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POWER-SPECTRAL-DENSITY RELATIONSHIP FOR RETARDED DIFFERENTIAL EQUATIONS

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16. Abstract

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SUMMARY

The power-spectral-density (PSD) relationship between input and output of a set of linear differential-difference equations of the retarded type with real constant coefficients and delays is discussed. The form of the PSD relationship is identical with that applicable to unretarded equations. Since the PSD relationship is useful if and only if the system described by the equations is stable, the stability must be determined before applying the PSD relationship. Since it is sometimes difficult to determine the stability of retarded equations, such equations are often approximated by simpler forms. It is pointed out that some common approximations can lead to erroneous conclusions regarding the stability of a system and, therefore, to the possibility of obtaining PSD results which are not valid.

INTRODUCTION

The relationship between input and output of physical systems which can be described in terms of linear differential equations with constant coefficients can be described in several ways. This relationship may be expressed either in terms of a response to a unit impulse or in terms of a transfer function. For sinusoidal inputs of various frequencies the response may be expressed in terms of a frequency-response function. For random inputs the power-spectral-density (PSD) relationship between the input and output may be obtained by use of the techniques of generalized harmonic analyses. In order for the frequency response or the PSD relationship to be meaningful, however, the system described by the equations must be stable. The purpose of the present study is to discuss the PSD relationship between input and output for physical systems which can be described by retarded differential-difference equations. These systems occur when there is a delay in the action of a state variable on the system.

Basically, a differential-difference equation of the retarded type is a differential equation in which the highest order derivative of the dependent variable contains no delay in its argument (time), whereas in any of its other derivatives or in the dependent variable itself, delays may occur. Some specific applications of these equations are contained in references 1 to 7.

SYMBOLS

A, A_l	$N \times N$ matrices of real constants
$a_{11}, a_{12}, a_{21}, a_{22}$ $a_{011}, a_{012}, a_{021}, a_{022}$	real constants
$\vec{b}$	$N \times 1$ vector of real constants
C_m	pitching-moment coefficient, $\dfrac{\text{Pitching moment}}{\bar{q} S_W \bar{c}}$
C_Z	Z-force coefficient, $\dfrac{\text{Force in Z-direction}}{\bar{q} S_W}$
c_0, c_1, c_2, c_3	real constants
$\bar{c}$	mean aerodynamic chord, m
d_0, d_1	real constants
$\vec{G_x}$	transfer function between $\vec{x}$ and u
$\vec{H_x}$	frequency-response function (see eq. (8))
$\vec{h_x}$	impulse-response function (see eq. (6))
I	$N \times N$ identity matrix
$i = \sqrt{-1}$	
j, i	integers
k_Y	radius of gyration about Y-axis, m
l	integer
m	mass, kg
N	dimension of system

Q	number of delays in system
q	pitch rate, rad/sec
$\bar{q}$	dynamic air pressure, N/m^2
S_W	wing area, m^2
s	characteristic root (complex variable, $\sigma + i\omega$)
t	time, sec
u	scalar forcing function
V	airspeed, m/sec
X	$N \times N$ real transition matrix
$\vec{x}$	N-dimensional state vector
α	angle of attack, rad
α_g	gust angle of attack, rad
α_1,α_2	real constants
β_1,β_2	real constants
γ	dummy integration variable
$\frac{\partial\epsilon}{\partial\alpha}$	change in downwash at tail per unit change of wing angle of attack
θ,θ_l	real and constant time delays, respectively
σ	real part of s
τ	transport time lag, $\frac{\text{Tail length}}{V}$, sec
ϕ_u	power spectral density of u, $\frac{(\text{Amplitude})^2 - \text{sec}}{\text{rad}}$

ϕ_x power spectral density of $\vec{x}$, $\frac{(\text{Amplitude})^2 \text{ - sec}}{\text{rad}}$

ω imaginary part of s, rad/sec

$\left(C_{Z_\alpha}\right)_0 = \left(\frac{\partial C_Z}{\partial \alpha}\right)_{\text{wing + fuselage}}$ $\left(C_{m_\alpha}\right)_0 = \left(\frac{\partial C_m}{\partial \alpha}\right)_{\text{wing + fuselage}}$

$\left(C_{Z_\alpha}\right)_t = \left(\frac{\partial C_Z}{\partial \alpha}\right)_{\text{tail}}$ $\left(C_{m_\alpha}\right)_t = \left(\frac{\partial C_m}{\partial \alpha}\right)_{\text{tail}}$

Superscripts:

* critical or touch-point value

T transpose

Dots over symbols denote derivatives with respect to time. An arrow over a symbol denotes a vector. (The same symbol without the arrow is scalar.) The notation $\| \;\|$ represents the norm of a matrix.

