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A VECTOR-DYADIC DEVELOPMENT OF THE EQUATIONS OF MOTION FOR N-COUPLED RIGID BODIES AND POINT MASSES

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CONTENTS

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INTRODUCTION

The design and analysis of most spacecraft attitude-control systems is usually based upon the assumption that the spacecraft itself can be modeled adequately as one body or as a collection of contiguous rigid bodies.

When, in the analyst's judgment, two or more bodies are required for representative modeling, the derivation of the system's equations of motion becomes laborious and subject to numerous errors in SIGN and judgment. The general-purpose digital program N-BOD has been developed to relieve the analyst of this time-consuming and error-prone task.

N-BOD assumes that the spacecraft can be modeled as a topological tree of rigid bodies, momentum wheels, and point masses. It also assumes that:

- All momentum wheels are symmetric and embedded within rigid bodies and that point masses exist only at limb ends.
- Unless otherwise directed by the user, all gyroscopic interaction torques are significant.
- All contiguous rigid bodies are connected by either zero-, one-, two-, or three-axis gimbals and point masses have either zero, one, two, or three degrees of relative translational freedom.

N-BOD provides the user with two modes of output:

- It may be used to derive and to output in vector-dyadic form on a line printer the complete nonlinear equations of motion of the system described to it.
- It may be used to solve numerically the equations of motion and to output any set of system-state variables.

Several techniques are given in the literature for deriving the equations of motion for an arbitrary number of interconnected rigid bodies: namely, References 1 through 6.

In each of the formalisms cited, the N-coupled rigid bodies must form a topological tree; that is, no closed paths are permitted in the system.

Two basic approaches are developed in the literature which show how formalistic methods can be evolved which will ultimately define the torque-free dynamics of an arbitrary N-coupled-body system.

Hooker and Margulies (Reference 2), and Roberson and Wittenburg (Reference 4) choose to write the equation of motion of each individual body of the connected system, taking into account in a very formalistic manner the interaction forces between adjacent bodies. Velman (Reference 1) uses what he refers to as a nested-body approach; that is, rather than having N-vector equations, one for each of the N bodies, the bodies are grouped into N nests and the equation of motion of each nest is then defined.

While the first method is well documented in the open literature, the nested-body method of Velman is outlined only in Reference 1. The equations of the nested bodies can, however, be derived from Velman's discussion and basic principles defined in most texts on rigid-body motion; for example, see Reference 7.

In Reference 3, Hooker comes to the realization that by making use of the nested-body concept, the method used by both himself in his first paper and by Velman in Reference 1 for handling constraints can be vastly simplified. Rather than switch from the discrete-body to the nested-body approach as Hooker does in Reference 3. one may attain considerable simplification simply by starting from the nested-body concept.

The purpose of this report is to combine the best features of Hooker's discrete-body approach and Velman's nested-body approach into a computationally efficient formalism. Material in this report will concentrate exclusively upon the theoretical development of the equations programed in N-BOD.

BASIC SYSTEM

The basic system to be studied can be defined as an arbitrary number of rigid bodies, momentum wheels, and point masses coupled together in such a manner as to form a topological tree (no closed paths).

Contiguous rigid bodies are assumed to be connected together by either a zero-, one-, two-, or three-axis gimbal; point masses are assumed to have either zero, one, two, or three degrees of relative translational freedom; momentum wheels are assumed to be symmetric about their spin axes and to be imbedded within rigid bodies. Rigid bodies which contain momentum wheels are referred to as gyrostats.

Let

N = total number of rigid bodies, gyrostats, and point masses

M = total number of imbedded momentum wheels.

To define the system mathematically, each body, momentum wheel, point mass, and hinge point must be assigned a unique label. The set of consecutive integers will be used. Let body 1 be the principal body of the system. All other bodies are given distinct integer labels ranging from 2 to N inclusive. For simplicity of computation, it is convenient for the labeling of the bodies to be such that, along any topological path from body 1 to its end, the body labels are of increasing numerical magnitude.

As a direct consequence of this labeling convention, the connection topology of the N-body system can be uniquely defined by the N X 1 connection matrix J (λ), where

$J(1) = 0$, and

$J(\lambda)$ = label of the body contiguous to and inboard of body λ, $\lambda = 2, 3, \ldots, N$.

All hinge points must be assigned unique labels. It is efficient to define them such that:

Hinge point $\lambda - 1$ is the point of connection between the contiguous bodies $J(\lambda)$ and λ.

All momentum wheels are given distinct integer labels from 1 to M inclusive. These may be randomly assigned.

Let

MO(m) = body label of the gyrostat in which momentum wheel m is imbedded.

In order to distinguish between rigid bodies and point masses in the system, two sets of body labels are defined:

S_R = set of all body labels associated with rigid bodies, and

S_L = set of all body labels associated with point masses.

The union of S_R and S_L is the set of all body labels S; that is,

$$S = S_R \cup S_L . \tag{1}$$

To define the equations of motion of the entire N-body system, the system is broken up into N distinct nests of bodies. The equation of motion for each nest is then defined and solved simultaneously with the equations for all other nests.

The nests are each given distinct integer labels from 0 to N - 1, inclusive, such that:

S_{k-1} = set of all body labels associated with those bodies which, relative to body 1, are outboard of hinge point k - 1 of body k. The elements of S_{k-1} define the bodies which make up the nest k - 1.

To define the motion of body λ relative to a reference frame fixed at the origin of the nest S_{k-1}, the bodies lying along the topological path from hinge point k - 1 to the center-of-mass of body λ must be known. This information is given by the connection

matrix $J(\lambda)$; however, it is notationally efficient to define the sets of body labels $S_{k-1,\lambda-1}$ such that

$S_{k-1,\lambda-1}$ = set of all body labels associated with those bodies lying on the topological path from hinge point k - 1 to the center-of-mass of body λ.

As an aid to the visualization of the contents of each set of body labels defined, a particular example is given which is general enough to bring out the salient features of the notation defined.

Figure 1 defines a particular 10-body system with each body and hinge point given a distinct integer label. Note in particular, that the body labels are of increasing numerical value along any topological path beginning at hinge point 0; furthermore, momentum wheels can be numbered randomly.

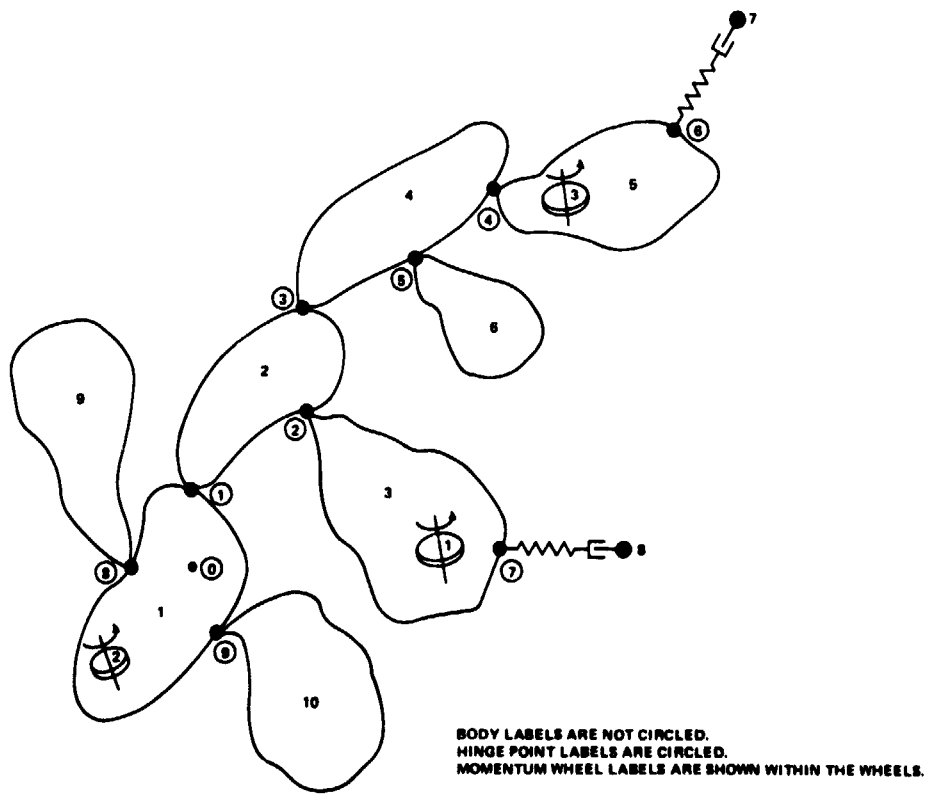

Figure 1. Labeling Scheme for 10-body Example

From the figure it can be seen that the contents of each set defined are:

$S_R = \{1, 2, 3, 4, 5, 6, 9, 10\}$

$S_L = \{7, 8\}$

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$S_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$S_1 = \{2, 3, 4, 5, 6, 7, 8\}$

$S_2 = \{3, 8\}$

$S_3 = \{4, 5, 6, 7\}$

$S_4 = \{5, 7\}$

$S_5 = \{6\}$

$S_6 = \{7\}$

$S_7 = \{8\}$

$S_8 = \{9\}$

$S_9 = \{10\}$

MO(1) = 3

MO(2) = 1

MO(3) = 5

The elements of the set $S_{k-1,\lambda-1}$ are given in the following example.

k - 1 \ λ - 1	0	1	2	3	4	5	6	7	8	9
0	1	1,2	1,2,3	1,2,4	1,2,4,5	1,2,4,6	1,2,4,5,7	1,2,3,8	1,9	1,10
1		2	2,3	2,4	2,4,5	2,4,6	2,4,5,7	2,3,8		
2			3					3,8		
3				4	4,5	4,6	4,5,7			
4					5		5,7			
5						6				
6							7			
7								8		
8									9	
9										10

All undefined sets in the above table are to be taken as empty sets. Physically, $S_{k-1,\lambda-1} = \{\ \}$ implies that body λ is not contained in the nest k - 1.

Each rigid body of the system is assumed to have nonzero mass and rotational inertia. Each point mass is assumed to have nonzero mass and zero rotational inertia.

Let

m_λ = total mass of body λ plus that of all momentum wheels imbedded in it,

Φ_λ = total inertia tensor of body λ plus that of all despun momentum wheels imbedded in it, about the composite body center-of-mass, and

I_{W_m} = inertia tensor of momentum wheel m about its center-of-mass.

To define the linear or angular momentum of each body or nest of bodies in the system, position vectors locating hinge points and centers of mass must be given.

Let

$\vec{\beta}_1$ = position vector from origin of inertial reference to hinge point 0 of body 1,

$\vec{\beta}_\lambda$ = position vector from hinge point J (λ) - 1 to the hinge point λ - 1 of body λ, and

$\vec{\alpha}_\lambda$ = position vector from hinge point λ - 1 to the center-of-mass of body λ.

The exact position and labeling of each of these vectors is shown in figure 2 for the standard 10-body example.

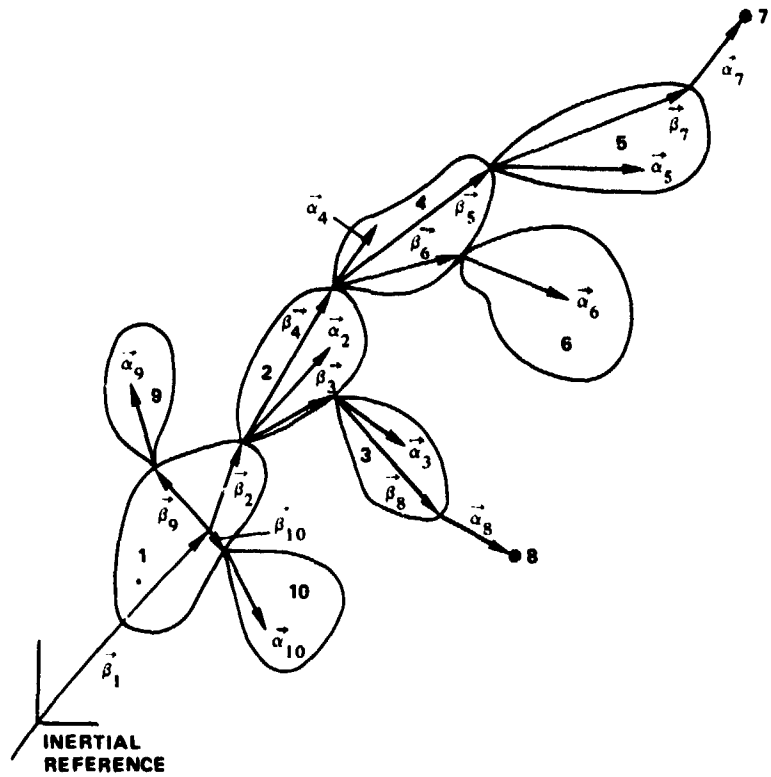

Figure 2. Positions and Labeling of Hinge-point and Center-of-mass Location Vectors

The vectors and dyads appearing in the equations of motion are most naturally defined relative to certain body fixed-reference frames.

Let

Reference frame 0 = inertially fixed frame of reference having its origin at the inertial origin,

Reference frame λ = body λ fixed frame of reference having its origin at the hinge point $\lambda - 1$. If body λ is a point mass, the reference frame λ is taken to be fixed relative to reference frame $J(\lambda)$ and have its origin at hinge point $\lambda - 1$.

To avoid explicit definition of coordinate systems and their associated kinematics, all equations are written in terms of vectors and dyads. Inevitably, this leads to the necessity of differentiating, relative to an inertial reference frame, vectors which are more easily

expressed in some other moving frame. Extensive use is made of the identities from vector differential calculus that

$$\dot{\vec{v}}_I = \dot{\vec{\rho}}_i + \overset{\textcircled{i}}{\vec{v}} + \vec{\omega}_i \times \vec{v}, \tag{2}$$

where

$\vec{v}$ = vector to arbitrary point P from origin of reference frame i,

$\dot{\vec{v}}_I$ = velocity of P with respect to the inertial frame,

$\overset{\textcircled{i}}{\vec{v}}$ = velocity of P with respect to moving reference frame i,

$\vec{\omega}_i$ = angular velocity of moving reference frame i with respect to the inertial frame,

$\dot{\vec{\rho}}_i$ = inertial velocity of the origin of reference frame i,

and

$$\ddot{\vec{v}}_I = \ddot{\vec{\rho}}_i + \dot{\vec{\omega}}_i \times \vec{v} + \vec{\omega}_i \times (\vec{\omega}_i \times \vec{v}) + 2\vec{\omega}_i \times \overset{\textcircled{i}}{\vec{v}} + \overset{\textcircled{i}\,\textcircled{i}}{\vec{v}}, \tag{3}$$

where

$\ddot{\vec{v}}_I$ = acceleration of P with respect to the inertial frame,

$\ddot{\vec{\rho}}_i$ = inertial acceleration of the origin of reference frame i,

$\dot{\vec{\omega}}_i \times \vec{v}$ = linear acceleration of P due to angular acceleration of reference frame i,

$\vec{\omega}_i \times (\vec{\omega}_i \times \vec{v})$ = centripetal acceleration of P due to rotation of reference frame i,

$2\vec{\omega}_i \times \overset{\textcircled{i}}{\vec{v}}$ = Coriolis acceleration of P, and

$\overset{\textcircled{i}\,\textcircled{i}}{\vec{v}}$ = apparent acceleration of P relative to reference frame i.

EQUATIONS OF MOTION (VECTOR-DYADIC FORM)

The entire N-body system is free to translate and rotate relative to a fixed inertial origin. All contiguous rigid bodies are free to rotate relative to each other, point masses are free to translate relative to their respective contiguous bodies, and all momentum wheels are free to rotate about their axis of symmetry relative to the gyrostat in which they are imbedded.

Let

$\vec{G}_{I,\lambda}$ = linear momentum of body λ relative to the inertial origin,

$\vec{L}_{I,\lambda}$ = angular momentum of body λ relative to the inertial origin,

$\vec{H}_m$ = angular momentum of momentum wheel m relative to the inertial origin,

$\vec{F}_\lambda$ = resultant of external forces acting on body λ,

$\vec{\phi}_\lambda$ = resultant of external torques acting on body λ, and

$\vec{\phi}_{W_m}$ = resultant of external torques acting on momentum wheel m.

Then from Newton's three fundamental laws of motion, the equations of motion for each body λ and each momentum wheel imbedded therein are given by

$$\dot{\vec{G}}_{I,\lambda} = \vec{F}_\lambda , \tag{4}$$

$$\dot{\vec{L}}_{I,\lambda} = \vec{\phi}_\lambda , \tag{5}$$

$$\dot{\vec{H}}_m = \vec{\phi}_{W_m} , \qquad m:MO(m) = \lambda , \tag{6}$$

where

$m:MO(m) = \lambda$ = all m such that $MO(m) = \lambda$.

