FRACTURE AND CONTACT PROBLEMS

FOR AN ELASTIC WEDGE

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Abstract

The paper deals with the plane elastostatic contact problem for an infinite elastic wedge of arbitrary angle. The medium is loaded through a frictionless rigid wedge of a given symmetric profile. Using the Mellin transform formulation the mixed boundary value problem is reduced to a singular integral equation with the contact stress as the unknown function. With the application of the results to the fracture of the medium in mind, the main emphasis in the study has been on the investigation of the singular nature of the stress state around the apex of the wedge and on the determination of the contact pressure.

1. INTRODUCTION

The importance of the stress concentration around internal sharp corners in design has long been recognized and the related problems have been extensively studied (e.g., [1]). In most of these studies concerning the "notch effect" it is generally assumed that the radius of curvature of the notch is greater than zero. Considering the sound design practice, this assumption is fully justified. However, in components where the expected mode of failure is brittle fracture, for the application of the related fracture criterion it may be more convenient and perhaps even necessary to study the limiting case of the problem in which the notch radius of curvature is zero. This would mean approximating the medium around the notch by a wedge-shaped domain with a given angle. The traction boundary value problem for elastic wedges also has been rather extensively studied (e.g., [2 - 11]).

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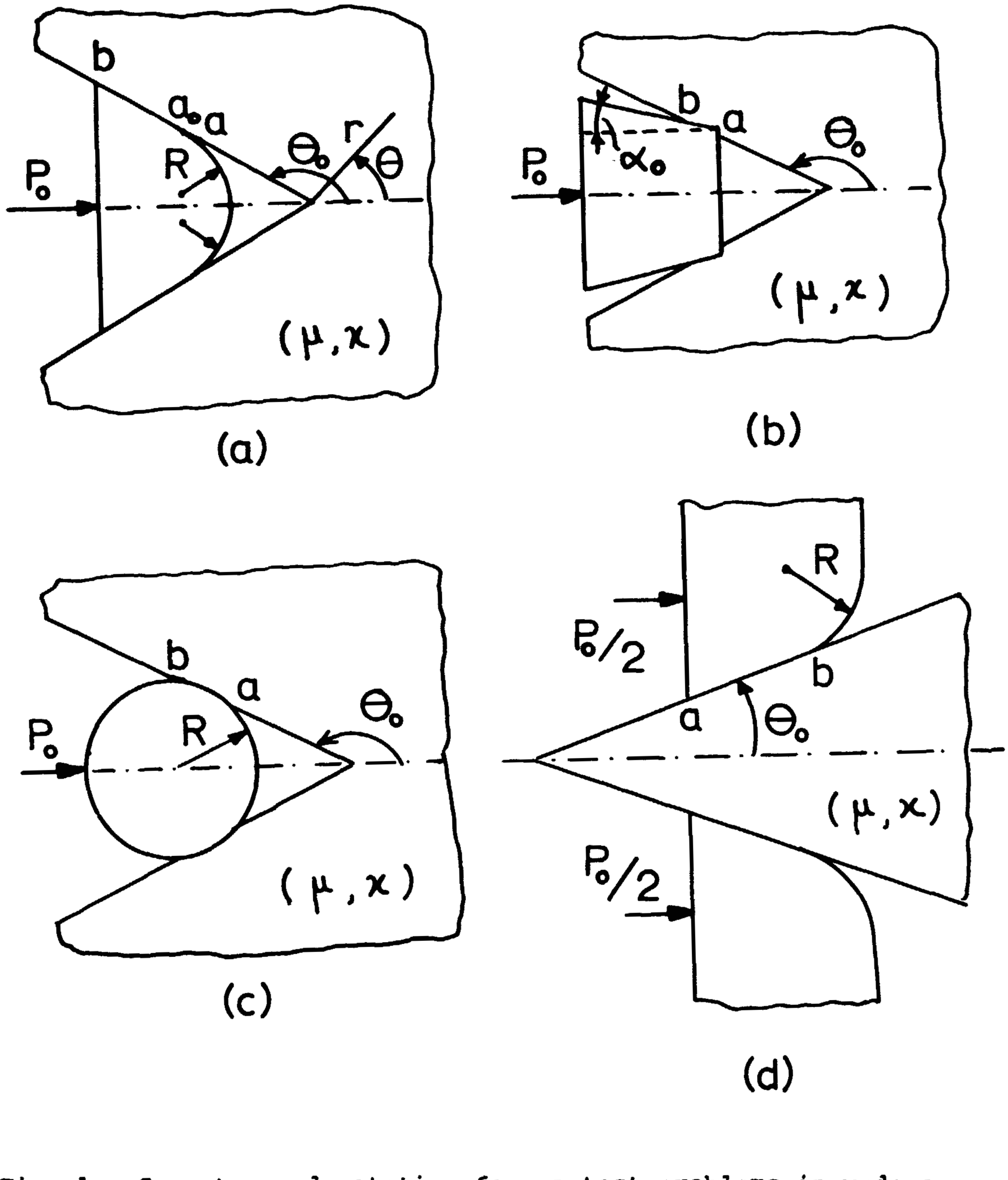

Fig. 1. Geometry and notation for contact problems in wedges.

The problems which have not been looked into are the so-called contact problems in wedge-shaped domains where the external loads are applied to the medium through a rigid wedge of a given profile (Fig. 1). Typically, these are mixed boundary value problems in which in addition to the contact stresses the contact area itself may be unknown. In this paper we will consider the plane elastostatic contact problem for a wedge with arbitrary angle loaded by a rigid frictionless symmetric wedge with an arbitrary profile. The main objective of the study is the determination of the contact pressure and the singular nature of the stress state in the immediate neighborhood of the wedge apex.

