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CONTROL OPTIMIZATION OF A LIFTING BODY ENTRY PROBLEM BY AN IMPROVED AND A MODIFIED METHOD OF PERTURBATION FUNCTIONS

A Dissertation Presented to the Faculty
of the Department of Mechanical
Engineering of the University of Houston
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy

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16. Abstract

The present investigation is a study of the solution of a complex entry optimization problem. The problem is transformed into a two-point boundary value problem by using classical calculus of variation methods. Two-point boundary value problems usually require iterative numerical methods for solution, and thus, two perturbation methods were devised. These methods attempted to desensitize the contingency of the solution of this type of problem on the required initial co-state estimates.

Numerical results are presented for the optimal solution resulting from a number of different initial co-states estimates. The perturbation methods are compared, and it is found that they are an improvement over existing methods.

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Frank Garcia, Jr.

May 1974

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j	Number of coefficients in atmospheric model
m	Number of control variables
n	Number of state and co-state variables
p	Number of initial constraints
q	Number of terminal constraints

Scalars

a_j	j-coefficients for logarithmic form
A	Aerodynamic reference area
C_D	Newtonian aerodynamic drag coefficient
C_L	Newtonian aerodynamic lift coefficient
D	Aerodynamic drag force
G	Total aerodynamic force
$\bar{g}$	Acceleration of gravity
g_o	Acceleration of gravity at sea level
H	Variational Hamiltonian function
$\bar{h}$	Altitude from the earth's surface to vehicle
h_o	Reference altitude
L_V	Vertical component of aerodynamic lift force
L_H	Horizontal component of aerodynamic lift force
m	Mass of reentry vehicle
P	Objective functional
Q	Convective stagnation heat flux referenced to a one-foot radius sphere

r	Radius from center of earth to vehicle
r_e	Radius of the earth
S	Optimum step
t	Independent variable, time
t_o	Initial time
t_f	Final time
U	Norm of dissatisfaction function squared
V	Velocity of the reentry vehicle
V_c	Reference circular orbital velocity
W	Weight of reentry vehicle
α	Reentry vehicle angle of attack
β	Constant in exponential atmosphere
β_j	j-coefficients for exponential form
Γ	Flight path angle of the reentry vehicle
γ	Coordinate system longitude
δ	Correction for optimum U
ε	Convergence criteria, 1×10^{-8}
η	Correction factor
θ	Coordinate system co-latitude
Λ	Heading angle of the reentry vehicle
ρ	Atmospheric density
ρ_o	Reference atmospheric density
$\bar{\sigma}$	Standard deviation
τ	Non-dimensionalized time
ϕ	Terminal value function for extremizing
ψ	Argument of the objective functional
Ω	Stopping function

Matrices

(The brackets indicate the dimensionality of the symbol.)

g [n x 1]	Initial boundary conditions
h [n+1 x 1]	Terminal boundary conditions
I [2n x 2n]	Identity matrix
L [q x 1]	Initial conditions
M [q x 1]	Terminal conditions
T [n+1 x n+1]	Correction transformation
u [m x 1]	Control variables
x [n x 1]	State variables
x_o [n x 1]	Initial state variables
x_f [n x 1]	Terminal state variables
Y [2n x 1]	Random numbers of uniform distribution
Z [2n x 1]	Combined state and co-state variables
λ [n x 1]	Co-state variables
π [2n x 2n]	Fundamental or state transition matrix
$\pi_{\delta\lambda}$ [2n x n]	Reduced state transition matrix
σ [2n x 1]	Davidon's correction factor

Special Symbols

$( )'$	Indicates $\frac{d( )}{d\tau}$
$(\dot{})$	Indicates $\frac{d( )}{dt}$
$\delta( )$	First variation of $( )$
$d( )$	Total differential of $( )$
$( )^T$	Transpose of matrix $( )$
$( )^{-1}$	Inverse of matrix $( )$

$( )_w$	Partial derivative of ( ) with respect to w
$( )_{wv}$	Second partial derivative of ( ) with respect to w and v
$(\bar{\ })$	Indicates that ( ) is a non-dimensionalized variable
$\|\|w\|\|$	Indicates the square root of the sum of the squares of w (norm)
$\Big\|_{t_f}^{t_o}$	Indicates evaluation of a variable at t_o and t_f
$( )\big\|_k$	Indicates the variable (, at the k^{th} iteration
$( )\big\|_{t_f}$	Indicates the variable ( ) evaluated at t_f

CHAPTER I

INTRODUCTION

In many non-linear and/or time-varying problems, solutions are often sought for the "best" or optimal way to control or guide a system subject to some criterion. Typically, this criterion consists of an objective functional (pay-off function) that is to be extremized (maximum or minimum), the desired terminal conditions, and the possible constraints on the state and/or control. After the problem has been mathematically formulated, there are various conceptual techniques available to obtain the conditions required to numerically solve the optimization problem. Most of these conceptual techniques may be considered in one of two main categories, direct or indirect. Direct methods are dependent on the direct evaluation of an objective functional in order to arrive at control changes, which in turn, result in an improvement in the objective functional. Indirect methods, on the other hand, use variational analysis to arrive at a set of conditions that the optimal solution must satisfy without actually evaluating the objective functional. Thus, the problem becomes one of finding solutions which satisfy the conditions of a resulting two-point boundary value (TPBV) problem.

The most attractive of the direct methods are the gradient methods developed by Kelley [1]* simultaneously with those of Bryson and

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*Numbers in brackets refer to the references in the Bibliography at the end of this thesis.

and Denham [2]. These methods have the advantage that convergence is not contingent upon good initial guesses and that improvement in the criterion to which it is subject is guaranteed by each step. The methods have the great disadvantage that convergence often deteriorates in the vicinity of the optimum, since the gradient goes to zero. However, Lasdon, Mitter, and Warren [3] have used second derivative information to improve the convergence near the optimum. Numerous other extensions have been made to the basic gradient methods, and these are lucidly discussed by Sage [4] and Wilde and Beightler [5].

Among the more usual indirect approaches are the calculus of variations as explained among others by Bryson and Ho [6] and Bliss [7]; the dynamic programming of Dreyfus and Bellman [8,9]; and Pontryagin's Principle [10]. A disadvantage of the majority of these methods is that a solution is often contingent upon good initial guesses, particularly when complex problems are involved. These methods, however, do possess the great advantage that whenever a solution is obtained, its associated control is considered to be optimal because of the necessary conditions which must be satisfied to obtain a solution. The calculus of variations concept, in particular, offers a powerful tool for solving optimization problems because of the conditions resulting from this classical theory.

This thesis presents a solution to a complex lifting reentry three-degree-of-freedom problem by using the concept of the calculus of variations to generate a set of necessary conditions. Then solving

numerically the resulting TPBV problem by an improved existing technique and a new "direct-indirect" technique. Both of these techniques are designed to decrease the strong dependency of the solution upon good initial guesses. Since such mathematical treatment leads to a TPBV problem whose solution requires usage of large memory high-speed computers, the solution of practical problems has been rather limited until the last decade.

The atmospheric entry of a lifting body is a formidable physical problem. When optimal solutions are sought to entry problems, several simplifying dynamic and kinematic assumptions must be made if results are to be obtained with a reasonable amount of computational effort. This study addresses itself to the motion of the center of gravity of the vehicle as it passes through the atmosphere of a non-rotating spherical earth. The only external forces considered are the gravitational and the dissipative aerodynamic forces; vehicle control is effected through lift vector modulation. An approximate expression due to Detra, Kemp, and Riddell [11] is used to calculate the stagnation convective heat rate, neglecting other heat transfer processes. The three-dimensional trajectory formulation is tailored for ease of numerical solution and proper representation of the problem at hand. Previous entry studies conducted by the author and vacuum trajectory studies by Tapley, Szebehely, and Lewallen [12] and Lewallen, Schwausch, and Tapley [13] show that effort in formulating differential equations for numerical solutions often pays dividends by way of increased accuracy and speed. For this reason, the equations of motion describing

the entry of a lifting body are in a spherical coordinate system and are non-dimensionalized. In most dissipative force problems such as this one, the medium is an important factor in the resulting motion. With this in mind, the atmosphere was not modeled in the conventional manner shown in the literature (such as in Chapman [14], Loh [15], and Klafin and Barnhard [16]) $\rho = \rho_0 \exp(-\bar{h}/h_0)$, but was modeled by a logarithmic form such as $\ln\rho = a_0 + \sum_{i=1}^{j} a_i \bar{h}^{i+1}$. In this manner, a very good model of the atmosphere is obtained that represents a continuous atmosphere between any desired two altitudes.

In many problems in the field of flight mechanics, it is important to restrict the total heat input to a vehicle during entry into a planetary atmosphere along with the total aerodynamic load. The optimization problem may now be stated as follows: Given a set of differential equations which describe the motion of a lifting vehicle entering through the earth's atmosphere, find the combination of state and control variables that will minimize the integral on time of the sum of the aerodynamic force and the heat flux (pay-off function), and at the same time, satisfy specified terminal conditions which will determine the terminal trajectory time. As posed, this problem falls into a class of control optimization problems due to Bolza [17]. The entry problem considered here does not have state and/or control constraints along the trajectory. Problems with such constraints are fully discussed by Bryson and Ho [6], Hestenes [18], and Lastman and Tapley [19]. Proceeding with the calculus of variations on the posed

problem results in a TPBV problem with the appearance of a set of co-states (Lagrange multipliers) which have little physical significance and often are a discouraging factor in optimization studies. This is due to the fact that an initial guess of the co-states must be made to start the numerical integration of the differential equations associated with the TPBV problem. More often than not, the result is an initial solution which is totally unacceptable from both the numerical viewpoint and the physical viewpoint. Thus, it should be obvious that the optimal solution of a complex problem is dependent upon good initial guesses and that it follows that a major effort of this study is to attempt to desensitize the dependence of an optimal solution on the initial guesses.

Having formulated the TPBV problem, it now remains to choose a method to solve it. Specifically, the problem at hand is to systematically meet the boundary conditions while estimating the initial co-states. Numerical techniques to accomplish this have been considered by Hestenes [20] as early as 1949 and have been improved steadily since then. Some of the popular methods include the method of adjoint functions of Jarovics and McIntyre [21] which uses the equations adjoint to the linearized co-state equations; the quasilinearization methods formulated through a concept as presented by Kalaba [22] along with the modified quasilinearization method by Lewallen [23]; the perturbation methods based on the work of Breakwell [24] and Breakwell et al [25]; and, more recently, the method of perturbation functions (MPF) of Lewallen [23]. Based on the work of

Lewallen [23], which compares the convergence characteristics of the MPF, the modified quasilinearization method, and the method of adjoint functions, among others on an orbital transfer problem, the author chose the MPF because of its reported superior performance.

The MPF method was modified to improve the iteration philosophy for correcting the terminal dissatisfaction (terminal boundary errors). The iteration philosophy of Lewallen [23] was based on a norm reduction method only. More recently, Lastman and Tapley [19] have compared the MPF method using a norm reduction method and a magnitude limitation method separately with limited success. Also recently Doiron [47] has improved the MPF method by essentially employing a correction factor based on the norm and the desired number of iterations. In this thesis, the author uses a combination norm reduction and magnitude limitation method defined as an improved method of perturbation functions (IMPF). A new method which uses one of three function minimization techniques on a terminal dissatisfaction function during the early iterations and then switches to the IMPF is also exercised on the entry problem. This method is referred to as the modified method of perturbation functions (MMPF). The three function minimization techniques used are the pattern search method of Hooke and Jeeves [26], Davidon's [27] conjugate gradient search method of Fletcher and Powell [28], and an accelerated random search technique [29]. In order to test the ability of the IMPF and the three MMPF techniques to converge when started with different guesses, the entry optimization problem was started with six different sets of guesses.

There have been several optimization techniques applied to the problem of reentry trajectories with partial success. Bryson et al [30] used the gradient method to solve a planar entry problem for minimum entry heat, and Breakwell et al [25] solved a planar entry problem for maximum terminal speed at a given altitude. The Pontryagin maximum principle was used by Leondes and Niemann [31] on a planar entry problem. Lastman and Tapley [19] considered a two-degree-of-freedom entry using an objective functional similar to the one used in this study and employing the MPF technique. Tapley and Williamson [32], along with Colunga [33], considered several Apollo entry problems using MPF with an objective functional similar to the one used here. No where in the literature has the author been able to find an entry problem using the same features as the one presented in this thesis; one that could be used directly as a basis of comparison for the IMPF and MMPF strategies. Therefore, an MPF strategy similar to Lewallen's [23] is used on several different initial guesses to serve as a comparison basis.

In the following chapter, the entry problem is completely defined from a general statement of the problem through the formulation of the pay-off function. Chapter III presents the formulation of an optimization problem using the calculus of variations to arrive at a set of conditions which guarantee optimality. In addition, the resulting TPBV problem is defined. In Chapter IV a solution of a TPBV problem by the MPF is outlined that includes the differential correction of norm reduction and magnitude limitation which give rise to the IMPF.

The MMPF using each of the three function minimization schemes on a terminal dissatisfaction function is presented in Chapter V. Chapters III through V address an optimization problem in general. Chapter VI specifically focuses on the entry problem stated in Chapter II and presents its boundary conditions along with a discussion and analysis of the attempt to desensitize the effect of initial guesses on convergence. Finally, a short summary and the conclusions of this study appear in the last chapter.

CHAPTER II

PROBLEM DEFINITION

2.1 Statement of the Entry Problem

The statement of the problem for the present study is to find the path which a spacecraft, using roll modulation for control with given aerodynamic characteristics and entering through the earth's atmosphere, must follow in going from one point in state space to another with a certain objective functional. The objective functional requires that the total integral of the sum of the stagnation convective heat flux and the total aerodynamic forces experienced by the vehicle during its trajectory between the two points in space be minimized, at least in a local sense. The objective function, as stated above, results in a design requirement which favors the spacecraft thermal protection system weight and exposes the crew to short-duration, high-level aerodynamic force pulses.

