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DETERMINATION OF THE EFFECTS OF NOZZLE NONLINEARITIES UPON
NONLINEAR STABILITY OF LIQUID PROPELLANT ROCKET MOTORS

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INTRODUCTION

This report is a summary of work completed under NASA grant NGR 11-002-179 entitled "Determination of the Effects of Nozzle Nonlinearities Upon the Nonlinear Stability of Liquid Propellant Rocket Motors". Research activities supported by this grant were begun in August 1973, and satisfactory progress has been made toward meeting the research objectives during the first year of effort. Before giving a description of this progress, the motivations and objectives of this research project will be briefly reviewed.

Various aerospace propulsion devices, such as liquid and solid propellant rocket motors and air breathing jet engines, are often subject to combustion instabilities which are detrimental to the performance and safety of operation of these devices. In order to design stable engines, capabilities for a priori determination of the linear and nonlinear characteristics of the instability and the range of operating conditions for which these engines are dynamically stable must be acquired. In order to perform such an analysis, the behavior of the exhaust nozzle under oscillatory flow conditions must be understood. In particular, it is necessary to know how a wave generated in the combustion chamber is partially transmitted and partially reflected at the nozzle entrance. This information is usually expressed as a boundary condition (usually referred to as a Nozzle Admittance Relation) that must be satisfied at the nozzle entrance.

Before such a boundary condition can be derived, the nature of the wave motion inside the nozzle must be investigated. The behavior of oscillations in a converging-diverging supercritical nozzle was first treated by Tsien¹ who considered the case in which the oscillation of the incoming flow is one-dimensional and isothermal. Crocco^{2,3} extended Tsien's work to cover the more general cases of non-isothermal one- and three-dimensional oscillations. The analyses of Tsien and Crocco are both restricted to small-amplitude (i.e., linear) oscillations. More recently, a nonlinear nozzle theory has been developed by Zinn and Crocco^{4,5,6} who extended the previous linear theories to the investigation of the

behavior of finite-amplitude waves.

In recent studies (supported under NASA grant NGL 11-002-083) conducted by Zinn, Powell, and Lores, theories were developed which describe the nonlinear behavior of longitudinal^{7,8} and transverse^{9,10} instabilities in liquid-propellant rocket chambers with quasi-steady nozzles. These theories have now been extended to situations in which the instabilities are three-dimensional and the rocket combustors are attached to conventional nozzles¹¹. All of these theories have successfully predicted the transient behavior, nonlinear waveforms, and limit-cycle amplitudes of longitudinal and tangential instabilities in unstable motors.

A new nonlinear nozzle theory is needed for the following reasons. First, the nonlinear analysis of Zinn^{5,6} is mathematically complicated and requires considerable computer time. For this reason, Zinn's analysis has never been used to perform actual computations of the wave structure in the nozzle or the nonlinear nozzle response. Secondly, the nonlinear nozzle admittance relation developed by Zinn is not compatible with the recently developed nonlinear combustion theories (see References 7 through 11). Consequently, a linear nozzle boundary condition or short nozzle (quasi-steady) assumption had to be used in all of the combustion instability theories developed to date. With the exception of a few special cases, where the amplitude of the instability is assumed to be moderate and the mean flow Mach number is small (e.g., see Reference 9), the use of a linear nozzle admittance relation in a nonlinear stability analysis is obviously inconsistent. Furthermore, in the case of transverse instabilities the "linear" nozzle has been known to exert a destabilizing effect; in these cases it is especially important to know how nonlinearities affect the nozzle behavior.

The objective of this research program is to develop a three-dimensional, nonlinear nozzle admittance relation to be used as a boundary condition in the recently-developed nonlinear combustion instability theories. This objective will be accomplished by performing the following four tasks:

- Task I: Development of the theory
- Task II: Calculation of the nozzle response
- Task III: Application of the nozzle theory to combustion
 instability problems

Task IV: Preparation of the final technical report

During the first six months of this project, considerable progress was made toward completing the first of the above tasks. However, unforeseen difficulties in the mathematical formulation of the problem arose in December, and most of the first year was needed to complete Task I. Once the theory and computer programs were developed, Task II was completed during the remaining time. A one-year extension of support has been granted by NASA to complete Tasks III and IV. A summary of the work completed on Tasks I and II is given in the remainder of this report.

TASK I: DEVELOPMENT OF THEORY

Derivation of the Nozzle Wave Equation

As in the Zinn-Crocco analysis,^{5,6} finite-amplitude, periodic oscillations inside the slowly convergent, subsonic portion of an axisymmetric nozzle operating in the supercritical range were investigated. The flow in the nozzle was assumed to be adiabatic and inviscid and to have no body forces or chemical reactions. The fluid was also assumed to be calorically perfect.

The nondimensional equations describing the gas motion in the nozzle were written in the following form:

$$\frac{\partial p}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0 \quad (1)$$

$$\frac{\partial \underline{V}}{\partial t} + \frac{1}{2} \nabla (\underline{V} \cdot \underline{V}) + (\nabla \times \underline{V}) \times \underline{V} + \frac{1}{\gamma p} \nabla p = 0 \quad (2)$$

$$\frac{\partial S}{\partial t} + \underline{V} \cdot \nabla S = 0 \quad (3)$$

$$S = \frac{1}{\gamma} \ln p - \ln p + \text{constant} \quad (4)$$

where γ is the specific heat ratio; \bar{V} , p , ρ , and S are the dimensionless velocity, pressure, density and entropy respectively and t is the dimensionless time.

It was also assumed that the nozzle flow is isentropic and irrotational. Under these conditions the energy equation (i.e., Equation (3)) is no longer needed, the state equation (i.e., Equation (4)) reduces to the isentropic flow relation, $p = \rho^\gamma$, and a velocity potential exists such that $\nabla\bar{\phi} = \bar{V}$. The continuity and momentum equations were combined, with the aid of the isentropic relation, to yield the following equation which describes the behavior of the velocity potential:

$$\begin{aligned} \nabla^2\bar{\phi} - \bar{\phi}_{tt} = 2\nabla\bar{\phi} \cdot \nabla\bar{\phi}_t + (\gamma - 1) \bar{\phi}_t \nabla^2\bar{\phi} \\ + \frac{\gamma - 1}{2} (\nabla\bar{\phi} \cdot \nabla\bar{\phi}) \nabla^2\bar{\phi} + \frac{1}{2} \nabla\bar{\phi} \cdot \nabla(\nabla\bar{\phi} \cdot \nabla\bar{\phi}) \end{aligned} \quad (5)$$

This equation is consistent with the wave equation used in the second-order nonlinear combustion instability theory developed by Powell, Zinn, and Lores (see References 7 and 10).

In the nonlinear combustion instability theories developed by Powell and Zinn, each variable was expressed as the sum of a space-dependent steady state quantity and a time- and space-dependent perturbation quantity. In order to obtain a nozzle admittance relation compatible with these theories, the velocity potential was expressed as follows:

$$\bar{\phi} = \bar{\bar{\phi}} + \bar{\phi}' \quad (6)$$

where the prime denotes the perturbation quantity and the bar denotes the steady-state quantity. Using the relation $\nabla\bar{\bar{\phi}} = \bar{V}$, Equation (6) was substituted into Equation (5) to obtain the following wave equation for the nozzle:

$$\left[1 - \frac{\gamma - 1}{2} \bar{V}^2 \right] \nabla^2\bar{\phi}' - \bar{\phi}'_{tt} = 2 \bar{V} \cdot \left[\nabla\bar{\phi}'_t + \frac{1}{2} \nabla(\bar{V} \cdot \nabla\bar{\phi}') \right] + \quad (7)$$

$$\begin{aligned}
& + (\gamma - 1) (\nabla \cdot \bar{V}) \left[\bar{\Phi}'_t + \bar{V} \cdot \nabla \bar{\Phi}' \right] + \frac{1}{2} \nabla (\bar{V}^2) \cdot \nabla \bar{\Phi}' \\
& + 2 \nabla \bar{\Phi}' \cdot \left[\nabla \bar{\Phi}'_t + \frac{1}{2} \nabla (\bar{V} \cdot \nabla \bar{\Phi}') \right] + \frac{1}{2} \bar{V} \cdot \nabla (\nabla \bar{\Phi}' \cdot \nabla \bar{\Phi}') \\
& + (\gamma - 1) \nabla^2 \bar{\Phi}' \left[\bar{\Phi}'_t + \bar{V} \cdot \nabla \bar{\Phi}' \right] + \frac{\gamma - 1}{2} (\nabla \cdot \bar{V}) (\nabla \bar{\Phi}' \cdot \nabla \bar{\Phi}') \\
& + \left\{ \frac{\gamma - 1}{2} \nabla^2 \bar{\Phi}' (\nabla \bar{\Phi}' \cdot \nabla \bar{\Phi}') + \frac{1}{2} \nabla \bar{\Phi}' \cdot \nabla (\nabla \bar{\Phi}' \cdot \nabla \bar{\Phi}') \right\} .
\end{aligned}$$

