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# CONTRIBUTION TO THE THEORY OF TIDAL OSCILLATIONS OF AN ELASTIC EARTH. EXTERNAL TIDAL POTENTIAL

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"It is completely idle to say that the tides are caused by the conflict arising between the motion of the earth and the motion of the lunar sphere, not only because it is neither obvious nor has it been explained how this must follow, but because its glaring falsity is revealed by the rotation of the earth being not contrary to the motion of the moon, but in the same direction. Thus everything that has been previously conjectured by others seems to me completely invalid. But among all the great men who have philosophized about this remarkable effects, I am more astonished at Kepler than at any other. Despite his open and acute mind, and though he has at his fingertips the motions attributed to the earth, he has nevertheless lent his ear and his assent to the moon's dominion over waters, to occult properties, and to such puerilities"

Galileo Galilei,

Dialogue concerning the two chief world systems, p. 462. Translation by S. Drake. Univ. of California press. 1953.

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ABSTRACT

In the present article we establish the differential equations of the tidal oscillations of the Earth under the assumption that the interior of the Earth is laterally inhomogeneous. We develop the theory using vectorial and dyadic symbolism to shorten the exposition and to reduce the differential equations to a symmetric form convenient for programming and for numerical integration. The formation of tidal "bulldges" on the surfaces of discontinuity and the changes in the internal density produce small periodic variations in the exterior geopotential which are reflected in the motion of artificial satellites. The analogues of Love elastic parameters in the expansion of exterior tidal potential reflect the asymmetric and inhomogeneous structure of the interior of the Earth.

## Introduction

In the present article we establish the differential equations of the tidal oscillations of the Earth under the assumption that the interior of the Earth is laterally inhomogeneous. We develop the theory using vectorial and dyadic symbolism, partly to shorten the exposition and, partly, to reduce the differential equations to a symmetric form convenient for programming and numerical integration.

It has been customary in the investigations of the oscillations of the Earth to assume the spherical symmetry of internal density and of Lamé elastic parameters. In addition, the undisturbed Earth has been considered to be in hydrostatic equilibrium. However, we have already reached the stage where the hypothesis of spherical symmetry cannot explain the results of modern seismic observations. The analysis of the seismic information provided by the Peru-Bolivian border earthquake (Dziewonski, 1970), shows that the measurements of the group velocities along the world circling path indicate the existence of lateral inhomogeneities in the crust and mantle. Dziewonski also finds that the measurement of the surface wave dispersion provides the means for the determination of the internal density distribution. From the analysis of the information supplied by the Alaska earthquake Toksöz and Anderson (1966), conclude that the available phase velocity data are sufficiently accurate to indicate the regional variations and the heterogeneity of the upper 400 km mantle.

Arkani-Hamed (1970), suggests the lateral variations of density in the upper mantle are approx. 3% and of the shear modulus approx. 20%. It is of importance to extend our understanding of the mechanism of the Earth's tidal oscillations, partly for the sake of the interesting geophysical problem itself and partly because the tidal oscillations produce small periodic changes in the geopotential which are reflected in the motion of artificial satellites. The problem of tidal perturbations in the motion of artificial satellites recently attracted the attention of a number of authors (Kozai, 1965), (Newton, 1968), (Kaula, 1969), (Smith and al. 1971, 1973), (Musen and Estes, 1972), (Musen and Felsentreger, 1973), (Douglas and al., 1972, 1973), (Lambeck and Cazenave, 1973), (Musen, 1973). The periodic variations in the geopotential are caused by changes in the internal density and by the formation of tidal "bulldges" on the surfaces of discontinuity inside the Earth and on the free surface (Biot, 1965). Both changes are small, but their totality produces appreciable and observable perturbations in the motion of artificial satellites. A periodic expansion of the tidal perturbations in the height of satellite GEOS-A above the Earth, in terms of the arguments of lunar theory, was obtained by R. Estes using the Musen and Estes semi-analytical theory of tidal effects (1972) in the motion of artificial satellites. They are quite observable. Estes' expansion contains two long period terms with amplitudes of 3.8 m. and 5.0 m., respectively.

A proper theory of the Earth's tidal oscillations, bodily and surface, can help us to understand some characteristics of satellite motion.

On the other hand, a proper theory of the tidal perturbations of satellites can give us some insight into the Earth's internal structure and its elastic properties.

The average elastic parameters of the Earth for different satellites were recently obtained by Kozai (1965), Newton (1968), Smith and al. (1971, 1973) and Douglas and al. (1972). Typically the same value of the elastic factor is used for all harmonics of a given degree in the expansion of the exterior tidal potential, irrespectively of the tidal frequency. However, the work of Alterman and al. (1959) indicates that the elastic response of the Earth is different for different tidal frequencies. More computational work is required in order to understand the full implication of this idea on the theory of motion of artificial satellites. The parameter of the Earth's elastic response (Love number) to the tidal force of a given frequency, as it appears in the expansion of the exterior tidal potential, can be obtained as a by-product of the numerical integration of the partial differential equation controlling the tidal oscillations. The integration should be extended over the whole Earth.