2. FORMULATION OF THE PROBLEM

The problem under consideration is described in Fig. 1. It is assumed that an infinite elastic wedge ($0 \leq r < \infty$, $-\theta_o \leq \theta \leq \theta_o$, $0 < \theta_o \leq \pi$) is subjected to an external resultant load of magnitude P_o through a rigid splitting wedge (or vise) of known profile, P_o acts along the line $\theta = \pi$, the profile of the wedge is symmetrical with respect to the $\theta = \pi$ plane, and the contact between the rigid wedge and the elastic medium is frictionless. Thus, because of symmetry it is sufficient to consider one half of the medium. Defining the following stress and displacement combinations

$$\sigma(r,\theta) = \sigma_{r\theta} + i\sigma_{\theta\theta} \ ,$$

$$v(r,\theta) = \frac{\partial}{\partial r} (u_r + iu_\theta) \ , \qquad \text{(1a,b)}$$

using the standard Mellin transform, and, for example, referring to the derivation given in [11], for an infinite wedge one may easily obtain

$$\mathcal{M}[r^2\sigma] = 2i(s+1)[Ase^{is\theta} + B(s+1)e^{i(s+2)\theta} - \bar{B}e^{-i(s+2)\theta}] ,$$

$$\mathcal{M}[r^2 v] = -\frac{s+1}{\mu}[Ase^{is\theta} + B(s+1)e^{i(s+2)\theta} + \kappa\bar{B}e^{-i(s+2)\theta}] , \quad (2a,b)$$

where (r,θ) are the polar coordinates (Fig. 1a), and the Mellin transform of a function $f(r)$, $(0 \le r < \infty)$ and its derivatives, and the inversion integral are defined by

$$\mathcal{M}[f] = \int_0^\infty f(r) r^{s-1} dr ,$$

$$\int_0^\infty \frac{d^n f}{dr^n} r^{n+s-1} dr = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} \mathcal{M}[f] ,$$

$$f(r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}[f] r^{-s} ds , \quad (3a\text{-}c)$$

provided the strip of regularity containing c is selected in such a way that, considered together with the behavior of $f(r)$ as $r \to 0$ and $r \to \infty$, the integrals in (3) exist. This means that the regularity conditions as $r \to 0$ and $r \to \infty$ determine the strip of regularity and the constant c. In (2) μ and κ are the elastic constants of the medium (μ: shear modulus, $\kappa = 3-4\nu$ for plane strain, $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress, ν: Poisson's ratio). The complex "integration constants" A and B are functions of the transform variable s, and are determined from the boundary conditions given along $\theta = \mp\theta_o$ (or along $\theta = 0$ and $\theta = \theta_o$ if the problem has symmetry).

In particular consider the elastic wedge of angle $2\theta_o$ subjected to (symmetric concentrated) tractions

$$\sigma(r,\theta_o) = (f_1 + if_2)\delta(r - r_o) . \quad (4)$$

The functions A(s) and B(s) may be determined from the boundary conditions given by (4) and by the following conditions specified along $\theta = 0$:

$$\mathrm{Im}[v(r,0)] = 0 \ , \qquad \mathrm{Re}[\sigma(r,0)] = 0 \ . \tag{5a,b}$$

After some manipulations it may easily be shown that A and B are real and are given by

$$sA(s) = \frac{g_1(s+2)\sin(s+2)\theta_o - g_2 s\cos(s+2)\theta_o}{\Delta(s)} \ ,$$

$$B(s) = \frac{g_2\cos s\theta_o - g_1\sin s\theta_o}{\Delta(s)} \ ,$$

$$\Delta(s) = (s+1)\sin 2\theta_o + \sin 2(s+1)\theta_o \ ,$$

$$g_1 + ig_2 = \frac{r_o^{s+1}(f_1 + if_2)}{2i(s+1)} \ . \tag{6a-d}$$

Since the friction is neglected, letting $f_1 = 0$, the basic solution found above may be used as the Green's function to find the solution for distributed tractions $\sigma_{\theta\theta}(r,\theta_o) = f_2(r)$, $(a < r < b)$. In the formulation of the present mixed boundary value problem the quantity of primary interest is the displacement derivative $\partial u_\theta/\partial r$ for $\theta = \theta_o$ which is found to be

$$-\frac{2\mu}{1+\kappa}\, r\,\frac{\partial}{\partial r}\, u_\theta(r,\theta_o) = \int_a^b f_2(r_o)dr_o\,\frac{1}{2\pi i}\ *$$

$$*\int_{c-i\infty}^{c+i\infty}\left(\frac{r_o}{r}\right)^{s+1}\frac{\sin s\theta_o\sin(s+2)\theta_o}{\Delta(s)}\,ds \ , \qquad (0 < r < \infty) \ . \tag{7}$$

