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OPTIMAL OUTPUT FEEDBACK CONTROL OF LINEAR  
SYSTEMS IN PRESENCE OF FORCING AND MEASUREMENT NOISE

By  
Suresh M. Joshi

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OPTIMAL OUTPUT FEEDBACK CONTROL OF LINEAR  
SYSTEMS IN PRESENCE OF FORCING AND MEASUREMENT NOISE

By Suresh M. Joshi

SUMMARY

The problem of obtaining an optimal control law, which is constrained to be a feedback of the available measurements, is considered for both continuous and discrete-time linear systems subjected to additive white process noise and measurement noise. Necessary conditions are obtained for minimizing a quadratic performance function for both finite and infinite terminal time cases. The feedback gain matrices are constrained to be constant for the infinite terminal time cases. For all the cases considered, algorithms are derived for generating sequences of feedback gain matrices which successively improve the performance function. A numerical example is given for the purpose of demonstration.

INTRODUCTION

Control system design for processes which suffer from process and measurement noise is an important problem. If the process dynamics are linear, and if the process noise and the measurement noise are Gaussian and white, it is well known that the control law which minimizes a quadratic performance function is a linear function of the optimal estimate of the state. Thus, the application of this control law necessitates the use of an online state estimator. This design approach may not be attractive from a practical viewpoint because of the high cost of the additional equipment required. In addition, there exists a dangerous possibility of divergence of the state estimator when the noise input matrix, and/or the means of the forcing and measurement noise processes are not

known accurately.

For this reason, the design of a satisfactory controller, which needs the feedback only of the available plant measurements, is a problem of immense practical importance. There have been numerous recent attempts in this area. These have proceeded mainly in two directions: pole shifting techniques using output feedback; and minimization of a quadratic performance function using output feedback. The former procedure is necessarily confined to deterministic systems, while the latter procedure is also useful for the practically important stochastic case, in the sense that the a priori knowledge of the noise statistics can be used to advantage. This paper considers the latter approach.

The noise-free version of the linear quadratic optimal output feedback control problem was considered in references (1) through (4). The basic philosophy was to minimize the performance degradation caused by the constraint on the control law. The constrained optimal control law depends on the initial state; therefore, the approach was to minimize the value of the performance function averaged over the initial state (with a priori known statistics), (references (3) and (4)); or to minimize the "worst" value of the performance function, when the initial state is known to lie within a hyperellipsoid in the state space (ref. (2)); or to minimize the maximum ratio of suboptimal and optimal values of the performance functions (ref. (1)). In all these cases, the problem was finally reduced to a complex nonlinear optimization problem.

In references (3) and (4), the noise-free case was considered, and the algebraic necessary conditions were derived. In addition, design of optimal dynamic compensators of a prespecified order was also discussed in reference (4). In reference (5), Axsäter considered the stochastic problem, with white process noise, but no measurement noise, and derived the necessary conditions for optimality. McLane (ref. (6)), considered the above problem, with state and control dependent forcing noise, but no measurement noise. All the efforts described above considered only continuous time systems. In addition, measurement noise was assumed to be absent. In practice, however, measurement noise is almost always present; therefore, it is important to consider the degrading effect of the direct transmittal of measurement noise through output feedback. In reference (7), the discrete-time, finite terminal time problem was considered and necessary conditions were obtained using dynamic programming. However, the infinite terminal time problem was not considered, and a minimizing algorithm

was not developed.

The purpose of this paper is to consider both continuous-time and discrete-time linear systems which are subjected to both forcing and measurement noise. The necessary conditions for optimality are derived for continuous-time systems in the first part, using the matrix minimum principle (ref. (8)). Both finite and infinite terminal time cases are considered, the control law for the latter case being constrained to be a constant feedback of the noisy measurements. Algorithms are derived to give sequences of successively better control gains, for both finite and infinite terminal time cases. The discrete-time case is next considered. The necessary conditions are derived, using the discrete matrix minimum principle, for both finite and infinite terminal time cases. For the finite terminal time case, it is shown that the necessary conditions coincide with those derived in reference (7) using dynamic programming, while for the infinite terminal time case, the feedback gains are constrained to be constant. Algorithms are developed for all cases to generate sequences of feedback gains which successively improve the performance function. A continuous-time numerical example is given for the purpose of demonstration.

