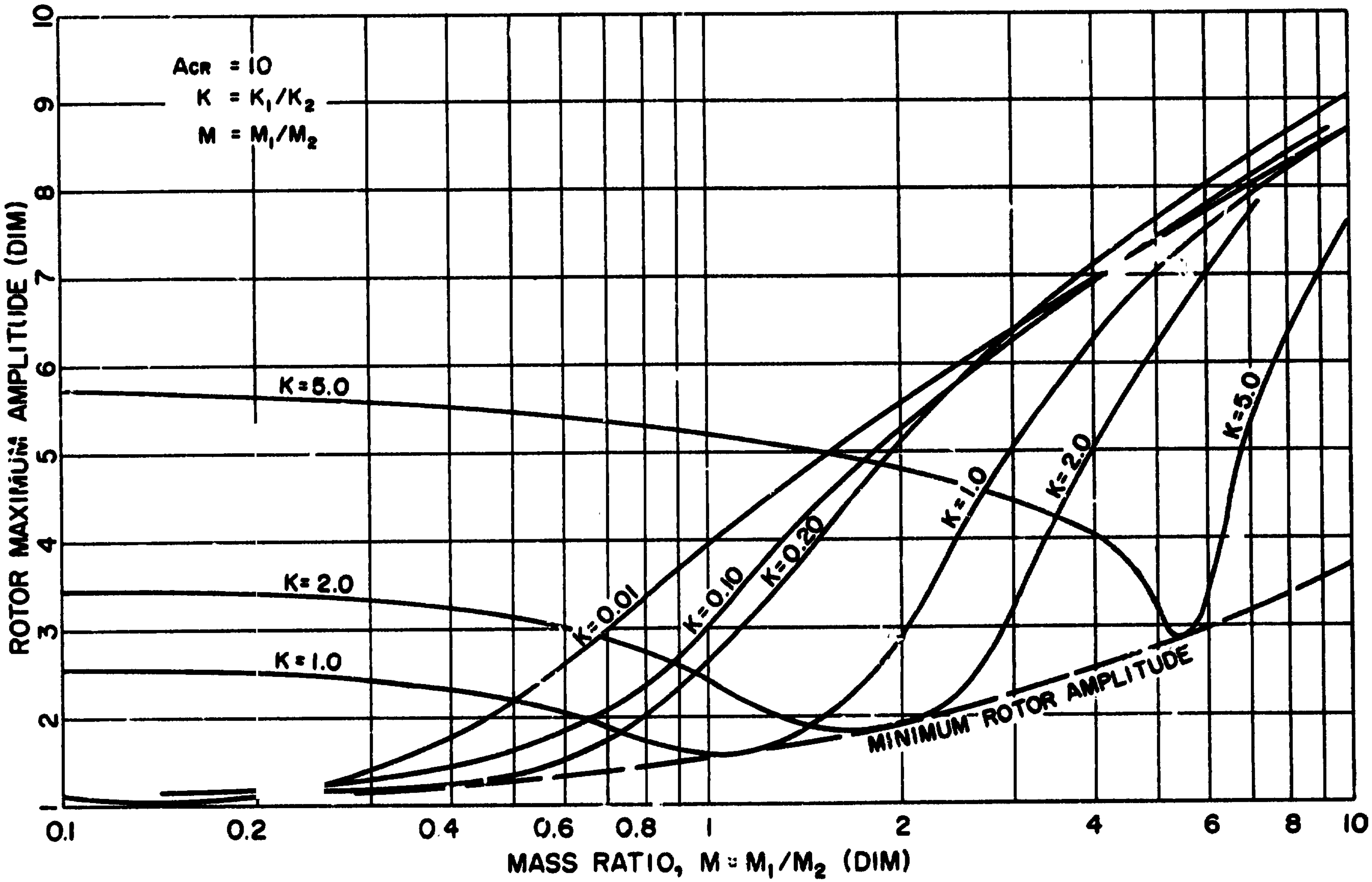

Figure 20 Rotor Maximum Amplitude for Various Values of Stiffness and Mass Ratio with Optimum Support Damping

