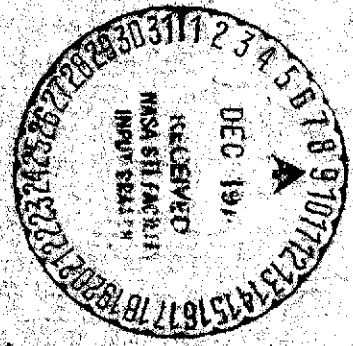Technical Report No. 1
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APPLICATION OF WAVE MECHANICS THEORY

TO FLUID DYNAMICS PROBLEMS--FUNDAMENTALS

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INTRODUCTION

The primary goal of this report is to explain the application of the basic formalistic elements of wave mechanics theory, usually considered as being a proper tool describing the physical phenomena on the microscopic level, to fluid dynamics of gases and liquids, usually considered as being a proper tool to describe the physical phenomena on the macroscopic level (visually observable). The practical advantages of relating the two fields (wave mechanics and fluid mechanics) through the use of the Schroedinger equation will constitute the approach to this relationship.

1. FUNDAMENTAL CONSIDERATIONS RELATING WAVE MECHANICS (MICROSCOPIC) TO FLUID DYNAMICS (MACROSCOPIC)

1.1. The Mathematical and Physical Features

Before a discussion and description of many important, particular aspects of the association of wave mechanics theory to fluid dynamics, a few general remarks are in order. Some particular aspects are immediately obvious from the statement of the general goal:

(a) Considerations of a mathematical nature: As a general (unwritten) rule, the majority of fields in the domain of the mechanics of solids and fluids (liquids and gases) are governed by systems of equations of a nonlinear (sometimes highly nonlinear) nature (Newtonian mechanics). The wave mechanics equation of Schroedinger is a linear equation. A set of Schroedinger equations can be added (summed) or subtracted; their solutions can be multiplied by a constant or a set of constants; this is one of the advantages of linear partial differential equations (the class of equations to which the Schroedinger wave equation belongs). The highly nonlinear terms which appear in the classical equations of Euler and Navier-Stokes (based upon Newtonian mechanics) cannot be compared with the advantages of linear equations. As is well known, there does not exist an exact or a correct definition of the nonlinear aspects of partial differential equations. There exists no knowledge of the characteristic properties and behavior

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of nonlinear partial differential equations (much less of their solutions).

All characteristic features of the linear, as contrasted to the nonlinear partial differential equations, are basic when dealing with equations which describe the important characteristic, physical aspects and physical behavior of liquid or gaseous media. In physical results, more often than not, there arises the necessity of adding the results of the numerical calculations or of multiplying them by a constant. These are only a few examples for illustrative purposes.

(b) <u>Characteristics of a physical nature:</u> The approach in determining the behavior and fundamental characteristics of liquids and gases of macroscopic fluid dynamics is based, among others, upon the notion of density. This means that the smallest amount of the medium which can be considered is the amount of mass per unit volume, e.g. cm^3. The physical considerations are adjusted to this concept and the macroscopic measurements are adjusted to this concept as well. However, in reality, the smallest amount of fluid which can be put in motion due to disturbances which may be introduced into the medium may be smaller (even much smaller) than a cubic centimeter. These and similar problems are not simple when treated exclusively in the domain of macroscopic fluid dynamics; but their solutions become simpler when treated by means of wave mechanics theory. In the Schroedinger equation, the mass "m" of the element in question refers basically to the mass of the electron (m = 0.9107 x 10^{-27} gram or m = 0.911 x 10^{-27} gram, (F. K. Richtmyer and E. H. Kennard, <u>Introduction to Modern Physics</u>, McGraw-Hill Book Company, Inc., 1947, pp. 85 and 216). In practice, one may use the concept of the "cluster" of electrons or the concept of the cluster of molecules guided by the electron under consideration as their "leader" in place of "m" in the Schroedinger equation. The cluster may have a mass with

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reference to the volume less than that of a cubic centimeter. It should obviously refer to a volume corresponding to the mass greater than that of an electron but smaller than the mass of the medium in question with reference to a cubic centimeter. As a matter of fact, the investigator is of the impression that during the phenomenon of perturbations due to disturbances introduced in some cases into a medium, it is not the mass of the medium with reference to the cubic centimeter which takes part in disturbed motion, but rather the mass with reference to the volume smaller than that of a cubic centimeter. This conjecture, based upon visual observations of photographs of disturbed media, requires verification by tests and experimental measurements. The concept of the mass of the medium in question obviously appears in the final calculated or tested results after the mean values are calculated and included into the numerical scheme. The proposition of Madelung (1926) allows one to deal with the wave equation of Schroedinger.

The approach to fluid dynamics based upon the wave mechanics equation of Schroedinger allows one to take into account the mass of the fluid with reference to a volume, which is smaller than that of a cubic centimeter. Consequently, the Schroedinger equation allows one to deal with the phenomena of flow on the scale above the microscopic level (electron level) but below the fully macroscopic level (the level which can be tested by the use of macroscopic instrumentation -- classical, deterministic fluid dynamics). All the remarks made above refer to the state of the fluid above that of super- fluidity. This means that the state of the fluid is above the phase transition phenomenon and above the λ - point (nor- mally at a temperature much above absolute zero).

One additional remark is appropriate. The following description is used with reference to some physical phenomena: nuclear level, nuclear phenomena, extension of quantum theory to the domain of nuclear dimensions, and the like.

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A question may arise as to the kind of particles the work "nuclear" should or might refer to. Since it is desired to avoid a presentation of "Theoretical Physics," the reader is asked to turn to the appropriate literature with regard to these problems of the nomenclature. The most important matter in the presentation of this quantum approach is the "understanding" of the new concept of the new idea.

For the purposes presented in this report it is sufficient to mention that a matter (a solid or a fluid, liquid or gas) (or briefly a substance) is built of molecules. A molecule, in turn, is built of atoms (example: molecule of water vapor is a cluster of two hydrogen atoms and one oxygen atom.) The smaller entities than an atom are: proton, neutron, electron, meson (see O. M. Stewart, Physics, Ginn and Company, 1944, p. 189). The electron has a negative charge and a mass of $\approx 0.917 \times 10^{-27}$ gram (F. K. Richtmayer and E. H. Kennard, Introduction to Modern Physics, McGraw-Hill Comp., 1947, p. 85). One of the fundamental equations describing the behavior of small entities in various physical phenomena is Schroedinger wave equation (1926). This will be used below by the writer. The element appearing in Schroedinger equation is an electron. The elements heavier than an electron are built from a combination of protons and neutrons. Consequently, in the approach, used below by the writer, the notion of an electron is generalized to the notion of a cluster of protons, neutrons, atoms and molecules gathered around the electron in question and guided by it. In such a way, the author is allowed to apply the Schroedinger equation to any mass element which ever it may appear in the physical phenomenon in question, starting from the electron on one side and ending with classical, macroscopic concept of density on the other.

In previous centuries only one approach was used in the description and application of the theory of fluid dynamics; i.e., the approach based upon the Newtonian mechanics. This approach was, and still remains very powerful. From the mathematical point of view, it is based upon the deterministic mathematics; i.e., the theory of functions (in particular, theory of analytic functions). Many problems in the past were solved using this approach which gave good agreement with experimental

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data. The fundamental equations used in these approaches were Euler's equation, Navier-Stokes equation and equations based upon Boltzmann kinetic theory.

The beginning of the twentieth century witnessed the appearance of another approach to describe the dynamic-mechanical problems; viz, the quantum approach. This approach uses probabilistic mathematics as its tool, and is primarily used to describe phenomena which are so small (such as the motions of an electron), that they cannot be described by the macroscopic, deterministic equations. For a number of years science tried to make a definite distinction between the fields of science operating in the small (molecular) and in the large (macroscopic) phenomena. The first was referred to as physics, and the second as mechanics (of solids, of fluids, of gases, etc.). Over the course of years it became obvious that the phenomena in the "small" influenced the phenomena in the "large", and vice versa, and that distinctions above should be abolished. Thus physics entered the field of the mechanics of solids, thereby explaining such phenomena as "superfluidity." This, however, is not yet sufficient since there are many practical phenomena, such as the interaction of the mechanics of fluids or gases with electromagnetic wave propagation, which need explanation of the interaction phase.

Such an approach can belong to both the mechanics of electro-magnetic wave propagation and to the mechanics of fluids. There are some instances where immediate application to both electromagnetic wave propagation and turbulent fluid behavior is in order. Nature does not and cannot care about any such distinction between phenomena. All such phenomena are "natural" phenomena, and distinctions were made by scientists alone. This enables the scientist to provide an easier description so that others may better understand.

The results of the "interference" and of the "interaction" described previously appear in the phenomenon of "refractivity", which is very important in considering the problem of wave propagation through fluid media (radio waves). The analytical approach to the problem of refractivity can be theoretically approached from the viewpoint of Newtonian fluid mechanics or from the viewpoint of electromagnetic wave propagation

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(quantum). The problem is important to physicists, applied mathematicians, and engineers who are engaged in the field of propagation. In the past the first approach produced useful results, but in the last twenty or thirty years it has become obvious that the approach from the quantum side can possibly give even better results. Thus, in the last decades the American and Soviet schools of physics developed the so-called "methods of quantum field theory in statistical physics." These became very powerful in explaining such phenomena as superconductivity, theory of the Fermi liquid, electromagnetic radiation in absorbing media, refractivity, etc. Since the methods are relatively new, a decision has not as yet been reached as to which particular field is the appropriate one for having them included; whether classical physics, modern physics, applied physics, theoretical mechanics, applied mechanics, fundamental mathematics, applied mathematics, physico-mathematics, or some new special field that might be developed.

Another question is at which university level these methods should be given to students -- first year graduate students, second year graduate students, or to only Ph.D. candidates in their last year of work -- and should exposure be in regular courses or seminars only. It is clear that the above concepts cannot be treated as rules but only as propositions and they are not general propositions, but are such that they should be applied only in special cases, namely to cases in which they might give better practical results than those presently obtainable. It is generally known that the methods of quantum field theory in statistical physics have recently become (in some fields and for some problems) the strongest methods presently available to scientists.

As mentioned above, the purpose of the present work is a two-fold one: formalism of the application of the wave mechanics to macroscopic fluid dynamics and a discussion of practical advantages of the association of wave mechanics with macroscopic fluid dynamics. The first step towards this goal will involve laminar flow having visible streamlines. Disturbances in the form of a curl of the velocity vector are introduced into this system. Due to the fact that the entire flow system is described in terms of wave functions and by means of the Schroedinger equation, the above disturbances can be geometrically

added (superimposed) upon the laminar flow system.

An analogous situation occurs in the phenomenon of turbulence. Truly speaking, in the case of turbulence, there may also appear the phenomenon of resonance which is an item to be discussed separately. In order to avoid confusion in the nomenclature, the words "disturbed flow" will be used to describe the status of the flow system in which turbulence may appear, as well as any other kind of mechanical or thermal disturbance. As an example of a disturbed flow system, turbulent flow, turbulence, theories of turbulence, classical theories of turbulence, and so on, will occasionally be cited.

The proposition of applying wave mechanics theory to turbulence has an advantage over the classical statistical theories of turbulence which use highly nonlinear (unsolvable) classical, deterministic Navier-Stoker equations; the wave mechanics theory uses Schroedinger equation (or a system of linear equations and solutions). In the classical statistical theories, the concept of mean values is defined according to Reynold's rules (1883), which refer to macroscopic observations in gaseous and liquid flows. These have been accepted by subsequent investigators. In wave mechanics theory the assumption that turbulence is a natural phenomenon, which becomes observable at Re_{cr} but exists above the transition point, is the important factor which enables one to apply this "natural" approach to turbulence. All systems which oscillate "are quantized, whether they are material oscillators, sound waves, or electromagnetic waves." In quantum theory, the mean values are defined without ambiguity. In classical statistical theories, mathematics (namely algebra) could not adequately solve the question of Renold's mean values (see the works of G. Birkhoff, J. Kampe' de Fe'riet, and others). In wave mechanics theory the propositions of the mean values in the probability calculus are generally accepted and successful. In classical statistical theory, the existing theories of turbulence are not too successful, whereas in the wave mechanics approach it will be demonstrated that wave mechanics can be used to attack the problem of "self-excitable" turbulence (C.A.T.). In the classical statistical approach the use of Navier-Stokes equations violates the fundamental theorem and proof of John von Neumann on "hidden variables." In classical statistical theory,

the notion of correlation functions between two (or more) points was introduced, while in wave mechanics the same can be done with the use of the wave function. In classical statistics the trend is to solve the deterministic, causal (or stochastic) system (nonlinear) in a classical manner; in wave mechanics the wave functions and equations give only the probabilistic value, but in the realistic (classical) limit the observation is so rough that the difference between "the probable and actual behavior is never detected." In classical statistics seldom reference is made to the phenomenon of resonance. In wave mechanics the writer calls attention to this phenomenon in C. A. T., and it is possible that the phenomenona appear in several other kinds of disturbances as well. Actually, a "laminar" flow does not exist. That which is referred to as macroscopically and observable laminar flow is, in reality, microscopically turbulent flow. The value of the observable property of laminar flow is actually the mean value of the property of a submacroscopic (or microscopic) turbulent flow. In reality, it was W. Heisenberg in 1948 who stated that fluid dynamicists should give greater research emphasis to laminar flow rather than to turbulent flow.

1.2. Elementary Considerations Relating to the Submacroscopic Nature

In recent years, remarkable success has been achieved in many domains of statistical physics due to the extensive use of methods based upon the quantum field theory. Statistical physics studies the behavior of systems consisting of a very large number of particles. In the last analysis, the macroscopic properties of liquids and gases are due to microscopic interactions between the particles making up the system. The overall macroscopic characteristics are determined by certain average properties of the system. The macroscopic state of a system is specified by the pressure P, the temperature T, and the average number of particles N in the system. A closed system of N particles is characterized by its energy levels. Due to the wavy structure of matter, the smallest elements of the medium are subject to vibrations and systems are described as a superposition of monochromatic plane waves. Each wave is characterized by a wave vector, $\vec{k}$ and consists of several branches, $\omega_s(\vec{k})$, the total number of branches equal to 3r, where r is the number of particles belonging to the unit volume of the medium. For small

momenta, the three (accoustic) branches are characterized by the fact that the frequency depends linearly on the wave vector:

$$\omega_s(\vec{k}) = u_s \, |\vec{k}| \,, \quad u_s = \text{velocity of sound.} \tag{1.2.1}$$

Unless the contrary is explicitly stated, a system of units in which both Planck's constant and the velocity of light equal 1 will be used. From a knowledge of the frequency spectrum, the energy levels, and the coordinates, one can calculate, in principle, the thermodynamic and kinetic characteristics of the vibrating elements. Another model can be obtained by applying the "correspondence principle," which states that every plane wave corresponds to a set of moving particles (called phonons) with momentum determined by the wave vector $\vec{k}$ and energy determined by the frequency $\omega_s(\vec{k})$. This leads to an expression for the energy levels of the system which is analogous to that for an ideal gas. With n_i interpreted as the number of phonons in the state $i = (\vec{k}, s)$ where n_i covers the range of all integers, including zero, the energy spectrum of a system is given by the formula:

$$E = \sum_{i=1}^{3N} \omega_i(n_i + 1/2) \quad \text{(Bose statistics)} \tag{1.2.2}$$

The phonons and neutral gases obey Bose statistics.

Since this work is restricted solely to the question of the application of wave mechanics to macroscopic fluid dynamics, it will be impossible to present many details of the fundamental nature of wave or quantum mechanics. Consequently, for more details on the fundamentals of wave or quantum mechanics and of possible models of the nature of quantum, the reader should refer to the appropriate literature.

Two more items which are pertinent to the general considerations of the physical and geometrical nature of the association of wave mechanics with the macroscopic fluid dynamics are phase transition and the stress-strain deformation system. As is seen from the introductory remarks, the general field of fluid dynamics (liquids and gases) can be divided roughly into two subfields: (a) fluids (both liquids and gases) which obey the laws of Newtonian (1642 - 1727) mechanics, and may be described

with sufficient accuracy by means of the language of classical, deterministic Newtonian mechanics, and (b) fluids (both liquids and gases) in which the physical behavior demonstrates very clearly a lack of complete determinism of elementary processes. This compels science to accept far-reaching changes in the general concepts concerning the fundamental nature of matter and of energy. In the description of the phenomena in these kinds of fluids one has to use the language of modern mechanics, wave and quantum mechanics, originated by Bohr (1913). In particular, when describing the behavior of some fluids at low temperatures, one must use the language of modern mechanics. For example, the research was initiated with the very elementary concept of super-fluidity (or superfluid fluids) where the fluids in question belong to the class of Bose statistics fluids (liquids and gases). The most interesting property of a Bose liquid is the property of "superfluidity," i.e., the possibility of flowing through capillary tubes without friction (Landau, 1941) (liquid helium).

Consider a Bose liquid at absolute zero, flowing with velocity v in a capillary. In the coordinate system fixed with respect to the liquid, the liquid is at rest and the capillary moves with velocity -v. As a result of friction between the liquid and the wall of the capillary, the liquid begins to be "carried along" by the wall. This means that the liquid begins to have non zero energy and momentum, which is possible only if elementary excitations appear in the liquid. As soon as a single such excitation appears, the liquid acquires momentum p and energy $\epsilon(p)$.

Now, suppose one is to go back to the coordinate system fixed with respect to the capillary. In this system, the energy of the liquid equals:

$$\epsilon + p \cdot v + 1/2\ M \nu^2.$$

(1.2.3)

Thus, the appearance of an excitation changes the energy by an amount $\epsilon + p \cdot v$. In order for such an excitation to appear, the change in energy must be negative, i.e.,

$$\epsilon + p \cdot v < 0.$$

(1.2.4)

The quantity $\epsilon + p \cdot v$ takes its minimum value when p and v have opposite directions. Thus, in any case, one must have:

$$\epsilon - p\nu < 0 \text{ or } \nu > \epsilon/p. \tag{1.2.5}$$

See: (A. A. Abrikosov, L. P. Gorkov, J. E. Dzyloshinski , Methods of Quantum Field Theory in Statistical Physics, 1963, p. 11).

This means that in order for it to be possible for any excitations to appear in the fluid, the velocity must satisfy the condition:

$$\nu > (\epsilon/p)_{min}. \tag{1.2.6}$$

The minimum value of ϵ/p corresponds to the point of the curve $\epsilon(p)$ where:

$$\frac{d\epsilon}{dp} = \frac{\epsilon}{p}, \tag{1.2.7}$$

i.e., the point where a line drawn from the origin of coordinates is tangent to the curve $\epsilon(p)$. Thus, superfluid flow can occur only in the case where the velocity of the liquid is less than the velocity of the elementary excitation at the points satisfying the condition (1.2.7). It is recalled that $d\epsilon/dp$ is the velocity of the elementary excitation. For every Bose liquid, these always exists at least one point where the condition (1.2.7) is satisfied: namely, the origin of coordinates $P = 0$. Since for values of p near zero, the excitations move with the velocity of sound, the superfluidity condition is certainly not satisfied for flow velocities exceeding the velocity of sound u.

Thus, one obtains the following general picture of the motion of a Bose liquid when the velocity is such that the superfluidity condition holds. First, the temperature $T = 0$ (absolute zero) is considered. If the liquid is initially in the ground state, i.e., if it contains no elementary excitations, then no excitations can appear later and the motion is superfluid. For $T \neq 0$, the picture essentially changes in such a way that the fluid contains excitations whose number is determined by the appropriate statistical formulas. Although new excitations cannot appear, nothing, as noted above, can prevent the excitations already present from colliding with the walls, thereby exchanging momentum with the walls. Only a part of the mass of the liquid participates in this viscous motion. The remaining part of the mass of the liquid moves as before, with no friction between it and the walls or between it and the part of the fluid participating in the viscous flow. Thus, at $T \neq 0$, a Bose liquid represents a kind of mixture of two liquids, one which is "superfluid" and the other which is "normal", moving with no friction between them.

Of course, in reality no such separation occurs, and there are simply two motions in the liquid, each of which has its own effective mass or density. First, there is the "normal" density; this is denoted by ρ_n. The remaining part of the density of the fluid, denoted by ρ_s, corresponds to superfluid motion, and hence:

$$\rho = \rho_n + \rho_s. \tag{1.2.8}$$

Let v_n denote the macroscopic velocity of the gas of excitations, and let v_s denote the velocity of the superfluid liquid. Then the velocity v_s has the following basic property: If the Bose liquid is put in a cylinder and the cylinder rotates about its axis, the normal part is "carried along" by the walls of the cylinder and the liquid itself begins to rotate. On the other hand, the superfluid part remains at rest, and hence does not have to be taken into account. In other words, the motion of the superfluid part is always irrotational, a fact which is expressed mathematically by the condition

$$\text{curl } v_s = 0 . \tag{1.2.9}$$

The motion of the superfluid part of the liquid imposes certain conditions on the excitations. In fact, it is noted that it is precisely in the reference system fixed with respect to the superfluid part that the function $\epsilon(p)$ has the form discussed above. In the rest system, obviously one has:

$$\epsilon' = \epsilon(p) + p \cdot v_s , \tag{1.2.10}$$

where p is the momentum in the reference system fixed with respect to the superfluid liquid. This has to be taken into account in writing the transport equation for the excitations.

The fact that a Bose liquid contains two types of motions with different velocities leads to a very distinctive kind of hydrodynamics whose equations can be derived from the transport equation. The "two-velocity hydrodynamics" of a Bose liquid differs from ordinary hydrodynamics in many ways. In particular, it turns out that two different kinds of oscillations can occur in a Bose liquid, with two different velocities of propagation. The oscillations of the first kind represent ordinary sound, or what is called "first sound," with velocity of propagation equal to u. In a sound wave of this kind, the liquid moves as a

whole, i.e., the normal and superfluid parts do not separate. The oscillations of the second kind, the so-called "second sound," propagate with velocity of a different value. In a wave of this kind, the oscillations of the normal and superfluid parts of the liquid have opposite phases, and hence the total flow vector of the liquid is:

$$\vec{j} = \rho_n \vec{v}_n + \rho_s \vec{v}_s \approx 0.$$ (1.2.11)

1.3. Transition Point

Let us now consider what can be said about the behavior of a Bose liquid at higher temperatures, when the number of excitations in the liquid becomes large. In this case, interaction between excitations can no longer be neglected, and it can be assumed that this picture is preserved for relatively high temperatures. The same applies to the hydrodynamical equations, since they are actually consequences of conservation laws. As the temperature increases, the normal density ρ_n increases until it reaches a value equal to ρ. At this point, called the λ-point, a phase transition occurs in the medium. Below the transition point, superfluid motion is possible. However, above the transition point, superfluid motion is no longer possible, and the hydrodynamics of the Bose liquid do not differ from ordinary hydrodynamics.

In principle, the transition from $\rho_n \neq \rho$ to $\rho_n = \rho$ might take place either continuously or discontinuously. It follows from an experiment with helium that the normal density ρ_n grows continuously as the temperature increases and becomes equal to ρ at the λ-point. Considerably above the λ-point, helium has no peculiarities of behavior as compared with an ordinary liquid. As for the neighborhood of the λ-point, there is good reason to expect a number of essentially new properties. The problem of the behavior of various characteristics of systems, especially their thermodynamic properties, in the neighborhood of a point where a phase transition occurs, remains partly unsolved at present and represents one of the most interesting problems of the physics of matter in the condensed state.

The macroscopic conditions may be expressed in terms of the Navier-Stokes equations. The present task is to associate the Navier-Stokes equations with the Schroedinger equation. In such a manner, a

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link between a flow domain in a viscous gas and the mathematical for-
malism of the wave mechanics theory is to be constructed. The coeffi-
cients of viscosity and heat conductivity are supposed to be known and
given from the tables of standard atmosphere. It bears repetition to
state the the present approach with the use of the wave mechanics refers
to the fluid above the λ-point.

All the above assumptions, which use the Navier-Stokes equations,
enables the writer to treat the fluid system as a Bose liquid above the
phase-transition λ-point. Below this point the fluid is moving as a two-
phase fluid (super-fluidity and super-conductivity phenomena may be
present). Above this point the fluid dynamics of a Bose liquid does not
differ from the ordinary, classical hydrodynamics.

1.4. General Stress-Strain System in Macroscopic Viscous Fluids

The characteristic features of macroscopic fluid mechanics
which were essentially developed prior to the twentieth century to describe
the phenomena occuring in inviscid and nonheat-conducting liquids and
gases will not be discussed. It can be said that the actual development
of the field of viscous, heat-conducting liquids and gases took place in
the twentieth century. Ludwig Prandtl (Goettingen, Germany, 1904)
and his school are responsible for developing the field in a modern sense.
They based their development on previous results, particularly those of
Euler (1752 - 1755). These earlier results usually referred to inviscid,
nonheat-conducting (i.e., ideal) liquids and gases. When approaching
the theory of viscous, heat-conducting liquids and gases, one of the most
characteristic aspects which must be taken into account is the influence
of the viscosity and heat conductivity upon the dynamics of the system.
Prandtl based his approach upon the theory of solids simply because
nothing better was available at that time in either the field of solids or
fluid media.

In passing from the theory of ideal fluids to the theory of viscous
fluids, Prandtl assumed the validity of the concept of the existence of
stresses and strains in the fluid body (analogous to solids). Next, there
appeared the concept of deformable bodies in which deformations,
elongations, displacements, angular displacements, expansions, angular
deformations, and others were taken into consideration. In the case of

elastic solid bodies the well-known Hooke's law was used. In the case of viscous fluids, Stokes' law of friction was substituted. These two laws are so intimately related to one another that in deriving either one of them, the other is simultaneously obtained. The only difference between them is that in Hooke's law for elastic bodies, the forces which oppose the deformation of a body are proportional to the magnitude of the strain, whereas in Stokes' law of friction in fluids, these forces are proportional to the rate-of-strain.

Omitting the details of this development in the post-Prandtl era, the fact is that this proposition was and still is very successful. It has solved many problems in applied physics and engineering. As can be readily seen from the general outline above, the approach is characterized by its "macroscopic" nature. A volume of the fluid in question is treated as a solid. The internal interaction of intermolecular forces "in the small" are taken into account by means of the coefficients of viscosity and heat conductivity where these coefficients have to be taken from the experimental data. To obtain better agreement between the analysis and experimental data, two coefficients of viscosity were introduced. In order to find a relation between these two coefficients of viscosity, the kinetic theory of gases was used. This furnished the most elementary solution -- namely zero, and allowed one to derive the relation sought between the two coefficients of viscosity, $\mu_1 = -2/3\mu$. This appears to have been verified experimentally by measurements of acoustic absorptions which have been made only for perfect monatomic gases. Similar experiments with liquids, however, have shown that values for the viscosity ratio differ from 2/3. It was decided to take the Navier-Stokes equation (1822, 1826, 1845) as the fundamental equation governing the dynamics behavior of viscous fluids.

The above outline clearly demonstrates the fact that present day "macroscopic" fluid dynamics treats liquids and gases as large macroscopic entities. Consequently, the phenomena which refer to these media are also treated in a crude, macroscopic manner. No refinements can be introduced because it is highly questionable whether one can gain insight into the phenomena taking place inside the cubic volume (borrowed from the theory of mechanics of solids). The numerical

coefficients of viscosity, first and second, are very approximate and do not take into account the phenomena appearing inside the cubic volume, such as inter-molecular forces, intermolecular action and reaction when external forces are acting upon the external walls of the cubic volume. Nothing can be said about the interference phenomena between the particular molecules or about the dynamic action of molecules upon each other. There is not even a possibility of accomplishing this since no equation in the macroscopic theory of fluids exists which takes into account the intermolecular action between particular molecules. However, it is a known fact to modern physicists that phenomena in "the small" -- on the microscopic scale--influence, and to some degree may even control, phenomena in "the large"--on the macroscopic scale. From the known phenomena in the field of fluid dynamics, the problem of turbulence, discovered by Reynolds in 1883, was investigated by many scientists using macroscopic fluid dynamics. The analytical system used is, of course, the Navier-Stokes system. It can be stated now that, after many decades of research on the problem, the macroscopic approach to turbulence in viscous gases and liquids through the Navier-Stokes system, statistical specification of the turbulent field, velocity correlation functions at n-points, joint probability density function, joint probability averages, correlation tensors, spectrum tensors, and many other items, has not been too successful. It must be admitted, however, that some of the results obtained were promising, but even today the problem of turbulence remains unsolved and not well understood.

In contrast, the field of quantum fluid dynamics concerns phenomena "in the small", the microscopic scale, as, for example, the phenomena in liquids at low temperature such as liquid helium and other similar, very difficult problems. Almost from the beginning of its operation, quantum fluid dynamics proved to be very successful. This success seems to be either partly or predominantly due to restrictions superimposed by methods used in the field. The problems are usually restricted to single particles or small numbers of particles. In using the methods of quantum field theory in statistical physics, one may be interested in interactions between two particles (electron-phonon interactions) and in similar interparticle phenomena. The mathematical tool used is restricted to the Schroedinger equation for one particle

(electron) or to the Schroedinger representation (operator) of the changes in time of the state of the system (density) of particles. The N particles in question are originally in identical states, normalized by the proper relation. In some cases, one may use the Heisenberg representation in place of the Schroedinger representation.

The above discussion demonstrates very clearly that, in the description of some dynamic phenomena in fluids, it may be better in some cases to resign from the macroscopic representation (stress-strain deformation system in a cubic volume) and to restrict one's attention to the investigation of single small particles and to the interaction of one or two single small particles. After such small-scale phenomena are understood, one may then investigate the phenomena in groups of three, four, six, or ten particles, after which one can generalize the results obtained to the "n" similar groups (clusters) of elements in the domain under consideration.

It can now be clearly seen that the methods of operation in both fields are different. Quantum or wave mechanics starts from small, microscopic elements; includes a small number of interactions between these elements (one, two, or three), in which all are contained inside a large, macroscopic volume; and generalizes the results obtained in this manner to a larger number of such groups of microscopic elements with a small number of interactions. The classical, deterministic fluid dynamics starts from a large, macroscopic volume, containing many small, interacting microscopic elements; disregards the interaction phenomena between the great number of small, microscopic elements inside the large macroscopic volume; and generalizes the results obtained in this manner to a larger number of such macroscopic elements, always disregarding the exact and precise calculation of the influence of the interactions of small, microscopic elements upon the physical status of large macroscopic elements. The coefficients of viscosity (first and second), which supposedly should take into account the influence of interactions between small microscopic elements upon the physical status of large macroscopic elements, have values obtained from obsolete macroscopic experimental test data and consequently cannot represent precisely the results (of primary importance) of interactions between

microscopic elements and their influence upon the physical status of large macroscopic elements.

1.5. Experimental Coefficients

The previous section demonstrated that exact knowledge of the numerical values of the physical coefficients is extremely important. The coefficients are: first coefficient of viscosity, second coefficient of viscosity, and coefficient of heat transfer. One should also include here the precise values of the coefficient of kinematic viscosity. Moreover, in investigating the phenomena in the atmosphere, particularly the upper atmosphere, one should have an excellent understanding of the heat phenomena in the upper atmosphere, such as forced and natural convection. Knowledge of the physical coefficients in all of the above-mentioned phenomena is of the greatest importance. Unfortunately, our knowledge in this respect is not only very limited, but it is obsolete. The ratio of the two coefficients of viscosity is always given in the form $\mu_1 \mu^{-1} = -2/3$, although it is well known that this value is true in only one particular case, a case which does not always occur in practice.

The conclusion is that more precise and more exact values of the physical coefficients should be supplied for all the phenomena, particularly when the variations in pressure, temperature, vapor coefficients and coefficients of viscosity due to changes in altitude have to be taken into account. This is particularly important for the conditions conductive to the phenomenon of C. A. T.

2. FUNDAMENTAL ASPECTS OF WAVE MECHANICS THEORY

2.1. Schroedinger Equation and Its Characteristic Properties

This research omits all the explanations and details referring to the physical characteristics of wave mechanics theory, in particular the characteristic quantum theoretical features such as wave-particle duality, wavy nature of matter, and so on. The description will be limited to the formal aspects and to the formal association between wave mechanics theory and classical, deterministic fluid dynamics.

The fundamental equation which describes wave mechanics phenomena in the formal, mathematical language is the Schroedinger wave equation of the form

$$\nabla^2 \psi - 8\pi^2 mh^{-2}\Phi\psi - i\,4\pi mh^{-1}\partial\psi/\partial t = 0 , \qquad (2.1.1)$$

where

∇^2 = Laplace operator (Laplacian) in the three-dimensional Cartesian coordinate space (x,y,z);

ψ = the wave function, $\psi = \psi(x,y,z,t)$;

m = the mass of the electron;

h = Planck's constant = 6.6×10^{-27} erg-sec.;

t = time (sec);

Φ = the external potential energy (function of the position of the electron);

$i = \sqrt{-1}$.

From the formalistic point of view, the basic concepts of wave mechanics employ probability theory as the main tool in its application for the solution of various problems. Although not generally realized, the treatment of wave mechanics as a field of mechanics has been receiving increased attention during the past several decades. This is because the results of wave mechanics, when applied to many fields of mechanics, correspond closely with those obtained from many of the tests and experiments in mechanics. Thus, it is very difficult to talk about the differences in values obtained from numerical analysis or from experiments. As a matter of fact, in recent years remarkable success has been achieved in physics by the extensive use of methods borrowed from quantum field theory.

The success of these methods is associated in some cases with the application of "Feynman diagrams." The basic advantage of the diagram technique lies in its intuitive character. Operating with one-particle concepts, one can use the technique to determine the structure of any approximation and the required expressions can then be written with the aid of correspondence rules. These new methods make it possible not only to solve a large number of problems which do not yield to the old formulation of the theory, but also to provide many new relations of a general character.

From the wave equation, Equation (2.1.1) and using the wave function in the form $\psi = a \exp(i\,\beta)$, with (a,β) dependant upon both

time and space, in 1926 Madelung obtained two equations, both real, $L_{R,I}$, which denote the real and imaginary parts, R and I respectively:

$$L_R (\psi) = \frac{\partial\phi}{\partial t} + \frac{1}{2} (\nabla\phi)^2 + \Phi m^{-1} - (\nabla^2 a)\, a^{-1} h^2 (8\pi^2 m^2)^{-1} = 0 \quad (2.1.2)$$

$$L_I (\psi) = \nabla \cdot (a^2 \nabla\phi) + \partial (a^2)/\partial t = 0 \quad (2.1.3)$$

where: $\phi = -\beta h (2\pi m)^{-1}$.

Applying the ∇ operator to the equation (2.1.2) and using

$$\vec{U} = \nabla\phi \text{ (velocity vector) and}$$
$$a^2 = \rho \text{ (density of the medium)}, \quad (2.1.4)$$

Modelung obtained

$$\partial\vec{U}/\partial t + \frac{1}{2} \nabla(\vec{U}^2) + m^{-1} \nabla\Phi - \nabla[\, a^{-1} \nabla^2 ah^2 (8\pi^2 m^2)^{-1}] = 0, \quad (2.1.5)$$

$$\nabla \cdot (\rho\vec{U}) + \partial\rho/\partial t = 0, \quad (2.1.6)$$

$$\nabla(\nabla\phi)^2 = 2[\, \vec{U} \cdot \nabla\vec{U} + \vec{U} \times (\nabla \times \vec{U})], \quad (2.1.7)$$

where $\nabla \times \vec{U} = 0$,

with equation (2.1.5) corresponding to the Euler equation of motion and equation (2.1.6) to the equation of the conservation of mass (continuity). The following interpretation of Madelung's development can be proposed:

$m^{-1}\nabla\Phi$ represents $\rho^{-1}\vec{f}$ = the action of the extraneous force field, and $a^{-1} \nabla^2 a h^2 (8\pi^2 m^2)^{-1}$ = the action of the static pressure, which is equivalent to the term $[\, -\int \rho^{-1} dp]$ in classical macroscopic fluid dynamics, where p denotes the static pressure in the medium.

Equations (2.1.5) and (2.1.6) are equivalent to equation (2.1.1) and represent a form of transformation of the wave equation, (equation (2.1.1),), into two parts, one real and the other one imaginary. They are obtained by elementary operations, such as the gradient operator ∇, the decomposition of the Schroedinger wave equation, and so on. Hence, equations (2.1.5) and (2.1.6), which use different terms and definitions, essentially represent the Schroedinger wave equation, (equation (2.1.1),). From equation (2.1.7) it is clear that only the term $\vec{U} \cdot \nabla\vec{U}$ was retained during the operations. The other term, curl $\nabla \times \vec{U}$, was assumed to be equal to zero. Since the wave equation is a linear equation, various

particular solutions of it can be added, thus providing a general solution of equation (2.1.1). Consequently, one is justified in assuming that $\nabla \vec{U} = 0$ and $\nabla \times \vec{U} \neq 0$ is another solution of the wave equation, which provides a particular solution to equation (2.1.1) for a flow field in which there appears only the curl $\vec{U}$ distributed according to the boundary (or other) conditions of the domain. The particular solutions obtained in this manner can be added, due to the linearity of equation (2.1.1). This enables the construction of various geometrical (topological) and mechanical (dynamical) aspects of the real fluid dynamic flow pattern, which can be solved part by part and then added to provide the answer to the problem (analysis and synthesis).

In using this wave mechanics approach as the main tool to solutions of practical problems and applying analysis and synthesis, one has to use the formalism of probability theory so as to be satisfied with the "probable results." This is somewhat parallel to the results obtained in the classical theory of fluid dynamics, which uses the deterministic, classical, and rigorous mathematics of function theory and analysis as the tool. The fundamentals of quantum theory were excellently developed by John von Neumann in his book Mathematische Grundlagen der Quantenmechanik, Berlin, Juilius Springer, 1932.

2.2. Scale Magnification Factor

These investigations are developed in the microscopic domain; i.e., in wave mechanics theory where the Planck constant, h, the mass of the electron and the velocity of light are the most characteristic factors. One looks upon the phenomena described above as through an enormous magnifying glass. This is all possible due to the validity of the hypothesis of linearity in quantum theory. To elevate the phenomenon from the microscopic domain to the macroscopic reality, a new number, the Planck K-number, in place of Planck constant is proposed. The Planck K-number may be considered to be a parameter which varies from point to point. The value of the Planck K-number (PKN) is defined as:

$$PKN = h \, (1 + INFh^{-1}); \quad INF = \rho \ U_\infty L \times \text{unit volume} \qquad (2.2.1)$$

or: $PKN \approx \rho \ U_\infty L \times \text{unit volume}$, where U_∞ = velocity,

often denoted by the symbol V.

The Planck constant (mass x velocity x length) and the quantity INF defined as the numerator in the Reynolds number multiplied by a unit volume have the same dimensions.

The problem of the extension of wave mechanics (quantum theory) to the domain of the macroscopic scale is a subject of never-ending discussion. In this section only a few items of immediate importance can be discussed. In the formulation (by Madelung) of the quantum theory in the hydrodynamic form there appears two constants, the Planck constant and the mass of the electron, and consequently three possible cases and three possible terms can appear in problems under consideration:

(1) terms in which there appears a ratio of $h\,m^{-1}$, or $m\,h^{-1}$;

(2) terms in which there appears only the factor m;

(3) terms in which there appears only the factor h.