If now the profile of the rigid stamp is a given function $U(r)$ along the contact area $a < r < b$, substituting

$$\frac{\partial}{\partial r}\, u_\theta(r,\theta_o) = \frac{\partial}{\partial r}\, U(r) = h(r) \ , \qquad (a < r < b) \tag{8}$$

in (7) we obtain an integral equation to determine the unknown function $f_2(r)$. In order to evaluate the kernel in (7) the strip of regularity containing c has to be determined. Let s_k be the roots of the characteristic equation $\Delta(s) = 0$, (6c), i.e., let

$$\Delta(s_k) = 0 , \qquad (k = \mp 1, \mp 2, \ldots),$$

$$\mathrm{Re}(s_{-j-1}) < \mathrm{Re}(s_{-j}) < c < \mathrm{Re}(s_j) < \mathrm{Re}(s_{j+1}), \qquad (j = 1,2,\ldots). \tag{9}$$

Thus, the kernel may be expressed as the sum of residues at s_k, $(k = 1,2,\ldots)$ for $r_o < r$ and at s_{-k}, $(k = 1,2,\ldots)$ for $r_o > r$, the strip of regularity containing c being $\mathrm{Re}(s_{-1}) < \mathrm{Re}(s) < \mathrm{Re}(s_{+1})$. On the other hand from the physical conditions that

$$\frac{\partial u_\theta}{\partial r} \sim \frac{1}{r^\alpha} , \qquad (\alpha < 1) \text{ for } r \to 0 ,$$

$$\frac{\partial u_\theta}{\partial r} \sim \frac{1}{r^\beta} , \qquad (\beta \geq 1) \text{ for } r \to \infty , \tag{10}$$

and from (7) it is easily seen that $s_{+1} = -1$ and s_{-1} is the first root of $\Delta(s)$ to the left of $\mathrm{Re}(s) = -1$ line.

Since the resulting infinite series giving the kernel in (7) cannot be summed in closed form, and since the kernel is expected to have a singular part, in order to bring out the distinctive features of the integral equation the following technique will be used to evaluate the kernel. Noting that $\mathrm{Re}(s_{-1}) < c < -1$, for numerical convenience the line of integration $\mathrm{Re}(s) = c$ in (7) may be replaced by letting $c = -1$ and indenting the contour to the left. Thus defining

$$s+1 = iy , \qquad \log\left(\frac{r_o}{r}\right) = \rho , \tag{11}$$

and using (8), after some routine manipulations from (7) we obtain

$$-\frac{4\mu}{1+\kappa}\, rh(r) = \frac{1}{\pi}\int_a^b f(r_o)\,dr_o \Big[\frac{\pi \sin^2\theta_o}{2\theta_o + \sin 2\theta_o}$$

$$- \int_0^\infty \frac{\cosh 2\theta_o y - \cos 2\theta_o}{D(y)} \sin\rho y\, dy \Big], \quad (a < r < b) \tag{12}$$

where

$$f(r) = f_2(r) \; ,$$

$$D(y) = y\sin 2\theta_o + \sinh 2\theta_o y \; . \qquad (13a,b)$$

It may be shown that as $r_o \to r$ the infinite integral in (12) becomes divergent. Since the integrand is bounded everywhere in $(0 \le r < \infty)$, the divergence will be due to the behavior of the integrand at infinity and the divergent part may easily be separated by considering the asymptotic behavior of the integrand as $y \to \infty$, giving

$$\frac{4\mu}{1+\kappa} h(r) = \frac{1}{\pi} \int_a^b f(r_o) dr_o \left[\frac{1}{r\log(r_o/r)} - \frac{\pi \sin^2\theta_o}{r(2\theta_o + \sin 2\theta_o)} + \int_o^\infty \left(\frac{\cosh 2\theta_o y - \cos 2\theta_o}{D(y)} - 1 \right) \frac{1}{r} \sin\rho y \, dy \right] ,$$

$$(a < r < b) . \qquad (14)$$

From

$$\frac{1}{r\log(r_o/r)} = \frac{1}{r(\frac{r_o}{r} - 1)} \left[1 + \sum_1^\infty \frac{(-1)^n}{n+1} \left(\frac{r_o}{r} - 1\right)^n\right]^{-1}$$

$$= \frac{1}{r_o - r} \left[1 + 0\left(\frac{r_o}{r} - 1\right)\right] \qquad (15)$$

it is seen that at $r_o = r$ the kernel in (14) has a simple Cauchy-type singularity. The singular integral equation (14) must be solved under the condition that

$$\int_a^b f(r) dr = P = - \frac{P_o}{2\sin\theta_o} \qquad (16)$$

which will have to be used in the determination of the unknown constants arising from the solution of (14).

3. THE STRESS INTENSITY FACTOR

From the viewpoint of fracture of the material an important aspect of the problem is the study of the singular behavior of the stress state in the close neighborhood of the apex $r=0$. In particular, it is important to evaluate the cleavage stress around $r=0$ as a function of the external load and the geometrical parameters. In the symmetric problem under consideration the plane of the maximum cleavage stress $\sigma_{\theta\theta}$ is known to be $\theta=0$ [10]. Thus, using $f_1=0$, $f_2(r)=f(r)$ from (1a), (2a) and (6) we obtain

$$r\sigma_{\theta\theta}(r,0) = \int_a^b f(r_o)dr_o \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{r_o}{r}\right)^{s+1} \frac{G(s)}{\Delta(s)} ds , \quad (0<r<\infty),$$

$$G(s) = (s+2)\sin(s+2)\theta_o - s\sin s\theta_o . \qquad (17a,b)$$

For $r<a$ in (17) closing the contour to the left and considering (9) we find

$$r\sigma_{\theta\theta}(r,0) = \sum_{k=1}^{\infty} \frac{G(s_{-k})}{\Delta'(s_{-k})} \int_a^b f(r_o)\left(\frac{r_o}{r}\right)^{1+s_{-k}} dr_o , \quad (0<r<a),$$

$$\Delta'(s) = \sin 2\theta_o + 2\theta_o\cos 2(s+1)\theta_o . \qquad (18a,b)$$

We now observe that (18a) is of the following form

$$\sigma_{\theta\theta}(r,0) = \sum_1^{\infty} C_k r^{-(2+s_{-k})} \qquad (19)$$

where $C_1, C_2, \ldots$ are known constants. s_{-k}, $(k=1,2,\ldots)$ are the roots of the following characteristic equation:

$$\Delta(s) = (s+1)\sin 2\theta_o + \sin 2(s+1)\theta_o = 0 , \quad (\mathrm{Re}(s)<-1). \qquad (20)$$