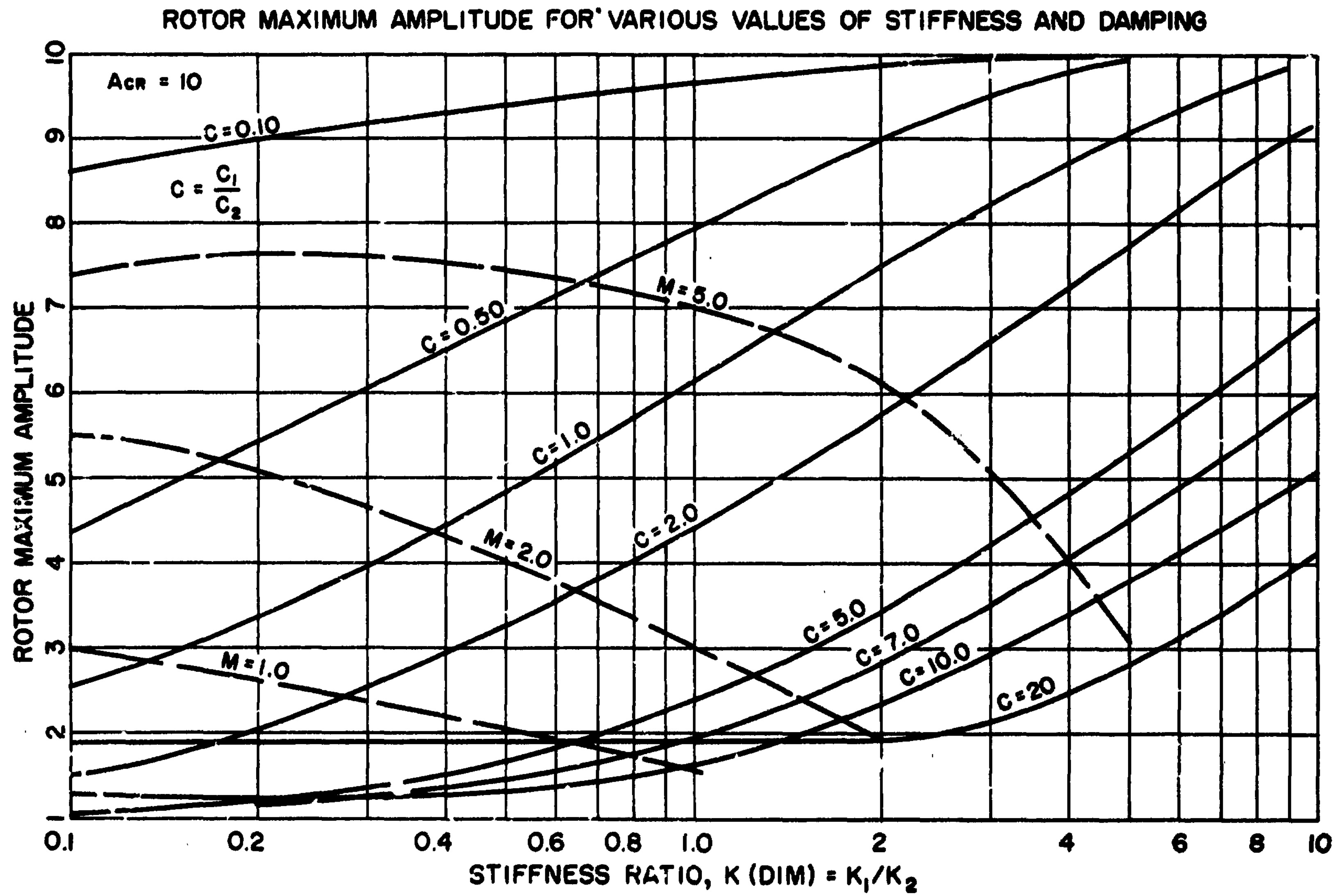

Figure 21 Rotor Maximum Amplitude vs. Stiffness Ratio for Various Values of Damping and Support Mass Ratios

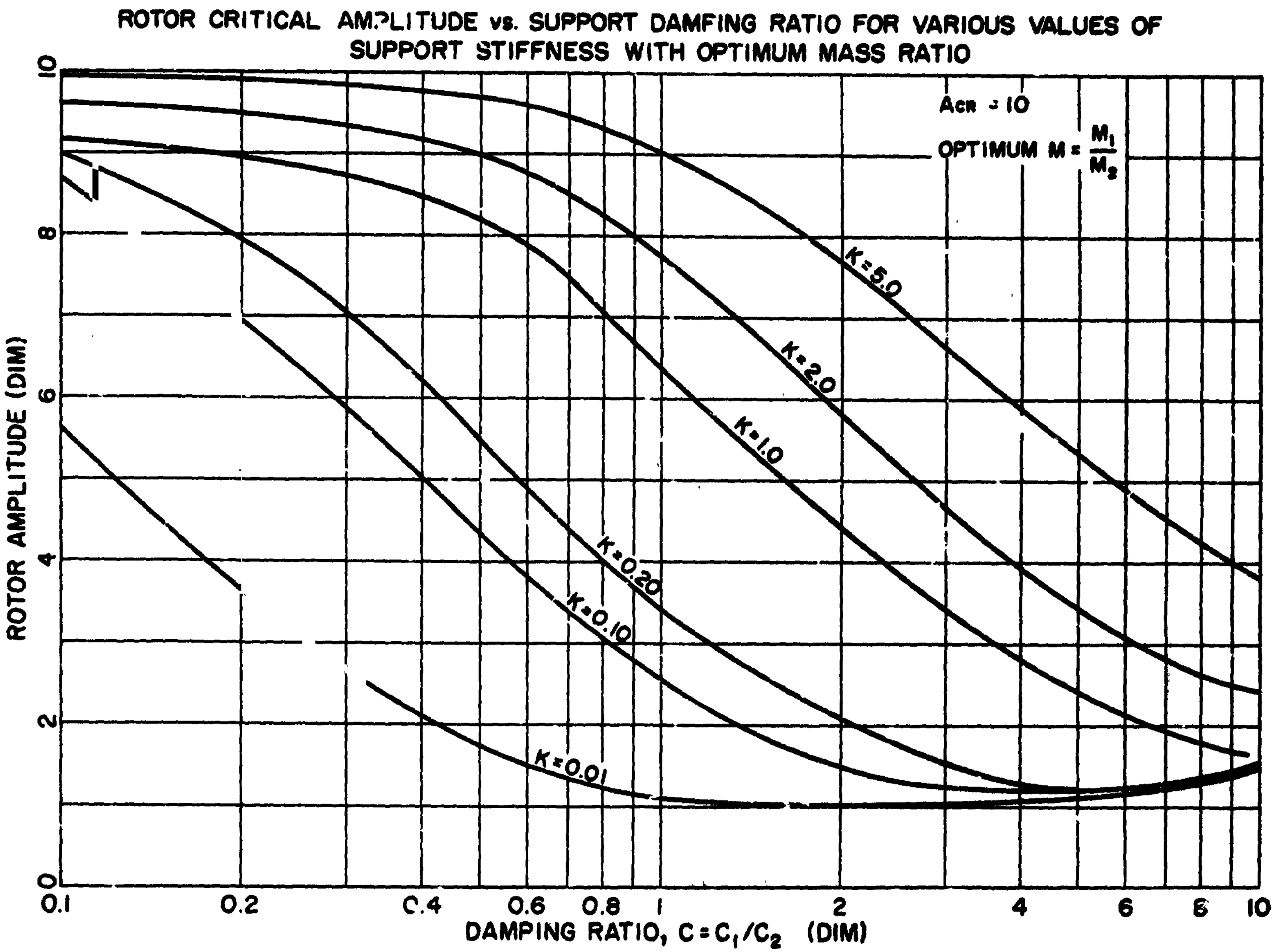

Figure 22 Rotor Critical Amplitude vs. Support Damping Ratio for Various Values of Support Stiffness with Optimum Mass Ratio