Similarly, the equations of motion for the nest of bodies k - 1 are

$$\sum_{\lambda \epsilon S_{k-1}} \dot{\vec{G}}_{I,\lambda} = \sum_{\lambda \epsilon S_{k-1}} \vec{F}_\lambda , \tag{7}$$

$$\sum_{\lambda \epsilon S_{k-1}} \dot{\vec{L}}_{I,\lambda} = \sum_{\lambda \epsilon S_{k-1}} \vec{\phi}_\lambda , \tag{8}$$

$$\dot{\vec{H}}_m = \dot{\vec{\phi}}_{W_m} , \quad m: MO(m) \epsilon S_{k-1} , \tag{9}$$

where

$\sum_{\lambda \epsilon S_{k-1}}$ = sum over all bodies λ contained in the nest k - 1, and

$m:MO(m) \in S_{k-1}$ = all m such that MO(m) is contained in the nest k - 1.

The equations of motion for the entire N-body system are obtained by solving simultaneously 2N + M vector equations. These equations may be either in the discrete-body form of Equations (4), (5), and (6) or in the nested-body form of Equations (7), (8), and (9). This analysis follows the nested-body form.

Between each pair of connected contiguous bodies, forces and torques of constraint exist which limit the number of degrees of relative freedom from six to, at most, three.

Let

$\vec{F}^{c}_{k-1}$ = resultant force of constraint acting on body k through hinge point k - 1,

$\vec{\phi}^{c}_{k-1}$ = resultant torque of constraint acting on body k through hinge point k - 1, and

$\vec{\phi}^{c}_{W_m}$ = resultant torque of constraint acting on wheel m preventing motion about any axis normal to its spin axis.

For most problems of practical interest, mechanisms such as springs, dampers, motors, and so forth exist between the contiguous bodies.

Let

$\vec{F}^{H}_{k-1}$ = resultant force acting on body k due to all mechanisms existing between bodies J(k) and k at hinge point k - 1,

$\vec{\phi}^{H}_{k-1}$ = resultant torque acting on body k due to all mechanisms existing between bodies J(k) and k at hinge point k - 1, and

$\vec{CL}_{m}$ = resultant torque acting on momentum wheel m due to all mechanisms existing between it and body MO(m).

It should be noted that at every hinge point of the system (with the exception of hinge point 0), the forces and torques defined above exist in equal and opposite pairs. Thus, for any nest of bodies containing the hinge point k - 1, these forces and torques are internal to the nest and have a resultant effect of zero. At the hinge point of the nest, however, they are external.

External forces may be present which act on the N-body system. These may be locally applied, distributed throughout, or applied only over selected portions of the system. (It is assumed that external forces are not applied directly to momentum wheels.)

Let

$\vec{F}^{(e)}_{\lambda,i}$ = resultant external force applied to point i of body λ, and

$\vec{R}_{\lambda,i}$ = vector from center-of-mass of body λ to the point i at which the force $\vec{F}^{(e)}_{\lambda,i}$ is applied.

For Newton's laws of motion to be applied, motion must be defined relative to the inertial origin.

Let

$\vec{\gamma}_{I,\lambda}$ = position vector from inertial origin to center-of-mass of body λ, and

$\vec{\gamma}_{k-1,\lambda}$ = position vector from hinge point k-1 to center-of-mass of body λ.

From the definitions given in the previous section, it follows that

$$\vec{\gamma}_{I,\lambda} = \vec{\beta}_I + \vec{\gamma}_{0,\lambda} \tag{10}$$

and

$$\vec{\gamma}_{k-1,\lambda} = \sum_{\substack{i \in S_{k-1,\lambda-1} \\ i \neq k}} \vec{\beta}_i + \vec{\alpha}_\lambda \,. \tag{11}$$

From these definitions, the resultant of the external forces acting upon the nest k-1 is given by

$$\sum_{\lambda \in S_{k-1}} \vec{F}_\lambda = \vec{F}^c_{k-1} + \vec{F}^H_{k-1} + \sum_{\lambda \in S_{k-1}} \sum_{i \in \lambda} \vec{F}^{(e)}_{\lambda,i} \,, \tag{12}$$

while the resultant of the external torques acting upon the nest k-1, measured relative to the inertial origin, is

$$\sum_{\lambda \in S_{k-1}} \vec{\phi}_\lambda = \vec{\phi}^c_{k-1} + \vec{\phi}^H_{k-1} + (\vec{\gamma}_{I,k} - \vec{\alpha}_k) \times (\vec{F}^c_{k-1} + \vec{F}^H_{k-1}) + \sum_{\lambda \in S_{k-1}} \sum_{i \in \lambda} (\vec{\gamma}_{I,\lambda} + \vec{R}_{\lambda,i}) \times \vec{F}^{(e)}_{\lambda,i} \,, \tag{13}$$

and the resultant torque acting upon momentum wheel m is

$$\vec{\phi}_{W_m} = \vec{\phi}^c_{W_m} + \vec{CL}_m \,. \tag{14}$$

The vector equations which completely define the motion of the nest k-1 are obtained by direct substitution of Equations (12), (13), and (14) into Equations (7), (8), and (9).

$$\sum_{\lambda \in S_{k-1}} \dot{\vec{G}}_{I,\lambda} = \vec{F}^{ai}_{k-1} + \vec{F}^c_{k-1} + \sum_{\lambda \in S_{k-1}} \vec{F}^{(e)}_\lambda \,, \tag{15}$$

$$\sum_{\lambda \epsilon S_{k-1}} \dot{\vec{L}}_{I,\lambda} = \vec{\phi}^{H}_{k-1} + \vec{\phi}^{c}_{k-1} + (\vec{\gamma}_{I,k} - \vec{\alpha}_{k}) \times (\vec{F}^{c}_{k-1} + \vec{F}^{H}_{k-1})$$

$$+ \sum_{\lambda \epsilon S_{k-1}} (\vec{\gamma}_{I,\lambda} \times \vec{F}^{(e)}_{\lambda} + \vec{\phi}^{(e)}_{\lambda}), \tag{16}$$

and

$$\dot{\vec{H}}_{m} = \vec{\phi}^{c}_{W_m} + \dot{\vec{CL}}_{m} \qquad m: MO(m) \epsilon S_{k-1}, \tag{17}$$

where, for the sake of notation compression, the following definitions have been made:

$$\vec{F}^{(e)}_{\lambda} = \sum_{i \epsilon \lambda} \vec{F}^{(e)}_{\lambda,i}, \tag{18}$$

and

$$\vec{\phi}^{(e)}_{\lambda} = \sum_{i \epsilon \lambda} \vec{R}_{\lambda,i} \times \vec{F}^{(e)}_{\lambda,i}. \tag{19}$$

The inertial momentum for each body λ and wheel m may be defined in terms of their respective mass and inertia properties and the system-state variables.

Let

$\vec{\omega}_{\lambda}$ = inertial angular velocity of body λ,

$\vec{\tilde{\omega}}_{\lambda}$ = angular velocity of body λ relative to body $J(\lambda)$,

$\vec{\omega}_{W_m}$ = inertial angular velocity of momentum wheel m,

$\vec{\tilde{\omega}}_{W_m}$ = angular velocity of momentum wheel m relative to body MO(m), and

$\vec{\tilde{H}}_{m}$ = angular momentum of wheel m about its own center-of-mass, relative to body MO(m).

From definitions provided in virtually all texts on rigid-body dynamics, it follows that:

a. Body λ, a point mass, rigid body, or gyrostat

$$\vec{G}_{I,\lambda} = m_{\lambda} \dot{\vec{\gamma}}_{I,\lambda} \tag{20}$$

$$\dot{\vec{G}}_{I,\lambda} = m_{\lambda} \ddot{\vec{\gamma}}_{I,\lambda}. \tag{21}$$

b. Body λ, a point mass

$$\vec{L}_{I,\lambda} = \vec{\gamma}_{I,\lambda} \times m_\lambda \dot{\vec{\gamma}}_{I,\lambda} \tag{22}$$

$$\dot{\vec{L}}_{I,\lambda} = \vec{\gamma}_{I,\lambda} \times m_\lambda \ddot{\vec{\gamma}}_{I,\lambda}. \tag{23}$$

c. Body λ, a rigid body

$$\vec{L}_{I,\lambda} = \Phi_\lambda \cdot \vec{\omega}_\lambda + \vec{\gamma}_{I,\lambda} \times m_\lambda \dot{\vec{\gamma}}_{I,\lambda} \tag{24}$$

$$\dot{\vec{L}}_{I,\lambda} = \Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + \vec{\omega}_\lambda \times \Phi_\lambda \cdot \vec{\omega}_\lambda + \vec{\gamma}_{I,\lambda} \times m_\lambda \ddot{\vec{\gamma}}_{I,\lambda}. \tag{25}$$

d. Body λ, a gyrostat

$$\vec{L}_{I,\lambda} = \Phi_\lambda \cdot \vec{\omega}_\lambda + \vec{\gamma}_{I,\lambda} \times m_\lambda \dot{\vec{\gamma}}_{I,\lambda} + \sum_{\substack{m: \\ MO(m)=\lambda}} \vec{\vec{H}}_m \tag{26}$$

$$\dot{\vec{L}}_{I,\lambda} = \Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + \vec{\omega}_\lambda \times \Phi_\lambda \cdot \vec{\omega}_\lambda + \vec{\gamma}_{I,\lambda} \times m_\lambda \ddot{\vec{\gamma}}_{I,\lambda} + \sum_{\substack{m: \\ MO(m)=\lambda}} \dot{\vec{\vec{H}}}_m. \tag{27}$$

e. Momentum wheel m

$$\vec{\vec{H}}_m = I_{W_m} \cdot \vec{\vec{\omega}}_{W_m} \tag{28}$$

$$\vec{H}_m = \vec{\vec{H}}_m + I_{W_m} \cdot \vec{\omega}_\lambda \tag{29}$$

$$\dot{\vec{H}}_m = \dot{\vec{\vec{H}}}_m + I_{W_m} \cdot \dot{\vec{\omega}}_\lambda + \vec{\omega}_\lambda \times I_{W_m} \cdot \vec{\omega}_\lambda. \tag{30}$$

It has been assumed that the composite N-body system has at most six degrees of freedom relative to an inertial reference, contiguous bodies have at most three degrees of relative freedom, and momentum wheels have only one degree of relative freedom. Thus, there exists at most $3(N + 1) + M$ degrees of freedom for the N-body system, and its motion can be completely defined by exactly $N + 1 + M$ coupled-vector equations.

These coupled-vector equations are given by the following:

a. For $k \epsilon S_R$

This condition implies that body k is either a rigid body or a gyrostat and that the nest k-1 has at most three degrees of relative rotational freedom. The interaction force between bodies k and J(k), which constrains relative translational motion, is given from Equation (15) by

$$\vec{F}^c_{k-1} + \vec{F}^H_{k-1} = \sum_{\lambda \epsilon S_{k-1}} (\dot{\vec{G}}_{I,\lambda} - \vec{F}^{(e)}_{\lambda}). \tag{31}$$

The rotation equation for all nests k-1 having $k \epsilon S_R$ is obtained by substituting Equations (21), (27), and (31) into Equation (16):

$$\sum_{\lambda \epsilon S_{k-1}} \left[\Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \ddot{\vec{\gamma}}_{I,\lambda} + \sum_{\substack{m: \\ MO(m)=\lambda}} \dot{\vec{\vec{H}}}_m \right]$$
$$= \vec{\phi}^c_{k-1} + \vec{\phi}^H_{k-1} + \sum_{\lambda \epsilon S_{k-1}} \left[-\vec{\omega}_\lambda \times \Phi_\lambda \cdot \vec{\omega}_\lambda + \vec{\gamma}_{k-1,\lambda} \times \vec{F}^{(e)}_{\lambda} + \vec{\phi}^{(e)}_{\lambda} \right]. \tag{32}$$

b. For $k \epsilon S_L$

This condition implies that the nest k-1 contains only the point mass labeled body k, and that the nest k-1 has at most three translational degrees of relative freedom. The translational equation for the nest k-1 is from Equations (15) and (21):

$$m_k \ddot{\vec{\gamma}}_{I,k} = \vec{F}^H_{k-1} + \vec{F}^c_{k-1} + \vec{F}^{(e)}_k. \tag{33}$$

c. The composite system is free to translate relative to the inertial reference. The translation equation for the composite system is from Equations (15) and (21):

$$\sum_{\lambda \epsilon S_0} m_\lambda \ddot{\vec{\gamma}}_{I,\lambda} = \vec{F}^H_0 + \vec{F}^c_0 + \sum_{\lambda \epsilon S_0} \vec{F}^{(e)}_{\lambda}. \tag{34}$$

d. Momentum wheels are defined to have one degree of relative rotational freedom. The relative momentum equation for momentum wheel m contained in body λ, derived from Equations (17) and (30), is given by

$$\dot{\vec{\vec{H}}}_m + I_{W_m} \cdot \dot{\vec{\omega}}_\lambda = -\vec{\omega}_\lambda \times I_{W_m} \cdot \vec{\omega}_\lambda + \vec{CL}_m + \vec{\phi}^c_{W_m}. \tag{35}$$

These equations of motion must now be put into a computationally efficient form.

Accordingly, the inertial acceleration of the center-of-mass of body λ can, by making use of Equations (2), (3), (10), and (11), be written as

$$\ddot{\vec{\gamma}}_{I,\lambda} = \ddot{\vec{\beta}}_1 + \sum_{\substack{i \epsilon S_{0,\lambda-1} \\ i \neq 1}} [\dot{\vec{\omega}}_{J(i)} \times \vec{\beta}_i + \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i)]$$

$$+ \begin{cases} \dot{\vec{\omega}}_\lambda \times \vec{\alpha}_\lambda + \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) & \lambda \epsilon S_R \\ \ddot{\vec{\alpha}}_\lambda + \dot{\vec{\omega}}_{J(\lambda)} \times \vec{\alpha}_\lambda + 2\vec{\omega}_{J(\lambda)} \times \mathring{\vec{\alpha}}_\lambda + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) & \lambda \epsilon S_L , \end{cases} \tag{36}$$

where the open dot implies differentiation with respect to the reference frame λ fixed at hinge point λ-1.

Considerable symmetry in the final form of the equations of motion is obtainable if the inertial angular acceleration vectors are expressed in terms of the relative angular acceleration vectors. From the definitions of inertial and relative angular velocities

$$\vec{\omega}_\lambda = \sum_{i \epsilon S_{0,\lambda-1}} \vec{\tilde{\omega}}_i \qquad \lambda \epsilon S_R , \tag{37}$$

and hence

$$\dot{\vec{\omega}}_\lambda = \sum_{i \epsilon S_{0,\lambda-1}} \dot{\vec{\tilde{\omega}}}_i . \tag{38}$$

Substitution of Equation (38) into (36) along with a rearrangement of terms leads to

$$\ddot{\vec{\gamma}}_{I,\lambda} = \ddot{\vec{\beta}}_1 + \begin{cases} \displaystyle\sum_{i \epsilon S_{0,\lambda-1}} \dot{\vec{\tilde{\omega}}}_i \times \vec{\gamma}_{i-1,\lambda} & \lambda \epsilon S_R \\ \displaystyle\sum_{\substack{i \epsilon S_{0,\lambda-1} \\ i \neq \lambda}} \dot{\vec{\tilde{\omega}}}_i \times \vec{\gamma}_{i-1,\lambda} + \mathring{\mathring{\vec{\alpha}}}_\lambda & \lambda \epsilon S_L \end{cases}$$

$$+\sum_{\substack{i\epsilon S_{0,\lambda-1}\\ i\neq 1}} \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i)$$

$$+ \begin{cases} \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) & \lambda\epsilon S_R \\ 2\vec{\omega}_{J(\lambda)} \times \overset{\circ}{\vec{\alpha}}_\lambda + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) & \lambda\epsilon S_L . \end{cases} \qquad (39)$$

It is also convenient to make use of Equations (2) and (28) to write

$$\dot{\vec{\tilde{H}}}_m = I_{W_m^*} \overset{\circ}{\vec{\tilde{\omega}}}_{W_m} + \vec{\omega}_\lambda \times \vec{\tilde{H}}_m , \qquad (40)$$

where MO(m) = λ and, again, the open dot implies differentiation with respect to the body λ fixed reference frame.