2.2 Basic Assumptions

The aerodynamic properties, mainly the lift and drag coefficients (C_L, C_D), are assumed to remain constant for a given angle of attack in the hypersonic flight regime [34]. For purposes of this study, the angle of attack was assumed constant; consequently, trajectory control is available through roll angle modulation only. The gravitational field is an inverse square, gravitational force field, and the earth and its atmosphere are non-rotating with respect to inertial space.

In most dissipative force problems, such as the one being presented, the dissipative medium is an important factor in the ensuing motion. With this in mind, the atmosphere [35] was not modeled in the classical manner shown in the literature, i.e. $\rho = \rho_0 \exp(-\bar{h}/h_0)$, but was modeled by a logarithmic form such as $\ln\rho = a_0 + \sum_{i=1}^{j} a_i \bar{h}^{i+1}$. The advantage of choosing the latter functional form over the classical form is that a continuous atmosphere may be defined over any altitude range to any desired degree of accuracy rather than a piecewise continuation. Thus, a very accurate model of an exponential atmosphere was generated by obtaining the "a" coefficients from a least squares fit to the j^{th} order of an actual atmosphere [36]. For this study, j=6 was found to be sufficiently accurate. Taking the exponential of the logarithmic form mentioned earlier, the exponential density, $\rho = \beta \exp\left[-\sum_{i=0}^{6} (\beta_i \bar{h}^i)\,\bar{h}\right]$, results.

2.3 Governing Differential Equations of Motion

The mathematical model used in deriving the differential equations of motion assumes, in addition to the assumptions in Section 2.2, a point mass with three degrees-of-freedom whose motion is referenced to an inertial X, Y, Z coordinate system and is expressed in a spherical coordinate system (r, γ, θ) as shown in Figure 2-1. Control of the spacecraft is effected by rolling the stability axis about the velocity vector (Figure 2-2), thus using the vehicle's lift to perform out-of-plane maneuvers (roll modulation). The two

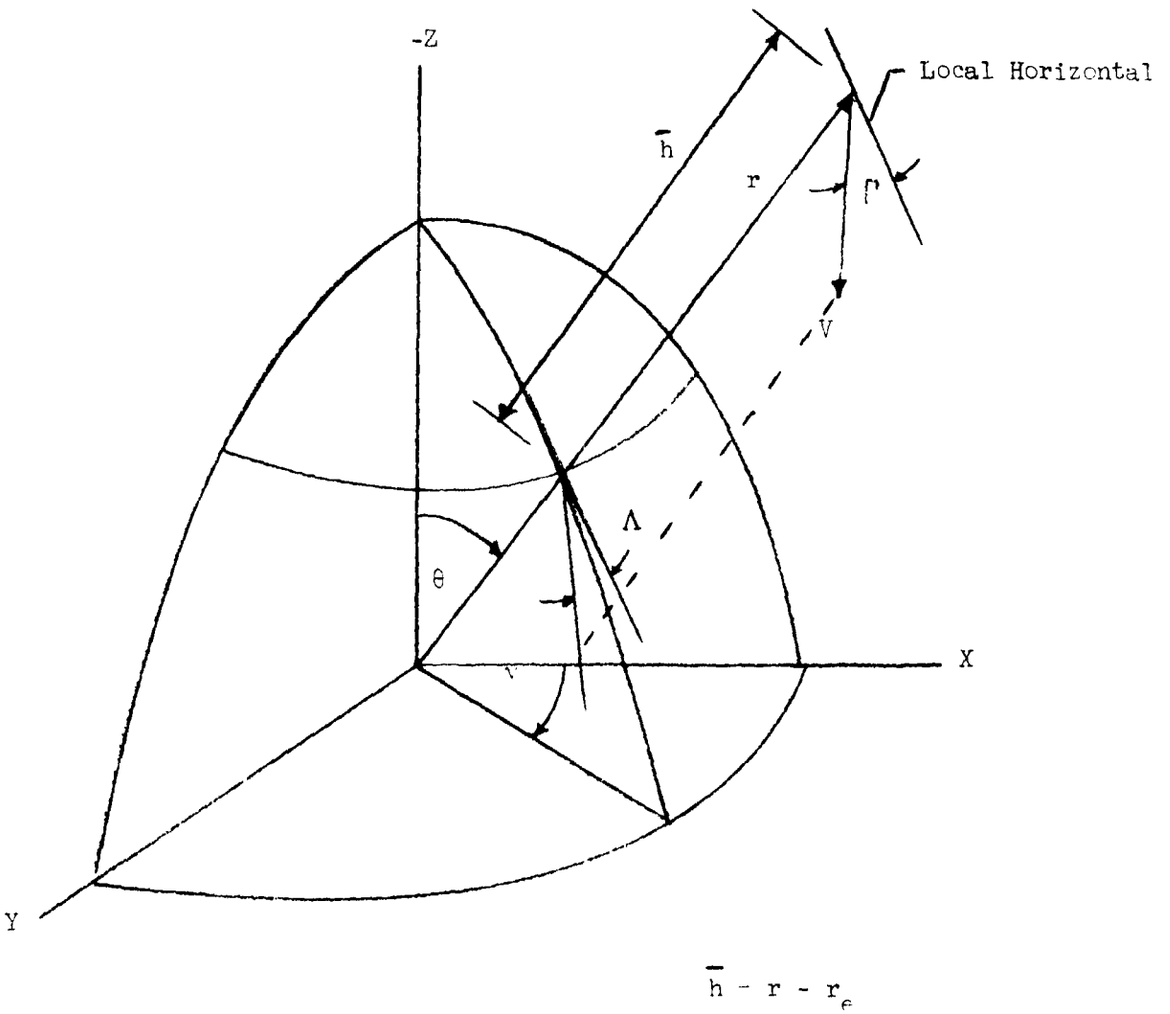


Figure 2-1. Entry Geometry and Coordinate System.

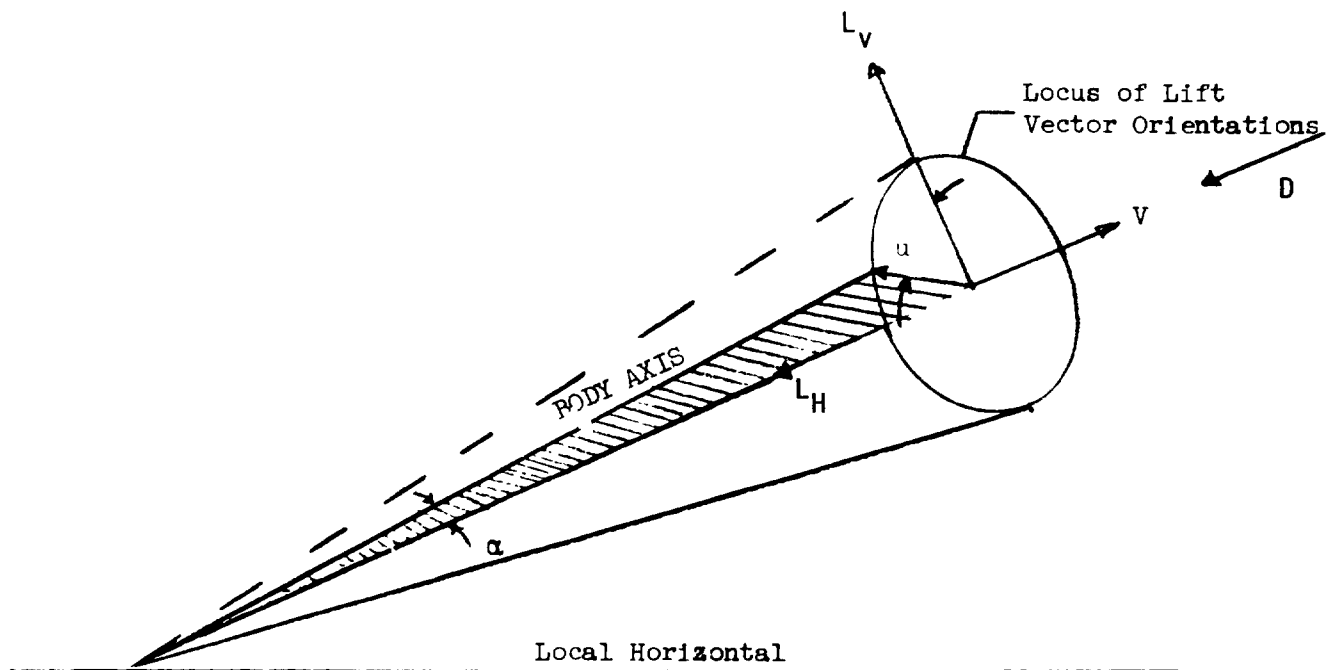


Figure 2-2. Vehicle Lift, Drag, and Control Angle Definition.

aerodynamic lift force components resulting from a roll maneuver u, are defined as

$$L_V = \frac{1}{2} \rho V^2 A C_L \cos u \tag{2-1a}$$

$$L_H = \frac{1}{2} \rho V^2 A C_L \sin u \tag{2-1b}$$

where $-\pi \quad u \quad \pi$ and the aerodynamic drag force as

$$D = \frac{1}{2} \rho V^2 A C_D \tag{2-1b}$$

Then using the inverse square gravitational field

$$\bar{g} = g_o \left(\frac{r_e}{r}\right)^2 \tag{2-1d}$$

and vehicle mass

$$m = \frac{W}{g_o} \tag{2-1e}$$

it may be shown [37,38] that the six first-order differential equations describing the motion are

$$\dot{V} = - \frac{1}{2} \rho V^2 C_D A \frac{g_o}{W} + g_o \left(\frac{r_e}{r}\right)^2 \sin \Gamma \tag{2-2a}$$

$$\dot{r} = -V \sin \Gamma \tag{2-2b}$$

$$\dot{\Gamma} = - \frac{V}{r} \cos \Gamma - \frac{1}{2} \rho V^2 C_L A \, g_o \frac{\cos u}{WV} + \left(\frac{r_e}{r}\right)^2 g_o \frac{\cos \Gamma}{V} \tag{2-2c}$$

$$\dot{\Lambda} = \frac{1}{2} \rho V^2 C_L A \frac{\sin u}{WV \cos \Gamma} g_o - \frac{V}{r} \cot \theta \cos \Gamma \sin \Lambda \tag{2-2d}$$

$$\dot{\theta} = \frac{V \cos \Gamma \cos \Lambda}{r} \tag{2-2e}$$

$$\dot{\gamma} = \frac{V \cos \Gamma \sin \Lambda}{r \sin \theta} \tag{2-2f}$$

The above set of equations (2-2) form an n set of non-linear differential equations whose solution in general is obtained by a numerical integration process. Numerical solutions are also quite dependent on the range of values which the dependent as well as the independent variables assume. Studies presented in the literature suggest that a normalization scheme of some sort aids in the solution of this type of equations. Therefore, the procedure used to non-dimensionalize the equations of motion (Equation 2-2) involved derivatives as well as dependent variables. The transformation applied in going from dimensional to non-dimensional derivatives was

$$\frac{d(\ )}{d\tau} = \frac{r_e}{V_c} \frac{d(\ )}{dt}$$

where

$$V_c = \sqrt{g_o r_e}$$

In addition, the following length, velocity, density, and energy constants were used to non-dimensionalize the dependent variables:

Length - Radius of the earth, r_e	20.925738×10^6 ft
Gravity - Sea level gravity, g_o	32.174 ft/sec^2
Density - Sea level density, ρ_o	0.2378×10^{-2} slug/ft^3
Energy - Mechanical to thermal energy conversion factor, J	778.0 ft-lb/Btu

Thus, the set of n equations (2-2) normalized according to the above

scheme with a scalar control u are as follows:

$$\tilde{V}' = -\frac{1}{2}\tilde{\rho}\tilde{V}^2\frac{C_D A}{\tilde{W}} + \frac{\sin\Gamma}{\tilde{r}^2} \tag{2-3a}$$

$$\tilde{r}' = -\tilde{V}\sin\Gamma \tag{2-3b}$$

$$\Gamma' = -\frac{\tilde{V}}{\tilde{r}}\cos\Gamma - \frac{1}{2}\tilde{\rho}\tilde{V}C_L\cos u\frac{\tilde{A}}{\tilde{W}} + \frac{\cos\Gamma}{\tilde{V}\tilde{r}^2} \tag{2-3c}$$

$$\Lambda' = -\frac{\tilde{V}}{\tilde{r}}\cos\Gamma - \frac{1}{2}\tilde{\rho}\tilde{V}C_L\sin u\frac{\tilde{A}}{\tilde{W}} + \frac{\cos\Gamma}{\tilde{V}\tilde{r}^2} \tag{2-3d}$$

$$\theta' = \frac{\tilde{V}\cos\Gamma\cos\Lambda}{\tilde{r}} \tag{2-3e}$$

$$\gamma' = \frac{V\cos\Gamma\sin\Lambda}{\tilde{r}\sin\theta} \tag{2-3f}$$

2.4 Formulation of the Objective Functional Argument

As mentioned in Section 2.1, the objective functional is the minimum integral of the sum of the convective stagnation heat flux and the aerodynamic forces evaluated between the initial and final trajectory points. The argument of this objective functional is then the heat flux and the aerodynamic forces. The convective stagnation heat flux expression commonly used in the literature to describe the heat input to a spacecraft is that due to Detra, Kemp, and Riddel [11]. Referenced to a one-foot radius sphere, the heat flux can be expressed as

$$A = 17{,}600\left(\frac{V}{26{,}000}\right)^{3.15}\left(\frac{\rho}{0.002378}\right)^{C.5} \tag{2-4}$$

and the total aerodynamic load as

$$G = \left(\frac{D}{W}\right)^2 + \left(\frac{L_V}{W}\right)^2 + \left(\frac{L_H}{W}\right)^2 \tag{2-5}$$

Normalizing Equations (2-4) and (2-5) as specified in Section 2.3, the following is obtained:

$$\tilde{Q} = C_1 \tilde{V}^{3.15} \tilde{\rho}^{0.5} \tag{2-6}$$

$$\tilde{G} = \frac{1}{2} \frac{\tilde{A}}{\tilde{W}} \left(C_L^2 + C_D^2\right)^{1/2} \tilde{\rho}\tilde{V}^2 \tag{2-7}$$

$$\psi = \tilde{Q} + \mu\tilde{G} \tag{2-8}$$

where μ is a weighing parameter, and ψ is the argument of the objective functional.