Before proceeding with the analysis, a coordinate system, appropriate for the introduction of the boundary condition at the nozzle walls, was chosen. Following the approach used by Zinn and Crocco^{5,6} for an axi-symmetric nozzle, the axial variable z was replaced by the steady-state potential function φ , and the radial variable r was replaced by the steady-state stream function ψ . The potential and stream functions are defined by:

$$r \bar{u} = \frac{d\psi}{\delta n} \quad ; \quad \bar{u} = \frac{d\varphi}{\delta s} \quad (8)$$

where δs and δn respectively represent elementary (non-dimensional) lengths in the directions of the unperturbed streamlines and of their normals on the meridional planes (see Figure 1) and \bar{u} is the steady-state velocity. A third independent variable, θ , measures the azimuthal variation. In the new coordinate system, the perturbation velocity is expressed in terms of its components along the coordinate directions as:

$$V' = u' \underline{e}_\varphi + v' \underline{e}_\psi + w' \underline{e}_\theta \quad (9)$$

where the e 's are unit vectors.

The transformation of Equation (7) to (φ, ψ, θ) coordinates was greatly simplified by assuming that the steady-state flow is one-dimensional, which

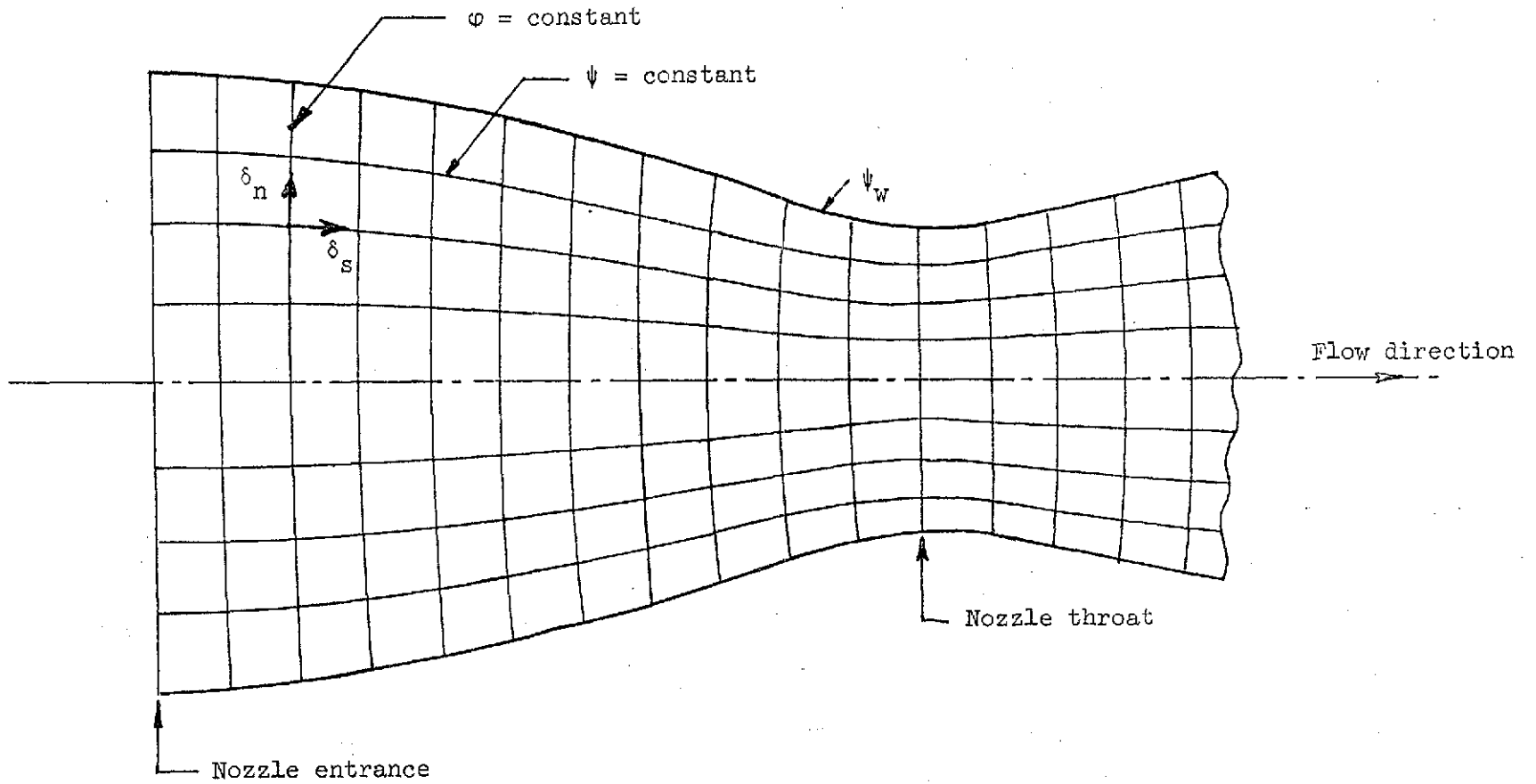


Figure 1. Coordinate System used for the Solution of the Oscillatory Nozzle Flow.

is a good approximation for slowly convergent nozzles. Under these conditions the dependence of $\bar{\rho}$ and \bar{u} on ψ and θ can be neglected, so that they are considered to be practically uniform on each surface $\varphi = \text{constant}$. Also the angle of obliquity of the stream-lines to the axis of symmetry is sufficiently small so that its cosine is practically 1 and the element of normal δn along the surface $\varphi = \text{constant}$ can be identified with dr . Hence the first of Equations (8) was integrated to obtain:

$$r^2 = \frac{2}{\bar{\rho}\bar{u}} \psi \dots \tag{10}$$

In addition the mean flow velocity (vector appearing in Equation (7) is given by:

$$\vec{V} = \bar{u}(\varphi) \underline{e}_\varphi \tag{11}$$

With the aid of Equations (10) and (11) and the expressions for the Laplacian, divergence, and gradient in a (φ, ψ, θ) coordinate system, Equation (7) was transformed to the following equation:

$$\begin{aligned}
 & f_1(\varphi) \bar{\Phi}'_{\varphi\varphi} - f_2(\varphi) \bar{\Phi}'_{\varphi} + f_3(\varphi) \left[2(\psi \bar{\Phi}'_{\psi\psi} + \bar{\Phi}'_{\psi}) + \frac{1}{2\psi} \bar{\Phi}'_{\theta\theta} \right] \\
 & - 2 \bar{\Phi}'_{\varphi t} + f_4(\varphi) \bar{\Phi}'_t - \frac{1}{\bar{u}^2} \bar{\Phi}'_{tt} \\
 & = 2 \bar{\Phi}'_{\varphi} \bar{\Phi}'_{\varphi t} + \frac{4\bar{\rho}}{\bar{u}} \psi \bar{\Phi}'_{\psi} \bar{\Phi}'_{\psi t} + \frac{\bar{\rho}}{\bar{u}\psi} \bar{\Phi}'_{\theta} \bar{\Phi}'_{\theta t} \\
 & + (\gamma + 1) \bar{u}^2 \bar{\Phi}'_{\varphi} \bar{\Phi}'_{\varphi\varphi} + 2 \bar{\rho}\bar{u} \psi \bar{\Phi}'_{\psi} \bar{\Phi}'_{\psi\varphi} \\
 & + \frac{\bar{\rho}\bar{u}}{2\psi} \bar{\Phi}'_{\theta} \bar{\Phi}'_{\theta\varphi} + f_5(\varphi) (\bar{\Phi}'_{\varphi})^2 \\
 & + f_6(\varphi) \psi (\bar{\Phi}'_{\psi})^2 + f_6(\varphi) \frac{1}{4\psi} (\bar{\Phi}'_{\theta})^2 + (\gamma - 1) \bar{\Phi}'_{\varphi\varphi} \bar{\Phi}'_t
 \end{aligned} \tag{12}$$

$$\begin{aligned}
& - f_4(\varphi) \bar{\Phi}'_{\varphi} \bar{\Phi}'_t + (\gamma - 1) \frac{\bar{\rho}}{\bar{u}} \left[2 \left(\psi \bar{\Phi}'_{\psi\psi} + \bar{\Phi}'_{\psi} \right) \right. \\
& \left. + \frac{1}{2\psi} \bar{\Phi}'_{\theta\theta} \right] \bar{\Phi}'_t + (\gamma - 1) \bar{\rho} \bar{u} \left[2 \left(\psi \bar{\Phi}'_{\psi\psi} + \bar{\Phi}'_{\psi} \right) + \frac{1}{2\psi} \bar{\Phi}'_{\theta\theta} \right] \bar{\Phi}'_{\varphi}
\end{aligned}$$