Substitution of the vector identities given by Equations (38), (39), and (40) into the system equations of motion, along with a rearrangement of terms yields:

a. Rotation equation for the nest k-1, $k\epsilon S_R$

$$\sum_{\lambda\epsilon S_{k-1}} \left[\vec{\gamma}_{k-1,\lambda} \times m_\lambda \ddot{\vec{\beta}}_1 + \sum_{\substack{m:\\ MO(m)=\lambda}} I_{W_m^*} \overset{\circ}{\vec{\tilde{\omega}}}_{W_m} \right]$$

$$+ \sum_{\substack{i\epsilon S_{0,k-1}\\ i\neq k}} \left[\sum_{\substack{\lambda\epsilon S_{k-1}\\ \lambda\epsilon S_L}} \Phi_\lambda \cdot \dot{\vec{\tilde{\omega}}}_i + \sum_{\lambda\epsilon S_{k-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda (\dot{\vec{\tilde{\omega}}}_i \times \vec{\gamma}_{i-1,\lambda}) \right]$$

$$+ \sum_{i\epsilon S_{k-1}} \begin{cases} \sum_{\substack{\lambda\epsilon S_{i-1}\\ \lambda\epsilon S_L}} \Phi_\lambda \cdot \dot{\vec{\tilde{\omega}}}_i + \sum_{\lambda\epsilon S_{i-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda (\dot{\vec{\tilde{\omega}}}_i \times \vec{\gamma}_{i-1,\lambda}) & i\epsilon S_R \\ \sum_{\lambda\epsilon S_{i-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \overset{\circ\circ}{\vec{\alpha}}_\lambda & i\epsilon S_L \end{cases}$$

$$= - \sum_{\substack{\lambda \in S_{k-1} \\ \lambda \notin S_L}} \left[\vec{\omega}_\lambda \times \Phi_\lambda \cdot \vec{\omega}_\lambda + \sum_{\substack{m: \\ MO(m)=\lambda}} \vec{\omega}_\lambda \times \vec{\vec{H}}_m \right]$$

$$- \sum_{\lambda \in S_{k-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left[\sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i) \right.$$

$$+ \left\{ \begin{array}{ll} \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) & \lambda \in S_R \\ 2\,\vec{\omega}_{J(\lambda)} \times \overset{\circ}{\vec{\alpha}}_\lambda + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) & \lambda \in S_L \end{array} \right]$$

$$+ \vec{\phi}^{c}_{k-1} + \vec{\phi}^{H}_{k-1} + \sum_{\lambda \in S_{k-1}} \left[\vec{\gamma}_{k-1,\lambda} \times \vec{F}^{(e)}_\lambda + \vec{\phi}^{(e)}_\lambda \right] . \qquad (41)$$

b. Translation equation for the nest $\lambda - 1$, $\lambda \in S_L$

$$m_\lambda \left[\ddot{\vec{\beta}}_1 + \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq \lambda}} \dot{\vec{\vec{\omega}}}_i \times \vec{\gamma}_{i-1,\lambda} + \overset{\circ\circ}{\vec{\alpha}}_\lambda \right]$$

$$= -m_\lambda \left[\sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i) + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) + 2\,\vec{\omega}_{J(\lambda)} \times \overset{\circ}{\vec{\alpha}}_\lambda \right] \qquad (42)$$

$$+ \vec{F}^{c}_{\lambda-1} + \vec{F}^{H}_{\lambda-1} + \vec{F}^{(e)}_\lambda .$$

c. Translation equation for composite system

$$\sum_{\lambda \epsilon S_0} m_\lambda \ddot{\vec{\beta}}_1 + \sum_{i \epsilon S_0} \begin{Bmatrix} \sum_{\lambda \epsilon S_{i-1}} m_\lambda (\dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda}) & i \epsilon S_R \\ \sum_{\lambda \epsilon S_{i-1}} m_\lambda \overset{\circ\circ}{\vec{\alpha}}_\lambda & i \epsilon S_L \end{Bmatrix}$$

$$= - \sum_{\lambda \epsilon S_0} m_\lambda \left[\sum_{\substack{i \epsilon S_{0,\lambda-1} \\ i \neq 1}} \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i) + \begin{Bmatrix} \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) & \lambda \epsilon S_R \\ 2 \vec{\omega}_{J(\lambda)} \times \overset{\circ}{\vec{\alpha}}_\lambda + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) & \lambda \epsilon S_L \end{Bmatrix} \right]$$

$$+ \vec{F}_0^H + \vec{F}_0^c + \sum_{\lambda \epsilon S_0} \vec{F}_\lambda^{(e)} . \tag{43}$$

d. Momentum equation for wheel m: MO(m) = λ

$$I_{W_m} \cdot \overset{\circ}{\dot{\vec{\omega}}}_{W_m} + \sum_{i \epsilon S_{0,\lambda-1}} I_{W_m} \cdot \dot{\vec{\omega}}_i = -\vec{\omega}_\lambda \times (I_{W_m} \cdot \vec{\omega}_\lambda + \vec{H}_m) + \vec{CL}_m + \vec{\phi}_{W_m}^c . \tag{44}$$

It is desirable to condense these vector equations eventually into a concise matrix format. To do this, however, several vector operations must be replaced by equivalent vector-dyadic, scalar-product operations. In particular, one can make use of the following:

a. Multiplication of a scalar and a vector

$$m_\lambda \ddot{\vec{\beta}}_1 = m_\lambda 1 \cdot \ddot{\vec{\beta}}_1 , \tag{45}$$

$$m_\lambda \overset{\circ\circ}{\vec{\alpha}}_\lambda = m_\lambda 1 \cdot \overset{\circ\circ}{\vec{\alpha}}_\lambda , \tag{46}$$

where

$1 \equiv$ unit dyad.

b. Vector product of two vectors

$$\dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} = \Gamma_{i-1,\lambda} \cdot \dot{\vec{\omega}}_i , \tag{47}$$

$$\vec{\gamma}_{k-1,\lambda} \times \ddot{\vec{\beta}}_1 = -\Gamma_{k-1,\lambda} \cdot \ddot{\vec{\beta}}_1 \tag{48}$$

$$\vec{\gamma}_{k-1,\lambda} \times \overset{\circ\circ}{\vec{\alpha}}_\lambda = -\Gamma_{k-1,\lambda} \cdot \overset{\circ\circ}{\vec{\alpha}}_\lambda, \tag{49}$$

where

$$\Gamma_{k-1,\lambda} = \mathscr{S}(\vec{\gamma}_{k-1,\lambda}), \tag{50}$$

and $\mathscr{S}$ is the tensor operator which transforms vectors into skew-symmetric tensors of rank two (dyads). If $\vec{v}$ is an arbitrary vector with components $\{v_1, v_2, v_3\}$, then

$$\mathscr{S}(\vec{v}) = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}. \tag{51}$$

c. Vector triple product

$$\vec{\gamma}_{k-1,\lambda} \times m_\lambda (\dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda}) = G^\lambda_{k-1,i-1} \cdot \dot{\vec{\omega}}_i, \tag{52}$$

where

$$G^\lambda_{k-1,i-1} = m_\lambda [(\vec{\gamma}_{k-1,\lambda} \cdot \vec{\gamma}_{i-1,\lambda}) 1 - \vec{\gamma}_{i-1,\lambda} \vec{\gamma}_{k-1,\lambda}]. \tag{53}$$

To establish symmetry conditions, it must be noted that the components of the pseudo-inertia tensors $G^\lambda_{k-1,i-1}$ and $G^\lambda_{i-1,k-1}$, when written in the same coordinate system, obey the following matrix transpose relationship:

$$[G^\lambda_{k-1,i-1}] = [G^\lambda_{i-1,k-1}]^T. \tag{54}$$

The vector-dyadic form of the equations of motion is obtainable by substitution of Equations (45) through (52) into Equations (41) through (44). That is:

a. Rotation equation for the nest k-1, $k \epsilon S_R$

$$-\sum_{\lambda \epsilon S_{k-1}} m_\lambda \Gamma_{k-1,\lambda} \cdot \ddot{\vec{\beta}}_1 + \sum_{\substack{i \epsilon S_{0,k-1} \\ i \neq k}} \left[\sum_{\substack{\lambda \epsilon S_{k-1} \\ \lambda \notin S_L}} \Phi_\lambda + \sum_{\lambda \epsilon S_{k-1}} G^\lambda_{k-1,i-1} \right] \cdot \dot{\vec{\omega}}_i$$

$$+\sum_{i\epsilon S_{k-1}}\left\{\begin{array}{ll}\left[\sum_{\substack{\lambda\epsilon S_{i-1}\\ \lambda\notin S_L}}\Phi_\lambda+\sum_{\lambda\epsilon S_{i-1}}G^{\lambda}_{k-1,i-1}\right]\cdot\ddot{\vec{\omega}}_i & i\epsilon S_R\\ -\sum_{\lambda\epsilon S_{i-1}}m_\lambda\,\Gamma_{k-1,\lambda}\cdot\overset{\circ\circ}{\vec{\alpha}}_\lambda & i\epsilon S_L\end{array}\right\}+\sum_{\lambda\epsilon S_{k-1}}\sum_{\substack{m:\\ MO(m)=\lambda}}I_{W_m}\cdot\overset{\circ}{\vec{\omega}}_{W_m}$$

$$=-\sum_{\substack{\lambda\epsilon S_{k-1}\\ \lambda\notin S_L}}\left[\vec{\omega}_\lambda\times\Phi_\lambda\cdot\vec{\omega}_\lambda+\sum_{\substack{m:\\ MO(m)=\lambda}}\vec{\omega}_\lambda\times\vec{\vec{H}}_m\right]$$

$$-\sum_{\lambda\epsilon S_{k-1}}\vec{\gamma}_{k-1,\lambda}\times m_\lambda\left[\sum_{\substack{i\epsilon S_{0,\lambda-1}\\ i\neq 1}}\vec{\omega}_{J(i)}\times(\vec{\omega}_{J(i)}\times\vec{\beta}_i)\right.$$

$$\left.+\left\{\begin{array}{ll}\vec{\omega}_\lambda\times(\vec{\omega}_\lambda\times\vec{\alpha}_\lambda) & \lambda\epsilon S_R\\ 2\,\vec{\omega}_{J(\lambda)}\times\overset{\circ}{\vec{\alpha}}_\lambda+\vec{\omega}_{J(\lambda)}\times(\vec{\omega}_{J(\lambda)}\times\vec{\alpha}_\lambda) & \lambda\epsilon S_L\end{array}\right.\right]$$

$$+\vec{\phi}^{c}_{k-1}+\vec{\phi}^{H}_{k-1}+\sum_{\lambda\epsilon S_{k-1}}\left[\vec{\gamma}_{k-1,\lambda}\times\vec{F}^{(e)}_\lambda+\phi^{(e)}_\lambda\right]. \qquad (55)$$

b. Translation equation for the nest λ-1, $\lambda\epsilon S_L$

$$m_\lambda\left[1\cdot\ddot{\vec{\beta}}_1+\sum_{\substack{i\epsilon S_{0,\lambda-1}\\ i\neq\lambda}}\Gamma_{i-1,\lambda}\cdot\ddot{\vec{\omega}}_i+1\cdot\overset{\circ\circ}{\vec{\alpha}}_\lambda\right]$$

$$=-m_\lambda\left[\sum_{\substack{i\epsilon S_{0,\lambda-1}\\ i\neq 1}}\vec{\omega}_{J(i)}\times(\vec{\omega}_{J(i)}\times\vec{\beta}_i)+\vec{\omega}_{J(\lambda)}\times(\vec{\omega}_{J(\lambda)}\times\vec{\alpha}_\lambda)+2\,\vec{\omega}_{J(\lambda)}\times\overset{\circ}{\vec{\alpha}}_\lambda\right]$$

$$+\vec{F}^{c}_{\lambda-1}+\vec{F}^{H}_{\lambda-1}+\vec{F}^{(e)}_\lambda. \qquad (56)$$

c. Translation equation for composite system

$$\sum_{\lambda \epsilon S_0} m_\lambda \, 1 \cdot \ddot{\vec{\beta}}_1 + \sum_{i \epsilon S_0} \left\{ \begin{array}{ll} \displaystyle\sum_{\lambda \epsilon S_{i-1}} m_\lambda \, \Gamma_{i-1,\lambda} \cdot \dot{\tilde{\tilde{\omega}}}_i & i \epsilon S_R \\ \displaystyle\sum_{\lambda \epsilon S_{i-1}} m_\lambda \, 1 \cdot \overset{\circ\circ}{\vec{\alpha}}_\lambda & i \epsilon S_L \end{array} \right\} = - \sum_{\lambda \epsilon S_0} m_\lambda \left[\sum_{\substack{i \epsilon S_{0,\lambda-1} \\ i \neq 1}} \vec{\omega}_{J(i)} \right.$$

$$\left. \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i) + \left\{ \begin{array}{ll} \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) & \lambda \epsilon S_R \\ 2 \vec{\omega}_{J(\lambda)} \times \overset{\circ}{\vec{\alpha}}_\lambda + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) & \lambda \epsilon S_L \end{array} \right] \right.$$

$$+ \vec{F}_0^c + \vec{F}_0^H + \sum_{\lambda \epsilon S_0} \vec{F}_\lambda^{(e)}. \qquad (57)$$

d. Momentum equation for wheel m: $MO(m) = \lambda$

$$I_{W_m} \cdot \dot{\tilde{\tilde{\omega}}}_{W_m} + \sum_{i \epsilon S_{0,\lambda-1}} I_{W_m} \cdot \dot{\tilde{\tilde{\omega}}}_i$$

$$= - \vec{\omega}_\lambda \times (I_{W_m} \cdot \vec{\omega}_\lambda + \tilde{\tilde{H}}_m) + \vec{CL}_m + \vec{\phi}_{W_m}^c. \qquad (58)$$

EQUATIONS OF MOTION (MATRIX FORM)

In the previous section, the equations of motion of the coupled N-body system have been derived in vector-dyadic form from the principles of linear and angular momentum. A cursory examination of Equations (55) through (58) is sufficient to note that terms associated with the same gyroscopic effects have been grouped together. In putting the equations into a matrix format it is convenient to make use of partitioned matrices so as to retain this separation of effects.

Accordingly, one may define:

$\{\dot{\tilde{\omega}}\}$ = $(N + 1) \times 1$ column matrix of relative acceleration vectors. The vector element $\dot{\tilde{\omega}}_k$ in row k is

a. Body k a rigid body $k \epsilon S_R$

$$\dot{\tilde{\omega}}_k = \dot{\tilde{\tilde{\omega}}}_k . \qquad (59)$$

b. Body k a point mass $k \epsilon S_L$

$$\dot{\tilde{\omega}}_k = \overset{\circ\circ}{\vec{\alpha}}_k . \tag{60}$$

c. Total system $k = N + 1$

$$\dot{\tilde{\omega}}_k = \ddot{\vec{\beta}}_1 . \tag{61}$$

$\left\{ \overset{\circ}{\tilde{\omega}}_W \right\}$ = M X 1 column matrix of momentum-wheel relative angular acceleration vectors. The vector element in row m is

$$\overset{\circ}{\tilde{\omega}}_{W_m} = \overset{\circ}{\vec{\omega}}_{W_m} . \tag{62}$$

$[X]$ = (N + 1) X (N + 1) symmetric matrix of pseudo-inertia tensors. The tensor element in row k, column i (i ≥ k) is from Equations (55), (56), and (57), given by

a. $k \epsilon S_R$, $i \epsilon S_{k-1}$, $i \epsilon S_R$ (Nest i-1 is a nest of at least one rigid body within the nest k-1).

$$X_{k,i} = \sum_{\substack{\lambda \epsilon S_{i-1} \\ \lambda \notin S_L}} \Phi_\lambda + \sum_{\lambda \epsilon S_{i-1}} G^\lambda_{k-1,i-1} . \tag{63}$$

For $k = i$, $X_{k,i}$ is the inertia tensor of the nest k-1 about the hinge point k-1. For $k \neq i$, it defines an inertia cross-coupling tensor between nests k-1 and i-1.

b. $k \epsilon S_R$, $i \epsilon S_{k-1}$, $i \epsilon S_L$ (Nest i-1 is a nest of one point mass within the nest k-1.)

$$X_{k,i} = -m_i \Gamma_{k-1,i} . \tag{64}$$

$X_{k,i}$ is the tensor form of the mass moment of the nest i-1 relative to the hinge point k-1.

c. $k \epsilon S_R$, $i \notin S_{k-1}$ (Nest i-1 is not contained within the nest k-1.)

$$X_{k,i} = 0. \tag{65}$$

d. $k \epsilon S_L$, $i = k$ (Nest i-1 is identical to nest k-1 and contains one point mass.)

$$X_{k,i} = m_i 1 . \tag{66}$$

$X_{k,i}$ is the tensor form of the mass of the nest i-1.

e. $k \epsilon S_L$, $i \notin S_{k-1}$ (Nest i-1 is not contained within the nest k-1.)

$$X_{k,i} = 0. \tag{67}$$

f. $k \epsilon S_R$, $i = N + 1$ (Nest k-1 has at least one rigid body.)

$$X_{k,i} = -\sum_{\lambda \epsilon S_{k-1}} m_\lambda \, \Gamma_{k-1,\lambda}. \tag{68}$$

$X_{k,i}$ is the tensor form of the mass moment of the nest k-1 relative to the hinge point k-1.

g. $k \epsilon S_L$, $i = N + 1$ (Nest k-1 has one point mass.)

$$X_{k,i} = m_k \, 1. \tag{69}$$

$X_{k,i}$ is the tensor form of the mass of the nest k-1.

h. $k = N + 1$, $i = N + 1$

$$X_{k,i} = \sum_{\lambda \epsilon S_0} m_\lambda \, 1. \tag{70}$$

$X_{k,i}$ is the tensor form of the total system mass.