The problem as defined in this chapter may be classified as an optimization problem whose necessary conditions and their applications are derived and shown respectively in Chapter III.

2.5 Specified Initial and Terminal Conditions

As pointed out in Section 2.1, the problem under study is to find the path a spacecraft must follow to satisfy some objective functional as described in Section 2.4 while going from some specified initial state to a specified final state. The specified initial state is defined as the initial conditions of the problem and the final state as the terminal conditions. For the problem under study, the initial boundary conditions specified are n+1; that is, all of the independent variables n as well as the dependent variable time, which is zero, are

specified. The terminal conditions specified are all of the independent variables except the heading (azimuth) Λ; thus, at the terminal time there are n-1 specified terminal conditions.

CHAPTER III

FORMULATION OF AN OPTIMIZATION PROBLEM

3.1 Necessary Conditions for Optimal Trajectories

Trajectory optimization problems classically require the satisfaction of various conditions. These evolve from consideration of the following problem: Given a set of non-linear differential equations, determine the history of its variables so that some objective functional is optimized while satisfying specified initial and terminal constraints. The objective functional is, in general, some combination of the variables describing the problem. Variables in optimization problems are classically divided into state, co-state, independent (normally time), and control variables.

The non-linear set of differential equations which describe the trajectory are

$$\dot{x} = f(x, u, t) \tag{3-1}$$

where, for the problem under study, x is an n-vector of state variables; u is an m-vector of control variables; and t is the independent variable, time. The initially specified initial conditions are

$$L(x_0, t_0) = 0 \tag{3-2}$$

where L is a p-vector, and the initially specified terminal conditions with M as a q-vector are

$$M(x_f, t_f) = 0 \tag{3-3}$$

The objective functional, the quantity to be extremized, is a scalar and, in general, is formulated as a problem of Bolza [17]

$$P = \phi\ (x_f,\ t_f) + \int_{t_o}^{t_f} \psi\ (x,\ u,\ t)\ dt \tag{3-4}$$

where $\phi = \phi\ (x_f,\ t_f)$ allows the introduction of functions whose terminal values must be extremized. The non-linear differential equation (3-1) may be adjoined to P with the co-state λ:

$$P = \phi + \int_{t_o}^{t_f} \left[\psi + \lambda^T\ (f - \dot{x})\right]\ dt \tag{3-5}$$

For convenience, the scalar term H is defined as follows:

$$H = \psi + \lambda^T\ f \tag{3-6}$$

where H is commonly referred to as the variational Hamiltonian function. The term H takes its name from the well known "Hamiltonian function" of classical mechanics [39] because of its functional similarity. Substituting Equation (3-6) into (3-5) results in the following form of P:

$$P = \phi + \int_{t_o}^{t_f} (H - \lambda^T\ \dot{x})\ dt \tag{3-7}$$

Equations (3-1) through (3-3) and (3-7) describe in general many optimization problems for deterministic, non-linear, time-dependent systems.

Integrating the second term under the integral of the above expression (3-7) by parts, then

$$P = \phi - \lambda^T x \Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} (H + \dot{\lambda}^T x)\, dt \tag{3-8}$$

The first necessary condition classically required throughout the literature [6] for an optimal trajectory by using calculus of variations techniques is that the first variation of P vanish. Citron [40] shows that in the calculus of variations theory, the first variation is identical to the total differential in optimization of functions. The conditions are discussed by Tapley and Lewallen [41] and a derivation is included for completeness. Taking the first variation of P in the form of (3-8), the result is given by

$$dP = d\phi \Big|_{t_f} - (a\lambda^T x + \lambda^T dx) \Big|_{t_o}^{t_f} + d\left[\int_{t_o}^{t_f} (H + \dot{\lambda}^T x)\, dt\right] \tag{3-9}$$

Taking the indicated differentials in the above expression and using the Liebnitz rule on the last term, Equation (3-9) becomes

$$\begin{aligned} dP = {} & (\phi_x^T dx + \phi_t\, dt) \Big|_{t_f} - (d\lambda^T x + \lambda^T dx) \Big|_{t_o}^{t_f} \\ & + (H + \dot{\lambda}^T x)\, dt \Big|_{t_o}^{t_f} + \int_{t_o}^{t_f} \left[(H_x^T + \lambda^T f_x)\, \delta x + H_u^T\, \delta u \right. \\ & \left. + H_\lambda^T\, \delta\lambda + \dot{\lambda}^T\, \delta x + \delta\dot{\lambda}^T x\right] dt \end{aligned} \tag{3-10}$$

Performing the indicated integration on the last term in (3-10) and defining the total differential of λ to the first order as $d\lambda^T = \delta\lambda^T + \dot{\lambda}^T\, dt_f$ where the variation in λ, $\delta\lambda^T$, takes place during a fixed time period, the necessary condition that dP vanish now becomes

$$dP = (\phi_x^T - \lambda^T)\, dx \Big|_{t_f} + (\phi_t + H)\, dt \Big|_{t_f} + (\lambda^T\, dx - H dt) \Big|_{t_o}$$

$$+ \int_{t_o}^{t_f} \left[(H_x^T + \dot{\lambda}^T)\, \delta x + H_u^T\, \delta u + \delta\lambda^T\, (f - x)\right] dt = 0 \qquad (3\text{-}11)$$

The condition that dP vanish implies that each term in (3-11) goes to zero provided that all of the variations are independent. The resulting necessary conditions may be classified as initial transversality conditions, terminal transversality conditions, and conditions to be satisfied for all time t between t_o and t_f.

The initial transversality condition

$$\lambda^T\, dx - H\, dt \Big|_{t_o} = 0 \qquad (3\text{-}12)$$

will be identically satisfied if the initial state and time are specified. However, if the initial state and time are not specified and dx and dt are independent, then (3-12) implies that λ^T and H must vanish at t_o. On the other hand, if dx and dt are not independent, use must be made of (3-2) to satisfy (3-12).

The remaining two transversality conditions which must be satisfied at the terminal time t_f are

$$(\phi_t + H)\, dt \,\Big|_{t_f} = 0 \tag{3-13}$$

and

$$(\phi_x^T - \lambda^T)\, dx \,\Big|_{t_f} = 0 \tag{3-14}$$

The above two equations imply that if dt and dx are not specified, their respective coefficients, $(\phi_t + H)$ and $(\phi_x^T - \lambda^T)$, must be zero.

The only remaining necessary conditions derived from dP = 0 (3-11) come from within the integral term in that expression and must be satisfied for all $t_o \leq t \leq t_f$. The first condition within the integral of (3-11) is

$$H_x^T + \dot{\lambda}^T = 0 \tag{3-15}$$

which provides the co-state differential equations and is the classical Euler Lagrange equation. The next condition is the classical optimality condition

$$H_u = 0 \tag{3-16}$$

The third and last condition resulting from the integrand of (3-11) is merely that the original differential equations of motion be satisfied, that is

$$\dot{x} - f = 0 \tag{3-17}$$

A trajectory satisfying the necessary conditions expressed by Equations (3-13) through (3-17) defines a stationary trajectory, these conditions are not sufficient to guarantee an optimal trajectory. A fourth condition, and one that is sufficient to insure an optimum, is known as the Legendre-Clebsch condition in the calculus of variations. This condition involves the Weierstrass E-Function as explained by Gelfand and Fomin [42] as well as Bryson and Ho [6] and must be equal to, or greater than, zero for a minimum. The Legendre-Clebsch condition states that, for $t_o \leq t \leq t_f$, H_{uu} must be non-singular positive definite

$$H_{uu} \geq 0 \tag{3-18}$$

3.2 Reduction of an Optimization Problem to a Two-Point Boundary Value Problem

A trajectory optimization problem may now be reduced to a TPBV problem by considering all of the conditions for optimality derived in Section 3.1. The conditions that must be satisfied for all $t_o \leq t \leq t_f$ are as follows:

$$\dot{x} = f = H_{\lambda}^{T} \tag{3-19}$$

where x is an n vector first-order, non-linear, differential equation of motion;

$$\dot{\lambda} = - H_{x}^{T} \tag{3-20}$$

where λ is also an n vector first order non-linear differential equation of co-states and is called the Euler-Lagrange equation;

$$H_u = 0 \tag{3-21}$$

which is the classical optimality condition. Since u is an m vector, (3-21) defines m algebraic equations. These m algebraic equations may be solved for m control variables, u, as a function of the state, co-state, and independent variable time and thus u may be eliminated from Equations (3-19) and (3-20) if desired;

$$H_{uu} \geq 0 \tag{3-22}$$

this inequality states that H_{uu} must be non-singular positive definite. It is commonly used in trajectory optimization problems to resolve sign ambiguities arising out of the condition for u from the optimality condition $H_u = 0$.

Equations (3-19) and (3-20), the state and co-state equations, may be re-written in a more compact form as

$$\dot{Z} = \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} H_{\lambda}^{T}(x,\lambda,u,t) \\ -H_{x}^{T}(x,\lambda,u,t) \end{bmatrix} = F(Z,u,t) \tag{3-23}$$

where Z is a 2n vector composed of state and co-state variables.

The initial boundary conditions which must be satisfied at $t = t_o$ are

$$g(x_o, t_o) = 0 \tag{3-24}$$

which is an n vector made up of the initially specified initial conditions L from Equation (3-2) and the initial transversality conditions from Equation (3-12). As pointed out in Section 3.1, if dx and dt in Equation (3-12) are not independent, which is generally the case, then dL = 0 is used to eliminate p variations from the n+1

variations of Equation (3-12), leaving n+1-p independent variations whose coefficients are equated to zero to yield n+1-p relations. Thus, when these n+1-p relations are combined with the p relations arising out of L, there results a set of n+1 relations, g = 0 and t_0.

$$h\ (Z_f,\ t_f) = 0 \tag{3-25}$$

is an n+1 terminal boundary condition vector which is made up of the initially specified terminal conditions M from Equation (3-3) and the terminal transversality conditions from Equations (3-13) and (3-14). An argument parallel to that used above to insure that dx and dt are independent on the initial boundary also applies here on the terminal boundary. Thus, n+1 independent coefficients are assured.

Equations (3-24) and (3-25) completely define the 2n+2 conditions necessary to solve a TPBV problem. In many problems, the initial time t_0 is usually specified as zero; therefore, under this assumption, we have defined n initial conditions, g, and n+1 terminal conditions, h. In summary, the TPBV problem to be solved is defined by a system of 2n non-linear, first-order, differential equations (3-23) with n initial boundary conditions, g, and n+1 terminal boundary conditions, h. The solution of this problem results in a minimum of the objective functional (3-4) with satisfaction of all the constraints defined by (3-2) and (3-3). The general solution of these problems is so complex that closed-form solutions are only realized under a multitude of simplifying assumptions, if at all; thus, solutions must be obtained numerically. In fact even numerical solutions are complex.

The following two chapters will discuss this problem and use perturbation methods to obtain solutions. Chapter IV will deal with an improved version of the popular method of perturbation functions (IMPF) and Chapter V will address a new approach, a modified method of perturbation functions (MMPF). Chapter IV also outlines an algorithm employing the improved version of MPF which, with minor modification, is also applicable with the MMPF outlined in Chapter V.

CHAPTER IV

SOLUTION OF A TPBV PROBLEM BY AN IMPROVED METHOD OF PERTURBATION FUNCTIONS

4.1 Introduction

Equations (3-2[illegible] through (3-25) are the transformation of the entry problem as stated in Chapter II into an optimal control problem. This resulting optimal control problem is a TPBV problem whose numerical solution may be obtained in general only through some sophisticated iteration algorithm. The need for an iteration algorithm arises from the fact that there are 2n non-linear, first-order differential equations (3-23) which must be numerically integrated either forward or backwards in time; however, there are only either n or n+1 conditions known for the forward and backwards integration, respectively. Thus, it is necessary to guess either n or n+1 conditions to allow integration of the differential equations. An improved method of perturbation functions (IMPF) is used to update the assumed conditions and to continue iteratively the updating procedure until the terminal boundary conditions h go to zero. Figure 4-1 shows the general outline of such an algorithm.