where

$$\begin{aligned}
f_1(\varphi) &= \bar{c}^2 - \bar{u}^2 \\
f_2(\varphi) &= \frac{1}{\bar{c}^2} \frac{d\bar{u}^2}{d\varphi} \\
f_3(\varphi) &= \frac{\bar{\rho} \bar{c}^2}{\bar{u}} \\
f_4(\varphi) &= \frac{-(\gamma - 1)}{2\bar{c}^2} \frac{d\bar{u}^2}{d\varphi} \\
f_5(\varphi) &= \frac{3}{2} \left[1 + \frac{\gamma - 1}{2} \frac{\bar{u}^2}{\bar{c}^2} \right] \frac{d\bar{u}^2}{d\varphi} \\
f_6(\varphi) &= \frac{\rho}{2\bar{u}} \left[1 - (2 - \gamma) \frac{\bar{u}^2}{\bar{c}^2} \right] \frac{d\bar{u}^2}{d\varphi}
\end{aligned} \tag{13}$$

In Equations (13) \bar{c} is the steady-state sonic velocity given by:

$$\bar{c}^2 = 1 - \frac{\gamma - 1}{2} \bar{u}^2 \tag{14}$$

In deriving Equation (12) the third-order terms in Equation (7) (i.e., the last two terms on the right-hand side) have been neglected, thus Equation (12) is correct to second order.

Application of the Galerkin Method

The equations obtained by the above procedure have no known closed-form mathematical solutions. Consequently, it is necessary to resort to

the use of either numerical solution techniques or approximate analytical techniques. Since the numerical solution techniques generally require excessive computer time, the latter approach was used. In the nonlinear combustion instability theories developed by Powell and Zinn (see References 7 - 11) the governing equations were solved by means of an approximate solution technique known as the Galerkin Method, which is a special case of the Method of Weighted Residuals^{12,13}. In these investigations it was shown that the Galerkin Method could be successfully applied in the solution of nonlinear combustion instability problems; its application was straightforward and it required relatively little computation time. Thus the Galerkin Method was also used in the present analysis to determine the nonlinear nozzle admittance relation.

In order to employ the Galerkin Method in the solution of the wave equation (i.e., Equation (12)), it was first necessary to express the velocity potential, $\tilde{\Phi}'$, as an approximating series expansion. The structure of this series expansion was guided by the experience gained in the nonlinear nozzle admittance studies performed by Zinn and Crocco (see Reference 5) as well as in the nonlinear combustion instability analyses of Powell and Zinn (see Reference 10). Thus the velocity potential was expressed as follows:

$$\tilde{\Phi}' = \sum_{m=0}^M \sum_{n=1}^N \left\{ A_{mn}(\varphi) \cos m\theta J_m \left[S_{mn} \left(\frac{\psi}{\psi_w} \right)^{\frac{1}{2}} \right] e^{ik_{mn} \omega t} \right\} \quad (15)$$

In Equation (15), the functions $A_{mn}(\varphi)$ are unknown complex functions of the axial variable φ . The θ - and ψ -dependent eigenfunctions were determined from the first-order (i.e., linear) solutions by Zinn⁵. In these functions m is the transverse mode number, n is the radial mode number, J_m is a Bessel function of order m , ψ_w is the value of the steady-state stream function evaluated at the nozzle wall, and S_{mn} is a root of the equation $dJ_m(x)/dx = 0$. The expansions given above describe standing wave motion; they can be easily modified to describe spinning wave motion. In the time-dependence, ω is the fundamental frequency which must be specified and the integer k_{mn} gives the frequency of the higher harmonics. The values of k_{mn} for the various modes appearing in Equation (15) must be determined from

the results of the nonlinear combustion instability analysis of Powell and Zinn¹⁰. For example it was found that, due to nonlinear coupling between modes, the second tangential ($m = 2, n = 1$) and first radial ($m = 0, n = 1$) modes oscillated with twice the frequency of the first tangential ($m = 1, n = 1$) mode. Thus in Equation (15) $k_{11} = 1$ for the first tangential mode and $k_{mn} = 2$ for the second tangential and the first radial modes. The amplitudes and phases of the various modes depend on the axial location (i.e., φ) in the nozzle through the unknown functions $A_{mn}(\varphi)$.

In order to simplify the algebra involved in the application of the Galerkin Method, the approximating series expansion for $\tilde{\Phi}'$ is written as a single summation as follows:

$$\tilde{\Phi}' = \sum_{p=1}^N A_p(\varphi) \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t} \quad (16)$$

where to each value of the index p , there corresponds the mode numbers $m(p)$ and $n(p)$, which determine the value of k_p . In Eq. (16) $\Theta_p(\theta)$ and $\Psi_p(\psi)$ are the θ - and ψ -dependent functions while N is the number of terms in the series expansion. In the present analysis, a three-term expansion consisting of the first tangential ($p = 1; m = 1, n = 1$), second tangential ($p = 2; m = 2, n = 1$) and first radial ($p = 3; m = 0, n = 1$) modes was used, but the theory is applicable to any number of modes.

In order to obtain the solution, the unknown φ -dependent functions, $A_p(\varphi)$, were determined by the Galerkin Method as follows. The assumed series expansion for the velocity potential (i.e., Eq. (16)) was substituted into the wave equation to form the residual, $E(\tilde{\Phi}')$. In the event that this residual is identically zero, the assumed solution is an exact solution. The residual, therefore, represents the error incurred by using the approximate solutions given by Eq. (16). The Galerkin Method determines the amplitudes $A_p(\varphi)$ that minimizes the residual $E(\tilde{\Phi}')$.

Applying the Galerkin Method, the residual $E(\tilde{\Phi}')$ was required to satisfy the following Galerkin orthogonality conditions:

$$\int_0^T \int_S E(\tilde{\Phi}') T_j(t) \Theta_j(\theta) \Psi_j(\psi) dS dt = 0, \quad j = 1, 2, \dots, N. \quad (17)$$

The weighting functions $T_j(t)$, $\Theta_j(\theta)$ and $\Psi_j(\psi)$ correspond to the terms that appear in the assumed series expansion. The temporal weighting function, $T_j(t)$, is the complex conjugate of the assumed time dependence, and thus

$$T_j(t) = e^{-ik \omega t} \quad (18)$$

The azimuthal weighting functions, $\Theta_j(\theta)$, are given by

$$\Theta_j(\theta) = \cos m\theta \quad (19)$$

while the radial weighting functions, $\Psi_j(\psi)$, are given by

$$\Psi_j(\psi) = J_m \left[S_j \left(\frac{\psi}{\psi_w} \right)^{\frac{1}{2}} \right] \quad (20)$$

The time integration is performed over one period of oscillation, $T = \frac{2\pi}{\omega}$, while the spatial integration is performed over any surface of $\varphi = \text{constant}$ in the nozzle (in Eq. (17) dS indicates an incremental area on this surface).

Evaluating the spatial and temporal integrals in Eq. (17) yields the following system of N nonlinear, second order, coupled, complex ordinary differential equations to be solved for the complex amplitude functions, $A_p(\varphi)$:

$$\begin{aligned} & \sum_{p=1}^N \left\{ C_1 \frac{d^2 A_p(\varphi)}{d\varphi} + C_2 \frac{dA_p(\varphi)}{d\varphi} + C_3 A_p(\varphi) \right. \\ & + \sum_{p=1}^N \sum_{q=1}^N \left\{ D_1 \frac{d^2 A_p(\varphi)}{d\varphi^2} A_q(\varphi) + D_2 \frac{d^2 A_p(\varphi)}{d\varphi^2} \frac{dA_q(\varphi)}{d\varphi} \right. \\ & \left. \left. + D_3 \frac{dA_p(\varphi)}{d\varphi} \frac{dA_q(\varphi)}{d\varphi} + D_4 A_p(\varphi) \frac{dA_q(\varphi)}{d\varphi} + D_5 A_p(\varphi) A_q(\varphi) \right\} \right. \\ & \left. + Q = 0 \quad , \quad j = 1, 2, \dots, N \quad (21) \right. \end{aligned}$$

In the above equations, Q represents the additional nonlinear terms that arise when a complex solution (i.e. Eq. (16)) is used to solve the nonlinear wave equation (i.e. Eq. (12)). These terms are similar in form to the nonlinear terms shown, but they involve the complex conjugates of the amplitude functions. The procedure for deriving these terms is given in Appendix B of Ref. 11. The coefficients C_k and D_k are functions of the axial variable φ as well as the indices j, p and q . Analytical expressions for these coefficients contain integrals involving trigonometric and Bessel functions. In the absence of closed-form expressions for the integrals of Bessel functions, these integrals were computed numerically.