#### SYMBOLS

A	$n \times n$ system matrix
B	$n \times m$ input matrix
C	$l \times n$ output matrix
E	expected value operator
G	$m \times l$ feedback matrix
g	$m \times l$ matrix defined in the text
H	Hamiltonian
$\bar{H}$	Lagrangian
i,j,k	indices
J, $\bar{J}$	performance functions for the finite and infinite terminal-time cases
K	$n \times n$ costate matrix
$\bar{K}$	$n \times n$ Lagrange multiplier matrix
N	terminal time for the discrete case
P	$n \times n$ Riccati-type matrix

Q	n x n weighting matrix for the state vector
R	m x m weighting matrix for the input vector
S	n x n terminal state weighting matrix
Tr	Trace operator
t	time
$t_f$	terminal time
u	m-vector input
V	n x n forcing noise covariance matrix
v	n-vector forcing noise
W	l x l measurement noise covariance matrix
w	l-vector measurement noise
x	n-dimensional state vector
y	l-dimensional output vector
$\delta(t)$	Dirac delta function
$\delta(k)$	Kronecker delta function
$\Sigma$	n x n state covariance matrix
$\bar{\Sigma}$	steady-state value of $\Sigma$

The superscript "T" denotes the transpose of a matrix, and "-1" denotes the inverse.

## CONTINUOUS-TIME SYSTEMS

### Finite Terminal Time Case

The system is given by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + v(t) \quad (1)$$

$$y(t) = C(t)x(t) + w(t) \quad (2)$$

where  $A(t)$ ,  $B(t)$ ,  $C(t)$  are  $n \times n$ ,  $n \times m$ , and  $l \times n$  matrices and  $x(t)$ ,  $u(t)$ , and  $y(t)$  are respectively  $n \times 1$ ,  $m \times 1$ , and  $l \times 1$  state vector, input vector, and output vector.  $v(t)$  and  $w(t)$  are white noise processes, with zero means, and

$$E[v(t)v^T(\tau)] = V(t)\delta(t-\tau) \quad (3)$$

$$E[w(t)w^T(\tau)] = W(t)\delta(t-\tau) \quad (4)$$

where  $V(t) \geq 0$ ,  $W(t) \geq 0$  are  $n \times n$  and  $l \times l$  matrices, and  $\delta(\cdot)$  is the Dirac delta function. Also,

$$E[v(t)w^T(\tau)] = 0 \quad (5)$$

The initial state covariance matrix is assumed to be known:

$$E[x(0)x^T(0)] = \Sigma(0) = \Sigma_0$$

Consider the problem of minimizing the functional

$$\frac{1}{2} E[x^T(t_f)Sx(t_f)] + \frac{1}{2} E \int_0^{t_f} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)] dt \quad (6)$$

where  $Q(t) \geq 0$ ,  $R(t) > 0$ ,  $S \geq 0$  are  $n \times n$ ,  $m \times m$ , and  $n \times n$  matrices. Suppose the control law is restricted to be a feedback of the output.

$$\begin{aligned} u(t) &= G(t)y(t) \\ &= G(t)C(t)x(t) + G(t)w(t) \end{aligned} \quad (7)$$

where  $G(t)$  is the  $m \times l$  output feedback matrix.

This formulation allows for direct transmittal of white noise into the system via feedback. It is clearly seen that the term  $E[u^T(t)R(t)u(t)]$  is infinite, since  $E[w(t)w^T(\tau)] = W(t)\delta(t-\tau)$ . This can be avoided by including only that part of the control energy which does not contain white noise. Thus the following modified performance function is considered:

$$\begin{aligned} J &= \frac{1}{2} E[x^T(t_f)Sx(t_f)] \\ &+ \frac{1}{2} E \int_0^{t_f} \left\{ x^T(t)Q(t)x(t) + (G(t)C(t)x(t))^T R(t) (G(t)C(t)x(t)) \right\} dt \end{aligned} \quad (8)$$

The measurement noise is not directly included in the performance function, but the degrading effect because of its feedback is indeed taken into account via the state covariance evolution equation. Thus, since the state covariance matrix depends on the product of the feedback gain matrix and the measurement noise, the measurement noise-dependent part of the input signal will be auto-

matically penalized in proportion to its covariance matrix  $W$ , as will be seen later. This formulation thus provides a design technique in which the knowledge of the noise statistics can be used to advantage.

The evolution of the covariance matrix of the state  $x(t)$  can be easily derived as

$$\frac{d\Sigma(t)}{dt} = (A + BGC)\Sigma + \Sigma(A + BGC)^T + BGWG^T B^T + V \quad (9)$$

where  $\Sigma(t) = E[x(t)x^T(t)]$ ,  $\Sigma(0) = \Sigma_0$  and the symbol  $\cdot(t)$  is dropped for convenience.