4.2 Method of Perturbation Functions

As the name itself implies, the method consists of perturbing the $\dot{Z}$ function. This function, Equation (3-23), with the control u eliminated by using Equations (3-21) and (3-22) is perturbed linearly about the k^{th} trajectory at the end of each iteration. This perturbation then provides an estimate of the value of the unknown variables

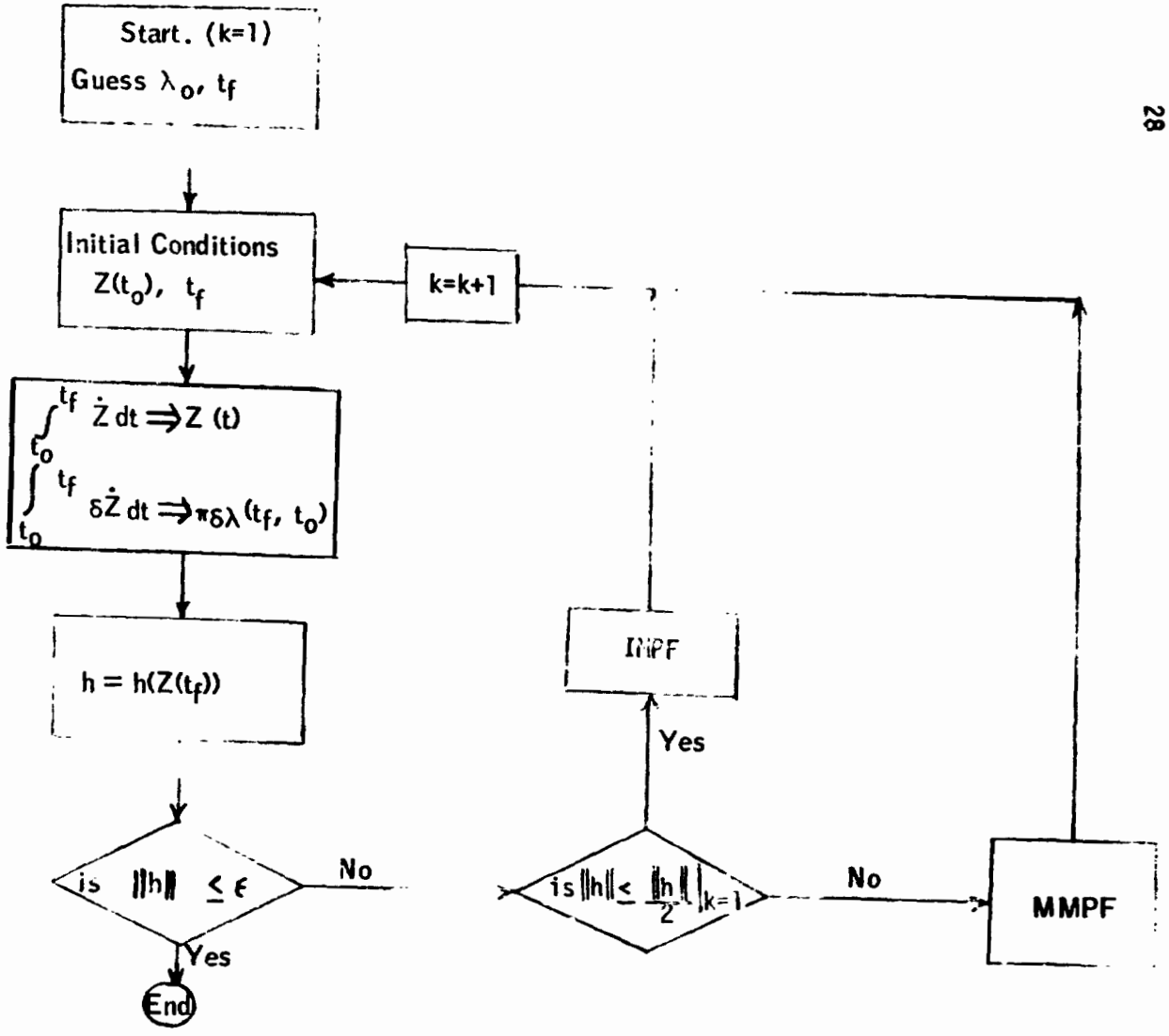


Figure 4-1. A General Computational Iterative Outline of the Direct-indirect Method for Solution of a TPBV Problem.

needed to begin the integration process at the next iteration and, at the same time, drive the dissatisfaction, h, to zero. In general terms, this process is illustrated in Figure 4-1. Practical use of this method has shown that it possesses second order convergence properties [41] when it is near a solution.

If the initial boundary conditions satisfy g = 0, Equation (3-24), then $\dot{Z} = F(Z,t)$. Equation (3-23), may be integrated forward in time starting at $t_0 = 0$ to some assumed t_f. Generally, $h \neq 0$ on the initial iteration so that Equation (3-25) is not satisfied, mainly because the n assumed co-states along with the final time estimate, t_f, are not the optimal values. The terminal dissatisfaction function h is then used to correct the assumed values by the procedure described below.

Consider a perturbed trajectory terminating on a perturbed set of terminal conditions. If only the linear terms of a Taylor Series expansion about some reference trajectory are retained, then

$$\dot{Z}^* = \dot{Z}_k + \frac{\partial F}{\partial Z}\Big|_k [Z^* - Z_k] \qquad (4\text{-}1)$$

and if $\delta Z = Z^*_{k+1} - Z_k$ then the above results can be expressed as

$$\delta\dot{Z} = \frac{\partial F}{\partial Z}\Big|_k \delta Z \qquad (4\text{-}2)$$

which is a set of 2n linear perturbation equation with $\frac{\partial F}{\partial Z}$ being a 2n x 2n matrix evaluated on the k^{th} or reference trajectory. Since, on the reference trajectory $h \neq 0$, a perturbation equation is also obtained for the terminal constraint h which is evaluated at the

terminal time t_f. The differential constraint $dh\ (Z_f, t_f)$ is

$$dh = \frac{\partial h}{\partial Z}\Big|_{tf} dZ_f + \frac{\partial h}{\partial t}\Big|_{t_f} dt_f \qquad (4\text{-}3)$$

and represents a change in the terminal error due to changes in Z_f and t_f. If $dZ_f = \delta Z_f + \dot{Z}_f\, dt_f$ is used to allow terminal Z variations due to t_f, then Equation (4-3) becomes

$$dh = \frac{\partial h}{\partial Z}\Big|_{t_f} \delta Z_f + \dot{h}\, dt_f \qquad (4\text{-}4)$$

The desired linear algebraic correction equation is derived under the conditions that there exists a fundamental matrix π, as defined in the literature, so that the terminal variations in Z (δZ_f) can be related to the initial variations in Z (δZ_o) according to

$$\delta Z_f = \pi\ (t_f, t_o)\ \delta Z_o \qquad (4\text{-}5)$$

The above expression is the well-known solution to a differential equation of the form of Equation (4-2). Equation (4-5) implies that δZ_f is some linear function of the initial variation δZ_o, where π is a 2n x 2n matrix. Thus, Equation (4-4) can now be written in a more useful form as

$$dh = \frac{\partial h}{\partial Z}\Big|_{t_f} \pi\ \delta Z_o + \dot{h}\, dt_f \qquad (4\text{-}6)$$

by making use of Equation (4-5). As stated earlier, n of the 2n elements in δZ_o are known; therefore, no corrections are required for these states. Thus, only the right half of the 2n x 2n π matrix in

Equation (4-5) needs to be known, and it may be rewritten as

$$\delta Z_f = \pi_{\delta\lambda}\, \delta\lambda_o \tag{4-7}$$

where $\pi_{\delta\lambda}$ is a 2n x n matrix. Using this reduced form, Equation (4-6) in partitioned matrix form now becomes

$$dh = \left[\frac{\partial h}{\partial Z}\Big|_{t_f}\ \pi_{\delta\lambda}\ \vdots\ \dot{h}\right]\begin{bmatrix}\delta\lambda_o \\ \hline dt_f\end{bmatrix} = T\begin{bmatrix}\delta\lambda_o \\ \hline dt_f\end{bmatrix} \tag{4-8a}$$

or, upon solving for the correction vector and assuming that T is non-singular,

$$\begin{bmatrix}\delta\lambda_o \\ \hline dt_f\end{bmatrix} = T^{-1}\, dh \tag{4-8b}$$

There remains two variables to be determined in the above expression, namely, $\pi_{\delta\lambda}$ and dh. Since the estimate of dh will be covered in the following section, the determination of $\pi_{\delta\lambda}$ can be made. Evaluation methods for π or $\pi_{\delta\lambda}$ are well covered in the literature [43] but will be summarized here for the sake of completeness. Consider Equation (4-5), which is a solution to Equation (4-2), and relates the final perturbations δZ_f to the initial perturbations δZ_o. Assume that a 2n x 2n identity matrix I is used to express the initial perturbations where each column of I

is used as δZ_{o_i} in the following manner:

$$I = \begin{bmatrix} 1 & 0 & 0 & & & 0 \\ 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ \cdot & \cdot & 0 & \cdots & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ 0 & 0 & 0 & & & 1 \end{bmatrix} \tag{4-9}$$

where i = 1,2n refers to each column of I. Then each successive application of δZ_{o_i} results in a δZ_{f_i} which is an i^{th} column of the transition (fundamental) matrix π. Therefore, if Equation (4-2) is integrated 2n times with δZ_{o_i} as an initial vector from $t_o = 0$ to t_f, the transition matrix π would be available. Furthermore, for the case at hand where only $\pi_{\delta\lambda}$ is needed, the integration of Equation (4-2) is required only n times, and the initial vector used is δZ_{o_ℓ} where $\ell = n+1, 2n$.

4.3 Determination of the Differential Correction

The usual procedure found in the literature (Lewallen [23] is an example) for the evaluation of dh is to let

$$dh = -\eta h \tag{4-10}$$

with the restriction that the scalar correction factor $0 < \eta \leq 1$ and the values that η then assumes are chosen as some step function of $\|h\|_k$. Equation (4-8b) may now be rewritten by substituting for dh with

Equation (4-10) as

$$\begin{bmatrix} \delta\lambda_0 \\ \hline dt_f \end{bmatrix} = - \eta T^{-1} h \qquad (4\text{-}11)$$

For problems whose initial solutions result in large $\|h\|$, η is normally chosen close to zero; as the norm of the dissatisfaction $\|h\|$ begins to vanish η approaches one. The successful solution to a problem often depends to a large degree on the logic used for selecting η.

Thus, the logic used to develop a process for selecting η should concentrate on the fact that an attempt is being made to solve a highly complex non-linear problem by at best a quasi-second order technique. This means that particular attention should be directed toward the following:

a) The case where $\eta = 1$ which indicates 100 percent of the correction vector $\begin{bmatrix} \delta\lambda_0 \\ \hline dt_f \end{bmatrix}$ is requested. Large corrections should in general be suspect and not allowed.

b) The maximum (100 percent) correction should be based on the expected range of values of the state. The comparison can be made by comparing the norms of the correction vector and the range of values of the state.

c) The norm of the terminal dissatisfaction should also be monitored. Steps to reduce large percentages of h should not be allowed basically because elimination of large errors in a non-linear system which is based on linear estimates has little guarantee for success.

The general philosophy of the correction factor using the logic stated in (a), (b), and (c) above is to avoid overshooting the envelope of convergence; improve the area of convergence; and, at the same time, speed up the convergence process. It will later be shown in Chapter VI that this philosophy does, in fact, improve the envelope of convergence as compared to results in the literature. The functional relationship for the correction factor is

$$\eta = \begin{cases} f(\|h\|) & \text{if} \quad \|d\| \leq \|e\| \\ 1/\|d\| & \text{if} \quad \|d\| > \|e\| \end{cases} \tag{4-12}$$

where $f(\|h\|)$ is defined as a step function, $\|d\|$ is the norm of 100 percent of the correction vector (Equation (4-11) evaluated with $\eta = 1$), and $\|e\|$ is a constant representing the norm of the expected range of the states and time of the particular problem whose solution is being sought. In the event that the present norm is greater than the past, η is reduced by a factor of two.

4.4 Iteration Scheme for the Improved Method of Perturbation Functions

Now that the initial correction vector equation $\begin{bmatrix} \delta\lambda_0 \\ --- \\ dt_f \end{bmatrix} = -\eta T^{-1} h$ has been completely defined, an iteration scheme for solving the TPBV problem may be stated. The scheme requires that an initial vector of λ and an estimate of the terminal time, t_f, be made available so that the 2n equations, $\dot{Z} = F(Z,t)$, may be integrated forward in time to $t = t_f$. Simultaneously with this integration, the 2n perturbation equations $\delta\dot{Z} = \frac{\partial F}{\partial Z}\delta Z$ are integrated to the same t_f n times using the

starting vector defined by the right half of Equation (4-9). At the terminal time, the terminal dissatisfaction h as well as its norm, is evaluated. Subsequently, the algebraic equations which determine the initial corrections in λ and t_f to be applied at the next iteration (Equation 4-11) are solved. This iteration procedure is continued until some small predetermined value of $\|h\|$ is reached, when it is assumed that the problem is solved or has converged. However, it is not always possible to achieve convergence under ordinary circumstances. If convergence is not achieved within a reasonable number of iterations and divergence is diagnosed, it is best to start again with a different estimate on the unknowns.

The described iteration scheme for the IMPF may be sequentially applied in the following manner:

a) Choose initial estimates for λ_0, t_f, and $\|e\|$.

b) Simultaneously integrate forward in time to t_f the 2n non-linear differential equations, $\dot{Z} = F(Z,t)$, and the 2n linear perturbation equations, $\delta\dot{Z} = \frac{\partial F}{\partial Z}\delta Z$.

c) At the terminal time, evaluate h, $\|h\|$, and $\|d\|$ to select the scalar correction factor η.

d) Solve for $\delta\lambda_0$ and dt_f to update previous initial estimates.

e) Examine $\|h\|$ and number of iterations to determine whether to continue iterating or stop. If the process is continued, return to step (a).

The following chapter will examine some alternate approaches to the solution of a TPBV problem which does not make use of the perturbation on h initially, Equation (4-4), but first considers some function of h as a quantity to be minimized. Minimization of this function then results in a solution to the TPBV problem by switching to IMPF at some predetermined small value of h. This is done to take full advantage of the IMPF convergent characteristics which are very rapid whenever the initial guesses fall within the envelope of convergence. This technique may be classified as a direct-indirect approach; direct in the sense that some function of h is extremized directly, and indirect in the sense that IMPF does not extremize h to arrive at a solution.

CHAPTER V

MODIFIED METHOD OF PERTURBATION FUNCTIONS

5.1 Introduction

In the previous chapter, an improved method of perturbation functions (IMPF) was used to solve a TPBV problem arising out of the calculus of variations. This method essentially used the linear perturbations of $\dot{Z} = F(Z,t)$, Equation (3-23), and the dissatisfaction vector h, Equation (3-24), to arrive at an initial correction vector $\begin{bmatrix} \delta\lambda_0 \\ --- \\ dt_f \end{bmatrix}$, Equation (4-8b), with which to correct the previous initial λ and terminal time estimates. The algorithm then makes these corrections and the process is continued as described in the last section of the previous chapter. Although this method is one of the most popular for solving TPBV problems, it is by no means independent of the initial estimates of co-states and terminal time. For this reason, three new strategies under the general name of modified method of perturbation functions (MMPF) are explored. These new strategies make use of function minimization methods which have shown good results in the literature and are used with the MPF method to form a direct-indirect approach to the solution of a TPBV problem. Essentially they are used to define an initial correction vector replacing Equation (4-8b). The name MMPF is derived from the fact that some of the features of MPF are used in MMPF. Most of the methods surveyed throughout the literature may be classified as either gradient methods, pattern search methods, or random search methods (an example is found in [44]). Of the three classifications, gradient methods generally use the most

intelligence and random search the least. The three particular methods chosen are (1) the Davidon Method [28] which is a proven gradient technique; (2) the method of Hooke and Jeeves [26], a pattern search concept; and (3) an accelerated random search strategy [29]. The above three methods are identified as MMPF-D, MMPF-HJ, and MMPF-ARS, where the letter or letters after the hyphen in each case refers to Davidon, Hooke and Jeeves, and accelerated random search, respectively. For the sake of completeness, these individual methods will be briefly explained in the following three sections.