As a check on the above analysis, a single mode series consisting of the first tangential mode was used in deriving Eq. (21). For this case, all the coefficients of the nonlinear terms vanish and the resulting linear equation is:

$$\begin{aligned} \bar{u}^{-2}(\bar{c}^{-2} - \bar{u}^{-2}) \frac{d^2 A}{d\varphi^2} - \bar{u}^{-2} \left[\frac{1}{\bar{c}^{-2}} \frac{d\bar{u}^{-2}}{d\varphi} + 2i\omega \right] \frac{dA}{d\varphi} \\ + \left\{ -\frac{s_{11}^2}{2\psi_w} \bar{u}^{-2} - \frac{\gamma - 1}{2} i\omega \frac{\bar{u}^{-2}}{\bar{c}^{-2}} \frac{d\bar{u}^{-2}}{d\varphi} + \omega^2 \right\} A(\varphi) = 0 \end{aligned} \quad (22)$$

which is identical to Crocco and Sirignano's equation³ for the isentropic and irrotational case.

Dominance of the 1T Mode

The well known fact that most transverse instabilities behave like the first tangential (1T) mode was used to further simplify Eq. (21). Based on the results of the recent combustion instability theory,¹¹ it was assumed that the amplitude of the 1T mode was considerably larger than the amplitudes of the remaining modes in the series solution. Through an order of magnitude analysis, correct to the second order, Eq. (21) reduced to the following system of equations:

$$\begin{aligned} & \bar{u}^{-2} (\bar{c}^{-2} - \bar{u}^{-2}) \frac{d^2 A_1}{d\varphi^2} - \bar{u}^{-2} \left[\frac{1}{\bar{c}^{-2}} \frac{d\bar{u}^{-2}}{d\varphi} + 2i\omega \right] \frac{dA_1}{d\varphi} \\ & + \left[-\frac{S_1^2}{2\psi_w} \frac{\bar{u}^{-2}}{\rho u \bar{c}^2} - \frac{\gamma - 1}{2} i\omega \frac{\bar{u}^{-2}}{\bar{c}^{-2}} \frac{d\bar{u}^{-2}}{d\varphi} + \omega^2 \right] A_1(\varphi) = 0 \end{aligned} \quad (23a)$$

$$\begin{aligned} & \bar{u}^{-2} (\bar{c}^{-2} - \bar{u}^{-2}) \frac{d^2 A_p}{d\varphi^2} - \bar{u}^{-2} \left[\frac{1}{\bar{c}^{-2}} \frac{d\bar{u}^{-2}}{d\varphi} + 2ik_p \omega \right] \frac{dA_p}{d\varphi} \\ & + \left[-\frac{S_p^2}{2\psi_w} \frac{\bar{u}^{-2}}{\rho u \bar{c}^2} - \frac{\gamma - 1}{2} ik_p \omega \frac{\bar{u}^{-2}}{\bar{c}^{-2}} \frac{d\bar{u}^{-2}}{d\varphi} + k_p^2 \omega^2 \right] A_p(\varphi) \\ & = -D_1(\varphi, p) \frac{d^2 A_1}{d\varphi^2} - D_2(\varphi, p) \frac{d^2 A_1}{d\varphi^2} \frac{dA_1}{d\varphi} \\ & - D_3(\varphi, p) \left(\frac{dA_1}{d\varphi} \right)^2 - D_4(\varphi, p) \frac{dA_1}{d\varphi} A_1 - D_5(\varphi, p) A_1^2 \\ & - Q_p = 0, \end{aligned} \quad (23b)$$

$$p = 2, 3, \dots, N$$

The above equations can be written concisely as follows:

$$H_p(\omega) \frac{d^2 A_p(\varphi)}{d\varphi^2} + M_p(\omega) \frac{dA_p(\varphi)}{d\varphi} + N_p(\omega) A_p(\varphi) = I_p(\omega) \quad (24)$$

$$p = 1, 2, \dots, N,$$

where $I_1(\omega) = 0$.

It can be seen that the above equations are decoupled with respect to the 1T mode; that is, the solution for A_1 can be obtained independently of the amplitudes of the other modes. Thus, to second order, the nonlinearities of the problem do not affect the 1T mode. On the other hand the nonlinearities influence the amplitudes of the higher modes

(i.e., $A_2, A_3 \dots$) by means of the inhomogeneous terms in the equations for the other modes.

Homogeneous and Particular Solutions

Equation (24) is a second order, linear ordinary differential equation and its general solution is a combination of the homogeneous solution that satisfies the homogeneous part of Eq. (24), i.e.,

$$L\{A_p^{(h)}\} = H_p \frac{d^2 A_p^{(h)}}{d\varphi^2} + M_p \frac{dA_p^{(h)}}{d\varphi} + N_p A_p^{(h)} = 0 \quad (25)$$

and the particular solution that satisfies Eq. (24). The general solution can be written in the following form:

$$A_p(\varphi) = K_1 A_p^{(h)} + K_2 \tilde{A}_p^{(h)} + A_p^{(i)}$$

where $A_p^{(h)}$ and $\tilde{A}_p^{(h)}$ are two independent solutions of Eq. (25), K_1 and K_2 are arbitrary constants, and $A_p^{(i)}$ is a particular solution of the inhomogeneous equation.

Examination of the coefficients of Eq. (24) show that this equation has the following singular points:

$$\bar{u} = 0$$

$$\bar{u} = \bar{c} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{2}} = \bar{c}_{\text{throat}}$$

$$\bar{u} = \infty$$

For a supercritical nozzle with a finite area entrance, only the singularity at the throat is of concern to us. Assuming that the singularity of the solution appears in $\tilde{A}_p^{(h)}$, the condition requiring the regularity of the solution at the throat can be expressed by requiring $K_2 = 0$. Consequently, the required solution of Eq. (24) is of the form

$$A_p(\varphi) = K_1 A_p^{(h)}(\varphi) + A_p^{(i)}(\varphi) \quad (26)$$

Derivation of Admittance Relations

Using the above result, a nonlinear admittance relation to be used as a boundary condition in nonlinear combustion instability analyses can be derived. Denoting the terms of Eq. (16) by

$$\Phi'_p = A_p(\omega) \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t}, \quad (27)$$

taking partial derivatives with respect to z and t , and using Eq. (26) gives

$$\begin{aligned} \frac{\partial \Phi'_p}{\partial z} - \bar{u} \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t} \frac{dA_p^{(i)}}{d\varphi} \\ = K_1 \bar{u} \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t} \frac{dA_p^{(h)}}{d\varphi} \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{\partial \Phi'_p}{\partial t} - ik_p \omega \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t} A_p^{(i)} \\ = K_1 ik_p \omega \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t} A_p^{(h)}. \end{aligned} \quad (29)$$

Eliminating K_1 between Eqs. (28) and (29) and defining

$$\zeta_p = \frac{dA_p^{(h)}/d\varphi}{A_p^{(h)}} \quad (30)$$

$$\Gamma_p = \frac{1}{c^2 A_p^{(h)}} \left[A_p^{(i)} \frac{dA_p^{(h)}}{d\varphi} - A_p^{(h)} \frac{dA_p^{(i)}}{d\varphi} \right] \quad (31)$$

$$Y_p = \frac{i\bar{u}\zeta_p}{\gamma k_p \omega} \quad (32)$$

yields

$$\frac{\partial \Phi'_p}{\partial z} + \gamma Y_p \frac{\partial \Phi'_p}{\partial t} = -\bar{u} c^2 \Theta_p(\theta) \Psi_p(\psi) e^{ik_p \omega t} \Gamma_p, \quad (33)$$

$$p = 1, 2, \dots, N$$

Equation (33) is the nonlinear nozzle admittance relation, to be used as the boundary condition at the nozzle entrance in nonlinear combustion instability analyses. The right-hand-side of this equation arises from the nonlinear terms in the nozzle wave equation. The quantities Y_p and Γ_p are respectively the linear and nonlinear admittance coefficients for the p^{th} mode. The nonlinear admittance, Γ_p , represents the effect of nozzle nonlinearities upon the nozzle admittance and it is identically zero when nonlinearities are not present.