ANALYSIS

Response of a Retarded System

In this study a retarded system means a system described by a set of linear differential-difference equations of the retarded type with real constant coefficients and delays; that is,

$$\dot{\vec{x}}(t) = A\,\vec{x}(t) + \sum_{l=0}^{Q} A_l\,\vec{x}(t-\theta_l) + \vec{b}\,u(t) \tag{1}$$

where A and A_l are $N \times N$ constant matrices, $\vec{x}(t)$ is an $N \times 1$ state vector, $\vec{b}$ is an $N \times 1$ constant vector, $u(t)$ is a scalar input function of time, and θ_l is a constant time delay (transport lag).

The general solution of equation (1) is (see ref. 8)

$$\vec{x}(t) = X(t)\,\vec{x}(0) + \sum_{l=0}^{Q} \int_{-\theta_l}^{0} A_l\, X(t-\gamma-\theta_l)\,\vec{x}(\gamma)\,d\gamma + \int_0^t X(t-\gamma)\,\vec{b}\,u(\gamma)\,d\gamma \tag{2}$$

where $X(t)$ is the state transition matrix of equation (1) and is the solution of the equation

$$\dot{X}(t) = A\,X(t) + \sum_{l=0}^{Q} A_l\,X(t-\theta_l) \tag{3}$$

with

$$X(0) = I$$

$$X(t) = 0 \qquad (t < 0)$$

Solving equation (3) can be thought of as sequentially solving a set of ordinary differential equations with constant coefficients. The existence and continuity of $X(t)$ follow from the well-developed theory of ordinary differential equations. (See ref. 9.)

The separate responses due to the initial conditions on $\vec{x}(t)$ and due to the forcing function $u(t)$ are clearly shown in equation (2). The response due to the forcing function is

$$\vec{x}(t) = \int_0^t X(t-\gamma)\,\vec{b}\,u(\gamma)\,d\gamma \tag{4}$$

It is assumed that $u(t) = 0$ for $t < 0$, so that equation (4) can be transformed into the familiar convolution or Duhamel integral

$$\vec{x}(t) = \int_0^\infty \vec{h_x}(\gamma)\,u(t-\gamma)\,d\gamma \tag{5}$$

where

$$\vec{h_x}(\gamma) = X(\gamma)\,\vec{b} \tag{6}$$

is known as the impulse-response function.

PSD Relationship Between Input and Output of the Retarded System

The power-spectral-density (PSD) relationship between output $\vec{x}(t)$ and input $u(t)$ is derived in the usual manner by means of equation (5) (see refs. 10 and 11, for example) and the derivation will not be given here. The relationship is

$$\phi_x(\omega) = \vec{H}_x(i\omega)\,\phi_u(\omega)\left[\vec{H}_x(-i\omega)\right]^T \tag{7}$$

where $\phi_x(\omega)$ and $\phi_u(\omega)$ are the power spectral densities of $\vec{x}(t)$ and $u(t)$, respectively, and

$$\begin{aligned}\vec{H}_x(i\omega) &= \int_{-\infty}^{\infty} \vec{h}_x(t)\, e^{-i\omega t}\, dt \\ &= \int_{0}^{\infty} \vec{h}_x(t)\, e^{-i\omega t}\, dt\end{aligned} \tag{8}$$

is called the frequency-response function. The lower limit of zero on the latter integral in equation (8) occurs because $\vec{h}_x(t) = 0$ for $t < 0$.