Case 1.   Term with the factor $h\,m^{-1}$

The writer discusses case (1) first. The ratio $(h\,m^{-1})$ appears in equation for the function ϕ:

$$h = 6.62517 \times 10^{-27}\ gr\ cm\ sec, \qquad (2.2.2)$$

$$m = 0.9107 \times 10^{-27}\ gr; \qquad (2.2.3)$$

where both constants contain mass in gr units and

$$h\,m^{-1} = 6.62517\ (0.9107)^{-1}\ cm\ sec. \qquad (2.2.4)$$

Consequently the corresponding functions become:

$$\phi = -\beta \times 6.62517\ [\ 2\pi \times 0.9107]^{-1}\ cm\ sec. , \qquad (2.2.5)$$

$$\beta = -2\pi \times 0.9107\ [\ 6.62517]^{-1}\ \phi. \qquad (2.2.6)$$

The wave function in the Madelung proposition has the form

$$\psi = a\ \exp\ (i\ \beta) = a\ \exp\ i[\ -2\pi \times 0.9107\ (6.6251)^{-1}\ \phi]\ ; \qquad (2.2.7)$$

$$\vec{U} = grad\ \phi = \vec{U}\ (u,v). \qquad (2.2.8)$$

Choosing the flow along an infinitely long, flat plate as an example, then in the boundary layer under consideration, the tables (from Schlichting and Howarth), give the values of f, f', f'', for various values of the composite variable, η, with $\eta = \eta\ (x,y)$ for the horizontal component as follows:

$$\eta = y\ (U_\infty\ \nu^{-1}x^{-1})^{1/2};\quad u = U_\infty f' = U_\infty \frac{df}{d\eta};\ f = f(\eta); \qquad (2.2.9)$$

$$\psi = \psi(x,y);\ \psi = \nu^{1/2}x^{1/2}\ U_\infty^{1/2}f(\eta); \qquad (2.2.10)$$

$$u = \partial \psi / \partial y = (\partial \psi / \partial \eta)(\partial \eta / \partial y) = U_\infty f'(\eta); \tag{2.2.11}$$

$$v = -\partial \psi / \partial x = 1/2 \, \nu^{1/2} U_\infty^{1/2} x^{-1/2} (\eta f' - f). \tag{2.2.12}$$

The reader is reminded that symbol ψ as used has two different meanings: the wave function in the Schroedinger equation, equation (2.1.1), and the stream function in the Blasius-Prandtl boundary layer equation, equation (2.2.10). Hereafter these two functions will be denoted by different subscripts:

ψ_S = wave function (Schroedinger);

ψ_{BP} = stream function (Blasius-Prandtl; $f(\eta) = f_{BP}(\eta)$.

In the present case ρ = constant, $a^2 = \rho$ (density) = constant; $a = \rho^{1/2}$, $\phi = \phi(\eta)$ and:

$$\nabla \phi = \text{grad} \, \phi = d\phi(\eta); \tag{2.2.13}$$

$$(\partial \phi / \partial x) = (d\phi / d\eta)(\partial \eta / \partial x); \tag{2.2.14}$$

$$(\partial \phi / \partial y) = (d\phi / d\eta)(\partial \eta / \partial y). \tag{2.2.15}$$

The above proposition is a link between the functions

$$\psi_S = \psi_S(x, y, t); \, \psi_{BP} = \psi_{BP}(x, y) = \psi_{BP}(\eta) \text{ and}$$

$$f(x, y) = f(\eta); \, \beta = \beta(x, y, t); \, a(x, y, t); \, a^2 = \rho(x, y, t)$$

From the Blasius approach:

$$u = U_\infty f'(\eta); \tag{2.2.16}$$

$$v = \frac{1}{2} \nu^{1/2} U_\infty^{1/2} x^{-1/2} (\eta f' - f). \tag{2.2.17}$$

In a two-dimensional flow the space dependence involves two coordinates (x, y) and time. Equation (2.2.6) gives:

$$\beta = -2\pi \times 0.9107 \, (6.62517)^{-1} \, \phi = A\phi; \, \phi = f(\eta) \, U_\infty \tag{2.2.18}$$

$$A = -2\pi \times 0.9107 \, (6.62517)^{-1}. \tag{2.2.19}$$

The reader's attention is called to the fact that passage from the microscopic domain of quantum theory (Planck constant $\sim 10^{-27}$ and mass of the electron $\sim 10^{-27}$) to the macroscopic domain of classical or diabatic flow involves preservation of variables such as the density of the medium which are measured, tested, and subjected to macroscopic experimentation. Each particle of the medium, however small it may be, is subjected

directly to the action of the external force fields of action in the same manner as is accomplished in the ordinary, macroscopic mechanics, through the use of Avogadro's Number. For all practical purposes, no reference has been made to the possibility of using any existing or proposed extension of the quantum theory to the nuclear domain. Let it quote from D. Bohm, Quantum Theory, Prentice Hall, 1964, p. 627, "We state that quantum theory has actually evolved in such a way that it implies the need for a new concept or the relation between large scale and small scale properties of a given system. Between others, one may discuss aspects of this new concept: 1. Quantum theory presupposes a classical level and the correctness of classical concepts in describing this level. 2. The classically definite aspects of large scale systems cannot be deduced from the quantum-mechanical relationships of assumed small-scale elements. Instead, classical definiteness and quantum potentialities complement each other in providing a complete description of the system as a whole. Although these ideas are only implicit in the present form of the quantum theory, we wish to suggest here in a speculative way that the successful extension of quantum theory to the domain of nuclear dimensions may perhaps introduce more explicitly the idea that the nature of what can exist at the nuclear level depends to some extent on the macroscopic environment. In this connection it was shown that the definition of small scale properties of a system is possible only as a result of interaction with large scale systems undergoing irreversible processes. In line with the above suggestion, we propose also that irreversible processes taking place in the large scale environment may also have to appear explicitly in the fundamental equations describing phenomena at the nuclear level. "

The part of Bohm's suggestion which was followed by the investigator almost literally is underlined. Instead of upgrading the fundamental equations and results of quantum theory from the microscopic, quantum level, to the level of the macroscopic with observable and measurable data of technical fluid dynamics, the equations of Prandtl, Blasius, and others of the Navier-Stokes class included in category of the macroscopic level, have been down-graded to the level of the quantum theory of microscopic nature and character. This has been done directly, without the creation of any special philosophy of general transfer theory.

Since, in the concept of the Planck K-Number, which represents some sort of ideological and philosophical extension of the Planck constant, the most important variable quantity seems to be the density of the fluid medium in question expressed in terms of gram-mass units, the process of the transformation of the density of a medium from one system of units to another, and vice versa is below repeated. Given ρ in (M, L, θ) system (gram-mass), the problem is to construct the set of operations for expressing this in gram weight. Since mass = force/acceleration, and force = mass x acceleration, one has the following set of operations for a given ρ in gram-mass in order to refer the mass to the acceleration due to gravity: divide ρ (gram-mass cm^{-3}) by the acceleration g_L expressed in cm sec^{-2}; example: given ρ in gram mass cm^{-3}, divide it by g_L (L_{cm}, θ sec):

$$\rho \ [\text{ gram mass/acceleration of gravity}] \ cm^{-3}$$

$$= \rho \ [\text{gr mass}/g_L] \ L^{-1} \ cm^{-3} \ \theta^2$$

$$= \rho \ [\text{gr weight}] \ cm^{-4} \theta^2$$

$$= \rho \ \text{gr-weight} \ cm^{-4} \theta^2, \ \theta \ \text{sec};$$

thus:

$$\rho \ (\text{gram mass } cm^{-3})/(980 \ cm \ sec^{-2}) = 0.0012 \ \text{gr } cm^{-3}/(980 \ cm \ sec^{-2})$$

$$= 0.122448979591 \times 10^{-5} \ (\text{gr } cm^{-4} \ sec^2) \ \text{in gram weight.} \quad (2.2.20)$$

If ρ is given in gram-weight, i.e., the mass of unit volume of fluid is referred to the weight (force due to gravity), then the set of operations is as follows, where $\rho \ [\text{gram weight-}cm^{-4} \text{ -sec}^2]$ is given:

$$\rho \ [\text{gram weight x acceleration of gravity}] \ cm^{-4} sec^2$$

$$= \rho \ [\text{gram weight x } g_L \ L\theta^{-2}] \ cm^{-4} sec^2$$

$$= \rho \ [\ \frac{\text{gram mass}}{\text{gravity acceleration}} \ x \ g_L] \ L\theta^{-2} \ cm^{-4} \ sec^2$$

$$= \rho \ [\ \frac{\text{gram mass}}{g_L} \ x \ g_L] \ cm \ sec^{-2} \ cm^{-4} \ sec^2$$

$$= \rho \ [\text{gram-mass}] \ cm^{-3} = \rho \ [\text{gr-mass-}cm^{-3}] \ .$$

Example:

$$\rho \ (\text{gram weight-cm}^{-4} \ \text{sec}^2) \times 980 \ \text{cm sec}^{-2}$$

$$= 0.122 \times 10^{-5} \ \text{gr cm}^{-4} \ \text{sec}^2 \ (\text{in gram weight}) \times 980 \ \text{cm sec}^{-2}$$

$$= \sim 0.0012 \ \text{gr cm}^{-3} \ \text{in gram-mass}. \tag{2.2.21}$$

In some problems the necessity of using the coefficients of dynamic viscosity, μ, or kinematic viscosity, ν, may arise. The writer assumes that both coefficients have the same value in both the microscopic and macroscopic domains. The PKN number, equation (2.2.1), (ρ V L x unit volume) is calculated in a similar manner. As an example, consider the flow in the boundary later along an infinitely long flat plate:

$\rho = 0.0012$ gram-mass; $V \equiv u$ in the boundary layer $= f' U_\infty$, where f' is taken from Schlichting's tables; $U_\infty = 200$ km/hour, L = 10 meter $= 10^3$ cm;

$$\rho = 0.12244897591 \times 10^{-5} \ (\text{gr-cm}^{-4} \text{-sec}^2) \ \text{in gram weight}.$$

The density ρ may be expressed in gram-mass or in gram-weight:

$\rho = 0.0012$ gram-mass-cm^{-3};

$\rho = 0.123$ gr-cm^{-4} -sec^2, gram-weight;

$L = 10$ me $= 10^3$ cm;

$V = u$ in the boundary layer from Schlichting's tables.

The calculated values of PKN oscillate between the quantities

$$\text{PKN} = 0.4427 \times 10^3 \ \text{gr cm sec}$$

to

$$\text{PKN} = 0.7223 \times 10^4 \ \text{gr cm sec (gr mass); similarly in gram-weight}$$
units:

$$\text{PKN} = 0.4517 \times 10^0 \ \text{gr-cm}^{-4} \ \text{sec}^2$$

to

$$\text{PKN} = 0.6803 \times 10^1 \ \text{gr-cm}^{-4} \text{-sec}^2 \ (\text{gr-weight}).$$

The oscillations are due to various values of the velocity $u \equiv V$ taken from Schlichting's tables. As is seen, there is a difference between the two values of PKN equal approximately to 10^3. This is so because the acceleration due to gravity of the earth is approximately equal to $980 \approx 10^3$ cm-sec^{-2}. Consequently, if one wants to use PKN as an approach to the magnification factor, one should first transform ρ (gram-weight) to ρ (gram-mass) and after that apply the result to the indicated operations to obtain PKN in gram-mass (the same units in which h is used).

It has been demonstrated above that the effect of the ratio of the two quantum numbers, h and m^{-1}, influences only the macroscopic results, the quantum effects being cancelled. This is certainly a splendid result of the Madelung proposition.

Case 2. Term with the factor m or m^{-1}

The term with the factor m appears in equation (2.1.5), where Φ denotes the potential energy in the field. Again the problem is how to solve the quantum effects of the quantity "m" in this equation where m is the electron mass

$$m = \text{electron mass} = 0.9107 \times 10^{-27} \text{ gr (gr-mass)}.$$

The term of this kind in equation (2.1.5) is:

$$m^{-1} \nabla \phi = (0.9107 \times 10^{-27})^{-1} \nabla \phi. \tag{2.2.22}$$

The sequence of operations is discussed and given below. If a gram molecule (mole) of gas or any other substance (R. B. Lindsay, General Physics, John Wiley and Sons, Inc., 1947, p. 108) whose mass is equal to the molecular weight in grams is assumed, one may obtain the answer from the elementary kinetic theory of gases or liquids. According to this theory, a gas is composed of a large number of very small material particles -- molecules -- which obey the laws of mechanics (Lindsay, p. 192). One can consider such physical aspects as: molecule mass, molecular weight, volume of one mole, etc. For example,

$$\text{the mass of the hydrogen molecule } m = 3.32 \times 10^{-24} \text{ gram.} \tag{2.2.23}$$

A very significant number is the so-called Avogadro's number which is equal to

$$A_v = 6.06 \times 10^{23}, \tag{2.2.24}$$

which gives the number of molecules in a gram molecule or mole in all the classes of substances. Similarly, one can obtain the volume of one mole, which is the same for all perfect gases at 0° C and normal atmospheric pressure (Lindsay, p. 198) from

$$V_o = 22.41 \times 10^3 \text{ cm}^3/\text{mole.} \tag{2.2.25}$$

Also the number of molecules per cubic centimeter for a perfect gas under standard conditions (Loschmidt Number) (Lindsay, p. 198) can be obtained from

$$L = 2.71 \times 10^{19}/\text{cm}^3. \tag{2.2.26}$$

Some recommend the use of the value for the number $L = 2.69 \times 10^{19}$ at atmospheric pressure and $0^{\circ}C$ (Stewart, p. 191). Using the mass of a hydrogen atom as 1.66×10^{-24} (gram mass) and the mass of the hydrogen molecule as twice that of the atom (Stewart, p. 191); then one gets:

$$\text{molecule} = 2m = 2 \times 1.66 \times 10^{-24} = 3.32 \times 10^{-24} \text{ (gram mass)}, \quad (2.2.27)$$

which is in agreement with equation (2.2.23). Returning to equation (2.2.22), and assuming that the action of the potential energy, Φ, refers to all the molecules in a mole (or gram molecule), i.e., to the (Avogadro) number, 6.06×10^{23}, then the action of Φ on each molecule is, on the average, the same.

Summarizing, one can state that if only one element equal to a single electron appears in the problem, then m has the value:

$$m = 0.9107 \times 10^{-27} \text{ (gr-mass)}.$$

If there is 6.06×10^{23} number of molecules (gram molecules) in the volume in question and if each molecule corresponds approximately to one electron, then, hypothetically, the quantity m in equation (2.2.22) may be substituted by the quantity:

$$(0.9107 \times 10^{-27})(6.06 \times 10^{23}) = 5.518842 \times 10^{-4}$$
$$= 0.0005518842 \text{ (gram mass)}. \quad (2.2.28)$$

Consider an illustrative example: in the macroscopic fluid dynamics of viscous fluids, the well-known equation of Blasius is assumed and is used as the first example to illustrate the application of Prandtl's boundary layer theory (referred to in this investigation as the Blasius-Prandtl or the Prandtl-Blasius equations, or briefly the Prandtl equation):

$$u \, \partial u/\partial x + v \, \partial u/\partial y = \nu \, \partial^2 u/\partial y^2; \quad \nu = \mu \rho^{-1}; \quad (2.2.29)$$

$$\partial u/\partial x + \partial v/\partial y = 0. \quad (2.2.30)$$

Where the boundary conditions are,

$$y = 0: u = v = 0; \quad y = \infty: u = U_{\infty}. \quad (2.2.31)$$

In the equations above, the symbols used denote:

u = horizontal velocity component in the boundary layer along an infinitely long flat plate, with $u = u(x, y)$;

v = corresponding vertical velocity component in the same boundary layer with $v = v(x, y)$;

ν = coefficient of the kinematic viscosity of the fluid (liquid or gas) medium in question.

In the microscopic domain of wave mechanics, the validity of the Schroedinger equation is assumed and is given in equation (2.1.1).

Generalizing the Madelung proposition, one can propose a generalization of the concept of the diabatic flow (NASA, 1946) from the macroscopic flow domain to the microscopic flow domain and vice versa. The above is actually a generalization of the quotation of Bohm in his Quantum Theory. The approach to the solution of the system of equations (2.2.29) and (2.2.30) was achieved by Prandtl's and his followers by means of the stream function, ψ_{BP}, and the well-known relations from equation (2.2.10):

$$\psi_{BP} = \psi(x, y) \qquad (2.2.32)$$

$$u = \partial\psi/\partial y = (\partial\psi/\partial\eta)(\partial\eta/\partial y) = U_\infty f'(\eta); \qquad (2.2.33)$$

$$v = \partial\psi/\partial x = -(\partial\psi/\partial\eta)(\partial\eta/\partial x) =$$

$$1/2 \ (\nu U_\infty x^{-1})^{1/2} (\eta f' - f). \qquad (2.2.34)$$

Combining this result with the hypothesis on the validity of the phenomena of the nature and of the diabatic flow relations, one can state that "the Prandtl equation, equations (2.2.29) and (2.2.30) are formalistically fully equivalent to the equation deduced from the Schroedinger wave equation in the Madelung formulation, equation (2.1.5). Certainly this is a remarkable result obtained from wave mechanics applying Madelung's generalization idea from quantum to classical mechanics.

Case 3. Term with the factor h

As the last possible case of the discussion on possible scale magnification factors, there remains case (3), i.e., the case in which the terms contain only the factor h, the Planck constant. The proposition is made that the scale magnification factor is to elevate the phenomena appearing in the microscopic domain (the domain of wave mechanics theory) to the macroscopic domain (macroscopic reality), where all the phenomena can be visually observed. The characteristic constants or factors appearing in the microscopic, quantum approach are usually the Planck constant, h, and the mass of the electron. The velocity of light, which is an absolute constant, appears in some problems whereas in other particular problems the velocity of sound appears as the characteristic property of the system in question. Again, it is usually assumed

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that the velocity of sound is an absolute constant. The idea of using some "scale magnification factors" should not be considered as a completely new idea. Indirectly, the factors used by scientists of the Soviet school may be considered to be "scale magnification factors."

The Soviet school of quantum theoretic physics often assumes (unless the contrary is explicitly stated) that the operations are performed in a system of units in which both Planck's constant, $\not{h} = 2\pi^{-1}h$, and the velocity of light, c, equal 1. Moreover, the temperature is often expressed in such energy units that some special values are superimposed upon the wave vector, k. In the case of fluid dynamics and the variable phenomena associated with it, the writer has proposed the Planck K-Number, which may serve as a "scale magnification factor" at those points of the domain of the flow and in those problems where it becomes necessary to use it. In general, in the Madelung idea of the generalization of wave mechanics (quantum) theory (i. e. , the Schroedinger Equation (2.1.1), the terms containing the Planck constant as the single coefficient do not appear. But, for the sake of completeness, case (3), which is theoretically possible, is discussed below.

The approach involves an inclusion of such sophisticated and generally little-known aspects of modern, macroscopic fluid dynamics theory as the "diabatic flow theory," (NASA, 1944).

The scale magnification factor was proposed in equation (2.2.1) in the form:

$$PKN = h\ (1 + INF\ h^{-1}); \quad INF = \rho\ U_\infty L \times \text{unit volume}; \tag{2.2.35}$$

or approximately:

$$PKN = h + INF = \sim INF; \tag{2.2.36}$$

$$h = 6.625 \times 10^{-27}\ \text{gr-cm-sec; (gram-mass)} \tag{2.2.37}$$

The notion of the Reynolds number, which is one of the most characteristic dimensionless numbers in the theory of dynamics of fluids appears partly in the above proposition. The Reynolds number is given by the equation $R_e = \rho U_\infty L\ \mu^{-1}$ where the density ρ is usually expressed in units of gram-weight. The numerator of R_e is equal to $\rho\ U_\infty L$ and when multiplied by the unit volume, has the same dimensions as the Planck constant if the proper dimensions of ρ are preserved. Consequently, one may combine these two numbers; i. e. , Planck's constant and the Planck K-Number

to obtain a factor which may serve as the scale magnification factor from the microscopic scale to the macroscopic scale in which the phenomena in question can be visually observed.

The quantity PKN, called the Planck K-Number, has the same dimensions as the Planck number, h, and may be used in place of it:

$$PKN = h \, (1 + INF \, h^{-1}); \quad INF = \rho \, U_\infty \, L \times \text{unit volume.} \qquad (2.2.38)$$

One can assume, due to the fact that h is very small that

$$PKN \sim INF; \quad h = 6.625 \times 10^{-27} \text{ gr-cm-sec;} \qquad (2.2.39)$$

where gram-mass is used in h.

Since h has dimensions in gram-mass, the Planck number also has to be used, calculated, and tested in dimensions of gram-mass. The density ρ of the medium (fluid, liquid, or gas) under consideration is usually given, calculated, and measured in units of gram-weight. The three systems of units generally used are: (1) absolute or dynamical or physical system also denoted as a mass-length-time (M, L, θ) system; (2) gravitational or technical system also denoted as a force-length-time (F, L, θ) system; (3) unnamed or force-mass-length-time system also denoted as a (F, M, L, θ) system. The present project deals only with the (M, L, θ) system where the Planck number is expressed naturally in the (M, L, θ) system. Engineers often express their values (such as the density of a medium) in the units of the (F, L, θ) system and in gram-weight units; the symbol θ denotes time in seconds.

One more item has to be mentioned in connection with the Schroedinger equation; namely, this equation has been used to explain the resonance phenomena between various particles of the liquid. Such resonance phenomena appear in the C.A.T. problem. Only an introduction to this phenomenon will be presented in the next section (below).

2.3. Resonance

The following equation was obtained above:

$$\partial \vec{U}/\partial t + \frac{1}{2} \nabla(\vec{U}^2) + m^{-1}\nabla\Phi - \nabla[\, a^{-1}\vec{\nabla}^2 \, ah^2 \, (8\pi^2 m^2)^{-1}] = 0 \qquad (2.3.1)$$

where

$$\vec{U} = \nabla\phi; \quad \phi = -\beta h \, (2\pi m)^{-1}; \quad \psi = a \exp(i\beta); \quad a^2 = \rho \, ;$$
$$\text{div} \, (a^2 \, \text{grad} \, \phi) + \partial/\partial t \, a^2 = 0 \, ,$$

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After some straightforward operations, one obtains the identity:

$$(\nabla^2 a)\, a^{-1} = \frac{1}{2}\,[\,\nabla^2(a^2)\,]\,(a^2)^{-1} - [\,a^{-1}\nabla a\,]^2;\ \nabla^2 = \text{Laplacian.} \quad (2.3.2)$$

Introducing the concept of bulk modulus, E, $E\rho^{-1} = dp/d\rho$, one obtains:

$$(\nabla^2 a)\, a^{-1} = \frac{1}{2}\,(\nabla^2 \rho)\, \rho^{-1} - \frac{1}{4}\,(\nabla \rho)^2\, E^{-2}; \quad (2.3.3)$$

$$\partial \vec{U}/\partial t + \frac{1}{2}\,\nabla(\vec{U}^2) + m^{-1}\nabla\Phi - \nabla[\,\frac{1}{2}\,\rho^{-1}\nabla^2\rho - \frac{1}{4}\,E^{-2}\,(\nabla p)^2\,]; \quad (2.3.4)$$

$$\partial \vec{U}/\partial t + \frac{1}{2}\,(\vec{U}^2) + m^{-1}\nabla\Phi - \nabla[\,\frac{1}{2}\,E\rho^{-1}\,(\rho c^2)^{-1}\nabla^2\rho - \frac{1}{4}\,(\rho c^2)^{-2}\,(\nabla p)^2\,] = 0: \quad (2.3.5)$$

$$E = \rho\,(dp/d\rho);\ (dp/d\rho) = c^2;\ E = \rho c^2; \quad (2.3.6)$$

where the variable $(dp/d\rho) = c^2$ denotes (by definition) the velocity of sound in the medium and for the particular condition (isotropic, adiabatic, isothermal, etc). Consider the C.A.T. phenomenon in an ideal case, in free air, without any boundaries whatsoever and without any extraneous forces acting upon the medium in question, then the only acting force is the pressure, p, which according to the fundamental concepts of quantum theory is an oscillating quantity, (harmonic oscillation). In the first approximation, in equation (2.3.5), the third term, $m^{-1}\nabla\Phi$, may be related to the phenomenon of the diabatic flow (heat addition flow, irreversible heat addition, non-conservative heat flow, or dissipative heat flow). The fourth term refers to "spring" characteristic properties of the gaseous or liquid medium in question (bulk modulus), in which the greater the compressibility of the gas, the lower the speed of sound; the quantity c denotes the speed of sound. The last term in equation (2.3.5) refers to the pressure "p," considered to be the only acting "force" in the present condition. The sequence of terms of equation (2.3.5) is similar to the sequence of terms in the case of the vibration of a single mass element "m" (one-dimensional):

$$m\ddot{x} + \bar{c}\,\dot{x} + kx = F_o \sin \omega t;\ \dot{x} = dx/dt;\ t = \text{time} \quad (2.3.7)$$

where: x is the deflection (coordinate); $\bar{c}$, the viscous damping coefficient; k, the spring stiffness in lb/in.; $F_o \sin \omega t$, the harmonic force; and ω, the frequency of the harmonic excitations. An analogy seems to be obvious -- the phenomenon of turbulence (C.A.T.) may be considered to belong to the family of self-excitable flutter (vibration) phenomena in which the

damping exists due to the viscosity phenomenon. Viscosity is defined as the natural transverse transport of momentum in gases and liquids (analogously the heat conduction coefficient is defined as the transverse transport of energy). At a certain moment (critical Reynolds' Number), a gaseous or liquid medium which is moving (even uniformly) in one direction, gets a tendency to expand in all directions (including transverse directions) and to transport momentum and energy in transverse directions as well due to the natural expansion (characteristic properties of the medium). The phenomenon of flutter is considered to be a phenomenon in which a transformation of the longitudinal transport of energy of the motion of the medium (gas or liquid) into the transverse transport of the energy of the oscillations of the "fluttering" element takes place.

In 1940, a bridge which spanned the strait between the Tacoma and Olympic peninsulas in the state of Washington collapsed. Film taken during the collapse of the bridge distinctly shows the effect of the flutter phenomenon. The one-directional flow of the strong wind in the narrow strait clearly shows the transformation of the longitudinal energy of the air into the transverse energy of the vibrational character (flutter) of the oscillations of the bridge.

It seems that the two phenomena, turbulence and flutter, have some important common physical characteristic properties: the energy in the longitudinal flow of a gaseous or liquid medium, moving uniformly in one direction, develops at a particular moment a strong tendency to transform a certain part of its energy into a transverse direction. In the case of the C. A. T. , this is due to the "natural" tendency of the medium in question to spread in all possible directions, since actually the so-called "static" pressure in a gas or liquid acts uniformly in all directions, at least in the condition of rest (or when in very slow motion). The so-called coefficients of viscosity and heat conductivity (transverse transports of momentum and energy) seem to be natural results of this "natural" tendency; i. e. , the expansion phenomenon due to viscosity is a natural tendency which can cause other phenomena. In the case of the phenomenon of flutter, the "natural" tendency of the gaseous (or liquid) medium to expand causes a deflection of an elastic element, usually located transversely to the direction of the main, incoming flow stream in the medium.

Due to this fact, elastic forces appear in the element and oscillations (flutter) develop progressively. The above physico-geometrical description of turbulence agrees very well with the analytical description of it by means of the wave mechanics. Specifically, the longitudinal term $\nabla(\vec{U})$ always appears in conjunction with the "transverse" term, $\nabla \times \vec{U}$. The wave equation furnishes (and the vector calculus, as well):

$$\nabla(\vec{U}^2) = 2[(\vec{U} \cdot \nabla) \vec{U} + \vec{U} \times \nabla \times \vec{U}] . \qquad (2.3.8)$$

As was emphasized previously, due to the validity of the hypothesis of linear superposition, one may add the results of two or more wave equations. A film of the Tacoma Bridge made before its final collapse demonstrates many instructional principles during the final period of the flutter phenomenon. The association of the turbulence phenomenon with the flutter phenomenon as proposed above calls for paying more attention to the flutter problem. Unfortunately, our knowledge of the problem is very limited. Omitting the phenomena of flutter of certain rotary elements in machines such as steam engines and diesel engines, for the time being, the theory of flight turned to the investigation of flutter problems around 1928. In subsequent years, investigations of airplane accidents like those of Puss Moth, the results of physicists and mathematicians like Frazer, Duncan, and others were used to formulate a pattern of how to prevent the appearance of the flutter phenomenon in certain elements of an airplane, but mostly by means of mass balancing.

3. LAMINARITY OF THE FLOW

3.1. Presentation of the Problem

The present day macroscopic fluid dynamics is divided into two fields: laminar and turbulent flows. Based on this division an investigator must decide what method he will choose in order to solve the equation (Navier-Stokes or Schroedinger) for a particular problem. In simple language, one may say that in a laminar flow there are no vortexes present, whereas they are present in turbulent flow. In addition, there may be other physical phenomena present in turbulent flow.

A vortex has two characteristic properties: (a) value of the curl, ω, and (b) the radius. In this research a considerable amount of time and energy has been devoted to this latter problem. To deal with a specific problem, as an example, the flow in the laminar boundary layer along an

infinitely long flat plate was chosen. This kind of flow is well known and the macroscopic results are tabulated in the literature by Schlichting, Goldstein and Howarth. The results obtained in the present investigation were unexpected, and one can state without reservation that the theory of viscous fluids still requires considerable research effort.

The existence of vortexes in the boundary layer requires a knowledge of the lengths of the radii of these vortexes. This is of importance for future consideration of the problem. In the calculation of the radii one has to ignore, for the time being at least, the influence of the viscosity and possibly heat conductivity on the lengths of the radii. Consequently, the calculations are restricted to an inviscid gas. A comparison of the attempts at calculating the radii of the vortexes will allow one to choose the lengths of the radii which will be closest to reality by means of possible tests and reasoning.

As usual, the fundamental aspects of the items in question serve as the beginning.

(a) In fluid mechanics the term circulation means the value of the instantaneous line integral of the flow velocity, $\vec{V}$, taken in the positive direction along a closed curve in space (Owczarek, pp. 43-44) and is denoted by Γ; thus:

$$\Gamma = \oint_C \vec{V} \cdot d\vec{r} . \tag{3.1.1}$$

Kelvin's equation states that:

$$D\Gamma/Dt = \frac{D}{Dt} \oint_C \vec{V} \cdot d\vec{r} ; \tag{3.1.2}$$

where for functions $\vec{V}$ and $\vec{r}$, one gets:

$$\frac{D}{Dt} \oint_C \vec{V} \cdot d\vec{r} = \oint_C D \ \vec{V}/Dt \cdot d\vec{r} + \oint_C \vec{V} \cdot D \ (d\vec{r})/Dt; \tag{3.1.3}$$

(b) The instantaneous vorticity, $\vec{\omega}$, is defined as the curl of the velocity field, $\vec{V}$ thus:

$$\omega = \nabla \times \vec{V}; \quad \vec{\omega} = (\omega_x, \ \omega_y, \ \omega_z). \tag{3.1.4}$$

According to Cauchy's interpretation, the component of vorticity at P in the z-direction is (Owczarek, p. 47):

$$(\text{curl } \vec{V})_z = \partial v/\partial x - \partial u/\partial y = \omega_z; \quad \vec{V} \ [u(x,y); v(x,y)] . \tag{3.1.5}$$

(c) Stoke's theorem allows one to relate circulation Γ with the component of vorticity $\vec{\omega}$ as follows: consider a piecewise smooth surface S having as its edge (boundary) a simple smooth closed curve C (which does not have multiple points) or a piecewise closed curve C (which does not have a unique tangent at each point). The area bounded by the curve C is denoted by the symbol A. Then:

$$dΓ/dA = comp_n \; \vec{\omega} .$$ (3.1.6)

In other words, the component of vorticity at some point P in the direction of the line n normal to a surface A bounded by a simple closed curve C is equal to the ratio of the circulation around C to the area of the surface in the limit as the area approaches zero, shrinking to point P (Owczarek, p. 47). This interpretation of vorticity is credited to Hankel (1861), Roch (1862), and Kelvin (1869) and is quoted in Truesdell.

The above theorem, valid in an ideal, inviscid fluid (gas), is applied to the present problem, assuming that the effects of viscosity are negligibly small. In turn, the simplest possible case of two-dimensional vortex flow (Howarth, p. 158) is used, in which the streamlines are circles with their centers on the straight axes and the speed is constant along any streamline. Let v be the speed along a streamline of radius r in the two-dimensional Prandtl-Blasius boundary layer (by assumption); then the motion will be irrotational (viscosity effects are neglected) if the circulation has the same value, K, say, along every streamline; i.e.,

$$2\pi rv = K.$$ (3.1.7)

3.2. Curl, Vortex Frequency, and Radius of the Vortex

In the Blasius-Prandtl boundary layer theory, the concept of the stream function appears in two possible forms: dimensional and dimensionless. In many cases one can use the dimensionless stream function to obtain reasonable answers to some problems. One has

$$\psi = \psi_{PB} = U_\infty^{1/2} \nu^{1/2} x^{1/2} f(\eta);$$ (3.2.1)

$$\eta = y \, U_\infty^{1/2} \, \nu^{-1/2} \, x^{-1/2} \; (cm \; cm^{1/2} \; sec^{-1/2} \; cm^{-1} sec^{1/2} cm^{-1/2}$$ (3.2.2)

η being dimensionless;

$$\partial f/\partial y = (\partial f/\partial \eta)(\partial \eta/\partial y) = f' \ U_\infty^{1/2} \ \nu^{-1/2} x^{-1/2}; \qquad (3.2.3)$$

$$\partial f/\partial x = (\partial f/\partial \eta)(\partial \eta/\partial x) = f' \ (-\tfrac{1}{2} \ \eta \ x^{-1}). \qquad (3.2.4)$$

One can denote a dimensionless velocity component by $\vec{V}_d = (u_d, \ v_d)$ where

$$u_d = \partial f/\partial y; \quad v_d = -\partial f/\partial x \qquad (3.2.5)$$

One may investigate which fundamental laws are satisfied by the velocity components, as proposed above:

(a) Continuity: with

$$\partial \eta/\partial y = U_\infty^{1/2} \ \nu^{-1/2} x^{-1/2}; \quad \partial \eta/\partial x = -\tfrac{1}{2} \ \eta x^{-1}. \qquad (3.2.6)$$

The coordinate η is a dimensionless variable whereas all the other arguments (x, y, ν, U_∞) have dimensions. Consequently, the partial derivatives, obtained in equation (2.6.6), have the dimensions:

$$\partial \eta/\partial y \approx cm^{1/2} \ sec^{-1/2} \ cm^{-1} \ sec^{1/2} cm^{-1/2} \approx cm^{-1}; \qquad (3.2.6a)$$

$$\partial \eta/\partial x \approx cm^{-1}; \qquad (3.2.6b)$$

$$u_d = f'(\eta) \ U_\infty^{1/2} \ \nu^{-1/2} x^{-1/2}, \quad (\approx cm^{-1}); \qquad (3.2.7)$$

$$v_d = \tfrac{1}{2} \ f'(\eta) \ [\ \eta x^{-1}], \quad (\approx cm^{-1}) \ ; \qquad (3.2.8)$$

$$\partial u_d/\partial x = (U_\infty^{1/2} \ \nu^{-1/2} x^{-1/2}) \ f'' \ (\partial \eta/\partial x) - \tfrac{1}{2} \ U_\infty^{1/2} \ \nu^{-1/2} x^{-3/2} f'$$

$$= -\tfrac{1}{2} \ U_\infty^{1/2} \ \nu^{-1/2} x^{-3/2} \ (f\eta f'' + f'), \quad (\approx cm^{-2}); \qquad (3.2.9)$$

$$\partial v_d/\partial y = \tfrac{1}{2} \ f''(\eta) \ (\eta x^{-1}) \ \partial \eta/\partial y + \tfrac{1}{2} \ f' \ x^{-1} \ \partial \eta/\partial y$$

$$= \tfrac{1}{2} \ U_\infty^{1/2} \ \nu^{-1/2} x^{-3/2} \ (\eta f'' + f'), \quad (\approx cm^{-2}). \qquad (3.2.10)$$

Therefore, the continuity equation:

$$div \ \vec{V}_d = \partial u_d/\partial x + \partial v_d/\partial y = 0 \qquad (3.2.11)$$

is satisfied by the function f (η).

(b) Irrotationality:

When the vorticity vector in the flow is reduced to one component about the z-axis, then

$$\omega_{dz} = \frac{1}{2} \text{curl } \vec{\omega}_d = \frac{1}{2} (\partial v_d / \partial x - \partial u_d / \partial y). \qquad (3.2.12)$$

Starting with equations (3.2.7) and (3.2.8) one obtains,

$$\partial v_d / \partial x = - \frac{1}{4} f'' \eta^2 x^{-2} - \frac{3}{4} f' \eta x^{-2} = - \frac{1}{4} f'' y^2 U_\infty \nu^{-1} x^{-3}$$

$$- \frac{3}{4} f' y U_\infty^{1/2} \nu^{-1/2} x^{-5/2}$$

$$(\approx cm^{-2}) ; \qquad (3.2.13)$$

$$\partial u_d / \partial_y = U_\infty \nu^{-1} x^{-1} f'' \quad (\approx cm^{-2}). \qquad (3.2.14)$$

This implies that:

$$\omega_{dz} = \frac{1}{2} (\partial v_d / \partial_x - \partial u_d / \partial y) \neq 0, \quad (\approx cm^{-2}), \qquad (3.2.15)$$

where both terms have dimensions $\approx cm^{-2}$ and are not equal to zero. The value of ω_{dz} was calculated at 2430 points of the domain with the following results:

the values oscillate around the numbers in cm^{-2};

$$-0.4151 \times 10^{-11}; -0.8896 \times 10^{-22}; -0.6866 \times 10^{-25}; -0.8440 \times 10^{-27}$$
$$(3.2.16)$$

$$-0.8708 \times 10^{-28}; -0.6727 \times 10^{-28}; -0.8937 \times 10^{-29}. \qquad (3.2.17)$$

It is to be remembered that the Planck No. is $\sim 10^{-27}$. Thus in the laminar boundary layer, vortexes with the values of ω_d of the order of the Planck number appear. Up to the present, this fact was unnoticed and certainly it was never observed by Prandtl and his followers.

In general, it is difficult to locate in the literature the definition of the vortex strength or of the vortex strength distribution. The definition given is taken from Samaras: "Assume an incompressible vortex-free velocity field without any volume source intensity; for such a source-free vortex velocity field, the volume distribution of vortices may be calculated from the vortex potential $\vec{V}_p$; the author [Samaras] assumes that the circumferential velocity vector, $\vec{v}_c$, can be calculated in the following manner:

$$\vec{v}_c = \text{rot } \vec{V}_p \; ; \tag{3.2.18}$$

$$\vec{w} = \text{rot rot } \vec{V}_p = \text{rot } \vec{v}_c \; ; \tag{3.2.19}$$

where the vortex potential $\vec{V}_p$ has dimensions $\approx \text{cm}^2 \text{ sec}^{-1}$; the quantity $\frac{1}{2} \vec{w}$ is called vortex strength distribution in volume."