It can easily be shown that Eq. (33) can be written in terms of the pressure and axial velocity perturbations as:

$$U_p - Y_p P_p = - \bar{u} \bar{c}^2 \Gamma_p, \quad p = 1, 2, \dots, N \quad (34)$$

where U_p and P_p are the amplitudes of the axial velocity and pressure perturbations respectively as given by:

$$p' = \sum_{p=1}^N P_p(\varphi) \Theta_p(\theta) \Psi_p(\psi) e^{ik\omega t} \quad (35)$$

$$u' = \sum_{p=1}^N U_p(\varphi) \Theta_p(\theta) \Psi_p(\psi) e^{ik\omega t} \quad (36)$$

Equation (34) is equivalent to Eq. (33) to second order only when the Mach number at the nozzle entrance, \bar{u}_e , is small.

In order to use the admittance relation (Eq. (33) or (34)) in the combustion instability theories, the admittance coefficients Y_p (or ζ_p) and Γ_p must be determined for a given nozzle. The equations governing these quantities are readily derived from Eq. (24) using the definitions for ζ_p (i.e., Eqs.(30) and (31)). The resulting equations are:

$$H_p \frac{d\zeta_p}{d\varphi} = - M_p \zeta_p - N_p - H_p \zeta_p^2 \quad (37)$$

$$H_p \frac{d\Gamma_p}{d\varphi} = \left(-H_p \zeta_p + H_p \frac{Y_p - 1}{2c^2} \frac{du^2}{d\varphi} - M_p \right) \Gamma_p - \frac{I_p}{c^2}, \quad (38)$$

$$p = 1, 2, \dots, N$$

TASK II: CALCULATION OF THE NOZZLE RESPONSE

To obtain the nozzle response for any specified nozzle, Eqs. (37) and (38) are solved in the following manner. As pointed out earlier, the nonlinear terms vanish for the 1T mode (i.e., $\Gamma_1 = 0$, $I_1 = 0$) and it is only necessary to solve Eq. (37) to obtain ζ_1 (and hence Y_1) at the nozzle entrance. Since Eq. (37) does not depend on the higher modes, it can be solved independently for ζ_1 . Once ζ_1 has been determined, both Eqs. (37) and (38) must be solved for the other modes. In order to do this, the amplitude $A_1(\varphi)$ must be determined since Eq. (38) depends on $A_1(\varphi)$ and its derivatives through $I_p(\varphi)$. Once $\zeta_1(\varphi)$ is known, $A_1(\varphi)$ is determined by numerically integrating Eq. (30) where the constant of integration is determined by the specified value of the pressure amplitude P_1 (of the 1T mode) at the nozzle entrance. The value of A_1 thus found is introduced into Eq. (38) which is then solved for Γ_p .

It may be observed that Eq. (37) and (38) have singularities at the same points as Eq. (24). As before, the only singularity of interest is the throat. Since Eqs. (37) and (38) are first order ordinary differential equations, the numerical integration of these equations must start at some initial point where the initial conditions are known, and terminate at the nozzle entrance where the admittance coefficients Y_p and Γ_p are needed. Since the equations are singular at the throat, the integration is initiated at a point that is located a short distance upstream of the throat. The needed initial conditions are obtained by expanding the dependent variables in a Taylor series about the throat ($\varphi = 0$); thus,

$$\zeta_p(\varphi) = \zeta_p(0) + \varphi \zeta_p'(0) + \dots \quad (39a)$$

$$\Gamma_p(\varphi) = \Gamma_p(0) + \varphi \Gamma_p'(0) + \dots \quad (39b)$$

The coefficients $\zeta_p(0)$ and $\zeta'_p(0)$ can be determined by substituting Eq. (39a) in Eq. (37), and taking the limit as $\varphi \rightarrow 0$. The results are:

$$\zeta_p(0) = -\frac{N_p(0)}{M_p(0)} \quad (40a)$$

$$\zeta'_p(0) = \frac{-M'_p(0) \zeta_p(0) - H'_p(0) \zeta_p^2(0) - N'_k(0)}{H'_p(0) + M'_p(0)}, \quad (40b)$$

$$p = 1, 2, \dots, N$$

Similarly, $\Gamma_k(0)$ and $\Gamma'_k(0)$ can be determined by substituting Eq. (39b) in Eq. (38), and taking the limit as $\varphi \rightarrow 0$. The results are:

$$\Gamma_p(0) = -\frac{I_p(0)}{\bar{c}^2(0) M_p(0)} \quad (41a)$$

$$\begin{aligned} \Gamma'_p(0) = & \left\{ -\bar{c}^2(0) H'_p(0) \zeta_p(0) \Gamma_p(0) + \frac{\gamma-1}{2} \frac{d\bar{u}^2}{d\varphi}(0) H'_p(0) \Gamma_p(0) \right. \\ & - \bar{c}^2(0) M'_p(0) \Gamma_p(0) + \frac{\gamma-1}{2} \frac{d\bar{u}^2}{d\varphi}(0) M'_p(0) \Gamma_p(0) \\ & \left. - I'_p(0) \right\} / \left\{ \bar{c}^2(0) H'_p(0) + \bar{c}^2(0) M'_p(0) \right\} \quad (41b) \end{aligned}$$

In Eqs. (37) and (38), the quantities H_p , M_p , N_p and I_p are functions of the steady-state flow variables in the nozzle and these must be computed before performing the numerical integration to obtain ζ_p and Γ_p . For a specified nozzle profile, the steady-state quantities are computed by solving the quasi-one-dimensional isentropic steady-state equations for nozzle flow. Figure 2 shows the nozzle profile used in our computations. All of the length variables have been non-dimensionalized with respect to the radius of the combustion chamber, to which the nozzle is attached, and hence $r_c = 1$. At the throat r_{th} is fixed by the Mach number at the nozzle entrance plane. The nozzle profile is smooth and is

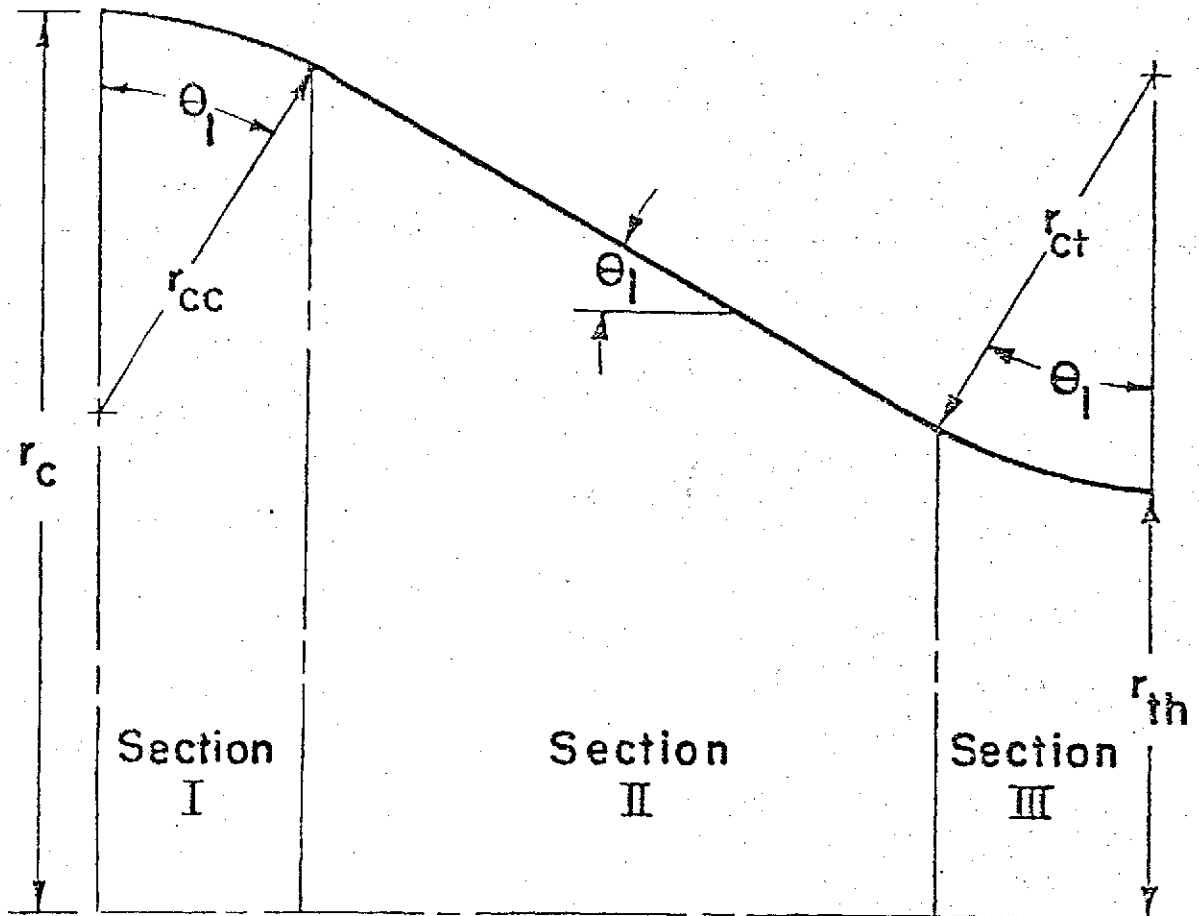


Figure 2. Nozzle Profile Used in Calculating Admittances.

completely specified by r_{cc} , r_{ct} and θ_1 , which are respectively the radius of curvature at the chamber, radius of curvature at the throat and slope of the central conical section. The steady-state equations are integrated using equal steps in steady-state potential ϕ by beginning at the throat and continuing to the nozzle entrance where the radius of the wall equals 1.