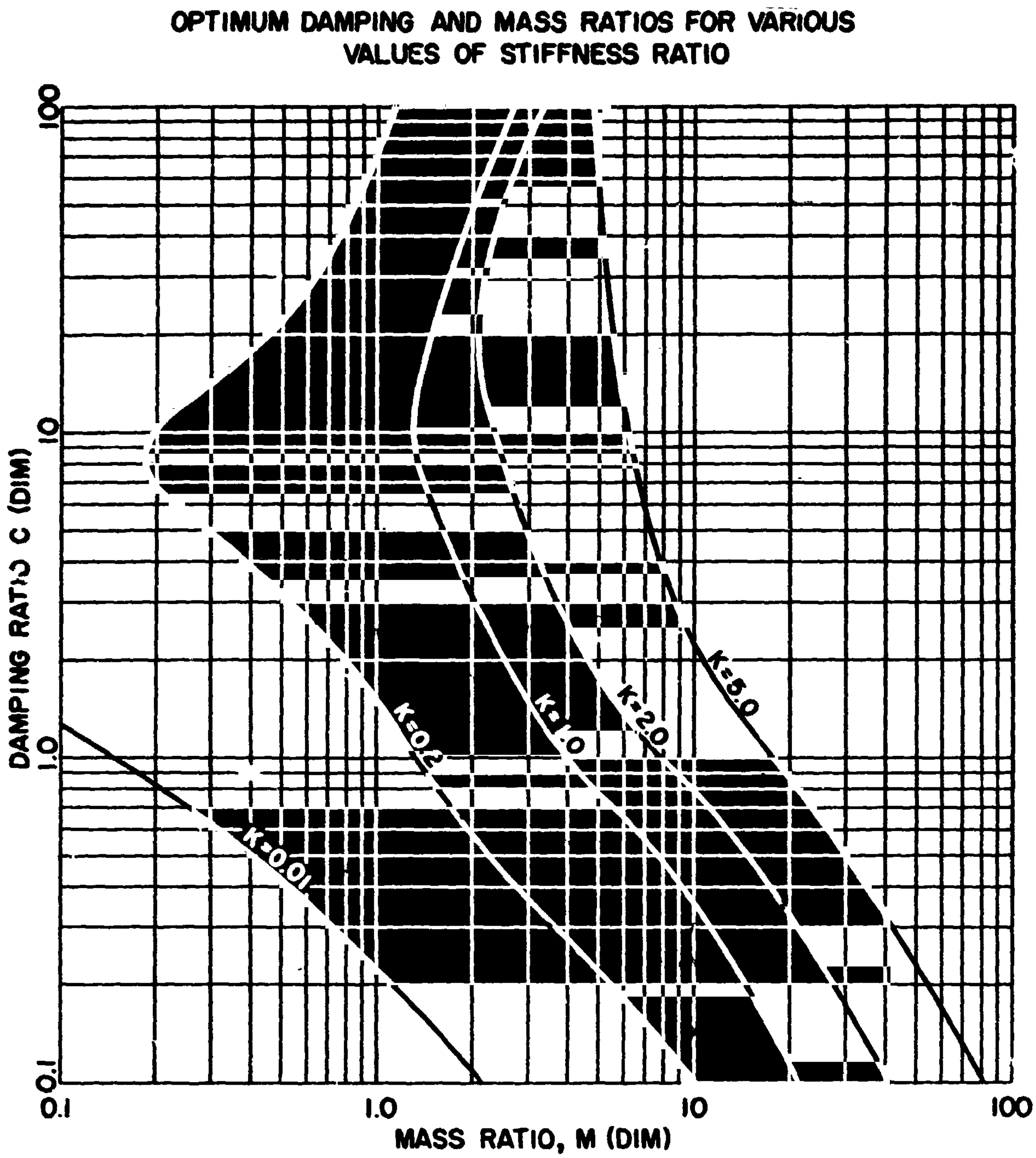

Figure 23 Optimum Damping and Mass Ratios for Various Values of Stiffness Ratio

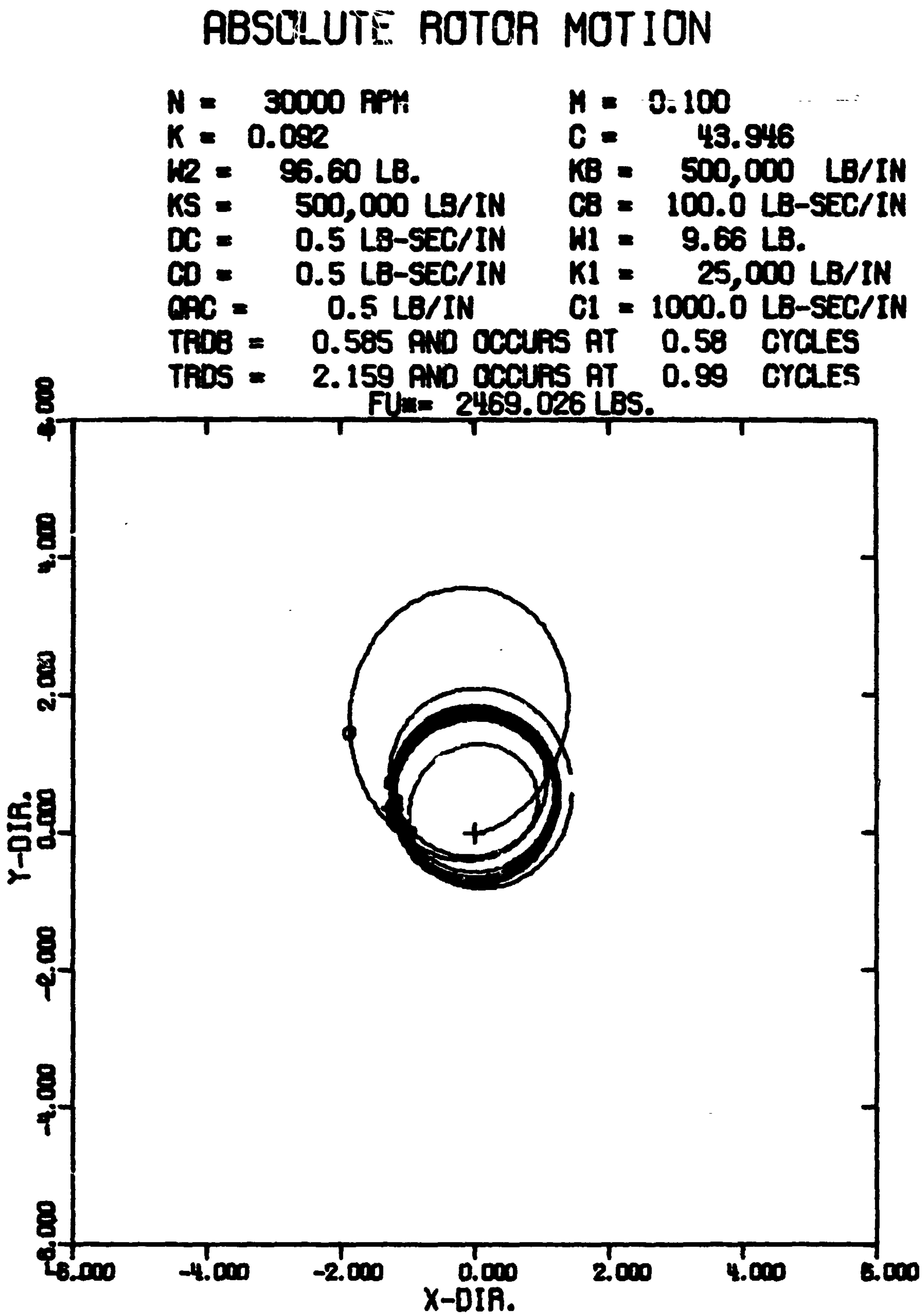

Figure 24 Dimensionless Transient Motion of an Unbalanced Rotor for