The performance function of equation (8) can also be written as

$$J = \frac{1}{2} \text{Tr}[\Sigma(t_f)] + \frac{1}{2} \int_0^{t_f} \text{Tr}[(Q + C^T G^T R G C)\Sigma] dt \quad (10)$$

It is required to minimize the  $J$  in (10) with respect to  $G$  subject to the constraint (9).

It is now possible to apply the matrix minimum principle (ref. (8)). Define the Hamiltonian

$$H(\Sigma, G, K) = \frac{1}{2} \left\{ \text{Tr}[(Q + C^T G^T R G C)\Sigma] + \text{Tr} \left[ \left\{ (A + BGC)\Sigma + \Sigma(A + BGC)^T + BGWG^T B^T + V \right\} K^T \right] \right\} \quad (11)$$

where  $K(t)$  is an  $n \times n$  matrix of costate variables. The initial state covariance  $\Sigma(0) = \Sigma_0$ , and the matrices  $V(t)$  and  $W(t)$  are assumed to be known a priori. Using the matrix minimum principle, and the gradient matrices derived in reference (8), the necessary conditions for optimality can be derived as

$$RGC^T + B^T P \Sigma C^T + B^T P B G W = 0 \quad (12)$$

$$\frac{dP}{dt} = - \left[ (A + BGC)^T P + P(A + BGC) + (Q + C^T G^T R G C) \right] \quad (13)$$



$$\frac{d\Sigma}{dt} = (A + BGC)\Sigma + \Sigma(A + BGC)^T + BGWG^T B^T + V \quad (14)$$

$$\Sigma(0) = \Sigma_0 \quad (15)$$

$$P(t_f) = S \quad (16)$$

In these equations,  $P(t) = 2K(t)$ , and the symmetry of  $P(t)$  and  $K(t)$  is obtained during the derivation because of the symmetry of the right-hand side of (13), and because  $S = S^T$ . Equations (12) to (16) describe a nonlinear two-point boundary value problem. For a given  $G(t)$ , equations (13) and (14) are linear in  $P$  and  $\Sigma$ , while for given  $P$  and  $\Sigma$ , equation (12) is linear in  $G$ . It is interesting to note that, although the measurement noise-dependent portion of the control signal was not weighted in the performance function, the measurement covariance matrix  $W$  does tend to reduce the feedback gain  $G$  in equation (12), as expected. For the case with perfect measurements ( $W = 0$ ), the necessary conditions reduce to those of Åxsäter (ref. (5)), while for the noise-free case ( $V = 0, W = 0$ ), the necessary conditions reduce to those due to Levine, Johnson, and Athans (ref. (4)).

#### Infinite Duration Case

The necessary conditions obtained above require the solution of a complex, two-point boundary value problem, and the feedback gain  $G(t)$  obtained is time-varying. A natural extension of this problem is the constant-coefficient, infinite terminal time case, with  $G$  constrained to be a time-invariant matrix. Under these circumstances, with  $A, B, C, V, W$  constant, for a stabilizing output feedback gain matrix  $G$ , it is well known that the covariance matrix  $\Sigma(t)$  tends to a constant matrix  $\bar{\Sigma} \geq 0$  as  $t \rightarrow \infty$ . In the treatment below, it is assumed that, for the  $V, G, W$ , and the dynamics under consideration,  $\bar{\Sigma}$  exists and is a positive definite matrix. This would indeed be true if  $(V + BGWG^T B^T)$  is positive definite.

In this case, if the terminal time  $t_f \rightarrow \infty$ , the performance function of (10) will be infinite, since the integrand tends to a constant value. Therefore, it is meaningful to minimize the "cost rate":

$$\bar{J} = \lim_{t_f \rightarrow \infty} \frac{1}{t_f} \int_0^{t_f} \text{Tr} \left\{ (Q + C^T G^T R G C) \Sigma \right\} dt \quad (17)$$

or

$$\bar{J} = \frac{1}{2} \text{Tr} \left\{ (Q + C^T G^T R G C) \bar{\Sigma} \right\} \quad (18)$$

where  $\bar{\Sigma}$  is the steady state (positive definite by assumption) solution of the covariance equation (9). Thus, the problem reduces to a static optimization problem. Defining the Lagrangian

$$\begin{aligned} \bar{H}(\bar{\Sigma}, G, \bar{K}) = & \frac{1}{2} \text{Tr} \left[ (Q + C^T G^T R G C) \bar{\Sigma} \right] \\ & + \text{Tr} \left[ \left\{ (A + B G C) \bar{\Sigma} + \bar{\Sigma} (A + B G C)^T \right. \right. \\ & \left. \left. + B G W G^T B^T + V \right\} \bar{K}^T \right] \end{aligned} \quad (19)$$