Computations of the admittance coefficients have been performed using a three-term series expansion consisting of the first tangential, second tangential and first radial modes. An Adam-Bashforth predictor-corrector scheme was used to perform the numerical integration, while the starting values needed to apply this method were obtained using a fourth order Runge-Kutta integration scheme. The integration computer program has been written so that the integration can be performed up to the nozzle entrance and also inside the combustion chamber for any desired distance. Thus, the admittance relation is obtained at the nozzle entrance section or at any station inside the chamber. Computations have been performed for several nozzles, at different frequencies and pressure amplitudes of the first tangential mode.

Figures 3 and 4 show the frequency dependence of the linear admittance coefficients for the 1T, 2T, and 1R modes for a typical nozzle ($\theta_1 = 20^\circ$, $r_{cc} = 1.0$, $r_{ct} = 0.9234$; $M = 0.2$). Here, ω is the frequency of the 1T mode, while the frequency of the 2T and 1R modes is 2ω due to nonlinear coupling. Hence the real parts of the linear admittance coefficients for the 2T and 1R modes attain their peak values at a higher frequency than that for the 1T mode. The linear admittance coefficients for the 1T mode are in complete agreement with those calculated previously by Bell and Zinn¹⁴ as expected from Eq. (22).

The frequency dependence of the nonlinear admittance coefficient for the 2T mode is plotted in Fig. 5 with pressure amplitude of the 1T mode as a parameter. While the behavior of the linear admittance coefficient depends only upon the frequency of oscillations, the behavior of the nonlinear admittance coefficient is seen to depend on the amplitude of the 1T mode. This result is expected, since in Eq. (38), I_p is a function of the amplitude of the 1T mode. As expected the absolute values of both Γ_r and Γ_i increase with increasing pressure amplitude of

the 1T mode, which acts as a driving force. It is observed that the absolute values of Γ_r and Γ_i vary similarly with frequency as the absolute values of Y_r and Y_i . The frequency dependence of the nonlinear admittance coefficient for the 1R mode is plotted in Fig. 6 with pressure amplitude of the 1T mode as a parameter.

Figures 7 and 8 show the effect of pressure amplitude upon the magnitude of the ratio of nonlinear admittance coefficient to the linear admittance coefficient for the 2T and 1R modes respectively. These results clearly indicate that the nonlinear contribution to the nozzle admittance is significant and should be included in nonlinear combustion stability analyses.