It should be noted that when the tensors defined above are all expressed relative to a common frame of reference, the matrix $[X]$ is symmetric. This is easily proven by application of Equations (50), (51), and (54).

$\lfloor I^c \rfloor$ = $(N + 1) \times M$, rectangular matrix of momentum wheel inertia tensors. The tensor element in row k, column m is from Equations (55) and (56) given by

a. $MO(m) \epsilon S_{k-1}$ (The rigid body in which momentum wheel m is imbedded is within the nest k-1.)

$$I^c_{k,m} = I_{W_m}. \tag{71}$$

b. $MO(m) \notin S_{k-1}$ or $k = N+1$ (Momentum wheel m is not within the nest k-1.)

$$I^c_{k,m} = 0. \tag{72}$$

$\lfloor I^c \rfloor^T$ = $M \times (N + 1)$, rectangular matrix, the transpose of $\lfloor I^c \rfloor$.

$[I^r]$ = $M \times M$, square matrix of momentum wheel inertia tensors. From Equation (58), the tensor element in row m, column n is given by

a. $m = n$

$$I^s_{m,n} = I_{W_m}, \tag{73}$$

b. $m \neq n$

$$I^s_{m,n} = 0. \tag{74}$$

$\{\eta^c\}$ = $(N + 1) \times 1$, column matrix of forces and torques associated with centripetal and Coriolis acceleration effects. Note that the force associated with the mass of body λ and its combined centripetal and Coriolis acceleration can be expressed as

$$\vec{C}_\lambda = m_\lambda \left[\sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i) + \begin{cases} \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) & \lambda \epsilon S_R \\ 2\vec{\omega}_{J(\lambda)} \times \dot{\vec{\alpha}}_\lambda + \vec{\omega}_{J(\lambda)} \times (\vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda) & \lambda \epsilon S_L \end{cases} \right]. \tag{75}$$

From Equations (55), (56), and (57) the vector element η^c_k in row k is

a. $k \epsilon S_R$ (The nest k-1 has at least one rigid body.)

$$\eta^c_k = -\sum_{\lambda \epsilon S_{k-1}} \vec{\gamma}_{k-1,\lambda} \times \vec{C}_\lambda. \tag{76}$$

η^c_k is the resultant torque at hinge point k-1 due to the centripetal and Coriolis acceleration of each body in the nest k-1.

b. $k \epsilon S_L$ (The nest k-1 contains one point mass.)

$$\eta^c_k = -\vec{C}_\lambda. \tag{77}$$

c. $k = N + 1$

$$\eta^c_k = -\sum_{\lambda \epsilon S_0} \vec{C}_\lambda. \tag{78}$$

η^c_{N+1} is the resultant force acting on the composite system due to the centripetal and Coriolis acceleration of each body.

$\{\eta^I\}$ = $(N + 1) \times 1$, column matrix of torques associated with the inertial angular velocity of the body fixed reference frames and the inertial angular momentum of the rigid bodies and gyrostats about their respective centers-of-mass. Note that the inertial angular momentum of body λ about its own center-of-mass is given by

$$\vec{L}_{\lambda,\lambda} = \Phi_\lambda \cdot \vec{\omega}_\lambda + \sum_{\substack{m: \\ MO(m)=\lambda}} \vec{\vec{H}}_m . \tag{79}$$

From Equation (55) the vector element η^I_R in row k is

a. $k \epsilon S_R$ (Nest k-1 contains at least one rigid body.)

$$\eta^I_k = -\sum_{\substack{\lambda \epsilon S_{k-1} \\ \lambda \notin S_L}} \vec{\omega}_\lambda \times \vec{L}_{\lambda,\lambda} . \tag{80}$$

η^I_k is the rate of change of angular momentum of all rigid bodies and gyrostats in the nest k-1 about their own centers-of-mass due to the inertial angular velocity of their respective body fixed-reference frames.

b. $k \epsilon S_L$ or $k = N + 1$

$$\eta^I_k = 0. \tag{81}$$

$\{\eta^W\}$ = $M \times 1$, column matrix of torques associated with the inertial angular velocity of the gyrostats fixed reference frames and the inertial angular momentum of each momentum wheel about its respective center-of-mass. Note that the inertial angular momentum of momentum wheel m imbedded in body λ about its own center-of-mass is given by

$$\vec{L}_{W_m,W_m} = I_{W_m} \cdot \vec{\omega}_\lambda + \vec{\vec{H}}_m . \tag{82}$$

From Equation (58) the vector element in row m is

$$\eta^W_m = -\vec{\omega}_\lambda \times \vec{L}_{W_m,W_m} . \tag{83}$$

η^W_m is the rate of change of angular momentum of momentum wheel m about its own center-of-mass due to the inertial angular velocity of the reference frame fixed in the gyrostat in which it is imbedded.

Also

$\left\{\phi^{C_1}\right\}$ = (N + 1) X 1, column matrix of the forces and torques of constraint,

$\left\{\phi^{H_1}\right\}$ = (N + 1) X 1, column matrix of the forces and torques associated with mechanisms existing between contiguous bodies, and

$\left\{\phi^{E_1}\right\}$ = (N + 1) X 1, column matrix of the forces and torques associated with causes external to the N-body system.

The vector element in row k of

$$\left\{\phi^{C_1}\right\} + \left\{\phi^{H_1}\right\} + \left\{\phi^{E_1}\right\}$$

is

a. $k \epsilon S_R$ (Nest k-1 contains at least one rigid body.)

$$\phi_k^{c1} + \phi_k^{H1} + \phi_k^{E1} = \vec{\phi}_{k-1}^{c} + \vec{\phi}_{k-1}^{H} + \sum_{\lambda \epsilon S_{k-1}} \left[\vec{\gamma}_{k-1,\lambda} \times \vec{F}_{\lambda}^{(e)} + \vec{\phi}_{\lambda}^{(e)} \right]. \tag{84}$$

This is the resultant torque external to the nest k-1 about the hinge point k-1.

b. $k \epsilon S_L$ (Nest k-1 contains one point mass.)

$$\phi_k^{c1} + \phi_k^{H1} + \phi_k^{E1} = \vec{F}_{k-1}^{c} + \vec{F}_{k-1}^{H} + \vec{F}_{k}^{(e)}. \tag{85}$$

This is the resultant force external to the nest k-1.

c. k = N + 1

$$\phi_k^{c1} + \phi_k^{H1} + \phi_k^{E1} = \vec{F}_0^{c} + \vec{F}_0^{H} + \sum_{\lambda \epsilon S_0} \vec{F}_{\lambda}^{(e)}. \tag{86}$$

This is the resultant force external to the composite system.

$\left\{\phi^{C_2}\right\}$ = M X 1, column matrix of the constraint torques acting on the momentum wheels, and

$\left\{\phi^{H_2}\right\}$ = M X 1, column matrix of the torques associated with mechanisms existing between the momentum wheels and the bodies in which they are imbedded.

The vector element in row m of

$$\left\{\phi^{C_2}\right\} + \left\{\phi^{H_2}\right\}$$

is

$$\phi_m^{c_2} + \phi_m^{H_2} = \vec{\phi}^{c}_{W_m} + \vec{CL}_m \,. \tag{87}$$

This is the resultant torque external to and acting on the momentum wheel m.

Making use of the above notation, the simultaneous vector-dyadic equations of motion given by Equations (55) through (58) may be expressed in partitioned matrix form as

$$\left[\begin{array}{c|c} X & I^c \\ \hline I^{c^T} & I^s \end{array}\right] \cdot \left\{\begin{array}{c} \dot{\tilde{\omega}} \\ \hline \dot{\tilde{\omega}}_W \end{array}\right\} = \left\{\begin{array}{c} \eta^c \\ \hline 0 \end{array}\right\} + \left\{\begin{array}{c} \eta^I \\ \hline \eta^W \end{array}\right\} + \left\{\begin{array}{c} \phi^{c_1} \\ \hline \phi^{c_2} \end{array}\right\} + \left\{\begin{array}{c} \phi^{H_1} \\ \hline \phi^{H_2} \end{array}\right\} + \left\{\begin{array}{c} \phi^{E_1} \\ \hline 0 \end{array}\right\}. \tag{88}$$

To solve these equations numerically, they must be put in the form of a set of simultaneous, scalar-differential equations. The difficulty with this step is in solving for or eliminating the unknown forces and torques of constraint.

In theory, if the N-body system has a total of N_T degrees of freedom and N_L holonomic conditions of constraint, then it is possible to generate a set of

$$N_F = N_T - N_L \tag{89}$$

generalized coordinate equations which are independent and completely define the system dynamics.

In Reference 2, Hooker and Margulies present a method for constraint elimination. The method essentially derives expressions for the constraint torques in terms of system parameters and then substitutes them into the equations of motion. From a computational viewpoint, this method is cumbersome since the order of the system of equations remains unchanged and several additional matrix inversions are required.

In Reference 3, Hooker presents another method for constraint elimination which is well suited for digital computation. The method defines a set of N_F free coordinate vectors which span the N_F dimensional vector space in which motion is possible. By application of orthogonality conditions, forces and torques of constraint can be eliminated and a set of N_F simultaneous scalar equations derived which completely define the system's motion in the N_F dimensional vector space.

To accomplish this, Hooker notes that at every hinge point it is possible to define a triad of linearly-independent free and locked coordinate vectors. Physically, if one imagines that contiguous rigid bodies are connected by either a zero-, one-, two-, or three-axis gimbal, the free coordinate vectors correspond to unit vectors fixed along the gimbal axes, and the locked coordinate vectors correspond to unit vectors fixed along axes about which motion is totally constrained.

Since the equations of relative motion are vector equations, at each hinge point the relative velocity vector is expressible in terms of a linear combination of the free coordinate vectors while the constraint torque is expressible in terms of a linear combination of the locked coordinate vectors. These vectors are orthogonal to each other. By forming the appropriate vector scalar products, the scalar equations which define the components of motion about or along each free coordinate vector can be obtained and the need for evaluating the constraint torques circumvented.

To apply Hooker's technique, several kinematic related definitions must be introduced.

KINEMATICS OF THE N-BODY SYSTEM

The subject of kinematics to which this section addresses itself is concerned exclusively with determination of the relative orientation and rate of each body in the N-body system. The causes to which relative motion may be ascribed are not of interest. User convenience and computational efficiency have been the primary factors used in determining which kinematic methods should be applied.

It is the author's opinion that there is no one kinematic technique which is best for all possible problems. Accordingly, the program N-BOD has been programmed to give the user a limited selection of kinematic options. Unless directed otherwise, direction cosine methods are applied. No attempt is made to artificially orthonormalize the computed transformation matrices. At the user's option, an algebraic quaternion method can be used. This is simply a generalization of standard Euler angle techniques. Selective use of both direction cosine and algebraic quaternion methods should permit the user to circumvent the common kinematic problems of orthogonalization and gimbal lock.

At the inertial origin and at every hinge point of the N-body system a reference frame has been defined;

$\left\{X^0_1, X^0_2, X^0_3\right\}$ = reference frame 0, fixed inertially at the inertial origin, and

$\left\{X^\lambda_1, X^\lambda_2, X^\lambda_3\right\}$ = reference frame λ, fixed at hinge point λ-1 of body λ; if body λ is a point mass, the coordinate axes are respectively parallel to those of reference frame $J(\lambda)$.

It is desirable to provide the user with the ability to specify an arbitrary "at rest," or "nominal zero internal stress" orientation for each reference frame. Accordingly let

$\left[{}^{N}_{\lambda}\mathcal{T}_{J(\lambda)}\right]$ = transformation matrix which takes vectors from body $J(\lambda)$ fixed coordinates into body λ fixed coordinates when the system is at rest in the nominal zero internal stress state.

To completely define the at-rest state of the N-body system, translation and momentum wheel orientation conditions also must be defined.

a. The entire N-body system is free to translate relative to the inertial origin;

$\vec{\beta}_1$ = vector from the inertial origin to the origin of reference frame 1 (the center-of-mass of body 1).

b. Point masses may be given an at-rest position relative to their respective contiguous bodies;

$^N\vec{\alpha}_\lambda$ = vector from hinge point λ-1 to the at-rest position of the point mass λ.

c. The rotation angle through which a momentum wheel has rotated can be important for special cases. Rather than introduce a wheel-fixed reference frame, it is defined that the at-rest angle of the wheel, relative to the body in which it is imbedded, is zero. There is no loss of generality here since all momentum wheels are by definition symmetric.

In order to eliminate the forces and torques of constraint in the equations of motion, a triad of unit coordinate vectors is defined at every point in the system about or from which relative motion is measured.

Let

N_0 = total number of coordinate vector triads. For a system of N coupled bodies and M momentum wheels

$$N_0 = N + 1 + M. \tag{90}$$

N_F = total number of free coordinate vectors; these span the N_F dimensional vector space in which motion is possible.

N_L = total number of locked coordinate vectors; these span the N_L dimensional vector space in which motion is totally constrained.

$\left\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_{N_F}\right\}$ = set of unit free coordinate vectors which span the N_F dimensional vector space in which motion is possible.

$\left\{\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_{N_L}\right\}$ = set of unit locked coordinate vectors which span the N_L dimensional vector space in which motion is totally constrained.

$\left\{\theta_1, \theta_2, \ldots, \theta_{N_F}\right\}$ = set of scalar parameters which at any instant of time define the relative displacement about or along each respective free coordinate vector (analogous to Euler angles).

$\left\{\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_{N_F}\right\}$ = set of scalar parameters which define the time rate of change of the respective displacement parameters (analogous to Euler angle rates).

To uniquely define each of the free and locked coordinate vectors, several definitions must be made. In principle the vectors may be randomly labeled; however, for computational purposes it is advantageous to choose a particular numbering sequence.

Each triad of coordinate vectors may be assigned a label. The origin of the triad is the point from or about which relative motion is measured. That is:

a. To measure the relative rotation of rigid body k, the triad k is defined to have its origin at hinge point k-1.

b. To measure the relative translation of point mass k, the triad k is defined to have its origin defined at its at-rest position.

c. To measure the relative translation of the center-of-mass of body 1 relative to the inertial reference, the triad N + 1 is defined to have its origin at the inertial origin.

d. To measure the relative rotation of momentum wheel m, the triad N + 1 + m is defined to have its origin at the center-of-mass of momentum wheel m.

The mix of free and locked coordinate vectors in each triad is defined by the integer function P (λ) where

$P(\lambda)$ = total number of locked coordinate vectors in the triad λ.

The free and locked coordinate vectors which make up triad λ are:

a. $P(\lambda) = 0$ Three degrees of freedom

$$\left\{\vec{q}_j, \vec{q}_{j+1}, \vec{q}_{j+2}\right\},$$

b. $P(\lambda) = 1$ Two degrees of freedom

$$\left\{\vec{q}_j, \vec{q}_{j+1}, \vec{p}_i\right\},$$

c. $P(\lambda) = 2$ One degree of freedom

$$\left\{\vec{q}_j, \vec{p}_i, \vec{p}_{i+1}\right\}, \qquad \text{and}$$

d. $P(\lambda) = 3$ Zero degrees of freedom

$$\left\{\vec{p}_i, \vec{p}_{i+1}, \vec{p}_{i+2}\right\},$$

where for any λ, $\lambda = 1, 2, \ldots, N_0$

$$j = 1 + \sum_{\ell=1}^{\lambda-1} (3 - P(\ell)), \tag{91}$$

$$i = 1 + \sum_{\ell=1}^{\lambda-1} P(\ell), \tag{92}$$

and

$$\sum_{\ell=1}^{0} \equiv 0 . \tag{93}$$

If body λ is a point mass, the free and locked coordinate vectors of the triad λ are mutually orthogonal and fixed with respect to the reference frame λ.

If body λ is a rigid body, then by definition it is connected at hinge point λ-1 to body $J(\lambda)$ by either a zero-, one-, two-, or three-axis gimbal. The numerical ordering of the free coordinate vectors is defined by the following convention:

a. For a one-, two-, or three-axis gimbal, the first rotation is about $\vec{q}_j$ through the angle θ_j. The free vector $\vec{q}_j$ is fixed with respect to the reference frame $J(\lambda)$.

b. For a two- or three-axis gimbal, the second rotation is about $\vec{q}_{j+1}$ through the angle θ_{j+1}. For a two-axis gimbal, the free vector $\vec{q}_{j+1}$ is fixed with respect to reference frame λ. For a three-axis gimbal, the free vector $\vec{q}_{j+1}$ is defined by the vector cross-product

$$\vec{q}_{j+1} = \frac{\vec{q}_{j+2} \times \vec{q}_j}{\left|\vec{q}_{j+2} \times \vec{q}_j\right|} . \tag{94}$$

c. For a three-axis gimbal, the third rotation is about $\vec{q}_{j+2}$ through the angle θ_{j+2}. The free vector $\vec{q}_{j+2}$ is fixed with respect to the reference frame λ.