In the present article we obtain, starting from the principle of D'Alembert, a modified form of the Biot (1965) differential equation for the elastic oscillations, using the apparatus of vector and dyadic analysis. We find the form of the perturbative terms which disappear together with the initial deviatoric stress.

The method of Biot includes the effect of the initial stress, as well as the effect of the additional stress as created by the vorticity of the displacement field.

Biot's formulation is Lagrangian. The displacement of a material point is given relative to its "initial" position. We incorporate into the differential equation of the tidal oscillations of the solid Earth the effect of the semi-diurnal nutation and of the geostrophic force (Molodensky, 1953), (Melchior and Georis, 1968), (Melchior, 1971), and also the effect of the initial deviatoric stress. It is of interest to note that in the Biot formulation the density factor, associated with the forces of inertia, retains its initial value and there is no necessity to consider its time variation. The geostrophic force in tidal theory is more important than in the theory of free oscillations of the Earth, where it is only a perturbative term. The rotation of the Earth causes only a "fine splitting" of the spectrum of free oscillations' frequencies and the analytical perturbations technique, similar to one in quantum-mechanics, can be applied (Madariaga, 1972), (Dahlen, 1968, 1972) (Luh, 1973).

In the tidal theory the forced oscillations take place and a large number of significant tidal frequencies are of the same order of magnitude as the angular speed of the Earth's rotation itself. As a consequence the geostrophic term in the differential equation for tidal oscillations is no longer a perturbative term. In our exposition we remain in the frame of ideas of classical mechanics. Like Biot, we make use of the principle of D'Alembert, and we assume that the stress



depends on the instantaneous strain and vorticity. We wish to mention briefly that another, non-classical, approach to tides is also possible. In modern books on Elasticity (Eringen, 1968) the principle of heredity is re-introduced. It was first formulated by Boltzmann (1874) and Volterra (1930). Its basic assumption is that the present state of the system, including its stress, depends on all of the system's past history. The constitutive equations must then contain a convolution with respect to time. Surely, the principle of heredity falls outside of the frame of Analytical Mechanics, but it was recently developed for the sea tides by Munk and Cartwright (1966), (Cartwright, 1968) and successfully applied to the spectroscopic analysis of tides at Honolulu and on the coast of East Britain. Despite this we do not use the principle of heredity in the present work, although we recognize that it deserves considerable attention, because the Earth might represent a system with memory.

### Basic Notations

We introduce the following basic notations:

$\mathbf{i}, \mathbf{j}, \mathbf{k}$  - the basic unit vectors of a rotating system of rectangular coordinates with the origin and directions rigidly fixed relative to the "initial" (mean) position of the mantle,

$\mathbf{I} = \mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j} + \mathbf{k}\mathbf{k}$  - the idemfactor,

$\mathbf{r}$  - the position vector of the Earth particle in its "initial" (mean) state  $P_0$ ,

$x, y, z$  - the rectangular coordinates of  $P_0$ ,

$\Omega$  - the instantaneous angular speed of rotation of the Earth,

$\Omega_0 \mathbf{k}$  the constant part of  $\Omega$ ,

$\Omega_0 \mathbf{N}$  - the effect of nutation in  $\Omega$ ,

$$\Omega = \Omega_0 (\mathbf{k} + \mathbf{N}), \quad (1)$$

$\rho$  - the initial density at  $P_0$ ,

$S$  - the initial stress at  $P_0$ ; we assume that  $S$  is a symmetric dyadic,

$p$  - the hydrostatic pressure at  $P_0$ ,

$\tau$  - the initial deviatoric stress at  $P_0$ ,

$$S = -p\mathbf{I} + \tau, \quad (2)$$

$U(\mathbf{r})$  - the initial force function of selfgravitation (per unit of mass) at  $P_0$ ; the force function of the centrifugal force is not included,

$\mathbf{R}$  - the position vector of the displaced point  $P$  at the moment  $t$ ,

$\mathbf{w}$  - the absolute acceleration of the displaced point  $P$  at the moment  $t$ ,

$\mathbf{u} = \mathbf{R} - \mathbf{r}$  - the elastic displacement of  $P_0$  at the moment  $t$ ,