Returning to the problem of vortex characteristics according to Howarth (p. 158), one has:

$$2\pi r v = K, \quad K \approx \text{cm}^2 \text{ sec}^{-1}, \tag{3.2.20}$$

where v is the speed along a streamline of radius r; the streamlines in a two-dimensional vortex flow are circles with their centers on the straight axis, and the speed is constant along any streamline. The motion is an irrotational one if the circulation has the same value, K, along every streamline. Assuming that the circulation is equal to the vortex strength distribution in volume or in area, equation (3.2.20), and that the values used are those calculated previously, then one gets the same equation for ω_z:

$$\omega_z = \frac{1}{2} (\partial v / \partial x - \partial u / \partial y), \tag{3.2.21}$$

or

$$\omega_{dz} = \frac{1}{2} (\partial v_d / \partial x - \partial u_d / \partial y), \tag{3.2.22}$$

which may be more applicable to the problem under consideration. One may write:

$$\vec{V} = (u, v) = (\vec{i} \; u, \; \vec{j} \; v); \tag{3.2.23}$$

$$u = U_\infty f'(\eta), \quad (\approx \text{cm sec}^{-1}) \; ; \tag{3.2.24}$$

$$v = \frac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2}) (\eta f' - f), \quad (\approx \text{cm sec}^{-1}) \; ; \tag{3.2.25}$$

where (u, v) may be substituted for by (u_d, v_d). Suppose the motion is simply a rotation with angular velocity;

$$\vec{\Omega} = \vec{i} \; \omega_1 + \vec{j} \; \omega_2 + \vec{k} \; \omega_3 \tag{3.2.26}$$

about an axis through the origin, and that,

$$\vec{R} = \vec{i} \; x + \vec{j} \; y + \vec{k} \; z \tag{3.2.27}$$

be the vector from the origin to a point $P(x, y, z)$, then the velocity at P in such a rotation is:

$$\vec{U} = \vec{\Omega} \times \vec{R} ; \tag{3.2.28}$$

$$\vec{U} = \vec{\Omega} \times \vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\,(\omega_2 z - \omega_3 y) + \vec{j}\,(\omega_3 x - \omega_1 z) + \vec{k}\,(\omega_1 y - \omega_2 x); \tag{3.2.29}$$

where the velocity vector $\vec{U}$, equation (3.2.29), corresponds to the velocity vector $\vec{v}$ in equation (3.2.20). Using equation (3.2.19) one can identify the vectors $\vec{U}$ and $\vec{v}$ (in the first approximation) from:

$$\vec{\omega} = \text{rot}\,\vec{v} \cong \text{rot}\,\vec{U}; \quad \vec{U} = \vec{i}\,U_1 + \vec{j}\,U_2 + \vec{k}\,U_3; \tag{3.2.30}$$

$$U_1 = \omega_2 z - \omega_3 y; \quad U_2 = \omega_3 x - \omega_1 z; \quad U_3 = \omega_1 y - \omega_2 x ; \tag{3.2.31}$$

$$\vec{\Omega} = \text{rot}\,\vec{U} = \nabla \times \vec{U} = \vec{i}\,\left(\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z}\right) + \vec{j}\,\left(\frac{\partial U_1}{\partial z} - \frac{\partial U_3}{\partial x}\right) + \vec{k}$$

$$\left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial y}\right) ; \tag{3.2.32}$$

the vortex strength distribution in volume or in area is denoted by

$$\frac{1}{2}\,\vec{\omega} = \text{vortex strength distribution in volume or in area } (\approx \sec^{-1}).$$

In the above derivation one has to assume that $\vec{\Omega}$ is constant (at least temporarily) and then one gets:

$$\nabla \times \vec{U} = \vec{k}\,(\partial v/\partial x - \partial u/\partial y); \tag{3.2.33}$$

$$\frac{1}{2}\,\vec{\omega} = C\,\frac{1}{2}\,(\partial v/\partial x - \partial u/\partial y), \quad (\approx \sec^{-1}), \tag{3.2.34}$$

where C must have dimensions of cm^2 in order to be in agreement with the concept of "circulation" as used by Howarth, equation (3.2.20). If, in place of the components (u,v) one uses the components (u_d, v_d), then the dimensions of the constant C must be properly adjusted. In the problem under consideration, one can notice that K in Howarth's approach corresponds to $\vec{V}_p$ in Samaras approach;

$$K \approx \vec{V}_p \approx \vec{K}. \tag{3.2.35}$$

One can notice further that the variable $\frac{1}{2}\,\vec{\omega}$ must be multiplied by the area $(\approx cm^2)$ in order to incorporate this variable into the notion of the

entity called the vortex potential function, $\vec{V}_p$, ($\approx cm^2 \, sec^{-1}$). Continuing with the present example, it is proposed that the vortex potential function, $\vec{V}_p$, contains a uniformly distributed vortex strength (a hypothetical case) and that consequently one can write:

$$\vec{V}_p = K = C \frac{1}{2} \vec{w} \; , \; C \approx cm^2, \tag{3.2.36}$$

and that from equation (3.2.20) one gets [from Eqs. (3.2.35) and other above] :

$$2\pi \, r \, v = \frac{1}{2} \; C \, \vec{w} \; ; \; C = 1 \, x \, cm^2 \; ; \; v \stackrel{\sim}{=} v_d. \tag{3.2.37}$$

Equation (3.2.37) was programmed for a computer in order to calculate the value of "r" at 2430 points for the boundary layer in question. The results are discussed below. The values used are $v \stackrel{\sim}{=} v_d$ and $\vec{w} \stackrel{\sim}{=} \vec{\omega}_{dz}$.

Some agreement by means of assumptions, or logical conjectures has to be proposed and constructed between the existing theories of ideal fluids (inviscid, non-heat conducting) on one side, and of real viscous, heat-conducting fluid on the other. A strictly laminar flow can exist only in ideal fluid. A real fluid, above the inflection point (above super-fluidity regime) is always a viscous one. By the definition, viscosity is defined as a transverse transport of momentum (analogously, heat-conductivity is defined as a transverse transport of energy). But, the transverse actions cause the appearance of transverse motions, i.e., transverse disturbances (turbulence amongst others). Consequently, a viscous (real) fluid cannot move in form of a laminar flow. The movement can be in form of a disturbed (turbulent) flow, unobservable by naked human eye. This may be called a "quasi-laminar" flow. The mean value of the disturbed flow, due to small disturbances, may make a visual appearance of a laminar flow. That, which was called by L. Prandtl a laminar flow in 1904, is actually a "quasi-laminar flow", or a flow in which the mean values of the velocity disturbances are so small that they cannot be observed by the naked human eye. For this reason, Prandtl and his School divided the fluid flows into laminar and turbulent ones. The proper division should be: observable and unobservable (by the naked human eye.) The approach by Prandtl to the "laminar flow along an infinitely long flat plate) should give in the final result only one velocity component, i.e., the horizontal one. But the results of the

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calculations by Blasius (see Schlichting) are that there exists two velocity components in such a boundary layer: horizontal and vertical, both dependent on (x, y) directions in two-dimensional flow domain. Obviously, this fact causes, naturally and automatically, the appearance of the "vortex lines" with their axes parallel to the z-direction (which is perpendicular to both, (x, y) directions). Even, if the vortexes are very small, such a flow is not a "laminar" one. Consequently again, some agreement by means of assumptions or conjectures has to be proposed between the existing values of the components (u, v) in the laminar boundary layer (along a flat plate), as calculated by Blasius (Schlichting), and the (undisputable) physical fact of existence of vortex lines (however small and unobservable by the naked human eye) in the same boundary layer flow. This is being done in the present paragraph.

The data proves that the "laminar" flow in the "laminar" boundary layer contains vortexes and vortex lines and that the flow is not laminar at all. The movement of the layers of the fluid medium takes place by means of "rolling" of the layers of the medium on "rolls" of vortex lines. This idea is not a new one, and the investigator may quote several authors on the rolling phenomenon appearing in layers of a viscous medium. This research attempts to calculate the radius of vortexes and results in varying degrees of success. In general, it seems the value of the radius is large at the front of the boundary layer, depending upon the method of calculating the radius. Its size may be of the order 10^{+a}, where "a" may be a significant number. The larger the distance (measured along the flat plate from the leading edge of the flat plate in the direction of the flow), the smaller the radius of the vortexes, with values dropping down to become comparable with the Planck constant. This implies that at the leading edge and at the front of the flat plate and of the boundary layer, the "layers" of the fluid medium are "rolling" in big rolls -- big vortexes in the medium (due to vorticity effects). Intuitively, such a geometric interpretation of the physical phenomenon seems to make sense. With the increasing distance, the boundary layer becomes better organized, when the first impulse of the incoming flow becomes smaller, The "rolls" become smaller, and their number increases when their size decreases until

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finally the radius of the rolls reaches the length comparable to the magnitude of the Planck constant. Since Planck's constant is one of the characteristic quantities in the theory of quantum mechanics (and of wave mechanics, naturally), the following physical picture emerges as the final conclusion of the reasoning presented immediately before. The field of fluid dynamics and mechanics has been treated up to now from the macroscopic point of view, as a field belonging to the domain and entity of Newtonian mechanics. It now has been demonstrated from the foregoing that the formalism and the operations in the field of macroscopic fluid dynamics and mechanics go far beyond the domain of Newtonian mechanics and must include a new level, the level of the magnitude of Planck's constant in quantum mechanics. At this level much of the macroscopic instrumentation and experimental devices may have little application.

The problem of the possible interrelation and interference between the phenomena in a fluid medium, such as the "rolls" and "rolling" discussed previously, and similar characteristics along the solid flat plate or a solid body which constitute the surface and are the origin of the phenomenon of friction and of the consequent rolling phenoneman, are not treated in the present research. These phenomena may be discussed from various points of view: drag of a moving body; vibrations of a flat or almost flat wing; flutter of a body; vibrations of a cylindrical shell; resonance phenomenon between "rolls" of a fluid medium and their frequency; the elastoplastic characteristic properties of a moving solid body; etc.

The full expression for the value of the curl (vortex) with the use of the Blasius stream function, ψ, is given in equation (3.2.1.)):

$$\omega_z = \partial v/\partial x - \partial u/\partial y; \quad \psi = (U_\infty \nu x)^{1/2} f(\eta); \tag{3.2.38}$$

$$\partial v/\partial x = (\partial v/\partial \eta)(\partial \eta/\partial x); \quad \eta = y (U_\infty x^{-1} \nu^{-1})^{1/2}; \tag{3.2.39}$$

$$u = (\partial \psi/\partial y) = (\partial \psi/\partial \eta)(\partial \eta/\partial y) = U_\infty f'(\eta); \tag{3.2.40}$$

$$v = -\partial \psi/\partial x = 1/2 (U_\infty \nu x^{-1})^{1/2} (\eta f' - f); \tag{3.2.41}$$

$$\partial \eta/\partial x = (-1/2) y (U_\infty \nu^{-1})^{1/2} x^{-3/2} = (-1/2) \eta x^{-1}; \tag{3.2.42}$$

$$\partial v/\partial x = -1/4 \ (U_\infty \nu)^{1/2} \ x^{-3/2} \ (\eta \, f' - f) + 1/2 \ (U_\infty \nu)^{-1/2} \ .$$

$$. \ x^{-1/2} \ (f' + \eta f'' - f') \ (\partial \eta/\partial x) = -1/4 \ (U_\infty \nu)^{1/2} \ x^{-3/2} \ .$$

$$. \ (\eta f' - f) \ -1/4 \ (U_\infty \nu)^{1/2} \ x^{-3/2} \ (\eta^2 f'') = \ . \ -1/4 \ (U_\infty \nu)^{1/2} \ .$$

$$x^{-3/2} \ (\eta^2 f'' + \eta f' - f); \ (\approx \sec^{-1}); \tag{3.2.43}$$

$$\partial u/\partial y = U_\infty^{3/2} \ \nu^{-1/2} \ x^{-1/2} \ f''(\eta); \ (\approx \sec^{-1}); \tag{3.2.44}$$

$$\omega_z = -1/4 \ (U_\infty \nu)^{1/2} x^{-3/2} \ (\eta^2 f'' + \eta f' - f) - U_\infty^{3/2} \nu^{-1/2}.$$

$$. \ x^{-1/2} f'' \ (\eta). \tag{3.2.45}$$

These forms include the dependence upon altitude through the function ν. Results of the calculations for the altitudes between 200 met. and 10,600 met. are:

$$\eta = 0.2; \ \omega_z = -0.9224 \times 10^{-2} \ (\sec^{-1});$$

$$\eta = 8.2; \ \omega_z = -0.1139 \times 10^{-4} \ \text{to} = -0.2149 \times 10^{-6} \ , \ \text{depending upon}$$

the altitude (200 met to 10,600 met);

$$\eta = 8.4; \ \omega_z = -0.6373 \times 10^{-16} \ \text{to} = -0.4281 \times 10^{-21} \ , \ \text{depending}$$

upon the altitude (200 met to 10,600 met);

$$\eta = 8.8; \ \omega_z = -0.4922 \times 10^{-21} \ (\sec^{-1}).$$

3.3. Stream Function and Velocity Potential Function

This section will discuss the notion and fundamental aspects of two basic functions - the stream function and the velocity potential function - and the notion of streamlines. As examples, the simplest possible cases of flow are used such as flow along a flat plate.

The definitions of a potential and potential field are known from the fundamentals of physics and theoretical mechanics and consequently do not need to be repeated here. The remarks which follow refer primarily to the problem under consideration in this report.

In classical hydrodynamics of a nonviscous and incompressible fluid, one of the assumptions usually made is that the flow is irrotational and that there exists a velocity potential Φ such that:

$$u_1 = \partial\Phi/\partial x_1 \ ; \ u_2 = \partial\Phi/\partial x_2; \ u_3 = \partial\Phi/\partial x_3 \ ; \ u_j = \partial\Phi/\partial x_j. \tag{3.3.1}$$

The basis of this assumption is Kelvin's theorem, which states that the rate of change of circulation with respect to time is zero for an ideal

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fluid. If there exists a velocity potential Φ, the equation of continuity gives:

$$\partial u_j / \partial x_j = 0, \text{ or } \partial^2 \Phi / \partial x_k^2 = 0; \quad j, k = 1, 2, \text{ or } 3; \qquad (3.3.2)$$

which states that Laplace's equation of the velocity potential function, Φ, must be equal to zero:

$$\partial^2 \Phi / \partial x^2 + \partial^2 \Phi / \partial y^2 + \partial^2 \Phi / \partial z^2 = 0 \qquad (3.3.3)$$

The operator defined by

$$\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2 \qquad (3.3.4)$$

is the Laplace operator in three dimensions. Thus in an ideal fluid the Laplace operator applied to the velocity potential function must be equal to zero.

The velocity potential function should satisfy the important condition of zero vorticity:

$$\omega_{ki} = \partial u_i / \partial x_k - \partial u_k / \partial x_i = 0 \qquad (3.3.5)$$

This obviously is satisfied if one introduces the continuous velocity potential function

$$u_i = \partial \Phi / \partial x_i. \qquad (3.3.6)$$

In this case

$$\partial u_i / \partial x_k - \partial u_k / \partial x_i$$
$$= (\partial / \partial x_k)(\partial \Phi / \partial x_i) - (\partial / \partial x_i)(\partial \Phi / \partial x_k) \equiv 0 \qquad (3.3.7)$$

since the above mixed partial derivatives must be equal if the function Φ is a continuous function with respect to both independent variables (coordinates), (x_i, x_k). This is the reason that irrotational flow is called potential flow (Liepmann, Puckett, p. 118). Calculation of the vorticity in the present problem shows that the flow in the boundary layer is rotational everywhere and that consequently the flow cannot be potential. The value of vorticity, calculated in the present two-dimensional flow, is equal to the z-component of the vorticity vector:

$$\omega_z = \frac{1}{2} \text{curl } \vec{V} = \frac{1}{2} (\partial v / \partial x - \partial u / \partial y). \qquad (3.3.8)$$

The formulas which appear in the calculation of the quantity ω_z are as follows (the concept of the stream function is used):

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$$\psi = (\nu^{1/2} x^{1/2} U_\infty^{1/2}) \, f(\eta); \quad \eta = y \, U_\infty^{1/2} \, \nu^{-1/2} x^{-1/2}; \tag{3.3.9}$$

$$\partial \eta / \partial x = - \left(\frac{1}{2}\right) \eta x^{-1}; \quad \partial \eta / \partial y = U_\infty^{1/2} \nu^{-1/2} x^{-1/2}; \tag{3.3.10}$$

$$\partial \psi / \partial \eta = (\nu^{1/2} x^{1/2} U_\infty^{1/2}) \, f'(\eta); \tag{3.3.11}$$

$$u = U_\infty f'(\eta); \tag{3.3.12}$$

$$v = \frac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2}) \, [\, \eta f'(\eta) - f(\eta)\,]; \tag{3.3.13}$$

$$\partial u / \partial y = (U_\infty^{3/2} \nu^{-1/2} x^{-1/2}) f''(\eta); \tag{3.3.14}$$

$$\partial v / \partial x = -\frac{1}{4} (\nu^{1/2} U_\infty^{1/2} x^{-3/2})[\, \eta^2 f''(\eta) + \eta f'(\eta) - f(\eta)\,]. \tag{3.3.15}$$

The introduction of the stream function implies the fact that the equation of continuity should be satisfied (under ordinary conditions). In a two-dimensional steady motion of an incompressible fluid medium with $\vec{V} = (u, v)$ in (x, y) - space, the continuity equation is:

$$\partial u / \partial x + \partial v / \partial y = 0 \; ; \; u = \partial \psi / \partial y \; ; \; v = - \partial \psi / \partial x \; ; \; \psi = \psi_{PB} \tag{3.3.16}$$

Consequently one gets:

$$\partial^2 \psi / \partial x \partial y - \partial^2 \psi / \partial y \partial x = 0, \tag{3.3.17}$$

provided that the mixed second order partial derivatives of the stream function $\psi \equiv \psi_{PB}$ are equal; i. e. , and the value of the second order partial derivative does not depend upon the sequence of the partial differentiation. This implies, of course, that the stream function $\psi = \psi_{PB}$ is a continuous function at least up to the second order differentiation (or higher). It should be remembered that (1) the velocity potential functions, $\Phi = \Phi(x, y)$, exists, if, and only if, the motion of a fluid (liquid or gas) is an irrotational one, however no assumption has been made regarding steady or two-dimensional flow; and (2) $\psi \equiv \psi_{PB}$ exists in two-dimensional (and in some special cases of axially symmetric) steady motion, with no assumption of irrotationality (Liepmann-Puckett, p. 119.) Concerning the existence of these functions in n-dimensional space (higher than 2), the question requires special consideration since in higher dimensions, the characteristic lines, like streamlines, require the application of a much more sophisticated and complicated geometry (topology). In the problem of a two-dimensional boundary layer, only two characteristic functions,

Φ and ψ appear. All the known properties of these two functions are collected and investigated below whether or not they satisfy the required conditions.

Stream function: In a two-dimensional, steady fluid motion (with absolutely no assumption of irrotationality), the stream function has to satisfy the equation of continuity. The function ψ_{PB} is:

$$\psi \equiv \psi_{PB} = \nu^{1/2} x^{1/2} U_\infty^{1/2} f(\eta); \quad (\nu, U_\infty); \tag{3.3.18}$$

where

$$\eta = y (U_\infty^{1/2} \nu^{-1/2} x^{-1/2}) = \eta(x, y); \tag{3.3.19}$$

$$\partial\eta/\partial x = -(\tfrac{1}{2}) \eta x^{-1}; \quad \partial\eta/\partial y = U_\infty^{1/2} \nu^{-1/2} x^{-1/2}; \tag{3.3.20}$$

$$u = U_\infty f'(\eta); \tag{3.3.21}$$

$$v = \tfrac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2}) [\eta f'(\eta) - f(\eta)]; \tag{3.3.22}$$

Then the equation of continuity should be satisfied in a steady, two dimensional incompressible flow:

$$\partial u/\partial x = \partial v/\partial y = 0 \tag{3.3.23}$$

or

$$\partial^2/\psi/\partial y \partial x - \partial^2 \psi/\partial x \partial y = -\tfrac{1}{2} U_\infty x^{-1} \eta f''(\eta) + \tfrac{1}{2} U_\infty x^{-1} \eta f''(\eta) = 0. \tag{3.3.24}$$

This is correct. Consequently, the stream function proposed and introduced by Prandtl and Blasius satisfies the continuity equation. The next step is to calculate the rotationality. By definition, the vorticity vector component existing in the two-dimensional flow in question, ω_z, is equal to (with $\psi \equiv \psi_{PB}$):

$$\omega_z = \tfrac{1}{2} \operatorname{curl} \vec{V} = \tfrac{1}{2} (\partial v/\partial x - \partial u/\partial y); \tag{3.3.25}$$

or with

$$v = -\partial\psi/\partial x; \quad u = \partial\psi/\partial y \tag{3.3.26}$$

one gets:

$$\omega_z = -\tfrac{1}{2} [\partial^2/\partial x^2 + \partial^2/\partial y^2] \psi. \tag{3.3.27}$$

The calculations show that ω_z is not equal to zero. Thus the stream function proposed by Prandtl-Blasius, $\psi = \psi_{PB}$, satisfies the continuity equation, but does not satisfy the irrotationality condition; i.e., the flow is rotational.

Velocity potential flow: The fundamental condition for the existence of the velocity potential function, Φ, is the condition of zero vorticity; i.e., ω_z must be equal to zero. Since in the present case, $\omega_z \neq 0$, a velocity potential in the present case does not exist. Another condition for Φ, $\nabla \Phi = \vec{V}$, is preserved, and the concept of the velocity quasi-potential function is introduced.

3.4. Streamlines

Particular attention will be given to the concept and physical aspects of streamlines for a selected problem. The question is how to locate streamlines in the flow under discussion. As the stream function, ψ ($\equiv \psi_{PB}$), satisfies the continuity equation, the curves $\psi = $ constant represent streamlines in the (x, y) - domain:

$$d\psi = (\partial \psi / \partial y)\, dy + (\partial \psi / \partial x)\, dx = u\, dy - v\, dx, \qquad (3.4.1)$$

or

$$(dy/dx) = v/u. \qquad (3.4.2)$$

Consequently,

$$\psi_{PB} = (U_\infty^{1/2} \nu^{1/2} x^{1/2})\, f(\eta) = \text{constant} \qquad (3.4.3)$$

defines a streamline in a two-dimensional (x, y) domain. Another form of the above condition is equation (3.4.1), which can be rewritten in the form

$$d\psi = U_\infty f'(\eta)\, dy - \frac{1}{2}(U_\infty^{1/2} \nu^{1/2} x^{-1/2})\,(\eta\, f' - f)\, dx = 0 , \qquad (3.4.4)$$

or

$$dy/dx = (U_\infty^{1/2} \nu^{1/2})\,(\eta\, f' - f)\,[\, 2\, U_\infty\, f'(\eta)\, x^{1/2}\,]^{-1} , \qquad (3.4.5)$$

the latter equation determining the slope of a streamline at each point. To calculate the slope, i.e., to use equation (3.4.5) **properly,** one may notice that:

(a) The flow refers to the flow along an infinitely long flat plate located along the x-axis; i.e., located horizontally. However, at first glance it seems that the streamlines are not horizontal lines. Equation (3.4.5) does not show at once that the slope dy/dx is zero everywhere. As a matter of fact, the functions appearing on the right side of equation (3.4.5) are functions of the composite coordinate $\eta = \eta(x, y)$ and of the two independent coordinates (x, y). A result, dy/dx = 0, would mean that

the streamlines have zero slope everywhere and that they are determined by equation $y = y(x)$ = constant; i.e., lines parallel to the x-axis. This is obviously not the case since the slope varies from point to point.

(b) It was proven above that the flow in the boundary layer under discussion is completely filled with vortexes and line vortexes. Consequently, a relationship and correlation between the streamlines, as defined by the condition $\psi \equiv \psi_{PB}$ = constant, and the structure of vortexes should be the subject of closer investigation.

4. VELOCITY POTENTIAL

The next problem to be discussed, is the problem of calculating the velocity potential function which is necessary to continue the quantum approach according to the Madelung proposition. There can be proposed two ways of calculating the velocity potential function in the present case: The first way is the use of the so-called quasi-potential function, i.e., a function which does not satisfy all the conventional requirements of the classical velocity potential function but only a part of these requirements. The second way is the use of the complex variable functions.

4.1. Velocity Potential Function: Quasi-Potential Function

In applying wave mechanics theory to macroscopic fluid dynamics, the function ϕ appears such that the gradient of this function is equal to the velocity vector; i.e., $\vec{U} = \Delta\phi$. This implies that the function ϕ is interpreted as a velocity potential function. Consequently, this section is devoted to the description of the characteristic properties of the velocity potential function.

There exist two basic, fundamental functions in the geometry of the flow of a liquid or gaseous medium (considerations are restricted to a two-dimensional space only): (a) stream function, $\Psi = \Psi(x,y)$; $\partial\Psi/\partial y = u(x,y)$; $\partial\Psi/\partial x = -v(x,y)$. The stream function has to satisfy the (mass) continuity equation; it is the origin of the streamlines, since by the definition, the condition Ψ = constant, gives the equation of a streamline: $\Psi = \Psi(x,y)$ = constant = equation of a streamline as a function of (x,y); the second function of interest is:

(b) velocity potential function, $\phi = \phi(x,y)$. This is explained very thoroughly below.

The formalistic tool applied by the wave mechanics is, in general, in the form of the Schroedinger wave equation. In the present case, as the next step, there is accepted the interpretation (and the corresponding modification) of the Schroedinger equation in the "hydrodynamic form" proposed by Irving Madelung, a quantum physicist, in 1926, (the same year in which Schroedinger published his paper):Quantum Theorie In Hydrodynamischer Form", Zeit, f. Phys., 40, pp. 322-325, 1926. Madelung has proposed a decomposition of the wave function into two factors and next a modification of these factors and the association of these factors with the density of the fluid and the velocity potential function respectively.

The wave function is chosen in the form:

$$\psi = a \exp (i\beta); \quad \phi = -\beta h (2\pi m)^{-1}; \tag{4.1.1}$$

where both functions $a = a(x,y,z,t)$ and $\beta = \beta(x,y,z,t)$ are functions of position and time. Then

$$\text{grad } \phi = \vec{U}; \quad a^2 = \rho = \text{density}; \quad \rho = \text{constant}, \tag{4.1.2}$$

in the case of incompressible fluid flow. In the case of two-dimensional boundary layer flow, $\vec{U} = (u,v)$.

Using the set of relations proposed by Madelung:

$$\phi = -\beta h (2\pi m)^{-1}; \quad \beta = -2\pi m\phi h^{-1}; \quad \text{grad } \phi = \vec{U} (u,v), \quad \partial\phi/\partial x = u;$$
$$\partial\phi/\partial y = v. \tag{4.1.3}$$

Available data relative to the problem of the association of wave mechanics with fluid dynamics show that one of the functions which is known and can be used is the "dimensionless stream function," $f(\eta)$, which is found as a solution of the partial differential equation of the flow in the boundary layer along an infinitely long flat plate. The value of this function was, at first, found and calculated by Blasius and later by Howarth and Schlichting (see Schlichting book, p. 121) who were able to supply improved numerical values. Schlichting's table contains the values of the function for both the independent and dependent variables:

$$\eta = \eta(x,y); \quad f = f(\eta); \quad f'(\eta); \quad f''(\eta). \tag{4.1.4}$$

From the formalism of the velocity component u, one has:

$$u = U_\infty f'(\eta), \tag{4.1.5}$$

where $f'(\eta)$ denotes the first derivative of the dependent function, $f(\eta)$, with respect to the independent variable, η.

The foregoing relationships cannot be considered to be a one-to-one set of transformations and relationships for the simple reason that the function $\eta = \eta(x,y)$ is a one-to-two relationship. As a matter of fact, it was proven previously that the variable η, considered to be a function of two independent variables x and y, is a discontinuous function in the variable y. This discontinuity is of the second order and higher, which means that the first partial derivative of η with respect to y and all higher order partial derivatives of η with respect to y are equal to zero. Thus, the function η is a discontinuous function with respect to the y-coordinate. In addition to the characteristic properties of the function η, one has to remember that there is a fundamental difference between the definitions, physical meanings and geometrical structures of the two functions under consideration. The scalar point function, Φ, is called the velocity potential function and $\vec{U} \equiv \nabla\Phi$, (vorticity in the flow is $\vec{\omega} = 0$) and can be termed potential flow, where the surfaces Φ = constant are the equipotential surfaces. An irrotational fluid motion is characterized by the absence of vortex lines and by the fact that the streamlines are normal to the equipotential surfaces (Owczarek, p. 53). The velocity potential function is basically associated with the dynamics of the fluid motion. The other characteristic function in any fluid dynamics is the well-known stream function which satisfies the continuity equation in a two-dimensional domain (x,y) and gives the equation of the streamlines.

In order to attack the problem of the velocity potential function, one can introduce the "dimensionless" quantities. All lengths, including the coordinates (x,y) are reduced with the aid of a suitable reference length L, assuming L = 1 cm in the cm-gr-sec system. All velocities are made dimensionless with the aid of a suitable velocity, U_{∞}, (cm sec^{-1}). The density ρ is made dimensionless with the aid of the density ρ_o for cm^{-4} - gram-weight-sec^2 or gram-mass-cm^{-3}. The coefficient of the viscosity, μ, is made dimensionless with the aid of μ_o for the conditions at rest at T = 20°C and cm^{-2}-gram-weight-sec. The coefficient of the kinematic viscosity, ν, is made dimensionless using the coefficient of

the kinematic fiscosity, ν_o at $T = 20^\circ C$ and $cm^2 \ sec^{-1}$. Regarding the form of the auxiliary parametric coordinate, η, a wide variety of choices for this coordinate exists from the literature. The velocity potential function, ϕ, may be introduced in the present report although the boundary layer in question is only "macroscopically laminar," i.e., in spite of the fact (see Sections below) that both Prandtl and Blasius assumed "laminar boundary layer flow" but the investigator had previously proven that vortex lines exist in the flow which in itself is sufficient proof that the flow is rotational. The reason for introducing the velocity quasi-potential function, ϕ, even in a hypothetical and abstractive sense, is the necessity for obtaining the function ϕ for future applications.

For the case of a steady flow of a two-dimensional boundary later along a flat plate even without a longitudinal pressure gradient, one could assume that the velocity u is a function of $\eta_1 = y/\sqrt{x} = yx^{-1/2}$ (Pai, p. 154, bottom). From the structure of the function $\eta_1 = \eta_1(x,y) = yx^{-1/2}$, it can be immediately seen that the function η_1 (like the function η in the Prandtl-Blasius boundary layer theory) is a discontinuous function in the second and higher order in the coordinate y. Obviously, many other functions of this nature could be proposed. The dimensionless coordinate η and the dimensionless form of the velocity quasi-potential function $g = g(\eta)$ are assumed; ϕ denotes the dimensional velocity quasi-potential function such that

$$\phi = -(\nu^{1/2} X^{1/2} U_\infty^{1/2}) g(\eta) \ ; \ \eta = \eta(x,y); \ \phi = \phi(\eta). \qquad (4.1.6)$$

X denotes a certain constant value in the x-direction which may in reality be equal to the finite length of the flat plate; i.e., $X \equiv L$.

At this point there may be some advantage in repeating briefly the kinds of functions generally appearing in the field of fluid dynamics, starting, for the sake of simplicity, with the two-dimensional flow domain.

(a) The velocity potential function: The condition of zero vorticity is satisfied if one introduces such a function Φ so that the velocity components (u, v) in a two-dimensional space are related to the function Φ by means of the relationship

$$u = \partial\Phi/\partial x \ ; \ v = \partial\Phi/\partial y \ ; \ \vec{U} \ (u,v) = \nabla\Phi \equiv grad \ \Phi. \qquad (4.1.7)$$

If a function Φ exists in the given flow domain, both from the physical and mathematical point of view, then the condition of zero vorticity is automatically satisfied (Liepmann-Puckett, p. 118). Conversely, it can be shown that if the vorticity in the flow in question is equal to zero, one can always find a potential function, Φ, satisfying equation (4.1.7). This is the reason that irrotational flow is called potential flow and that the two terms are synonymous (Liepmann-Puckett, p. 118).

(b) <u>The stream function</u>: The equation of continuity can be satisfied, under certain conditions, by the introduction of a function, ψ, called the stream function. This function is well known and needs no further discussion here. If the stream function is equal to a constant then the condition for a streamline is satisfied.

It should be remembered that the velocity potential function, Φ, exists if the motion is irrotational. No similar assumption has been made regarding steady flow and no assumption is necessary regarding two-dimensional flow. Regarding the stream function, it exists in two-dimensional (and in some cases in axially symmetric) steady flow motion.

(c) The velocity quasi-potential function, ϕ, is a function which satisfies the condition:

$$\vec{U} = (u, v) = \text{grad } \phi \equiv \nabla\phi; \phi = \phi(x, y, z). \tag{4.1.8}$$

Irrotationality is not superimposed upon the velocity quasi-potential function or upon the physical system in question. The difference between the velocity potential function, Φ, which is subject to the two strong conditions, (a) $\vec{U} = (u, v) = \nabla\Phi \equiv \text{grad } \Phi$ and (b) curl $\vec{U} = 0$, and the velocity quasi-potential function, $\phi = \phi(x, y, z)$, which is subject to one only strong condition, (a) $\vec{U} = (u, v) = \nabla\phi \equiv \text{grad } \phi$, is that the function Φ is a "stronger" velocity potential function and the function ϕ is a "weaker" velocity potential function. It is tacitly assumed that both functions, $\Phi = \Phi(x, y, z)$ and $\phi = \phi(x, y, z)$ are three dimensional.

The two terms, "laminar" and "streamline," need to be discussed. Prandtl-Tietjens state that "this (i. e., solutions found in practical hydraulics, such as flows through pipes and channels), is due to the fact that there exist two radically different kinds of flow: (a) turbulent flow where the particles of the fluid do not move in paths parallel to the walls of the tube but flow in a very irregular manner. In addition

to motion in the direction of the axis of the tube (the principal motion)
secondary motions can be observed which are perpendicular to the axis.
(b) laminar flow where there exists a certain (usually small) velocity
at which the individual particles of fluid start moving rapidly in paths
parallel to the walls of the tube. (Prandtl-Tietjens, pp. 14-15)."

Liepmann-Puckett, in general, do not introduce any direct
definition or definitions of laminar or non-laminar (possibly turbulent)
flow. They state only (Liepmann-Puckett, p. 239) that "it is well
known that at high Reynolds numbers laminar flow becomes unstable
and is replaced by turbulent flow. This latter flow is characterized
by the fact that the velocity is steady only if averaged over a certain
length of time." Owczarek makes no distinction between laminar and
non-laminar flow and discusses only solenoidal and lamellar vector
fields where a vector field $\vec{F}$ in a region in space is

$$\nabla \cdot \vec{F} = 0. \tag{4.1.9}$$

The vector tubes of $\vec{F}$ are closed or start and end at the boundary of the
region. Such a vector field is called solenoidal. At any cross section of
the vector tube there is preserved the condition

$$\int_S d\vec{A} \cdot \vec{F} = \text{constant.} \tag{4.1.10}$$

A typical example of a solenoidal vector point function, $\vec{F}$, is the velocity
of an incompressible fluid. A lamellar vector field is a field in which the
curl of the vector point function vanishes. If $\vec{F}$ represents the vector
point function, then, in a lamellar field,

$$\nabla \times \vec{F} = 0, \tag{4.1.11}$$

or

$$\oint_C \vec{F} \cdot d\vec{r} = 0, \tag{4.1.12}$$

throughout the region having C as its boundary (simply connected region
is chosen for the sake of simplicity). Typical examples of lamellar vector
point functions are the velocity in an irrotational flow and the gravita-
tional field intensity due to the distribution of matter (Owczarek, pp. 595-
596). Glauert states: "Consider the steady motion in layers normal to

the axis of y. The layer of fluid between the planes y and (y + dy) will have a velocity u at all points and u will be a function of y only. When the fluid moves in layers in this manner, it is said to be in laminar motion. When two parallel layers of fluid are moving in the same direction with different velocities, the surface of separation is a vortex sheet and the elementary vortices of this sheet act as roller bearings between the two layers of the fluid. The work which must be done against the tangential stress is represented by the dissipation of energy which occurs in the vortices. The tangential stress at the surface of separation of layers is intimately related to the vortex sheet." (Glauert, pp. 99 and 100).

Writers like von Mises, Patterson, Shkarofsky, et al., Howarth, Goldstein, and others do not discuss the question of a division of a flow of a medium into a laminar and/or non-laminar. They simply treat the conditions of flows without any specific definitions of such flows. The above presentation seems to indicate very clearly that, in the majority of real viscous fluids, there exists the following situation: let one begin the investigation of the fluid from the macroscopic point of view at $T = 0$ (absolute zero) and around (a little above) this point (see Abrikosow, et al., p. 15). With the increase of the temperature, there appears in some liquids (like in liquid Helium He^3) a region in which there exist the (so-called) superfluidity phenomena, i.e., the fluid does not demonstrate the effects of viscosity action. With a further increase of the temperature, there appears the "transition, λ, point". Below this point, a superfluid motion and superfluid status are possible; above that point, the superfluid motion is not possible. Above λ point, the hydrodynamics of the liquid do not differ from ordinary hydronamics and are known under the name of macroscopic (classical) hydrodynamics.

Since in macroscopic hydrodynamics every fluid (not superfluid) possesses a viscosity, consequently very often the motion of the existing real fluid occurs by means of a "roller bearing" effect (vortex sheets and elementary vortices) mentioned by Glauert. The existence of vortices, however, proves that the fluid motion is a rotational (or turbulent) one and cannot be an irrotational one. The problem can thus be reduced to the following distinction between the flows: due to the viscosity action the flow of a liquid or gaseous medium is always (above the transition λ point) a rotational one; this means that the curl of velocity is always

different from zero, curl $\vec{V} \neq 0$, and is never equal to zero. It is always different from zero in the <u>microscopic</u> domain (above the critical λ-point) but this fact is unobservable by the naked human eye, as is also the <u>macroscopic</u> domain at low speeds, at low Reynolds numbers. With the increasing values of the velocity of the flow, there appears the action of the kinetic energy of the flow following the vector equality, $\nabla (\vec{U}^2) = 2 \vec{U} \cdot (\text{grad } \vec{U}) + 2 \vec{U} \times (\text{curl } U)$. As it is seen, the increasing kinetic energy always increases the action of the curl, increasing simultaneously the action of the gradient of $\vec{U}$. A flow in which curl $\vec{U}$ is small and is unobservable by naked human eyes is called often macroscopically "laminar" flow (laminar boundary layer). However, "de facto", the flow is not a laminar one, with curl being an unobservable phenomenon and with the increasing value of the kinetic energy the value of the curl; $\nabla \times \vec{U}$, is becoming so strong that the curl can be seen by the naked human eyes. Such a flow is usually called ("macroscopically") turbulent flow. In the past, the writer's used to distinguish between laminar and turbulent flows based upon the macroscopic observations of the flow by the naked human eyes. The writer would propose a different distinction: flows and phenomena observable and unobservable macroscopically or microscopically by naked human eyes. A flow can be judged to be a laminar one according to the macroscopic observations. But, in fact, the same flow may be a turbulent one if judged from the "microscopic" point of view -- the disturbances due to curl may be unobservable by naked human eyes. Obviously, at some future time, one will propose a better distinction between the "laminar and turbulent" flows than the presently existing one.

To summarize the results, and to have a clear picture of them, the following collection of the results is presented.