Twelve Cycles on Over-Damped Supports [K = M = 0.1, C = 44]

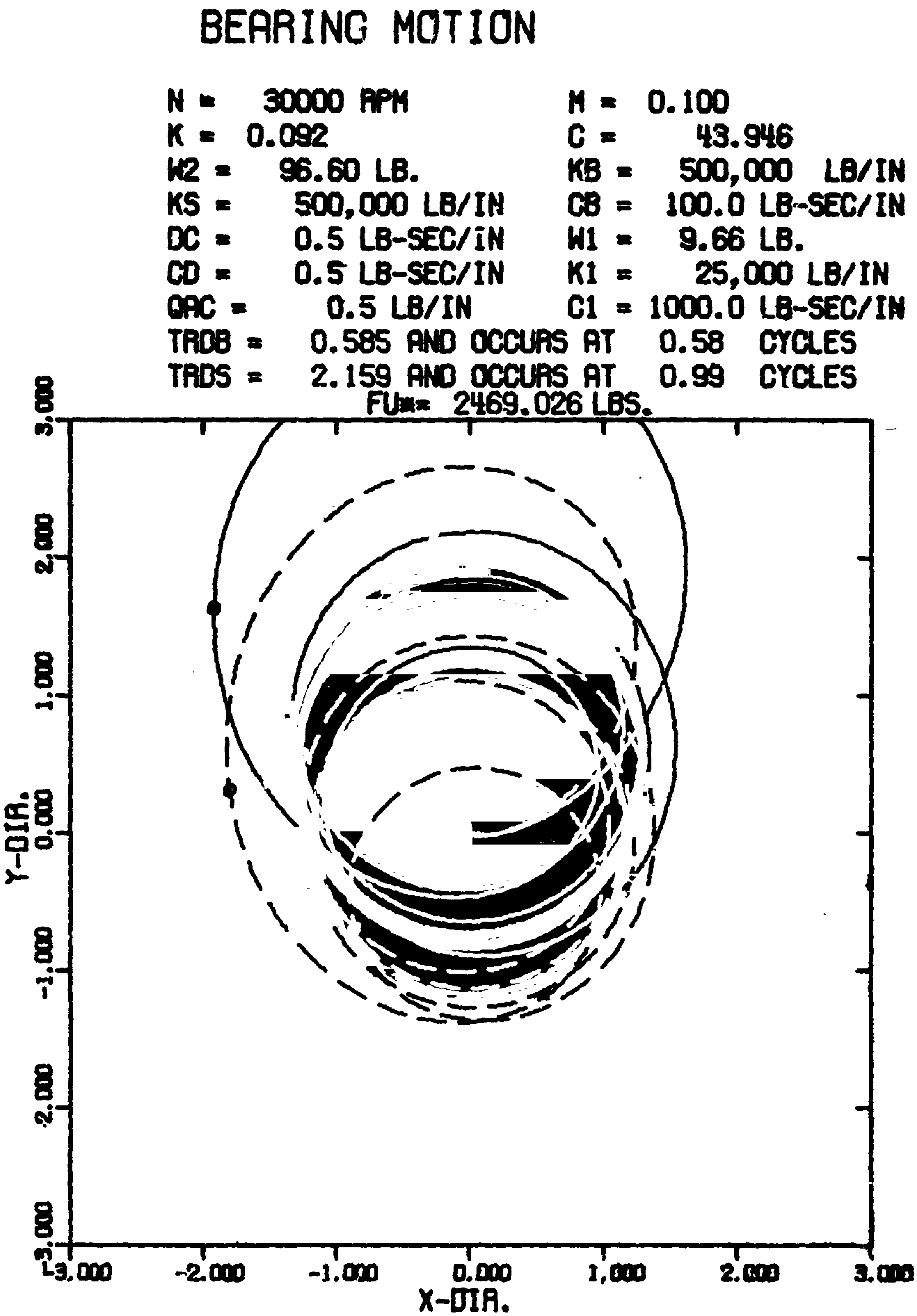

Figure 25 Dimensionless Bearing Absolute and Relative Transient Motion for Twelve Cycles on Over-Damped Supports [K = M = 0.1, C = 44]