The characteristic equation corresponding to equation (3) is

$$\det\left(sI - A - \sum_{l=0}^{Q} A_l e^{-\theta_l s}\right) = 0 \tag{9}$$

It can be seen from equation (3) that any column vector of $X(t)$, say $\vec{X}_j(t)$, satisfies the retarded equation

$$\dot{\vec{X}}_j(t) = A\vec{X}_j(t) + \sum_{l=0}^{Q} A_l \vec{X}_j\left(t-\theta_l\right) \tag{10}$$

which also has the characteristic equation (9). To each characteristic root $s_r = \sigma_r + i\omega_r$ of equation (9), there corresponds a nonzero vector $\vec{C}_r$ such that $\vec{C}_r e^{s_r t}$ is a solution of equation (10). (See ref. 9.) Therefore, unless $\sigma_r < 0$, equation (8) is not meaningful. If $\sigma_r \leqq -\mu < 0$, then it can be shown that

$$\|X(t)\| \leqq Ke^{-\mu t} \tag{11}$$

where K and μ are positive constants (refs. 8 and 9). It follows by using equation (11) that $\vec{H}_x(i\omega)$ is bounded. If the integral in equation (8) exists for each element of $X(t)$, then $\vec{H}_x(i\omega)$ exists. This existence is easily established by using the norm of reference 9, equation (11), and a theorem on page 29 of reference 12.

Before proceeding, consider the following practice, which can lead to erroneous results. The transfer function of equation (1) is defined as

$$\vec{G}_x(s) = \int_0^\infty \vec{h}_x(t)\, e^{-st}\, dt \tag{12}$$

Notice that equation (12) with $s = i\omega$ is identical in form with equation (8); consequently, a common practice of getting $\vec{H}_x$ is to obtain the transfer function and then set $s = i\omega$. This procedure will lead to invalid PSD results if the system is not stable, because in this case the magnitude of the transfer function can be finite, whereas the magnitude of the frequency-response function can be infinite. For example, consider the scalar differential equation (zero lags)

$$\dot{x}(t) - x(t) = u(t) \tag{13}$$

In this case, the impulse-response function is

$$h_x(t) = e^t \tag{14}$$

The transfer function is, if $\sigma > 1$,

$$G_x(s) = \int_0^\infty e^t\, e^{-st}\, dt = \frac{1}{s - 1} \tag{15}$$

and the frequency-response function is

$$H_x(i\omega) = \int_0^\infty e^t\, e^{-i\omega t}\, dt = \int_0^\infty \left[\cos(\omega t) + i \sin(\omega t)\right] e^t\, dt \tag{16}$$

Formally, setting $s = i\omega$ in equation (15) and substituting the resulting expression $\left(G_x(i\omega) = \frac{1}{i\omega - 1}\right)$ into equation (7) for $\vec{H}_x(i\omega)$ result in

$$\phi_x(\omega) = \frac{1}{\omega^2 + 1}\, \phi_u(\omega) \tag{17}$$

However, $\vec{H}_x(i\omega)\left[\vec{H}_x(-i\omega)\right]^T$ in equation (7) is infinite, as can be seen by examining the magnitude of equation (16). The point is: Make sure the system is stable before setting $s = i\omega$ in the transfer function to get the frequency-response function.

Approximating the Stability of a Second-Order Retarded System

Since the PSD relationship (eq. (7)) is used only if the system is stable, it is necessary to determine the stability of the retarded system prior to applying the relationship. There are several methods of doing this. (See refs. 13 and 14.)

Retarded systems are often approximated with ordinary differential equations for simplification. However, some of the more common approximations can lead to erroneous results concerning stability, and, therefore, application of the PSD relationship (eq. (7)) can be invalid.

If any delay θ_l in equation (9) is not zero, the resulting exponential term $e^{-\theta_l s}$ causes equation (9) to be transcendental, so that it has an infinite number of roots. Approximations are often used to simplify the analysis of such problems. The following second-order system is examined to show the variation in results which can occur for various approximations:

$$\ddot{x}(t) + \dot{x}(t-\theta) + x(t-\theta) = f(t) \tag{18}$$

where $x(t)$ and $f(t)$ are scalar functions.