<u>The dimensional stream function</u> in the form proposed by Prandtl-Blasius is:

$$\psi = \psi(x, y) = \psi(x, \eta) = (\nu^{1/2} x^{1/2} U_\infty^{1/2}) f(\eta) ; \qquad (4.1.13)$$

$$\eta = \eta(x, y) = y U_\infty^{1/2} \nu^{-1/2} x^{-1/2} ; \qquad (4.1.14)$$

$$\partial\eta/\partial x = - (\tfrac{1}{2}) \eta x^{-1}; \qquad (4.1.15)$$

$$\partial\eta/\partial y = U_\infty^{1/2} \, \nu^{-1/2} \, x^{-1/2} \; ; \tag{4.1.16}$$

$$u = \partial\psi/\partial y = (\partial\psi/\partial\eta)(\partial\eta/\partial y) = U_\infty f'(\eta) \; ; \tag{4.1.17}$$

$$v = -\partial\psi/\partial x = \frac{1}{2}(\nu^{1/2} U_\infty^{1/2} x^{-1/2})(\eta f' - f) \; ; \tag{4.1.18}$$

where the function $f(\eta)$ denotes the dimensionless stream function. The stream function ψ (or ψ_{PB}) must satisfy the equation of continuity in a two-dimensional flow, which it does. No condition on irrotationality is required.

The velocity potential function, Φ, must satisfy the condition of irrotationality (zero vorticity) and in this case

$$\vec{V} = (u,v,w) = \nabla\Phi = \text{grad } \Phi \tag{4.1.19}$$

$$\omega = \nabla \times \vec{V} = 0$$

where the velocity vector is the gradient of the velocity function, Φ, and the curl of velocity vector must be equal to zero. One can introduce, additionally, one more function called quasi-potential function, satisfying only partly the above stated conditions.

The dimensional velocity quasi-potential function, ϕ, has only one property namely, the velocity vector is the gradient of the scalar function ϕ, (no condition of irrotationality superimposed upon the function ϕ) and is

$$\vec{V} = (u,v,w) = \nabla\phi = \text{grad } \phi \; ,$$

where the curl of the velocity vector may or may not be equal to zero. The dimensional velocity quasi-potential function is proposed in the form

$$\phi = -(\nu^{1/2} X^{1/2} U_\infty^{1/2}) g(\eta) = \phi(x,y) = \phi(x,\eta) \; ,$$

$$\eta = y(U_\infty \nu^{-1} x^{-1})^{1/2}; \quad \partial\eta/\partial x = -(\tfrac{1}{2})(\eta x^{-1}); \quad \partial\eta/\partial y = U_\infty^{1/2}(\nu x)^{-1/2}$$

$$\tag{4.1.21}$$

where $g(\eta)$ is called the dimensionless velocity quasi-potential function; then

$$u = \partial\phi/\partial x = (\partial\phi/\partial\eta)(\partial\eta/\partial x) = -(\nu^{1/2} X^{1/2} U_\infty^{1/2}) g'(\eta)(-\tfrac{1}{2})\,\eta\, x^{-1}$$

$$= (\tfrac{1}{2})(\eta x^{-1})(\nu^{1/2} X^{1/2} U_\infty^{1/2}) g'(\eta) \; ; \tag{4.1.22}$$

$$v = (\partial\phi/\partial y) = (\partial\phi/\partial\eta)(\partial\eta/\partial y) = (\nu^{1/2} X^{1/2} U_\infty^{1/2}) \, g'(\eta) \, (U_\infty^{1/2}\nu^{-1/2}x^{-1/2})$$

$$= U_\infty (X x^{-1})^{1/2} \, g'(\eta) \; . \tag{4.1.23}$$

One can verify that, with $\nu \approx cm^2 \, sec^{-1}$, $X \approx cm$, $U_\infty \approx cm \, sec^{-1}$, the proper dimensions are preserved in the above equations for u and v, Equations (4.1.22) and (4.1.23). For the velocity component u, Equation (4.1.22) becomes

$$(\eta x^{-1}) \, \nu^{1/2} X^{1/2} U_\infty^{1/2} \approx (cm \, sec^{-1/2})(cm^{1/2})(cm^{1/2} \, sec^{-1/2})(cm^{-1})$$

$$\approx (cm^2 \, sec^{-1})(cm^{-1}) \approx (cm \, sec^{-1}),$$

which is correct. The velocity v also has dimensions $\approx cm \, sec^{-1}$.

The definition of streamline can be found in von Mises, Glauert, Owczarek and others. At each fixed time t, there is a (two-parameter) family of streamlines; i.e., space curves described by moving particles. For $t = t_o$ each streamline is tangent at each point to the velocity vector at this point or: $dx: dy: dz = q_x: q_y: q_z$, where q_x denotes the velocity component in x-direction and $q_x = q_x(x,y,z,t_o)$, etc. Clearly the streamline through any point $\vec{r}$ at $t = t_o$, is tangent to the trajectory of the particle through $\vec{r}$ at that instant [von Mises, pp. 4-5]. As the stream function ψ satisfies the continuity equation, the curves of $\psi = $ constant represent streamlines. The relation

$$d\psi = (\partial\psi/\partial x) \, dx + (\partial\psi/\partial y) \, dy = -v \, dx + u \, dy = 0 \text{ for } \psi = \text{constant}$$
$$\tag{4.1.24}$$

(Owczarek, pp. 62-63);

is valid in a two-dimensional (temporarily fixed) flow at any period of time due to the fact that

$$u = \partial\psi/\partial y \, ; \quad v = -\partial\psi/\partial x \; . \tag{4.1.25}$$

Equations derived above may allow one to pass from the notion of the streamfunction, $\psi(x,y)$, (dimensional), or $f(\eta)$, (dimensionless), to the concept of velocity potential functions, Φ, or velocity functions $\phi(x,y)$, (dimensional), or $g(\eta)$, (dimensionless), and vice versa. From Equations (4.1.22) and (4.1.5) one obtains

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$$u = (\tfrac{1}{2})\,(\eta\,x^{-1})\,(\nu^{1/2}X^{1/2}U_\infty^{1/2})\,g'(\eta) = U_\infty f'(\eta)\,. \qquad (4.1.26)$$

From Equations (4. 1. 23) and the Prandtl-Blasius approach one obtains

$$v = U_\infty(X\,x^{-1})^{1/2}\,g'(\eta) = \tfrac{1}{2}\,U_\infty^{1/2}\,\nu^{1/2}\,x^{-1/2}\,[\eta\,f'(\eta) - f(\eta)]\,. \qquad (4.1.27)$$

Verification of the dimensions of various terms and factors in the present work is repeated occasionally but done for precautionary purposes so as to avoid possible mistakes.

In the above, the $g'(\eta)$ denotes a differentiation with respect to η. One may verify the dimensions of the variables appearing in the problem, if and when the dimensional variables are used:

$u \approx cm\ sec^{-1}$; $v \approx cm\ sec^{-1}$, $U_\infty \approx cm\ sec^{-1}$;

$f = f(\eta) \approx$ dimensionless streamfunction;

$\eta \approx$ dimensionless coordinate; $(x,y) \approx$ dimensional coordinate $\approx cm$;

$\psi =$ stream function $\approx cm^2\ sec^{-1}$, since $u = \partial\psi/\partial y$;

$\phi =$ velocity potential function $\approx cm^2\ sec^{-1}$, since:

$u = \partial\phi/\partial x.$

The variables and the coefficients appearing in the functions in question are

$$\eta = y\,U_\infty^{1/2}\,\nu^{-1/2}\,x^{-1/2}\ ;\quad \text{(dimensionless);}$$

$\nu =$ coefficient of kinematic viscosity $\approx cm^2\ sec^{-1}$;

$$\eta \approx cm\,(cm^{1/2}\,sec^{-1/2})(cm^{-1}\,sec^{1/2})\,cm^{-1/2} = \text{dimensionless;}$$

$$\psi = \nu^{1/2}\,x^{1/2}\,U_\infty^{1/2}\,f(\eta) \approx (cm\,sec^{-1/2})\,cm^{1/2}\,cm^{1/2}\,sec^{-1/2}$$

$$\approx cm^2\ sec^{-1}.$$

The proposed form of the velocity qusi-potential function, Equation (4. 1. 6) is

$$\phi = -\,(\nu^{1/2}\,X^{1/2}\,U_\infty^{1/2})\,g(\eta)\,, \qquad (4.1.28)$$

where

$X =$ constant length $\approx cm$;

and where $g(\eta)$ may be called the dimensionless velocity quasi-potential function. Thus

$$\phi \approx \nu^{1/2} X^{1/2} U_\infty^{1/2} \approx (cm\ sec^{-1/2})\ cm^{1/2}\ (cm^{1/2}\ sec^{-1/2})$$

$$\approx cm^2\ sec^{-1}. \qquad (4.1.29)$$

Verification of the coefficients of the velocity components, u and v (Schlichting, p. 117) furnishes:

coefficient of u: $\quad u = U_\infty f'(\eta); \quad U_\infty \approx cm\ sec^{-1}; \qquad (4.1.30)$

coefficient of v: $\quad v = \dfrac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2})(\eta\ f' - f); \qquad (4.1.31)$

then

$$\nu^{1/2} U_\infty^{1/2} x^{-1/2} \approx (cm\ sec^{-1/2})(cm^{1/2}\ sec^{-1/2})(cm^{-1/2}) \approx cm\ sec^{-1}.$$

$$(4.1.32)$$

$\nabla\phi = grad\ \phi = \vec{V} (u,v) = $ velocity vector. From the definition of the velocity quasi-potential function,

$$\partial\phi/\partial x = (\partial\phi/\partial\eta)(\partial\eta/\partial x) = u; \qquad (4.1.33)$$

$$\partial\phi/\partial y = (\partial\phi/\partial\eta)(\partial\eta/\partial y) = v; \qquad (4.1.34)$$

where the derivatives, appearing in Equations (4.1.33) and (4.1.34), are:

$$\partial\phi/\partial\eta = - (\nu^{1/2} X^{1/2} U_\infty^{1/2})\ d\ g(\eta)/\partial\eta; \qquad (4.1.35)$$

$$\eta = y\ U_\infty^{1/2} \nu^{-1/2} x^{-1/2}; \quad (\partial\eta/\partial x) = (-\tfrac{1}{2})\ \eta\ x^{-1}; \qquad (4.1.36)$$

$$\partial\eta/\partial y = U_\infty^{1/2} \nu^{-1/2} x^{-1/2}; \qquad (4.1.37)$$

thereby verifying the correctioness of the dimensions:

$$u = \partial\phi/\partial\eta)(\partial\eta/\partial x) = - (\nu^{1/2} X^{1/2} U_\infty^{1/2})(d\ g(\eta)/d\eta)(-\tfrac{1}{2}\ \eta\ x^{-1})$$

$$= \dfrac{1}{2} (\nu^{1/2} X^{1/2} U_\infty^{1/2})\ (\eta\ x^{-1})\ (d\ g(\eta)/d\eta)$$

$$\approx (cm\ sec^{-1/2})(cm^{1/2})(cm^{1/2}\ sec^{-1/2})(cm^{-1}) \approx cm\ sec^{-1},$$

$$(4.1.38)$$

since η is a dimensionless coordinate; and

$$v = (\partial\phi/\partial\eta)(\partial\eta/\partial y)$$

$$= -(\nu^{1/2} X^{1/2} U_\infty^{1/2})(d\ g(\eta)/d\eta)(\partial\eta/\partial y)$$

$$= -(\nu^{1/2} X^{1/2} U_\infty^{1/2})(d\ g(\eta)/d\eta)(U_\infty^{1/2} \nu^{-1/2} x^{-1/2})$$

$$= \approx (cm\ sec^{-1/2})(cm^{1/2}\ sec^{-1/2})(g'(\eta))(cm^{1/2}\ sec^{-1/2})$$

$$\cdot (cm^{-1}\ sec^{+1/2})(cm^{-1/2}) \approx U_\infty(d\ g(\eta)/d\eta) \approx cm\ sec^{-1}\ .$$

$$\text{(4. 1. 39)}$$

The relationship between the velocity quasi-potential function and stream function f (f calculated numerically by Blasius) is proposed in the form

$$\frac{1}{2}(\nu X\ U_\infty)^{1/2}(\eta\ x^{-1})\ g'(\eta) = U_\infty f'(\eta)\ , \qquad \text{(4. 1. 40)}$$

with the functions $f = f(\eta)$, $f'(\eta) = df/d\eta$, being calculated and tabulated numerically in Schlichting's book (p. 121).

The approach to the calculation of the dimensionless velocity quasi-potential function is done for illustrative purposes only. The reason for the example is to demonstrate that anyone attempting to solve a problem of this character may be unaware at the outset that there are great differences in the absolute values of the velocity components u and v. These functions must be used to calculate the value of the potential function. The first step of the approach toward achieving this goal is demonstrated below and consists of a process of multiplication quantitatively large functions of u by quantitatively small functions of v. If these quantitative values are not known initially, one may proceed in the manner given below by trying to seek an average value.

From Equation (4. 1. 40) one gets:

$$g'(\eta) = 2(\nu^{-1/2} U_\infty^{1/2} X^{-1/2})(x\ \eta^{-1})\ f'(\eta)\ , \qquad \text{(4. 1. 41)}$$

or

$$\int dg(\eta) = 2(\nu^{-1/2} X^{-1/2} U_\infty^{1/2} x)\int \eta^{-1}\ f'(\eta)\ d\eta\ . \qquad \text{(4. 1. 42)}$$

Verifying the coefficient

$$(\nu^{-1/2} X^{-1/2} U_\infty^{1/2} x) \approx (cm^2\ sec^{-1})^{-1/2}(cm^{-1/2})(cm^{1/2}\ sec^{-1/2})\ cm$$

$$\approx dimensionless.$$

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This is correct; because the function $f'(\eta)$ is not given in a closed form (only in tabulated form) and thus the function $g(\eta)$ can only be calculated by a numerical integration. An analogous approach is followed with respect to the component v:

$$v = \partial\phi/\partial y = - (\nu^{1/2} X^{1/2} U_\infty^{1/2})(d\, g(\eta)/d\eta)(\partial\eta/\partial y)$$

$$= - (\nu^{1/2} X^{1/2} U_\infty^{1/2})(g'(\eta))(U_\infty^{1/2} \nu^{-1/2} x^{-1/2})$$

$$= - U_\infty (X\, x^{-1})^{1/2} g'(\eta) . \qquad (4.1.43)$$

On the other hand the variable v is given in the following form from the Prandtl-Blasius theory (Schlichting, p. 117):

$$v = - \partial\psi_{PB}/\partial x = \frac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2})(\eta\, f'-f) . \qquad (4.1.44)$$

A comparison of the equations gives ((4.1.43) = (4.1.44)):

$$- U_\infty (X\, x^{-1/2})^{1/2} g'(\eta) = \frac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2})(\eta\, f'-f) , \quad (4.1.45)$$

where $g'(\eta)$, η, $f'(\eta)$, f, are dimensionless, and the coefficients on both sides of the equation have dimensions since dimensional quantities for the coordinates (x, X) and for the coefficient of kinematic viscosity are used.
Thus

$$\mathrm{cm\ sec}^{-1} \approx (\mathrm{cm}^2\ \mathrm{sec}^{-1})^{1/2} (\mathrm{cm\ sec}^{-1})^{1/2} (\mathrm{cm}^{-1/2})$$

$$\approx (\mathrm{cm\ sec}^{-1/2}) (\mathrm{cm}^{1/2}\ \mathrm{sec}^{-1/2})(\mathrm{cm}^{-1/2}) \approx \mathrm{cm\ sec}^{-1}$$

$$(4.1.46)$$

which is correct; Equation (4.1.45) gives:

$$- U_\infty (X\, x^{-1})^{1/2} g'(\eta) = \frac{1}{2} (\nu^{1/2} U_\infty^{1/2} x^{-1/2}) (\eta\, f'-f) ; \qquad (4.1.47)$$

or

$$g'(\eta) = - \frac{1}{2} (\nu^{1/2} U_\infty^{-1/2} X^{-1/2}) (\eta\, f'-f) \qquad (4.1.48)$$

where the coefficient on the right hand side has no dimensions. In reality, in the case dimensional quantities are used the value becomes

$$(\mathrm{cm}^2\ \mathrm{sec}^{-1})^{1/2} (\mathrm{cm\ sec}^{-1})^{-1/2} (\mathrm{cm}^{-1/2}) \approx \text{dimensionless.} \quad (4.1.49)$$

This is correct due to the fact that η, $f'(\eta)$, $f(\eta)$, $g'(\eta)$, are dimensionless functions.

From Equation (4.1.48) one obtains:

$$g'(\eta) = \frac{1}{2} (\nu^{1/2}) \ U_\infty^{-1/2} \ x^{-1/2}) \ (f - \eta \ f'). \qquad (4.1.50)$$

This obviously must be equal to Equation (4.1.41). When both sides of Equations (4.1.41) and (4.1.50) are multiplied (i.e., left by left, right by right), respectively, one gets:

$$[g'(\eta)]^2 = X^{-1} \ (x \ \eta^{-1}) \ f' \ (f - \eta \ f') . \qquad (4.1.51)$$

Or:

$$g'(\eta) = X^{-1/2} \ (x \ \eta^{-1})^{1/2} \ [f'(\eta)]^{1/2} \ [f(\eta) - \eta \ f'(\eta)]^{1/2} \qquad (4.1.52)$$

to find the function $g(\eta)$, Equation (4.1.52) must be integrated numerically. In Equation (4.1.52) the quantities (x, X) must be measured in the same units of length (cm) if one uses dimensional quantities.

The velocity quasi-potential function, $\phi(\eta)$, contains in its structure, the dimensionless quasi-potential function $g(\eta)$. The function $g(\eta)$ has to be calculated numerically by means of the dimensionless stream function $f(\eta)$ and its derivatives f' and f'' (if necessary). The function f is a function of one independent, dimensionless coordinate, η. Its derivatives involve only ordinary derivatives f', $f'' = d^2 f/d\eta^2$. The relationship between the dimensionless velocity potential function $g(\eta)$ and the dimensionless stream function f and its derivative f' is given in Equation (4.1.52). The involved integration has to be done numerically. The following integration procedure is used:

The functions $\eta = \eta(x, y)$; $f = f(\eta)$; $f' = f'(\eta)$; $f'' = f''(\eta)$, are tabulated in the Schlichting book, p. 121.

First Step

These functions can be plotted graphically as functions of η. The numerical tables of Schlichting are ordered according to the stepwise variations of η, which is the independent variable (coordinate).

Second Step

The second step of the numerical integration is to calculate the

values of the function $f(\eta)$, $f'(\eta)$, $f''(\eta)$ at intermediate points; i. e. , at: $\eta = 0. 1$; $0. 3$; $0. 5$; $0. 7$; $0. 9$.   This procedure has to be done by the use of interpolation calculus.   The functions (f, f', f'') are continuous and smooth functions, their slopes do not exhibit any particular or peculiar behavior or singularities; and, consequently, the values of these functions at intermediate points are calculated as arithmetic means; for example,

$$F(\eta)_{\eta = 3. 5} = \frac{1}{2} \left[F(\eta)_{\eta=3} + F(\eta)_{\eta=4} \right] .$$

Since an integral is equivalent to a sum and an integration procedure is equivalent to a summation procedure, the integration procedure is substituted by the summation procedure of partial areas, where the partial area is of the form(as an example)

$$\text{partial area} = F(\eta)_{n=3. 5} \cdot \left[(\eta)_{n=4} - (\eta)_{n=3} \right] \tag{4. 1.53}$$

where the η-axis is located horizontally and its coordinates vary from (0) to ($+ \infty$).

Third Step

After the dimensionless velocity potential function, $g(\eta)$, is calculated, the next step is to calculate the dimensional velocity quasi-potential function, ϕ, Equation (4. 1. 28):

$$\begin{aligned} \phi &= - (\nu^{1/2} X^{1/2} U_{\infty}^{1/2}) \, g(\eta) \\ &= - (\nu^{1/2} X^{1/2} U_{\infty}^{1/2}) \, g(y \, U_{\infty}^{1/2} \, \nu^{-1/2} \, x^{1/2}) \\ &= \phi(x, y) . \end{aligned} \tag{4. 1. 54}$$

Fourth Step

The fourth step consists of the construction of the function β, (Madelung) where β is of the form:

$$\beta = - 2 \pi m \phi h^{-1} , \tag{4. 1. 55}$$

m = mass of the electron = $\sim 0. 9107 \times 10^{-27}$;

h = Planck's constant = $6. 62517 \times 10^{-27}$;

$\phi = \phi(x, y, z, t)$; $\beta = \beta(x, y, z, t)$.

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The function β enables one to calculate the value of the wave function, (Madelung); i. e. , $\psi = \psi(x,y,z,t) = a \exp(i\beta)$. Obviously, this will be a numerical calculation since the original and starting functions, such as the functions $f(\eta)$, $g(\eta)$ were only calculated numerically and not analytically (in closed form). The wave function ψ will be used and discussed very thoroughly in a separate section of this report. The above presentation and derivation of the wave function is successful since the derivation of ψ originated from the given stream function, ψ_{PB}, the value of which was calculated by Prandtl and Blasius and which is tabulated in Schlichting's book.

Fifth Step

As the next step, one may try to find a direct relationship and transformation between the functions appearing in the wave mechanics on one side (like the functions β, ϕ, and g), and the functions appearing in the macroscopic fluid dynamics on another (like the functions ψ, f). The first correspond to the microscopic domain of quantum dimensions, the second correspond to the macroscopic domain of well-known classical dimensions. As demonstrated here, the practical application of the idea is expressed very nicely by D. Bohm (Quantum Theory, p. 628): ..."it was shown that the definition of small scale properties of a system is possible only as a result of interaction with large scale systems undergoing irreversible processes. In line with the above suggestion, we propose also that irreversible processes taking place in the large scale environment may also have to appear explicitly in the fundamental equations describing phenomena at the nuclear level." The pertinent parts of the sentence are underlined. The above quotation will be used to calculate the function $g(\eta)$ valid in the microscopic (quantum) domain but with the use of the functions $f(\eta)$, $f'(\eta)$, $f''(\eta)$, which are known, calculated and used in the macroscopic, real domain.

During the numerical integration of the integral, Equation (4.1.52), it came out that the last factor in Equation (4.1.52) under the integral sign was always negative and that, consequently, the square root was an imaginary number. To avoid this inconvenience the following procedure was adopted: Equation (4.1.52) is the product of Equation (4.1.41) and Equation (4.1.50); from which it was apparent that the right hand side of Equation (4.1.50) produced the negative value. Consequently, Equation

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(4. 1. 50) is multiplied by (-1) and written in the following form

$$(-1) \, g'(\eta) = \frac{1}{2} (\nu^{1/2} \, U_\infty^{-1/2} \, X^{-1/2}) \, (\eta f' - f) \quad \text{from Eq.} \qquad (4.1.56)$$
$$(4.1.50).$$

Next, Equation (4. 1. 41) is multiplied by Equation (4. 1. 56); i. e. , left hand side of Equation (4. 1. 41) is multiplied by the left hand side of Equation (4. 1. 56) and right hand side of Equation (4. 1. 41) is multiplied by the right hand side of Equation (4. 1. 56) giving as the result

$$(-1) \, [g'(\eta)]^2 = (X^{-1}) \, (x \, \eta^{-1}) \, f'(\eta f' - f) \ . \qquad (4.1.57)$$

The next operation involves taking the square root:

$$(\sqrt{-1}) \, g'(\eta) = (X^{-1/2}) \, (x \, \eta^{-1})^{1/2} \, (f')^{1/2} \, (\eta \, f' - f)^{1/2} \ . \qquad (4.1.58)$$

Equation (4. 1. 58) corresponds to Equation (4. 1. 52) and the integration furnishes:

$$i \, g(\eta) = (X^{-1/2}) \int (x \, \eta^{-1})^{1/2} \, (f')^{1/2} \, (\eta f' - f)^{1/2} \, d\eta, \ i = \sqrt{-1} \ . \qquad (4.1.59)$$

This equation corresponds to integrated Equation (4. 1. 52) but no longer has the negative values in the last factor which item causes some difficulty when performing numerical integration on the computer.

The above calculation of the function $g(\eta)$ is equivalent to taking a geometric mean value from the two values of $g(\eta)$: one given by Equation (4. 1. 41) and the other given by Equation (4. 1. 50), respectively. An analogous or similar result could be obtained by taking the arithmetic mean from the two values of $g(\eta)$, one given by Equation (4. 1. 41) and another by Equation (4. 1. 50), respectively. After performing the integration on the right-hand side of Equation (4. 1. 59), one ignores the fact that the left-hand side is an imaginary one and considers only the real part of the result.

The foregoing example demonstrates an approximate manner of the calculation of the velocity potential function. The numerical integration can be achieved according to one of two schemes as follows: suppose that one wants to integrate the area under the curve $y = f(x)$; then the area can be divided into a certain number of strips having the y-coordinates y_0, y_1, y_2, and so on. The area of one strip is

$$A_1 = y_{1/2} (b-a) ,$$

(4.1.60)

or

$$A_2 = \frac{1}{2} (y_0 + y_1) [b-a] ;$$

(4.1.61)

where (b,a) are the x-coordinates of the limits (end points) of the strip and where y_0, y_1, correspond to (a,b), respectively; $y_{1/2}$ corresponds to the point $(a+b)/2$, respectively. The error of the A_2 approach is usually greater one than that of the A_1 approach.

4.2. Velocity Potential Function: Complex Variable Function.

The second way of calculating the velocity potential function involves the elements of the theory of complex variables. This approach is not always appreciated but can be very valuable.

A single complex coordinate of a point is defined by the equation

$$z = x + iy = r(\cos\theta + i\sin\theta) ; \quad i = \sqrt{-1} ;$$

(4.2.1)

$$\cos\theta + i\sin\theta = \exp(i\theta) ;$$

(4.2.2)

$$z = r\exp(i\theta); \quad r = \text{modulus of } z;$$

(4.2.3)

$$r = \text{mod } z = |z| ; \quad \theta = \text{argument of } z .$$

(4.2.4)

Any function $f(z)$ of the complex variable z can be separated into its real and imaginary parts, and can be expressed in the form $(X + iY)$, where X and Y are real. Consider a function of the complex variable z which has a single valued differential coefficient at every point and let

$$f(z) = \xi + i\eta; \quad df/dz = p + iq$$

(4.2.5)

then

$$p + iq = df/dz = \partial f/\partial x = \partial\xi/\partial x + i\,\partial\eta/\partial x ,$$

(4.2.6)

and

$$df/dz = i^{-1}\,\partial f/\partial y = -i\,\partial\xi/\partial y + \partial\eta/\partial y .$$

(4.2.7)

Hence

$$\partial\xi/\partial x = \partial\eta/\partial y = p ,$$

(4.2.8)

and

$$\partial\xi/\partial y = -\partial\eta/\partial x = -q ,$$

(4.2.9)

$$\nabla^2\xi = \nabla^2\eta = 0 .$$

(4.2.10)

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Introducing the velocity potential function, Φ, and the stream function, ψ, one can combine these two into a complex velocity potential function:

$$w = f(z) = \Phi + i\,\psi; \quad dw/dz = u - i\,v; \tag{4.2.11}$$

$$\Phi \equiv \xi; \quad \psi \equiv \eta; \quad \partial\xi/\partial x = \partial\eta/\partial y = \partial\Phi/\partial x = \partial\psi/\partial y; \tag{4.2.12}$$

$$\eta \equiv \psi; \quad \xi \equiv \Phi; \quad \partial\xi/\partial y = -\partial\eta/\partial x = \partial\Phi/\partial y = -\partial\psi/\partial x; \tag{4.2.13}$$

$$\partial\Phi/\partial x = \partial\psi/\partial y = u; \tag{4.2.14}$$

$$\partial\Phi/\partial y = -\partial\psi/\partial x = -v. \tag{4.2.15}$$

The above formalism presents the fundamentals of the association of the velocity potential function with the stream function [Glauert, pp. 53-55] in a two-dimensional fluid motion. Concerning the higher dimensional space (three or more) some special propositions have to be used. These will not be discussed at this time. Equations (4.2.14) and (4.2.15) give an association between the function ψ, stream function, discussed in previous sections, and the sought-after function, velocity potential, Φ. As stated previously, the function ψ is known, and has been calculated and tabulated in Schlichting's book. The function Φ must be found by a numerical integration. The manner of achieving this will follow.

In general, when the function ψ is assumed to be known, the values of the velocity components, u,v, can be found and calculated by the analytical process of differentiation. For the sake of clarity, the process of the detailed differentiation of the function ψ will be reproduced below. The task will therefore be to calculate and to find the correct value, or as close as possible, of the function Φ; i.e., the velocity potential function. This task is associated with an inverse process; e.g., process of integration.

(A) The stream function ψ must satisfy the continuity equation. The proof is very simple, as taken from Schlichting, pp. 117, one has

$$u = U_\infty f'(\eta); \quad v = \frac{1}{2} U_\infty^{1/2} \nu^{1/2} x^{-1/2} (\eta f' - f); \tag{4.2.16}$$

$$\partial u/\partial x = (\partial u/\partial \eta)(\partial \eta/\partial x) = U_\infty f''(\eta)\left(-\frac{1}{2} y\, U_\infty^{1/2} \nu^{-1/2} x^{-3/2}\right)$$

$$= U_\infty f''(\eta)\left(-\frac{1}{2} \eta\, x^{-1}\right); \quad \eta = y\, U_\infty^{1/2} \nu^{-1/2} x^{-1/2}; \tag{4.2.17}$$

$$\partial v/\partial y = (\partial v/\partial \eta)(\partial \eta/\partial y) = \tfrac{1}{2} U_\infty^{1/2} \nu^{1/2} x^{-1/2}(f'\eta f'' - f') U_\infty^{1/2} \nu^{-1/2} x^{-1/2}$$

$$= \tfrac{1}{2} U_\infty \eta x^{-1} f'' . \tag{4.2.18}$$

The sum of the two equations (4.2.17) and (4.2.18) is zero which is correct.

(B) The velocity potential function stipulates that the flow is is irrotational and the vorticity vanishes; i.e., $\omega = \nabla \times \vec{V} = 0$. In simply connected regions the flow velocity vector in such flow is the gradient of a single-valued scalar point function, Φ , called the velocity potential function.

Thus the scalar point function, Φ , should be constructed in such a manner as to furnish an irrotational or potential flow and its gradient should be equal to the flow velocity vector. The velocity quasi-potential function ϕ and the dimensionless velocity quasi-potential function, $g(\eta)$ have been introduced to fulfill only one condition mentioned above; namely, that the gradient should be equal to the flow velocity vector.

Returning to the problem of the complex velocity potential function, Equations (4.2.5) to (4.2.15), respectively, one may repeat:

$$w = \Phi + i \psi; \quad dw/dz = u - i v; \quad u = \partial \psi/\partial y ; \tag{4.2.19}$$

and

$$v = - \partial \psi/\partial x; \quad \eta = y U_\infty^{1/2} \nu^{-1/2} x^{-1/2} ; \tag{4.2.20}$$

or:

$$(\partial \psi)/\partial y = (\partial \psi/\partial \eta)(\partial \eta/\partial y) ; \tag{4.2.21}$$

$$\partial \psi/\partial x = (\partial \psi/\partial \eta)(\partial \eta/\partial x) . \tag{4.2.22}$$

From the set of relations, derived above, one gets

$$\partial \Phi/\partial x = \partial \psi/\partial y = u ; \quad \partial \Phi/\partial y = - \partial \psi/\partial x = - v , \tag{4.2.23}$$

which is in agreement with the statement that the real part must be equal to the real part and the imaginary part must be equal to the imaginary part as

$$df/dz = \partial f/\partial x = \partial \Phi/\partial x + i \partial \psi/\partial x$$

$$= i^{-1} \partial f/\partial y = - i \partial \Phi/\partial y + \partial \psi/\partial y ; \tag{4.2.24}$$

or

$$\partial \Phi/\partial x = \partial \psi/\partial y ; \quad \partial \psi/\partial x = - \partial \Phi/\partial y . \tag{4.2.25}$$

Another form of the above stated in equational form may be

$$\partial\Phi/\partial x + i\, \partial\Phi/\partial y = \partial\psi/\partial y - i\, \partial\psi/\partial x \; ; \qquad (4.2.26)$$

or

$$\partial\Phi/\partial x = u \; ; \quad \partial\Phi/\partial y = v \; ; \qquad (4.2.27)$$

$$\partial\psi/\partial x = -v \; ; \quad \partial\psi/\partial y = u \; . \qquad (4.2.28)$$

Thus

$$u + i\,v = u + iv \qquad (4.2.29)$$

is obviously true. For purely formalistic purposes one can write:

$$
\begin{aligned}
d\Phi \; &= \; (\partial\Phi/\partial x)\, dx + (\partial\Phi/\partial y)\, dy \\[4pt]
&= \; u\, dx + v\, dy \\[4pt]
&= \; (\partial\psi/\partial y)\, dx - (\partial\psi/\partial x)\, dy \\[4pt]
&= \; (\partial\psi/\partial\eta)(\partial\eta/\partial y)\, dx - (\partial\psi/\partial\eta)(\partial\eta/\partial x)\, dy \\[4pt]
&= \; U_{\infty} f'(\eta)\, dx + \tfrac{1}{2}\,(U_{\infty}^{1/2}\, \nu^{1/2}\, x^{-1/2})\,(\eta\, f'-f)\, dy \; . \qquad (4.2.30)
\end{aligned}
$$

Or:

$$\int d\Phi \; = \; U_{\infty} \int f'(\eta)\, dx + \tfrac{1}{2} \int (U_{\infty}^{1/2}\, \nu^{1/2}\, x^{-1/2})(\eta f'-f)\, dy \qquad (4.2.31)$$

where one can establish the limits of integration from a to b, provided that the upper and lower limits refer simultaneously to the composite coordinate $\eta = \eta(x,y)$, and the chosen Cartesian coordinates (x,y).
Thus:

$$\Phi\Big|_{a}^{b} \; = \; U_{\infty} \int_{a}^{b} f'(\eta)\, dx + \tfrac{1}{2}\int_{a}^{b} (U_{\infty}^{1/2}\, \nu^{1/2}\, x^{-1/2})(\eta f'-f)\, dy \; . \qquad (4.2.32)$$

Neglecting the finite integrals and returning to the indefinite integrals in Equation (4.2.32), and differentiating, one obtains

$$d\Phi \; = \; U_{\infty} f'(\eta)\, dx + \tfrac{1}{2}\,(U_{\infty}^{1/2}\, \nu^{1/2}\, x^{-1/2})(\eta\, f'-f)\, dy \; , \qquad (4.2.33)$$

which may be written in the form

$$d\Phi \; = \; (\partial\Phi/\partial x)\, dx + (\partial\Phi/\partial y)\, dy \; . \qquad (4.2.34)$$

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According to the definition of the velocity potential; i. e., $\nabla\Phi = \vec{V}$, one obtains

$$\partial\Phi/\partial x = u ; \quad \partial\Phi/\partial y = v . \tag{4.2.35}$$

A comparison of Equations (4.2.33), (4.2.34), and (4.2.35), with the results obtained previously in Equations (4.2.1) to (4.2.15)proves that the approach is correct.

One may verify the last point of the previous analysis, namely, irrotationality. According to the definition of the velocity potential function, Φ, the scalar function Φ should furnish an irrotational flow; i. e.,

$$\vec{V} = \nabla\Phi; \quad \omega = \nabla \times \vec{V} = 0 . \tag{4.2.36}$$

However, by means of numerical calculations it has been demonstrated very thoroughly that curl $\vec{V} \neq 0$; i. e., the curl $\vec{V}$ is never equal to zero in the domain of the boundary layer under consideration (along flat plate). Thus, the flow under discussion is quasi-potential $[\nabla\Phi = \vec{V}; \nabla \times \vec{V} \neq 0]$ and the proper nomenclature for the function Φ was introduced earlier, i. e., it was called "the velocity quasi-potential function.

(Equation (4.2.32) has to be solved numerically and the numerical integration has to be achieved according to the scheme presented in Section 4.1. For a given value of y = 200 meters, 400 meters, etc., one obtains values of η and $x(x^{1/2}, x^{-1/2}$, etc.), and consequently the values of x, y, and $\eta = \eta(x,y)$ are known. The knowledge of these independent variables, (x,y) and of the composite independent variable $\eta = \eta(x,y)$, allows one to use the formula (4.1.54) to obtain the approximate values of a function $\phi = \phi(\eta)$. Thus, in the final step, one calculates the sum of the two integrals, Equation (4.2.32). In place of the numerical double integration process, it is proposed in the first approximation that the sum of the two partial integrals, Equation (4.2.32) be used: one numerically calculated in the x-direction and the other numerically calculated in the y-direction.

4.3. Diabatic Flow

In practice, the first assumption which has to be made is the assumption referring to the kind of the fluid and the kind of the flow which takes place in the phenomenon under consideration. The phenomenon

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under the present consideration refers to the existence of the "laminar" boundary layer in a flow of a viscous (consequently heat conducting following the fundamental results of the kinetic theory of gases and liquids) fluid medium. Due to the fact that the entire above approach is applied to the domain of the flow of the fluid with the irreversible phenomena (change of the kinetic energy into heat), the actual physical picture is completely different from the mathematical abstraction where there exists the concept of an ideal, reversible fluid flow with the concept of the stream function. Some assumption and hypothesis must be proposed in order to bring the above picture from the mathematical, idealistic and even purely abstractive level back to the level of the real fluid flow (irreversible phenomena, heat and energy dissipation). This is done by a straightforward assumption which relates the flow to the so-called "diabatic flow regime," where the irreversible phenomena-heat and energy dissipation-can take place. The fundamentals and the theory of "diabatic flow" were developed by NASA beginning in about 1944, and since then many papers and reports have been published. The reader is referred to the literature.

The term $(m^{-1} \nabla \Phi)$ in the equation proposed by Madelung represents, according to the writer's assumption, the viscous irreversible forces and dissipative phenomena. A more generalized form of the equation of motion must include the action of the extraneous forces and action of the dissipative forces as well. Put in such a form, the equation proposed by Madelung may be considered to be representative of a class of generalized equations of motion describing the motion of viscous, dissipative fluids as described by the Navier-Stokes equation. The above simple and easy explanation may also serve as an elementary explanation of the term of "hidden variables" introduced by the late John von Neumann. Under this term one should understand the physical action and the results of the physical "forces," which are necessarily not only of a "constructive" or of a "positive" or of an "accelerating" nature but may be also be of a "destructive" or "negative" or "decelerative" nature (dissipative). The late John von Neumann, in his book "Mathematische Grundlagen der Quanten-Mechanik," cites the Navier-Stokes equation and the Boltzmann-Maxwell equation as being representatives of the equations of motion being expressed in terms of "hidden variables." The "hidden variables" express exactly the forces

which may be destructive, or negative, or decelerative or dissipative in nature. In the present case the term $m^{-1}\nabla\Phi$ will represent the action and the effects of all the terms containing the coefficient of viscosity.

Let one compare the equation proposed by Madelung with the classical Euler's momentum equation in steady flow:

$$\vec{U}\cdot\nabla\vec{U} + \rho^{-1}\nabla P - \vec{f} = 0\,; \quad \vec{f} = \text{extraneous force per unit mass of the fluid;} \quad (4.3.1)$$

where P = static pressure.