where  $\bar{K}$  is a  $n \times n$  constant matrix of Lagrange multipliers, the necessary conditions for a minimum are

$$\frac{\partial \bar{H}}{\partial \bar{\Sigma}} = 0, \quad \frac{\partial \bar{H}}{\partial G} = 0, \quad \frac{\partial \bar{H}}{\partial \bar{K}} = 0 \quad (20)$$

which finally reduce to the steady-state forms of equations (12), (13), and (14), with  $\Sigma$  replaced by  $\bar{\Sigma}$ , and  $P = 2\bar{K} = P^T$ . Thus the necessary conditions for the infinite terminal time problem, with control gains constrained to be constant, are simultaneous nonlinear matrix algebraic equations. It should be noted that the noise-free case is not a simple extension of these results since, in that case,  $\bar{\Sigma} = 0$  for a stable  $G$ .

#### A Numerical Algorithm

Considering first the finite terminal time case, let  $G^0(t)$  and  $G^1(t)$  be two feedback gains, such that

$$\frac{d\Sigma^i}{dt} = (A + B G^i C) \Sigma^i + \Sigma^i (A + B G^i C)^T + B G^i W G^{iT} B^T + V \quad (21)$$

$$i = 0, 1$$

$$\frac{dP^0}{dt} = - \left[ (A + B G^0 C)^T P^0 + P^0 (A + B G^0 C) + Q + C^T G^{0T} R G^0 C \right] \quad (22)$$

After a lengthy algebraic manipulation as outlined in Appendix A, it can be proved that

$$\begin{aligned}
J(G^0) - J(G^1) = & \\
\frac{1}{2} \text{Tr} \int_0^t & \left[ \left\{ RG^0 C \Sigma^1 C^T + B^T P^0 \Sigma^1 C^T + B^T P^0 B G^1 W \right\}^T R^{-1} \right. \\
& \left. \left\{ RG^0 C \Sigma^1 C^T + B^T P^0 \Sigma^1 C^T + B^T P^0 B G^1 W \right\} \right. \\
& - \left. \left\{ RG^1 C \Sigma^1 C^T + B^T P^0 \Sigma^1 C^T + B^T P^0 B G^1 W \right\}^T R^{-1} \right. \\
& \left. \left\{ RG^1 C \Sigma^1 C^T + B^T P^0 \Sigma^1 C^T + B^T P^0 B G^1 W \right\} \right] (C \Sigma^1 C^T)^{-1} dt \\
& + \frac{1}{2} \text{Tr} \int_0^t \left[ (G^0 - G^1)^T B^T P^0 B (G^0 - G^1) \right] W dt \quad (23)
\end{aligned}$$

Thus, if  $G^1$  is chosen to satisfy

$$RG^1 C \Sigma^1 C^T + B^T P^0 \Sigma^1 C^T + B^T P^0 B G^1 W = 0 \quad (24)$$

we have

$$J(G^0) - J(G^1) \geq 0 \quad (25)$$

Thus, a minimizing algorithm is

- (a) choose an initial  $G^0(t)$
- (b) obtain  $P^0(t)$  using (22)
- (c) solve (24) and (21) simultaneously for  $\Sigma^1$  and  $G^1$
- (d) go to (b) after increasing the superscript by unity

Thus, a successive reduction in  $J$  is obtained. In the proof of the algorithm, it has been assumed that  $\Sigma(t) > 0$ . This will be true if  $\Sigma(0) > 0$ ; however, it is not necessary that  $\Sigma(0)$  be positive definite for  $\Sigma(t)$  to be positive definite.

For the infinite terminal time case, after a similar manipulation (given in the Appendix A), it can be shown that equation (23) holds with the integral signs removed and  $\bar{\Sigma}^1$  replacing  $\Sigma^1$ . Therefore, the algorithm given in steps (a) - (d) will still hold, with equations (21) and (22) replaced by their steady-state versions, if each  $G$  is stable,  $\bar{\Sigma} > 0$  at each stage.

The following continuous-time system is considered for the purpose of demonstration of the technique developed:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 4 & -5 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

with

$$V = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}$$

$$W = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

The performance function to be minimized was that in equation (17), with

$$Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The algorithm developed in the text was used. At each iteration, the simultaneous nonlinear equations in  $\bar{\Sigma}$  and  $G$  (equation (12) and the steady-state version of equation (14)) were solved using Newton's method. The initial feedback gain matrix was:

$$G^0 = \begin{bmatrix} -9.9 & -6.7 \\ -0.279 & 0.02 \end{bmatrix}$$

The choice was arbitrary, the only restriction being the stability of the matrix  $(A + BG^0C)$ .