In all cases of the modified method of perturbation functions, three changes had to be made to the method of perturbation functions concept described in Chapter IV to implement the direct-indirect concept of solution. The first change was the selection of a stopping condition on which to end the integration process since, with function minimization processes, there is no way of obtaining a terminal time estimate. This was accomplished by defining the terminal time as the time which satisfied one of the selected terminal boundary conditions in h; preferably, one which is monotonic decreasing. Therefore, the satisfaction of

$$\Omega = 0 \tag{5-1}$$

is used as the stopping condition. Next came the selection of the function evaluated at the terminal time which is to be minimized. Since the TPBV problem is solved whenever the norm $\|h\|$ reaches some small specified value, the logical choice for a function is some

$U(h) \geq 0$. Thus, the function chosen to be minimized was

$$U = h^T h \geq 0 \tag{5-2}$$

It should be pointed out that the gradient information of U which is required in the MMPF-D would already be available from dh, Equation (4-4). Finally, since the set of the variables Z which minimize U are at the terminal time, they must be expressed as corrections to the initial Z's. This is accomplished by considering the solution of $\delta\lambda_o$ from $\delta Z_f = \pi_{\delta\lambda}\delta\lambda_o$, Equation (4-7), that minimizes the sum of squares of the residuals. This solution is given by

$$\delta\lambda_o = (\pi_{\delta\lambda}^T \pi_{\delta\lambda})^{-1} \pi_{\delta\lambda}^T \delta Z_f \tag{5-3}$$

where δZ_f is defined by the difference between Z at t_f and the desired Z which results from the minimization of U.

By making use of Equations (5-1) through (5-3), the MMPF using one of the three previously mentioned function optimization methods would proceed until $\|h\|$ reached some small predetermined value. At this time, a switch would be made to the IMPF strategy described in Chapter IV. A general computational outline of the direct-indirect method for solution using any of the three MMPF methods with the IMPF method as a final strategy is shown in Figure 4-1.

5.2 Modified Method of Perturbation Functions Using the Davidon Method

The method of Davidon as described by [28] involves the use of the first and second partial derivative information on the function being extremized and is derived assuming a quadratic function. Generally speaking, this method uses the idea of generating the

direction of search by multipling the gradient by a matrix that is in some manner related to the inverse of the second partial derivative matrix. This method appears particularly attractive to solve TPBV problems when used in conjunction with IMPF. Methods using gradient information (first partial derivatives) become extremely slow as the optimum is approached. Therefore, it is necessary to switch to a method which converges rapidly as the optimum is approached. As pointed out in Chapter IV, the IMPF procedure converges rapidly in the vicinity of the optimum, and a natural transition would be to this method.

Davidon's method described here briefly for a general function starts by examining a quadratic objective function, for example,

$$U(Z) = a + b^T Z + \frac{1}{2} Z^T cZ \tag{5-4}$$

where the Hessian of U(Z) (c) is a non-singular symmetric positive definite matrix. Thus, U(Z) has a local minimum which, in fact, is also a global minimum. The gradient of (5-4) at any point Z is

$$U'(Z) = b + cZ \tag{5-5a}$$

from which

$$Z = c^{-1} (U'(Z) - b) \tag{5-5b}$$

At the optimum of a differentiable function, all first derivatives are zero, and Equation (5-5b) may be written as

$$Z^* = -c^{-1} b \tag{5-6}$$

The above equation and (5-5b) may be subtracted to obtain the correction Δ to be made to Z in order to have the optimum U,

$$Z^* - Z = \Delta = -c^{-1} U' \tag{5-7}$$

According to Davidon's method, if this inverse were available and the U under consideration were quadratic, the optimum could be found in one step. In most practical applications, this is not the case, and Equation (5-7), as such, is not applicable directly to most functions. In Davidon's method, Equation (5-7) is used only to define a direction of search in the space of the variables Z. And thus, a new correction vector, σ, must be defined so that

$$\sigma = S\Delta = -S\, c^{-1} U' \tag{5-8}$$

where the step size, S, is found by obtaining the optimum along a line of search using quadratic interpolation. The evaluation of c^{-1}, as outlined in [28], is performed indirectly by using the change in gradients from iteration to iteration, the last c^{-1}, and the correction vector σ.

An iterative scheme for a minimum under MMPF-D may be described as follows:

a) Begin with the identity matrix, I, as the initial guess for c_1^{-1}, and assume some λ_o.

b) Integrate forward both the non-linear and the perturbation differential equations, as described in Section 4.4, observing Equations (5-1) through (5-3).

c) Evaluate the gradient of U, U'.

d) Compute a direction using Equation (5-7).

e) Evaluate the step size, S, by using a quadratic interpolation scheme or some other suitable scheme.

f) Compute the correction vector, σ, as $\sigma_k = S_k \Delta_k$.

g) Evaluate the desired terminal variables as $Z_{k+1} = Z_k + \sigma_k$ and compute $\delta\lambda_o$ using Equation (5-3).

h) Compute the Q_{k+1}^{-1} for the next iteration, which must be be positive definite so that the new direction will give local improvement of U(Z).

i) If $\|h\| \leq 1/2 \|h\|_{k=1}$, switch to IMPF; otherwise, start over at (b) with new estimates for λ_o.

5.3 Modified Method of Perturbation Functions Using the Method of Hooke and Jeeves

The method of Hooke and Jeeves [26] is a direct pattern search technique requiring no derivative information but involving a sequential examination of trial solutions. This method is comprised of a series of exploratory moves about each variable in the function under consideration. These exploratory moves are performed on one variable at a time insisting that the value of the function be less than, or equal to, its previous value if a minimum is being sought. At the end of the total exploration in the variable space, it is assumed that a pattern may be established. Thus, a new exploration point is set by simultaneously stepping off a distance equivalent to that of the initial start point and that at which the search stopped.

This technique is continued until the function reaches a desired value. The technique has strategies which handle situations where no improvement is detected whenever the establishment of a pattern is being sought.

For the sake of completeness, an example of an application of this method in two-variable space (x,y) as proposed by Hooke and Jeeves is presented. It is assumed that the search is started at the x and y corresponding to b_1 in Figure 5-1, where b represents a pattern resulting from an exploratory search at E, and the subscripts indicate the trial. The resulting exploration about point 1 in Figure 5-1 becomes b_2, and, by using b_1 and b_2 and reasoning that a similar exploration about b_2 would give similar results, a new exploration point, E_2, is selected. Its location in space is given by 2 (b_2 - b_1); using this formula, the pattern will continue to grow. The search continues until a point where improvement in the desired function is not realized (point E_4 in Figure 5-1). At this point, a new strategy is needed to continue. The point E_4 is discarded, and the last good pattern point now becomes the new exploration point. Figure 5-2 shows that the new exploration point has required a decrease in step to reach a new base b_6. Now using b_5 and b_6, it is possible to start as before. The strategy continues until a solution is reached.

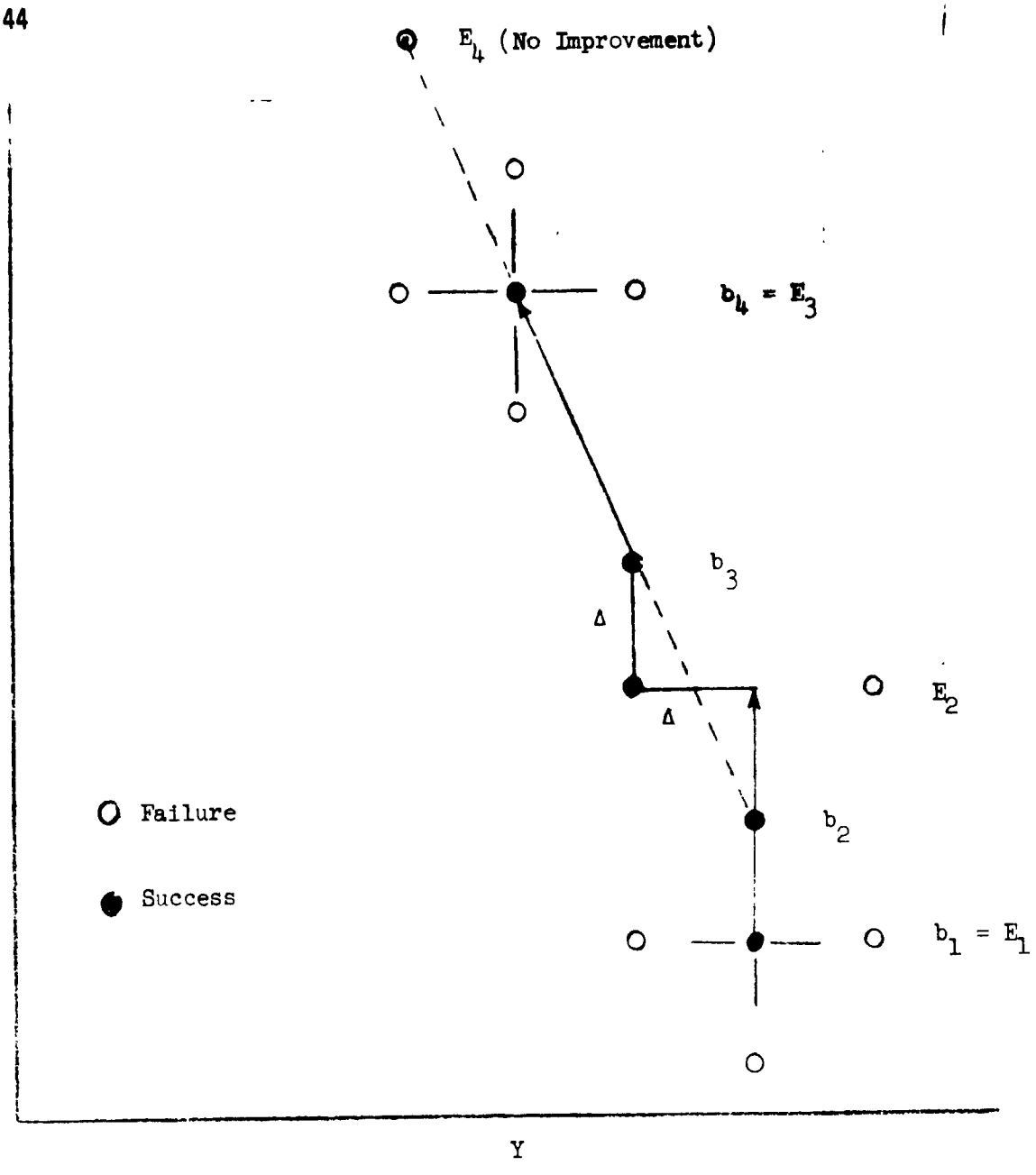


Figure 5-1. Schematic of Hooke and Jeeves Pattern Search Showing Improvement.

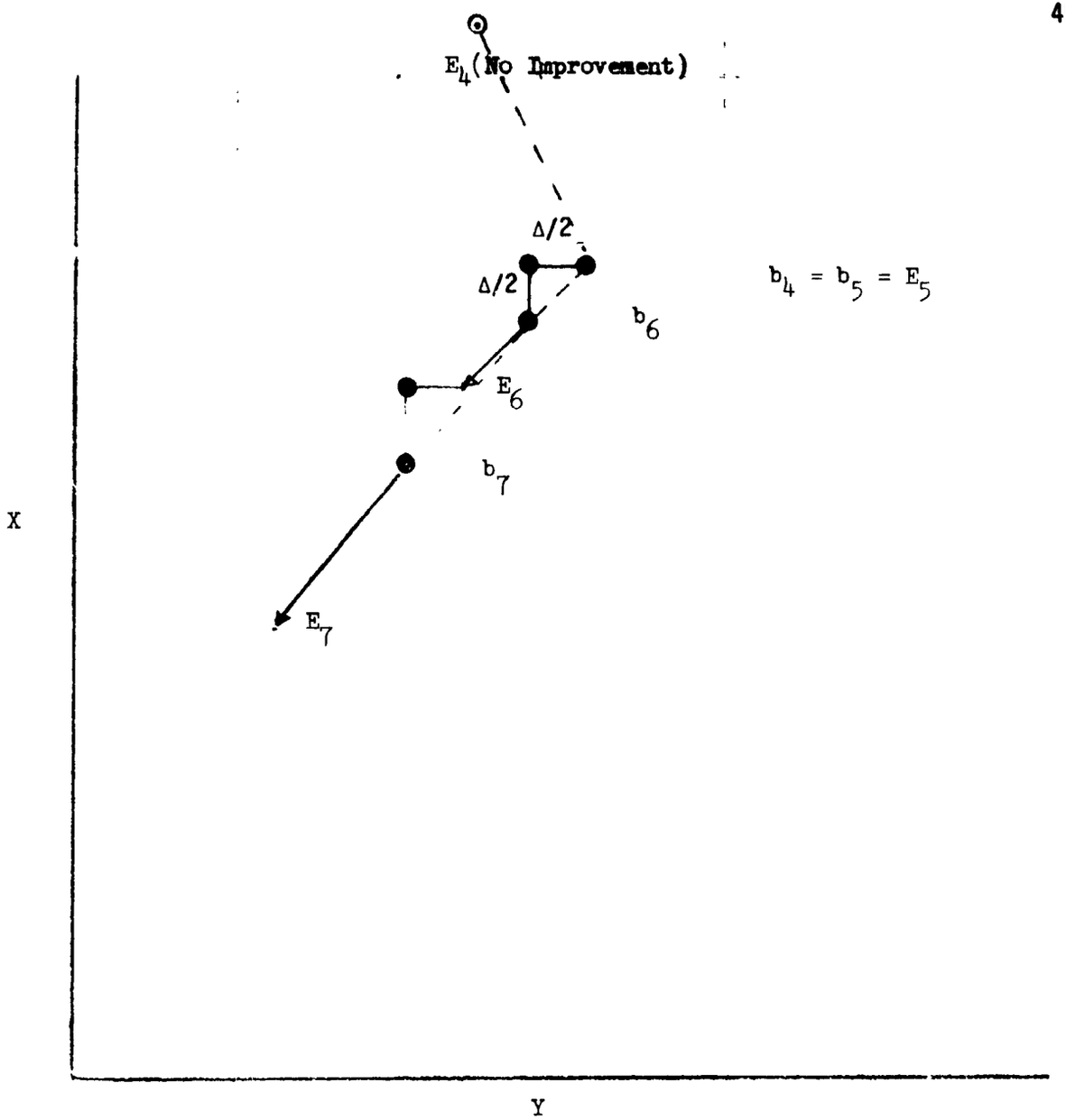


Figure 5-2. Schematic of Hooke and Jeeves Pattern Search Showing Pattern Destruction.