For the sake of comparison, the reduced form of the Navier-Stokes equation is used in describing the flow along an infinitely long flat plate in which the fluid is assumed to be incompressible, and the conditions of the flow are stationary:

$$u\,\partial u/\partial x + v\,\partial u/\partial y = -\rho^{-1}\,dp/dx + \nu\,(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2)\,. \quad (4.3.2)$$

In the flow in the boundary layer along a flat plate Prandtl-Blasius assumed that the terms dp/dx and $\partial^2 u/\partial x^2$ are so small that they can be neglected. Thus one gets:

$$u\,\partial u/\partial x + v\,\partial u/\partial y - \nu\,\partial^2 u/\partial y^2 = 0\,. \quad (4.3.3)$$

Introducing the concept of the stream function in the classical sense, one obtains

$$\psi = \psi(x,y,t)\,; \quad u = \partial\psi/\partial y\,; \quad v = -\partial\psi/\partial x\,;\,(\text{stream function}). \quad (4.3.4)$$

Then Equation (4.3.3) in the case of a steady flow with $\partial p/\partial x = 0$ takes the form [Schlichting, p. 114, Equation (7.21)]:

$$\partial\psi/\partial y\,\,\partial^2\psi/\partial x\partial y - \partial\psi/\partial x\,\,\partial^2\psi/\partial y^2 - \nu\,\partial^3\psi/\partial y^3 = 0\,. \quad (4.3.5)$$

To refresh the memory of the reader the wave equation and the sequence of the equations are concurrently presented with the appropriate operations obtained from the wave equation. The wave equation is:

$$\nabla^2\psi - 8\pi^2 m h^{-2}\Phi\psi - i\,4\pi m h^{-1}\,\partial\psi/\partial t = 0\,; \quad (4.3.6)$$

$$\psi = a\exp(i\beta)\,; \quad a = a(x,y,z,t)\,; \quad \beta = \beta(x,y,z,t)\,; \quad (4.3.7)$$

$$\phi = -\beta h(2\pi m)^{-1} = -\beta \times \frac{6.62517}{(2\pi \times 9.107)} \times 10^1 \text{ cm-sec};$$ (4.3.8)

$$h = (6.62517 \pm 0.00023) \times 1.020 \times 10^{-27} \text{ gr cm sec};$$

$$m = \sim 0.9107 \times 10^{-27} \text{ gr};$$

$$\partial\phi/\partial t + \frac{1}{2}(\nabla\phi)^2 + \Phi m^{-1} - (\nabla^2 a) a^{-1} h^2 (8\pi^2 m^2)^{-1} = 0;$$

$$\vec{U} = \nabla\phi \ ;$$ (4.3.9)

$$\nabla \cdot (a^2 \nabla\phi) + \partial(a^2)/\partial t = 0 \ ; \qquad a^2 = \rho \ ;$$ (4.3.10)

∇ operator applied to (4.3.9):

$$\partial\vec{U}/\partial t + \frac{1}{2}\nabla(\vec{U}^2) + m^{-1}\nabla\Phi - \nabla[a^{-1}\nabla^2 a \, h^2 (8\pi^2 m^2)^{-1}] = 0;$$ (4.3.11)

$$\nabla(\vec{U}^2) = 2[\vec{U}\cdot\nabla\vec{U} + \vec{U}\times(\nabla\times\vec{U})].$$ (4.3.12)

In the first approach Madelung assumed $\nabla\times\vec{U} = 0$. The remaining part of Equation (4.3.12) is inserted into Equation (4.3.9) thus giving the equation corresponding to the equation of momentum in the reduced form:

$$(\vec{U}\cdot\nabla)\vec{U} + m^{-1}\nabla\Phi = 0 .$$ (4.3.13)

The problem which now presents itself is to propose the scale magnification factors which should enable one to associate the equations describing the macroscopic phenomena with the equations describing the microscopic phenomena (on the level of molecules and electrons). One may compare two equations: one describing the phenomena on the macroscopic scale as discussed in the boundary layer flow along a flat plate under stationary conditions:

$$u \, \partial u/\partial x + v \, \partial u/\partial y + \rho^{-1} \, dp/dx - \nu \, \partial^2 u/\partial y^2 = 0 \ ; \quad \nu = \mu\rho^{-1} \ ;$$

(4.3.14)

the second one describing the same physical phenomena on the microscopic scale:

$$(\vec{U}\cdot\nabla)\vec{U} + m^{-1}\nabla\Phi = 0 .$$ (4.3.15)

Since the symbol m denotes the mass of the electron, $m = 0.9107 \times 10^{-27}$ gram, one has to propose a scale magnification factor for "m," as discussed above.

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5. SPECIAL MATHEMATICAL CONSIDERATIONS

5.1. Some Characteristic Properties of Linear Systems

The advantages of the application of wave mechanics theory to problems in fluid dynamics are the following: (1) the wave, Schroedinger, equation is a linear partial differential equation, wherease the classical Navier-Stokes equation is a highly nonlinear partial differential equation; (2) the wave, Schroedinger, equation, initially and fundamentally, refers to an electron which can be easily generalized, by means of an assumption, to a cluster or an ensemble of particles grouped together around an electron and guided by it; the concept of density appears usually in the classical, macroscopic approach to the fluids. The cluster of particles may be smaller in size than the number of particles in a cubic centimeter, the factors defining density in any fluid or gaseous medium. It is our impression that the natural phenomena in fluids and gases do not always behave according to the rules established by classical, deterministic fluid dynamics which are built around the concept of the mass density. Occasionally, for some physical phenomena, the behavior of liquids and gases may follow the rules of microscopic (wave, quantum mechanics) fluid dynamics which are built on the concept of the cluster of particles, i. e. , mass smaller than that referring to a cubic centimeter. These phenomena may be investigated either by approaching a solution from the macroscopic domain, mass per cubic centimeter (density) or from the microscopic domain (the element mass such as the electron or a cluster of particles grouped around the electron). Either approach is appropriate; however, the present approach employes the method common to the electron or the cluster of particles when grouped around a specific electron, and then applying the concept to the density of the liquid. This allows the use of wave mechanics from the very beginning. The material in this section is intended to provide the background necessary for the person who may not be fully acquainted with the characteristic properties of linear partial differential systems or as a refresher for the person who is fully acquainted with these properties. Obviously, no such properties can be discussed for the nonlinear partial differential system, since, in general, the present state of mathematics finds it difficult to define correctly nonlinearity and nonlinear systems.

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In particular the fundamental aspects of linearity and the law of superposition are discussed. Differential equations are divided into various categories and classes: ordinary differential equation refers to an equation containing one or more ordinary derivatives; partial differential equation refers to an equation in which the derivatives are partial derivatives. From S. A. Havanessian, Louis A. Pipes, Digital Computer Methods in Engineering, pp. 380-381, the ordinary differential equation is, for example,

$$d^2x/dt^2 + A^{dx}/dt + B x = 0, \quad x = x(t) ;$$

whereas the partial differential equation is:

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0, \quad V = V(x,y,z) .$$

Linearity and Superposition

Assume that a function $u = u(x,t)$ is a sufficient differentiable function of the arguments x and t. An operator represents an assignment or transformation of one function to another by means of a set of manipulations like differentiation, multiplication, addition, and subtraction. Very often, the basic process is partial differentiation and the operator in question is called a partial differential operator. For brevity, such an operator is denoted by L, and the function it assigns to a particular u by $L[u]$. The above operator has a very special property. For example, it is noted that if $u(x,t)$ and $\nu(x,t)$ are any twice-differentiable functions, then the same is true of a linear combination of these functions:

$$\alpha u(x,t) + \beta \nu(x,t) , \quad \alpha = \text{constant}, \beta = \text{constant}; \qquad (5.1.1)$$

where α and β are any constants. Thus, the operator $L[\alpha u + \beta \nu]$ and some of its properties are defined below. By the rules of partial differentiation one has:

$$L[\alpha u + \beta \nu] = \alpha L[u] + \beta L[\nu] . \qquad (5.1.2)$$

An operator having these properties is called a linear operator. It is well-known that all operators (differential, algebraic, and possibly others) are not linear. As a matter of fact, most physical problems involve nonlinear operators; various simplifying assumptions have to be made in order to replace a nonlinear operator by a linear one. This is typical, in the sense that it is possible to approximate the solutions of many problems by replacing nonlinear

by linear operators. An equation which equates $L[u]$, where L is a linear partial differential operator, to a given function F

$$L[u] = F \tag{5.1.3}$$

is called a "<u>linear partial differential equation</u>." A linear partial differential equation with $F \equiv 0$ is called a "homogeneous equation." An example of a linear partial differential operator is

$$L[u] = [\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}] u(x,t) , \tag{5.1.4}$$

where c^2 is a constant (Weinberger, pp. 30-31).

It is known that the unknown function $u(x,t)$ is, in general, not determined by the differential equation alone. One must prescribe initial and/or boundary conditions. The transformation associating a $u(x,t)$ with its initial values $u(x,\theta)$ is also a linear operator, which one may denote, say, by $L_1[u]$.
Then

$$L_1[u] \equiv u(x,0) . \tag{5.1.5}$$

Similarly:

$$L_2[u] \equiv \frac{\partial u}{\partial t} (x,0) \tag{5.1.6}$$

is a linear operator, as are:

$$L_3[u] \equiv u(0,t) ; \quad L_4[u] \equiv u(\ell,t) . \tag{5.1.7}$$

Assume the need to solve the initial-boundary value problem of the form

$$\partial^2 u/\partial t^2 - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t), \quad \text{for } 0 < x < \ell, \, t > 0 ; \tag{5.1.8}$$

$$u(x,0) = f(x) , \quad \text{for } 0 \leq x \leq \ell ; \tag{5.1.9}$$

$$\frac{\partial u}{\partial t} (x,0) = g(x) , \quad \text{for } 0 \leq x \leq \ell , \tag{5.1.10}$$

$$u(0,t) = 0 , \tag{5.1.11}$$

$$u(\ell,t) = 0 , \quad \ell = \text{constant} . \tag{5.1.12}$$

One may easily recognize that the above equation is the one-dimensional wave equation, also called the one-dimensional vibrating string equation.

It is one of the few partial differential equations whose general solution can be found explicitly. The above linear partial differential equation, Equation (5.1.8), with all the initial and boundary conditions super-imposed upon it in the form of Equations (5.1.9) to (5.1.12) will now be presented with the use of linear operators:

$$u = u(x,t); \quad L[u] \equiv [\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}] u = F(x,t) ; \tag{5.1.13}$$

$$L_1[u] \equiv u(x,0) = f(x) ; \tag{5.1.14}$$

$$L_2[u] \equiv \partial u/\partial t \,(x,0) = g(x) ; \tag{5.1.15}$$

$$L_3[u] \equiv u(0,t) = 0 ; \tag{5.1.16}$$

$$L_4[u] \equiv u(\ell,t) = 0 ; \quad \ell = \text{length of the string;} \tag{5.1.17}$$

This is a system of linear partial differential equations with subsidiary conditions. Such a system of a (one) linear partial differential equation together with a set of linear subsidiary conditions is called a "linear problem." Some of the implications that occur by working with problems which involve only linear operators (see Weinberger, pp. 30-32) will be demonstrated below. Consider a problem of the form

$$L[u] = F ; \qquad u = u(x,t) ; \tag{5.1.18}$$

$$L_1[u] = f_1 ; \tag{5.1.19}$$

$$L_2[u] = f_2 ; \tag{5.1.20}$$

$$\qquad \text{(I)}$$

$$\cdot \qquad \cdot$$
$$\cdot \qquad \cdot$$
$$\cdot \qquad \cdot$$

$$L_k[u] = f_k ; \tag{5.1.21}$$

where the first equation is a linear partial differential equation while the others are linear with initial or boundary conditions. Suppose that one is able to find a particular solution (call it v) of the differential equation (5.1.18),

$$L(v) = F , \tag{5.1.22}$$

which need **not** satisfy any of the other conditions. Then the new variable can be defined:

$$w = u - \nu.$$ (5. 1. 23)

From the linearity of the operator L one obtains the equation

$$L[w] = L[u] - L[\nu] = F - F = 0 ,$$ (5. 1. 24)

which states that the variable w satisfies a homogeneous differential equation. Thus it has been shown that any solution u of the equation $L[u] = F$ can be written as the sum of any particular solution ν of this equation and a solution w of the corresponding homogeneous equation, e. g. , from Equation (5. 1. 24):

$$u = w + \nu ,$$ (5. 1. 25)

$$L[u] = L[w] + L[\nu] . \quad \text{Q. E. D.}$$ (5. 1. 26)

Any solution of a homogeneous linear <u>ordinary</u> differential equation of order "n" is a linear combination of n linearly independent solutions of this equation. This is no longer true in the case of partial differential equations. This fact accounts to a large extent for the vastly greater difficulty in solving linear partial differential equations (Weinberg, p. 30). The order and degree of a differential equation refer to the derivative of highest order after the equation has been rationalized. Thus, the equation

$$d^3y/dx^3 + x (dy/dx)^{1/2} + x^2 y = 0$$ (5. 1. 27)

is of the third order and of the second degree, since when it is rationalized it will contain the term $(d^3y/dx^3)^2$. If one now has a particular solution, ν, Equations (5. 1. 22), (5. 1. 23), and (5. 1. 24), i. e. , $L[\nu] = F$, the problem as discussed above, i. e. , Equations (5. 1. 18) to (5. 1. 21), can be reduced to a new problem of the same kind, but with $F = 0$. By putting $w = u - \nu$, one has from the linearity,

$$L(w) = 0 ;$$ (5. 1. 28)

$$L_1(w) = f_1 - L_1(\nu) ;$$ (5. 1. 29)

$$L_2(w) = f_2 - L_2(\nu) ; \qquad \text{(II)}$$ (5. 1. 30)

$$- - - - - - - - - -$$

$$L_k(w) = f_k - L_k(\nu) .$$ (5. 1. 31)

In a similar manner, one may replace the system of Equations (5.1.18) to (5.1.21) by a similar problem where some of the functions f_1, f_2, ..., f_k are zero by subtracting from u a function which satisfies some of the boundary conditions.

The linearity of problem (I) can also be used to split it into a number of simpler subproblems. Suppose that the function u_o is the solution of the system:

$$L[u_o] = F ; \qquad (5.1.32)$$

$$L_1[u_o] = 0 ; \qquad (5.1.33)$$

(III)

$$L_2[u_o] = 0 ; \qquad (5.1.34)$$

$$L_k[u_o] = 0 ; \qquad (5.1.35)$$

and also that u_1 is the solution to an analogous system with some variations:

$$L[u_1] = 0 ; \qquad (5.1.36)$$

$$L_1[u_1] = f_1 ; \qquad (5.1.37)$$

$$L_2[u_1] = 0 ; \qquad \text{(IV)} \qquad (5.1.38)$$

$$L_k[u_1] = 0 ; \qquad (5.1.39)$$

and that there are solutions for analogous conditions for u_2, u_3, ..., u_k, then each of the functions $u_o, u_1, \ldots, u_k$, involves only one piece of the data F, $f_1, f_2, \ldots, f_k$. By linearity, one finds that the function,

$$u = u_o + u_1 + u_2 + \ldots + u_k , \qquad (5.1.40)$$

satisfies System (I), Equations (5.1.18) to (5.1.21). Thus, u is obtained as a sum of terms each of which represents the effect of only one piece of the data. In a particular case the final solution may represent a decomposition into the effects of initial position, initial velocity, and force (see Weinberger, p. 32).

Decomposition of Subproblems, Principle of Superposition

Each of the subproblems, discussed above, may in itself be decomposed. Suppose there is a set of functions $v^1, v^2, \ldots,$ all of which satisfy the same set of homogeneous equations, say:

$$L_1[\nu^{(i)}] = 0 ; \qquad\qquad\qquad\qquad\qquad (5.1.41)$$

$$L_2[\nu^{(i)}] = 0 ; \qquad\qquad\qquad (V) \qquad\qquad\qquad (5.1.42)$$

$$----------$$

$$L_k[\nu^{(i)}] = 0 ; \quad i = 1, 2, \ldots, \qquad\qquad\qquad (5.1.43)$$

then, if f_1 can be represented as a finite linear combination of the functions $L_1[\nu^{(1)}]$, $L_1[\nu^{(2)}]$, ..., that is, if equation (5.1.44) becomes

$$f_1 = a_1 L_1[\nu^{(1)}] + a_2 L_1[\nu^{(2)}] + a_3 L_1[\nu^{(3)}] + \ldots + a_n L_1[\nu^{(n)}] ,$$

$$(5.1.44)$$

and if it is valid and can be used, then the function

$$u_1 = a_1 \nu^{(1)} + a_2 \nu^{(2)} + \ldots + a_n \nu^{(n)} , \qquad\qquad (5.1.45)$$

is the solution of the problem (IV). This is called the "principle of super-position" (Weinberger, p. 33).

This principle can frequently still be applied if f_1 is the limit of a sequence of such linear combinations. In particular, it could be an infinite series:

$$f_1 = \sum_1^\infty a_i L_1[\nu^{(i)}] = \lim_{n \to \infty} \sum_1^n a_i L_1[\nu^{(i)}] . \qquad (5.1.46)$$

In this case the function

$$u_1 = \sum_{i=1}^n a_i \nu^{(i)} , \qquad\qquad\qquad (5.1.47)$$

will satisfy the system (IV), Equation (5.1.36) to (5.1.39), provided the operators L, L_1, ..., L_k can be applied to the series term by term. Since the operators L, L_1, L_2, ..., L_k are differential operators, this will be the case if all the series obtained by applying each of the derivatives that appear in these operators to each term of Equation (5.1.47) converge uniformly (Weinberger, p. 33).

It appears appropriate to digress a bit to the field of the fundamentals of analysis to explain the term "uniform convergence." The series:

$$\sum_{i=1}^{\infty} a_i \, L[\, \nu^{(i)}(x)\,] \, , \qquad\qquad (5.1.48)$$

is said to converge uniformly on the interval $a \le x \le b$ if for any $\epsilon > 0$ there exists an integer $N(\epsilon)$ independent of x such that the following inequality

$$\left| \sum_{i=n+1}^{m} a_i \, L[\, \nu^{(i)}(x)]\, \right| < \epsilon \, , \qquad\qquad (5.1.49)$$

is preserved throughout the interval whenever n and m are larger than $N(\epsilon)$.

Uniqueness

A final simplification present in linear but not in nonlinear problems occurs in the formulation of uniqueness and continuity theorems. There is no intention to go deeply into the fundamentals of the analysis of the basic fundamental theorems; consequently only a few remarks on this subject will be given. Suppose that u and $\overline{u}$ are both solutions of problem (I), Equations (5.1.18) to (5.1.21), then by linearity the function $\nu = (u - \overline{u})$ satisfies the homogeneous system:

$$L[\nu] = 0 \qquad\qquad (5.1.50)$$

$$L_1[\nu] = 0 \qquad\qquad \text{(VI)} \qquad\qquad (5.1.51)$$

$$- - - - - - - - -$$

$$L_k[\nu] = 0 \, . \qquad\qquad (5.1.52)$$

System (VI) has the trivial solution $\nu \equiv 0$. If this is its only solution, then $(u - \overline{u})$ must be identically zero; that is, the set (VI) has at most one solution, but, of course, it could also have none. Suppose, on the other hand, that there is a solution "ν" of the set (VI) which is not identically equal to zero, then if (VI) has a solution say, u, the function $(u + a\nu)$, where a is any constant, is also a solution. That is, the system (VI) cannot possibly have exactly one solution. It may have no solution at all, or infinitely many solutions. Thus the uniqueness problem for System (I), Equations (5.1.18) to (5.1.21), is reduced

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to that for the homogeneous (homogeneity of the problem set) problem the latter System (VI), Equations (5.1.50) to (5.1.52). System (VI) is independent of the particular data F, $f_1, f_2, \ldots, f_k$, which appear in System (I), Equations (5.1.18) to (5.1.21) (see Weingerger, pp. 30 to 34).

Continuity

The question of continuity of the linear systems of equations is by no means trivial in nature but is of importance. Let u be a solution of System (I), Equations (5.1.18) to (5.1.21), and let $\bar{u}$ be a solution

$$L[\bar{u}] = \bar{F} \; ; \qquad (5.1.53)$$

$$L_1[\bar{u}] = \bar{f}_1 \; ; \qquad (5.1.54)$$

(VII)

$$L_2[\bar{u}] = \bar{f}_2 \; ; \qquad (5.1.55)$$

$$L_k[\bar{u}] = \bar{f}_k \; ;$$

The question: is it true that if

$$(F - \bar{F}), \quad (f_1 - \bar{f}_1), \quad \ldots, \quad (f_k - \bar{f}_k) \qquad (5.1.56)$$

are small, the difference $(u - \bar{u})$ is also small? If we let:

$$\nu = u - \bar{u} \; ; \qquad (5.1.57)$$

$$G = F - \bar{F} \; ; \qquad (5.1.58)$$

$$g_1 = f_1 - \bar{f}_1 \; ; \qquad (5.1.59)$$

(VIII)

$$g_2 = g_1 - \bar{g}_1 \; ; \qquad (5.1.60)$$

$$g_i = g_k - \bar{g}_k \; , \qquad (5.1.61)$$

then we see that the function ν satisfies the problem (below):

$$L(\nu) = G \; ; \qquad (5.1.62)$$

$$L_1(\nu) = g_1 \; ; \qquad (5.1.63)$$

(IX)

$$L_2(\nu) = g_2 \; ; \qquad (5.1.64)$$

$$L_k(\nu) = g_k \; . \qquad (5.1.65)$$

The question, proposed above, now becomes: Is it true that the solution

ν of (IX) is small if the data G, g_1, g_2, ..., g_k are small? This question is a special case of the original one with

$$u = F = f_1 = f_2 = \ldots = f_k = 0 . \tag{5.1.66}$$

For linear problems we need only treat this special case (Weinberger, pp. 34-35).

The discussion, presented above, implies very clearly that in each particular case and in each particular problem in question, the indicated operations and manipulations must be performed in order to find out whether the required conditions for the <u>uniqueness</u> <u>theorem</u> and for the <u>continuity theorem</u> are separately satisfied for each particular problem. Some examples of such investigations are given in the literature (see Weinberger, pp. 35 and subsequent.)

In closing this brief discussion on the subject of linear operators, one remark more can be made; namely, that the one-dimensional wave equation, Equation (5.1.4), can be easily transformed by means of the transformation of coordinates of the form $(x,t) \rightarrow (\xi,\eta)$:

$$\partial^2 u/\partial t^2 - c^2 \, \partial^2 u/\partial x^2 = 0 ; \tag{5.1.67}$$

$$x - ct = \xi ; \quad x + ct = \eta ; \tag{5.1.68}$$

into the so-called "normal form" of the wave equation:

$$\partial^2 u/\partial \xi \, \partial \eta = 0 . \tag{5.1.69}$$

The solution of Equation (5.1.69) is immediate:

$$u = f(\xi) + g(\eta) , \tag{5.1.70}$$

where $f(\xi)$ and $g(\eta)$ are arbitrary functions of their arguments. One can now immediately recognize that the two families of straight lines given by:

$$\xi = \text{constant}; \quad \eta = \text{constant}, \tag{5.1.71}$$

are the equations of the characteristics of the one-dimensional wave equation, Equation (5.1.67).

5.2. ENSS (Euler-Navier-Stokes-Schroedinger) Operators

In previous sections various characteristic aspects and properties

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of linear differential equations (ordinary and partial) and of linear operators were discussed. This section discusses the operations involving special kind of operators, "wave mechanics operators" or "Schroedinger operators," which are based primarily upon the fundamentals of wave mechanics and the Schroedinger equation. The subject will be presented very systematically, logically and orderly beginning with the very fundamentals.

The medium under consideration, the fluid, is assumed to be a neutral one obeying the Bose statistics. The fundamental equation upon which, the entire formalism of the present wave mechanics approach is based is the Schroedinger wave equation of the form

$$\nabla^2 \psi - 8 \pi^2 m h^{-2} \Phi \psi - i 4 \pi m h^{-1} \partial\psi/\partial t = 0 , \qquad (5.2.1)$$

where the symbols used denote the following:

∇^2 = Laplacian operator;

h = Planck's constant = 6.54×10^{-27} cm gram sec;

m = mass of the electron = 0.9107×10^{-27} gram;

ψ = wave function; $\psi = \psi(t, x, y, z)$;

Φ = potential energy of the field in which the entire above system is located; $\Phi = \Phi(t, x, y, z)$ is actually a function of the position of the electron.

Assuming, that

$$\psi = a \exp(i\beta); \quad a = a(x, y, z, t); \quad \beta = \beta(x, y, z, t) , \qquad (5.2.2)$$

Madelung obtained two equations (real and imaginary parts, (R), (I)), respectively:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \Phi m^{-1} - (\nabla^2 a) a^{-1} h^2 (8 \pi^2 m^2)^{-1} = 0 ; \quad (R) ; \qquad (5.2.3)$$

$$\nabla \cdot (a^2 \nabla\phi) + \partial(a^2)/\partial t = 0 ; \quad (I) \qquad (5.2.4)$$

$$\phi = -\beta h(2 \pi m)^{-1} = -C_1 \beta ; \quad C_1 = h(2 \pi m)^{-1} ; \qquad (5.2.5)$$

where C_1 was a constant. With $\vec{U} = \nabla\phi$ and $\nabla \times \vec{U} = 0$, Equation (5.2.3) was modeled by Madelung to be equivalent to the Euler equation of motion, and Equation (5.2.4) with $a^2 = \rho$ (density of the fluid) as being equivalent to the equation of the conservation of mass of the fluid

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(continuity). In order to obtain the full form of the equations in question, one has to apply one more formal operation; i. e. , a gradient operation to Equation (5. 2. 3);

$$\nabla(\nabla\phi)^2 = 2\left[\vec{U} \cdot \nabla\vec{U} + \vec{U} \times (\nabla \times \vec{U})\right] ; \tag{5.2.6}$$

$$\nabla(\nabla\phi)^2 = 2\,\vec{U} \cdot \nabla\vec{U} \quad \text{if } (\nabla \times \vec{U}) = 0 ; \tag{5.2.7}$$

$$\nabla(\nabla\phi)^2 = 2\,\vec{U} \times (\nabla \times \vec{U}) \quad \text{if } \nabla\vec{U} = 0 . \tag{5.2.8}$$

Of course, one could preserve both terms in Equation (5. 2. 6); e. g. , gradient and curl. In order to associate the field of Quantum Mechanics with the field of Fluid Mechanics in the macroscopic sense, Madelung accepted Equation (5. 2. 7) as the guide for his approach. In this way he was able to associate the Schroedinger equation, Equation (5. 2. 1), with the Euler equation of motion,

$$\partial\vec{U}/\partial t + \frac{1}{2}\,\nabla(\vec{U}^2) + m^{-1}\,\nabla\Phi - \nabla(a^{-1}\,\nabla^2 a\, h^2(8\pi^2 m^2)^{-1} = 0 . \tag{5.2.9}$$

Madelung proposed the following interpretation of various terms in Equation (5. 2. 9): the term $m^{-1}\nabla\Phi$ represents the extraneous force field, $\rho^{-1}\vec{f}$, acting upon the medium in question; and the term $a^{-1}\nabla^2 a\, h^2(8\pi^2 m^2)^{-1}$ is associated with the action of the static pressure; i. e. , $\int \rho^{-1}\,dp$. In the conventional approach to the mechanics and/or dynamics of viscous fluids in the macroscopic domain, one is inclined to apply the Navier-Stokes system of equations to describe and to deal with the phenomena in the macroscopic domain. Although several other forms of equations such as the Burnett equations, attempt to describe more or less accurately, the physical phenomena in viscous liquids and gases. The Navier-Stokes system of equations still seems to be considered as being the best one for these purposes. Some mathematicians and physicists have devoted lifetimes to selecting what they felt was possibly the best system of equations from all the existing systems of equations from one or more fields of mechanics and physics (mechanics of continuous media, mechanics of liquids, of gases, kinetic theory of gases, molecular theory of gases and possibly others Almost unaminously they agree that the Navier-Stokes system of equations best fulfills the needs in the macroscopic; i. e. , observable, domain for describing the physical phenomena of viscous fluids.

Professor George Uhlenbeck, well-known physicist, previously from the University of Michigan, Ann Arbor, now Rockefeller Institute, New York, expressed this opinion during the Eighth International Symposium on Rarefied Gas Dynamics, Stanford University, Stanford, California, July 10-14, 1972.

Another item which is important to the reader is the fact that the Schroedinger equation is a linear equation in the complex domain in the fully mathematical, analytical sense of this word. After the choice of the wave function was made by Irving Madelung (1926) in the form of Equation (5.2.2) and following some formal operations of the purely analytical nature such as taking the gradient, and after the decomposition of the Schroedinger equation into the real part (R) and the imaginary part (I), he obtained the Euler equation of motion which is a nonlinear equation. Such behavior of the partial differential equation in passing from one field of physics such as microscopic quantum mechanics to another field such as macroscopic fluid mechanics is not new. In the past it was signaled more than once that the passage (transfer) of the Schroedinger equation from one field of physics to another field could be associated with the change of the associated classification of the partial differential equations concerning linearity or nonlinearity. This fact must be kept in mind when changes, like the quantum mechanics to mechanics or the reverse, are considered. This assumption (or conjecture) is the basis for the proposed association and relationship between quantum mechanics of viscous fluids based on the acceptance of the validity of the systems obeying the rules of diabatic flows.

The ideas of diabatic flow systems and of extending the Madelung "quantum theory modification and application to macroscopic fluid dynamics," including the domain of ideal fluid systems and viscous fluid systems, are fundamentally and deeply rooted in ideas expressed by John von Neumann in his immortal work, "Mathematical Foundations of Quantum Mechanics." In this work von Neumann proposed the idea of "hidden variables" as one of the concepts of modern mechanics. As examples, he quoted the Navier-Stokes system of equations and the Maxwell-Boltzmann (Burnett) system of equations as being in some way representative of his concept of "hidden variables" in the field of fluid dynamics.

Returning to the problem of the association of the quantum theory,

Schroedinger equation with the macroscopic fluid dynamics, Navier-Stokes equation, one may demonstrate that the results obtained previously can be obtained in another manner as well. Let one assume that the starting point is through use of the Euler equation of motion (Owczarek, p. 65):

$$\partial \vec{V}/\partial t + \vec{V} \cdot \nabla \vec{V} + \frac{1}{\rho} \nabla P - \vec{f} = 0 \; ; \tag{5.2.10}$$

where the vector $\vec{f}$ denotes the vector force field acting upon the particle in question. Making use of the vector relation (Owczarek, p. 52)

$$\vec{V} \cdot \nabla \vec{V} = \vec{\omega} \times \vec{V} + \nabla(\frac{1}{2} V^2) \; ; \quad \vec{\omega} = \nabla \times \vec{V} \; , \tag{5.2.11}$$

if $\vec{\omega}$ denotes the vorticity vector, one gets another form of the equation where additionally one may include the action of the static pressure:

$$\partial \vec{V}/\partial t + \vec{\omega} \times \vec{V} + \nabla(\frac{1}{2} V^2) + \frac{1}{\rho} \nabla P - \vec{f} = 0 \; , \tag{5.2.12}$$

where the symbol P denotes the static pressure in the fluid. When the dissipative action and forces are taken into account and are superimposed upon the field of an inviscid (non-heat conducting) fluid, the Euler equation transforms into the Navier-Stokes equation (or an analogous one). One of the acceptable forms of the Navier-Stokes equation is the following one (Owczarek, p. 542):

$$\rho \left[\partial \vec{V}/\partial t + \vec{\omega} \times \vec{V} + \nabla(\frac{1}{2} V^2) \right] + \nabla P - \rho \vec{f} + \frac{2}{3} \nabla(\mu \nabla \cdot \vec{V})$$

$$- 2 (\nabla \mu) \cdot \nabla \vec{V} - (\nabla \mu) \times (\nabla \times \vec{V}) - \mu [\nabla^2 \vec{V} + \nabla(\nabla \cdot \vec{V})] = 0 \; ,$$

$$\tag{5.2.13}$$

where μ is the coefficient of viscosity which usually depends upon the temperature field and $\vec{f}$, the gravitational force field.

The problem is to associate equation (5.2.13) with the Schroedinger equation and its transformation as was performed by Madelung, Equations (5.2.1) to (5.2.7). The guiding points are the following:

(a) The finite system to be developed has to be a linear system for the simple reason that the mathematical analysis, at the present time at least, can only handle linear systems.

(b) The Madelung proposition will be used to associate, i. e. , to construct a link or a bridge between linear wave mechanics, the Schroedinger equation, and the nonlinear equation of motion of an inviscid, non-heat conducting (ideal) fluid, the Euler equation.

(c) In general, the association presented above allows one to generalize the above fundamental association,(Schroedinger-Euler) or (Euler-Schroedinger),to other kinds of equations like Navier-Stokes. Thus a bridge is proposed between the (Navier-Stokes-Schroedinger) or (Schroedinger-Navier-Stokes), i.e.,the linear wave mechanics equation and the nonlinear equation of viscous and heat conducting fluid (real fluid).

(d) To achieve this goal from the fundamental point of view, an assumption has to be made that the fluid system under consideration follows the rules of the diabatic flow system, proposed by NASA around 1944 and later. In this sytem the dissipative phenomena due to the factor of viscosity and heat conductivity appear; the system, so proposed, denoted as (Euler-Navier-Stokes-Schroedinger) is treated in an analogous (but not identical) manner to the previously mentioned (Euler-Schroedinger) system.

(e) The added system, the Navier-Stokes system, attached to the chain (Euler-Schroedinger) system, when treated alone is a nonlinear system of differential equations. The first terms of this system (see Equation (5. 2. 13)) (disregarding the terms containing the viscosity coefficient μ), are identical with the terms existing in the Madelung remodeling of the Schroedinger equation, and the Euler form of the equation of motion (Equations (5. 2. 1) to (5. 2. 7)) (the Euler equation is not linear). By his proposition, Madelung succeeded in obtaining a chain (an association) between the Euler (nonlinear) and Schroedinger (linear) equations, thereby obtaining some sort of "linearization" of the nonlinear Euler equation of motion. This will now be called the "Madelung linearization process" of the Euler equation of motion.

(f) The terms in the Navier-Stokes equation, containing the viscosity coefficient μ, Equation (5. 2. 13) , are, in general, nonlinear terms, since the coefficient μ is a function of the temperature field. In thermodynamics the pressure, temperature, and volume of a given system are measured and are referred to as the thermodynamic

characteristic quantities of any given system. In very small regions of space, especially near the "critical" point, change or passage from the super-fluid state to the ordinary hydrodynamics status, it is found that these thermodynamic quantities no longer obey exactly an equation of state regardless of form in which it is considered. Instead they exhibit large random fluctuations about a mean value that is predicted by the equation of state. Hence, the deterministic laws of thermodynamics break down and have to be replaced by laws of probability. This is so, because the thermodynamic variables are no longer appropriate for the particular problem and must be replaced by the position and velocity of each molecule, which turn out to be, from the viewpoint of thermodynamics, "hidden variables" (Bohm, p. 29). As stated before, the term "hidden variables" was introduced by John von Neumann (see references) and is used by Bohm and other quantum physicists. The thermodynamic quantities are, then, merely averages of "hidden variables" that cannot be observed by thermodynamic methods alone. To find the underlying "causal" laws, one must accept a description in terms of the individual molecules. Since this is usually a practical impossibility one has to refer to actual, physical experiments and to express the experimental ceofficients such as the viscosity coefficient and the heat conductivity coefficient in terms of thermodynamic quantities. These are, as stated above, merely averages of "hidden variables" that cannot be observed by thermodynamic methods alone. Thus, in the final conclusion, the coefficients of viscosity and heat conductivity are necessarily experimentally measured and tabulated. This is the reason why some quantum physicists and mathematicians call the fundamental equations used in the macroscopic theory of fluid dynamics, expressed in terms of the coefficients of viscosity and of heat conductivity, "the equations in terms of hidden variables." The family of equations in this category may include:

(1) The Navier-Stokes system of equations of viscous, heat-conducting fluid;

(2) The Burnett equations derived according to the molecular theory of gases (see Patterson, reference);

(3) Another possible set of equations based upon the method of Hilbert-Enskog-Chapman-Burnett (see Patterson, reference);

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(4) Any other set of equations derived in a similar way which uses both the above mentioned coefficients of viscosity and heat conductivity in its derivation.

Returning to the Navier-Stokes system, Equation (5.2.13) and a combined (Euler-Navier-Stokes-Schroedinger) system, one can start from the simplest possible case, where the density of the fluid, ρ, and its coefficient of viscosity, μ, are constant in magnitudes. In such a case, consider the terms which appear on the left-hand side; namely the two terms nonlinear in the function $\vec{V}$, as from Equation (5.2.11):

$$\vec{V} \cdot \nabla \vec{V} = \vec{\omega} \times \vec{V} + \nabla(\tfrac{1}{2} V^2) \text{ with } \vec{\omega} = \nabla \times \vec{V} . \tag{5.2.14}$$

In this equation all the terms, expressed in terms of the unknown vector function $\vec{V}$ are equivalent to the terms in the wave equation by using the chain of substitutions, Equations (5.2.1), (5.2.6), and (5.2.9). Basically, these substitutions do not change the physical nature and the physical character of the equation. The changes introduced are formalistic in nature, such as an operation on the square of the gradient, $(\nabla \phi)^2$, Equation (5.2.6), which have their origin in the particular kind of operation on the Schroedinger equation and the form and interpretation of the wave function, Equation (5.2.2) as chosen and proposed by Madelung. By this kind of selection and proposition, Madelung could not and did not change (or influence) the fact that the nonlinear terms mentioned are always associated with and refer to the linear wave equation.

This process cannot be considered to be a general method of linearization of the nonlinear differential equation, but in this special case, the nonlinear Navier-Stokes equation, Equation (5.2.13), under the assumed conditions (ρ = constant, μ = constant), can be associated with the combined (Euler-Navier-Stokes-Schroedinger) system of equations and can be treated as one of the members of the family of the (Euler-Navier-Stokes-Schroedinger) system. Consequently, Equation (5.2.13) can be treated as a linear equation. It should be emphasized again and again that Equation (5.2.13) is included in the family of the linear equations because, and only because of the association of Equation (5.2.13) with the Schroedinger linear wave equation by means of the Madelung process of linearization. Assuming that the static

pressure P is calculable by use of the equation of state, $P = R\rho T$, where R = gas constant, ρ = constant (given), T = given, or that P is constant or, a given function and that the vector function $\vec{f}$ is given and known, then one can safely assume that the Navier-Stokes equation, Equation (5.2.13), belonging to the family of the (Euler-Navier-Stokes-Schroedinger) system of equations, is a linear equation. This certainly is a major achievement of the Madelung proposition, (Quantentheorie in Hydro-dynamischer Form) published as long ago as 1926. A short-cut for the chain of names (Euler-Navier-Stokes-Schroedinger) in the form of the abbreviation (ENSS) will be used. The use of the symbolic notation for the terms in Equation (5.2.14) is proposed; namely, the (ENSS) differential operator, (ENSS)L :

$$(ENSS)L\vec{V} = [\vec{\omega} \times + (\vec{V} \cdot \nabla)]\vec{V} \quad . \tag{5.2.15}$$

Operating on the vector function $\vec{V}$, one easily obtains from Equation (5.2.13) for the constant values of ρ and μ:

$$\rho\{\partial/\partial t + (ENSS)L\} \vec{V} + \nabla P - \rho\vec{f} = \vec{F} ; \tag{5.2.16}$$

or

$$\rho\{\partial/\partial t + (ENSS)L\} \vec{V} + \nabla P - \rho\vec{f} - \vec{F} = 0 ; \tag{5.2.17}$$

$$F = \frac{2}{3}\mu(\nabla \cdot \vec{V}) - \mu[(\nabla^2\vec{V}) + \nabla(\nabla \cdot \vec{V})] ; \tag{5.2.18}$$

where μ = constant; ρ = constant. $\tag{5.2.19}$

Moreover, due to the law of the conservation of mass (the equation of continuity) for the media in which one assumes that ρ = constant, $\nabla \cdot \vec{V} = 0$.