The transfer function of equation (18) is

$$G_X(s) = \frac{1}{s^2 + se^{-\theta s} + e^{-\theta s}} \tag{19}$$

and the characteristic equation is

$$s^2 + (s + 1)e^{-\theta s} = 0 \tag{20}$$

To approximate the transfer function or characteristic equation with one that is not transcendental, the exponential term $e^{-\theta s}$ is replaced by various approximations, such as

$$e^{-\theta s} \approx 1 - \theta s \qquad \text{(from refs. 2 and 4)} \tag{21}$$

$$e^{-\theta s} \approx \frac{1}{1 + \theta s} \qquad \text{(from ref. 3)} \tag{22}$$

$$e^{-\theta s} \approx \frac{2 - \theta s}{2 + \theta s} \qquad \text{(first-order Padé approximation, ref. 15)} \tag{23}$$

Another approximation considered here is

$$e^{-\theta s} = 1 - \theta s + \sum_{l=2}^{j} \frac{(-\theta s)^l}{l!} \tag{24}$$

where $j \geqq 3$.

Using any one of the approximating equations (21), (22), (23), or (24) in equation (19) results in a modified transfer function with a new characteristic equation given by a polynomial equation in s. The negativeness of the real parts of the roots of this resulting characteristic equation can be examined by means of the Routh stability criteria. (See ref. 16.) The results are shown in table I.

Shown in table I are the values of the delay θ for which the resulting system is stable when the various approximations are used. The results vary from $\theta = 0$ (case 4) to $\theta < 1$ (cases 1 and 2). The correct range $\theta < 0.71$ was obtained from reference 13. It is interesting to note from case 4 that regardless of how accurately $e^{-\theta s}$ is approximated by a Maclaurin series beyond four terms, there is always a root with a nonnegative real part for any nonzero delay. Also note that the Padé approximation (case 3) is very close to the exact result of case 5.

TABLE I.- RANGES OF VALUES OF TIME DELAY θ WHICH YIELD ROOTS WITH NEGATIVE REAL PARTS WHEN VARIOUS APPROXIMATIONS ARE USED

Case	Expression used for $e^{-\theta s}$	Range of θ which results in $\sigma < 0$
1	$1 - \theta s$	$\theta < 1$
2	$\dfrac{1}{1 + \theta s}$	$\theta < 1$
3	$\dfrac{2 - \theta s}{2 + \theta s}$	$\theta < 0.76$
4	$1 - \theta s + \sum_{l=2}^{j} \dfrac{(-\theta s)^l}{l\,!} \quad (j \geqq 3)$	$\theta = 0$
5	$e^{-\theta s}$	$\theta < 0.71$

Stability of Retarded Short-Period-Mode Equations for Airplane (With Controls Fixed) in Turbulence

In reference 2 the short-period-mode equations of a rigid airplane flying in turbulence are formulated in such a manner that a retarded system results. In reference 4 these equations are used to make PSD computations, and it is assumed that the PSD computations are valid because the system appears to be stable when the approximating equation (21) is used. The exact stability of the controls-fixed case of reference 4 is examined in this section.

With $\theta = \tau$, the retarded short-period-mode equations for a rigid airplane flying in turbulence (with controls fixed) are

$$\begin{bmatrix} \dot{\alpha}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \alpha(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} a_{011} & a_{012} \\ a_{021} & a_{022} \end{bmatrix} \begin{bmatrix} \alpha(t - \theta) \\ q(t - \theta) \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \alpha_g(t) + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \alpha_g(t - \theta) \tag{25}$$

where

$$a_{11} = \frac{\bar{q}S_W}{mV}\left[\left(C_{Z_\alpha}\right)_0 + \left(C_{Z_\alpha}\right)_t\right]$$

$$a_{12} = 1 + \frac{\bar{q}S_W}{mV}\tau\left(C_{Z_\alpha}\right)_t$$

$$a_{21} = \frac{\bar{q}S_W\bar{c}}{mk_Y^2}\left[\left(C_{m_\alpha}\right)_0 + \left(C_{m_\alpha}\right)_t\right]$$

$$a_{22} = \frac{\bar{q}S_W\bar{c}\tau}{mk_Y^2}\left(C_{m_\alpha}\right)_t$$

$$a_{011} = -\frac{\bar{q}S_W}{mV}\left(C_{Z_\alpha}\right)_t \frac{\partial\epsilon}{\partial\alpha}$$