Extending the ideas presented by the example, an iterative scheme using MMPF-HJ for a minimum solution may be described as follows:

a) Integrate forward both the non-linear and the perturbation differential equations, as described in Section 4.4, observing Equations (5-1) and (5-2).

b) Using the Hooke and Jeeves algorithm outlined in [26], sequentially compute each desired Z resulting from either an exploratory step or a pattern step.

c) Compute $\delta\lambda_0$ according to Equation (5-3).

d) If $\|h\| \leq 1/2 \|h\|_{k=1}$ switch to IMPF; otherwise, start over at (a) using the new estimates for λ_0.

It should be pointed out that using the method of Hooke and Jeeves requires at least 2n iterations before a pattern point is established. This is due to the fact that, during the exploratory phases, the effects of changes on each variable must be sequentially examined.

5.4 Modified Method of Perturbation Functions Using an Accelerated Random Search Method

The accelerated random search method [29] is one which combines the ideas of both pattern search and random search with recalculation. This method has shown promise, particularly on multimodal hypersurfaces. The accelerated random search begins by making a random search in the vicinity of the initial estimate based on a normal distribution of probability. It follows with a step in

the opposite direction if the function is increased (failure), for example, in the case of search for a minimum. If this change in direction still produces no improvement, a new random step is generated (new set of δZ_k). As soon as the random search phase discovers a direction for improvement (success), a pattern search is begun doubling the step after each success. However, if the pattern search produces a failure, a retreat is made to the last improved step, and a random search is initiated. For practical reasons, the accelerated random search method confines its random search phase to an experimental region defined for each problem.

The normal distribution function which governs the magnitude of the increments, δZ, during the search phase is

$$|\delta Z_k| = \frac{1}{\bar{\sigma}_k \sqrt{2\pi}} \exp\left[-\frac{(Y_k/\bar{\sigma}_k)^2}{2}\right] \tag{5-9}$$

where Y represents 2n random numbers uniformly distributed between 0 and 1 and $\bar{\sigma}$ is their standard deviations about a mean of zero. In addition, the sign of δZ_k has equal probability of being plus or minus. During the pattern phase, the increments assume an arithmetic progression form as

$$\delta Z_{k+1} = 2\ \delta Z_k \tag{5-10}$$

and in the event of a failure during the random search phase, a reversal is executed by taking a step in the opposite direction as

$$\delta Z_{k+1} = -2\ \delta Z_k \tag{5-11}$$

An iterative scheme using the MMPF-ARS now may be described as follows:

a) Integrate forward both the non-linear and the perturbation differential equations, as described in Section 4.4, observing Equations (5-1) and (5-2).

b) Using Equation (5-9), compute the 2n desired terminal corrections where, for each correction, a different Y is obtained.

c) Transform the terminal corrections to initial corrections by using Equation (5-3) and proceed as in (a) with new initial λ estimates.

d) If U_{k+1} is an improvement over U_k by either a random forward step (Equation 5-9), a reversal step (Equation 5-11), or a new random search, compute the next terminal correction by using Equation (5-10) and proceed as in (c). Continue the pattern search as long as there is improvement in U. Whenever $||h|| \leq 1/2 \ ||h||_{k-1}$, switch to IMPF.

e) If there is no improvement in U during a pattern search, retreat to a random search mode, step (c), and start again.

Both this method and the Davidon method make all the corrections simultaneously instead of sequentially as in the method of Hooke and Jeeves.

CHAPTER VI

DISCUSSION AND ANALYSIS OF RESULTS

6.1 Introduction

The numerical aspects involved in the solution of the TPBV problem will be discussed. In the main, this will include those which are common to all four methods including the boundary conditions, numerical integration scheme used to integrate both the non-linear and linear differential equations, and finally the features which apply to each particular method. In this chapter, results are also shown which compare the IMPF and the three MMPF strategies outlined in Chapters 4 and 5, respectively, along with an additional strategy similar to that of [23]. Results are presented for the entry problem posed in Chapter II and transformed from an optimization problem into a TPBV problem as shown in Chapter III.

It should be pointed out that all quantities appearing in this chapter, except the co-states, are dimensional only for the sake of awareness. Internally, all computations involved non-dimensionalized quantities as stated earlier.

6.2 Initial and Terminal Boundary Conditions

The specification of the initial boundary conditions proceed as specified in Chapter III. First by selecting an initial state at $t = t_o = o$, an L (x_o, t_o) vector may be stated as

$$L = \begin{bmatrix} Z_1(t_o) - 25{,}975 \text{ ft/sec} \\ Z_2(t_o) - 400{,}000 \text{ ft} \\ Z_3(t_o) - 1.5 \text{ deg} \\ Z_4(t_o) - 90 \text{ deg} \\ Z_5(t_o) - 90 \text{ deg} \\ Z_6(t_o) - 0 \text{ deg} \end{bmatrix} = 0 \qquad (6\text{-}1)$$

using the form of Equation (3.23). Likewise, a specified terminal state vector may be selected at $t = t_f$ as

$$M = \begin{bmatrix} Z_1(t_f) - 25{,}306 \text{ ft/sec} \\ Z_2(t_f) - 252{,}784 \text{ ft} \\ Z_3(t_f) - 1.0568 \text{ deg} \\ Z_5(t_f) - 89.9676 \text{ deg} \\ Z_6(t_f) - 18.1897 \text{ deg} \end{bmatrix} = 0 \qquad (6\text{-}2)$$

Equation (6-1) and the fact that the initial state and time are specified result in no initial transversality conditions from Equation (3-12). Thus, at $t = t_o = o$, the initial boundary condition becomes

$$g = L = 0 \qquad (6\text{-}3)$$

On the other hand, since Equation (6-2) represents only five boundary conditions and seven are needed, the remaining two must come from the terminal transversality conditions, Equations (3-13) and (3-14). For the example problem, $\phi = 0$ and the terminal time t_f and $Z_4(t_f)$ have not been specified. Therefore, the remaining two boundary conditions

become $H(t_f) = 0$ and $\lambda_4 = Z_{10}(t_f) = 0$. Thus, the terminal boundary condition h becomes

$$h = \begin{bmatrix} Z_1\ (t_f) - 25{,}306 \text{ ft/sec} \\ Z_2\ (t_f) - 252{,}784 \text{ ft/sec} \\ Z_3\ (t_f) - 1.0568 \text{ deg} \\ Z_5\ (t_f) - 89.9676 \text{ deg} \\ Z_6\ (t_f) - 18.1897 \text{ deg} \\ Z_{10}\ (t_f) \\ H\ (t_f) \end{bmatrix} = 0 \qquad (6\text{-}4)$$

and as pointed out earlier it is the norm of h, $\|h\|$, that is used to determine convergence to a solution. Furthermore, since time does not appear explicitly in the generalized Hamiltonian

$$H\ (t) = 0 \qquad (6\text{-}5)$$

for $t_o \leq t \leq t_f$, where H is defined using Z variables as

$$H = C_1 Z_1^{3.15}\ \tilde{\rho}^{0.5} + \frac{1}{2}\ \frac{\tilde{A}}{\tilde{W}}\ (C_L^2 + C_D^2)^{0.5}\ \tilde{\rho}\ Z_1^2 + \sum_{i=1}^{i=6} Z_{i+6}\ F_i \qquad (6\text{-}6)$$

For this particular problem, the control, u, is eliminated from the $\dot{Z}$ equations by using the optimality condition, $H_u = 0$, and the Legendre-Clebsch condition for a minimum, $H_{uu} \geq 0$, as outlined in Section 3.2. Use of the optimality condition yields

$$\lambda_3 \sin u = -\lambda_4 \frac{\cos u}{\cos \Gamma} \qquad (6\text{-}7)$$

which may be used to solve for the functions sin u and cos u by

squaring (6-7) and using the trigonometric identity $\sin^2 u + \cos^2 u = 1$ to give

$$\sin u = \frac{\lambda_4}{\pm\sqrt{\lambda_4^2 + \lambda_3^2 \cos^2 \Gamma}} \tag{6-8a}$$

$$\cos u = \frac{\lambda_3 \cos \Gamma}{\pm\sqrt{\lambda_4^2 + \lambda_3^2 \cos^2 \Gamma}} \tag{6-8b}$$

The Legendre-Clebsch condition is then used to choose the proper sign for Equation (6-8). This condition requires that

$$H_{uu} = \lambda_3 \cos u - \lambda_4 \frac{\sin u}{\cos \Gamma} \geq 0 \tag{6-9}$$

and using Equation (6-8) to substitute for sin u and cos u in Equation (6-9) shows that for $-\pi/2 < \Gamma < \pi/2$ the negative sign is chosen for Equation (6-8a) and the positive sign for Equation (6-8b) or

$$\sin u = -\frac{\lambda_4}{\sqrt{\lambda_4^2 + \lambda_3^2 \cos^2 \Gamma}} \tag{6-10a}$$

$$\cos u = \frac{\lambda_3 \cos \Gamma}{\sqrt{\lambda_4^2 + \lambda_3^2 \cos^2 \Gamma}} \tag{6-10b}$$

It should be pointed out that, as a result of the terminal transversality conditions (6-4), $\lambda_4(t_f) = 0$ and therefore $u(t_f) = 180$ degrees on the optimal trajectory.

6.3 Numerical Integration Method

The numerical integration method used [45,46] to integrate the 2n non-linear differential equations, $\dot{Z} = F(Z,t)$, and the 2n linear perturbation equations, $\dot{\delta Z} = \frac{\partial F}{\partial Z}\,\delta Z$, is a variable step, fourth-order Runga-Kutta scheme. The $\dot{Z}$ equations of motion and the $\frac{\partial F}{\partial Z}$ are functionally presented in Appendix A. Since the entry problem chosen as an example is not for fixed terminal time, the generalized Hamiltonian becomes time-independent and, therefore, constant. This fact is used as an aid in selecting relative error bounds for the variable time step integration process. The task of selecting error bounds, particularly upper bounds, is a formidable one and often can be selected only according to experience and the type of problem at hand. The fact that the generalized Hamiltonian is a constant, and zero on the converged solution, allows observation of the effects of the error bound on the numerical accuracy and thus makes possible a logical choice. Table 6-1 shows the effects of the error bound for six cases on the variation of the generalized Hamiltonian during an iteration, as well as the value of the norm at the end of the iteration and the required UNIVAC 1108 time. The method of solution used was the IMPF described in Chapter IV; all six cases used the same identical boundary conditions and final time estimate. It is logical, therefore, to assume that this table gives a representative relative assessment of the effects of error bound on the accuracy of solution. Based on this information, a relative error bound of 5.0×10^{-3} - 5.0×10^{-5} was chosen based on a convergence criterion

Table 6-1

The Influence of Relative Error Bound Magnitude with an Error Bound Separation of 10^{-2} on an Iteration

Case	Allowable Relative Error	Amplitude of Hamiltonian*	Norm* of Terminal Dissatisfaction	UNIVAC 1108 Time, Sec
1	$10^{-1} - 10^{-3}$	4.848×10^{-8}	1.749×10^{-7}	10.30
2	$10^{-2} - 10^{-4}$	2.259×10^{-8}	6.016×10^{-7}	14.50
3	$10^{-3} - 10^{-5}$	3.060×10^{-9}	5.409×10^{-7}	24.45
4	$10^{-4} - 10^{-6}$	6.130×10^{-10}	5.552×10^{-7}	39.20
5	$10^{-5} - 10^{-7}$	3.000×10^{-11}	5.688×10^{-7}	100.50
6	$10^{-6} - 10^{-8}$	1.000×10^{-11}	5.689×10^{-7}	147.60

* Computed using non-dimensionalized variables.

of $h = 1 \times 10^{-8}$ compared to an expected Hamiltonian amplitude of about the same order of magnitude. A further consideration in the choice of the error bound was the exponential increase nature of the computer time with decrease in error bound.

6.4 Solution of the Entry Problem by IMPF

Solution of the entry problem using the terminal conditions of Section 6.2 and the integration method and constants stated in Section 6.3 was initiated by assuming a terminal time, t_f, of 260 seconds with which to begin the iteration process outlined in Chapter IV. The necessary constants needed for the correction process as described in Section 4.3 are as follows for $\|d\| \leq \|e\| = \sqrt{7}$:

$$\eta = \begin{cases} 0.4 \text{ if } \|h\| \geq 1 \times 10^{-3} \\ 0.4 \text{ if } \|h\| < 1 \times 10^{-3} \\ 0.8 \text{ if } \|h\| < 1 \times 10^{-4} \\ 1.0 \text{ if } \|h\| < 1 \times 10^{-5} \end{cases} \tag{6-11}$$

where h is assumed to be evaluated using non-dimensionalized variables. The choice of $\|e\| = \sqrt{7}$ comes from allowing an absolute range of unity for all the states and time since they are all non-dimensionalized and result in magnitudes close to one. The vector $\dot{h}$ and the $\frac{\partial h}{\partial Z}$ elements that are necessary to compute the corrections according to Equation (4-8a) are presented in Appendix A.