It should be noted that (20) is the same as the characteristic equation found in previous studies (e.g., [3, 4, 10, 11]). For $\frac{\pi}{2}\le\theta_o\le\pi$ from (20) it may be shown that s_{-1} is always real and

$$0 \le (2+s_{-1}) \le 0.5 , \quad \mathrm{Re}(2+s_{-k}) < 0 , \quad (k=2,3,\ldots). \qquad (21)$$

This means that in (19) only the first term may be singular at $r=0$. Thus, defining the stress intensity factor for the wedge as

$$k(0) = \lim_{r\to 0} r^{\omega}\sigma_{\theta\theta}(r,0) = C_1 , \quad \omega = 2+s_{-1} , \tag{22}$$

from (18) and (19) we obtain

$$k(0) = \frac{G(s_{-1})}{\Delta'(s_{-1})} \int_a^b f(r) r^{1+s_{-1}} dr . \tag{23}$$

For some selected half-wedge angles θ_o Table 1 shows the power ω of the stress singularity at the apex $r = 0$. In the special case of concentrated applied l s

$$\sigma_{\theta\theta}(r,\theta_o) = P\delta(r-r_1) \tag{24}$$

the stress intensity factor becomes

$$k(0) = Pr_1^{1+s_{-1}} \frac{G(s_{-1})}{\Delta'(s_{-1})} , \tag{24}$$

where G and Δ' are given by (17b) and (18b), respectively. After determining k(0) the results may be applied to study the fracture of brittle and semi-brittle solids by using an appropriate fracture criterion (such as, for example, that based on the notion of critical cleavage stress at a characteristic distance from the singular point described in [12]).

Table 1. The power ω of the stress singularity in an elastic wedge of angle $2\theta_o$ under symmetric loading.

θ_o(°)	90	99	108	117	120	126
ω	0	0.16631	0.28220	0.36327	0.38427	0.41886
θ_o(°)	135	144	150	165	172.5	180
ω	0.45552	0.47829	0.48778	0.49855	0.49982	0.5

4. SPECIAL CASES: $\theta_o = \pi$, $\theta_o = \pi/2$

In the special cases where the wedge degenerates into a cracked plane ($\theta_o = \pi$) or a half space ($\theta_o = \pi/2$) the kernel of the integral equation (14) may be evaluated in closed form. For example, for $\theta_o = \pi$ the kernel in (14) becomes

$$K(r,r_o) = \frac{1}{r\rho} + \int_0^\infty \left(\frac{\cosh 2\pi y - 1}{\sinh 2\pi y} - 1\right) \frac{1}{r} \sin\rho y dy$$

$$= \frac{1}{r_o - r} (r_o/r)^{1/2} , \qquad (26)$$

giving

$$\frac{4\mu}{1+\kappa} [h(r)\sqrt{r}] = \frac{1}{\pi} \int_a^b [\sqrt{r_o} f(r_o)] \frac{dr_o}{r_o - r} , \qquad (a < r < b). \qquad (27)$$

The solution of (27) may easily be obtained once the profile of the rigid wedge is specified. If this wedge has plane boundaries and sharp corners (a thin rigid block with rectangular cross-section), $h(r) = 0$, and the sc ution of (27) subject to the equilibrium condition (16) becomes

$$f(r) = \frac{P\sqrt{b}}{2K(k)\sqrt{r}\sqrt{(r-a)(b-r)}} , \qquad (a < r < b), \qquad (28)$$

where $K(k)$ is the complete elliptic integral of the first kind with the modulus

$$k^2 = \frac{b-a}{b} . \qquad (29)$$

In limit if we let $b \to \infty$, $a \to \infty$ for a fixed $b-a = 2c$, (28) reduces to the following known result of a rectangular punch on a half plane

$$f(r) = \frac{P}{\pi\sqrt{c^2 - t^2}} , \qquad (t = r - \frac{b+a}{2} , \quad -c < t < c). \qquad (30)$$

In this special case of semi-infinite crack $s_{-1} = -3/2$, and the stress intensity factors at $r = 0$ given by (23) and (25) become

$$k(0) = \lim_{r\to 0} \sqrt{r}\, \sigma_{\theta\theta}(r,0) = -\frac{1}{\pi} \int_a^b f(r) r^{-1/2} dr$$

$$= -\frac{P\sqrt{b}}{2\pi K(k)} \int_a^b \frac{dr}{r\sqrt{(r-a)(b-r)}} = -\frac{P}{2\sqrt{a}\, K(k)} , \qquad (31)$$

for the rectangular wedge problem, and

$$k(0) = -\frac{P}{\pi\sqrt{r_1}} \tag{32}$$

for the concentrated load P applied at $r = r_1$.