If body λ is a rigid body, the numerical ordering of the locked coordinate vectors at the hinge point is arbitrary. With the exception of the case of a two-axis gimbal, locked coordinate vectors are fixed in reference frame λ. For a two-axis gimbal, the locked vector $\vec{p}_i$ must be orthogonal to both free vectors $\vec{q}_j$ and $\vec{q}_{j+1}$. Thus, it is defined by the vector cross-product

$$\vec{p}_i = \frac{\vec{q}_j \times \vec{q}_{j+1}}{\left|\vec{q}_j \times \vec{q}_{j+1}\right|} . \tag{95}$$

If $\lambda = N + 1$, the free and locked coordinate vectors are fixed with respect to the inertial reference frame. They are defined by convention to be aligned respectively with the X_1^0, X_2^0, and X_3^0 coordinate axes.

If $\lambda = N + 1 + m$, the free vector is aligned with the spin axis of momentum wheel m which is fixed in reference frame MO(m). The two locked coordinate vectors are fixed normal to each other and to the spin axis in a wheel fixed-reference frame.

INITIAL ORIENTATION OF THE N-BODY SYSTEM

In defining the relative orientation of the bodies in an N-body system at time zero, it is undesirable to start from the hypothesis that the elements of the initial transformation matrices are given. If the response of a complex system under different initial conditions is to be studied, the computation of initial transformation matrices can be tedious. It is desirable to define a set of physically realizable, independent parameters which can be used to construct internally the initial conditions necessary for computation.

The scalar parameters which define relative displacement and rate about and along the free coordinate vectors provide a desirable set of parameters to work with.

Since relative motion takes place only about or along the free coordinate vectors, the initial orientation can be given by stipulating exactly N_F independent relative displacement parameters, one associated with each free coordinate vector. These parameters are taken relative to the nominal zero stress position.

Let

$\left[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}\right]$ = transformation matrix which takes vectors from body $J(\lambda)$ fixed coordinates to body λ fixed coordinates at time zero.

To compute all initial transformation matrices $\left[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}\right]$, $\lambda = 1, 2, \ldots, N$, it is not possible to assume that all gimbal axes will be parallel to body fixed-coordinate axes. It is therefore most convenient to make use of the quaternion techniques reviewed briefly in the appendix to compute the initial transformation matrices.

Making use of the matrix operator $\mathscr{F}$ and the quaternion operator $\mathscr{Q}$ defined in the appendix, the initial value of the transformation matrix $\left[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}\right]$, $\lambda = 1, 2, \ldots, N$ is given by

a. One-axis gimbal, reference frame I_1 aligned with body λ fixed axes

$$\left[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}\right] = \mathscr{F}\left({}_{I_N}\bar{\xi}_{I_1}\right)^{T} \left[{}^{N}_{\lambda}\mathscr{T}_{J(\lambda)}\right] \tag{96}$$

b. Two-axis gimbal, reference frame I_2 aligned with body λ fixed axes

$$\left[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}\right] = \mathscr{F}\left({}_{I_N}\bar{\xi}_{I_1}\ {}_{I_1}\bar{\xi}_{I_2}\right)^{T} \left[{}^{N}_{\lambda}\mathscr{T}_{J(\lambda)}\right], \tag{97}$$

c. Three-axis gimbal

$$\left[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}\right] = \mathscr{F}\left({}_{I_N}\bar{\xi}_{I_1}\ {}_{I_1}\bar{\xi}_{I_2}\ {}_{I_2}\bar{\xi}_{\lambda}\right)^{T}\ \left[{}^{N}_{\lambda}\mathscr{T}_{J(\lambda)}\right], \tag{98}$$

where

$${}_{I_N}\bar{\xi}_{I_1} = \cos\theta_m/2 + \mathscr{Q}\left(\left\{\vec{q}_m\right\}_{I_N}\right)\sin\theta_m/2 \tag{99}$$

$${}_{I_1}\bar{\xi}_{I_2} = \cos\theta_{m+1}/2 + \mathscr{Q}\left(\left\{\vec{q}_{m+1}\right\}_{I_1}\right)\sin\theta_{m+1}/2 \tag{100}$$

$${}_{I_2}\bar{\xi}_{\lambda} = \cos\theta_{m+2}/2 + \mathscr{Q}\left(\left\{\vec{q}_{m+2}\right\}_{I_2}\right)\sin\theta_{m+2}/2\,. \tag{101}$$

SYSTEM ORIENTATION VIA DIRECTION COSINES

To integrate the equations of motion of the system it is convenient to refer all vectors and dyads to a common frame of reference. Depending upon application, this may be either the inertially fixed frame of reference or the coordinate axes fixed in body 1. Generality is retained since the system can always be relabeled so that body 1 does contain the computing frame of reference.

Let

$\left[{}_{\lambda}\mathscr{T}_{c}\right]$ = transformation matrix which takes vectors from the computing frame of reference into body λ fixed coordinates at time t; $t \geqslant 0$, $\lambda = 1, 2, \ldots, N$.

To transform the vector $\vec{R}$ given relative to the computing frame coordinates into body λ fixed coordinates apply

$$\left\{\vec{R}\right\}_{\lambda} = \left[{}_{\lambda}\mathscr{T}_{c}\right]\left\{\vec{R}\right\}_{c}. \tag{102}$$

To transform the tensor **R** given relative to the computing frame coordinates into body λ fixed coordinates apply

$$[R]_{\lambda} = [{}_{\lambda}\mathscr{T}_{c}]\ [R]_{c}\ [{}_{c}\mathscr{T}_{\lambda}], \tag{103}$$

where

$$[{}_{c}\mathscr{T}_{\lambda}] = [{}_{\lambda}\mathscr{T}_{c}]^{T}. \tag{104}$$

If the computing frame is chosen to be the inertial-reference frame, then c = 0 and

$$[{}_{\lambda}\mathcal{T}_0] = \prod_{k\epsilon S_{0,\lambda-1}} [{}_{k}\mathcal{T}_{J(k)}], \tag{105}$$

where

$$\prod_{k\epsilon S_{0,\lambda-1}} [{}_{k}\mathcal{T}_{J(k)}] = [{}_{\lambda}\mathcal{T}_{J(\lambda)}][{}_{J(\lambda)}\mathcal{T}_{J(J(\lambda))}]\cdots[{}_{1}\mathcal{T}_0]. \tag{106}$$

If the computing frame is chosen to be the body 1 fixed-coordinate frame, then c = 1 and

$$[{}_{\lambda}\mathcal{T}_1] = \prod_{\substack{k\epsilon S_{0,\lambda-1} \\ k\neq 1}} [{}_{k}\mathcal{T}_{J(k)}]. \tag{107}$$

Since the components of the N-direction cosine transformation matrices $[{}_{\lambda}\mathcal{T}_c]$ are time varying, they must be continually updated. An expression is required to define the time rate of change of the components of $[{}_{\lambda}\mathcal{T}_c]$. These quantities can then be integrated and the time histories of the matrices defined.

Let

$\vec{R}$ = arbitrary vector fixed in body λ, and

${}_{c}\vec{\omega}_{\lambda}$ = angular velocity of the body λ fixed-reference frame relative to the computing frame of reference.

From vector differential calculus

$$\frac{{}^{c}d\vec{R}}{dt} = \frac{{}^{\lambda}d\vec{R}}{dt} + {}_{c}\vec{\omega}_{\lambda} \times \vec{R}, \tag{108}$$

where the presuperscript on the d/dt operator defines the reference frame in which differentiation is referenced. But $\vec{R}$ is fixed in body λ, therefore

$$\frac{{}^{c}d\vec{R}}{dt} = {}_{c}\vec{\omega}_{\lambda} \times \vec{R}. \tag{109}$$

Let

$\left\{\vec{R}\right\}_c$ = column matrix of the components of $\vec{R}$ relative to the computing frame, and

$\left\{{}_c\vec{\omega}_\lambda\right\}_c$ = column matrix of the components of ${}_c\vec{\omega}_\lambda$ relative to computing frame.

Recall that if

$$\left\{{}_c\vec{\omega}_\lambda\right\}_c = \begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix}_c , \tag{110}$$

then the matrix operator $\mathscr{S}$ can be defined such that

$$\mathscr{S}\left(\left\{{}_c\vec{\omega}_\lambda\right\}_c\right) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} , \tag{111}$$

and vector Equation (109) can be written in matrix notation as

$$\left\{\dot{\vec{R}}\right\}_c = -\mathscr{S}\left(\left\{{}_c\vec{\omega}_\lambda\right\}_c\right)\left\{\vec{R}\right\}_c , \tag{112}$$

where the dot over the matrix implies differentiation of each matrix element with respect to time.

From the definition of $[{}_\lambda\mathscr{T}_c]$ note that

$$\left\{\vec{R}\right\}_\lambda = [{}_\lambda\mathscr{T}_c]\left\{\vec{R}\right\}_c . \tag{113}$$

Since R is fixed in body λ

$$\left\{\dot{\vec{R}}\right\}_\lambda = 0, \tag{114}$$

and

$$\left\{\dot{\vec{R}}\right\}_c = [{}_\lambda\dot{\mathscr{T}}_c]^T\left\{\vec{R}\right\}_\lambda = [{}_\lambda\dot{\mathscr{T}}_c]^T\,[{}_\lambda\mathscr{T}_c]\left\{\vec{R}\right\}_c . \tag{115}$$

It follows from Equations (112) and (114), since the vector $\vec{R}$ is arbitrary and fixed in body λ, that

$$[{}_{\lambda}\mathscr{T}_{c}]^{T}\ [{}_{\lambda}\dot{\mathscr{T}}_{c}] = -\mathscr{S}(\{{}_{c}\vec{\omega}_{\lambda}\}_{c}). \tag{116}$$

Take the transpose of both sides and premultiply these by $[{}_{\lambda}\mathscr{T}_{c}]$ to obtain

$$[{}_{\lambda}\dot{\mathscr{T}}_{c}] = [{}_{\lambda}\mathscr{T}_{c}]\ \mathscr{S}(\{{}_{c}\vec{\omega}_{\lambda}\}_{c}). \tag{117}$$

Note that since

$$[{}_{\lambda}\mathscr{T}_{c}]^{T} \equiv [{}_{c}\mathscr{T}_{\lambda}], \tag{118}$$

Equation (117) can also be expressed as

$$[{}_{c}\dot{\mathscr{T}}_{\lambda}] = -\mathscr{S}(\{{}_{c}\vec{\omega}_{\lambda}\}_{c})\ [{}_{c}\mathscr{T}_{\lambda}]. \tag{119}$$

Equation (117) defines nine differential equations, one for each direction cosine. To completely define the relative orientation of the N-body system, this implies that 9N direction cosine equations must be solved simultaneously with the system equations of motion.

Due to the orthonormality of the direction cosine matrix, only six of the nine equations need be integrated. The solution to these equations can then be used to compute algebraically the three remaining quantities.

SYSTEM ORIENTATION VIA QUATERNION METHODS

The computation of transformation matrices may be accomplished by either integrating a set of kinematic differential equations, such as direction cosines, or algebraically by application of quaternion methods. Rather than set up a series of quaternion differential equations, as is commonly done, it is possible to make use of the free coordinate vectors and the respective angles of rotation to simply construct the required transformation matrices algebraically. It becomes a debatable question as to which technique is optimum for all problems.

To apply quaternion methods, the same procedure used to obtain the initial orientation of the N-body system is used.

That is, from

$$[{}_{\lambda}\mathscr{T}_{c}] = [{}_{\lambda}\mathscr{T}_{J(\lambda)}]\ [{}_{J(\lambda)}\mathscr{T}_{c}] \tag{120}$$

and Equations (96) through (101) it follows that

$$[{}_{\lambda}\mathscr{T}_{c}] = [\mathscr{T}({}_{I_N}\bar{\xi}_{\lambda})]^{T}\ [{}^{N}_{\lambda}\mathscr{T}_{J(\lambda)}]\ [{}_{J(\lambda)}\mathscr{T}_{c}], \tag{121}$$

where

$${}_{I_N}\bar{\xi}_{\lambda} = \begin{cases} {}_{I_N}\bar{\xi}_{I_1} & \text{one-axis gimbal } i_1 = \lambda \\ {}_{I_N}\bar{\xi}_{I_1}\ {}_{I_1}\bar{\xi}_{I_2} & \text{two-axis gimbal } I_2 = \lambda \\ {}_{I_N}\bar{\xi}_{I_1}\ {}_{I_1}\bar{\xi}_{I_2}\ {}_{I_2}\bar{\xi}_{\lambda} & \text{three-axis gimbal .} \end{cases} \tag{122}$$

It should be noted that the above equations reduce to one, two, or three successive Euler-angle rotations when the free coordinate vectors are aligned with coordinate axes in the at-rest state.

RELATIVE RATE AND DISPLACEMENT

At any hinge point k-1, the relative angular or linear rate between the contiguous bodies J(k) and k can be expressed as

$$\vec{\tilde{\omega}}_{k} = \sum_{m@k-1} \dot{\theta}_{m}\ \vec{q}_{m}\ , \tag{123}$$

where the summation is carried over all free coordinate vector indices defined at the origin of the triad k.

If body k is either a point mass having one, two, or three degrees of relative translational freedom or a rigid body having one or two degrees of rotational freedom, the relative displacement parameter θ_{m+i} along or about the free vector $\vec{q}_{m+i}$ is obtained from the equation

$$\theta_{m+i} = \int \dot{\theta}_{m+i}\ dt\ , \tag{124}$$

where

$$\dot{\theta}_{m+i} = \vec{q}_{m+i} \cdot \vec{\tilde{\omega}}_k \qquad i = 0, 1, 2, \tag{125}$$

If body k is a rigid body hinged to body J(k) by a three-axis gimbal, then since the three, free-coordinate vectors are not always mutually orthogonal, the relative rotation parameters θ_{m+i} (i = 0, 1, 2) must be derived as follows.

Let

$$\vec{q}_m \cdot \vec{\tilde{\omega}}_k = \Omega_m = \dot{\theta}_m + c\dot{\theta}_{m+2}, \tag{126}$$

$$\vec{q}_{m+1} \cdot \vec{\tilde{\omega}}_k = \Omega_{m+1} = \dot{\theta}_{m+1}, \tag{127}$$

$$\vec{q}_{m+2} \cdot \vec{\tilde{\omega}}_k = \Omega_{m+2} = c\dot{\theta}_m + \dot{\theta}_{m+2}, \tag{128}$$

and

$$\vec{q}_m \cdot \vec{q}_{m+2} = c. \tag{129}$$

From a simultaneous solution of the above

$$\theta_{m+i} = \int \dot{\theta}_{m+i}\, dt \qquad i = 0, 1, 2, \tag{130}$$

where

$$\dot{\theta}_m = \frac{\Omega_m - c\Omega_{m+2}}{1 - c^2}, \tag{131}$$

$$\dot{\theta}_{m+1} = \Omega_{m+1}, \tag{132}$$

and

$$\dot{\theta}_{m+2} = \frac{\Omega_{m+2} - c\Omega_m}{1 - c^2}. \tag{133}$$

ELIMINATION OF CONSTRAINT TORQUES

The partitioned matrix form of the simultaneous vector-dyadic equations of motion for the coupled N-body system are from Equation (88) given by

$$\left[\begin{array}{c|c} X & I^c \\ \hline I^{cT} & I^s \end{array}\right] \cdot \left\{\begin{array}{c} \dot{\tilde{\omega}} \\ \hline \dot{\tilde{\omega}}_w \end{array}\right\} = \left\{\begin{array}{c} \eta^c \\ \hline 0 \end{array}\right\} + \left\{\begin{array}{c} \eta^I \\ \hline \eta^w \end{array}\right\} + \left\{\begin{array}{c} \phi^{c_1} \\ \hline \phi^{c_2} \end{array}\right\} + \left\{\begin{array}{c} \phi^{H_1} \\ \hline \phi^{H_2} \end{array}\right\} + \left\{\begin{array}{c} \phi^{E_1} \\ \hline 0 \end{array}\right\}. \tag{88}$$

To obtain a solution of this equation, the forces and torques of constraint, defined by the elements of the matrices $\{\phi^{c_1}\}$ and $\{\phi^{c_2}\}$, must be either analytically defined or the equations restructured so as to eliminate them. The latter approach, used by Hooker, is adopted here.

To restructure the equations and eliminate the necessity for evaluating the forces and torques of constraint, the procedure outlined on page 27 in the section entitled Equations of Motion (Matrix Form) is followed.

From Equation (123),

$$\vec{\tilde{\omega}}_k = \sum_{m@k-1} \dot{\theta}_m \vec{q}_m \qquad k = 1, 2, \ldots, N, N+1, \tag{134}$$

where k = N + 1 implies the inertial origin. In a parallel manner

$$\vec{\tilde{\omega}}_{W_m} = \dot{\theta}_j \vec{q}_j \qquad m = 1, \ldots, M, \tag{135}$$

where the index j is defined by Equation (91).

Differentiation yields

$$\dot{\vec{\tilde{\omega}}}_k = \sum_{m@k-1} [\ddot{\theta}_m \vec{q}_m + \dot{\theta}_m \dot{\vec{q}}_m] \tag{136}$$

and

$$\mathring{\vec{\tilde{\omega}}}_{W_m} = \ddot{\theta}_j \vec{q}_j \tag{137}$$

where the closed and open dots imply, as before, differentiation with respect to the inertially fixed and locally fixed reference frames, respectively.