6.5 Solution of Entry Problem by MMPF

The stopping condition, Ω, chosen to end the integration process for each iteration in the MMPF solution was

$$\Omega = Z_2(t_f) - 252{,}784 \text{ ft} = h_2 = 0 \tag{6-12}$$

This variable was monotonic decreasing for all the cases studied and, therefore, served as an excellent stopping condition. The function chosen to be extremized resulted in a highly non-linear function involving all the states and co-states as well as the state differential equations. This function may be written as

$$U = h^T h = h_1^2 + h_2^2 + h_3^2 + h_4^2 + h_5^2 + h_6^2 + h_7^2 \qquad (6\text{-}13)$$

where h is defined by Equation (6-4). The value of $\|h\|$ at which the transition from MMPF to IMPF was executed was set at $\|h\| = \frac{\|h_i\|}{2}\Big|_{k=1}$. This particular value was chosen under the assumption that $\|h\| = \frac{\|h\|}{2}\Big|_{k=1}$ is sufficiently close to the minimum so that the IMPF can attempt to obtain a solution. Thus, at the above value of $\|h\|$, the IMPF is started with not only the co-state values at the particular iteration but also with the corresponding final time as the terminal time estimate. It can be expected that whenever the transition to IMPF is made, the final time estimate will be different than t_f=260 seconds, which was assumed to start the IMPF solutions of Section 6.4.

6.6 Results of Solution of the Entry Problem by IMPF and MMPF

In order to adequately explore the potential of the IMPF and the MMPF to solve the TPBV problem, a set of six initial co-state (λ) vectors were selected as shown in Table 6-2. The selection of the first four vectors made use of the information from the terminal boundary conditions which required the control u to be positive; thus, the first three co-states were chosen positive and the fourth, negative. Since it is apparent from Table 6-2 that the choice of the first vector

Table 6-2

Co-State Values Selected for Sensitivity Analysis of the TPBV Problem Solution Using IMPF, MMPF, and MPF

Initial Co-States and Terminal Time	Cases						Optimal Initial Co-States and Terminal Time
	1,7,11a 11b,11c	2,8	3,9	4,10,12a 12b,12c	,13a, 13b,13c	6	
λ_1	1.0	1.0	2.0	1.0	3.0	10.0	0.10897×10^{1}
λ_2	1.0	1.0	1.0	1.0	3.0	10.0	0.10064×10^{1}
λ_3	1.0	1.0	1.0	1.0	3.0	10.0	0.11167×10^{-1}
λ_4	-1.0	-1.0	-1.0	-1.0	3.0	10.0	-0.75931×10^{-2}
λ_5	0.0	0.0	0.0	1.0	3.0	10.0	0.23110×10^{-1}
λ_6	0.0	-3.0	0.0	1.0	3.0	10.0	-0.88073×10^{-3}
t_f^*, sec	260.0	260.0	260.0	260.0	260.0	260.0	0.25999×10^{3}

* This estimate applies only to cases 1-9.

is close to the optimal, the choice of the remaining five co-states was made to measure the effect on convergence of starting further away from the optimum. To further evaluate the improvement of these two methods, an additional MPF strategy based solely on the magnitude of the norm and similar to that of Lewallen [23] was used. The data in Table 6-2 show results for the following examples: Cases 1 through 6 used the IMPF with its correction factor based on norm and step size; Cases 7 through 10 used the MPF with a correction factor based only on norm and similar to that of Equation (6-11) with the exception that the first factor is 0.2 instead of 0.4; and finally, Cases 11 through 13 used the MMPF where a, b, and c stand for Davidon, Hooke-Jeeves, and accelerated random search, respectively, with a transition to the IMPF of Cases 1 through 6 as explained in Section 6.5.

The IMPF strategy using a correction factor dependent on the norm and the step size was used to solve the TPBV problem starting with each of the six co-state vectors of Table 6-2 as initial estimates and guessing an initial terminal time t_f = 260 seconds. This strategy was successful in obtaining a solution for each of the six candidate co-state vectors. The results depicted in Table 6-3 show essentially that the vector closest to the optimal solution required the least number of iterations. Cases 2 through 4, which used somewhat more difficult vectors, also converged in essentially the same number of iterations. Cases 5 and 6 are by far the most difficult but yet were driven to convergence by the strategy. In summary, the solution of the TPBV problem was very insensitive to the first four cases. However,

Table 6-3

Sensitivity of Solution to the TPBV Problem by IMPF

Case	Norms of Dissatisfaction			Iteration K
	First Iteration	K/2 Iteration	K^{th} Iteration	
1	0.95027×10^{0}	0.20766×10^{-2}	0.29430×10^{-10}	27*
2	0.31164×10^{1}	0.66015×10^{-1}	0.45041×10^{-11}	34*
3	0.95011×10^{0}	0.22286×10^{-1}	0.30175×10^{-10}	34*
4	0.11525×10^{1}	0.18221×10^{-1}	0.54636×10^{-11}	31*
5	0.47558×10^{1}	0.23621×10^{1}	0.36913×10^{-10}	107*
6	0.15852×10^{2}	0.71879×10^{1}	0.31400×10^{-9}	370*

* Converged, norm $\leq 0.1 \times 10^{-7}$.

as the progression was made to order of magnitude differences in the initial co-state (Case 6), the sensitivity increased, but the most important fact was that a solution was still obtained. This point is further emphasized when the MPF with a correction factor dependent on the norm only is discussed next.

Results using a correction factor philosophy similar to [23] and dependent on the norm only are presented in Table 6-4. As shown on Table 6-2, this scheme used only the first four vectors in Cases 7 through 10, respectively. The only solution possible out of the four cases was for Case 7 which required 50 percent more iterations than Case 1 for the same co-state vector and terminal time estimate. The remaining three cases diverged after four iterations even though the first two iterations gave a decrease from the first norm. This set of cases points out the power of a correction factor such as the one used in Cases 1 through 6.

Finally, results using the MMPF and starting with the first, fourth, and fifth co-state vectors are depicted in Table 6-5. The logic used to control the total number of iterations for these cases was as follows. A maximum of 50 iterations was allowed within which to halve the initial norm; once the transition was made to IMPF, a maximum of 30 iterations was allowed if the total number of iterations exceeded the companion case of Table 6-3. However, if the total number was less, iterations were continued up to an equivalent total as the companion case of Table 6-3. Table 6-5 exhibits two cases, 12a and 13c, which converged. Although Case 12a converged, it did so in

Table 6-4

Sensitivity of Solution to the TPBV Problem by MPF

Case	Norms of Dissatisfaction			Iteration K
	First Iteration	K/2 Iteration	K^{th} Iteration	
7	0.95027×10^{0}	0.41793×10^{-2}	0.14709×10^{-10}	40*
8	$0.3116^{4} \times 10^{1}$	0.24970×10^{1}	0.10840×10^{5}	4**
9	0.95011×10^{0}	0.67580×10^{0}	0.13330×10^{6}	4**
10	0.11525×10^{1}	0.89598×10^{0}	0.20436×10^{5}	4**

* Converged, norm 0.1×10^{-7}

** Diverged

Table 6-5

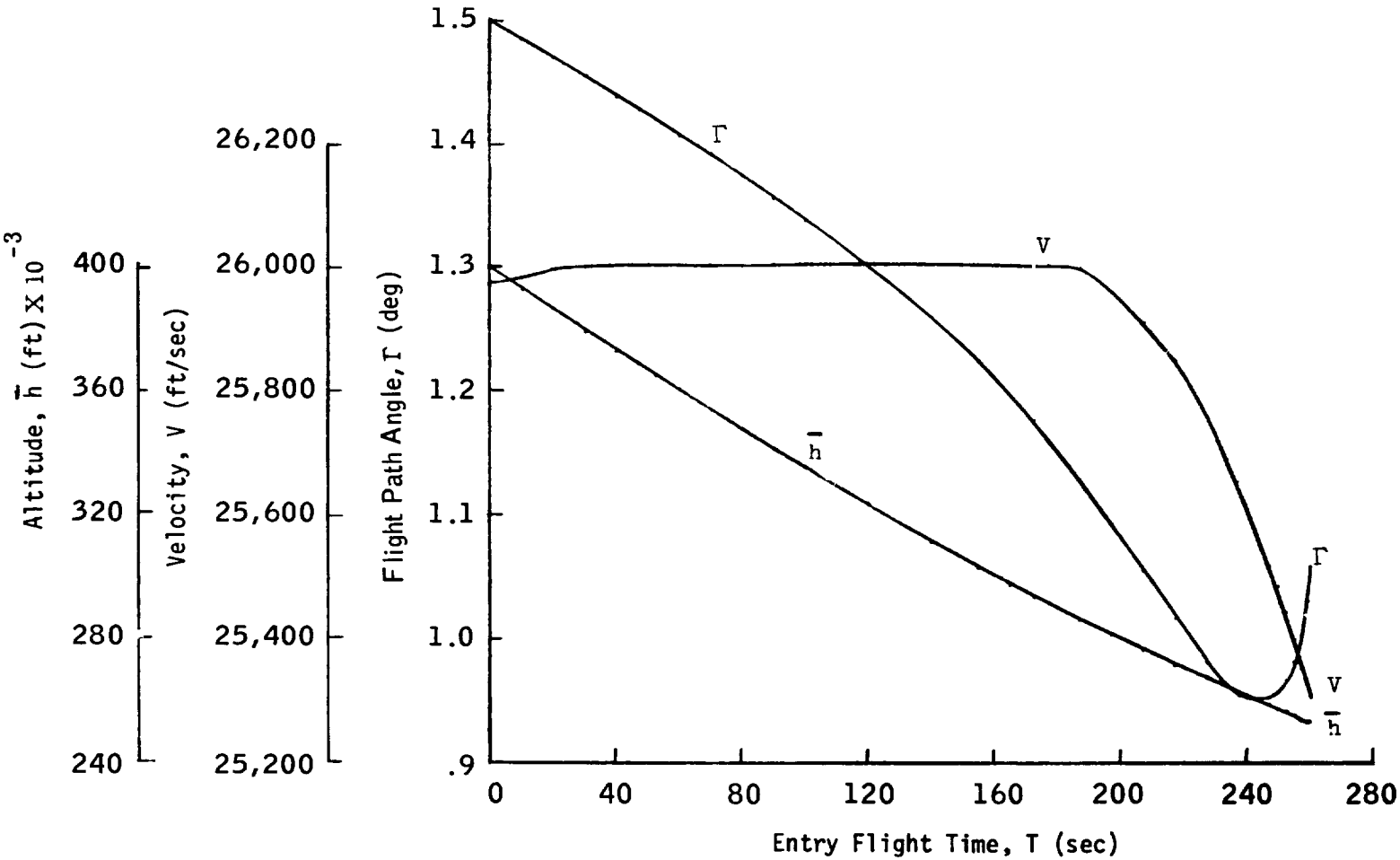
Sensitivity of Solution to the TPBV Problem by MMPF

Case	Norms of Dissatisfaction			Iteration K
	First Iteration	K/2 Iteration	K^{th} Iteration	
11a**	0.95027×10^{0}	———	———	———
11b		0.82234×10^{0}	0.43110×10^{0}	66
11c		0.47944×10^{0}	0.85521×10^{-3}	52
12a	0.11525×10^{1}	0.20664×10^{0}	0.97326×10^{-8}	40*
12b		0.60899×10^{0}	0.42272×10^{0}	70
12c		0.28758×10^{0}	0.19211×10^{-2}	38
13a**	0.47558×10^{1}	———	———	———
13b**		———	———	———
13c		0.11441×10^{1}	0.14190×10^{-11}	79*

* Converged, norm 0.1×10^{-7}.
** First iteration norm not halved within 50 iterations.