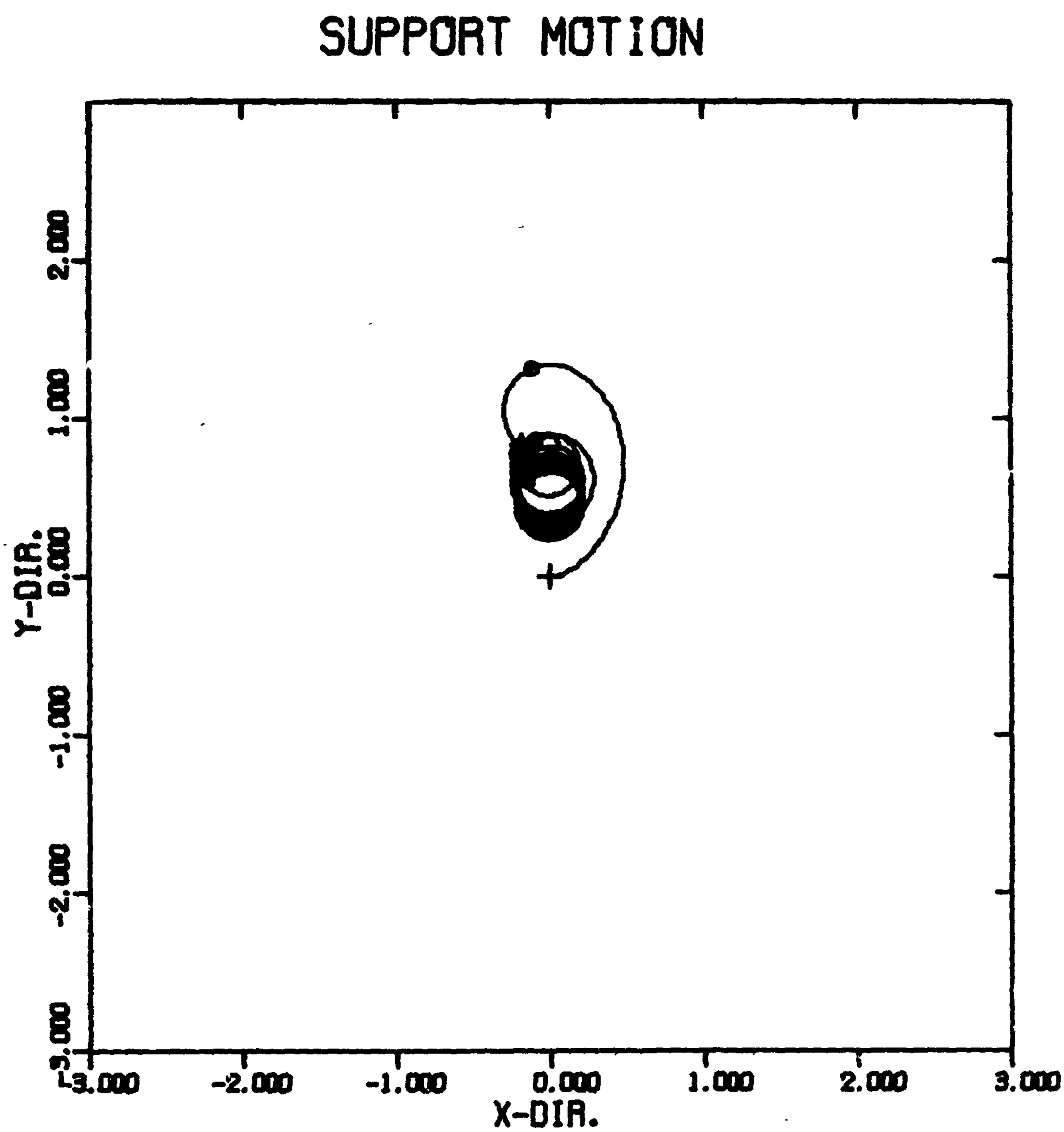

Figure 26 Dimensionless Transient Support Motion for Twelve Cycles with Excessive Damping [K = M = 0.1, C = 44]