Once again it should be emphasized that all of the above operations are purely formalistic operations and their sequence can be easily reversed in order to start with the equation of motion of Euler and return to the wave equation of Schroedinger. However, the wave equation is a linear equation and consequently, the entire chain of operations presented and discussed above must be considered as a chain of formalistic operations on a linear system of mutually reversible related operations. During these operations and due to the term $(\vec{V}^2)$, the term

$(\vec{V} \cdot \text{grad } \vec{V})$ appears which can be considered as the most important step in the entire sequence of operations. Thus one can easily go from the Euler equation of motion back to the wave equation by applying the chain of formal operations in the reverse direction. This whole sequence and system of operations is very logical and it proves one thing; namely, that in its nature the entire system and set of operations existing in the (ENSS) L operator is a linear system, because it has its origin in the Schroedinger linear wave equation. One link in the entire system having the form $(\vec{V} \cdot \text{grad } \vec{V})$, relates the Euler equation of motion to the field of hydrodynamics. The history of modern quantum field theory has demonstrated on more than one ocassion the very peculiar nature and behavior of the wave fundamental mechanics equation of Schroedinger; namely, that a chain of purely formal, very simple operations applied to the wave mechanics equation of Schroedinger (linear) culminates in the equation which, from the purely mathematical point of view, is formally a nonlinear differential equation; but which, from the purely physical point of view, expresses well the physical phenomenon at a given moment and for a given set of circumstances (such as the boundary conditions, the initial conditions, and so on).

Returning to the Navier-Stokes equation, Equation (5. 2. 13), or Equation (5. 2. 16), or Equation (5. 2. 17), the (ENSS) L - operator is instroduced in Equation (5. 2. 15);

$$\rho \left\{ \partial/\partial t + (ENSS) L \right\} \vec{V} + \nabla P - \rho \vec{f} + \vec{F} = 0 ; \tag{5. 2. 20}$$

$$(ENSS) L = [\vec{\omega} \times + (\vec{V} \cdot \nabla)] \vec{V} ; \tag{5. 2. 21}$$

$$F = \frac{2}{3} \mu \nabla(\nabla \cdot \vec{V}) - \mu[(\nabla^2 \vec{V} + \nabla(\nabla \cdot \vec{V})] ; \tag{5. 2. 22}$$

where ρ = constant; μ = constant; P, $\vec{f}$ = known, given; $\nabla \cdot \vec{V} = 0$.

It should be mentioned that, for the present at least, it is difficult to include any differential equation, ordinary or partial, directly without any discussion of the category of nonlinear or linear equations. In December 1971 at the Annual Meeting of the American Association for the Advancement of Sciences in Philadelphia, a special session on the "Nonlinear Problem" was organized at which the investigator

was one of the invited speakers. During this session he presented a discussion on "An Association Between the Wave Mechanics and the Nonlinear Phenomena in Classical Mechanics by Means of the Quantum Field Theoretic Methods in Statistical Physics." Among the many statements and conclusions, one result is particularly worthy of mention; vis., that up to the present time mathematics does not know exactly how to define and how to classify the concept of nonlinearity. One thing is certain, namely, that if a differential equation, ordinary or partial, or an algebraic equation is definitely and absolutely not linear, then it is certainly a nonlinear one." From the point of view of the concept of classifications, this appears to be a weak manner of classification.

In this research the class of (ENSS) differential L operators is included in the class of linear operators. One may consider also that the properties of Equation (5.2.20), the equation to be investigated, involve stationary conditions; i.e., (i.e., $(\partial/\partial t)$ is neglected. The assumption of the stationary conditions implies that

$$\nabla P = 0 \qquad\qquad (5.2.23)$$

and that the fluid mechanics system in question is a diabatic system, i.e., that the terms $(-\rho \vec{f} + \vec{F})$ in Equation (5.2.20) represent the dissipative forces. With $\mu\rho^{-1} = \nu$, and $\nabla \cdot \vec{V} = 0$ (incompressible flow medium), Equation (5.2.17) takes the form

$$[(ENSS) L] \vec{V} - \nu \nabla^2 \vec{V} = 0 . \qquad\qquad (5.2.24)$$

This is considered to be a linear system and as such it corresponds to Equation (5.1.3) above.

If $L[u] = F$, $u = u(x,t)$, x = vector in n-dimensional space, $\qquad$ (5.2.25)

and if Equation (5.2.24) is added to the "linear boundary problem" as expressed in Equation (5.1.21), then one gets

$$L[u] = F; \quad L_1[u] = f_1; \quad L_k[u] = f_k. \qquad\qquad (5.2.26)$$

Following the laws and rules of the linear problem, explained so thoroughly above, one can continue the discussion of the linear problem, Equations (5.2.25) and (5.2.26). Suppose one can find a particular solution "v"

of the differential equation

$$L(\nu) = F \ , \tag{5.1.22} = (5.2.27)$$

which need not satisfy any of the other conditions. Then from Equation (5.1.23) one gets:

$$w = u - \nu \ ; \ L[w] = 0 \ , \tag{5.2.28}$$

where w = satisfies homogeneous equation.

After one opens the (ENSS) L – operator in Equation (5.2.24) one gets with the use of Equation (5.2.15):

$$[\vec{\omega} \ x + (\vec{V} \cdot \nabla)] \ \vec{V} - \nu \ \nabla^2 \ \vec{V} = 0 \ , \tag{5.2.29}$$

and with:

$$\vec{V} = \vec{i} \ u + \vec{j} \ v + \vec{k} \ w \ ;$$

$$u = u(x, y); \ v = v(x, y); \ w = w(x, y); \ w = 0 \ ; \tag{5.2.30}$$

$$\vec{\omega} = \nabla \ x \ \vec{V} \ ; \tag{5.2.31}$$

$$\vec{\omega} = \nabla \ x \ \vec{V} = \vec{i}(\partial w / \partial y - \partial v / \partial z) + \vec{j} \ (\partial u / \partial z - \partial w / \partial x) + \vec{k} \ (\partial v / \partial x - \partial u / \partial y) \ ; \tag{5.2.32}$$

$$\vec{\omega} = (\omega_1, \omega_2, \omega_3) \ ; \ \omega_1 = 0 \ ; \ \omega_2 = 0; \ \omega_3 \neq 0 \ ; \tag{5.2.33}$$

$$\vec{\omega} \ x \ \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ u & v & w \end{vmatrix} \ ; \tag{5.2.34}$$

$$\vec{\omega} \ x \ \vec{V} = \vec{i} \ (\omega_2 w - \omega_3 v) + \vec{j} \ (\omega_3 u - \omega_1 w) + \vec{k} \ (\omega_1 v - \omega_2 u) \ ; \tag{5.2.36}$$

consequently in the present case one has:

$$\vec{\omega} \ x \ \vec{V} = \vec{i} \ (-\omega_3 v) + \vec{j} \ (\omega_3 u) + \vec{k} \cdot 0 \ . \tag{5.2.37}$$

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Since the differential operator ∇ is related to the gradient operator, one gets from Equation (5.2.21):

$$\nabla = \vec{i}\ \partial/\partial x + \vec{j}\ \partial/\partial y + \vec{k}\ \partial/\partial z\ ; \qquad (5.2.38)$$

$$(\vec{V} \cdot \nabla) = (\vec{i}\ u + \vec{j}\ v + \vec{k}\ w) \cdot (\vec{i}\ \partial/\partial x + \vec{j}\ \partial/\partial y + \vec{k}\ \partial/\partial z)$$

$$= u\ \partial/\partial x + v\ \partial/\partial y + w\ \partial/\partial z\ . \qquad (5.2.39)$$

Consequently, using Equation (5.2.30) one gets:

$$(\vec{V} \cdot \nabla)\ \vec{V} = (u\ \partial/\partial x + v\ \partial/\partial y + w\ \partial/\partial z)(\vec{i}\ u + \vec{j}\ v + \vec{k}\ w)\ . \qquad (5.2.40)$$

Expressed in another form:

$\vec{i}$ direction: $\quad u\ \partial u/\partial x + v\ \partial u/\partial y + w\ \partial u/\partial z\ ; \qquad (5.2.41)$

$\vec{j}$ direction: $\quad u\ \partial v/\partial x + v\ \partial v/\partial y + w\ \partial v/\partial z\ ; \qquad (5.2.42)$

$\vec{k}$ direction: $\quad u\ \partial w/\partial x + v\ \partial w/\partial y + w\ \partial w/\partial z\ . \qquad (5.2.43)$

Consequently in the case under consideration the (ENSS), L - operator, Equation (5.2.21), takes the following three-dimensional form:

$$(\text{ENSS})\ L = [\vec{\omega}\ x + (\vec{V} \cdot \nabla)]\ \vec{V} = [\vec{\omega}\ x\ \vec{V} + (\vec{V} \cdot \nabla)\ \vec{V}]\ : \qquad (5.2.44)$$

$\vec{i}$ direction: $\quad (-\omega_3 v) + u\ \partial u/\partial x + v\ \partial u/\partial y + w\ \partial u/\partial z\ ; \qquad (5.2.45)$

$\vec{j}$ direction: $\quad (\omega_3 u) + u\ \partial v/\partial x + v\ \partial v/\partial y + w\ \partial v/\partial z\ ; \qquad (5.2.46)$

$\vec{k}$ direction: $\quad 0 + u\ \partial w/\partial x + v\ \partial w/\partial y + w\ \partial w/\partial z \qquad (5.2.47)$

Moreover from Equation (5.2.32):

$$\omega_3 = \partial v/\partial x - \partial u/\partial y\ . \qquad (5.2.48)$$

The last term in Equation (5.2.29) takes the form:

$$\nabla^2 \vec{V} = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\ (\vec{i}\ u + \vec{j}\ v + \vec{k}\ w)\ : \qquad (5.2.49)$$

$\vec{i}$ direction: $\quad \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2\ ; \qquad (5.2.50)$

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$\vec{j}$ direction: $\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2$; (5.2.51)

$\vec{k}$ direction: $\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 + \partial^2 w/\partial z^2$. (5.2.52)

Thus, the full form of the vector equation, Equation (5.2.29), containing the (ENSS) L - operator has the following three components:

$\vec{i}$ direction: $(-\omega_3 v) + u\,\partial u/\partial x + v\,\partial u/\partial y + w\,\partial u/\partial z$

$$- \nu\,(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2) = 0 ; \qquad (5.2.53)$$

$\vec{j}$ direction: $(\omega_3 u) + u\,\partial v/\partial x + v\,\partial v/\partial y + w\,\partial v/\partial z$

$$- \nu\,(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2) = 0 ; \qquad (5.2.54)$$

$\vec{k}$ direction: $u\,\partial w/\partial x + v\,\partial w/\partial y + w\,\partial w/\partial z$

$$- \nu\,(\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 + \partial^2 w/\partial z^2) = 0 . \qquad (5.2.55)$$

The above components of the (ENSS) L - operator refer to the Navier-Stokes end of the chain of operations which obey the rules of the (ENSS) L - operator. It may be of interest to present the beginning; i.e., the first link in the chain in question, in this case the Schroedinger equation. To demonstrate this, we return to Equation (5.2.3), (5.2.6), and (5.2.9) in the stationary conditions, with the constant factor $a^2 = \rho$ (density) or to the system of equivalent equations, Equations (5.2.10), (5.2.11), (5.2.12), and (5.2.20) with $\nabla P = 0$ and with $(-\rho \vec{f} + \vec{F})$ representing the dissipative forces, then from Equation (5.2.20):

$$\vec{\omega} \times \vec{V} + \nabla(\tfrac{1}{2} V^2) - \rho\,\vec{f} + \vec{F} = 0. \qquad (5.2.56)$$

Thus we are back at the starting point; i.e., the Schroedinger form of equations, Equations (5.2.1), (5.2.3), where, in every case, one should use the identity $\vec{U} \equiv \vec{V}$. As is clearly seen, the nature of the mathematical operations (see Equations (5.2.11) and (5.2.14))is such that in the entire chain of operations and in the (ENSS) L - operator, the curl operation and the gradient operation, see Equation (5.2.11), always appear simultaneously:

$$\vec{V} \cdot \nabla\vec{V} = \vec{\omega} \times \vec{V} + \nabla(\tfrac{1}{2} V^2); \ \vec{\omega} = \nabla \times \vec{V}; \qquad (5.2.57)$$

or (5.2.11) $\equiv$ (5.2.57)

or grad $(\vec{V} \cdot \vec{V})$ = grad $(\vec{V}^2) \equiv 2(\vec{V} \cdot$ grad $\vec{V} + \vec{V} \times$ curl $\vec{V})$. (5.2.58)

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Thus, the appearance of the vorticity, circulation is a natural phenomenon in viscous fluid motions due to the physical characteristic properties of the coefficient of viscosity defined as the natural, transverse transport of momentum. All of the natural, existing fluids in the macroscopic, observable domain are viscous (inviscid fluids do not exist at normal temperatures. Consequently, the vorticity, curl, phenomena in viscous fluids often associated with the turbulence phenomena are "natural" phenomena. The difference between the turbulence phenomena in micro and macro circumstances is due to the varying degrees of observability, (the possibility of observing the turbulence phenomena) since turbulence phenomena cannot always be visually observed.

Equation (5.2.58) is an elementary, simple vector equation based upon the fundamental principles of mathematical vector analysis. It states that the two considered operations, grad and curl, always appear together, thus confirming the results obtained from the point of view of pure physics. Stated briefly, laminar flow does not and cannot exist in viscous fluids in the macroscopic flow domain at the temperatures and pressures above the phase transition, λ-point. The kind of flow which can exist under these circumstances can only be the turbulent flow of a varying degree of turbulence, visually observable or unobservable. Neither Reynolds in 1883, Prandtl in 1904, Blasius in 1908 nor G. I. Taylor in 1935 emphasized visually observable domain above the λ-point. Madelung in 1926 mentioned the association of quantum theory with the Euler equation of motion in inviscid fluid flow in the macroscopic domain. The explanation of the attitude of all other writers (Reynolds, Prandtl, et al.) is very simple; namely, at the time those writers were working and writing on the subject, neither they nor the science itself knew anything about superfluidity, λ-phase transition point, and so on.

Returning to the first equation derived above, i.e., Equation (5.2.24):

$$[\text{(ENSS) L}] \ \vec{V} - \nu \nabla^2 \vec{V} = 0, \ \text{(ENSS) L} \ \vec{V} = [\vec{\omega} \times + (\vec{V} \cdot \nabla)] \vec{V}; \quad (5.2.59)$$

when in particular, the x-component of this vector equation is:

$$- \omega_3 v + u \, \partial u/\partial x + v \, \partial u/\partial y + w \, \partial u/\partial z$$
$$- \nu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2) = 0 ; \quad (5.2.60)$$

then for a two-dimensional flow in the boundary layer along an infinitely long flat plate, Equation (5.2.60) is reduced to:

$$-\omega_3 v + u\, \partial u/\partial x + v\, \partial u/\partial y - \nu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = 0 \; ;$$

$$\omega_3 = \partial v/\partial x - \partial u/\partial y \; . \qquad\qquad (5.2.61)$$

Equation (5.2.61) is not identical to the reduced equation of Navier-Stokes used in the Prandtl-Blasius approach. But, for illustrative purposes it is proposed to begin with Equation (5.2.61) and the remaining elements of the system including the continuity equation and the boundary conditions. Then

$$\partial u/\partial x + \partial v/\partial y = 0 \; ; \; y = 0: u = v = 0; \; y = \infty: u = U_\infty. \qquad (5.2.62)$$

It is assumed that the system of Equations (5.2.61) and (5.2.62) is a "linear" system of equations similar to the System (I), Equations (5.1.18), (5.1.19), (5.1.20), and (5.1.21). In order to present the approach to a solution of this system in a systematic manner, the development begins with the original system of equations, even if this entails some repetition of the material.

Given a differential (ENSS) L linear system of equations including the boundary conditions:

$$(ENSS)L\, [u] - F = 0 \; ; \qquad\qquad (5.2.63)$$

$$-\omega_3 v + u\, \partial u/\partial x + v\, \partial u/\partial y - \nu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = 0 \; ; \qquad (5.2.64)$$

$$\omega_3 = \partial v/\partial x - \partial u/\partial y \; ; \qquad\qquad (5.2.65)$$

$$\partial u/\partial x + \partial v/\partial y = 0 \; ; \quad \nu = \mu\, \rho^{-1} \; ; \qquad\qquad \text{(AI)} \qquad (5.2.66)$$

$$y = 0: u = v = 0; \; y = \infty: u = U_\infty; \; \rho = \text{constant},$$

$$\mu = \text{constant.} \qquad\qquad (5.2.67)$$

The (ENSS) L - linear system (AI) corresponds to the linear system (I), Equations (5.1.18) to (5.1.21). Suppose that by the use of dimensional analysis, it is determined that some terms in the system (AI) are negligibly small, including the term containing ω_3, and may be omitted, then the system (AI) reduces to

(ENSS) $L[u] - F = 0$;   (5.2.68)

$$u \, \partial u/\partial x + v \, \partial u/\partial y - \nu \, \partial^2 u/\partial y^2 = 0 \; ;$$

                                                                    (AII)

$$\partial u/\partial x + \partial v/\partial y = 0 \; ;$$ (5.2.69)

where

$$y = 0: u = v = 0; \; y = \infty: u = U_\infty;$$ (5.2.70)

A particular solution of this system was found by Blasius and next by Howarth and is available in the form of tables. A particular solution of the Prandtl-Blasius form is denoted by the symbol ν_1, which corresponds to Equation (5.1.22). The new variable $w = u - \nu_1$ can be defined, which corresponds to Equation (5.1.23). By the linearity of the (ENSS) L - operator, one obtains the equation

$$(ENSS) \; L[w] \; = \; (ENSS) \; L[u] \; - F \; - \; \{ (ENSS) \; L[\nu_1] \; - F \} \; = 0. \quad (5.2.71)$$

This confirms the rule established by Equation (5.1.24) that "any solution u" of the equation in the system (AI) is the sum of any particular solution ν_1, of this system (now denoted by (AII)) and a solution "w" of the corresponding homogeneous equation. Because of the approximate forms of the systems discussed above, there may appear some undesired discrepancies between the results. In such cases it is recommended that the iterative method of Seidel (often called the Gauss-Seidel iterative method, see Hovanessian) be employed for a solution. The mathematical derivation of the convergence criteria of the Gauss-Seidel iteration method for solving sets of simultaneous equations is complicated since the criterion for convergence is related to the eigenvalues of a matrix derived from the matrix of the coefficients (see Hovanessian, p. 37, and Chapter 2). For this reason, in practical cases, the convergence criterion of the Gauss-Seidel method can be applied only in the case of sets of simultaneous equations consisting of very few equations, 2, 3 or 4 (Hovanessian, p. 38). According to the old hypothesis of Seidel (17th Century) the iterative process is a convergent process or at least an asymptotic one. Further, according to this proposition, solutions applicable to particular linear systems of wave equations can also be applied to nonlinear systems of partial differential equations by means of the (ENSS) L - operators or similar operators. For illustrative purposes, a solution of the n-th order linear system for ψ_n is used, with the corresponding initial (time) and boundary (space) value conditions as the given data for the solution of

(n+1)-th order system of ψ_{n+1}-function and the values so obtained are used to determine the values of the next functions as denoted by the symbols α_{n+1}, β_{n+1}, u_{n+1}, ρ_{n+1}, T_{n+1}, k_{n+1}, μ_{n+1}, and so on. The question may arise as to whether or not one may get some approximate knowledge and understanding about the convergence of the process described above.

Suppose that one treats the successive approximation steps as the transformation of dependent variables from one step, n, say, to the next step, (n+1)th, then from each step of such a process one can calculate the Jacobian of the dependent variables with respect to the independent variables:

$$J\left(\frac{u_{n1},\ u_{n2},\ u_{n3},\ \ldots,\ u_{nm}}{x_1,\ x_2,\ x_3,\ \ldots,\ x_m}\right) = \frac{\partial(u_{n1},u_{n2},u_{n3},\ \ldots,\ u_{nm})}{\partial(x_1,\ x_2,\ x_3,\ \ldots,\ x_m)} , \quad (5.2.72)$$

where the subscript "n" denotes the number of steps in the successive approximation procedure and the subscript "m" denotes the number of the coordinates in the Cartesian space in which the operations take place. Topologically, a Jacobian can be referred to as the volume in the x_m-dimensional hyper-space. This implies that a sequence of the decreasing values of the Jacobians $(J_1,\ J_2,\ J_3,\ \ldots,\ J_n)$ at a given point (x_m) would be an indication that the above iterative process is a convergent one or at least an asymptotic one. The investigator is deeply indebted to Professor J. Sutherland Frame, Department of Mathematics, Michigan State University, East Lansing, Michigan for a discussion on the subject of the convergence of the above successive approximation procedure. It was during this discussion that the concept of using the Jacobian as a certain measure of convergence or of the asymptotic behavior of the iterative process was conceived.

5.3. Elements of Probability Calculus

The wave equation gives a prediction of what happens to the wave function. The wave function, however, gives only the probability of where the element, the electron, can be found. In the classical limit, the observation is so gross that the difference between probable behavior and actual behavior is never detected. Hence the wave equation also determines the classical limit of the particle, electron, motion from the deterministic point of view. More generally, the wave equation determines the probable results of any process which the particle, electron, can undergo. It plays as fundamental a role in quantum theory as the Navier-Stokes equations of motion play in classical fluid dynamics.

Let us now introduce the concept of the propagation vector $\vec{k}$, defined as follows:

$$k_x = (2\pi\,\ell)\,L^{-1}; \quad k_y = (2\pi m)\,L^{-1}; \quad k_z = (2\pi n)\,L^{-1}; \tag{5.3.1}$$

$$k^2 = (2\pi\,L^{-1})^2\,(\ell^2 + m^2 + n^2)\,. \tag{5.3.2}$$

From this definition it follows that $k/2\pi$ is the number of waves in the distance L, the wave length is $\lambda = 2\pi\,k^{-1}$, and $k = 2\pi\,\lambda^{-1}$. $\vec{k}$ is a vector in the direction of propagation of the wave with magnitude $2\pi\,\lambda^{-1}$ which depends only upon the values of the integers ℓ, m, and n.

A more precise definition of the two probability functions significant in the present investigation is:

(1) $P(x)\,dx$, the probability that a particle can be found between x and $(x + dx)$;

(2) $P(k)\,dk$, the probability that the particle momentum $(m\dot{x} = m\,dx/dt)$ lies between k and $(k + dk)$: $k = m\dot{x}$.

Beginning with the definition $P(x)$, an acceptable definition of this quantity must satisfy at least the following requirements:

(1) The probability function $P(x)$ is never negative.

(2) The probability is large where $|\psi|$ is large and small where $|\psi|$ is small. The symbol ψ denotes the wave function.

(3) The significance of $P(x)$ must not depend in a critical way upon any physical quantity which is known to be irrelevant.

(4) The integrated probability of finding the particle, electron, or a cluster of particles somewhere in the system must be unity and remain unity for all time. The particle, electron, is neither emitted nor absorbed anywhere in the system); consequently, the function $P(k)$ must fulfill the condition

$$\int_{-\infty}^{\infty} P(k)\,dk = 1\,. \tag{5.3.3}$$

There are available proofs to demonstrate that there exists some sort of dependence between $P(x)$ and $P(k)$. The following relations are true:

$$\int_{-\infty}^{\infty} P(k)\,dk = \int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = \int_{-\infty}^{\infty} P(x)\,dx\,. \tag{5.3.4}$$

Above, the star * denotes the complex conjugate function. Hence, if P(x) is normalized to unity, then P(k) is automatically normalized, and since P(x) remains normalized for all time, P(k) remains so too.

Only the final results are shown since the derivation and justification of the results obtained are available in the literature.

Average value of a function of position

The average value of x by the definition is

$$\overline{x} = \int_{-\infty}^{\infty} P(x) \, x \, dx \, . \tag{5.3.5}$$

Since

$$P(x) = \psi^*(x) \, \psi(x) \, , \tag{5.3.6}$$

one can write:

$$\overline{x} = \int_{-\infty}^{\infty} \psi^*(x) \, x \, \psi(x) \, dx \, . \tag{5.3.7}$$

In a similar way, the average values of any function of x can be written:

$$\overline{f}(x) = \int_{-\infty}^{\infty} \psi^*(x) \, f(x) \, \psi(x) \, dx \, . \tag{5.3.8}$$

The generalization of this formalism to three dimensions is straightforward:

$$\overline{f}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^* \, f(x, y, z) \, \psi \, d\tau \, , \tag{5.3.9}$$

where $d\tau = dx \, dy \, dz$ represents the element of volume.

Average value of a function of momentum

The average value of momentum is

$$\overline{p} = \int_{-\infty}^{\infty} p \, P(p) \, dp = \int_{-\infty}^{\infty} \Phi^*(p) \, p \Phi(p) \, dp \, , \tag{5.3.10}$$

where $\Phi(p)$ is the normalized Fourier component of $\psi(x)$ with $p = \hbar k$, $\hbar = h(2\pi)^{-1}$, h being the Planck constant.

If $\psi(x)$ is normalized, then the corresponding function in the momentum space, $\phi(k)$, is automatically normalized and the following relations are valid:

$$\phi(k) = (\not{k})^{1/2} \Phi(p) ; \qquad\qquad (5.3.11)$$

$$\int_{-\infty}^{\infty} \phi*(k)\, \phi(k)\, dk = 1 ; \qquad\qquad (5.3.12)$$

$$\int_{-\infty}^{\infty} \Phi*(p)\, \Phi(p)\, dp = 1 . \qquad\qquad (5.3.13)$$

For any function of the momentum the average is given by

$$\overline{f}(p) = \int_{-\infty}^{\infty} f(p)\, P(p)\, dp = \int_{-\infty}^{\infty} \Phi*(p)\, f(p)\, \Phi(p)\, dp . \qquad (5.3.14)$$

Criterion for acceptable wave function

A basic requirement of any ψ is that it be quadratically integrable; i. e., that

$$\int_{-\infty}^{\infty} |\psi|^2\, dx = \text{a finite number} . \qquad\qquad (5.3.15)$$

Therefore, a necessary but not sufficient requirement for ψ is that $\psi \to 0$ as $x \to \pm \infty$, and $\Phi(p) \to 0$ as $p \to \pm \infty$.

5.4. Field Theory (Explanation)

The mathematical operations performed by John von Neumann in his proof on quantum theory, and the proof itself, became a pivotal argument for the ideology and the base of quantum physics. vonNeumann proved to virtually everyone's satisfaction that there could be no future discovery of parameters, previously "hidden" from quantum physics, which would permit precise measurements to be performed in violation of the "uncertainty law," thereby forever ruling out cause-effect from the scene of physics. From the known systems of equations, expressed in terms of "hidden" variables, following John von Neumann, one is able to include here the integro-differential equation of Boltzmann and the famous Navier-Stokes equation. The principle mathematical tool used by vonNeumann is classical Hamiltonian mechanics where the coordinates used are the conjugate coordinates (p, q), and the particles under consideration have constant mass. The mathematical operations are performed in the phase-space (p, q). The applied equation is the Schroedinger wave equation. However, in the field of fluid dynamics, usually (not the "rule") all the operations of a mathematical nature are performed in domains and

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systems having infinitely many degrees of freedom. The operations often
are performed on functions having boundary conditions located at points
or curves at infinity.

In this regard, the investigator proposed some time ago a conjecture
on a generalization of the vonNeumann proof, preserving all the principals
of logic and philosophy, from the phase-space to a function space, from
the classical mechanical system to the field theory system and from quantum
theory to quantum field theory. Except for some fairly nebulous ideas on
this and a similar nature, expressed at a few scientific meetings, nothing
else can be found on the subject in the modern literature. In the explanation
above, the term "field" in the title refers to a domain of infinitely many
degrees of freedom.

6. DISTURBANCES IN FLUIDS

Perhaps one of the most popular and best known disturbance
phenomenan in liquids and gases is the phenomenon called turbulence. The
present section deals with some disturbances treated as a general problem
in the field of mechanics. Particular aspects of turbulence, as for example
the statistical theory of turbulence, are not discussed in this section but
will be discussed separately in a future report on the subject of C.A.T.
(Clear Air Turbulence).

6.1. Geometrical and Mechanical Aspects of Disturbances (Turbulence -
A Special Case)

In the past, considerable effort has been devoted to research on the
problem of turbulence, particularly in the area known as the "statistical
theory of turbulence." Some definitions in this field can be expressed briefly
as follows (Hinze, p. 1): "In 1937 Taylor and von Kármán gave the following
definition: Turbulence is an irregular motion which in general makes its
appearance in fluids, gaseous or liquid when they flow past a solid surface
or even when neighboring streams of the same fluid flow pass over one
another." Indeed, this irregularity is a very important feature. Because
of irregularity, it is impossible to describe turbulent motion in all details
as a function of time and space coordinates, although it is possible to
describe it by laws of probability. It is important to note that it appears
possible to indicate distinct average values of various quantities, such as
velocity, pressure, temperature, etc. If turbulent motion were entirely
irregular, it would be inaccessible to any mathematical treatment.

Therefore, it is not sufficient just to say that turbulence is an irregular motion and to leave it at that. Perhaps the definition might be formulated somewhat more precisely as follows: "Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned."

The addition "with time and space coordinates" is necessary, since it is not sufficient to define turbulent motion as irregular in time alone. Take, for instance, the case in which a given quantity of a fluid is moved bodily in an irregular way; the motion of each part of the fluid is then irregular with respect to time to a stationary observer, but not to an observer moving with the fluid. Nor is turbulent motion a motion that is irregular in space alone, because a steady flow with an irregular flow pattern might then come under the definition of turbulence.

As Taylor and von Kármán have stated in their definition, turbulence can be generated by friction forces at fixed walls (flow through conduits, flow past bodies) or by the flow of layers of fluids with different velocities which pass over one another. As will be shown in what follows, there is a distinct difference between the kinds of turbulence generated in the two ways. Therefore it is convenient to indicate turbulence generated and continuously affected by fixed walls by the designation "wall turbulence" and to indicate turbulence in the absence of walls by "free turbulence." These are generally accepted terms.

In the case of real viscous fluids, viscosity effects will result in the conversion of kinetic energy of flow into heat; thus turbulent flow, like all flow of such fluids, is dissipative in nature. If there is no continuous external source of energy for the continuous generation of the turbulent motion, the motion will decay. Other effects of viscosity can make the turbulence more homogeneous and make it less dependent on direction. In the extreme case, the turbulence may have quantitatively the same structure in all parts of the flow field and is said to be homogeneous. Turbulence is called isotropic if its statistical features have no preference for any direction, so that perfect disorder reigns. As we shall see later, no average shear stress can occur and, consequently, there is no velocity gradient of the mean velocity. This mean velocity, if it occurs, is constant throughout the field.

In all other cases where the mean velocity shows a gradient, the

turbulence will be nonisotropic, or anisotropic. Since this gradient in mean velocity is associated with the occurrence of an average shear stress, the expression "shear-flow turbulence" is often used to designate this class of flow. Wall turbulence and anisotropic free turbulence fall into this class.

Von Kármán has introduced the concept of homologous turbulence for the case of constant average shear stress throughout the field, for instance, in plane Couette flow.

The investigator suggests that turbulence is a natural phenomenon in both liquids and gases, and as such it should be attacked by natural methods; i. e., quantum theoretical methods of statistical physics.

Let us pause for a moment and consider which geometrical-mechanical structures and configurations could enter into the entire geometry and mechanism of any disturbance regardless of the boundary conditions such as surrounding walls, flows past or over one another, and so on. Briefly, an in depth analysis of the disturbance in the free, unlimited sense from the standpoint of geometry, mechanics, kinematics and dynamics will be undertaken.

(A) <u>Gradient</u>: In any kind of flow there appears the gradient of the pressure or of the external force field. Only in the case of the most abstract field of an ideal fluid flow can one consider a fluid being absolutely inviscid and moving without any pressure gradient and/or any force field gradient. In a real fluid there must exist some element of a driving force. In the macroscopic sense only real fluids do exist although references have been made to superfluidity, super-viscosity and super-conductivity (on more than one occasion by the investigator). Such particular cases are, in general, disregarded in the present research effort.

(B) <u>Curl</u>: In general, in the past, the curl and its influence upon the entire geometry of fluid flow of any kind was of much less interest than the gradient. Keeping in mind that $\vec{U} \equiv \vec{V}$, from the form of Equation (5.2.58) one can see that actually $\vec{U} \cdot \nabla \vec{U}$ is always and inseparably associated with the curl $\vec{U} \times (\nabla \times \vec{U})$. This is the Euler type of equation for an ideal, inviscid fluid or, for some particular "idealistic" types of flow, where one can disregard the curl. Usually, this is done for purposes of enabling one to solve the system of equations in question which is complicated enough with only the gradient preserved and with the curl completely disregarded. There are exceptions to this statement which refer only to the rotational flows. These are particular cases which have to be treated independently.

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The primary problem in treating and dealing with the practical problems involving the curl is the fact that the curl is a somewhat evasive factor in fluid dynamics. As emphatically stated above, the "laminar" boundary layer in mathematical form as proposed by Prandtl-Blasius is not laminar at all. It is, in some sense, a turbulent boundary layer with the curl "hidden" from the macroscopic, visual observations (more than often the curl cannot be seen visually) and measurements. In spite of the facts presented above, the nomenclature and theory on the "laminar" boundary layer prevails. Since no fluid is ideal, a curl existing in the viscous boundary layer may be the origin of many or at least a great number of "wavy" phenomena. These often, after leaving the geometrical domain of the boundary layer, become visibly observable and can be tested macroscopically as in wakes and jet streams (even when observed far behind the "laminar" boundary layer). This is due to the factor of viscosity, since the viscosity acts as the "force" factor which may cause a slow but steady growth of the size and effective influences of vortexes in any geometrical domain, wherever they exist. In general, the curl has to be considered to be the main factor in any flow of a laminar or uniform nature, since its appearance may signal (due to its possibility of growth), the possibility of the appearance of a disturbed (turbulent) flow of an irreversible dissipative flow structure. This usually is some sort of an undesired flow phenomenon. Unfortunately, Equation (5.2.58) shows very clearly that, from the very nature and structure of the system, the gradient, which is very necessary for the continuation of the flow (it is a constructive element) is always connected with the curl. This is a sort of destructive element leading to the dissipative form of the phenomenon and to an irreversible transformation form of the kinetic energy into heat.

(C) Resonance: Configurations discussed above, the gradient, the curl, and the divergence operator, appear almost always in the mathematical representation of any force field. Thus, seemingly, the entire possible list of configurations has been exhausted. This is not quite true since for a long time the investigator has felt that one more additional factor appears important in the geometric-structural configuration of the phenomenon of the disturbances in fluids such as that of turbulence. The investigator has emphasized this in the past in meetings both in the United States and abroad. The factor in question is the factor of resonance. Under the assumption

that matter follows the rules established by modern quantum physics (and not by the classical physics) one is compelled to accept the quantum concepts as the guiding concepts of this approach. This leads to the necessity of accepting a physical picture of the quantum nature of matter as the fundamental base on which subsequently the entire mathematical structure of the system of disturbances should be built. One of the concepts used in the quantum theoretic methods in statistical physics is the concept of the association of material particles with oscillatory phenomena. A separate report may be devoted to the problem of an association of the wave mechanics (quantum) with the fluid dynamics in which the oscillatory phenomena and all other pertinent items will be thoroughly presented and discussed. For the time being it is sufficient to mention that the resonance phenomena may occur and according to the investigator, actually do occur in a disturbed (turbulent) field. In other words, more than one single wave appears in a problem ("beat" phenomena should be considered).

This report will approach disturbances in a very systematic manner. In order to achieve this the effort will be divided into separate steps. The first step, discussed in D below, refers to the sum of gradient and the curl, according to Equation (5.2.58). The resonance aspects will be treated in later reports.

(D) <u>Kinetic Energy Disturbances</u>: Equation (5.2.58) demonstrates very clearly that from a pure, formalistic standpoint the following is true when $\nabla\phi = \vec{b} \equiv \vec{U}$:

$$\text{grad} (\vec{b} \cdot \vec{b}) = \text{grad} (\vec{b}^2) = 2 \vec{b} \cdot (\text{grad } \vec{b}) + 2 \vec{b} \times (\text{curl } \vec{b}). \quad (6.1.1)$$

The rigorous formalism of mathematics states that grad $\vec{b}$ is always connected with curl $\vec{b}$. The question is: "Is there any physical explanation of this inter-correlation?". In the past the opinion on this question has been expressed; namely, that the factor which connects these two geometrical configurations is the viscosity which is always present in the earth's atmosphere. Only in the abstract case of an ideal fluid (viscosity equal to zero) can one assume that curl $\vec{b}$ is equal to zero, which is the case assumed by Madelung in 1926. This was done deliberately in order to obtain the Euler form of the equation in the wave mechanics approach to fluid dynamics.

Another interesting problem in the theory of disturbances is the

problem of the dimensions of the space in which the phenomenon of disturbances like that of turbulence may take place. Between the years 1930 - 1940 there was discussion in the scientific literature on the least number of dimensions of a space which are imperative for the existence of turbulence. In particular, G. I. Taylor and T. von Kármán were the main persons participating in that discussion. The conclusion was that the phenomenon of turbulence cannot exist in two-dimensional space and that the least number of dimensions necessary for the existence of turbulence is three (or more, including time). In the case of the Prandtl-Blasius boundary layer, this research considers a two-dimensional boundary layer. The approach involves the vorticity vector (curl) which possesses only one component, ω_z, in the direction perpendicular to the (x, y) plane. Consequently, one has to take into consideration the component ω_z, and necessarily one has to consider a three-dimensional geometrical configuration of the entire flow domain. Neglecting this aspect would be equivalent to rejecting the influence of the action of the curl upon the flow configuration. The author formerly achieved this in the Prandtl-Blasius approach. It is emphasized that the curl plays an important part in the present approach. The reason for this approach is that Equation (5.2.58) demonstrates very clearly that in real, viscous fluids (all fluids are real and viscous) there cannot exist a pure, laminar motion having only (grad $\vec{b}$), (Equation 6.1.1). (Grad $\vec{b}$) must be accompanied by the (curl $\vec{b}$), Equation (6.1.1). Briefly, a purely laminar flow cannot exist in real, viscous fluids. It is always related with the geometry of a rotational part due to the curl. This conjecture breaks down, and may even be false, when one operates in a domain with decreasing temperature and approaches the transition point. At the temperatures below the transition point, the super-viscosity and super-heat-conductivity phenomena which depend upon the kind and the characteristic natural properties of the fluid may appear which would introduce fundamental and basic changes in the picture presented above. The association between (grad $\vec{b}$) and (curl $\vec{b}$), Equation (6.1.1), may break down at the transition point.

Of course, any multi-dimensional domain and disturbance can be represented by means of a two-dimensional diagram obtained by means of ordinary projections (projective geometry). Actually, this can and was done for the case of the boundary layer along a flat plate. In an extreme case and under a considerable number of constraining conditional statements

one may possibly risk the statement that a particular disturbance in a flow could be represented as a two-dimensional disturbance. This may be referred to as a projection of the flow in question on a two-dimensional plane.