$\rho'$  - the density at  $P$  in the moment  $t$ ,

$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$  - the del operator relative to  $\mathbf{r}$ ,

$$\epsilon = \frac{1}{2} (\mathbf{u}\nabla + \nabla\mathbf{u}) \quad \text{- the strain dyadic,} \quad (3)$$

$$\boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{u} \quad \text{- the local vorticity} \quad (4)$$

$$\theta = \nabla \cdot \mathbf{u}, \quad (5)$$

then

$$\mathbf{I} \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{I} = \frac{1}{2} (\mathbf{u}\nabla - \nabla\mathbf{u}), \quad (6)$$

$$(\mathbf{I} \times \boldsymbol{\omega})' = -\mathbf{I} \times \boldsymbol{\omega}$$

and we deduce from (3) and (6):

$$\nabla\mathbf{u} = \epsilon - \mathbf{I} \times \boldsymbol{\omega}, \quad (7)$$

$$\mathbf{u}\nabla = \epsilon + \mathbf{I} \times \boldsymbol{\omega}, \quad (8)$$

$s$  - the stress increment added to  $S$  at the displaced point  
after the deformation caused by the vorticity,

$G$  - the constant of gravitation,

$a'$  - the mean radius of the Earth.

### Differential Equations of Tidal Oscillations

Let  $V$  be the volume of a portion of matter of the Earth in the initial state and  $A$  be its boundary. At time  $t$  they become  $V'$  and  $A'$ , respectively. The corresponding volume and the oriented surface elements we designate by:

$$dV, dA, dV', dA'.$$

Under the influence of the vorticity  $\omega$  a dyadic  $\mathbf{ab}$  receives the increment

$$(\omega \times \mathbf{a}) \mathbf{b} + \mathbf{a} (\omega \times \mathbf{b}) = \omega \times \mathbf{ab} - \mathbf{ab} \times \omega,$$

assuming that  $\omega$  is small. Consequently, under the influence of the vorticity the initial stress  $S$  receives at time  $t$  the increment:

$$\omega \times S - S \times \omega$$

and the total stress becomes:

$$T = s + S + \omega \times S - S \times \omega. \quad (9)$$

The principle of D'Alembert for the elastic tidal oscillations of the Earth in the rotating system coordinates takes the form:

$$\int_{V'} \rho' [\nabla_{\mathbf{R}} U(\mathbf{R}) + \nabla_{\mathbf{R}} (W + \psi) - \mathbf{w}] dV' + \int_{A'} d\mathbf{A}' \cdot \mathbf{T} = 0, \quad (10)$$

where

$$\mathbf{w} = \frac{d^2 \mathbf{R}}{dt^2} + 2\Omega \times \frac{d\mathbf{R}}{dt} + \frac{d\Omega}{dt} \times \mathbf{R} + \Omega \times (\Omega \times \mathbf{R}), \quad (11)$$

is the absolute acceleration of the displaced point,  $W$  is the lunisolar tidal force function, and  $\psi$  the force function of the additional attraction, as caused by the tidal changes of the internal density and by the formation of tidal "bulges" on the surfaces of discontinuity and on the free surface.

Taking into account:

$$d\mathbf{A}' = \frac{\partial \mathbf{R}}{\partial y} \times \frac{\partial \mathbf{R}}{\partial z} dydz + \frac{\partial \mathbf{R}}{\partial z} \times \frac{\partial \mathbf{R}}{\partial x} dzdx + \frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y} dx dy, \quad (12)$$

and

$$d\mathbf{A} = i dydz + j dzdx + k dx dy, \quad (13)$$

we obtain:

$$d\mathbf{A}' = d\mathbf{A} \cdot \mathbf{M}, \quad (14)$$

where the dyadic  $\mathbf{M}$  has the form:

$$\mathbf{M} = i \frac{\partial \mathbf{R}}{\partial y} \times \frac{\partial \mathbf{R}}{\partial z} + j \frac{\partial \mathbf{R}}{\partial z} \times \frac{\partial \mathbf{R}}{\partial x} + k \frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y}. \quad (15)$$

Substituting

$$\mathbf{R} = \mathbf{r} + \mathbf{u}$$

into (15) and taking the relations

$$\mathbf{I} \times \mathbf{i} = \mathbf{kj} - \mathbf{jk},$$

$$\mathbf{I} \times \mathbf{j} = \mathbf{ik} - \mathbf{ki}$$

$$\mathbf{I} \times \mathbf{k} = \mathbf{ji} - \mathbf{ij}$$

into account, we obtain after some easy transformations:

$$\mathbf{M} = \mathbf{I} - (\mathbf{I} \times \nabla) \times \mathbf{u}, \quad (16)$$

or, making use of the decomposition

$$(\mathbf{I} \times \nabla) \times \mathbf{u} = \mathbf{u} \nabla - \mathbf{I} \nabla \cdot \mathbf{u}$$

and of (8), we have

$$\mathbf{M} = (1 + \theta) \mathbf{I} - \epsilon - \mathbf{I} \times \omega. \quad (17)$$