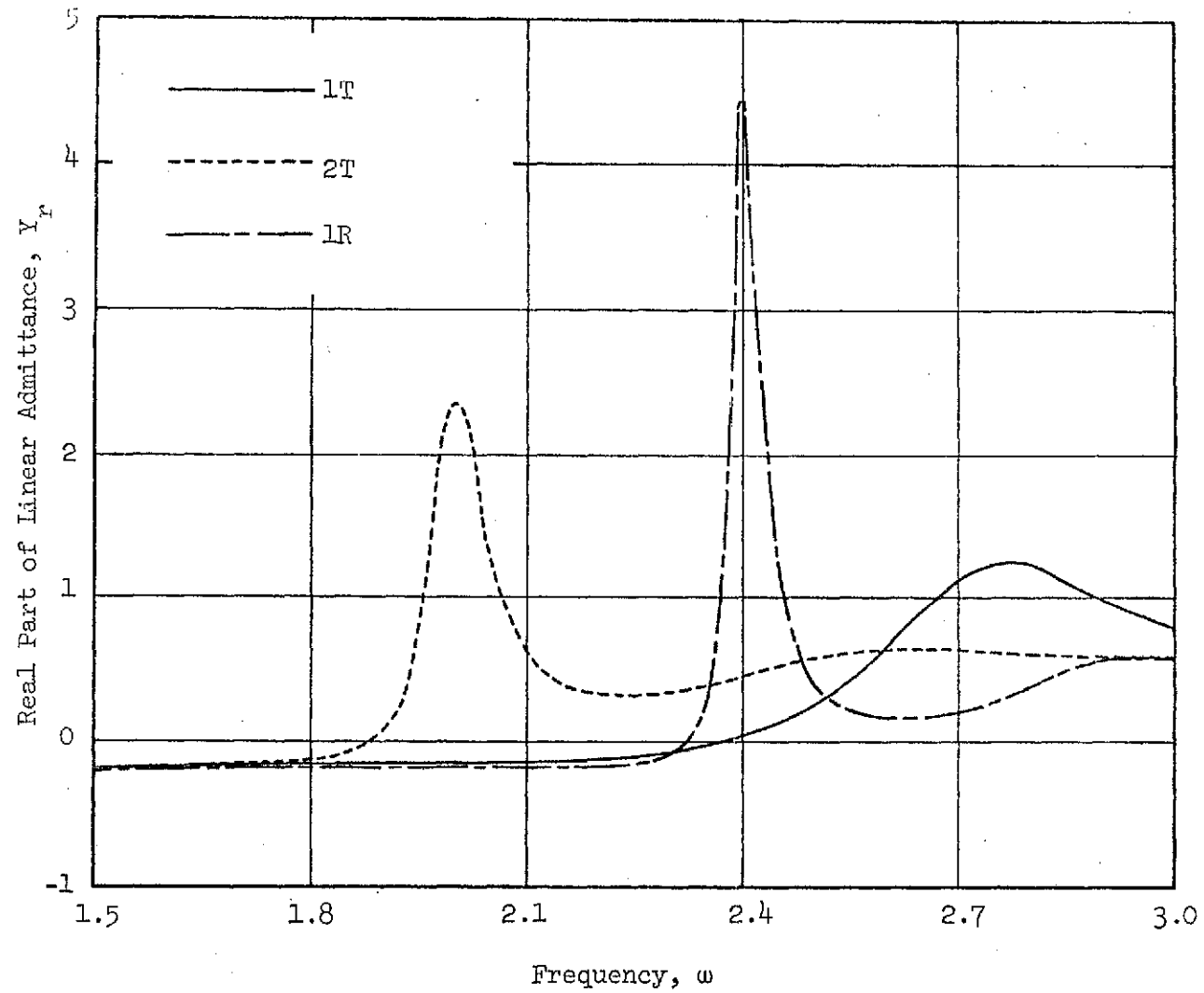


Figure 3. Linear Admittances for the 1T, 2T, and 1R Modes

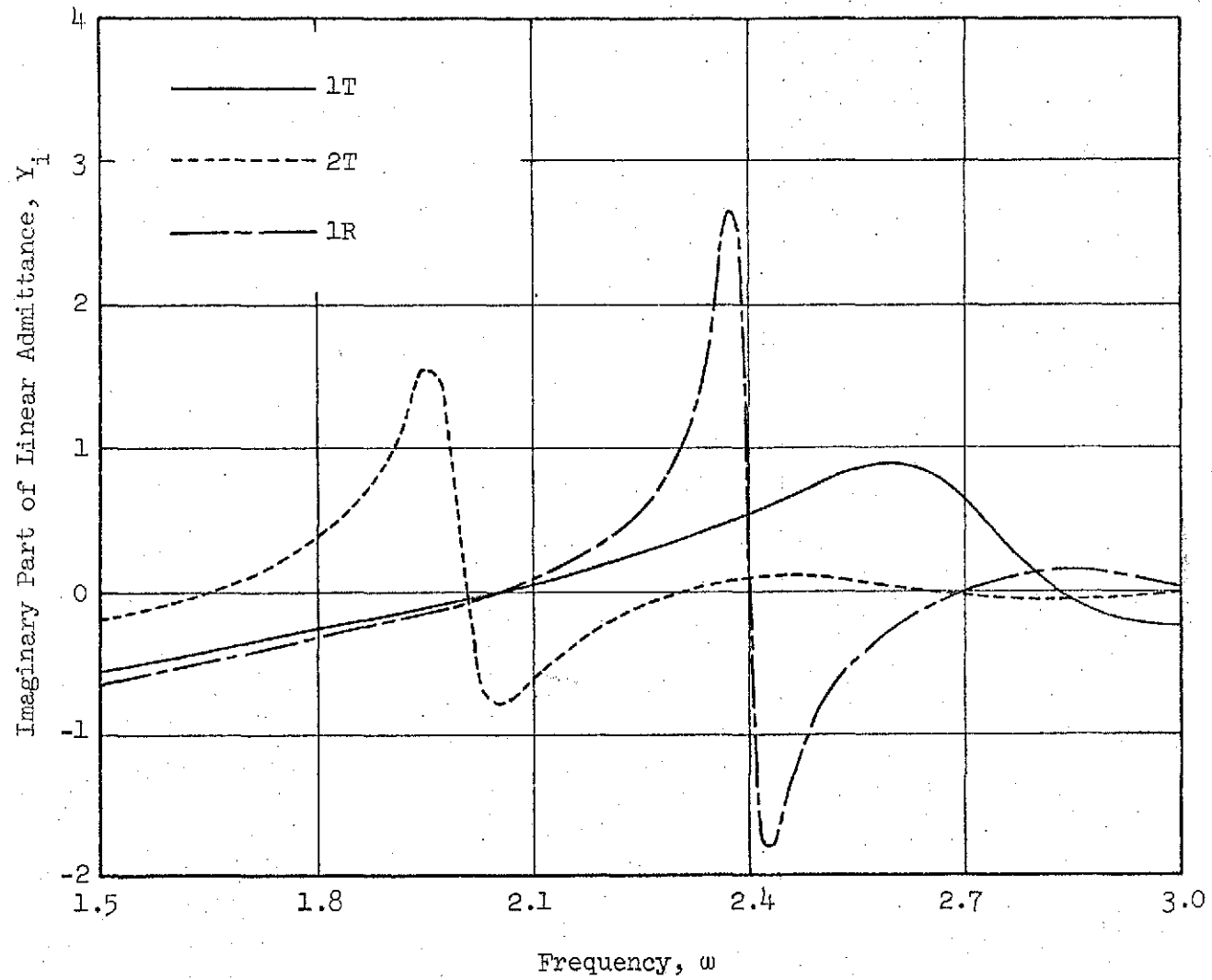


Figure 4. Linear Admittances for the 1T, 2T, and 1R Modes

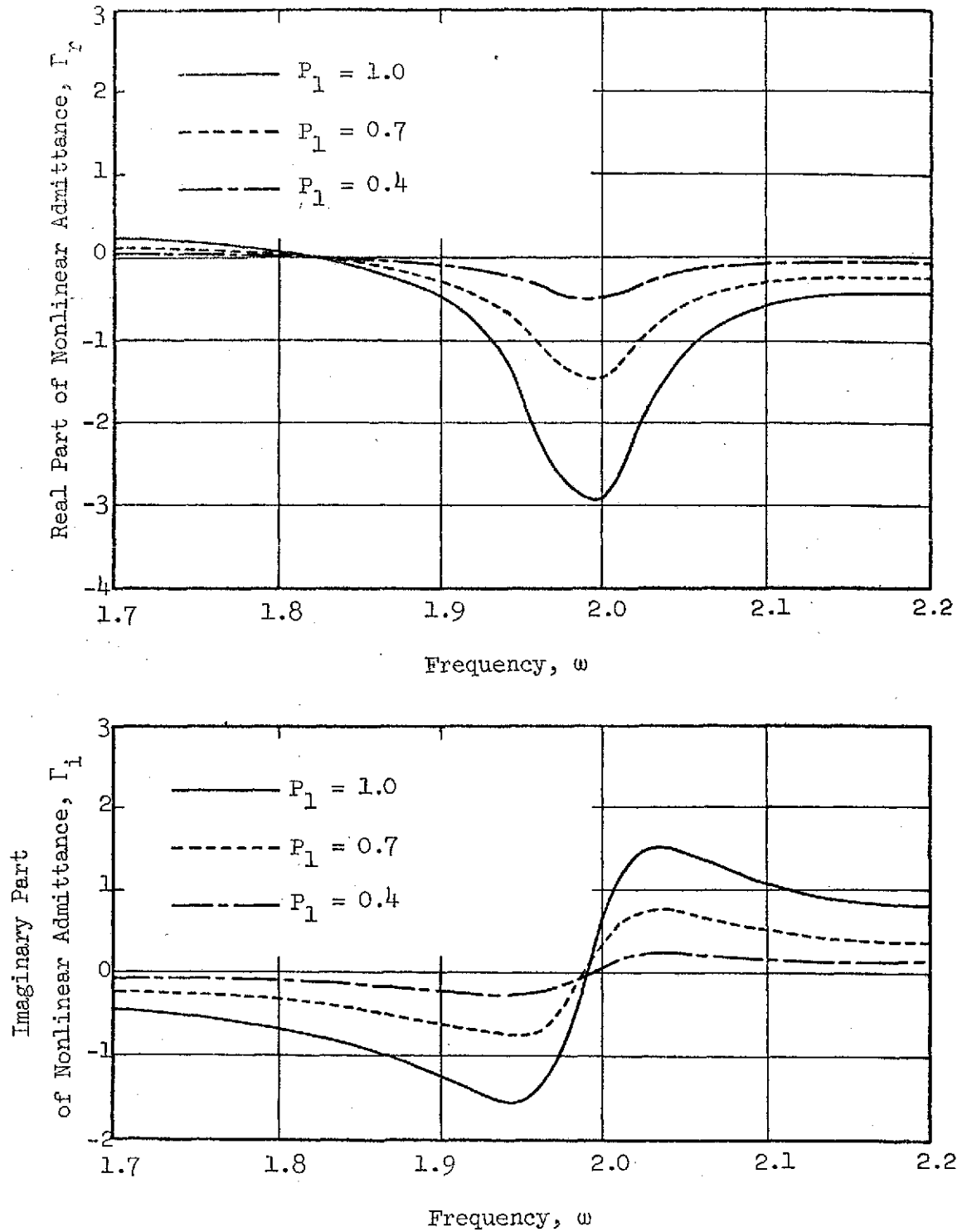


Figure 5. Nonlinear Admittances for the 2T Mode

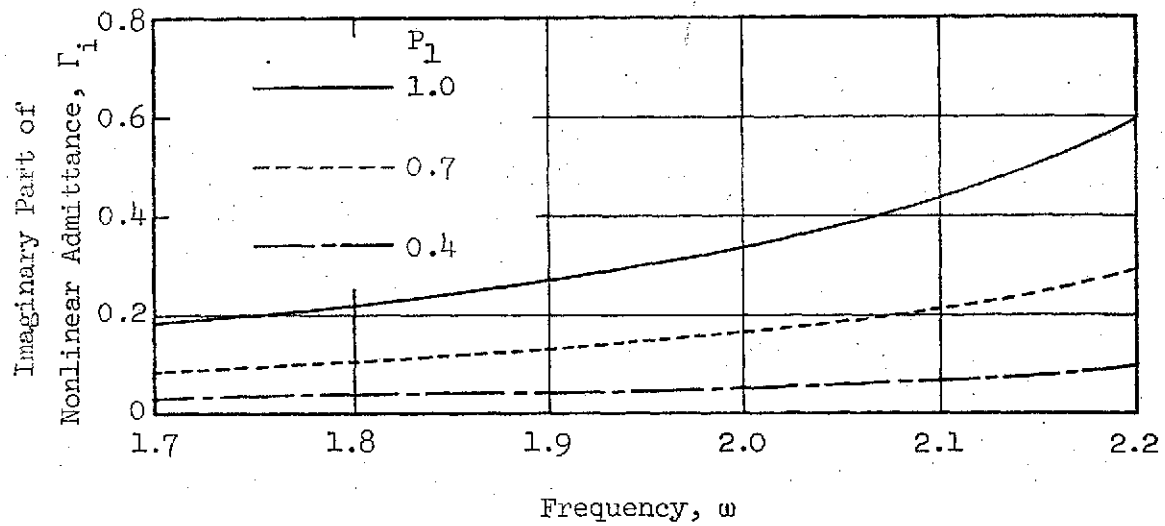
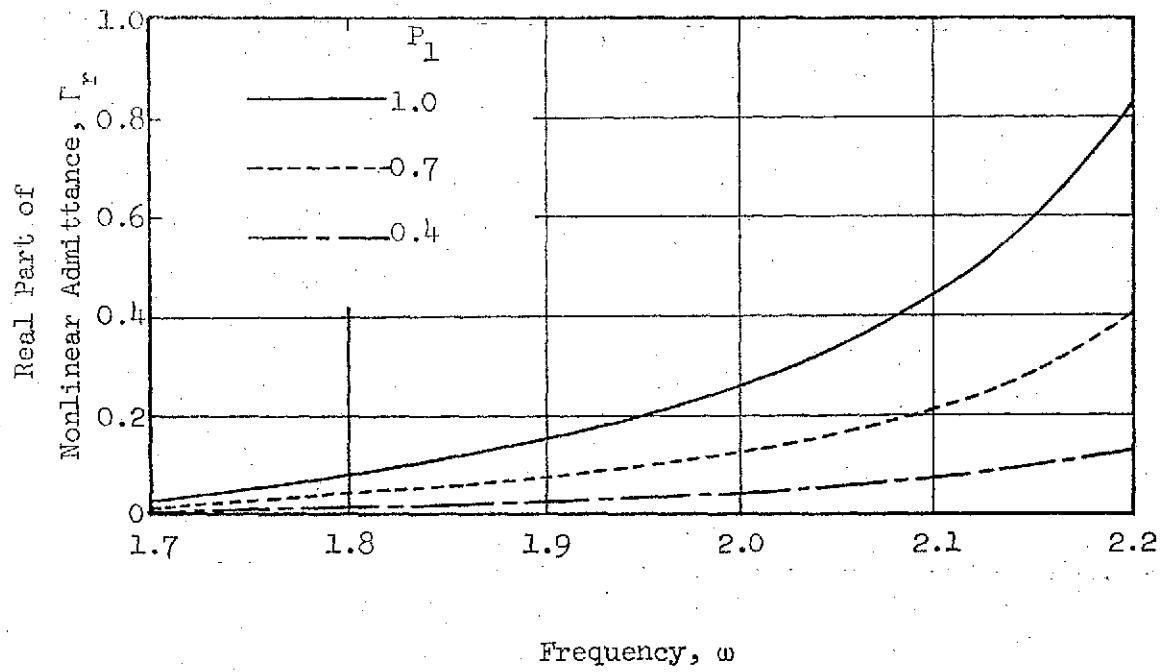