In the wedge problems in general and in the semi-infinite crack problem in particular another quantity of special physical interest may be the displacements on the wedge surface, particularly the rigid body displacement of the splitting wedge. Noting that the basic equations (7), (12), (14), or (27) give the displacement derivative $\partial u_\theta(r,\theta_o)/\partial r$ for $a < r < b$ as well as outside this interval, once the unknown function f(r) is determined, $\partial u_\theta/\partial r$ and $u_\theta(r,\theta_o)$ may easily be evaluated. For example, in the crack problem substituting from (28) into (27) and using (8) for $0 < r < a$ we find

$$\frac{\partial}{\partial r} u_\theta(r,\pi) = \frac{1+\kappa}{4\mu}\frac{P\sqrt{b}}{2K(k)}\frac{1}{\sqrt{r(a-r)(b-r)}}, \quad (0 < r < a),$$

$$u_\theta(r,\pi) = \frac{1+\kappa}{4\mu}\frac{P}{K(k)}F(k',\alpha), \quad (u_\theta(0,\pi) = 0), \tag{33a,b}$$

$$k^2 = 1 - \frac{a}{b}, \quad k'^2 = 1 - k^2 = \frac{a}{b}, \quad \sin\alpha = \sqrt{r/a}, \quad 0 < r < a,$$

where $F(k',\alpha)$ is the elliptic integral of the first kind. The relationship between the resultant wedging force P and the half thickness d_o of the rigid wedge may be obtained from (33b) by letting $r = a$ which gives

$$u_\theta(a,\pi) = d_o = \frac{1+\kappa}{4\mu}\frac{K(k')}{K(k)}P. \tag{34}$$

In terms of the half wedge thickness d_o the stress intensity factor given by (31) becomes

$$k(0) = -\frac{2\mu d_o}{(1+\kappa)\sqrt{a}\ K(k')} . \tag{35}$$

For $b \gg a$ (and if the wedge faces remain in contact) (35) may be approximated by

$$k(0) \simeq -\frac{4\mu d_o}{(1+\kappa)\pi\sqrt{a}} . \tag{36}$$

Similarly, for $\theta_o = \pi$, the integral equation (14) may be shown to reduce to

$$\frac{4\mu}{1+\kappa} h(r) = \frac{1}{\pi}\int_a^b \left(\frac{1}{r_o - r} - \frac{1}{r_o + r}\right) f(r_o)\,dr_o , \quad (a < r < b). \tag{37}$$

Solving (31) for a flat rectangular punch $(h(r) = 0)$ under the condition (16) we obtain

$$f(r) = \frac{2Pr}{\pi\sqrt{(r^2 - a^2)(b^2 - r^2)}} , \quad (a < r < b). \tag{38}$$

This is the known solution for the contact stress in a half plane loaded by two symmetrically located punches [13].

5. SOLUTION AND RESULTS

In this section the solution of the problem will be described and some results will be given for three different types of rigid splitting wedge profile, namely, (a) "the flat-ended wedge with sharp corners" for which a and b are known (insert in Fig. 2), (b) "the circular wedge" for which a and b are both unknown (Fig. 1c), and (c) "the wedge with one sharp corner" for which one end point of the contact region is known and the other is unknown (Fig. 1a, b, d). Mathematically these three cases are characterized by the index of the singular integral equation (14). The integral equation is the same for all three cases except for the input function h(r) which describes the profile of the rigid wedge (8). In the first case the index of the problem is +1, and the solution of the integral equation (14) contains an arbitrary constant which is determined from the equilibrium condition (16) [14]. For the case of the circular wedge the index is -1 and the unknowns a and b are determined from (16) and the consistency condition of the integral equation [14]. Finally in the third case the index is zero and the unknown a or b is determined from (16). In all three cases the integral equation is solved by using the simple numerical method described in [15] and [16]. Hence, in this paper we will present only the numerical results.

(a) The flat-ended wedge with sharp corners.

This problem is described by the insert in Fig. 2. In the problem, in addition to the stress intensity factor defined by (22) and given by (23), other quantities of interest are the contact stress f(r) and the constants describing the nature of the stress singularities at the corners a and b of the rigid wedge. For this problem h(r) = 0 and the solution of (14) may be expressed as

$$f(r) = \frac{f_o(r)}{\sqrt{(r-a)(b-r)}} = \frac{F(x)}{\sqrt{1-x^2}} ,$$

$$(r = cx+d,\ c = \frac{b-a}{2},\ d = \frac{b+a}{2},\ a < r < b,\ -1 < x < 1), \quad (39)$$

where $f_o(r)$ and $F(x)$ are bounded in the respective closed intervals for r and x. The constants characterizing the strength of the singularity of the contact stress $\sigma_{\theta\theta}(r,\theta_o) = f(r)$ at $r = a$ and $r = b$ may be defined by