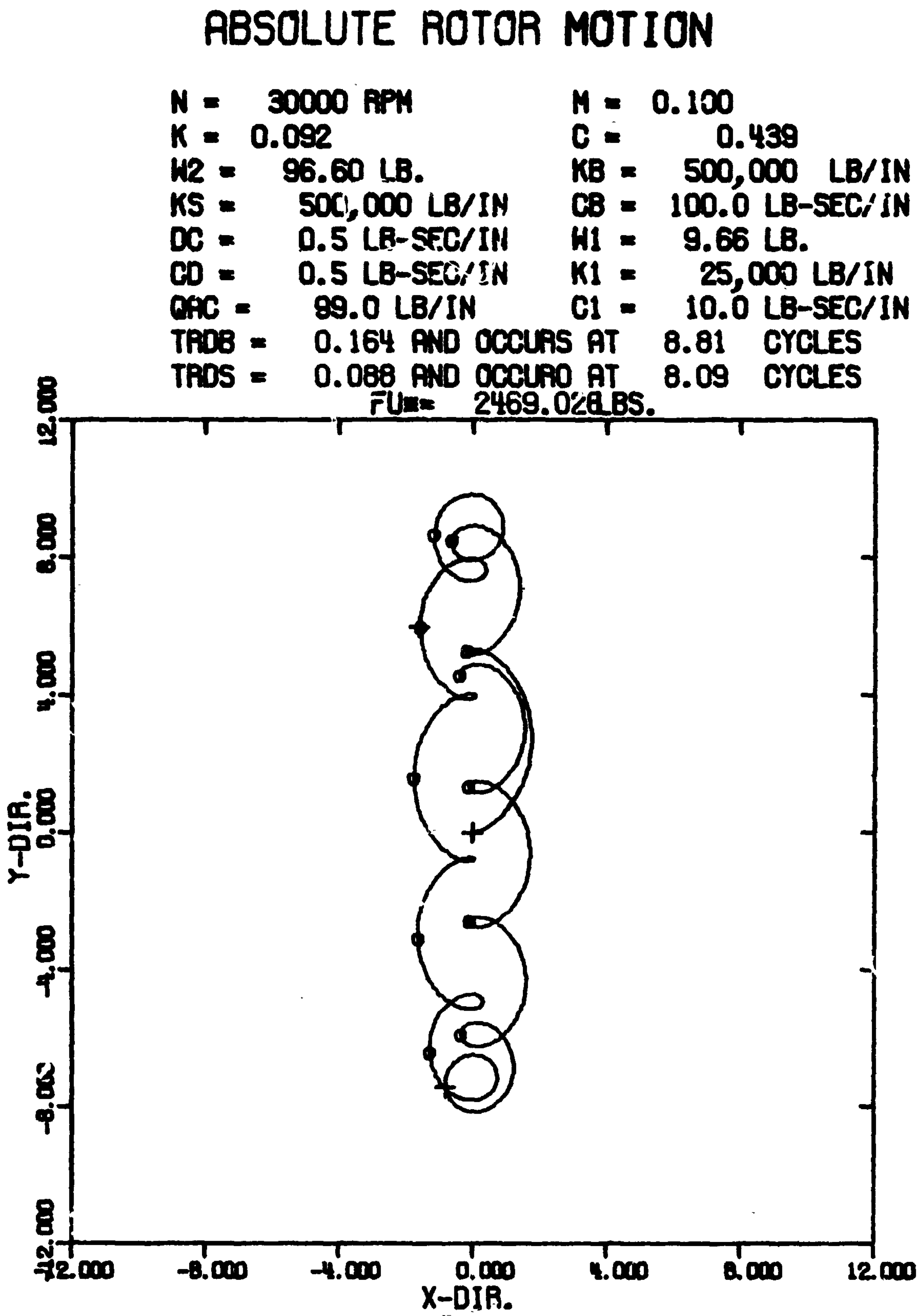

Figure 27 Dimensionless Transient Motion with Under-Damped Flexible Supports for Twelve Cycles [K = M = 0.10, C = 0.44]

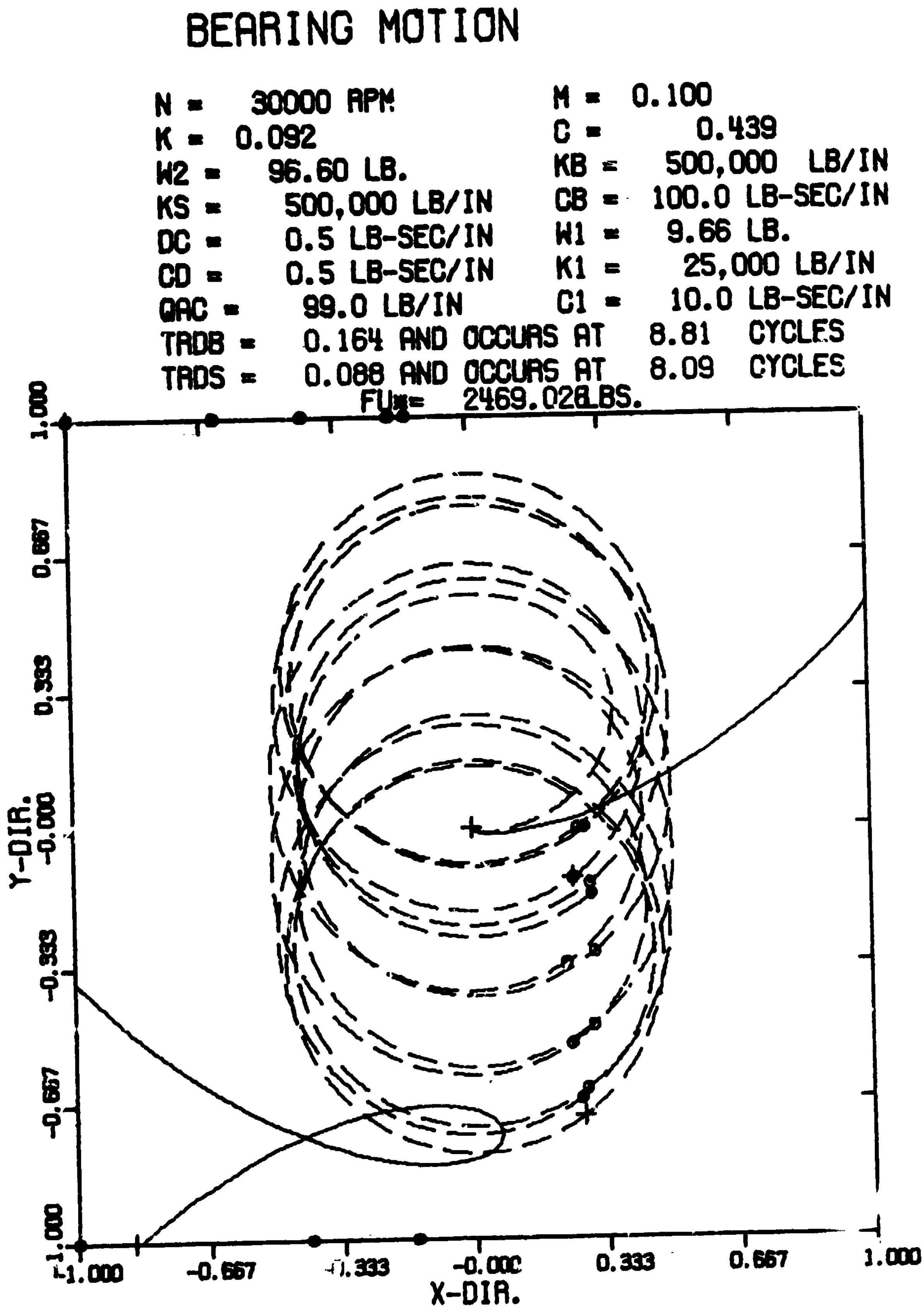

Figure 28 Dimensionless Bearing Absolute and Relative Transient Motion for Twelve Cycles on Under-Damped Supports [K = M = 0.1, C = 0.44]