$$a_{012} = 0$$

$$a_{021} = -\frac{\bar{q}S_W\bar{c}}{mk_Y^2}\left(C_{m_\alpha}\right)_t \frac{\partial\epsilon}{\partial\alpha}$$

$$a_{022} = 0$$

$$\alpha_1 = \frac{\bar{q}S_W}{mV}\left(C_{Z_\alpha}\right)_0$$

$$\alpha_2 = \frac{\bar{q}S_W\bar{c}}{mk_Y^2}\left(C_{m_\alpha}\right)_0$$

$$\beta_1 = \frac{\bar{q}S_W}{mV}\left(C_{Z_\alpha}\right)_t\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)$$

$$\beta_2 = \frac{\bar{q}S_W\bar{c}}{mk_Y^2}\left(C_{m_\alpha}\right)_t\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)$$

The characteristic equation of equation (25) is

$$s^2 + c_1 s + c_0 + d_0 e^{-\theta s} + d_1 s e^{-\theta s} = 0 \tag{26}$$

where

$$c_1 = -\left(a_{11} + a_{22}\right)$$

$$c_0 = a_{11}a_{22} - a_{12}a_{21}$$

$$d_0 = -a_{12}a_{021} + a_{22}a_{011}$$

$$d_1 = -a_{011}$$

The input-output relation (eq. (7)) can be applied to equation (25) if all roots of the characteristic equation (26) have negative real parts (i.e., if the system is stable). The difficulty, of course, is that the number of roots of equation (26) is infinite if $\theta \neq 0$.

With zero delay ($\theta = 0$), equation (26) becomes

$$s^2 + \left(c_1 + d_1\right)s + \left(c_0 + d_0\right) = 0 \tag{27}$$

The roots of equation (27) are easily determined. In order for the stability to change as the delay is continuously increased from $\theta = 0$ to $\theta = \tau$, a root-locus curve must touch the imaginary axis. The value of the delay θ^* and the touch point $(0,\omega^*)$, where the root-locus curve touches the imaginary axis, are easily computed in this case by means of the technique of reference 14. From equation (26), the values of ω^* and θ^* (for $s = i\omega$) are related by

$$e^{-i\omega^*\theta^*} = \frac{(\omega^*)^2 - c_0 - c_1\omega^* i}{d_0 + d_1\omega^* i} \tag{28}$$

This equation is equivalent to the following two equations:

$$(\omega^*)^4 + c_2(\omega^*)^2 + c_3 = 0 \tag{29}$$

and

$$\tan(-\omega^*\theta^*) = \frac{-d_1(\omega^*)^3 - \left(c_1d_0 - d_1c_0\right)\omega^*}{\left(d_0 - c_1d_1\right)(\omega^*)^2 - c_0d_0} \tag{30}$$

where

$$c_2 = c_1{}^2 - 2c_0 - d_1{}^2$$

and

$$c_3 = c_0{}^2 - d_0{}^2$$

Only the real values of ω^* in equation (29) are of interest; in addition, only positive values need be considered, since the root-locus curves are symmetric about the real axis. The values of ω^* which satisfy equation (29) are

$$\omega^* = \pm\left[\frac{-c_2 \pm \left(c_2^{\,2} - 4c_3\right)^{1/2}}{2}\right]^{1/2} \tag{31}$$

Any real positive values of ω^* are easily found from equation (31). If such values exist, equation (30) can be used to compute the value of the delay when a touch point occurs.

For many systems, either equation (31) does not yield a positive real root or equation (30) gives a value of delay larger than the desired delay. In either case, the system only has roots with negative real parts, and provided this is also true for equation (27), the PSD input-output relation holds.

Table II contains values of the various airplane characteristics used in reference 4. Using values from table II to compute the coefficients in equation (27) gives

$$c_1 + d_1 \approx 4 \text{ sec}^{-1}$$

$$c_0 + d_0 \approx 19 \text{ sec}^{-2}$$

Since $c_1 + d_1 > 0$, the system is stable when $\theta = 0$. Also, for c_2 and c_3 the values shown in table II give

$$c_2 \approx -37 \text{ sec}^{-2}$$

$$c_3 \approx 655 \text{ sec}^{-4}$$

Since $c_2^{\,2} - 4c_3 < 0$ in equation (31), there are no real values of ω^*, or no touch points. Thus, the input-output relation (eq. (7)) is valid for the desired case $\theta = \tau$.