Making use of the law of conservation of matter

$$\rho' dV' = \rho dV,$$

We can re-write the principle of D'Alembert (10) in the form:

$$\int_V [\nabla U(\mathbf{r}) + \mathbf{u} \cdot \nabla \nabla U(\mathbf{r}) + \nabla(W + \psi) - \mathbf{w}] dV \quad (18)$$

$$+ \int_A d\mathbf{A} \cdot (\mathbf{M} \cdot \mathbf{T}) = 0,$$

neglecting the terms of higher order in  $\mathbf{u}$ . Both integrals in (18) are now taken over the initial volume and surface, respectively. By applying Gauss' theorem to (18) we obtain the equation of motion:

$$\rho \mathbf{w} = \rho [\nabla(U + W + \psi) + \mathbf{u} \cdot \nabla \nabla U] + \nabla \cdot (\mathbf{M} \cdot \mathbf{T}). \quad (19)$$

It is of interest to note that  $\rho$  in (19) is the density corresponding to the initial state. From (9) and (17) we have, neglecting terms of higher order,

$$\mathbf{M} \cdot \mathbf{T} = (1 + \theta) \mathbf{S} + \mathbf{s} - \mathbf{S} \times \boldsymbol{\omega} - \boldsymbol{\epsilon} \cdot \mathbf{S} \quad (20)$$

and from this relation there follows:

$$\begin{aligned} \nabla \cdot (\mathbf{M} \cdot \mathbf{T}) &= (1 + \theta) \nabla \cdot \mathbf{S} - (\nabla \cdot \boldsymbol{\epsilon} - \nabla \theta) \cdot \mathbf{S} + \nabla \cdot \mathbf{s} \quad (21) \\ &\quad - (\mathbf{S} \cdot \nabla) \times \boldsymbol{\omega} - (\nabla \cdot \mathbf{S}) \times \boldsymbol{\omega} - \boldsymbol{\epsilon} \cdot \nabla \mathbf{S}. \end{aligned}$$

We have from (3) and (4)

$$\begin{aligned} \nabla \cdot \boldsymbol{\epsilon} - \nabla \theta &= + \frac{1}{2} (\nabla^2 \mathbf{u} - \nabla \nabla \cdot \mathbf{u}) \quad (22) \\ &= - \frac{1}{2} \nabla \times \nabla \times \mathbf{u} = - \nabla \times \boldsymbol{\omega}, \end{aligned}$$

and (21) becomes:

$$\begin{aligned} \rho \mathbf{w} = & \rho \nabla(U + W + \psi) + \rho \mathbf{u} \cdot \nabla \nabla U + \nabla \cdot \mathbf{s} \\ & + (1 + \theta) \nabla \cdot \mathbf{S} + \boldsymbol{\omega} \times (\nabla \cdot \mathbf{S}) + (\nabla \times \boldsymbol{\omega}) \cdot \mathbf{S} \\ & - (\mathbf{S} \cdot \nabla) \times \boldsymbol{\omega} - \epsilon \cdot \cdot \nabla \mathbf{S}. \end{aligned} \quad (23)$$

This last equation is the vectorial form of the Biot (1965) differential equation corresponding to the case of tidal oscillations.

We can set in (11):

$$\boldsymbol{\Omega} = \Omega_0 (\mathbf{k} + \mathbf{N}), \quad (24)$$

where  $\Omega_0$  is a constant and  $\Omega_0 \mathbf{N}$  is the effect of nutation. Substituting

$$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{u}}{dt}$$

and (24) into (11) and neglecting the insignificant terms, we have

$$\begin{aligned} \mathbf{w} = & \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} + \Omega_0 \frac{d\mathbf{N}}{dt} \times \mathbf{r} \\ & + \Omega_0^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{r}) + \Omega_0^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{u}) \\ & + \Omega_0^2 \mathbf{k} \times (\mathbf{N} \times \mathbf{r}) + \Omega_0^2 \mathbf{N} \times (\mathbf{k} \times \mathbf{r}), \end{aligned} \quad (25)$$

or, expanding the double products,

$$\begin{aligned} \mathbf{w} = & \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} + \Omega_0 \frac{d\mathbf{N}}{dt} \times \mathbf{r} - \nabla \phi \\ & - \Omega_0^2 (\mathbf{I} - \mathbf{k}\mathbf{k}) \cdot \mathbf{u} + \Omega_0^2 (\mathbf{k}\mathbf{N} \cdot \mathbf{r} + \mathbf{N}\mathbf{k} \cdot \mathbf{r} - 2\mathbf{r}\mathbf{k} \cdot \mathbf{N}) \end{aligned} \quad (26)$$

where

$$\phi = \frac{1}{2} \Omega_0^2 (x^2 + y^2). \quad (27)$$

There follows from (27):

$$\nabla \nabla \phi = \Omega_0^2 (\mathbf{I} - \mathbf{k} \mathbf{k})$$

and taking into account

$$\begin{aligned} \nabla [(\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r})] &= \nabla (\mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r} - r^2 \mathbf{k} \cdot \mathbf{N}) \\ &= \mathbf{k} \mathbf{N} \cdot \mathbf{r} + \mathbf{N} \mathbf{k} \cdot \mathbf{r} - 2 \mathbf{r} \mathbf{k} \cdot \mathbf{N}, \end{aligned} \quad (28)$$