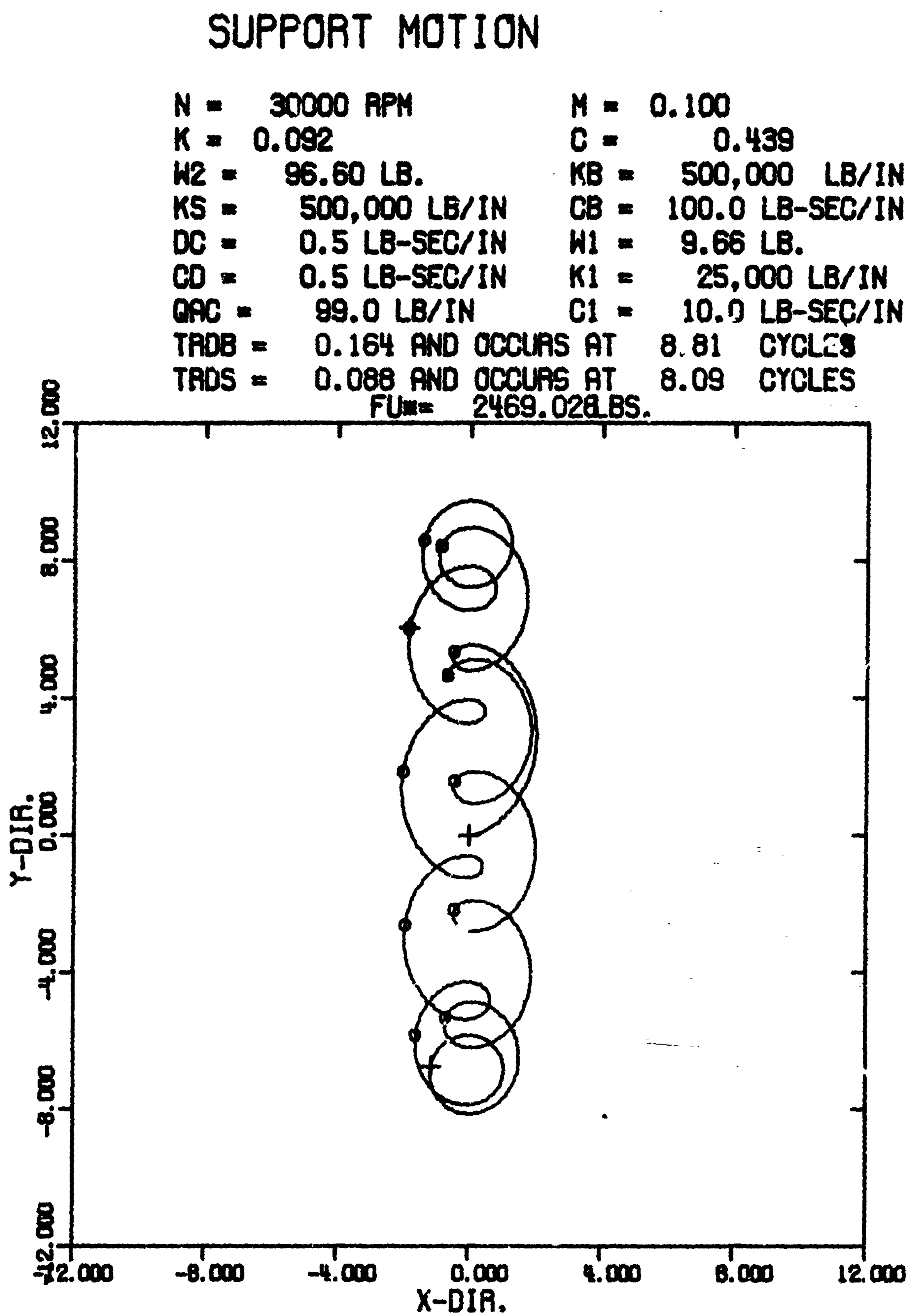

Figure 29 Dimensionless Transient Support Motion for Twelve Cycles with Under Damping [K = M = 0.1, C = 0.44]