(E) Turbulent versus Laminar Motion: The problem of the relationship between laminar and turbulent motions is as old as the problem of turbulence itself; i. e. , it was originated at least in 1883 by O. Reynolds. The reader who would like to get more information on this item should refer to the literature on turbulence (books by Pai, Hinze, etc.). Quoting from Pai, p. 7: "Another interesting point of view by Heisenberg as presented in the paper by Dryden is that it is not turbulence but laminar motion that requires explanation. The fluid without friction is a system with an infinite degree of freedom. Turbulence is essentially a statistical problem which results from a certain equilibrium distribution of energy among a very large number of degrees of freedom. It is the viscosity that reduces the number of degrees of freedom by damping the motion of small eddies."

The point of view expressed by Heisenberg is a most interesting one. The present investigation emphasizes and re-emphasizes the opinion that, in general, one has to question whether laminar motion can exist because of viscosity. Rather, one can maintain that any motion of a fluid which, in the macroscopic sense, always possesses viscosity, is not a laminar motion. This would automatically imply that the only possible motion existing in any real fluid is a turbulent motion (or a disturbed one). The only question left open for discussion is the degree of the disturbance, whether it is visibly observable or un-observable, whether it is of a microscopic or of a macroscopic nature, etc.

(F) Vortex Potential: It has been previously stated on several ocassions that the field of fluid dynamics, as a field of science, did not develop a strong background in developing the vortex potential function. In general, this has been avoided due to the fact that, up to now, the gradient of the force field has been considered as being the most important geometrical element in the dynamics of fluids. This remark is particularly important with reference to viscous, heat conducting fluids. The phenomena at the "core" of a vortex, exemplified by the eye of a tornado, are not very well known and have had little investigation.

(G) Statistical Theories of Turbulence: This item will be discussed in a future part of this research program.

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6.2. Gradient Plus Curl

As has been mentioned before, Equation (5.2.58) demonstrates clearly that the gradient of the velocity vector, $\vec{U}$, (grad $\vec{U} = \nabla\vec{U}$), is always associated with the curl of the velocity vector, $\vec{U}$, (curl $\vec{U} = \nabla \times \vec{U}$). In 1926, Irving Madelung, attacking the problem of the association of quantum mechanics theory with macroscopic hydro-dynamics theory, had proposed that curl $\vec{U} = 0$. This enabled him to obtain a pure form of the Euler equation of motion starting directly with the Schroedinger equation. The present research attacks the problem of the geometry of fluid dynamics flow and its resultant characteristic properties under the assumption that both geometrical aspects in the full form of Equation (5.2.58) are preserved; namely, $\vec{U} \times$ curl $\vec{U} + \vec{U} \cdot$ grad $\vec{U}$. Past and present literature has few examples of ordinary or partial differential equations in both geometrical elements, gradient and curl, for simultaneous treating and solving. Much less one can expect in the case of the Navier-Stokes differential equations (even when they are reduced to the ordinary differential equations, one cannot find reliable solutions in the literature for cases in which the gradient and the curl appear and interact simultaneously. For this reason, exact solutions in the case denoted by the symbolic notation (grad + curl) have not been possible and one has to be satisfied with only the approximate forms of solutions.

Below, the writer attacks the problem of the superposition of disturbances upon the laminar flow in the boundary layer along an infinitely long flat plate. This is considered as an illustrative example for other similar and analogous cases like flow along an airfoil, a propeller blade, jet engine blade, helicopter blade, and many others. Assume:

$$u\, \partial u/\partial x + v\, \partial u/\partial y = \nu\, \partial^2 u/\partial y^2; \quad \partial u/\partial x + \partial v/\partial y = 0 \, , \tag{6.2.1}$$

$$\text{B. C. :} \quad y = 0, \ u = v = 0; \quad y = \infty, \ u = U_\infty \cdot \tag{6.2.2}$$

With the use of the intermediate dimensionless coordinate, η :

$$\eta = y\ U_\infty^{1/2}\ \nu^{-1/2}\ x^{-1/2} \, , \tag{6.2.3}$$

and of the stream function, ψ ,

$$\psi = U_\infty^{1/2}\ \nu^{1/2}\ x^{1/2}\ f(\eta) \, , \tag{6.2.4}$$

where

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$f(\eta)$ denoting the dimensionless stream function, was solved in the form of a tabular representation. The solution of functions, f, f', f" appearing in the ordinary differential equation which correspond to and are associated with the system of Equations (6.2.1) and (6.2.2) give values of the velocity components (u,v) which are tabulated from

$$u = (\partial\psi/\partial\eta)(\partial\eta/\partial y) = U_\infty \, f'(\eta) ; \tag{6.2.5}$$

$$v = - (\partial\psi/\partial\eta)(\partial\eta/\partial x) = \frac{1}{2} (U_\infty^{1/2} \, \nu^{1/2} \, x^{-1/2})(\eta f' - f). \tag{6.2.6}$$

The ordinary differential equation, associated with the system of equations (6.2.1) and (6.2.2) is of the form:

$$f \, f'' + 2 f''' = 0; \quad \text{B.C.:} \quad \eta = 0, \; f = 0, \; f' = 0; \; \eta = \infty, \; f' = 1 . \tag{6.2.7}$$

(1) As the first step consider the classical, macroscopic approach. The above system of equations and solutions, Equations (6.2.1) to (6.2.7) refers to the special form (reduced) of the general Navier-Stokes system of equations of a viscous fluid in macroscopic fluid dynamics, in the two dimensional (plane) coordinate system (x,y). There is absolutely no association, whatsoever, to any three-dimensional-space geometrical configuration (x,y,z) or (x,y,t) including possibly the time-coordinate. In some cases it may be considered to represent a projection of a three-dimensional-space geometrical configuration (or a physical phenomenon tested in a three or more-dimensional space configuration). The possible projection would be done from a three-or more-dimensional-space configuration upon a plane in the two-dimensional (x,y) domain.

(2) As the next step consider the field of wave mechanics as starting with the Schroedinger equation in a three-dimensional geometry (x,y,z)-space and time, t:

$$\nabla^2 \, \psi_S - 8 \, \pi^2 \, m \, h^{-2} \, \Phi \, \psi_S - i \, 4 \, \pi \, m \, h^{-1} \, \partial\psi_S/\partial t = 0 , \tag{6.2.8}$$

where ψ_S = wave function; Φ = potential energy of the system as a function of the position of the electron; m = mass of the electron; ∇^2 the Laplacian; and h is the Planck constant. Assuming

$$\psi_S = a \exp (i\beta); \quad a = a(x,y,z,t); \quad \beta = \beta(x,y,z,t); \tag{6.2.9}$$

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and with

$$\phi = -\beta h (2 \pi m)^{-1}; \quad \vec{U} = \nabla \phi ; \quad a^2 = \rho ; \qquad (6.2.10)$$

after some operations, one gets

$$\partial \vec{U}/\partial t + \frac{1}{2} \nabla (\vec{U}^2) + m^{-1} \nabla \Phi - \nabla [a^{-1} \nabla^2 a \, h^2 (8 \pi^2 m^2)^{-1}] = 0. \qquad (6.2.11)$$

For a stationary flow, the first term in Equation (6.2.11) is zero and for an incompressible fluid the last term in Equation (6.2.11) is zero. The remaining terms are:

$$\frac{1}{2} \nabla (\vec{U}^2) + m^{-1} \nabla \Phi = 0 . \qquad (6.2.12)$$

The above was obtained by Madelung under the restrictive condition that the operations were done and are valid under the condition that curl $\vec{U} = 0$. The results obtained above in both macroscopic fluid dynamics and in microscopic quantum theory representation are considered as the first "iterative" approximations. The reader must keep in mind that the entire approach in the field of the wave mechanics (quantum) is based upon the tool of probability calculus. These first iterative approximation results can and will be used as the starting point for the second iteration; i. e. , as the initial conditions for the second iterative approximation. This section presents a detailed description of the initial stages of the second step iteration procedure. In steady conditions the Schroedinger equation in the Madelung interpretation reduces to the form of Equation (6.2.12):

$$\frac{1}{2} \nabla (\vec{U}^2) + m^{-1} \nabla \Phi = 0 , \qquad (6.2.12)$$

and Equation (6.1.1) with $\vec{b} = \vec{U}$ becomes

$$\nabla (\vec{U}^2) = \text{grad} (\vec{U}^2) = 2 \vec{U} \cdot \text{grad} \, \vec{U} + 2 \vec{U} \times \text{curl} \, \vec{U} . \qquad (6.2.13)$$

After inserting Equation (6.2.13) into (6.2.12) with $\vec{U}$ taken from the first iterative approximation, the goal is to investigate primarily the geometry of the second iterative approximation. Since the curl, $(\nabla \times \vec{U})$, has only one component, ω_z, in the z-direction, is perpendicular to the (x,y) plane in which the physical flow phenomenon takes place, and has the vector $\vec{U} = (u,v)$ with only two components, the problem is to calculate the term in Equation (6.2.13):

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$$\vec{U} \times \text{curl } \vec{U} = \begin{vmatrix} i & j & k \\ u & v & 0 \\ 0 & 0 & \omega_z \end{vmatrix} , \qquad (6.2.14)$$

which evidently is equal to

$$\vec{U} \times \nabla \times \vec{U} = \vec{i}\,(v\omega_z) - \vec{j}\,(u\omega_z) \cdot \qquad (6.2.15)$$

Equation (6.2.12) takes the form:

$$\vec{U} \cdot \text{grad } \vec{U} + i\,(v\omega_z) - \vec{j}\,(u\omega_z) + m^{-1}\nabla\Phi = 0, \quad \vec{U} = (u,v), \quad (6.2.16)$$

where the velocity components u and v are taken directly from the first iteration process and the term $m^{-1}\nabla\Phi$ represents the action of the diabatic force field. This includes the dissipative, irreversible phenomena of the viscous stresses. Assume, that the first iteration procedure followed exactly the Prandtl-Blasius boundary layer equations, then one has in the first iterative approximation:

$$\psi = U_\infty^{1/2}\,\nu^{1/2}x^{1/2}\,f(\eta); \quad \eta = y\,U_\infty^{1/2}\,\nu^{-1/2}x^{-1/2}; \qquad (6.2.17)$$

$$u = \partial\psi/\partial y = (\partial\psi/\partial\eta)(\partial\eta/\partial y) = U_\infty f'(\eta) ; \qquad (6.2.18)$$

$$v = -\partial\psi/\partial x = -(\partial\psi/\partial\eta)(\partial\eta/\partial x) = \tfrac{1}{2}(U_\infty^{1/2}\,\nu^{1/2}x^{-1/2})(\eta f' - f). \quad (6.2.19)$$

Inserting the values so obtained into the boundary layer equations, in the first iterative approximation:

$$u\,\partial u/\partial x + v\,\partial u/\partial y = \nu\,\partial^2 u/y^2; \quad \partial u/\partial x + \partial v/\partial y = 0 , \qquad (6.2.20)$$

$$\text{B.C.: } y = 0: \ u = v = 0; \ y = \infty: \ u = U_\infty, \qquad (6.2.21)$$

one obtains at first the same system where:

$$\partial/\partial x = (\partial/\partial\eta)(\partial\eta/\partial x); \ \partial/\partial y = (\partial/\partial\eta)(\partial\eta/\partial y); \ f = f(\eta); \ \eta = \eta(x,y);$$
$$f f'' + 2 f''' = 0; \ \text{B.C.: } \eta = 0: f = 0; \ f' = 0; \ \eta = \infty: f' = 1 . \qquad (6.2.22)$$

The second iterative approximation begins with Equations (6.2.12) and (6.2.13):

$$\vec{U} \cdot \text{grad } \vec{U} + \vec{U} \times \text{curl } \vec{U} + m^{-1} \nabla \Phi = 0; \quad \text{curl } \vec{U} \equiv \nabla \times \vec{U}. \quad (6.2.23)$$

Applying the idea of the scale magnification factors and of diabatic flow, discussed in great details in Sections 2.2 and 4.3, the term $(m^{-1} \Phi)$ corresponds to the right hand side of Equation (6.2.20). Using also Equation (6.2.15) and the decomposition into $\vec{i}$ and $\vec{j}$-directions, the first equation of the conservation of momentum is written in the form (the flow is stationary, $\partial p/\partial x = 0$):

$$u \, \partial u/\partial x + v \, \partial u/\partial y + v \, \omega_z - \nu \, \partial^2 u/\partial y^2 = 0 \,, \quad (6.2.24)$$

with the B.C. preserved as in Equation (6.2.21). The next step is to follow the reverse procedure and to find the equation in terms of the dimensionless stream function, $f = f(\eta)$, which corresponds to Equation (6.2.24). This will be called the ordinary differential equation corresponding to the second iterative approximation procedure. Equations (6.2.5) and (6.2.6) are used to determine the components (u,v) and for the expression ω_z:

$$\omega_z = \frac{1}{2}(\partial v/\partial x - \partial u/\partial y) \,; \quad (6.2.25)$$

$$\partial v/\partial x = -\frac{1}{4} U_\infty^{1/2} \nu^{1/2} x^{-3/2} (\eta^2 f'' + \eta f' - f) \,; \quad (6.2.26)$$

$$\partial u/\partial y = U_\infty^{3/2} \nu^{-1/2} x^{-1/2} f'' \,. \quad (6.2.27)$$

With η and f being dimensionless, $U_\infty \approx \text{cm sec}^{-1}$, $x \approx \text{cm}$, $\nu \approx \text{cm}^2 \text{ sec}^{-1}$, one gets:

$$u \approx \text{cm sec}^{-1}; \quad v \approx \text{cm}^{1/2} \text{ sec}^{-1/2} \text{ cm sec}^{-1/2} \text{ cm}^{-1/2} \approx \text{cm sec}^{-1}; \quad (6.2.28)$$

$$\partial v/\partial x \approx \text{cm}^{1/2} \text{ sec}^{-1/2} \text{ cm sec}^{-1/2} \text{ cm}^{-3/2} \approx \text{sec}^{-1}; \quad (6.2.28a)$$

$$\partial u/\partial y \approx \text{cm}^{3/2} \text{ sec}^{-3/2} \text{ cm}^{-1} \text{ sec}^{+1/2} \text{ cm}^{-1/2} \approx \text{sec}^{-1} \quad (6.2.28b)$$

The dimensions of each term in Equation (6.2.24) will be verified with the value of u taken from Equation (6.2.18), the value η from Equation (6.2.17), and the value of v from Equation (6.2.19).

$$\partial u/\partial x = U_\infty f''(\eta) \, \partial \eta/\partial x = -\frac{1}{2} U_\infty \eta x^{-1} f'' \,; \quad (\text{cm sec}^{-1} \text{ cm}^{-1} \approx \text{sec}^{-1}); \quad (6.2.29)$$

$$u \, \partial u/\partial x = -\frac{1}{2} U_\infty^2 \eta x^{-1} f' f'' \,; \quad (\text{cm}^2 \text{ sec}^{-2} \text{ cm}^{-1} \approx \text{cm sec}^{-2}); \quad (6.2.30)$$

$$v \, \partial u/\partial y = \frac{1}{2} \, U_\infty^2 \, x^{-1} \, (\eta f' - f) \, f''; \quad (cm^2 \, sec^{-2} \, cm^{-1} \approx cm \, sec^{-2}; \quad (6.2.31)$$

$$\nu \, \partial^2 u/\partial y^2 = \nu U_\infty^2 \, \nu^{-1} \, x^{-1} \, f''' = U_\infty^2 \, x^{-1} \, f''' \; ; \quad (cm^2 \, sec^{-2} \, cm^{-1} \approx cm \, sec^{-2}) \quad (6.2.32)$$

$$\omega_z = \frac{1}{2} \, (\partial v/\partial x - \partial u/\partial y) = -\frac{1}{2} \, U_\infty^{1/2} \, \nu^{1/2} \, x^{-1/2} [\frac{1}{4} \, x^{-1} \, (\eta^2 f'' + \eta f' - f)$$
$$+ \, U_\infty \, \nu^{-1} \, f'']$$

$$(cm^{1/2} \, sec^{-1/2} \, cm \, sec^{-1/2} \, cm^{-1/2} \, (cm^{-1} + cm \, sec^{-1} \, cm^{-2} \, sec)) \approx sec^{-1}$$
$$(6.2.33)$$

$$v \, \omega_z = -\frac{1}{4} \, U_\infty \, \nu x^{-1} (\eta f' - f) \, [\frac{1}{4} \, x^{-1} (\eta^2 f'' + \eta f' - f) + U_\infty \, \nu^{-1} \, f''] \, ,$$

$$(cm \, sec^{-1} \, cm^2 \, sec^{-1} \, cm^{-1} \, [cm^{-1} + cm \, sec^{-1} \, cm^{-2} \, sec])$$

$$\approx cm^2 \, sec^{-2} \, (cm^{-1}) \approx (cm \, sec^{-2}) \, , \quad (6.2.34)$$

with all the dimensions being correct. Inserting Equations (6.2.26) and (6.2.34) into Equation (6.2.24) gives the result of the second iterative approximation to the Prandtl-Blasius flow in the boundary layer along an infinitely long flat plate including the term $\vec{U} \times \nabla \times \vec{U}$, Equation (6.2.23), or $(v \, \omega_z)$, Equation (6.2.24), all the other terms being kept unaltered. Thus

$$-\frac{1}{2} \, U_\infty^2 \, x^{-1} \, \eta f' f'' + \frac{1}{2} \, U_\infty^2 \, x^{-1} \, (\eta f' - f) f''$$

$$-\frac{1}{4} \, U_\infty x^{-1} \, \nu (\eta f' - f) \, [\frac{1}{4} \, x^{-1} \, (\eta^2 f'' + \eta f' - f) + U_\infty \, \nu^{-1} \, f''')]$$

$$- \, U_\infty^2 \, x^{-1} \, f''' = 0 \; ; \quad (6.2.35)$$

This equation when multiplied by (2), (-1), and divided by $(U_\infty^2 \, x^{-1})$ gives the form:

$$f \, f'' + 2 \, f''' + \frac{1}{2} \, \nu (\eta f' - f)[\frac{1}{4} \, x^{-1} \, U_\infty^{-1} \, (\eta^2 f'' + \eta f' - f) + \nu^{-1} \, f''] = 0,$$

$$cm^2 \, sec^{-1} [cm^{-1} \, cm^{-1} \, sec + cm^{-2} \, sec] \approx \text{dimensionless,} \quad (6.2.36)$$

with all the terms being dimensionless. Let us now discuss Equation
(6.2.36). The first two terms in the equation are the terms obtained by
Prandtl and Blasius in 1908. The last term in the equation is a new term
due to the introduction of the term $\vec{U} \times (\nabla \times \vec{U})$ into the flow, see Equations
(6.2.13), (6.2.15), (6.2.16), and (6.2.24). Equation (6.2.36) can be
remodelled into the form:

$$ff'' + 2 f''' + \frac{1}{2} (\eta f' - f) f'' + \frac{1}{8} \nu x^{-1} U_{\infty}^{-1} (\eta f' - f)(\eta^2 f'' + \eta f' - f) = 0.$$
$$(6.2.37)$$

In Equation (6.2.37) the first three terms are functions of the dimensionless
coordinate η only. The function $f \equiv f(\eta)$ denotes the dimensionless stream
function.

From the manner in which Equation (6.2.37) was derived, it is
obvious that not only $(ff'' + 2 f''' = 0)$; i.e., the sum of two terms repre-
senting the original Prandtl-Blasius equation is equal to zero, but also
that the last two terms when summed as in Equation (6.2.37) must be equal
to zero (they are equal to: $v \omega_z$). In Equation (6.2.24) one has:

$$v \omega_z = v \left[\frac{1}{2} \{ \partial v/\partial x - (\partial u/\partial y) \} \right] = \frac{1}{2} v(\partial v/\partial x) - \frac{1}{2} v(\partial u/\partial y); \quad (6.2.38)$$

$$\frac{1}{2} v (\partial v/\partial x) = - \frac{1}{16} \nu U_{\infty} x^{-2} (\eta f' - f)(\eta^2 f'' + \eta f' - f), \text{ (cm sec}^{-2}), (6.2.39)$$

which when multiplied by (-2) and divided by $(U_{\infty}^2 x^{-1})$ gives as the result
the last term in Equation (6.2.37) in which the coefficient ($\sim \nu x^{-1} U_{\infty}^{-1}$) is
dimensionless. Analogously:

$$(-\frac{1}{2}) v(\partial u/\partial y) = (-\frac{1}{2}) \left[(\frac{1}{2} \nu^{1/2} U_{\infty}^{1/2} x^{-1/2} (\eta f' - f)) \right] \times$$

$$\left[U_{\infty}^{3/2} \nu^{-1/2} x^{-1/2} f''(\eta) \right] = - \frac{1}{4} U_{\infty}^2 x^{-1} (\eta f' - f) f'', \text{ (cm sec}^{-2}),$$
$$(6.2.40)$$

which when multiplied by (-2) and divided by $(U_{\infty}^2 x^{-1})$ gives, as the result,
the third term in Equation (6.2.37) which is dimensionless. The sum of
the third term and the fourth term, Equation (6.2.38), equal to $v\omega_z$, must
be equal to zero since in Equation (6.2.37) the sum of the first two terms
is equal to zero, $ff'' + 2''' = 0$, (representing the Prandtl-Blasius boundary
layer approach).

The results obtained up to now are summarized:

(a) In 1926, Madelung, applying the Schroedinger equation to fluid dynamic motion, assumed a rotationless flow geometry and obtained Equation (6.2.12) where only the gradient is preserved (Euler equation);

(b) This investigator, assuming that in any real fluid a gradient must always accompany the curl of the velocity vector, preserved the term $\vec{U} \times \text{curl } \vec{U}$, Equation (6.2.13), and obtained from the Schroedinger equation Equation (6.2.23);

(c) In going to classical, viscous flow fluid dynamics, Prandtl and Blasius, assuming a "laminar" flow in the boundary layer where only the gradient terms appear, obtained after certain simplifications the first equation of momentum in the form of Equation (6.2.20) or Equation (6.2.22);

(d) Correspondingly, this investigator, assuming that in addition to the gradient of the velocity vector there appears in the first equation of momentum the curl of the velocity vector, obtained as the final result Equation (6.2.24) and Equation (6.2.37);

(e) In both the above approaches, the final equations are expressed in terms of the dimensionless combined coordinate, η, and of the dimensionless stream function, $f = f(\eta)$, and its ordinary derivatives, $f'(\eta)$, $f''(\eta)$, and $f'''(\eta)$, of the first, second and third order, respectively;

(f) In order to avoid a new and special numerical solution of the highly nonlinear ordinary differential equation analogous to the equation solved numerically by Blasius (1908) and next by Howarth (1938), this investigator, following the idea of successive iterative approximation solutions and assuming that all the solutions obtained in the past are the first initial iterative approximations which are known and given, developed the formalism and the resulting equations to represent the second order iterative approximation.

(g) The developments in (f) above produced the result that the sum of the first two terms in Equation (6.2.37) is equal to zero, and the sum of the last two terms in Equation (6.2.37) is separately and independently equal to zero;

(h) All the terms in Equation (6.2.37) were originated and derived from the dimensional stream function $\psi \equiv \psi_{PB}$, in the form proposed by Prandtl and Blasius (1908) (Schlichting, p. 116);

(i) The sum of the last two terms in Equation (6.2.37) is equal to the single term, $v \, \omega_z$, Equation (6.2.24), with the B.C. preserved in exactly the same form as those in the Prandtl-Blasius form, Equation (6.2.21);

(j) In Equation (6.2.24), both factors in the term $(v \, \omega_z)$ are different from zero, $v \neq 0$, $\omega_z \neq 0$. The vertical velocity component, v, has its value given by Equation (6.2.6). The vorticity vector component, ω_z, is the only component in the vertical direction, z, in the two dimensional flow of the laminar boundary layer;

(k) Using the values of both the velocity components, u and v, calculated by Blasius and tabulated in Schlichting, p. 121, this investigator calculated the values of the vorticity vector component, ω_z, which exists in the Prandtl-Blasius "laminar" boundary layer. The values of the magnitude of ω_z were calculated at 2430 points of the boundary layer domain and were determined to oscillate around the values:

$$\omega_z = -0.1105 \times 10^{-1} \text{ to } \omega_z = -0.4922 \times 10^{-21} \ (\sim \sec^{-1}); \quad (6.2.41)$$

In addition to calculating the value of ω_z, the dimensionless stream function, $f = f(\eta)$ was also used from Schlichting book.

6.3. Operations on the Disturbed Flow

The investigation begins with Equation (6.2.37) rewritten in the form:

$$f \, f'' + 2 \, f''' + (\eta f' - f) \{ \tfrac{1}{8} \, \nu \, U_\infty^{-1} \, x^{-1} (\eta^2 f'' + \eta f' - f) + \tfrac{1}{2} \, f'' \} = 0, \quad (6.3.1)$$

where each term is dimensionless. The first two terms refer to the flow in the "laminar" boundary layer and the last term represents the disturbances superimposed upon the boundary layer. The next step is to find out the disturbances which must be superimposed upon the stream function ψ_{PB} to cause the appearance of the disturbances in the form of the last term in Equation (6.3.1). The same approach is used as in the previous section. The dimensional stream function, ψ_{PB}, is of the form

$$\psi_{PB} = U_\infty^{1/2} \, x^{1/2} \, \nu^{1/2} \, f(\eta), \quad (\approx cm^2 \ \sec^{-1}). \quad (6.3.2)$$

The disturbance superimposed upon the stream function has to result in the form

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$$2 \, v \, \omega_z = v \, (\partial v / \partial x - \partial u / \partial y). \qquad (6.3.3)$$

This was calculated and represented in its full form for the additional term and is equal to

$$(2 \, v \, \omega_z) = \frac{1}{2} \, (\eta f' - f) f'' + \frac{1}{8} \, \nu \, U_\infty^{-1} \, x^{-1} \, (\eta f' - f)(\eta^2 f'' + \eta f' - f), \qquad (6.3.4)$$

which is dimensionless. To make the function (6.3.4) summable with reference to the stream function, ψ_{PB}, Equation (6.3.4) is multiplied by a factor having dimensions $\approx cm^2 \, sec^{-1}$:

$$C_{add}(U_\infty^{1/2} \, \nu^{1/2} \, x^{1/2}) \, ; \, C_{add} = \text{dimensionless constant}, \qquad (6.3.5)$$

to obtain:

$$\psi_{add} = C_{add} \, (U_\infty^{1/2} \, \nu^{1/2} \, x^{1/2} \, [(\tfrac{1}{2})(\eta f' - f) f'' $$
$$+ \frac{1}{8} \, \nu \, U_\infty^{-1} \, x^{-1} \, (\eta f' - f)(\eta^2 f'' + \eta f' - f)], \qquad (6.3.6)$$

or

$$\psi_{add} = C_{add}(U_\infty^{1/2} \, \nu^{1/2} \, x^{1/2}) \frac{1}{2} \, (\eta f' - f) f'' $$
$$+ C_{add} \frac{1}{8} \, (U_\infty^{-1/2} \, \nu^{3/2} x^{-1/2})(\eta f' - f)(\eta^2 f'' + \eta f' - f), $$
$$(\approx cm^2 \, sec^{-1}) \, , \qquad (6.3.7)$$

where each term has dimensions ($\approx cm^2 \, sec^{-1}$). Consequently, the total stream function with the superimposed disturbances has the form:

$$\psi = \psi_{PB} + \psi_{add} = (U_\infty^{1/2} \, \nu^{1/2} \, x^{1/2}) \, \{[f(\eta) + C_{add} \frac{1}{2} (\eta f' - f) f'' $$
$$+ C_{add} \frac{1}{8} (U_\infty^{-1} \, \nu \, x^{-1})(\eta f' - f)(\eta^2 f'' + \eta f' - f)]\} \, , (6.3.8)$$

where:

$$(U_\infty^{1/2} \, \nu^{1/2} \, x^{1/2}) \approx cm^2 \, sec^{-1};$$
$$(U_\infty^{-1} \, \nu \, x^{-1}) \approx \text{dimensionless},$$
$$\eta = y \, (U_\infty \, \nu^{-1})^{1/2} \, x^{-1/2}. \qquad (6.3.9)$$

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All the other terms and factors are dimensionless, including all the terms in $\{[\quad]\}$.

The condition ψ = const., determines the disturbed streamlines. Another version of this condition, ψ = const., is $f(\eta)$ = constant where η = fixed (temporarily):

$$d\psi = 0 \; ; \; d\eta = (\partial\eta/\partial x) \, dx + (\partial\eta/\partial y) \, dy \; . \tag{6.3.10}$$

For each η = fixed, one obtains a family of parabolas. The condition of the variation of the function ψ is

$$\psi = (U_\infty^{1/2} \, \nu^{1/2} \, x^{1/2})[f(\eta) + H(\eta)] = \text{constant}, \tag{6.3.11}$$

$$H(\eta) = C_{add} \left[\frac{1}{2} \, (\eta f' - f)f'' + \frac{1}{8} \, (U_\infty^{-1} \, \nu \, x^{-1})(\eta f' - f)(\eta^2 f'' + \eta f' - f)\right] , \tag{6.3.12}$$

which leads to

$$f(\eta) + H(\eta) = \text{constant}; \; f'(\eta) + H'(\eta) = 0, \; H'(\eta) = \partial H/\partial\eta \; ; \tag{6.3.13}$$

$$H'(\eta) = C_{add}\{\frac{1}{2} \, [\eta f''f'' + (\eta f' - f)f''']$$

$$+ \frac{1}{8} \, (U_\infty^{-1} \, \nu \, x^{-1})[\eta f''(\eta^2 f'' + \eta f' - f) + (\eta f' - f)(3\eta f'' + \eta^2 f''')]\} \tag{6.3.14}$$

Without the loss of generality, the terms containing f''' are omitted due to the fact that values of f''' may not be tabulated and certainly are not tabulated in Schlichting's book.

After the function $H'(\eta)$ is calculated, it is superimposed upon the function $f'(\eta)$ and the sum is automatically plotted from computer calculations. Since one of the goals of the present research is the plotting of diagrams of streamlines in the physical (x, y) plane, there arises the problem of the transfer of the curves $f(\eta)$ and f', with $H(\eta)$ and $H'(\eta)$ superimposed upon them from one-dimensional η-space to the two-dimensional (x, y) space and the plotting of the resulting streamlines. The manner of achieving this is explained next.

Transfer of the Disturbed Streamlines from the η-Space to the (x, y) Space

The approach begins with plotting the curve $u \, U_\infty^{-1} = f'(\eta)$ according to the Prandtl-Blasius laminar boundary layer theory along an infinitely

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long flat plate. On this curve there is superimposed a small disturbance in form of the curve $H'(\eta)$, Equation (6.3.14). Plotting of these curves is done automatically by a mechanical plotter interfaced with the digital computer. It is now possible for the reader to obtain a clear picture of the disturbances superimposed upon the horizontal velocity u, in the boundary layer along an infinitely long flat plate according to Prandtl-Blasius formulation. The dimensionless horizontal velocity component in the boundary layer, $u\, U_\infty^{-1}$, is the most important velocity component in the two-dimensional boundary layer. Following the Prandtl-Blasius idea, the investigation operates in the two-dimensional, plane, Cartesian coordinates (x,y) system, with the two-velocity components, the horizontal component u (or the dimensionless $u\, U_\infty^{-1}$), and the vertical component v (or the dimensionless $v\, U_\infty^{-1}$). The next problem which is confronted is the transfer of the function expressed in the form of Equation (6.3.13), $(f'(\eta) + H'(\eta))$, in the η-space, from the η-space to the two-dimensional Cartesian space having the coordinates (x,y). This is achieved in the following manner where the relation between the composite coordinate, η, and the Cartesian coordinates, x and y, is given in the well-known form:

$$\eta = y\, U_\infty^{1/2}\, \nu^{-1/2}\, x^{-1/2} . \tag{6.3.15}$$

Squaring Equation (6.3.15) furnishes

$$\eta^2 = y^2\, U_\infty\, \nu^{-1}\, x^{-1} . \tag{6.3.16}$$

If on the curve $(H'(\eta) + f'(\eta))$, one selects a point, η = constant, then one obtains from Equation (6.3.16)

$$\text{constant} = y^2\, U_\infty\, \nu^{-1}\, x^{-1} . \tag{6.3.17}$$

This represents a parabola for each η^2 = constant which means that each point η = constant corresponds to a parabola. One can imagine that along the curve

$$f'(\eta) = u\, U_\infty^{-1} \tag{6.3.18}$$

or along the curve representing the disturbances in the horizontal velocity component, u or $u\, U_\infty^{-1}$; i.e., along the curve

$$f'(\eta) + H'(\eta) = 0 , \tag{6.3.19}$$

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there is moving Cartesian coordinates system with its axes located in the standard manner, where x is the horizontal axis and y is the vertical axis. In this moving coordinate system, each point, η = constant, corresponds to more than one parabola. A parabola has one point at infinity, the point of the tangency to a straight line located at infinity. Consequently, a parabola has two branches (so termed by the investigator for purely descriptive purposes). For each point η = constant, one can plot one parabola having both branches directed to the right and for the same point a second parabola having both branches directed to the left. Obviously, there exists a third possibility since the physical behavior of a particle with mass does not need to follow the mathematical principles of analysis. Thus, a moving particle of fluid with mass may at first follow one parabola until it reaches the point η = constant on the curve representing the disturbances where it can change its attitude and rotate due to the presence of the factor (curl $\vec{V}$) in the flow domain, thereby changing its directional pattern by following one and the same streamline or by jumping from one streamline (parabola) to another streamline (second parabola). Both streamlines, parabola one and parabola two, meet at one and the same point, $\rightarrow$ = constant. The change of streamlines, from one parabola to another cannot be visibly noticed. Thus there exists a third "parabola" which has both its branches directed in opposite directions, one of its branches directed to the right, another of its branches directed to the left.

Diagrams present the streamlines plotted at a considerable number of points located along the composite curve $(f'(\eta) + H'(\eta))$. All three possible kinds of parabolas are plotted at each of the selected points, η = constant. Since the curves under consideration, $f'(\eta)$ and $H'(\eta)$, are continuous curves, the reader can imagine three infinitely large classes of parabolas, first, second and third, intersecting each other at ∞^3 number of points. The curves are diagrammed with the horizontal x-axis plotted in the logarithmic scale and the vertical y-axis plotted on the normal length scale. The logarithmic scale enables one to plot the diagrams on an acceptable length of the paper. The continuous curve $f'(\eta)$ on the diagram with the horizontal, dimensionless velocity component, $u\, U_\infty^{-1}$, represents the undisturbed "laminar" flow conditions in the horizontal direction. The curve $H'(\eta)$ represents the disturbances

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superimposed upon the curve $u \ U_\infty^{-1}$ in the η-(one-dimensional) space. The parabolas represent the streamlines in the physical (x,y) space. The reader can easily follow the path of a particle. The main direction of the flow, U_∞, is from the left to the right. If one begins at the point $\eta = 0$, then one can follow the curve $f'(\eta)$ for a short distance and at some point, η = constant, one will enter a parabola. Due to the use of the logarthmic scale the section of a parabola at the beginning, i. e., near the point η = constant, will look like a straight line. At a certain point, when following this parabola, one enters the branch of this parabola, which turns to the side and intersects another parabola. At this point, the particle passes from one parabola to the other. The changing from one parabola to another is associated with the rotation of mass of particles which so characteristic of the physical appearance of the phenomenon of turbulence. The streamlines are plotted with the usual assumption that the flow is $\partial p/\partial x = 0$. Thus, the pressure gradient is zero and the flow consists of some sort of pressureless motion.

The value of the vorticity, calculated in Equation (6.2.33) is not the only one which may be used. Sometimes one may use another equation for the vorticity, where only the dimensionless functions are used.

Dimensionless Stream Function
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As was mentioned before, the concept of the streamlines actually reduces the discussion to that of the dimensionless stream function, $f(\eta)$. More details of this function will now be presented. One has

$$\psi \equiv \psi_{PB} = U_\infty^{1/2} \ \nu^{1/2} \ x^{1/2} \ f(\eta) ; \qquad (6.3.20)$$

$$\eta = y \ U_\infty^{1/2} \ \nu^{-1/2} \ x^{-1/2} \ (cm \ cm^{1/2} \ sec^{-1/2} \ cm^{-1} \ sec^{1/2} \ cm^{-1/2}),$$
$$(6.3.21)$$

where η is dimensionless.

If

$$\partial f/\partial y = (\partial f/\partial \eta)(\partial \eta/\partial y) = f' \ U_\infty^{1/2} \ \nu^{-1/2} \ x^{-1/2} ; \qquad (6.3.22)$$

$$\partial f/\partial x = (\partial f/\partial \eta)(\partial \eta/\partial x) = f'(-\frac{1}{2} \ \eta \ x^{-1}) ; \qquad (6.3.23)$$

then one can denote a dimensionless velocity component by $\vec{V}_d = (u_d, \ v_d)$:

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$$u_d = \partial f / \partial y \; ; \quad v_d = - \partial f / \partial x \; . \tag{6.3.24}$$

One can investigate which fundamental laws are satisfied by the velocity components, as shown above (this is a partial repetition of Section 3.2).

(A) Continuity

with $\partial \eta / \partial y = U_\infty^{1/2} \, \nu^{-1/2} \, x^{-1/2}$; $\partial \eta / \partial x = -\frac{1}{2} \eta \, x^{-1}$; (6.3.25)

the coordinate η is a dimensionless variable. However, all the other arguments (x, y, ν, U_∞) have dimensions and consequently the partial derivatives obtained above have the dimensions

$$\partial \eta / \partial y \approx cm^{1/2} \, sec^{-1/2} \, cm^{-1} \, sec^{1/2} \, cm^{-1/2} \approx cm^{-1} \; ; \tag{6.3.26}$$

$$\partial \eta / \partial x \approx cm^{-1} \; ; \tag{6.3.27}$$

$$u_d = f'(\eta) \, U_\infty^{1/2} \, \nu^{-1/2} \, x^{-1/2}, \quad (\approx cm^{-1}); \tag{6.3.28}$$

$$v_d = \frac{1}{2} \, f'(\eta) \, [\, \eta \, x^{-1} \,] \, , \quad (\approx cm^{-1}); \tag{6.3.29}$$

$$\partial u_d / \partial x = (U_\infty^{1/2} \, \nu^{-1/2} \, x^{-1/2}) \, f''(\partial \eta / \partial x) - \frac{1}{2} \, U_\infty^{1/2} \, \nu^{-1/2} \, x^{-3/2} \, f'$$

$$= - \frac{1}{2} \, U_\infty^{1/2} \, \nu^{-1/2} \, x^{-3/2} \, (f\eta f'' + f'), \quad (\approx cm^{-2}) \; ;$$

$$\partial v_d / \partial y = \frac{1}{2} \, f''(\eta) \, (\eta \, x^{-1}) \, \partial \eta / \partial y + \frac{1}{2} \, f' \, x^{-1} \, \partial \eta / \partial y \tag{6.3.30}$$

$$= \frac{1}{2} \, U_\infty^{1/2} \, \nu^{-1/2} \, x^{-3/2} \, (\eta f'' + f'), \quad (\approx cm^{-2}). \tag{6.3.31}$$

Consequently the continuity equation

$$\text{div} \, \vec{V}_d = \partial u_d / \partial x + \partial v_d / \partial y = 0 \, , \tag{6.3.32}$$

is satisfied by the function $f(\eta)$.