In order to put Equations (134) through (137) into the matrix format required for substitution into Equation (88), the following definitions are made.

$\lfloor q \rfloor$ = (N + 1) X (N_F - M) rectangular matrix of the free coordinate vectors. The vector element in row k, column m is

a. k = 1, 2, . . . N

$$q_{k,m} = \begin{cases} \vec{q}_m & \text{if free vector m is defined between bodies J(k) and k} \\ 0 & \text{if not.} \end{cases} \tag{138}$$

b. $k = N + 1$

$$q_{k,m} = \begin{cases} \vec{q}_m & \text{if free vector m is defined at the inertial origin} \\ 0 & \text{if not.} \end{cases} \tag{139}$$

$[h]$ = M X M square matrix of the free coordinate vectors existing along momentum-wheel spin axes. The vector element in row m, column n is

$$h_{m,n} = \begin{cases} \vec{q}_{N_F - M + m} & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases} \tag{140}$$

$\{\dot{\theta}\}$ = $(N_F - M) \times 1$ column matrix of scalar rates about or along the free coordinate vectors existing between bodies and at the inertial reference. The scalar element in row m is $\dot{\theta}_m$.

$\{\dot{\theta}_W\}$ = M X 1 column matrix of momentum-wheel relative rates. The scalar element in row m is $\dot{\theta}_{W_m}$.

Making use of the matrix notation, one may write

$$\begin{Bmatrix} \tilde{\omega} \\ \tilde{\omega}_W \end{Bmatrix} = \left[\begin{array}{c|c} q & 0 \\ \hline 0 & h \end{array}\right] \begin{Bmatrix} \dot{\theta} \\ \dot{\theta}_W \end{Bmatrix} \tag{141}$$

and

$$\begin{Bmatrix} \dot{\tilde{\omega}} \\ \dot{\tilde{\omega}}_W \end{Bmatrix} = \left[\begin{array}{c|c} q & 0 \\ \hline 0 & h \end{array}\right] \begin{Bmatrix} \ddot{\theta} \\ \ddot{\theta}_W \end{Bmatrix} + \left[\begin{array}{c|c} \dot{q} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{Bmatrix} \dot{\theta} \\ \dot{\theta}_W \end{Bmatrix}. \tag{142}$$

To eliminate the forces and torques of constraint, note that at every point from or about which relative motion is measured, the constraint vector is normal to all free coordinate vectors defined there. Thus it follows that the matrix vector-scalar product equation

$$\left[\begin{array}{c|c} q & 0 \\ \hline 0 & h \end{array}\right]^T \cdot \begin{Bmatrix} \phi^{C_1} \\ \hline \phi^{C_2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \hline 0 \end{Bmatrix} \tag{143}$$

can be used to eliminate the forces and torques of constraint from Equation (88) and restructure it into a set of simultaneous scalar equations.

Direct application of Equations (141) through (143) in Equation (88) yields

$$\left[\begin{array}{c|c} q & 0 \\ \hline 0 & h \end{array}\right]^T \cdot \left[\begin{array}{c|c} X & I^c \\ \hline I^{c^T} & I^s \end{array}\right] \cdot \left[\begin{array}{c|c} q & 0 \\ \hline 0 & h \end{array}\right] \left\{\begin{array}{c} \ddot{\theta} \\ \hline \ddot{\theta}_w \end{array}\right\}$$

$$= \left[\begin{array}{c|c} q & 0 \\ \hline 0 & h \end{array}\right]^T \cdot \left\{ -\left[\begin{array}{c|c} X & I^c \\ \hline I^{c^T} & I^s \end{array}\right] \cdot \left[\begin{array}{c|c} \dot{q} & 0 \\ \hline 0 & 0 \end{array}\right] \left\{\begin{array}{c} \dot{\theta} \\ \hline \dot{\theta}_w \end{array}\right\} + \left\{\begin{array}{c} \eta^c \\ \hline 0 \end{array}\right\} \right.$$

$$\left. + \left\{\begin{array}{c} \eta^I \\ \hline \eta^w \end{array}\right\} + \left\{\begin{array}{c} \phi^{H_1} \\ \hline \phi^{H_2} \end{array}\right\} + \left\{\begin{array}{c} \phi^{E_1} \\ \hline 0 \end{array}\right\} \right\}. \qquad (144)$$

Equation (144) defines the N_F scalar equations which completely define the motion of the coupled N-body system. These equations have been programmed for numerical solution and form the basis for the digital computer program N-BOD.

By intent, the only elements in Equation (144) which have not been extensively discussed are the elements of the matrices

$$\left\{\phi^{H_1}\right\}, \left\{\phi^{H_2}\right\}, \left\{\phi^{E_1}\right\}.$$

The elements of these matrices cannot in general be generalized. They define the forces and torques due to mechanisms existing between contiguous bodies, such as springs, dashpots, motors, and so forth and the forces and torques associated with effects external to the N-body system. The inclusion of several effects often encountered will be discussed in the following sections.

FORCES AND TORQUES DUE TO NONGYROSCOPIC EFFECTS

Most problems of practical interest involve the inclusion of one or more nongyroscopic effects into the equations of motion. For example:

a. Contiguous bodies may be viscoelastically coupled at their respective hinge points.

b. Control systems may exist which, based upon a given set of control laws, activate motors that drive contiguous bodies relative to each other or alter the angular rates of momentum wheels.

c. Thrusters controlled by an active control system may exist at various points in the N-body system.

d. The system may be disturbed by one or more forms of environmental loading such as that due to gravity, gravity gradient, thermal, solar pressure, aerodynamics, etc.

It is a relatively simple matter to include any or all of these effects in the matrices

$$\left\{\phi^{H_1}\right\}, \left\{\phi^{H_2}\right\}, \left\{\phi^{E_1}\right\}$$

defined in Equation (144).

ELASTIC COUPLING OF CONTIGUOUS BODIES

Springs can be placed between any pair of contiguous bodies. These restrict relative rotational motion for rigid bodies and relative translational motion for point masses. The springs may be either linear or nonlinear; however, for simplicity, this discussion considers only linear springs which restrict motion about or along free coordinate axes.

Consider the nest k - 1. Note that at every hinge point between bodies contained within the nest k - 1 spring forces and torques appear in equal and opposite pairs and thus exactly cancel each other. At hinge point k - 1, however, the reactive spring torque (force) is external to the nest k - 1 and must be treated as an external disturbance.

Assume that $\vec{q}_m$ is a free coordinate vector defined at hinge point k - 1 and that the motion about or along $\vec{q}_m$ is restrained by a linear spring.

Let

K_m = spring constant of the linear spring which restrains motion about or along the free coordinate vector $\vec{q}_m$;

then

$-K_m \theta_m \vec{q}_m$ = spring torque (force) associated with the relative angular (translational) displacement θ_m of the contiguous bodies k and J(k) about (or along) the free coordinate vector $\vec{q}_m$.

It follows that the net reactive spring torque (force) at hinge point k - 1 is

$$-\sum_{m@k-1} K_m \theta_m \vec{q}_m \qquad k = 1, 2, \ldots, N+1, \tag{145}$$

where the summation extends over all free coordinate indices m defined at hinge point k - 1 of body k, for k = N + i the free coordinate indices defined at the inertial origin are used.

This vector quantity must be added to the k^{th} row of $\left\{\phi^{H_1}\right\}$.

Similarly, if the momentum wheel n is elastically coupled to the body in which it is imbedded, the spring torque

$$-K_m \theta_{W_m} \vec{q}_m, \tag{146}$$

where

$$m = N_F - M + n, \tag{147}$$

must be added to the n^{th} row of $\left\{\phi^{H_2}\right\}$.

DISSIPATIVE COUPLING OF CONTIGUOUS BODIES

Dampers can be placed between any pair of contiguous bodies to retard relative motion. For simplicity, only linear viscous-type damping mechanisms are considered. Nonlinear dissipative devices such as hysterisis or Coulomb friction dampers can be incorporated in the formalism; however, their modeling can become quite complex.

Consider the nest k - 1. As in the case of springs, the only reactive damper force or torque acting external to the nest is the one defined at hinge point k - 1.

Assume that $\vec{q}_m$ is a free coordinate vector defined at hinge point k - 1 and that the relative motion about or along $\vec{q}_m$ is retarded by a linear viscous damper.

Let

D_m = damping coefficient of the linear viscous damper which retards motion about or along the free coordinate vector $\vec{q}_m$;

then

$-D_m \dot{\theta}_m \vec{q}_m$ = damping torque (force) associated with relative angular (translational) velocity $\dot{\theta}_m$ of the contiguous bodies k and J(k) about (along) the free coordinate vector $\vec{q}_m$.

It follows that the net damping torque (force) at hinge point k - 1 is

$$-\sum_{m@k-1} D_m \dot{\theta}_m \vec{q}_m \qquad k = 1, 2, \ldots, N+1. \tag{148}$$

This vector quantity must be added to the k^{th} row of $\left\{\phi^{H_1}\right\}$.

Similarly, if the momentum wheel n is dissipatively coupled to the body in which it is imbedded, the damping torque

$$-D_m \dot{\theta}_{W_m} \vec{q}_m, \tag{149}$$

where

$$m = N_F - M + n, \tag{150}$$

must be added to the n^{th} row of $\left\{\phi^{H_2}\right\}$.

MOTOR COUPLING OF CONTIGUOUS BODIES

Motors may be used to actively control the relative orientations of contiguous bodies; they may also be used to control the relative rates of momentum wheels. In most practical problems of interest, the motor torques are defined as the outputs of an active control system; the control system makes use of various system state variables and control laws to define the appropriate motor torques required to achieve a predefinable objective.

Let $\vec{q}_m$ be the free coordinate vector between bodies J(k) and k about which a motor exists. Furthermore, let

CL_m = scalar magnitude of the motor torque to be applied by the motor about the free coordinate vector $\vec{q}_m$;

then

$CL_m \vec{q}_m$ = motor torque applied about free vector $\vec{q}_m$ to body k.

It follows that the net motor torque at hinge point k-1 is

$$\sum_{m@k-1} CL_m \vec{q}_m \qquad k = 1, 2, \ldots, N+1. \tag{151}$$

This vector quantity must be added to the k^{th} row of $\left\{\phi^{H_1}\right\}$.

Similarly, if the momentum wheel n is coupled by a motor to the body in which it is imbedded, the motor torque

$$CL_m \vec{q}_m, \tag{152}$$

where

$$m = N_F - M + n, \tag{153}$$

must be added to the n^{th} row of $\left\{\phi^{H_2}\right\}$.

LOCALLY APPLIED FORCES

The type of locally applied forces which most often occur in satellite simulation are those attributable to gas jet firings. The location and force of these gas jets are definable. Let

$\vec{R}_{\lambda,J}$ = radius vector from the center-of-mass of body λ to gas jet J which is located on body λ.

$\vec{F}^{(e)}_{\lambda,J}$ = force associated with gas jet J of body λ when fired.

It is perfectly admissible for the firing of these jets to be governed by a control law which is a function of the relative attitude motion of the system. It is also admissible for the force to build up as some definable function of time.

From Equations (32), (33), and (34) it can be seen that the thruster acting on body λ produces a force which is external to every nest of bodies containing the body λ. The vector quantity which must be added to row k of $\left\{\phi^{E_1}\right\}$ is

$$\begin{cases} \vec{0} & \text{if } \lambda \notin S_{k-1} \\ \vec{F}^{(e)}_{\lambda,J} & \text{if } \lambda \epsilon S_{k-1} \text{ and } k \epsilon S_L \\ (\vec{\gamma}_{k-1,\lambda} + \vec{R}_{\lambda,J}) \times \vec{F}^{(e)}_{\lambda,J} & \text{if } \lambda \epsilon S_{k-1} \text{ and } k \epsilon S_R \\ \vec{F}^{(e)}_{\lambda,J} & \text{if } k = N + 1 \end{cases} \tag{154}$$

where $k = 1, 2, \ldots, N + 1$.

DISTRIBUTED FORCE FIELD

Environmental loading due to gravity, aerodynamics, solar pressure, and so forth produces a force field which is distributed over the N-body system.

From Equations (18) and (19) let

$\vec{F}^{(e)}_{\lambda}$ = resultant force acting on body λ due to the distributed force field, and

$\vec{\phi}_\lambda^{(e)}$ = resultant torque acting on body λ due to the distributed force field.

Again, it follows from Equations (32), (33), and (34) that the vector quantity which must be added to row k of $\{\phi^{E1}\}$ is

$$\begin{cases} F_\lambda^{(e)} & \lambda = k,\ k \epsilon S_L \\ \sum_{\lambda \epsilon S_{k-1}} \left[\vec{\gamma}_{k-1,\lambda} \times F_\lambda^{(e)} + \vec{\phi}_\lambda^{(e)} \right] & k \epsilon S_R \\ \sum_{\lambda \epsilon S_0} F_\lambda^{(e)} & k = N+1 . \end{cases} \tag{155}$$

For the particular example of an earth based system subject to a uniform gravitational force field, the gravity force acting on body λ is

$$\vec{F}_\lambda^{(e)} = - m_\lambda g \hat{\vec{\beta}}_0 , \tag{156}$$

where

g = acceleration of gravity,

$\hat{\vec{\beta}}_0$ = unit vector directed from the earth's center to the center-of-mass of the composite N-body system.

Furthermore, since the gravitation force is distributed uniformly through the entire volume of each body,

$$\vec{\phi}_\lambda^{(e)} = 0. \tag{157}$$

In this situation, $\vec{\beta}_1$ is measured from the inertial origin which does not necessarily have to be at the earth's center. $\hat{\vec{\beta}}_0$ is simply a unit vector, which can be expressed in any coordinate system that is constant in the inertial reference frame.

ORBIT DEFINITION

To incorporate into the formalism the ability to study gravity-gradient effects or to include earth, sun, or star sensors in an active on-board control system simulation, a rudimentary definition of the orbit must be made.

Assume that the satellite is in an elliptic orbit around a spherical earth (see figure 3). To define the orbit, the following quantities must be given:

a = semi-major axis of elliptic orbit,

e = orbit eccentricity, and

T_e = time of perihelion passage.

From these quantities and elementary orbital mechanics, one may compute

β_0 = distance from earth center to composite system center-of-mass,

v = true anomaly.

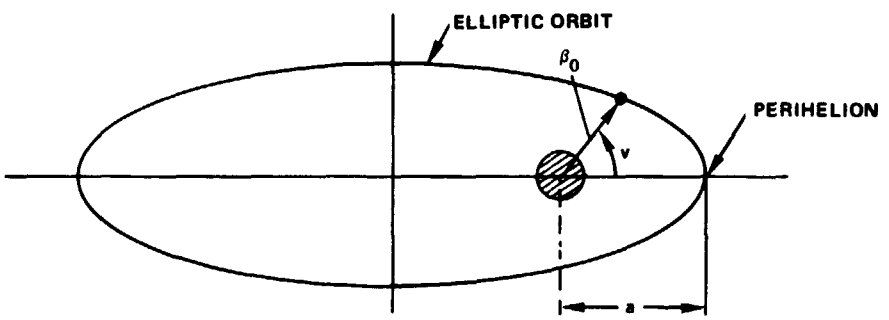

Figure 3. Elliptic Orbit Notation

Let

G_e = earth's gravitational constant;

then the orbit period P_e is

$$P_e = 2\pi\sqrt{\frac{a^3}{G_e}}, \tag{158}$$

the mean motion η_e is

$$\eta_e = \frac{2\pi}{P_e}, \tag{159}$$

and the mean anomaly M_e is

$$M_e = \eta_e (t - T_e). \tag{160}$$

From Kepler's law, the eccentric anomaly E_e is given by

$$M_e = E_e - e \sin E_e . \tag{161}$$

It has been shown in various texts that

$$\beta_0 = a(1 - e \cos E_e) = \frac{a(1 - e^2)}{1 + e \cos v}, \tag{162}$$

and

$$\tan \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E_e}{2}, \tag{163}$$

or

$$\cos v = \frac{\cos E_e - e}{1 - e \cos E_e}, \tag{164}$$

and

$$\sin v = \frac{\sqrt{1 - e^2} \sin E_e}{1 - e \cos E_e} . \tag{165}$$

The problem is to obtain E_e as a function of M_e. By the method of Lagrange, Moulten (Reference 8) shows that

$$\begin{aligned} E_e = M_e &+ e \sin M_e + \frac{e^2}{2} \sin 2M_e \\ &+ \frac{e^3}{3!\,2^2} (3^2 \sin 3M_e - 3 \sin M_e) \\ &+ \frac{e^4}{4!\,2^3} (4^3 \sin 4M_e - 4 \cdot 2^3 \sin 2M_e) \\ &+ \frac{e^5}{5!\,2^4} (5^4 \sin 5M_e - 5 \cdot 3^4 \sin 3M_e + 10 \sin M_e) \\ &+ \frac{e^6}{6!\,2^5} (6^5 \sin 6M_e - 6 \cdot 4^5 \sin 4M_e + 15 \cdot 2^5 \sin 2M_e) + \ldots . \end{aligned} \tag{166}$$

The expression converges rapidly for small values of eccentricity and can be suitably truncated in application.