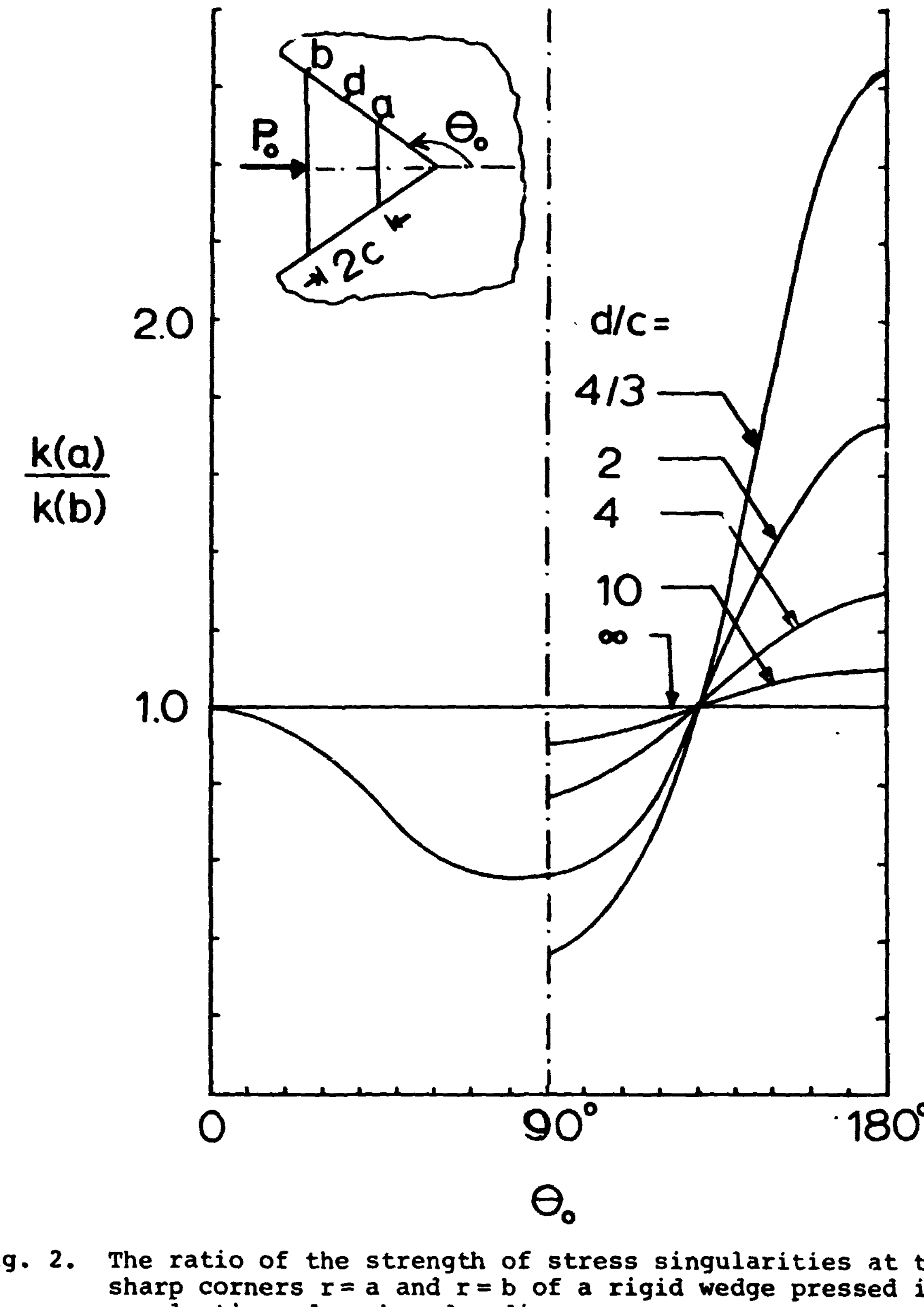

Fig. 2. The ratio of the strength of stress singularities at the sharp corners r = a and r = b of a rigid wedge pressed into an elastic wedge-shaped medium.

$$k(a) = \lim_{r \to a} \sqrt{2(r-a)}\, f(r) \; ,$$

$$k(b) = \lim_{r \to b} \sqrt{2(b-r)}\, f(r) \; . \qquad (40a,b)$$

Table 2 shows the stress intensity factor k(0) defined by (22) and the constants k(a) and k(b) for various wedge angles $2\theta_o$ and ratios d/c representing the relative dimensions. The normalizing factor $P/(\pi\sqrt{c})$ given for k(a) and k(b) is the strength of contact stress singularity for the half plane obtained from (30) and (40). The power of the stress singularity ω at $r = 0$ for the wedge angles $\pi/2 < \theta_o \leq \pi$ is given by Table 1. Most of the results given in Table 2 refer to the case $\pi/2 < \theta_o \leq \pi$ for which at $r = 0$ the stresses are

Table 2. Stress intensity factors for the flat-ended rigid wedge.

d/c	θ_o(°)	$-\dfrac{k(0)}{P/(\pi d^{1-\omega})}$	$\dfrac{k(b)}{P/(\pi\sqrt{c})}$	$\dfrac{k(a)}{P/(\pi\sqrt{c})}$
4/3	150	1.1168	0.7838	1.4637
	165	1.2582	0.6815	1.6651
	172.5	1.2918	0.6666	1.7108
	180	1.3239	0.6619	1.7513
2	10		0.5209	0.5156
	30		0.8610	0.7628
	60		1.1249	0.7831
	90		1.2251	0.7091
	120	0.4989	1.0965	0.8779
	150	0.9607	0.8575	1.2194
	165	1.0634	0.7931	1.3140
	172.5	1.0873	0.7777	1.3351
	180	1.0948	0.7741	1.3408
4	150	0.9032	0.9260	1.0906
	165	0.9934	0.8932	1.1297
	172.5	1.0138	0.8859	1.1384
	180	1.0204	0.8837	1.1408
10	150	0.8897	0.9691	1.0332
	165	0.9770	0.9556	1.0481
	172.5	0.9968	0.9526	1.0511
	180	1.0031	0.9516	1.0521

singular. However, for $d = 2c$ some results for $0 < \theta_o \leq \pi/2$ are also included (Fig. 1d, where both corners of the rigid "vise" are sharp).