The program converged after five iterations, to:

$$G = \begin{bmatrix} -6.879 \text{ E-01} & -5.196 \text{ E-01} \\ -9.861 \text{ E-01} & -3.993 \text{ E-01} \end{bmatrix}$$

At each iteration, it was verified that  $(A + BGC)$  was stable. Positive definiteness of  $\bar{\Sigma}$  followed because of the positive definiteness of  $V$ . The value of the performance function for the initial  $G$  was 3.466, which was reduced to 0.3443 at the final iteration.

## DISCRETE-TIME SYSTEMS

### Finite Terminal Time Case

Consider the discrete-time system

$$x(k+1) = A(k)x(k) + B(k)u(k) + v(k) \quad (26)$$

$$y(k) = C(k)x(k) + w(k) \quad (27)$$

Notations and dimensions are the same as in the continuous case.  $v(k)$  and  $w(k)$  are zero mean, white noise processes, with

$$E[v(k)v^T(i)] = V(k)\delta(k-i) \quad (28)$$

$$E[w(k)w^T(i)] = W(k)\delta(k-i) \quad (29)$$

and

$$E[v(k)w^T(i)] = 0 \quad (30)$$

$$E[x(0)x^T(0)] = \Sigma(0) = \Sigma_0 \quad (\text{given})$$

where  $\delta(\cdot)$  is the Kronecker delta function.

$$V(k) \geq 0, W(k) \geq 0$$

The performance function to be minimized is

$$J = \frac{1}{2} E \left[ \sum_{k=0}^{N-1} \left\{ x^T(k) Q(k) x(k) + u^T(k) R(k) u(k) \right\} \right] + \frac{1}{2} E[x^T(N) S x(N)] \quad (31)$$

subject to (26), (27), and the restriction that

$$\begin{aligned} u(k) &= G(k)y(k) \\ &= G(k)C(k)x(k) + G(k)w(k) \end{aligned}$$

$$Q(k) \geq 0, R(k) > 0, S \geq 0 \quad (32)$$

The Kronecker delta function is mathematically well defined; thus the performance function of equation (31) can be simplified as given below:

$$\begin{aligned} J &= \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr} \left[ \left\{ Q(k) + C^T(k) G^T(k) R(k) G(k) C(k) \right\} \Sigma(k) \right] \\ &+ \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr}[G^T(k) R(k) G(k) W(k)] \\ &+ \frac{1}{2} \text{Tr}[S \Sigma(N)] \end{aligned} \quad (33)$$

It is easy to show that the covariance matrix  $\Sigma(k) = E[x(k)x^T(k)]$  evolves according to

$$\Sigma(k+1) = (A + BGC)\Sigma(k)(A + BGC)^T + BGWG^T B^T + V \quad (34)$$

In writing (34), the symbol  $\cdot(k)$  has been dropped for convenience in most of the right-hand side variables. Thus, the problem reduces to the minimization of  $J$  in (33) subject to the constraint (34), with  $\Sigma(0)$ ,  $W(k)$  and  $V(k)$  known a priori. The "discrete matrix minimum principle" given in reference (8) can be readily used. Define the Hamiltonian

$$\begin{aligned} &H(\Sigma(k), G(k), K(k+1)) \\ &= \frac{1}{2} \text{Tr} \left\{ \left( Q(k) + C^T(k) G^T(k) R(k) G(k) C(k) \right) \Sigma(k) \right\} + \frac{1}{2} \text{Tr} \left\{ G^T(k) R(k) G(k) W(k) \right\} \\ &+ \text{Tr} \left[ \left\{ \left( A(k) + B(k) G(k) C(k) \right) \Sigma(k) \left( A(k) + B(k) G(k) C(k) \right)^T \right. \right. \\ &\left. \left. + B(k) G(k) W(k) G^T(k) B^T(k) + V(k) - \Sigma(k) \right\} K^T(k+1) \right] \end{aligned} \quad (35)$$

In the treatment below, all the variables at time  $k$  are denoted without  $\cdot(k)$ , whereas the variables at time  $(k+1)$  are denoted with  $\cdot(k+1)$ .