In any reference frame fixed in time, relative to an earth fixed-reference frame, the vector $\vec{\beta}_0$ from the earth's center to the composite center-of-mass of the N-body system may be written as

$$\vec{\beta}_0 = \beta_0 \hat{\vec{\beta}}_0, \tag{167}$$

where the magnitude of β_0 is defined above and the components of the unit vector $\hat{\vec{\beta}}_0$ are trigonometric functions of v, the true anomaly.

GRAVITATIONAL FORCE FIELD FOR ORBITING SPACECRAFT

For large orbiting spacecraft, it is improper to assume that the gravitational force is uniformly distributed through the entire structure. For a significant class of problems, gravity-gradient effects can influence the attitude dynamics of the spacecraft.

From Newton's law of gravitation, the gravitational force $\vec{F}^G_{\lambda,i}$ acting upon the mass element $m_{\lambda,i}$ of the body λ in the N-body model of the spacecraft is

$$\vec{F}^G_{\lambda,i} = -G_e m_{\lambda,i} \frac{\vec{\gamma}_{I,\lambda,i}}{(\vec{\gamma}_{I,\lambda,i} \cdot \vec{\gamma}_{I,\lambda,i})^{3/2}}, \tag{168}$$

where

G_e = earth's gravitational constant,

$\vec{\gamma}_{I,\lambda,i}$ = vector from the earth's center-of-mass to the mass element $m_{\lambda,i}$.

Let

$\vec{\beta}_0$ = vector from the earth's center-of-mass to the composite N-body system's center-of-mass,

$\vec{\delta}_\lambda$ = vector from composite N-body system's center-of-mass to the center of mass of body λ, and

$\vec{R}_{\lambda,i}$ = vector from the center-of-mass of body λ to the mass element $m_{\lambda,i}$.

Then

$$\vec{\gamma}_{I,\lambda,i} = \vec{\beta}_0 + \vec{\delta}_\lambda + \vec{R}_{\lambda,i}. \tag{169}$$

By substitution of a truncated binomial-series expansion for the denominator of Equation (168) it follows that

$$\vec{F}^{G}_{\lambda,i} = -G_e m_{\lambda,i} \beta_0^{-3} \left[1 - 3 \frac{(\vec{\beta}_0 \cdot (\vec{\delta}_\lambda + \vec{R}_{\lambda,i}))}{\beta_0^2}\right] \cdot (\vec{\beta}_0 + \vec{\delta}_\lambda + \vec{R}_{\lambda,i}) \tag{170}$$

Conversion to vector-dyadic notation and deletion of second-order terms yields

$$\vec{F}^{G}_{\lambda,i} = \vec{F}^{G0}_{\lambda,i} + \Delta\vec{F}^{G}_{\lambda,i}, \tag{171}$$

where

$$\vec{F}^{G0}_{\lambda,i} = -G_e m_{\lambda,i} \beta_0^{-2} \hat{\vec{\beta}}_0 \tag{172}$$

and

$$\Delta\vec{F}^{G}_{\lambda,i} = -G_e m_{\lambda,i} \beta_0^{-3} (1 - 3\hat{\vec{\beta}}_0 \hat{\vec{\beta}}_0) \cdot (\vec{\delta}_\lambda + \vec{R}_{\lambda,i}). \tag{173}$$

The resultant gravitational force and torque acting on body λ is obtained by a summation over all mass elements contained within the body.

$$\vec{F}^{G0}_{\lambda} = \sum_{i\epsilon\lambda} \vec{F}^{G0}_{\lambda,i} = -G_e m_\lambda \beta_0^{-2} \hat{\vec{\beta}}_0 \tag{174}$$

$$\Delta\vec{F}^{G}_{\lambda} = \sum_{i\epsilon\lambda} \Delta\vec{F}^{G}_{\lambda,i} = -G_e m_\lambda \beta_0^{-3} (1 - 3\hat{\vec{\beta}}_0 \hat{\vec{\beta}}_0) \cdot \vec{\delta}_\lambda \tag{175}$$

$$\vec{\phi}^{G0}_{\lambda} = \sum_{i\epsilon\lambda} \vec{R}_{\lambda,i} \times \vec{F}^{G0}_{\lambda,i} = 0 \tag{176}$$

$$\Delta\vec{\phi}^{G}_{\lambda} = \sum_{i\epsilon\lambda} \vec{R}_{\lambda,i} \times \Delta\vec{F}^{G}_{\lambda,i} = 3 G_e \beta_0^{-3} \hat{\vec{\beta}}_0 \times \Phi_\lambda \cdot \hat{\vec{\beta}}_0 \tag{177}$$

where

$$\Phi_\lambda = \sum_{i\epsilon\lambda} m_{\lambda,i} [(\vec{R}_{\lambda,i} \cdot \vec{R}_{\lambda,i}) 1 - \vec{R}_{\lambda,i} \vec{R}_{\lambda,i}] \tag{178}$$

and

$$m_\lambda = \sum_{l \epsilon \lambda} m_{\lambda,l} . \tag{179}$$

For any simulation problem in which gravitational effects must be considered, two approaches are available.

a. The inertially fixed reference is chosen to be fixed at the earth's center-of-mass.

Then recall

$\vec{\beta}_1$ = vector from inertial origin to center of mass of body 1,

hence

$$\vec{\beta}_0 = \vec{\beta}_1 - \vec{\delta}_1 . \tag{180}$$

From Equation (154), the vector quantity to be added to row k of $\left\{\phi^{E_1}\right\}$ is

$$\begin{cases} \vec{F}_\lambda^{G_0} + \Delta\vec{F}_\lambda^{G} & \lambda = k,\ k \epsilon S_L \\ \sum_{\lambda \epsilon S_{k-1}} [\vec{\gamma}_{k-1,\lambda} \times (\vec{F}_\lambda^{G_0} + \Delta\vec{F}_\lambda^{G}) + \Delta\vec{\phi}_\lambda^{G}] & k \epsilon S_R \\ \sum_{\lambda \epsilon S_0} \vec{F}_\lambda^{G_0} & k = N + 1 . \end{cases} \tag{181}$$

This approach simultaneously solves for both the orbit and the attitude dynamics of the modeled spacecraft. It should be recognized, however, that digital solutions may be subject to significant numerical error, since the orbit parameters will differ from the attitude-dynamics parameters by several orders of magnitude.

b. The inertially fixed reference is chosen to be fixed at the composite system center-of-mass. In this approach, the orbit position vector $\vec{\beta}_0$ is defined by Equations (162) through (166), and

$$\vec{\beta}_1 = \vec{\delta}_1 . \tag{182}$$

Since $\vec{\beta}_0$ is known, $\vec{F}_\lambda^{\,G_0}$ must be deleted from Equation (181) to obtain the perturbing force due to gravity-gradient effects alone. Then from Equation (181), the vector quantity to be added to row k of $\{\phi^{E1}\}$ is

$$\begin{cases} \Delta\vec{F}_\lambda^G & \lambda = k,\ k\epsilon S_L \\ \sum_{\lambda\epsilon S_{k-1}} [\,\vec{\gamma}_{k-1,\lambda} \times \Delta\vec{F}_\lambda^G + \Delta\vec{\phi}_\lambda^G\,] & k\epsilon S_R \\ 0 & k = N+1\,. \end{cases} \tag{183}$$

THERMALLY INDUCED MOTION

The effects of appendage deformation due to time-varying thermal gradients can under certain conditions adversely influence the attitude dynamics and at times even the attitude stability of spacecraft. For example see Reference 9.

In a directional (solar) thermal field thermal gradients across the diameter of an appendage are not established instantaneously, but grow exponentially to a steady state value with a specific thermal time constant. The thermal gradient distribution along with the thermal expansion properties of the appendage define an instantaneous position of thermal equilibrium. If the appendage has finite mass and stiffness characteristics the rate at which it will actually move to the thermal equilibrium position will be governed by its natural frequencies of vibration.

Virtually all spacecraft attitude dynamics problems attributable to the effects of thermally-induced deformation stem from the fact that, relative to the appendage, the direction and magnitude of the solar thermal field changes at a system natural frequency. This change in thermal input can be caused by such effects as three-axis rotational motion of the spacecraft, shadowing, or torsional motion of a torsionally weak boom.

To investigate such problems a crude model of thermal deformation is usually sufficient for worst case type analyses. The particular modeling tool which has been successfully employed by the author, within the confines of N-BOD, is a thermal spring; that is, a spring which has a time-varying thermal equilibrium position.

Assume that body k has a tendency to thermally deform and that the deformation can be adequately modeled as an angular rotation of the body about the free coordinate vector $\vec{q}_m$

Let

θ_m^T (t) = angular amount body k must rotate at time t about free coordinate vector $\vec{q}_m$ to be in a state of thermal equilibrium.

Furthermore let the position of thermal equilibrium θ_m^T be governed by the solution of the heat conduction equation

$$\frac{d}{dt}\theta_m^T + \frac{1}{\tau_m}\theta_m^T = \frac{\theta_m^{ST}}{\tau_m} f_m(\theta_1, \ldots, \theta_{N_F}) \qquad (184)$$

where

τ_m = thermal time constant;

θ_m^{ST} = a steady state angle of thermal deformation about $\vec{q}_m$ (calculated assuming constant thermal field normal to $\vec{q}_m$); and

$f_m(\theta_1, \ldots, \theta_{N_F})$ = a function of kinematics which defines the changing magnitude and direction of the thermal field relative to the body k fixed reference.

Then if

K_m = spring constant of the linear spring which restrains motion about the free coordinate vector $\vec{q}_m$,

$-K_m(\theta_m - \theta_m^T)\vec{q}_m$ = thermal spring torque about the free coordinate vector $\vec{q}_m$ which tends to drive body k to its position of thermal equilibrium.

It follows that the net reactive thermal spring torque at hinge point k-1 is

$$-\sum_{m\,@\,k\text{-}1} K_m(\theta_m - \theta_m^T)\vec{q}_m \qquad k = 1, 2, \ldots, N \qquad (185)$$

This vector quantity must be added to the k^{th} row of $\{\phi^{H_1}\}$.

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SYMBOLS

a	Semimajor axis of elliptic orbit
$\vec{C}_\lambda$	Force associated with the mass of body λ and its combined centripetal and Coriolis acceleration
$\vec{CL}_m$	Control torque about free coordinate vector $\vec{q}_m$
D_m	Damping coefficient of linear viscous damper which retards motion about free coordinate vector $\vec{q}_m$
E_e	Eccentric anomaly
e	Eccentricity of elliptic orbit
$\vec{F}_k$	Resultant of external forces acting on body k
$\vec{F}^c_{k-1}$	Resultant force of constraint acting on body k through hinge point k-1
$\vec{F}^H_{k-1}$	Resultant force acting on body k due to all mechanisms existing between bodies J(k) and k at hinge point k-1
$\vec{F}^{(e)}_{k,i}$	Resultant external force applied to point i of body k
$\vec{F}^{(e)}_k$	Resultant of all external forces applied to body k
$\mathscr{F}({}_\lambda\bar{\xi}_{J(\lambda)})$	Matrix operator which uses the components of quaternion ${}_\lambda\bar{\xi}_{J(\lambda)}$ to define transformation matrix $[{}_\lambda\mathscr{T}_{J(\lambda)}]$.
G_e	Earth's gravitational constant
$G^\lambda_{k-1,i-1}$	Pseudo-inertia tensor of body λ with respect to the origin of nest k - 1 and the hinge point i - 1
$\vec{G}_{I,\lambda}$	Linear momentum of body λ relative to the inertial origin
g	Acceleration of gravity
$\vec{H}_m$	Inertial angular momentum of momentum wheel m
$\vec{\tilde{H}}_m$	Relative angular momentum of momentum wheel m
$[h]$	(M X M), square matrix of free coordinate vectors along momentum wheel spin axes

I_{W_m}	Inertia tensor of momentum wheel m about its center-of-mass
$[I^c]$	(N + 1) X M, rectangular matrix of momentum-wheel inertia tensors
$[I^s]$	M X M, square matrix of momentum-wheel inertia tensors
$J(\lambda)$	Body label of the body to which body λ is attached at hinge point λ-1
K_m	Spring constant of linear spring which restrains motion about free coordinate vector $\vec{q}_m$
$\vec{L}_{I,\lambda}$	Inertial angular momentum of body λ relative to the inertial origin
$\vec{L}_{W_m,W_m}$	Inertial angular momentum of momentum wheel m relative to its center-of-mass
$\vec{L}_{\lambda,\lambda}$	Inertial angular momentum of body λ relative to its center-of-mass
M	Total number of momentum wheels
MO(m)	Body label of the gyrostat in which momentum wheel m is embedded
m_λ	Total mass of body λ
$m_{\lambda,i}$	Mass of mass element i of body λ
M_e	Mean anomaly
N	Total number of rigid bodies, gyrostats, and point masses
N_F	Total number of free coordinate axes
N_L	Total number of locked coordinate axes
N_o	Total number of coordinate vector triads
N_T	Total number of free and locked coordinate axes
P_e	Orbital period
$P(\lambda)$	Total number of constrained axes at hinge point λ-1 of body λ
$\{\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_{N_L}\}$	Set of unit locked coordinate vectors which span the N_L-dimensional vector space in which motion is totally constrained
$\{\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_{N_F}\}$	Set of unit free coordinate vectors which span the N_F-dimensional vector space in which motion is possible
$\mathcal{Q}$	Quaternion operator, which maps vectors into quaternions

$\lfloor q \rfloor$	$(N + 1) \times (N_F - M)$, rectangular matrix of free coordinate vectors
$\vec{R}_{\lambda,i}$	Position vector from center-of-mass of body λ to mass point i of body λ
S_R	Set of all rigid-body, body labels
S_L	Set of all point-mass body labels
S	Set of all body labels
S_{k-1}	Set of all body labels of those bodies outboard of hinge point k - 1, relative to body 1. Body labels of those bodies in nest k - 1
$S_{k-1,\ \lambda-1}$	Set of all body labels associated with those bodies lying on the topological path from hinge point k - 1 to the center-of-mass of body λ
$\mathscr{S}$	Tensor operator which maps vectors into skew symmetric tensors of rank 2, dyads
T_e	Time of perihelion passage
$[{}_{\lambda}\mathscr{T}_{c}\ ]$	Transformation matrix, takes vectors from computing frame to body λ fixed coordinates
$[{}^{N}_{\lambda}\mathscr{T}_{J(\lambda)}]$	Transformation matrix, takes vectors from body $J(\lambda)$ to body λ fixed coordinates in the nominal zero stress state
$[{}^{0}_{\lambda}\mathscr{T}_{J(\lambda)}]$	Transformation matrix, takes vectors from body $J(\lambda)$ to body λ fixed coordinates at time zero
v	True anomaly
$\mathbf{X}_{k,i}$	Dyad in the k^{th} row, i^{th} column of [X]
[X]	$(N + 1) \times (N + 1)$, matrix of inertia and pseudo-inertia dyads
(X^0_1, X^0_2, X^0_3)	Coordinate axes defined at inertial origin
$(X^\lambda_1, X^\lambda_2, X^\lambda_3)$	Coordinate axes fixed in body λ at hinge point λ - 1
$\vec{\alpha}_\lambda$	Position vector from hinge point λ - 1 of body λ to the center of mass of body λ
$\vec{\alpha}^N_\lambda$	Position vector from hinge point λ - 1 to nominal position of point mass λ
$\vec{\beta}_0$	Position vector from inertial origin to composite N-body system center-of-mass

$\hat{\vec{\beta}}_0$	Unit vector aligned with $\vec{\beta}_0$
$\vec{\beta}_1$	Position vector from inertial origin to hinge point 0 of body 1
$\vec{\beta}_\lambda$	Position vector from hinge point $J(\lambda) - 1$ to hinge point $\lambda - 1$ of body λ
$\vec{\gamma}_{I,\lambda}$	Position vector from inertial origin to center-of-mass of body λ
$\vec{\gamma}_{k,\lambda}$	Position vector from hinge point k to the center-of-mass of body λ
$\Gamma_{k,\lambda}$	Skew symmetric tensor form of the vector $\vec{\gamma}_{k,\lambda}$
$\vec{\delta}_\lambda$	Position vector from composite N-body system center-of-mass to the center-of-mass of body λ
${}_\lambda\bar{\xi}_{J(\lambda)}$	Rotation quaternion
η_e	Mean motion
$\{\eta^c\}$	$(N + 1) \times 1$, column matrix of forces and torques associated with centripetal and Coriolis acceleration effects
$\{\eta^I\}$	$(N + 1) \times 1$, column matrix of torques associated with the inertial angular momentum of rigid bodies about their own centers-of-mass
$\{\eta^W\}$	$M \times 1$, column matrix of torques associated with the inertial angular momentum of momentum wheels about their own centers-of-mass
θ_m	Displacement about or along free coordinate vector $\vec{q}_m$
$\{\dot{\theta}\}$	$(N_F - M) \times 1$, column matrix of all $\dot{\theta}_m$ $m = 1, 2, \ldots, N_F - M$
$\{\dot{\theta}_W\}$	$(M \times 1)$, column matrix of all $\dot{\theta}_m$ $m = N_F - M + 1, \ldots, N_F$
Φ_λ	Inertia tensor of body λ about its center-of-mass
$\vec{\phi}_k^{(e)}$	Resultant torque acting on body k due to external causes
$\vec{\phi}_k$	Resultant of external torques acting on body k
$\vec{\phi}_{W_m}$	Resultant of external torques acting on momentum wheel m
$\vec{\phi}_{k-1}^c$	Resultant torques of constraint acting on body k through hinge point k - 1