From the half plane solution given by (38) ($\theta_o = 90°$ in Table 2) it may be observed that $k(b) > k(a)$ and $k(a) = 0$ for $a = 0$. This means that if the punches are disconnected, they may tend to tilt and slide outward and away from each other. Figure 2 shows an interesting result obtained from Table 2, namely that $\theta_o \approx 130°$ seems to be a critical wedge angle. For $\theta_o > 130°$, $k(a) > k(b)$, and for $\theta_o < 130°$, $k(a) < k(b)$. For small and large angles this behavior is, of course, expected. However, it appears that (even though it is difficult to prove) this value of the wedge angle for which $k(a) = k(b)$ (implying an approximately symmetric pressure distribution under the wedge) is independent of the relative dimensions as characterized by d/c.

To give an idea about the contact stress distribution Table 3 shows some calculated results at a limited number of locations for various values of the wedge angle and for $d = 2c$. The variable x and the function $F(x)$ shown in the table are defined in (39).

Table 3. The measure of the distribution of contact stress, $F(x)/(P/d)$ for $d = 2c$ (see Eq. (39)).

θ \ x	1	0.6494	0.2334	-0.2334	-0.6494	-1
10°	0.3316	0.5961	0.8208	0.9062	0.8215	0.3282
30	0.5481	0.6226	0.7092	0.7612	0.6976	0.4857
60	0.7162	0.7022	0.6801	0.6415	0.5823	0.4985
90	0.7800	0.7429	0.6911	0.6190	0.5371	0.4515
120	0.6981	0.6819	0.6595	0.6285	0.5938	0.5589
150	0.5459	0.5653	0.5944	0.6392	0.6983	0.7763
165	0.5049	0.5336	0.5763	0.6414	0.7263	0.8365
172.5	0.4958	0.5266	0.5723	0.6419	0.7325	0.8499

(b) Rigid circular wedge.

This problem is described in Fig. 1c where the input function is (8)

$$h(r) = \frac{1}{R}\,[r - R\cot(\pi-\theta_o)]$$
$$= \frac{d}{R}\,(1+\gamma x) - \cot(\pi-\theta_o)\ , \qquad (\gamma = \frac{c}{d})$$
$$(a < r < b,\ -1 < x < 1)\,. \qquad (41)$$

The new variable x and the unknown constants c and d appearing in (41) are defined by (39). The problem as formulated by (14) is linear in f(r) and d and is highly nonlinear in γ. Hence it is solved in an inverse manner by pre-selecting γ, solving for f(r) and d, and then through (16) determining the corresponding P. Table 4 shows some calculated results for various values of γ.

Table 4. The results for the rigid circular wedge, $\gamma = (b-a)/(b+a)$, $\theta_o = 3\pi/4$ (Fig. 1c).

γ	$\frac{a}{R}$	$\frac{b}{R}$	$-\frac{1+\kappa}{4\mu R}\,P\times10^3$	$-\frac{k(0)}{P/(\pi R^{1-\omega})}$	$\frac{\pi c}{2P}\,f_{max}$
0.01	0.989996	1.009996	0.1571	0.7215	1.0000
0.05	0.949909	1.049899	3.9258	0.7217	1.0001
0.10	0.899662	1.099575	15.6891	0.7224	1.0004

The results confirm what one might have guessed prior to solving the problem, namely that in this case the contact pressure would be very close to that obtained from a rigid circular punch on a half plane and the stress intensity factor k(0) would be close to that obtained from a concentrated force solution as given by (25). The solution for the circular punch on half plane is

$$f(t) = \frac{2P}{\pi c^2}\sqrt{c^2 - x^2}\ , \qquad (-c < x < c)\,,$$

$$f_{max} = \frac{2P}{\pi c}\ . \qquad (42)$$

This is seen to be almost the same result given in the table. Also numerical results indicated that f(r) obtained from (14) was very nearly symmetric with respect to the point $r = d$. For $\theta_o = 135°$ the stress intensity factor for the concentrated load P acting at $r_1 = R\cot(\pi-\theta_o) = R$ may be evaluated from (25) and is found to be $k(0) = -0.7215\ P/(\pi R^{1-\omega})$ which is the same as that obtained for $\gamma = 0.01$. Even though in this case no extensive numerical work is warranted because of the availability of the closed form solution for the concentrated loads, for splitting wedges having a relatively large radius of curvature around the contact area $(b-a)/(b+a)$ may be sufficiently large so that the concentrated load solution may not be representative. In such cases the technique described in this paper would give the solution.

(c) Rigid wedge with one sharp corner.

The particular examples considered in this group are described by Figs. 1a, 1b, and 1d. Table 5 gives the results for the problem shown in Fig. 1b. Here α_o is the half angle of the rigid wedge. The half angle of the elastic wedge is assumed to be $\theta_o = 165°$. Thus, for $\alpha_o < 15°$, a is known, b is unknown and the contact stress is of the following form

$$f(r) = f_o(r)\sqrt{\frac{b-r}{r-a}} = F(x)\sqrt{\frac{1-x}{1+x}} ,$$

$$(r = cx+d,\ c = (b-a)/2,\ d = (b+a)/2). \quad (43)$$

For $\alpha_o > 15°$, b is known, a is unknown, and

$$f(r) = f_o(r)\sqrt{\frac{r-a}{b-r}} = F(x)\sqrt{\frac{1+x}{1-x}} . \quad (44)$$

The Table shows only one set of results obtained for $(2c/L) = 0.1$

Table 5. The results for a rigid wedge with one sharp corner. ($\theta_0 = 165°$, $(2c/L) = 0.1$).