TABLE II.- AIRPLANE MASS, DIMENSIONS, FLIGHT CONDITION, AND AERODYNAMIC CHARACTERISTICS

Mass, m, kg	5669.905
Wing area, S_W, m^2	39.019
Mean aerodynamic chord, $\bar{c}$, m	1.981
Radius of gyration about Y-axis, k_Y, m	2.572
Tail length, m	7.742
True airspeed, V, m/sec	108.893
Altitude, m	3048
Dynamic pressure, $\bar{q}$, N/m^2	5364.030
$\left(C_{Z_\alpha}\right)_0$, per radian	-5.3243
$\left(C_{Z_\alpha}\right)_t$, per radian	-0.682
$\left(C_{m_\alpha}\right)_0$, per radian	0.576
$\left(C_{m_\alpha}\right)_t$, per radian	-2.665
$\frac{\partial \epsilon}{\partial \alpha}$	0.2884
τ, sec	0.071

CONCLUDING REMARKS

The form of the power-spectral-density (PSD) relationship between input and output of a system of retarded linear differential-difference equations with real constant coefficients and delays is the same as that for unretarded equations, and, as usual, the PSD relationship is meaningful if and only if the system is stable.

The stability of retarded equations is generally difficult to determine; consequently, the equations are often approximated by ordinary differential equations to examine the stability. It is shown that some common approximations can lead to significantly different conclusions regarding system stability and, therefore, to the possibility of obtaining PSD results which are not valid.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., August 28, 1974.

REFERENCES

1. Schmitt, Klaus, ed.: Delay and Functional Differential Equations and Their Applications. Academic Press, Inc., 1972.
2. Phillips, William H.; and Kraft, Christopher C., Jr.: Theoretical Study of Some Methods for Increasing the Smoothness of Flight Through Rough Air. NACA TN 2416, 1951.
3. Boucher, Robert W.; and Kraft, Christopher C., Jr.: Analysis of a Vane-Controlled Gust-Alleviation System. NACA TN 3597, 1956.
4. Oehman, Waldo I.: Analytical Study of the Performance of a Gust Alleviation System for a STOL Airplane. NASA TN D-7201, 1973.
5. Wells, William Raymond: A Time Optimal Control Study of a Second-Order Linear System With Delay. Ph. D. Thesis, Virginia Polytech. Inst., 1968.
6. Hess, Ronald A.: Optimal Control Approximations for Time Delay Systems. AIAA J., vol. 10, no. 11, Nov. 1972, pp. 1536-1538.
7. Bellman, Richard; and Kalaba, Robert, eds.: Selected Papers on Mathematical Trends in Control Theory. Dover Publ., Inc., c.1964.
8. Halanay, A.: Differential Equations. Academic Press, Inc., 1966.
9. Bellman, Richard; and Cooke, Kenneth L.: Differential-Difference Equations. Academic Press, Inc., 1963.
10. Pratt, Kermit G.: Response of Flexible Airplanes to Atmospheric Turbulence. Performance and Dynamics of Aerospace Vehicles, NASA SP-258, 1971, pp. 439-503.
11. Papoulis, Athanasios: Probability, Random Variables, and Stochastic Processes. McGraw-Hill Book Co., Inc., c.1965.
12. Arsac, J. (Allen Nussbaum and Gretchen C. Heim, transl.): Fourier Transforms and the Theory of Distributions. Prentice-Hall, Inc., c.1966.
13. El'sgol'ts, L. E.: Introduction to the Theory of Differential Equations With Deviating Arguments. Holden-Day, Inc., 1966.
14. Shaughnessy, John D.; and Kashiwagi, Yasutada: The Determination of a Stability Indicative Function for Linear Systems With Multiple Delays. NASA TR R-301, 1969.
15. Rogers, A. E.; and Connolly, T. W.: Analog Computation in Engineering Design. McGraw-Hill Book Co., Inc., 1960.
16. Gupta, Someshwar C.; and Hasdorff, Lawrence: Fundamentals of Automatic Control. Johy Wiley & Sons, Inc., c.1970.