In (35),  $K(k+1)$  is the  $n \times n$  matrix of "costate variables". Applying the discrete matrix minimum principle and the gradient matrix formulae of reference (8), the necessary conditions are

$$G = -\left(R + B^T P(k+1)B\right)^{-1} B^T P(k+1)A \Sigma C^T (C \Sigma C^T + W)^{-1} \quad (36)$$

$$P(k) = Q + C^T G^T R G C + (A + BGC)^T P(k+1)(A + BGC) \quad (37)$$

$$\Sigma(k+1) = (A + BGC)\Sigma(k)(A + BGC)^T + B G W G^T B^T + V \quad (38)$$

$$P(N) = S \quad (39)$$

$$\Sigma(0) = \Sigma_0 \quad (40)$$

where, as in the continuous time case,  $P(k) = 2K(k) = P^T(k)$ . Note that in the discrete-time case, it has been possible to obtain  $G$  explicitly in (36). (It is assumed that  $\Sigma(k)$  is positive definite,  $C$  has rank  $l$ , so that  $C\Sigma(k)C^T$ , is positive definite. A sufficient condition for  $\Sigma(k)$  to be positive definite for  $k > 0$  is that  $\Sigma(0)$  is positive definite, although this is not necessary.)

Equations (36) to (40) define a nonlinear matrix two point boundary-value problem. It can be verified that these conditions are equivalent to those obtained in reference (7) using dynamic programming.

#### Infinite Duration Case

As in the continuous time case, if  $A$ ,  $B$ ,  $C$ ,  $V$ ,  $W$ , and  $G$  are constant and  $(A + BGC)$  is stable, it can be shown that the state covariance matrix  $\Sigma_k \rightarrow \bar{\Sigma} \geq 0$  as  $k \rightarrow \infty$  and satisfies (38) with  $\Sigma_{k+1} = \Sigma_k$ . Furthermore, it is assumed that  $\bar{\Sigma}$  is positive definite for the  $V$ ,  $W$ , and  $G$  and the dynamics under consideration. This would indeed be true if  $(V + B G W G^T B^T)$  is positive definite. As in the continuous case, the modified performance function is

$$\bar{J} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2} E[x^T(k) Q x(k) + u^T(k) R u(k)] \quad (41)$$

$$\text{or} \quad \bar{J} = \frac{1}{2} \text{Tr} \left\{ (Q + C^T G^T R G C) \bar{\Sigma} \right\} + \frac{1}{2} \text{Tr}(G^T R G W) \quad (42)$$

For this static optimization problem having the constraint (38) with  $\Sigma_{k+1} = \Sigma_k = \bar{\Sigma}$ , define the Lagrangian as

$$\begin{aligned} \bar{H}(\bar{\Sigma}, G, \bar{K}) = & \frac{1}{2} \text{Tr} \left[ (Q + C^T G^T R G C) \bar{\Sigma} \right] + \frac{1}{2} \text{Tr}[G^T R G W] \\ & + \text{Tr} \left[ \left\{ (A + B G C) \bar{\Sigma} (A + B G C)^T + B G W G^T B^T + V - \bar{\Sigma} \right\} \bar{K}^T \right] \end{aligned} \quad (43)$$

where  $\bar{K}$  is the  $n \times n$  matrix of Lagrange multipliers. The necessary conditions are obtained by equating  $\frac{\partial H}{\partial \Sigma}$ ,  $\frac{\partial H}{\partial G}$ , and  $\frac{\partial H}{\partial \bar{K}}$  to zero:

$$G = -(R + B^T P B)^{-1} B^T P A \bar{\Sigma} C^T (C \bar{\Sigma} C^T + W)^{-1} \quad (44)$$

$$P = Q + C^T G^T R G C + (A + B G C)^T P (A + B G C) \quad (45)$$

$$\bar{\Sigma} = (A + B G C) \bar{\Sigma} (A + B G C)^T + V + B G W G^T B^T \quad (46)$$

where, as in the continuous case,  $P = 2\bar{K} = P^T$ . This is a set of nonlinear matrix algebraic equations. Since  $R > 0$ ,  $\bar{\Sigma} > 0$  and  $\text{rank}(C) = l$ , the inverses in (44) exist.

#### A Numerical Algorithm

Considering first the finite terminal time case, let  $G^0(k)$  and  $G^1(k)$  be two feedback gains, such that

$$\Sigma^i(k+1) = (A + B G^i C) \Sigma^i(k) (A + B G^i C)^T = B G^i W G^{iT} B^T + V, \quad i = 0, 1 \quad (47)$$

and

$$P^0(k) = Q + C^T G^{0T} R G^0 C + (A + B G^0 C)^T P^0(k+1) (A + B G^0 C) \quad (48)$$

After a lengthy manipulation as outlined in Appendix B, it can be proved that

$$\begin{aligned} J(G^0) - J(G^1) = & \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr} \left[ (C \Sigma^1 C^T + W) \left\{ (G^0 + g)^T (R + B^T P^0(k+1) B) (G^0 + g) \right. \right. \\ & \left. \left. - (G^1 + g)^T (R + B^T P^0(k+1) B) (G^1 + g) \right\} \right] \end{aligned} \quad (49)$$

where



$$g = (R + B^T P^0(k+1)B)^{-1} B^T P^0(k+1) A \Sigma^1 C^T (C \Sigma^1 C^T + W)^{-1} \quad (50)$$