Figure 6. Nonlinear Admittances for the 1R Mode

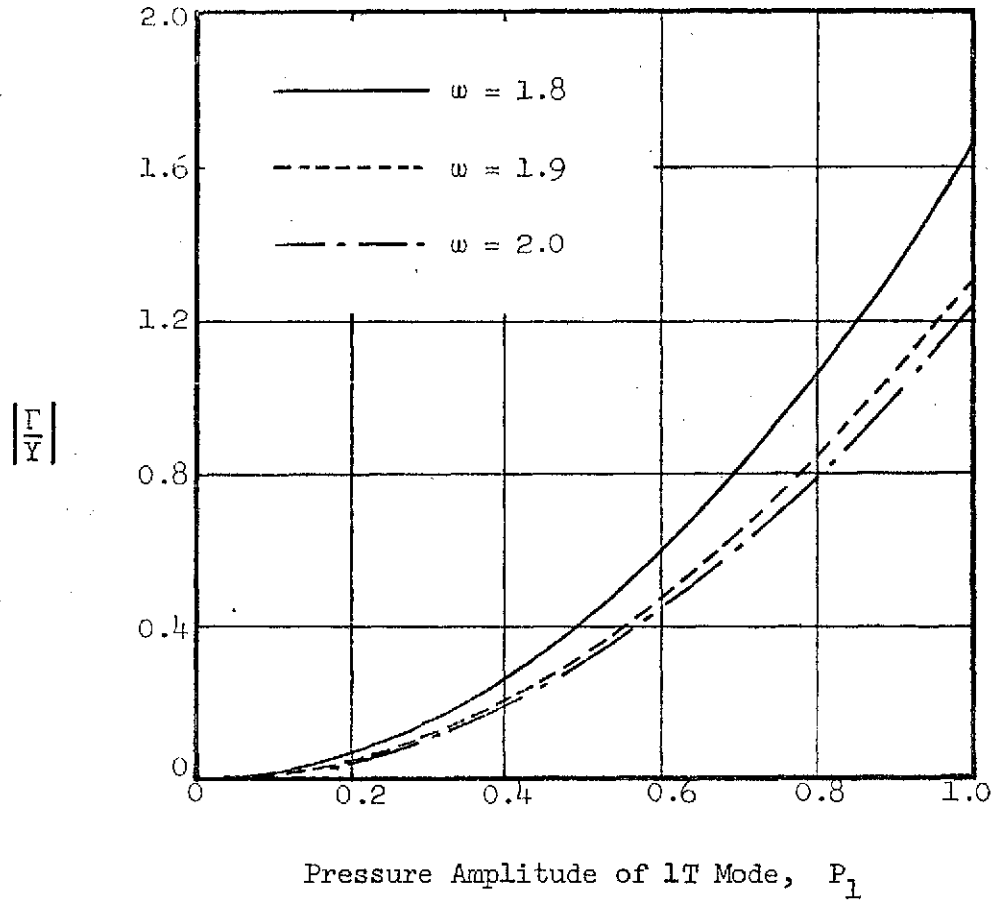


Figure 7. Relative Magnitudes of Linear and Nonlinear Admittances for 2T Mode.

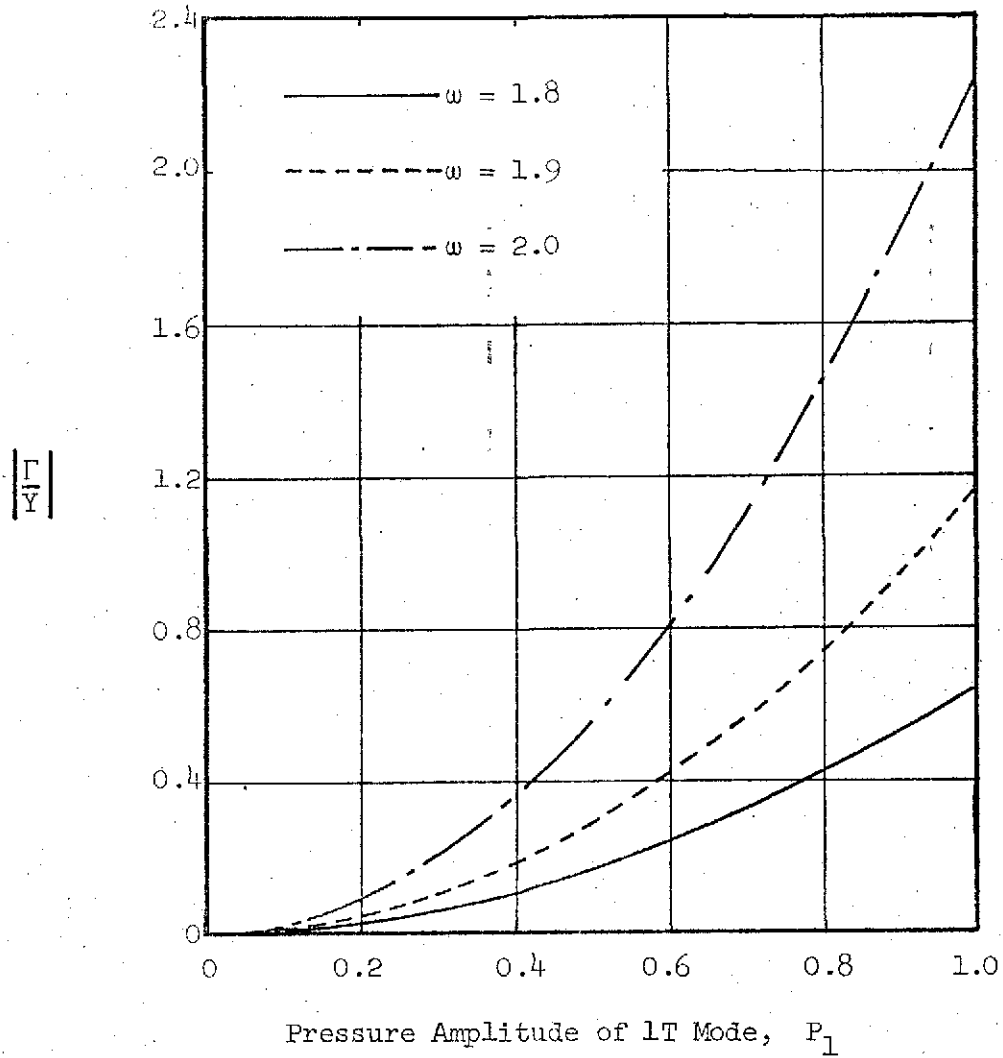


Figure 8. Relative Magnitudes of Linear and Nonlinear Admittances for lR Mode.

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