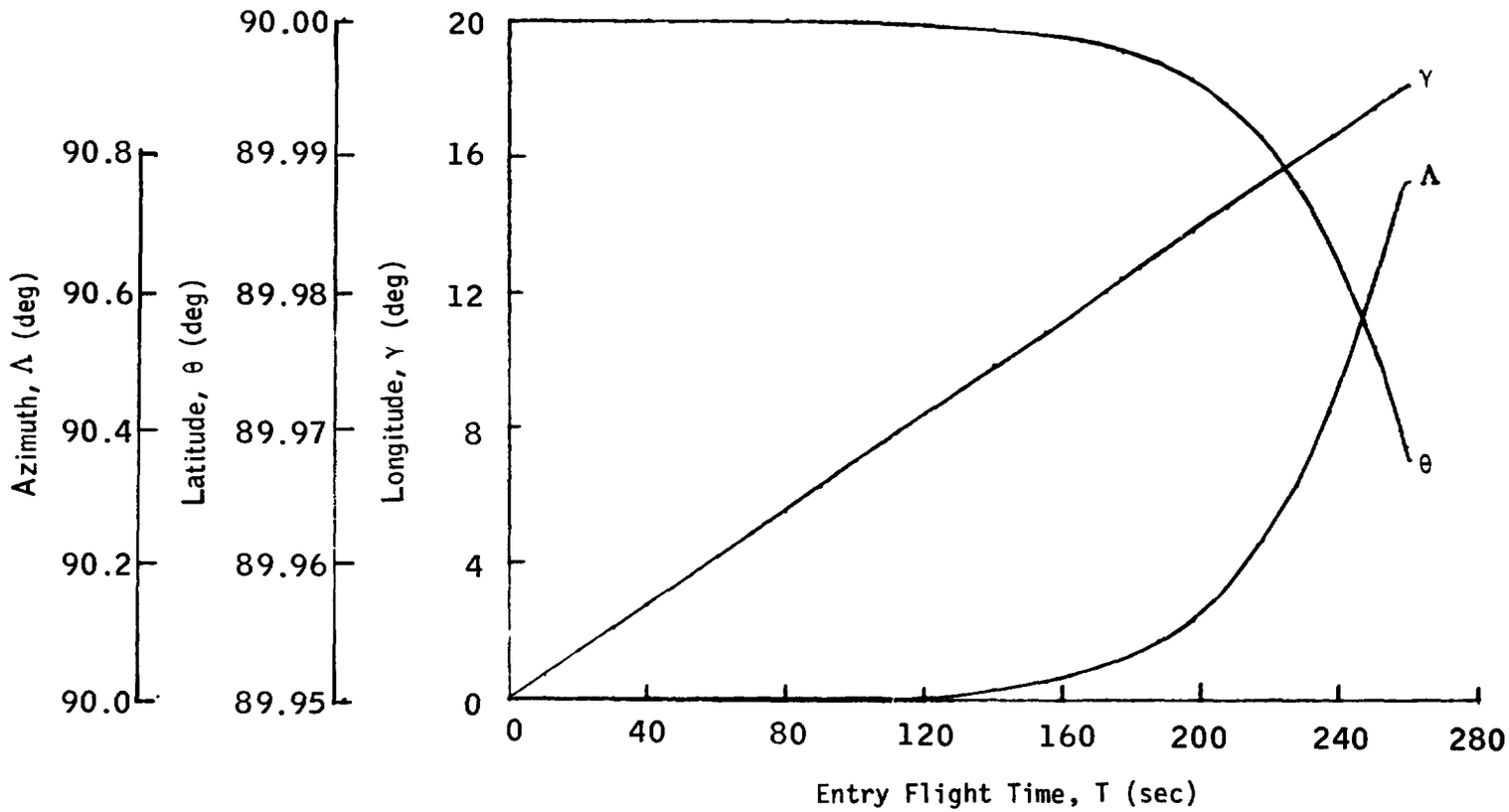
about one-third more iterations than Case 4. Case 13c, on the other hand, required approximately 26 percent fewer iterations than its companion case, Case 5. The remaining cases shown in Table 6-5 did not converge, nor did they diverge in the set number of iterations. It is felt that the cases which were evaluated will converge if allowed sufficient iterations but there was no reason for doing this. Based on the cases examined in Table 6-5, the MMPF using the accelerated random method seems the most powerful and it appears to get relatively better as it starts with co-states further from their optimum, as evidenced by Cases 11c, 12c, and 13c. The method of Hooke-Jeeves does not appear promising in comparison. Davidon's method converged on Case 12a but did not meet the basic criteria in 11a and 13a. This is not surprising because gradient methods are incapable of negotiating multimodal hypersurfaces and Equation (6-13), as evidenced by examination of Cases 11a and 13a, appears to be multimodal.

Figures 6-1 and 6-2 depict the optimum states, control variable, and the pay-off function variables Q and G as a function of the optimal time. It should be pointed out that, although the theory only guarantees a local optimum, all of the cases which converged do so to the identical solution. Thus, the statement may be made that the solution obtained and shown in Figure 6-1 is the optimum solution at least within the space defined by the six co-state vectors of Table 6-2.



(a) Altitude, Velocity, and Flight Path Angle

Figure 6-1. Optimal States for Solution of Entry Problem.



(b) Azimuth, Latitude, and Longitude

Figure 6-1. Continued.

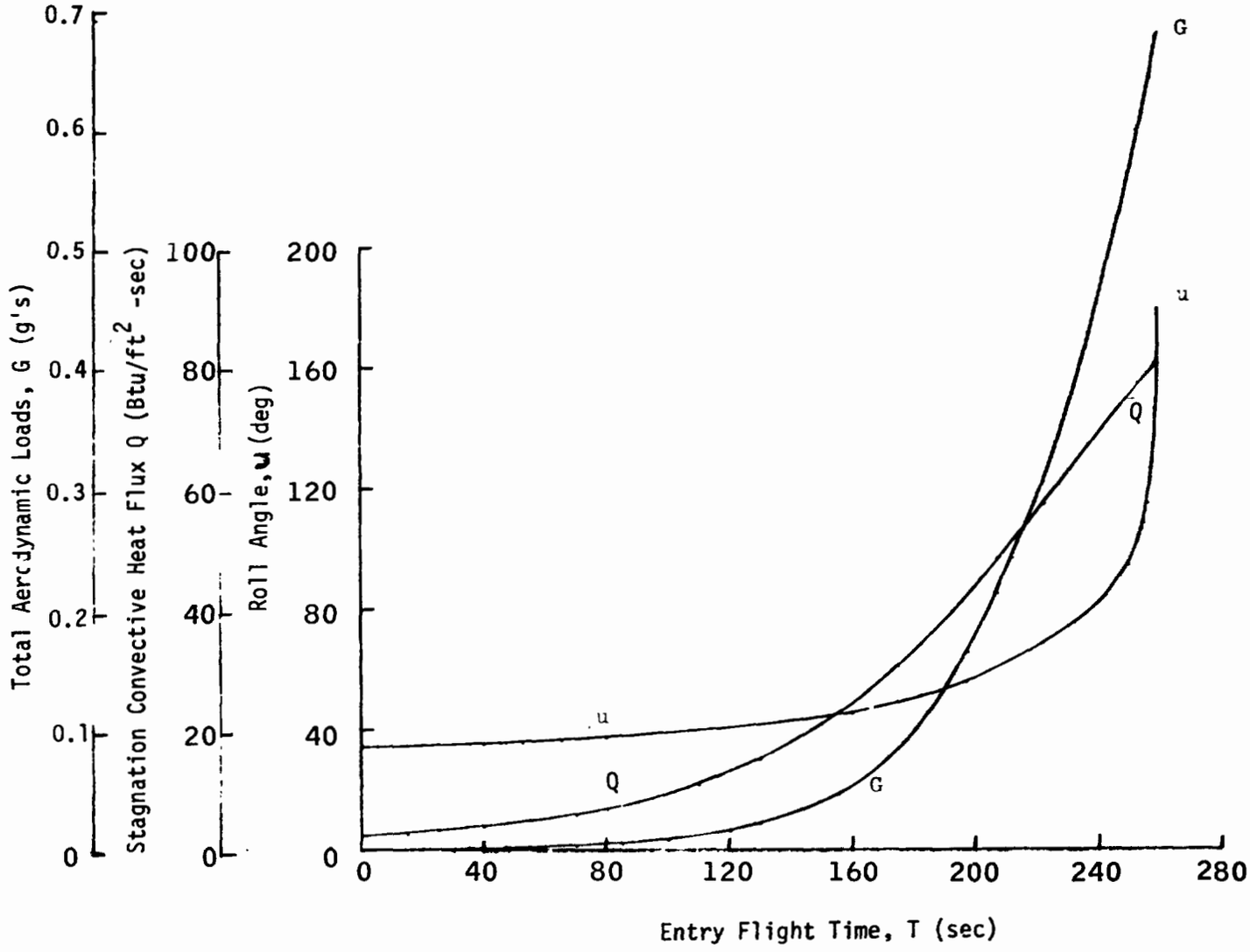


Figure 6-2. Optimal Control and Payoff Variables for Solution of Entry Problem.

CHAPTER VII

SUMMARY AND CONCLUSIONS

Two general iterative methods employing the basic method of perturbation functions to solve TPBV problems have been described. The first method (IMPF) uses a unique selection scheme to arrive at a correction factor which in essence sets the desired changes to be applied to the previous guessed co-states. The second method (MMPF) solves the problem in two phases. Phase 1, the start, uses function minimization techniques until the norm is subsequently reduced to a level equal to, or less than half of, that which it began with. At this point Phase 2 which is identical to the first method, takes over. To examine the capabilities of these two methods to obtain a solution, a complex entry optimization problem which was reduced to a TPBV problem through calculus of variations techniques was used. The two methods were then exercised using the above problem starting with several sets of different initial co-states. In addition, a method similar to that used in [23] on an Earth-Mars transfer problem was used to serve as an additional basis of comparison.

Based on this study, both of the techniques used must be judged superior to the one which is similar to that of [23]. Of the two techniques used, the IMPF with the correction factor based on norm and step size is by far superior as evidenced by its ability to obtain a solution for all the candidate cases. However, the MMPF using the accelerated random search strategy seems to become more attractive as the initial co-states get further away from their optimum values. Due

to the extreme complexity of the problem considered, it became computationally impractical to study cases further away from the optimum than those attempted.

Though convergent properties of iteration schemes are problem dependent, certain observations may be made in comparing the results of this study to those in the literature. Tapley and Lewallen [41] presented an Earth-Mars transfer problem which exhibited convergence from regions much closer to the optimum than those presented here. A similar statement may be made of studies on reentry problems such as the two-dimensional problem studied by Lastman and Tapley [19]. A recent study by Tapley and Williamson [32] on a three-degree-of-freedom reentry problem using MPF indicated 15 iterations were required for an optimal solution. However, based on the close proximity of their initial co-state estimates, indications are that the IMPF technique might improve on the iterations required for an optimal solution. In conclusion, it may be said that two methods have been formulated which have decreased the contingency of a solution to a complex entry problem on the initial co-state estimates.

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APPENDIX

APPENDIX A

THE STATE, CO-STATE, AND MPF EQUATIONS

The state (x) and co-state (λ) differential equations of motion for the entry problem form a set of 2n = 12 equations which are non-linear and ordinary in nature. Of this set of 2n equations, n are state equations and n are co-state equations. Both sets of equations may be expressed as functions of the variational Hamiltonian. The variational Hamiltonian (Equation (6-6)) can be written in terms of , λ, and u variables as

$$H = C_1 x_1^{3.15} \tilde{\rho}^{0.5} + \frac{1}{2} \frac{\tilde{A}}{W} (C_L^2 + C_D^2)^{1/2} \tilde{\rho} x_1^2 + \sum_{i=1}^{6} \lambda_i f_i \qquad (A\text{-}1)$$

where the n set of states x correspond to Equations (2-3a) through (2-3f) respectively; $\tilde{\rho}(x_2)$ is of the form explained in Section 2.2; and $F(x,u,t)$ represents the n differential equations of motion (2-3a) through (2-3f). Taking the partial derivative of H with respect to λ yields n differential equations of state

$$\dot{x} = H_\lambda = F(x,u,t) \qquad (A\text{-}2)$$

The n equations of co-state (λ) are then obtained by taking the partial derivative of H with respect to x

$$\dot{\lambda} = H_x = \dot{\lambda}(x,\lambda,u,t) \qquad (A\text{-}3)$$

Equations (A-2) and (A-3) may be combined to form a 2n set of differential equations

$$\dot{Z} = F(Z,u,t) \qquad (A\text{-}4)$$

where the variable Z is made up of the x and λ variables.

The perturbation equations, $\frac{\partial F}{\partial Z}$, result in a 2n X 2n matrix which may also conveniently be expressed as a function of H

$$\frac{\partial F}{\partial Z} = \left[\begin{array}{c|c} D_1(t) & D_2(t) \\ \hline D_3(t) & D_1^T(t) \end{array}\right] \tag{A-5}$$

$$D_1(t) = H_{\lambda x} - H_{\lambda u} H_{uu}^{-1} H_{ux} \tag{A-6}$$

$$D_2(t) = H_{\lambda u} H_{uu}^{-1} H_{u\lambda} \tag{A-7}$$

$$D_3(t) = -H_{xx} + H_{xu} H_{uu}^{-1} H_{ux} \tag{A-8}$$

For example, with the entry problem defined in this thesis, the top row of the D_1 partition is as follows, with u being a scalar since there is but one control variable:

$$D_1(1,1) = H_{\lambda_1} x_1 - \frac{1}{H_{uu}} H_{\lambda_1 u} H_{ux_1} \tag{A-9}$$

$$D_1(1,2) = H_{\lambda_1} x_2 - \frac{1}{H_{uu}} H_{\lambda_1 u} H_{ux_2} \tag{A-10}$$

$$D_1(1,3) = H_{\lambda_1} x_3 - \frac{1}{H_{uu}} H_{\lambda_1 u} H_{ux_3} \tag{A-11}$$

$$D_1(1,4) = H_{\lambda_1} x_4 - \frac{1}{H_{uu}} H_{\lambda_1 u} H_{ux_4} \tag{A-12}$$

$$D_1(1,5) = H_{\lambda_1} x_5 - H_{uu} H_{\lambda_1 u} H_{ux_5} \tag{A-13}$$

$$D_1(1,6) = H_{\lambda_1} x_6 - H_{uu} H_{\lambda_1 u} H_{ux_6} \tag{A-14}$$

Referring to Equation (6-4), the vector equation for the time derivatives of the terminal dissatisfaction h is

$$\dot{h} = \begin{bmatrix} \dot{Z}_1(t_f) \\ \dot{Z}_2(t_f) \\ \dot{Z}_3(t_f) \\ \dot{Z}_5(t_f) \\ \dot{Z}_6(t_f) \\ \dot{Z}_{10}(t_f) \\ \dot{H}(t_f) \end{bmatrix} \tag{A-15}$$

where $\dot{H}(t_f) = 0$ since $H(t)$ is a constant, as pointed out in Section 6.2, and $\dot{Z}(t_f)$ are the 2n differential equations of state and co-state. The $\frac{\partial h}{\partial Z}$ results in a 7 x 12 matrix which has zero for many of its elements. Only the non-zero elements will be specifically identified

$$\frac{\partial h_1}{\partial Z_1} = \frac{\partial h_2}{\partial Z_2} = \frac{\partial h_3}{\partial Z_3} = \frac{\partial h_4}{\partial Z_5} = \frac{\partial h_5}{\partial Z_6} = \frac{\partial h_6}{\partial Z_{10}} = 1 \tag{A-16}$$

The last row of the $\frac{\partial h}{\partial Z}$ matrix is $\frac{\partial h_7}{\partial Z}$ which is

$$\frac{\partial H}{\partial Z} = \begin{bmatrix} \dot{-\lambda} & \dot{x} \end{bmatrix} \tag{A-17}$$