In writing equations (47) through (50),  $\cdot(k)$  has been dropped in most places for convenience, except when required for clarity. Letting

$$G^1 = -g \quad (51)$$

in (49), we have

$$J(G^0) - J(G^1) \geq 0 \quad (52)$$

Thus, a minimizing algorithm is

- (a) choose initial  $G^0(k)$
- (b) obtain  $P^0(k)$ ,  $k = 0 \dots, N$  from the linear equation (48)
- (c) obtain  $\Sigma^1(k)$  by solving the nonlinear difference equation obtained by substitution of (51) in (47)
- (d) obtain  $G^1(k)$  using  $P^0(k+1)$  and  $\Sigma^1(k)$  determined in the above steps
- (e) go to (b), with the superscript raised by unity

Thus a successive reduction in  $J$  is obtained. For the infinite terminal time case, after a similar manipulation (given in the Appendix B), it can be shown that

$$J(G^0) - J(G^1) = \frac{1}{2} \text{Tr} \left[ (C \Sigma^1 C^T + W) \left\{ (G^0 + g)^T (R + B^T P^0 B) (G^0 + g) - (G^1 + g)^T (R + B^T P^0 B) (G^1 + g) \right\} \right] \quad (53)$$

where  $\bar{\Sigma}^1$  and  $P^0$  satisfy the static versions of (47) and (48), and

$$g = (R + B^T P^0 B)^{-1} B^T P^0 A \bar{\Sigma}^1 C^T (C \bar{\Sigma}^1 C^T + W)^{-1} \quad (54)$$

Thus  $J(G^1) \leq J(G^0)$  with  $G^1 = -g$ , and the resulting sequence of  $G^i$ 's improves the value of  $J$  successively if each  $(A + BGC)$  is stable, and each  $\Sigma > 0$ .

Thus, a numerical algorithm similar to that for the finite terminal time case is obtained.

## CONCLUSIONS

The practically important problem of obtaining an optimal output feedback control law for linear systems subjected to both forcing and measurement noise was considered. Necessary conditions for optimality were obtained for continuous-time and discrete-time systems, using the matrix minimum principle. Both finite and infinite terminal time cases were considered for each problem. Algorithms were derived for obtaining sequences of feedback gains which guarantee a monotonic improvement of the performance function. The method developed provides a powerful and practically feasible design technique in which the a-priori knowledge of the noise statistics is used to advantage.

Although the necessary conditions for a minimum were obtained, the question of existence is still unanswered, and needs further attention. Also, the convergence properties of the sequences of feedback gains generated deserve more attention.

For the infinite terminal time case, it was assumed that  $\bar{\Sigma} > 0$ . The case  $\bar{\Sigma} = 0$  will be a simple extension of the work of McLane (reference (6)); however, the case  $\bar{\Sigma} \geq 0$  needs further investigation. Also, the technique developed in this paper can be easily modified for the design of optimal dynamic compensators, by following a procedure similar to that by Levine et. al. (reference (4)).

APPENDIX A

ALGORITHM FOR SUCCESSIVE REDUCTION OF THE  
PERFORMANCE FUNCTION: CONTINUOUS TIME CASE

Since

$$\frac{d}{dt} \left\{ \text{Tr}(\Sigma^0 - \Sigma^1) P^0 \right\} = \text{Tr} \left[ \left\{ \frac{d}{dt}(\Sigma^0 - \Sigma^1) \right\} P^0 \right] + \text{Tr} \left[ (\Sigma^0 - \Sigma^1) \frac{dP^0}{dt} \right] \quad (\text{A-1})$$

After simplification using equation (21) and (22) and integrating both sides between 0 and  $t_f$ ,

$$\begin{aligned} J(G^0) - J(G^1) &= \text{Tr} \int_0^{t_f} [B(G^0 - G^1)C\Sigma^1 P^0] dt \\ &+ \frac{1}{2} \text{Tr} \int_0^{t_f} [\Sigma^1 (C^T G^{0T} R G^0 C - C^T G^{1T} R G^1 C)] dt \\ &+ \frac{1}{2} \text{Tr} \int_0^{t_f} B(G^0 W G^{0T} - G^1 W G^{1T}) B^T P^0 dt \end{aligned} \quad (\text{A-2})$$

Using the trace identities in reference (8), it can be proved that equations (A-2) and (23) are equivalent.