We can re-write (26) in the following form:

$$\begin{aligned} \mathbf{w} &= \frac{d^2 \mathbf{u}}{dt^2} + 2 \Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} - \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} \\ &\quad - \nabla [\phi - \Omega_0^2 (\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r})] - \mathbf{u} \cdot \nabla \nabla \phi. \end{aligned} \quad (29)$$

Substituting (29) into (19) and setting

$$\mathbf{V} = \mathbf{U} + \phi$$

We have:

$$\begin{aligned} \rho \left( \frac{d^2 \mathbf{u}}{dt^2} + 2 \Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} \right) &= \rho \nabla [\mathbf{V} + \mathbf{W} + \psi - \Omega_0^2 (\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r})] + \rho \mathbf{u} \cdot \nabla \nabla \mathbf{V} \\ &\quad + \rho \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} + \nabla \cdot \mathbf{s} + (1 + \theta) \nabla \cdot \mathbf{S} + \boldsymbol{\omega} \times (\nabla \cdot \mathbf{S}) + (\nabla \times \boldsymbol{\omega}) \cdot \mathbf{S} - (\mathbf{S} \cdot \nabla) \times \boldsymbol{\omega} - \boldsymbol{\epsilon} \cdot \nabla \mathbf{S}. \end{aligned} \quad (30)$$



Taking into account the condition of static equilibrium in the initial state

$$\nabla \cdot \mathbf{S} + \rho \nabla V = 0,$$

We obtain from (30):

$$\begin{aligned} \rho \left( \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} \right) = & -\rho \theta \nabla V + \rho (\mathbf{u} \cdot \nabla \nabla V - \boldsymbol{\omega} \times \nabla V) \quad (31) \\ & + \rho \nabla [W + \psi - \Omega_0^2 (\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r})] + \nabla \cdot \mathbf{s} \\ & + \rho \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} + (\nabla \times \boldsymbol{\omega}) \cdot \mathbf{S} - (\mathbf{S} \cdot \nabla) \times \boldsymbol{\omega} - \boldsymbol{\varepsilon} \cdot \nabla \mathbf{S}. \end{aligned}$$

We can split the right hand side of the last equation into the main and perturbative terms by making use of the identity:

$$\mathbf{u} \cdot \nabla \nabla V - \boldsymbol{\omega} \times \nabla V = \nabla (\mathbf{u} \cdot \nabla V) - \boldsymbol{\varepsilon} \cdot \nabla V, \quad (32)$$

which can be easily proved using (3)-(6). Substituting (32) into (31), we have:

$$\begin{aligned} \rho \left( \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} \right) = & \nabla \cdot \mathbf{s} - \rho \theta \nabla V \quad (33) \\ & + \rho \nabla K + \rho \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} - (\rho \boldsymbol{\varepsilon} \cdot \nabla V + \boldsymbol{\varepsilon} \cdot \nabla \mathbf{S}) \\ & + [(\nabla \times \boldsymbol{\omega}) \cdot \mathbf{S} - (\mathbf{S} \cdot \nabla) \times \boldsymbol{\omega}], \end{aligned}$$

where we set for brevity:

$$K = W + \psi + \mathbf{u} \cdot \nabla V - \Omega_0^2 (\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r}). \quad (33')$$

The last two terms disappear if the initial stress is reduced to the hydrostatic pressure.

Together with Biot (1965) we assume a linear relation between the additional stress  $s$  and the strain  $\epsilon$  of the form:

$$s = B \cdot \cdot \epsilon, \quad (34)$$

where  $B$  is a tensor of the fourth rank with some symmetry properties. Taking the second law of thermodynamics into consideration Dahlen (1972) obtains:

$$B = C + \frac{1}{2} (IS - SI), \quad (35)$$

where  $C$  is a tensor of the fourth rank, independent of the initial stress. The components of  $C$  define the set of linear isentropic coefficients.