$\vec{\phi}^{c}_{W_m}$	Resultant torque of constraint acting on momentum wheel m
$\vec{\phi}^{H}_{k-1}$	Resultant torque acting on body k due to all mechanisms existing between bodies J(k) and k at hinge point k - 1
$\{\phi^{c_1}\}$	(N + 1) X 1, column matrix of forces and torques of constraint acting between bodies
$\{\phi^{c_2}\}$	M X 1, column matrix of constraint torques acting on momentum wheels
$\{\phi^{H_1}\}$	(N + 1) X 1, column matrix of forces and torques due to mechanisms acting between bodies
$\{\phi^{H_2}\}$	M X 1, column matrix of torques due to mechanisms acting on momentum wheels
$\{\phi^{E_1}\}$	(N + 1) X 1, column matrix of forces and torques external to total system acting on rigid bodies and point masses
${}_c\vec{\omega}_\lambda$	Angular velocity of body λ fixed coordinates relative to the computing frame fixed coordinates
$\vec{\omega}_\lambda$	Angular velocity of body λ fixed coordinates relative to inertially fixed coordinates
$\vec{\tilde{\omega}}_\lambda$	Angular velocity of body λ fixed coordinates relative to body J(λ) fixed coordinates
$\{\omega\}$	(N + 1) X 1, column matrix of quantities $\vec{\omega}_\lambda$, λ = 1, . . . , N + 1
$\{\tilde{\omega}\}$	(N + 1) X 1, column matrix of quantities $\vec{\tilde{\omega}}_\lambda$, λ = 1, . . . , N + 1
$\vec{\omega}_{W_m}$	Inertial angular velocity of momentum wheel m
$\vec{\tilde{\omega}}_{W_m}$	Relative angular velocity of momentum wheel m
$\{\ \}$	Column matrix
$[\ ]$	Square matrix
$\lfloor\ \rfloor$	Rectangular matrix
$\vec{R}$	Vector $\vec{R}$
$\left\{\vec{R}\right\}_\lambda$	3 X 1 matrix of components of vector $\vec{R}$ relative to body λ fixed coordinates
$\bar{R}_\lambda$	Quaternion of vector $\vec{R}$ relative to body λ fixed coordinates

$\dot{\vec{R}}$	Time derivative of vector $\vec{R}$ relative to inertial fixed reference frame
$\overset{(i)}{\dot{\vec{R}}}$	Time derivative of vector $\vec{R}$ relative to local reference frame i
$\overset{\circ}{\vec{R}}$	Same as $\overset{(i)}{\dot{\vec{R}}}$ when no confusion as to which local reference frame differentiation is with respect to
$\mathbf{1}$	Unit dyad
$\sum_{i \epsilon S_{k-1}}$	Summation over all indices i contained in the set S_{k-1}
$\sum_{i @ k-1}$	Summation over all indices i of vectors defined at hinge point k - 1
$\prod_{k \epsilon S_{0,\lambda-1}}$	Multiple product over all indices k contained in set $S_{0,\lambda-1}$ in decreasing order of magnitude

REFERENCES

1. Velman, J. R., "Simulation Results for a Dual-spin Spacecraft," *Proceedings of the Symposium on Attitude Stabilization and Control of Dual-Spin Spacecraft, 1-2 August 1967*, SAMSO-TR-68-191, Air Force Systems Command, Space and Missile Systems Organization and Aerospace Corporation, November 1967.

2. Hooker, William W., and G. Margulies, "The Dynamical Attitude Equations for an n-Body Satellite," *J. Astronautical Sci.*, 12, 4, Winter 1965, pp. 123-128.

3. Hooker, William W., "A Set of r Dynamical Attitude Equations for an Arbitrary n-body Satellite Having r Rotational Degrees of Freedom," *AIAA, J.*, July 1970, pp. 1205-1207.

4. Roberson, Robert E., and Jens Wittenburg, "A Dynamical Formalism for an Arbitrary Number of Interconnected Rigid Bodies with Reference to the Problem of Satellite Attitude Control," Proc. Third Congress International Federation of Automatic Control, London, 1966.

5. Russell, W. J., *On the Formulation of Equations of Rotational Motion for an N-Body Spacecraft*, TR-0200(4133)-2, Aerospace Corp., El Segundo, California, February 1969.

6. Likens, P. W., *Dynamics and Control of Flexible Space Vehicles*, TR 32-1329, Rev. 1, Jet Propulsion Laboratories, January 15, 1970.

7. Goldstein, Herbert, *Classical Mechanics*, Addison-Wesley, 1959.

8. Moulton, F. R., *An Introduction to Celestial Mechanics*, MacMillan Co., 1914.

9. Frisch, H. P., "Thermally Induced Vibrations of Long Thin-Walled Cylinders of Open Section," *J. of Spacecraft and Rockets*, 7(8), August 1970, pp. 897-905.

APPENDIX

QUATERNION TECHNIQUES

The quaternions of Hamilton constitute a four-dimensional vector space, over the field of real numbers, with respect to a basis of four special vectors denoted by

$$\left\{1, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\right\}.$$

The algebraic operations for quaternions are the usual two vector operations of vector addition and scalar multiplication, plus the operation of quaternion multiplication.

In four dimension vector space the quaternion $\bar{\zeta}$ can be defined as

$$\bar{\zeta} = e_0 + e_1 \bar{\xi}_1 + e_2 \bar{\xi}_2 + e_3 \bar{\xi}_3 , \tag{A-1}$$

where

$$\left\{1, \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3\right\}$$

is the set of four linearly independent basis quaternions and

$$\left\{e_0, e_1, e_2, e_3\right\}$$

are real numbers. The product of any two of the basis quaternions is such that 1 acts as the identity and the multiplication table

$$\bar{\xi}_i^2 = -1 \quad i = 1, 2, 3 \tag{A-2}$$

$$\bar{\xi}_1 \bar{\xi}_2 = -\bar{\xi}_2 \bar{\xi}_1 = \bar{\xi}_3 \tag{A-3}$$

$$\bar{\xi}_2 \bar{\xi}_3 = -\bar{\xi}_3 \bar{\xi}_2 = \bar{\xi}_1 \tag{A-4}$$

$$\bar{\xi}_3 \bar{\xi}_1 = -\bar{\xi}_1 \bar{\xi}_3 = \bar{\xi}_2 \tag{A-5}$$

is satisfied.

Quaternions are most commonly used in conjunction with vectors to describe the effect of either a rotation of a vector or a transformation of coordinates. Accordingly, an operator is defined which takes vectors into quaternions and vice versa.

Let

$\mathcal{Q}$ = quaternion operator which takes vectors into quaternions

and

$\mathcal{Q}^{-1}$ = inverse quaternion operator which maps quaternions into vectors

If

$$\left\{\vec{R}\right\}_{J(\lambda)} = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}_{J(\lambda)}$$

= column matrix of the components of the vector $\vec{R}$ relative to the body $J(\lambda)$ fixed coordinate axes,

(A-6)

then

$$\mathcal{Q}\left(\left\{\vec{R}\right\}_{J(\lambda)}\right) = \bar{R}_{J(\lambda)}$$

= quaternion representation of the vector $\vec{R}$ given relative to body $J(\lambda)$ fixed coordinates (A-7)

where

$$\bar{R}_{J(\lambda)} = x_1 \bar{\xi}_1 + y_1 \bar{\xi}_2 + z_1 \bar{\xi}_3 \tag{A-8}$$

and

$$\mathcal{Q}^{-1}\left(\bar{R}_{J(\lambda)}\right) = \left\{\vec{R}\right\}_{J(\lambda)}. \tag{A-9}$$

Euler has shown that an arbitrary rotation of a rigid body about a fixed point is always equivalent to a rotation about a line passing through the point. It follows that the relative orientation of the coordinate axes fixed in body $J(\lambda)$ and in body λ can be completely defined by specifying the direction of the line about which and the angle through which the coordinate axes $\left\{X_1^{J(\lambda)}, X_2^{J(\lambda)}, X_3^{J(\lambda)}\right\}$ must be rotated so that its axes are respectively parallel to those of the coordinate axes $\left\{X_1^{\lambda}, X_2^{\lambda}, X_3^{\lambda}\right\}$.

Let

$\vec{q}$ = The unit vector aligned in a right handed sense along the axis about which the coordinate axes $\left\{X_1^{J(\lambda)}, X_2^{J(\lambda)}, X_3^{J(\lambda)}\right\}$ are rotated, and

θ = Angle through which the coordinate axes $\{X_1^{J(\lambda)}, X_2^{J(\lambda)}, X_3^{J(\lambda)}\}$ are rotated about $\vec{q}$ so as to be respectively aligned with the coordinate axes $\{X_1^{\lambda}, X_2^{\lambda}, X_3^{\lambda}\}$.

Several texts have shown that the components of $\vec{R}$ relative to body λ fixed coordinates can be obtained from the quaternion equation

$$\bar{R}_{J(\lambda)} = {}_{J(\lambda)}\bar{\xi}_{\lambda}\ \bar{R}_{\lambda}\ {}_{J(\lambda)}\bar{\xi}^{*}_{\lambda} \tag{A-10}$$

where

$${}_{J(\lambda)}\bar{\xi}_{\lambda} = \cos\theta/2 + \mathcal{Q}\left(\{\vec{q}\}_{J(\lambda)}\right)\sin\theta/2 \tag{A-11}$$

$${}_{J(\lambda)}\bar{\xi}^{*}_{\lambda} = \cos\theta/2 - \mathcal{Q}\left(\{\vec{q}\}_{J(\lambda)}\right)\sin\theta/2 \tag{A-12}$$

and

$$\mathcal{Q}^{-1}(\bar{R}_{\lambda}) = \{\vec{R}\}_{\lambda} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{\lambda} . \tag{A-13}$$

By expressing ${}_{J(\lambda)}\bar{\xi}_{\lambda}$ as

$${}_{J(\lambda)}\bar{\xi}_{\lambda} = e_0 + e_1\bar{\xi}_1 + e_2\bar{\xi}_2 + e_3\bar{\xi}_3 , \tag{A-14}$$

it is shown by the direct quaternion multiplication of Equation (A-10) that

$$\begin{aligned} x_1\bar{\xi}_1 + y_1\bar{\xi}_2 + z_1\bar{\xi}_3 = \\ &[(e_0^2 + e_1^2 - e_2^2 - e_3^2)\,x + 2(e_1e_2 - e_0e_3)\,y + 2(e_1e_3 + e_0e_2)\,z]\,\bar{\xi}_1 \\ &+ [2(e_0e_3 + e_1e_2)\,x + (e_0^2 - e_1^2 + e_2^2 - e_3^2)\,y + 2(e_2e_3 - e_0e_1)\,z]\,\bar{\xi}_2 \\ &+ [2(e_1e_3 - e_0e_2)\,x + 2(e_2e_3 + e_0e_1)\,y + (e_0^2 - e_1^2 - e_2^2 + e_3^2)\,z]\,\bar{\xi}_3 , \end{aligned} \tag{A-15}$$

or in matrix notation

$$\left\{\vec{R}\right\}_{J(\lambda)} = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_0 e_3 + e_1 e_2) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \left\{\vec{R}\right\}_{\lambda}$$

$$= \left[\mathcal{J}_{J(\lambda)\mathcal{J}_\lambda}\right] \left\{\vec{R}\right\}_{\lambda} \qquad \text{(A-16)}$$

where

$\left[\mathcal{J}_{J(\lambda)\mathcal{J}_\lambda}\right]$ = transformation matrix which takes vectors from body λ fixed coordinates into body $J(\lambda)$ fixed coordinates.

From this development it can be seen that a matrix operator $\mathcal{F}$ can be defined such that for the quaternion

$$\bar{\zeta} = e_0 + e_1 \bar{\xi}_1 + e_2 \bar{\xi}_2 + e_3 \bar{\xi}_3, \qquad \text{(A-17)}$$

$$\mathcal{F}(\bar{\xi}) = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_0 e_3 + e_1 e_2) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix}. \qquad \text{(A-18)}$$

Note that from the above definition

$$\mathcal{F}(\mathcal{J}_{J(\lambda)}\bar{\xi}_\lambda) = \left[\mathcal{J}_{J(\lambda)\mathcal{J}_\lambda}\right] \qquad \text{(A-19)}$$

In many practical problems it is infeasible to define the particular eigenvector q and angle θ which takes body $J(\lambda)$ fixed axes directly into body λ fixed axes. It is possible, however, to define several successive rotations through given angles and about given axes which will bring the two coordinate axes into alignment.

Suppose for example that m coordinate rotations are required to conveniently rotate body $J(\lambda)$ axes into body λ axes. The rotations are as follows:

a. Rotate body J(λ) axes about $\vec{q}_1$ thru angle θ_1 into intermediate reference frame I_1.

b. Rotate frame I_1 about $\vec{q}_2$ thru angle θ_2 into intermediate reference frame I_2.

c. Proceed sequentially until frame I_{m-1} is rotated about $\vec{q}_m$ thru angle $\vec{\theta}_m$ into body λ fixed frame.

In quaternion notation this rotation sequence is given by

$$\bar{R}_{J(\lambda)} = {}_{J(\lambda)}\bar{\xi}_{I_1}\ {}_{I_1}\bar{\xi}_{I_2} \cdots {}_{I_{m-1}}\bar{\xi}_{\lambda}\ \bar{R}_{\lambda}\ {}_{I_{m-1}}\bar{\xi}^*_{\lambda} \cdots {}_{I_1}\bar{\xi}^*_{I_2}\ {}_{J(\lambda)}\bar{\xi}^*_{I_1}, \tag{A-20}$$

where

$${}_{I_{i-1}}\bar{\xi}_{I_i} = \cos\theta_i/2 + \mathcal{Q}\left(\{\vec{q}_i\}_{I_{i-1}}\right)\sin\theta_i/2 \tag{A-21}$$

and

$\{\vec{q}_i\}_{I_{i-1}}$ = components of vector $\vec{q}_i$ relative to the intermediate reference frame I_{i-1}.

The resultant quaternion which takes body J(λ) axes directly into body λ axes is therefore given by the quaternion product

$${}_{J(\lambda)}\bar{\xi}_{\lambda} = {}_{J(\lambda)}\bar{\xi}_{I_1}\ {}_{I_1}\bar{\xi}_{I_2} \cdots {}_{I_{m-1}}\xi_{\lambda}. \tag{A-22}$$

The application of quaternions to the development of coordinate-transformation matrices has been demonstrated in the preceeding paragraphs. Quaternions are also used to define vector transformation matrices; that is, a quaternion equation can be defined which describes the effect of the rotation of an arbitrary vector $\vec{R}$ about an eigenvector $\vec{q}$ through the angle θ into a vector $\vec{R}_1$. If both $\vec{R}$ and $\vec{q}$ are defined in the body λ fixed coordinates, then

$$\bar{R}_{\lambda} = \bar{\xi}^*\ \bar{R}_{1_{\lambda}}\ \bar{\xi} \tag{A-23}$$

where

$$\bar{\xi} = \cos\theta/2 + \mathcal{Q}\left(\{\vec{q}\}_{\lambda}\right)\sin\theta/2, \tag{A-24}$$

$$\mathcal{L}^{-1}\ (\bar{R}_{1_\lambda}) = \left\{\vec{R}_1\right\}_\lambda\ , \tag{A-25}$$

and

$$\mathcal{L}^{-1}\ (\bar{R}_\lambda) = \left\{\vec{R}\right\}_\lambda . \tag{A-26}$$

Note that the components of the vector $\vec{R}_1$ are given relative to the same reference frame in which $\vec{R}$ and $\vec{q}$ are given. Furthermore, by making use of the matrix operator $\mathcal{F}(\bar{\xi})$ defined by Equations (A-18), Equation (A-23) can be expressed in matrix notation as

$$\left\{\vec{R}\right\}_\lambda = \mathcal{F}(\bar{\xi})^T \left\{\vec{R}_1\right\}_\lambda \tag{A-27}$$

Direct comparison of Equations (A-16) and (A-27) reveals that the transpose of the coordinate transformation matrix is the vector transformation matrix.