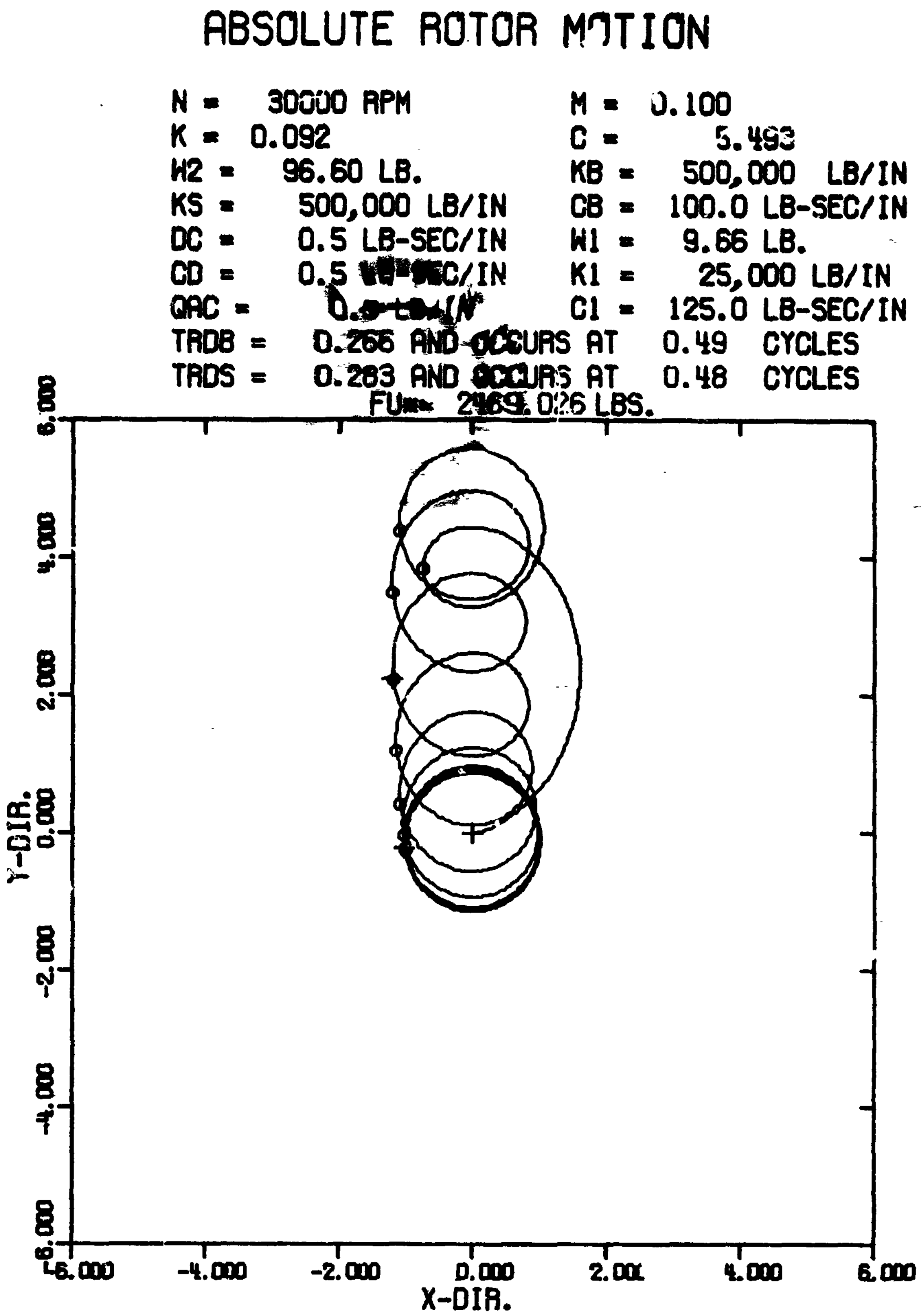

Figure 30 Dimensionless Rotor Motion with Optimum Steady-State Damping Showing the Steady-State Orbit After Seven Cycles of Running Speed [K = M = 0.1, C = 5.5]

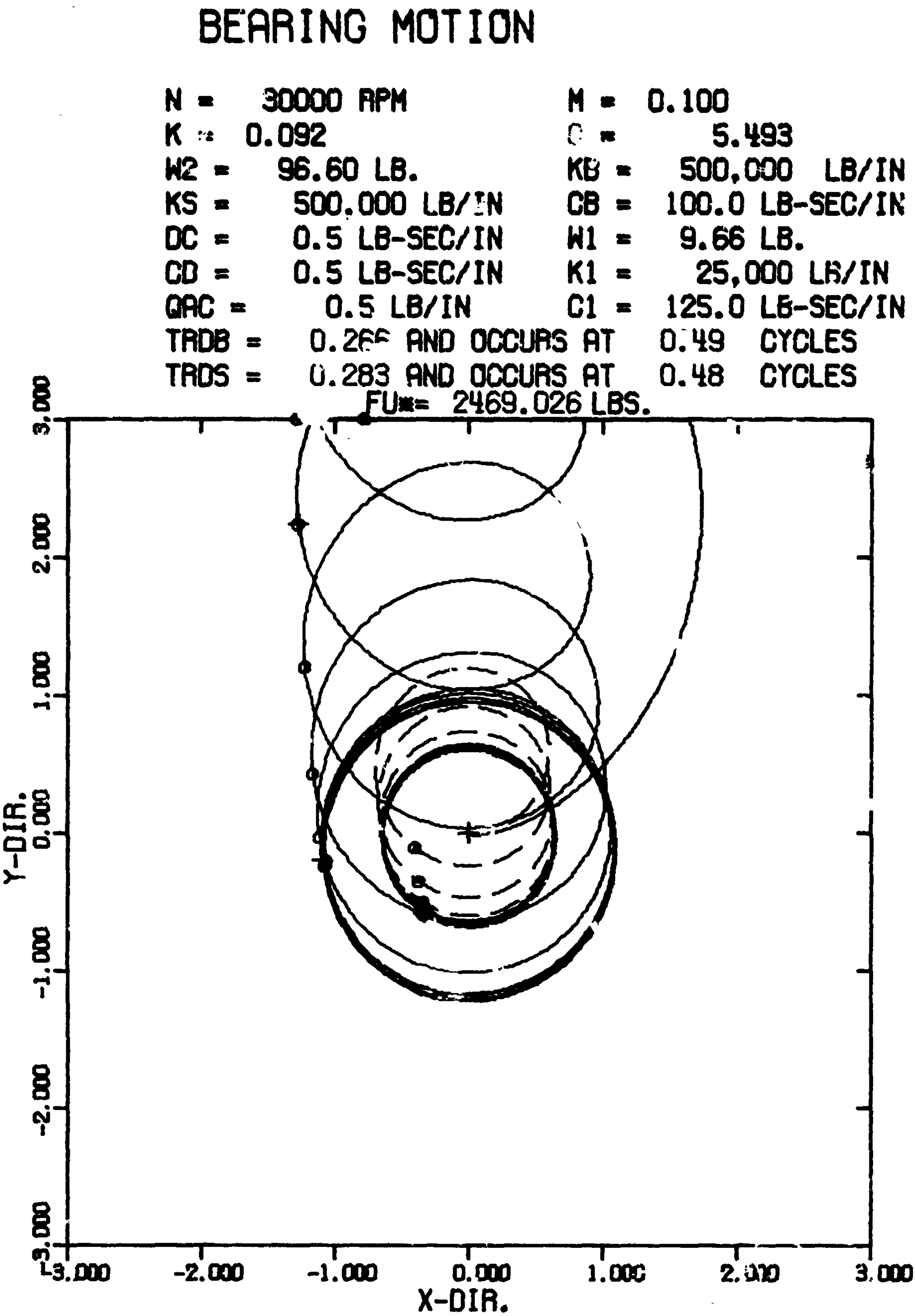

Figure 31 Dimensionless Bearing Absolute and Relative Transient Motion with Optimum Steady-State Damping [K = M = 0.1, C = 5.5]