Substituting (2) into (35) we have

$$B = C + \frac{1}{2} (I\tau - \tau I). \quad (36)$$

If we consider the Earth to be elastic and isotropic, the number of isentropic coefficients is reduced to two standard ones,  $\lambda(\mathbf{r})$  and  $\mu(\mathbf{r})$ . In this case we have

$$C = \lambda II + 2\mu(i Ii + j Ij + k Ik). \quad (37)$$

From (34), (36) and (37) we obtain:

$$s = \sigma + \frac{1}{2} (I\tau \cdot \cdot \epsilon - \theta\tau), \quad (38)$$

where  $\sigma$  is the standard stress dyadic,

$$\sigma = \lambda I\theta + 2\mu\epsilon. \quad (39)$$

The second term in (38) represents a correction for the deviatory prestress.

Substituting (38) into (33) and making use of (2), we obtain:

$$\rho \left( \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} \right) = \nabla \cdot \boldsymbol{\sigma} - \rho \theta \nabla \mathbf{V} + \rho \nabla \mathbf{K} \quad (40)$$

$$+ \rho \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} - \boldsymbol{\epsilon} \cdot (\rho \nabla \mathbf{V} - \nabla \mathbf{p}) + \mathbf{D},$$

where

$$\mathbf{D} = \frac{1}{2} [\nabla(\boldsymbol{\tau} \cdot \boldsymbol{\epsilon}) - \nabla(\theta \boldsymbol{\tau})] + (\nabla \times \boldsymbol{\omega}) \cdot \boldsymbol{\tau} - (\boldsymbol{\tau} \cdot \nabla) \times \boldsymbol{\omega} - \boldsymbol{\epsilon} \cdot \nabla \boldsymbol{\tau}. \quad (41)$$

represents the "cross-effects" between the strain and the deviatory pre-stress.

The last two terms in (40) are perturbative and disappear together with the initial deviatory stress. Thus, if we assume that the initial state is in hydrostatic equilibrium we have simply:

$$\frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} - \theta \nabla \mathbf{V} + \nabla \mathbf{K} + \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt}. \quad (42)$$

This last equation is a slightly modified form of the basic equation of the Molodensky tidal theory. The differential equation (40) serves as a foundation for the extension of Molodensky theory to the case when the Earth is laterally heterogeneous and the initial equilibrium is not a hydrostatic one.

Similarly, again making use of (2) and (38) and the identities

$$\rho \mathbf{u} \cdot \nabla \nabla \mathbf{V} = \nabla(\rho \mathbf{u} \cdot \nabla \mathbf{V}) - (\mathbf{u} \cdot \nabla \mathbf{V}) \nabla \rho - \rho \boldsymbol{\epsilon} \cdot \nabla \mathbf{V} + \rho \boldsymbol{\omega} \times \nabla \mathbf{V},$$

$$\rho \theta = \nabla \cdot (\rho \mathbf{u}) - \mathbf{u} \cdot \nabla \rho,$$

$$\mathbf{u} \times (\nabla \mathbf{V} \times \nabla \rho) = \nabla \mathbf{V}(\mathbf{u} \cdot \nabla \rho) - \nabla \rho(\mathbf{u} \cdot \nabla \mathbf{V}),$$

we reduce equation (31) to the form:

$$\rho \left( \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} \right) = -\nabla \cdot (\rho \mathbf{u}) \nabla V + \nabla(\rho \mathbf{u} \cdot \nabla V) \quad (43)$$

$$+ \rho \nabla \Psi + \rho \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} + \nabla \cdot \sigma$$

$$+ \mathbf{u} \times (\nabla V \times \nabla \rho) - \epsilon \cdot (\rho \nabla V - \nabla p) + \mathbf{D},$$

where

$$\Psi = W + \psi - \Omega_0 (\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r}).$$

If the initial equilibrium is hydrostatic then the perturbative terms disappear and equation (43) becomes:

$$\rho \left( \frac{d^2 \mathbf{u}}{dt^2} + 2\Omega_0 \mathbf{k} \times \frac{d\mathbf{u}}{dt} \right) = -\nabla \cdot (\rho \mathbf{u}) \nabla V + \nabla(\rho \mathbf{u} \cdot \nabla V) \quad (44)$$

$$+ \rho \nabla \Psi + \rho \Omega_0 \mathbf{r} \times \frac{d\mathbf{N}}{dt} + \nabla \cdot \sigma$$

If, in addition, the geostrophic force and the effects of nutation are neglected we have:

$$\rho \frac{d^2 \mathbf{u}}{dt^2} = -\nabla \cdot (\rho \mathbf{u}) \nabla V + \nabla(\rho \mathbf{u} \cdot \nabla V) + \rho \nabla \Psi. \quad (45)$$