α_0	$-\frac{1+\kappa}{4\mu L} P$	$-\frac{k(0)}{P/(\pi L^{1-\omega})}$	$\frac{F(1)}{P/(\pi c)}$	$\frac{F(-1)}{P/(\pi c)}$
5° (L=a)	0.02831	0.9622	0.9989	1.0004
10° (L=a)	0.01405	0.9622	0.9989	1.0004
20° (L=b)	0.01341	0.9867	1.0005	0.9984
25° (L=b)	0.02703	0.9867	1.0005	0.9984

and for various values of α_0. The stress intensity factor corresponding to the concentrated applied load P acting at $r = L$ is $k(0) = -0.9770\ P/(\pi L^{1-\omega})$. The solutions of the related problem of an inclined rigid punch on a half plane corresponding to (43) and (44) are

$$f(r) = \frac{P}{\pi c}\sqrt{\frac{b-r}{r-a}} = \frac{P}{\pi c}\sqrt{\frac{1-x}{1+x}} ,$$

$$f(r) = \frac{P}{\pi c}\sqrt{\frac{r-a}{b-r}} = \frac{P}{\pi c}\sqrt{\frac{1+x}{1-x}} , \tag{45a,b}$$

where in both cases $F(\mp 1) = P/(\pi c)$. It is seen that, unless the wedge angles α_0 and $\pi-\theta_0$ are very close to each other, in this case too the concentrated load solution (in conjunction with the half plane solution (45)) may be quite adequate.

The second example in this group is described by Fig. 1a. In this case it is assumed that the nose of the rigid wedge is blunted by two circular arcs tangent to the straight wedge boundaries. Note that even though the arcs have the same radius R, they are not necessarily concentric. In the example it is assumed that $\theta_0 = 165°$, and b, R and a_0 (the length of the straight line profile of the wedge) are known constants with $(a_0/R) = 2$ and $(b/R) = 3$. The contact stress is of the form

$$f(r) = f_o(r)\sqrt{\frac{r-a}{b-r}} = F(x)\sqrt{\frac{1+x}{1-x}},$$

$$(r = cx+d,\ 2c = b-a,\ 2d = b+a). \qquad (46)$$

The problem is again solved in an inverse manner by assuming the contact area as known and calculating the corresponding load P. The contact area is characterized by

$$\gamma = (a_o - a)/R . \qquad (47)$$

Table 6 shows some of the calculated results. The coordinate x and the contact stress coefficient F(x) shown in the Table are defined by (46).

Table 6. The results for a blunted rigid wedge. $\gamma = (a_o - a)/R$, $(a_o/R) = 2$, $(b/R) = 3$. The quantity shown for various values of x is the contact stress coefficient $10^2 F(x)(1+\kappa)/(4\mu R)$ (see (46)).

x \ γ	0	0.05	0.10	0.20	0.30	0.40
1	0	0.388	1.063	2.761	4.929	7.113
0.665	0	0.479	1.316	3.489	6.136	9.052
0.265	0	0.656	1.812	4.854	8.633	12.88
-0.190	0	1.082	3.022	8.286	15.15	23.33
-0.606	0	2.397	6.951	20.83	46.67	63.44
-1	0	32.39	43.10	62.79	80.99	97.37
$-\frac{1+\kappa}{4\mu R} P$	0	0.0139	0.0396	0.1130	0.2123	0.3312
$-\frac{k(0)}{P/(\pi R^{1-\omega})}$	0	0.6252	0.6278	0.6334	0.6390	0.6449

The last example in this group is the problem of an elastic wedge of angle $2\theta_o = 30°$ pressed into a rigid vise shown in Fig. 1d. In this example it is assumed that $(a/R) = 4$, $(b_o/R) = 6$, $\gamma = (b-b_o)/R$ and

$$f(r) = f_o(r)\sqrt{\frac{b-r}{r-a}} = F(x)\sqrt{\frac{1-x}{1+x}}, \quad (r = \frac{b-a}{2}x + \frac{b+a}{2}), \qquad (48)$$

where R again is the radius of the vise profile at the blunted end and b_o-a is the length of the straight line portion of the contact region. The results are shown in Table 7.

Table 7. The results for the elastic wedge pressed into a rigid vise. $\gamma = (b-b_o)/R$, $(a/R) = 4$, $(b_o/R) = 6$. The quantity shown for various values of x is the contact stress coefficient $10^4 F(x)(1+\kappa)/(4\mu R)$ (see (48)).

γ \ x	1	0.606	0.190	-0.265	-0.665	-1	$-\frac{1+\kappa}{4\mu R} P$
0	0	0	0	0	0	0	0
0.03	3822	93.83	52.57	35.11	24.13	16.02	0.0119
0.05	4550	219.1	122.3	81.62	56.07	37.13	0.0277
0.10	6006	619.2	342.5	228.6	156.7	103.6	0.0787
0.20	8845	1797	974.4	649.7	443.5	287.6	0.2297

It should be pointed out that in practical applications the friction between the rigid wedge and the elastic medium may not be negligible as assumed in this paper Also as a result of excessive wedging load cracking may take place at the apex of the wedge. These two aspects of the problem will be dealt with in a forthcoming paper.

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