Infinite Terminal Time Case

In this case, the equations (13) and (14) take on their steady-state forms with  $\bar{\Sigma}$  replacing  $\Sigma$ . Now,

$$\text{Tr}[O \cdot P^0] + \text{Tr}[(\bar{\Sigma}^0 - \bar{\Sigma}^1)O] = 0 \quad (\text{A-3})$$

But

$$\begin{aligned} (A + B G^0 C) \bar{\Sigma}^0 + \bar{\Sigma}^0 (A + B G^0 C)^T + B G^0 W G^{0T} B^T \\ - [(A + B G^1 C) \bar{\Sigma}^1 + \bar{\Sigma}^1 (A + B G^1 C)^T + B G^1 W G^{1T} B^T] = 0 \end{aligned} \quad (\text{A-4})$$

and

$$-(A + B G^0 C)^T P^0 - P^0 (A + B G^0 C) - (Q + C^T G^{0T} R G^0 C) = 0 \quad (\text{A-5})$$

Replacing the first and second null matrices in (A-3) by left-hand sides of (A-4) and (A-5), respectively, after simplification,

$$\begin{aligned}
 J(G^0) - J(G^1) = & \text{Tr}[B(G^0 - G^1)C\bar{\Sigma}^1 P^0] + \frac{1}{2} \text{Tr}[\bar{\Sigma}^1(C^T G^{0T} R G^0 C - C^T G^{1T} R G^1 C)] \\
 & + \frac{1}{2} \text{Tr} \{ [B(G^0 W G^{0T} - G^1 W G^{1T}) B^T] P^0 \} \quad (A-6)
 \end{aligned}$$

Similar to the finite terminal time case (A-6) can be shown to be equivalent to (23), with integral signs removed, and  $\bar{\Sigma}^1$  replaced by  $\bar{\Sigma}^1$ .

APPENDIX B

ALGORITHM FOR SUCCESSIVE REDUCTION OF THE PERFORMANCE  
FUNCTION: DISCRETE-TIME CASE

The expression

$$\text{Tr}[(\Sigma^0(k+1) - \Sigma^1(k+1))P^0(k+1)] - \text{Tr}[(\Sigma^0(k) - \Sigma^1(k))P^0(k)]$$

can be shown to be equal to the following expression, after substitution for  $[\Sigma^0(k+1) - \Sigma^1(k+1)]$  from (47) and for  $P^0(k)$  from (48), respectively, in its first and second terms, and after some manipulation:

$$\begin{aligned} & \text{Tr}[B(G^0 W G^{0T} - G^1 W G^{1T})B^T P^0(k+1)] - 2 \text{Tr}[A \Sigma^1 C^T (G^0 - G^1)^T B^T P^0(k+1)] \\ & - \text{Tr}[\Sigma^1 C^1 (G^{1T} B^T P^0(k+1) B G^1 - G^{0T} B^T P^0(k+1) B G^0) C] \\ & - \text{Tr}[(\Sigma^0 - \Sigma^1)(C^T G^{0T} R G^0 C + Q)] \end{aligned}$$

After summing the two expressions above from  $k = 0$  to  $N - 1$ , and after some manipulation, the following equation is obtained

$$\begin{aligned} J(G^0) - J(G^1) &= \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr}[(B G^0 W G^{0T} B^T - B G^1 W G^{1T} B^T) P^0(k+1)] \\ & - \sum_{k=0}^{N-1} \text{Tr}[A \Sigma^1 C^T (G^0 - G^1)^T B^T P^0(k+1)] \\ & + \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr}[\Sigma^1 C^T (G^{0T} B^T P^0(k+1) B G^0 - G^{1T} B^T P^0(k+1) B G^1) C] \\ & + \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr}[\Sigma^1 C^T (G^{0T} R G^0 - G^{1T} R G^1) C] \\ & + \frac{1}{2} \sum_{k=0}^{N-1} \text{Tr}[(G^{0T} R G^0 - G^{1T} R G^1) W] \end{aligned}$$

Using the trace identities in reference (8), it can be shown that (B-1) and (49) are equivalent.

#### Infinite Terminal Time Case

In this case equations (37) and (38) become equations (45) and (46), respectively. Now,

$$\text{Tr}[(\bar{\Sigma}^0 - \bar{\Sigma}^1)P^0] - \text{Tr}[(\bar{\Sigma}^0 - \bar{\Sigma}^1)P^0] = 0$$

Substitution for  $(\bar{\Sigma}^0 - \bar{\Sigma}^1)$  in the first term using equation (46), and for P in the second term, using equation (45), and proceeding exactly as for the infinite terminal time case, equation (53) can be obtained.

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