This last equation can serve as a foundation for the theories of oscillations of the Earth as developed by Takeuchi (1950), Alterman and al. (1959) and Saito (1971). A simplification of (40) and (42)-(44) is possible if we assume, together with Molodensky and Melchior, that the polar component of  $\mathbf{N}$  does not have any

influence on the tidal oscillations of the Earth. This takes place, for example, (Melchior and Georis, 1968) if the Earth is an ellipsoid of rotation and the internal density depends only on the radius-vector and the latitude. In this case

$$\mathbf{k} \cdot \mathbf{N} = 0$$

and, as a consequence,

$$(\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r}) = \mathbf{r} \cdot \mathbf{kN} \cdot \mathbf{r},$$

$$\Psi = W + \psi - \Omega_0^2 \mathbf{r} \cdot \mathbf{kN} \cdot \mathbf{r},$$

and  $\nabla\Psi$  to be substituted in the differential equations takes the form:

$$\nabla\Psi = \nabla(W + \psi) - \Omega_0^2 (\mathbf{kN} \cdot \mathbf{r} + \mathbf{Nk} \cdot \mathbf{r}).$$

If, in addition, we consider only terms of a given frequency  $\alpha$ , where  $\alpha$  is a linear combination of the basic tidal frequencies, then in accordance with the customary procedure we change the notations and replace

$$\mathbf{u} \text{ by } \mathbf{u}e^{i\alpha t}, \quad \mathbf{N} \text{ by } \mathbf{N}e^{i\alpha t},$$

$$\frac{d\mathbf{N}}{dt} \text{ by } i\alpha \mathbf{k} \times \mathbf{N}e^{i\alpha t},$$

$$\mathbf{r} \times \frac{d\mathbf{N}}{dt} \text{ by } i\alpha \mathbf{r} \times (\mathbf{k} \times \mathbf{N}) = 2i\alpha \mathbf{kr} \cdot \mathbf{N} - i\alpha \nabla(\mathbf{r} \cdot \mathbf{kN} \cdot \mathbf{r})$$

where the factors at  $e^{i\alpha t}$  depend upon the position only. Substituting these values into the differential equations, we obtain a linear partial differential equation satisfied by the new  $\mathbf{u}$ :

$$\begin{aligned} \rho(-\alpha^2 \mathbf{I} + 2i\alpha\Omega_0 \mathbf{k} \times \mathbf{I}) \cdot \mathbf{u} = \nabla \cdot \boldsymbol{\sigma} - \rho\theta\nabla\nabla + \rho\nabla Q & \quad (40') \\ + 2i\alpha\rho\Omega_0 \mathbf{kr} \cdot \mathbf{N} - \epsilon \cdot (\rho\nabla\nabla - \nabla\mathbf{p}) + \mathbf{D}, \end{aligned}$$

where

$$Q = W + \psi + \mathbf{u} \cdot \nabla V - \Omega_0 (\Omega_0 + i\alpha) \mathbf{r} \cdot \mathbf{k} \mathbf{N} \cdot \mathbf{r},$$

or

$$\begin{aligned} \rho(-a^2 \mathbf{I} + 2i\alpha\Omega_0 \mathbf{k} \times \mathbf{I}) \cdot \mathbf{u} &= \nabla \cdot \sigma - \nabla \cdot (\rho \mathbf{u}) \nabla V & (43') \\ &+ \nabla(\rho \mathbf{u} \cdot \nabla V) + \rho \nabla Q + 2\rho i\alpha\Omega_0 \mathbf{k} \mathbf{r} \cdot \mathbf{N} \\ &+ \mathbf{u} \times (\nabla V \times \nabla \rho) - \epsilon \cdot (\rho \nabla V - \nabla p) + \mathbf{D}. \end{aligned}$$

With the corresponding simplifications of (42) and (44) we have:

$$\begin{aligned} (-a^2 \mathbf{I} + 2i\alpha\Omega_0 \mathbf{k} \times \mathbf{I}) \cdot \mathbf{u} &= \frac{1}{\rho} \nabla \cdot \sigma - \theta \nabla V + \nabla Q & (42') \\ &+ 2i\alpha\Omega_0 \mathbf{k} \mathbf{r} \cdot \mathbf{N} \end{aligned}$$

for Molodensky theory, and

$$\begin{aligned} (-a^2 \mathbf{I} + 2i\alpha\Omega_0 \mathbf{k} \times \mathbf{I}) \cdot \mathbf{u} &= \frac{1}{\rho} [-\nabla \cdot (\rho \mathbf{u}) \nabla V + \nabla \cdot \sigma & (44') \\ &+ \nabla(\rho \mathbf{u} \cdot \nabla V)] + \nabla Q + 2i\alpha\Omega_0 \mathbf{k} \mathbf{r} \cdot \mathbf{N} \end{aligned}$$

for the extension of the Alterman-Jarosch-Pekeris theory. Multiplying (40')

and (42')-(44') by the dyadic

$$\Gamma = \mathbf{I} + i\beta \mathbf{k} \times \mathbf{I} - \beta^2 \mathbf{k} \mathbf{k},$$

$$\beta = \frac{2\Omega_0}{a},$$

we can solve with respect to  $\mathbf{u}$  and obtain:

$$-a^2(1 - \beta^2) \mathbf{u} = \Gamma \cdot \mathbf{P}. \quad (46)$$

where  $\mathbf{P}$  is the right hand side of one of the equations (40') or (42')-(44').

The differential equation (46), combined with the equation

$$\nabla^2 \psi = + 4\pi G \nabla \cdot (\rho \mathbf{u}), \quad (47)$$

can serve as a basis for the numerical integration of the tidal effects, starting from the surface and going "down." A number of well known continuity conditions must be satisfied in the transition for one layer to the next.

In addition, the form (46) helps to recognize the role of the divisor  $1 - \beta^2$  which becomes small for  $\alpha \sim 2\Omega_0$ .

#### On the Exterior Tidal Potential

The exterior tidal potential (Biot, 1965) acting on the satellite has the form:

$$\Omega' = - G \int_v \frac{\nabla \cdot (\rho \mathbf{u})}{|\mathbf{r} - \mathbf{r}'|} dv + \sum_S G \int_S \frac{\delta \rho (\mathbf{u} \cdot d\mathbf{A})}{|\mathbf{r} - \mathbf{r}'|}, \quad (48)$$

where  $\mathbf{r}$  is the position vector of the satellite and  $\mathbf{r}'$  is the position vector of the Earth particle. The first integral in (48) is taken over the volume of the whole tidally undisturbed Earth. The integrals under the summation sign are taken over the surfaces of discontinuity, and  $\delta \rho$  is the jump in density in transition from one layer to the other. We can assume, without loss of generality, that  $\mathbf{u}$  is the solution of the basic partial differential equation and that the periodic factor  $e^{i\alpha t}$  is omitted.

We have:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{n=0}^{+\infty} \frac{1}{2n+1} \frac{r'^n}{r^{n+1}} \sum_{m=-n}^{m=+n} Y_{nm}^* (\delta, \alpha - \theta) Y_{nm} (\theta', \lambda'), \quad (49)$$

where  $(r', \lambda', \theta')$  are the polar coordinates of the Earth particle relative to the system connected with the Earth,  $(r, \alpha, \delta)$  are the polar equatorial coordinates of the satellite, and  $\theta$  is the sidereal time, and  $Y_{nm}$  are the normalized complex spherical harmonics. Substituting (49) into (48) we obtain for the exterior tidal potential in the system connected with the Earth:

$$\Omega' = 4\pi G \sum_{n=0}^{+\infty} \frac{1}{2n+1} \cdot \frac{a'^n}{r^{n+1}} \sum_{m=-n}^{m=+n} K_{nm} Y_{nm}^*(\delta, \alpha - \theta),$$

where

$$K_{nm} = - \int_v \left(\frac{r'}{a'}\right)^n Y_{nm}(\theta', \lambda') \nabla \cdot (\rho \mathbf{u}) dv \quad (50)$$

$$+ \sum \int_S \left(\frac{r'}{a'}\right)^n Y_{nm}(\theta', \lambda') \delta \rho \mathbf{u} \cdot d\mathbf{A}.$$

The numbers  $K_{nm}$  are the analogues of Love elastic parameters. The choice of the initial conditions and of the coefficient of the periodic factor associated with  $\mathbf{u}$  can be arranged in such a manner, that  $K_{nm}$  will depend on the degree  $n$  stronger than on the frequency and the order  $n$ . The exact character of this dependence can be established only by the numerical integration of the differential equations of the tidal oscillations using an Earth model. Such an integration should be planned and performed in the near future.



### Conclusion

One of our goals in the present work was to show a connection between the exterior tidal potential acting on the satellite and the tidal bodily and surface displacements of the solid Earth. We also emphasized the role of static and tidal variations in internal density, as well as the role of layers of discontinuity in the formation of the exterior tidal potential. The parameters of the Earth's elastic response to tidal forces of a given frequency appear in the present expansion of the tidal potential in a natural way. They are represented as a sum of the two terms. The first term reflects the influence of the tidal change in the internal density and the second term carries the influence of the tidal "bulldges." The dependence of the elastic parameters (Love numbers) on the local coordinates on the Earth surface may, in fact, reflect the asymmetric and inhomogeneous structure of the interior of the Earth.\*) The differential equations of tidal oscillations we develop in the present article are valid for the laterally heterogeneous Earth and for the presence of initial deviatoric stress.

Like all equations of other theories of tidal oscillations they can be integrated only numerically.

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\*) D. E. Smith. Private communication. 1972.

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