(B) Irrotationality

The vorticity vector in the present case reduces to only one component about the z-axis:

$$\omega_{dz} = \frac{1}{2} \operatorname{curl} \vec{\omega}_d = \frac{1}{2} (\partial v_d / \partial x - \partial u_d / \partial y). \tag{6.3.33}$$

Starting with Equations (6.3.32) and (6.3.30), one gets

$$\partial v_d / \partial x = -\frac{1}{4} f'' \eta^2 x^{-2} - \frac{3}{4} f' \eta x^{-2} = -\frac{1}{4} f'' y^2 U_\infty \nu^{-1} x^{-3}$$

$$-\frac{3}{4} f' y U_\infty^{1/2} \nu^{-1/2} x^{-5/2},$$

$$(\approx cm^{-2}); \tag{6.3.34}$$

$$\partial u_d / \partial y = U_\infty \nu^{-1} x^{-1} f'' (\approx cm^{-2}) \tag{6.3.35}$$

This implies that:

$$\omega_{dz} = \frac{1}{2} (\partial v_d / \partial x - \partial u_d / \partial y) \neq 0 \quad (\approx cm^{-2}), \tag{6.3.36}$$

where both terms have dimensions $\approx cm^{-2}$ and are not equal to zero. The value of ω_{dz} was calculated at 2430 points with the following results:

the values oscillate around the numbers in cm^{-2};

-0.4151×10^{-11}; -0.8896×10^{-22}; -0.6866×10^{-25};
-0.8440×10^{-27}; $\tag{6.3.37}$

-0.8708×10^{-28}; -0.6727×10^{-28}; -0.8937×10^{-29}; $\tag{6.3.38}$

The reader is reminded that the Planck No. is $\sim 10^{-27}$ (!!!). Thus in the laminar boundary layer, there appear vortexes with the values of ω_d of the order of the Planck number. Up to now, this fact was unnoticed and quite clearly was never noticed by Prandtl and his followers.

In general, it is difficult, if not impossible, to locate in the literature the definition of the vortex strength or of the vortex strength distribution. The definition, listed below, is taken from Samaras: "Assume an incompressible vortex-free velocity field without any volume source intensity; for such a source-free vortex velocity field, the volume distribution of vortices may be calculated from the vortex potential $\vec{V}_p$. Samaras assumes that the circumferential velocity vector, $\vec{v}_c$, can be calculated in the following manner:

$$\vec{v}_c = \operatorname{rot} \vec{V}_p ; \tag{6.3.39}$$

$$\vec{w} = \operatorname{rot} \operatorname{rot} \vec{V}_p = \operatorname{rot} \vec{v}_c ; \tag{6.3.40}$$

where the vortex potential $\vec{V}_p$ has dimensions $\approx cm^2 \, sec^{-1}$.

Returning now to the problem of vortex characteristics according to Howarth (p. 158):

$$2\pi \, r \, \vec{v} = K, \quad K \approx cm^2 \, sec^{-1}, \tag{6.3.41}$$

it will be shown below that the variable K is equivalent to the vortex potential $\vec{V}_p$ (Samaras). In Equation (6.3.41) the velocity vector $\vec{v}$ is the speed along a streamline of radius r. The streamlines in two-dimensional vortex flow are circles with their centers on the straight axis, with the speed constant along any streamline. The motion is irrotational if the circulation has the same value, K, along every streamline. Assuming that the circulation is equal to the vortex strength distribution in volume or in area, Equation (6.3.41), and that the values used are those calculated before, one gets the equation for ω_z:

$$\omega_z = \frac{1}{2} \, (\partial v/\partial x - \partial u/\partial y) \,, \tag{6.3.42}$$

or

$$\omega_{dz} = \frac{1}{2} \, (\partial v_d/\partial x - \partial u_d/\partial y) \,, \tag{6.3.43}$$

which may better describe the problem under consideration. One can write

$$\vec{V} = (u, \, v) \equiv (\vec{i} \, u, \, \vec{j} \, v) \,; \tag{6.3.44}$$

$$u = U_\infty \, f'(\eta), \quad (\approx cm \, sec^{-1}) \,; \tag{6.3.45}$$

$$v = \frac{1}{2} \, (\nu^{1/2} \, U_\infty^{1/2} \, x^{-1/2}) \, (\eta f' - f) \quad (\approx cm \, sec^{-1}); \tag{6.3.46}$$

where $(u_d, \, v_d)$ may be substituted for (u, v). Suppose the motion is simply a rotation with angular velocity:

$$\vec{\Omega} = i\omega_1 + \vec{j}\omega_2 + \vec{k}\omega_3 \,; \tag{6.3.47}$$

about an axis through the origin; moreover, let:

$$\vec{R} = \vec{i} \, x + \vec{j} \, y + \vec{k} \, z \,, \tag{6.3.48}$$

be the vector from the origin to a point $P(x, y, z)$; then the velocity at P in such a rotation is:

$$\vec{U} = \vec{\Omega} \times \vec{R} \,; \tag{6.3.49}$$

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$$\vec{U} = \vec{\Omega} \times \vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\,(\omega_2 z - \omega_3 y) + \vec{j}\,(\omega_3 x - \omega_1 z) + \vec{k}\,(\omega_1 y - \omega_2 x). \qquad (6.3.50)$$

The velocity vector $\vec{U}$, Equation (6.3.50), corresponds to the velocity vector $\vec{v}$ in Equation (6.3.41). Using Equation (6.3.40), one can write the vectors $\vec{U}$ and $\vec{v}$ (in the first approximation):

$$\text{rot } \vec{v} \equiv \text{rot } \vec{U}; \quad \vec{U} = \vec{i}\,U_1 + \vec{j}\,U_2 + \vec{k}\,U_3 ; \qquad (6.3.51)$$

$$U_1 = \omega_2 z - \omega_3 y; \quad U_2 = \omega_3 x - \omega_1 z; \quad U_3 = \omega_1 y - \omega_2 x ; \qquad (6.3.52)$$

$$\text{rot } \vec{U} \equiv \nabla \times \vec{U} = \vec{i}\left(\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z}\right) + \vec{j}\left(\frac{\partial U_1}{\partial z} - \frac{\partial U_3}{\partial x}\right) + \vec{k}\left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial y}\right); \qquad (6.3.53)$$

where the vortex strength distribution in volume or in area is denoted by:

$$\frac{1}{2}\,\omega = \text{vortex strength distribution in volume or in area } (\approx \text{sec}^{-1}) . \qquad (6.3.54)$$

In the above derivation one has to assume that $\vec{\Omega}$ is constant (at least temporarily). Then one gets for the boundary layer along an infinitely long flat plate.

$$\nabla \times \vec{U} = \vec{k}\,(\partial v/\partial x - \partial u/\partial y) ;$$

$$\frac{1}{2}\,\omega = C\,\frac{1}{2}\,(\partial v/\partial x - \partial u/\partial y), \quad (\approx \text{sec}^{-1}) ,$$

where C must have dimensions of cm^2 in order to be in agreement with the concept of "circulation K" used in the Howarth sense, Equation (6.3.41). If, in place of the components (u, v) one uses the components (u_d, v_d), then the dimensions of the constant C must be properly adjusted. In the problem under consideration, one can observe that K in Howarth's approach corresponds to $\vec{V}_p$ in Samara's approach;

$$K \approx \vec{V}_p . \qquad (6.3.57)$$

From the discussion presented above, one can notice that the variable

$\frac{1}{2}\vec{w}$, Equation (6.3.54), must be multiplied by the area ($\approx cm^2$) in order for one to be able to incorporate this variable into the idea of the entity termed the vortex potential function, $\vec{V}_p$, ($\approx cm^2 sec^{-1}$). Continuing the present example, it is proposed that the vortex potential function, $\vec{V}_p$, Equation (6.3.57), contains a uniformly distributed vortex strength (a hypothetical case) and that consequently one can write:

$$\vec{V}_p = K = C\frac{1}{2}\vec{w}, \quad C \approx cm^2 \text{ (or any other properly chosen parameter)}, \tag{6.3.58}$$

and that from Equation (6.3.54) and Equation (6.3.58) one gets:

$$2\pi r\,\vec{v} = \frac{1}{2}C\,\vec{w}; \quad C = 1\,x\,cm^2; \quad v \equiv v_d. \tag{6.3.59}$$

Equation (6.3.59) was put on a computer to calculate the value of "r" at 2430 points of the boundary layer. The results are discussed below where the values used are $v \equiv v_d$, $\vec{w} \equiv \vec{\omega}_{dz}$.

The following remark is in order:"The components of the velocity vector, $\vec{U}$, can be evaluated using the quantity Ω, Equation (6.3.50). From Equation (6.3.59) it is seen that components of $\vec{U}$ were assumed to be equal to u and v, and it was further assumed that laminar flow components are composed of small arcs of vortexes (Ω). This was done deliberately, since the investigator believes that "laminar" flow does not exist in viscous fluids. A streamline is composed of small arcs of vortexes existing in any viscous fluid flow, visibly unobservable and unmeasurable by macroscopic test devices. The vortexes are of dimensions approaching to the value of the Planck constant."

A second remark which should be made is as follows: 'Past and present literature contains very little, if any, information on such topics as: calculations, operations, and manipulations dealing with the "vortex potential". In particular, the manner of calculating the circumferential velocity vector from the vortex potential is of primary importance, since the curl and not the gradient is responsible for the main aspects in the appearance of turbulence and for its results in both the microscopic (unobservable) and macroscopic (observable) sense. From the numerical data obtained by the writer, one could observe that the results of the calculations show clearly that the values of r at high values of $\eta = 8.4$ to

8.8, are relatively small. As an example, for η = 8.4 to 8.8, r = -0.9089 x 10^{-15} to r = -0.2292 x 10^{-15}, r in cm. Actually, the increase in the thickness of the boundary layer is due to the vertical motion in the layer, which, in turn, is due to the vorticity and viscosity phenomena. These phenomena become weak at the very large values of η. When all of the phenomena mentioned above become weak, then the graph of the horizontal velocity component, u $\approx$ f', becomes more flat and more horizontal, the radius of the vortexes becomes small and the flow in the boundary layer becomes more laminar."

Summarizing, it has been proved above that,beyond any doubt, "laminar" flow in the "laminar" boundary layer contains invisible vortexes and vortex lines, and the flow is not "laminar" at all. The movement of the layers of the fluid medium takes place by means of a "rolling" of the layers of the medium on invisible "rolls" of vortex lines. This idea is not a new one and the voices of other authors on the rolling phenomenon appearing in layers of a viscous medium have been cited. The (Glauert, p. 100) investigator has attempted to calculate the radii of vortexes, and in general it can be said that it appears that at the front of the boundary layer, the radius in question is very large. The larger the distance (measured along the flat plate from the leading edge of the flat plate in the direction of the flow), the smaller the radii of the vortexes become, dropping down to a value comparable to the Planck constant. This implies that, at the leading edge and at the front of the flat plate and of the boundary layer, the "layers" of the fluid medium are "rolling" on big rolls, on big vortexes in the medium (due to viscosity effects). Intuitively, such a geometric interpretation of the physical phenomenon in question seems to make sense. With increasing distance, when the first impulse of the incoming flow becomes smaller, the boundary layer becomes better organized, the "rollers" become smaller, their number increases when their size de- creases, until finally the radius of the rollers reaches the length comparable to the magnitude of the Planck constant. Since the Planck constant is one of the characteristic quantities in the theory of quantum mechanics (and naturally of wave mechanics),the following physical picture emerges as the final conclusion of the foregoing reasoning. Up to now, the field of fluid dynamics and mechanics was treated from the macroscopic point of view, as a field belonging to the domain of the Newtonian mechanics,

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whereas this research has demonstrated beyond any doubt that the formalism and the operations in the field of the macroscopic fluid dynamics and mechanics based on the Newtonian mechanics stands to profit by employing concepts involving the Planck constant, which means the application of microscopic methods of quantum mechanics. In this technique, a great number of microscopic instrumentation and test devices are urgently needed for application to fluid dynamics.

To be better acquainted with the influence of the disturbances upon the stream lines (parabolas in the present case), various cases of disturbances are considered below. These cases are put in form of plots included in this report. The problem of disturbances (variations) superimposed upon the function $f'(\eta)$ was attacked and the results plotted exactly according to the results obtained by Prandtl-Blasius and tabulated in Schlichting's work:

$$f'(\eta) \quad u \, U_\infty^{-1} ; \qquad \eta = y \, U_\infty^{1/2} \, \nu^{-1/2} \, x^{-1/2} ; \tag{6.3.60}$$

$$H'(\eta) = C_{add} \left\{ \frac{1}{2} \left[\eta \, f''f'' + (\eta f' - f) f''' \right] \right.$$

$$\left. + \frac{1}{8} (U_\infty^{-1} \, \nu x^{-1}) \left[\eta f'' (\eta^2 f'' + \eta f' - f) + (\eta f' - f)(3 \eta f'' + \eta^2 f''') \right] \right\} ; \tag{6.3.61}$$

$$f'(\eta) + H'(\eta) = \text{disturbed } f'(\eta) . \tag{6.3.62}$$

Without a loss of generality the terms containing f''' are neglected due to the fact that f''' may not be tabulated (is not tabulated in Schlichting's work).

The function $H'(\eta)$ is calculated and its value is superimposed upon the function $f'(\eta)$, and the sum is automatically plotted. The constant C_{add} is a parametric constant.

The parabolas plotted in the diagrams discussed above actually appear at each point of the curve $H'(\eta)$ or at each point of the sum of curves, $f'(\eta) + H'(\eta)$. The parabolas appear, theoretically, at ∞ number of points of the curve $[f'(\eta) + H'(\eta)]$. They are visibly unobservable but the streamlines, after perturbations, even in stationary (steady) conditions, intersect one another. The number of points of intersections in the disturbed laminar flow and in the stationary conditions is very large. When a fluid particle moves along one and the same streamline and next meets another streamline, it must adjust itself to the conditions prevailing on the "other" streamline. Among other things, it must adjust itself to the

direction of the velocity vector existing along the "new" streamline. This requires a change in the direction of the motion of the fluid particle (in the stationary conditions). As a result, at the points of intersections of streamlines, there appears a rotation phenomenon which is one of the most characteristic, if not the most characteristic physical feature of all the phenomena accompanying the physical appearance of the disturbance such as turbulence. On a film of turbulence one can clearly see the particle clusters of the fluid medium which move and rotate.

6.4. Disturbance in the Thermal Boundary Layer

This section discusses the problem of the disturbances in the thermal boundary layer. The concept of the dimensionless temperature distribution presented non-dimensionally in the equation below serves as the starting point (see Thermal Boundary Layers in Forced Flow, Schlichting, p. 311):

$$(T - T_\infty)(T_w - T_\infty)^{-1} = [\, 1 - \frac{1}{2} E \cdot b(P)]\theta_1(\eta, P) + \frac{1}{2} E\, \theta_2(\eta, P); \tag{6.4.1}$$

for illustrative purposes, various approximative assumptions were already made in Equation (6.4.1). Consequently, the use of various approximations seems to be fully justified. In Equation (6.4.1) the following approximation values of the product are applied:

$$E \cdot b = 6, 4, 2, 0, -2, -4 ; \tag{6.4.2}$$

as taken from Schlichting's book, p. 316. The value of the coefficient b is equal to:

$$b(P) = \theta_2(0, P) = \theta_2(0) ; \tag{6.4.3}$$

with $P \equiv 1$; which results in

$$\theta_2(0) = 1 - f'^2(0) = 1 , \quad f'(0) = 0 ; \tag{6.4.4}$$

The chosen value of $P = 1$ represents an approximation. The value of the function $\theta_1(\eta)$ is assumed always to be equal to $(1 - f'(\eta))$ for $P = 1$, even

if the value of P (the Prandtl Number) oscillates and take on values close to 1. The value of the dimensionless Eckert Number appearing in the last term in Equation (6.4.1) can be calculated as

$$E = U_\infty^2 \left[g\, c_p\, (T_w - T_\infty) \right]^{-1} \tag{6.4.5}$$

where the values of the other coefficients are

$$g = 978.05 \text{ cm sec}^{-2} ;$$

$$c_p = \text{specific heat at constant pressure of air} = 0.242.$$

The value of the coefficient c_p is taken from Stewart "Physics," p. 252. As is well-known, the definition of the specific heat is as follows, Stewart, p. 252: "The specific heat of a gas at constant pressure is numerically equal to the number of calories required to change the temperature of 1 gram of the gas 1 degree centigrade."

From the equivalence of the work and heat, it is known that 1 cal = 4.18 x 10^7 ergs, Stewart, p. 247, with the erg defined in the c.g.s. system in the well-known manner as: 1 erg is the work done when a force of 1 dyne is exerted on a body and there is a displacement of 1 centimeter in the direction of the force. British units, Btu(lb deg F)$^{-1}$ can also be used. Without going into too many details of a fundamental nature, the results of the calculations of Equation (6.4.1) are presented where the following value for the coefficient b is taken into account:

$$b = 0.835 \text{ (Schlichting, p. 316, Fig. 14.12); } P = 1. \tag{6.4.6}$$

If one prefers to use the values of the coefficient c_p in terms of Btu units, a short table of the thermal properties of substances as prepared by E. Schmidt as referenced is given in the work of Schlichting, Table 14.1, p. 294. Equation (6.4.1) is expressed in the dimensionless units, since the composite coordinate η is a dimensionless coordinate and the factor-coefficients $(E \cdot b)$ and E are dimensionless parameters. Next, the results of the calculations of Equation (6.4.1) were plotted for various values of the parametric coefficients $(E \cdot b)$, and E, $b = 0.835$, $P = 1$. These curves are denoted as the "thermal laminar flow or undisturbed flow curves" and will be subjected as a first step to small disturbances by means of superimposing

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upon them "small disturbances" in form of the first derivative of the function appearing on the right-hand side of Equation (6.4.1). The function under consideration is [Equation (6.4.1) with $(T - T_\infty)(T_w - T_\infty)^{-1} \equiv T(\eta)$] :

$$T(\eta) = [\ 1 - \frac{1}{2} E \cdot b(P)] \theta_1 (\eta, P) + \frac{1}{2} E\, \theta_2 (\eta, P), \qquad (6.4.7)$$

where the symbols used denote

E = Eckert Number from Equation (6.4.5) which is dimensionless);

b = dimensionless coefficient from Equation (6.4.6);

P = Prandtl Number (dimensionless), P = 1;

$$P = \nu\, a^{-1} = \mu\, g\, c_p\, k^{-1}\ ; \qquad (6.4.8)$$

$$\theta_1 (\eta) = (1 - f'(\eta))\ \text{with}\ P = 1\ ; \qquad (6.4.9)$$

$$\theta_2 (\eta) = (1 - f'^2(\eta))\ \text{with}\ P = 1\ . \qquad (6.4.10)$$

Then one gets by differentiation:

$$\theta_1'(\eta) = -f''(\eta)\ \ \text{with}\ P = 1\ ; \qquad (6.4.11)$$

$$\theta_2'(\eta) = -\,2\, f'\, f''\ \ \text{with}\ P = 1\ . \qquad (6.4.12)$$

The coefficients in Equation (6.4.7) are kept unchanged during the process of differentiation and consequently the derivative of the function $T(\eta)$ has the form:

$$T'(\eta) = [\ 1 - \frac{1}{2} E \cdot b(P)]\ \ (-f''(\eta) + \frac{1}{2} E\, [\ -2\, f'\, f''(\eta)]\ , \qquad (6.4.13)$$

with P always being equal to unity. Another form of Equation (6.4.13) is:

$$T'(\eta) = -\, f''(\eta)\{[\ 1 - \frac{1}{2} E \cdot b(P)]\ + \frac{1}{2} E\, 2\, f'(\eta)\ \}\ , \qquad (6.4.14)$$

where P = 1, b = 0.835, E · b takes on the sequence of values in Equation (6.4.2), and E is calculated each time from the assumed and chosen value of the product (E · b). Equation (6.4.7) presents a family of curves plotted as functions of the coordinate η. These are called the "undisturbed" curves on which are superimposed the first order disturbances in the form of Equation (6.4.14).

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Consequently, the final curves which are plotted and enclosed in this report as Appendix are the curves:

$$T_{tot}(\eta) = T(\eta) + T'(\eta) \; ; \tag{6.4.15}$$

$$\text{curve } T(\eta) + \text{disturbance } T'(\eta) = T_{tot}(\eta) \; ; \tag{6.4.16}$$

Equation (6.4.7) + Equation (6.4.14) = Equation (6.4.15).

These curves are presented for various values of the parameters E and b. Each plot represents a curve such as $f'(\eta)$, $T(\eta)$, $T(\eta) + T'(\eta)$, etc., which is always a function of the coordinate η in the η-space. Along these curves (functions of η) a rectangular coordinate system (x,y) moves such that the x-axis is always located horizontally and the y axis is always located vertically. Moreover, the horizontal axis, x, is sometimes plotted on the logarithmic scale whereas the vertical axis, y, is plotted on ordinary (linear) scale. For each point η on any curve in the η-space there corresponds a parabola since $\eta = $ constant $y \, x^{-1/2}$. Each time the parabola is located in the moving Cartesian coordinate system (x,y). These parabolas represent the streamlines in the thermal layer in the physical (x,y) space. Each point on any curve, $T(\eta)$, as a function of the coordinate η, is associated with the parabola in the physical space (x,y), and is located in the moving Cartesian coordinate system in (x,y) space. Obviously, these parabolas, i.e., their branches, intersect each other with the points of intersections clearly seen on the computer plotted diagrams.

In general, one can distinguish three cases of parabolas from

$$\eta = y \, x^{-1/2} \text{ constant } ; \tag{6.4.17}$$

$$\text{constant} = U_\infty^{1/2} \, \nu^{-1/2} \; ; \tag{6.4.18}$$

1st case: $\eta = $ positive; x = positive; $x^{-1/2} = $ positive; y = positive;

2nd case: $\eta = $ positive; $x^{-1/2} = $ negative; y = negative;

3rd case: $\eta = $ positive; one branch of the parabola has:

x = positive; $x^{-1/2} = $ positive; y = positive;

the other branch of the parabola has:

$x^{-1/2} = $ negative; y = negative .

The third case seems to be highly hypothetical but still, physically, it

seems to be possible and even practical. The path, streamline, of a particle consists of two parts of a parabola; one branch of the parabola is directed in one direction, another branch of the parabola is directed in another direction. When two branches of two different parabolas meet at a vertex which is common to both branches and common to both parabolas, the common vertex is located exactly at the point η, which is the origin of the running coordinate system (x, y) and consequently is located at the point $(0,0)$ of both parabolas. The various branches of various parabolas intersect. The final path of a mass particle is the curve connecting all the intersecting points.

At the points of the intersections of the two parabolas, i.e., the streamlines, the particles of the medium have to pass from one streamline to another. Since, in stationary conditions, the tangent to a streamline at any point gives us the direction of the velocity of the motion of the particle, the points of intersections of two streamlines are equivalent to the physical phenomenon of passing the particle from one streamline to another. Since the intersections usually take place at an angle, which is different from 0° or 180°, the intersections of streamlines (parabolas) causes an abrupt change of the paths of particles as a result of sudden rotation (macroscopically at a point). The rotations of particles are perhaps the most characteristic macroscopic optical phenomena of disturbed (turbulence) flow under physical observation. This report presents a certain number of computer plotted diagrams in the η-space and of the corresponding parabolas in the moving (x, y) Cartesian coordinate systems referring to the cases investigated; i.e., the disturbed flows in the three cases:

(a) boundary layer along an infinitely long flat plate;

(b) thermal boundary layer in forced flow;

(c) flow in the boundary layer around circular cylinder.

6.5. Computer - Analyzed - Geometrical - Graphical Plottering - Step-by-Step - Successive - Iterative - Approximation - Quantum - Theoretic - Method

From the formalistic point of view, the technique used in the present work involve the most modern aspects of the computer-age era: computer -analyzed, geometrical, graphical plottering, step-by-step, successive -iterative - approximation, quantum theoretic methods.

These methods are used in all the problems attacked in the present work, including the problem of C. A. T. A brief summary of the entire formalism and of the particular steps is demonstrated below on the simple problem of Prandtl -Blasius boundary layer flow along an infinitely long flat plate.

(1) As the first point one may discuss the dimensionless variable $\eta = \eta(x, y)$. From the structure of this function one can see that the condition η = constant and $d\eta$ = 0 leads to the relation $y = y(x)$. The constant in the relation η = constant has to be treated as a parametric constant (temporarily fixed). For various values of η one obtains a family of parabolas in the space (plane) of (x, y) coordinates, depending on various values of η = constant.

(2) The function $f = f(\eta)$, representing a function of one (composite) variable, η is given and is available in the tabular form. It is a dimensionless stream function. It is a differentiable function and there exist tables of the derivatives of this function up to the second order, i.e., f, f', f''. Due to the fact that f represents the dimensionless stream function, the condition f = constant represents the family of streamlines, the above constant being a parametric constant, i.e., a step-wise varying constant. In reality, the first derivate, $df/d\eta = f'$, represent the velocity, i.e., $f' = u\, U_\infty^{-1}$. The entire effort of Prandtl and Blasius was spent to get this result. They (wrongly) believed that their effort gave them the "laminar," one-directional flow. These functions are "undisturbed" functions.

(3) The next point is to introduce disturbances superimposed upon the function f. The following function is considered and plotted as function of η:

$$f(\eta);\quad H(\eta)$$

The function of $H(\eta)$ is used as function of η in the intervals of η = 0.2 from η = 0.0 to η = 8.8, with $\nu = 14.9 \times 10^{-6}$ met^2 sec^{-1}, $U_\infty = 200$ km hour^{-1}.

(4) The next point refers to the first derivative of the function f and disturbances superimposed upon f'. This means that the functions f' and H' are plotted as functions of η.   The function f' is a dimensionless horizontal velocity component function and the function H' is a dimensionless disturbance function superimposed upon the velocity

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f'. Since the function f''' was not calculated and tabulated, it was neglected (without a loss of generality).

(5) The next step in the proposed procedure is to pass from the space of the η-variable to the two-dimensional space of two independent variables (x,y). Since $\eta = \eta(x,y)$:

$$\eta = \eta(x,y) = y(U_\infty \, \nu^{-1} \, x^{-1})^{1/2},$$

any point η = fixed (constant), corresponds to a parabola. The following sequence of operations may be proposed:

(6) The curve f'(η) + H'(η) is plotted as a function of η. This operation is done automatically (by the plotter).

(7) A sequence of points is chosen on the plotted curve f'(η) + H'(η) for η having the following values: η = 0.1, 0.2, 0.3, 0.4, and so on, up to η = 8.8.

(8) Keeping η = fixed (temporarily), each η is associated with a parabola passing through the chosen η.

(9) Thus one can visualize a running, (x,y), Cartesian coordinate system along the curve f'(η) + H'(η). The axes of this running coordinate system are directed in the same directions: the x-axis is always directed horizontally, the y-axis is always directed vertically.

(10) These parabolas, passing through each point η-fixed, represent the streamlines in the (x-y) plane inside the boundary layer in question subject to some disturbances due to the superposition of the term containing the factor $\vec{U} \times \nabla \times \vec{U}$.

(11) Each point η = fixed is associated with two parabolas.

(i) one parabola with the corresponding point in infinity is located to the right. This parabola has both its "arms" directed to the right.

(ii) The second parabola with both its "arms" directed to the left.

(12) Both parabolas, discussed above, are plotted for each point of the curve:

$$f'(\eta) + H'(\eta) = f'(\eta) + \text{disturbance function } (\eta)$$

(13) Both parabolas represent the streamlines in the physical space (x-y plane) in the laminar boundary layer in question, in which there were introduced some disturbances due to the superposition

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upon the laminar boundary layer (in Prandtl-Blasius formulation) the additional term $v\,\omega_z$ (full or only a part of it). Intersection points determine the zig-zag pattern.

(14) The flow above the infinitely long flat plate takes place from the left to the right. The streamlines torn to pieces in the running (x-y) Cartesian coordinate system show clearly the zig-zag path of particles which prior were moving along the streamlines in question. The zig-zag path is due to the introduction of the curl $(\vec{U} \times \nabla \times \vec{U})$ into the flow pattern.

(15) The zig-zag path pattern obtained represents (in the first approximation) the first break-through with regard to the geometry of the turbulence phenomenon. One can notice the characteristic sharp corners at the intersections of parabolas which, obviously, exist actually when the particles of the fluid in question pass from one parabola to another: i.e., from one streamline to another. One has to keep in mind that these sharp corners exist in a realistic flow condition (as in the present case). Such sharp corners are, perhaps, the most characteristic features of the phenomenon of turbulence as seen from any oscillogram of turbulence.

(16) The plots of the streamlines are done entirely automatically by the computer and are enclosed in the present report.

(17) The plots use two coordinate systems: The system of η where the horizontal axis η is traced in the normal (linear) scale. The same is true for the vertical axis on which the function $f'(\eta) + H'(\eta)$ is measured; the second coordinate system is the running coordinate system. It is a rectangular (Cartesian) coordinate system with the x-axis directed horizontally and the y-axis directed vertically. This system moves on the $f'(\eta) + H'(\eta)$ curve and at each point of it the value of the η-coordinate is temporarily fixed, to enable one to plot the two parabolas. The horizontal x-axis is sometimes plotted to make it more adaptable to the existing conditions in the logarithmic scale. The vertical axis or the y-axis has the normal, linear scale.

(18) The plots of the streamlines, subject to the small variation (disturbance) refer sometimes only to the part of the circulation vector appearing in the problem; i.e., only $\partial/\partial x(v^2)$.

(19)   Usually the entire vorticity component $\omega_z = (\partial v/\partial x - \partial u/\partial y)$ was taken into account in the present report.

(20)   The results obtained as presented and discussed above, seem to be the first of this kind of break-through in the analytical representation of the phenomenon of turbulence.   The investigator presented his ideas to Professor von Kármán in Paris in June of 1955 while on sabbatical leave.   von Kármán was very much interested in these ideas.   At the time the investigator was unable to pursue these ideas because of the lack of high speed computers.

(21)   The investigator is convinced that the results obtained in the present research is primarily possible through application of the tool of quantum mechanics.   The use of the deterministic, classical mechanics would not allow for as deep a penetration into the fundamentals of the flow of a viscous fluid as has been possible with the application of the quantum field theoretic methods because the wave mechanics equation is a linear equation.

(22)   To justify partly the statement expressed in point (21) above , the author quotes a sentence from a private letter received from Dr. Ottmar Wehrman in Fokker-Werke, West Germany: "For your information I am preparing a paper on quantized vortex shedding and measurement of vortex phenomena behind very small cylinders.   These measurements indicate a critical quantized wave length of 2.78×10^{-2} cm for the vortex street formation."

(23)   The author has indicated many times that his evaluation of the possible radii of the vortices in the laminar boundary layer along an infinitely long flat plate oscillate around a value in the neighborhood of the Planck constant; i.e., around 10^{-27} and are visually unobservable.

(24)   The report includes plots of the curves $f'(\eta)$, $H'(\eta)$, the geometrical sum $[f'(\eta) + H'(\eta)]$, two parabolas at each point η = constant, intersection points of the streamlines (parabolas), and the zig-zag patterns representing, from the physical point of view, the paths of the particles.   From the geometrical point of view the zig-zag path consists of small parts of streamlines located between the intersection points.

Thus a particle moves along one streamline, meets another streamline at the intersection point, rotates at this point, passes from one streamline onto another streamline, moves for a short distance along the second streamline, up to the intersection point with a third streamline, rotates at the intersection point, passes along the third streamline, and so on. Graphically, the zig-zag patterns, obtained in a purely analytical-geometrical-graphical manner in the present report, are similar to oscillograms of turbulence obtained from measurements (windtunnel in the wake of a cylinder and other geometrical forms) obtained in a purely physical manner by means of oscillographs. The first iterative approximation using the analogy between the analysis and geometry on one side and the physical experimental tests on the other, presents very good agreement. The next steps to be discussed below should involve the correlation phenomena between two and more clusters of elementary particles, the results of the interference phenomena due to the inter-particle force action and the resonance phenomena; the results can be superimposed upon one another due to the fact that the systems are linear.

(25) The points (1) to (24), presented above, are the first steps which have to be undertaken by the investigator to solve the type of problems discussed. Due to the linearity of the applied equation and the possibility of summing of any arbitrary number of steps, the main emphasis is put upon the step-by-step successive iteration procedure, computer work, computer-plottering, plots, graphs, graphical representation of the results, and the interpretation of the graphical results. The number of points, (1) to (24), discussed above, is very modest. They serve only as the preparation for the main problem, to be attacked in the future, i.e., the problem of C.A.T. There the number of points will be much greater due to the fact that the writer has to take into account such phenomena like the interference between the clusters grouped around electrons, interference and correlation phenomena among them, resonance phenomena and many other items. Each such an additional phenomenon

has to be expressed in the language of a computer and solved by means of a computer plot. The final result is obtained by simple summing of all the steps on the computer.

7. INTRODUCTORY ELEMENTS OF THE BIFURCATION THEORY

7.1. Mathematical Elements of the Bifurcation Theory

To explain the elements of the bifurcation theory, the author begins with the well-known Blasius equation of the flow in the so-called laminar boundary layer along an infinitely long flat plate, as used in this investigation:

$$f\,f'' + 2f''' = 0; \quad f = f(\eta): \quad \eta = y\,(U_\infty\,\nu^{-1}\,x^{-1})^{1/2}; \tag{7.1.1}$$

boundary conditions are:

$$\eta = 0: \quad f = 0, \ f' = 0; \quad \eta = \infty \quad f' = 1. \tag{7.1.2}$$

Equation (7.1.1) originated from the boundary layer equations in the physical (x, y) plane of the form:

$$u\,\partial u/\partial x + v\,\partial u/\partial y = \nu\,\partial^2 u/\partial y^2; \quad \partial u/\partial x + \partial v/\partial y = 0; \tag{7.1.3}$$

with the boundary conditions:

$$y = 0: \quad u = v = 0; \quad y = \infty \quad u = U_\infty; \tag{7.1.4}$$

the following substitutions and functions were used:
composite dimensionless independent coordinate:

$$\eta = y\,(U_\infty\,\nu^{-1}\,x^{-1})^{1/2}; \tag{7.1.5}$$

stream function $\psi = \psi\,(x, y)$:

$$\psi = (U_\infty\,\nu x)^{1/2}\,f(\eta); \tag{7.1.6}$$

$f(\eta)$ denotes the dimensionless stream function;
the velocity components become:

longitudinal: $\quad u = \partial\psi/\partial y = (\partial\psi/\partial\eta)(\partial\eta/\partial y) = U_\infty f'(\eta); \tag{7.1.7}$

transverse: $\quad v = -\partial\psi/\partial x = \dfrac{1}{2}(U_\infty\nu x^{-1})^{1/2}\,(\eta f' - f). \tag{7.1.8}$

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Equation for the stream function $f(\eta)$, Equation (1.7.1) is an ordinary nonlinear differential equation in the function $f(\eta)$ of the third order. The three-boundary conditions (7.1.2) are, therefore, sufficient to determine the solution completely. Blasius especially looked for the function $f'(\eta)$, Equation (7.1.7), due to the fact that the function $f'(\eta)$ expresses physically in the two-dimensional (x, y) plane (the reader has to keep in mind that the xy-plane is the physical plane of the actual real flow) the horizontal velocity component in the laminar boundary layer along an infinitely long flat plate. Moreover, the three available boundary conditions, Equation (7.1.2) and Equation (7.1.4), refer specifically to the horizontal velocity component, u, usually expressed in the dimensionless form, uU_α^{-1}.

Equation (7.1.1) is a highly nonlinear ordinary equation in the function $f(\eta)$ and its ordinary derivatives; expressed, as it is, in the form of Equation (7.1.1) it obviously may have more than one solution. One of these solutions should satisfy the required boundary conditions given in the forms of Equation (7.1.2) and Equation (7.1.4). Boundary conditions expressed in form of Equation (7.1.4) refer to the physical (x, y) plane of the actual, real, macroscopic flow. The solution which satisfies the boundary condition given in Equation (7.1.2) and in Equation (7.1.4) is the well-known Blasius' solution, $uU_\infty^{-1} = f'(\eta)$. All the other possible solutions of Equation (7.1.1) do not need to satisfy the required boundary conditions expressed in form of Equation (7.1.2) and/or in the form of Equation (7.1.4).

One of the possible solutions of Equation (7.1.1) (not necessarily satisfying the required boundary conditions in the form of Equation (7.1.2)) may be chosen in the form:

$$f_1(\eta) = \eta; \quad f_1'(\eta) = 1; \quad f_1''(\eta) = 0; \quad f_1'''(\eta) = 0; \tag{7.1.9}$$

in such case, Equation (7.1.1) takes the form:

$$f_1(\eta) \times f_1''(\eta) + 2f_1'''(\eta) = \eta \times 0 + 2 \times 0 = 0, \tag{7.1.10}$$

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which is correct and consequently the function $f_1(\eta) = \eta$ satisfies Equation (7.1.1).

Equation (7.1.5) defines the association between the two spaces: (a) the physical space having two coordinate systems, (x, y) and (b) the "intermediary" η-space, having only one coordinate, η, as its independent variable. Thus it may be considered to act as the "transformation equation" between the physical (x, y) space and the intermediary η-space. Equation (7.1.5), as it now stands, contains the square root from the quantity x^{-1}. Under the definition of the "rationalization" process, one means the process of operations which may allow one to get rid of the terms containing square root, third root and similar, briefly, the terms which are not "rational" terms. The "rationalization" procedure, when applied to Equation (7.1.5), is equivalent to the process of squaring of Equation (7.1.5) and results in:

$$\eta^2 = y^2 U_\infty \nu^{-1} x^{-1};\qquad(7.1.11)$$

As a peculiar feature of the nonlinear equations to which class of equations Equation (7.1.1) belongs, one may notice that Equation (7.1.5), after the "rationalization" procedure, does not satisfy Equation (7.1.1). In reality, let:

$$f_2(\eta) = \eta^2; \; f_2'(\eta) = 2\eta; \; f_2''(\eta) = 2; \; f_2'''(\eta) = 0;\qquad(7.1.12)$$

Equation (7.1.1) now takes the form:

$$f_2(\eta) \, f_2''(\eta) + 2 f_2'''(\eta) = \eta^2 \times 2 + 2 \times 0 = 0,\qquad(7.1.13)$$

which obviously cannot be true, unless $\eta = 0$.

The fact that the function $f_1(\eta) = \eta$ satisfies Equation (7.1.1), and the same function after the rationalization procedure, $f_2(\eta) = \eta^2$, does not satisfy the same equation, Equation (7.1.1), is due to the peculiar properties of the nonlinear equation, Equation (7.1.1). As is well-known, due to the lack of the knowledge of the characteristic properties of the nonlinear differential equations (both ordinary and partial), only one statement can be made which is definitely true: if the equation in question is

definitely not a linear one, then it is (and may be called)   nonlinear.

The function ψ was introduced above, called the dimensional stream function $(\nu^{1/2} x^{1/2} U_\infty^{1/2} f(\eta) \approx cm \, sec^{-1/2} \, cm^{1/2} \, cm^{1/2} sec^{-1/2} = cm^2 sec^{-1})$ and the function $f(\eta)$, called the dimensionless stream function. According to the fundamental concepts of the fluid dynamics theory, the condition that a stream function is equal to a constant defines a stream line (or a set of stream lines).   This implies that the following equation must be true:

$$f_1(\eta) = \eta = constant; \text{ or rationalized } \eta^2 = constant; \quad (7.1.14)$$

or

$$\eta^2 = y^2 (U_\infty \nu^{-1}) x^{-1} = constant, \quad (7.1.15)$$

determines a set of stream lines.   Equation (7.1.15), i.e.:

$$y^2 (U_\infty \nu^{-1}) x^{-1} = constant, \quad (7.1.16)$$

in the space of the Cartesian coordinates, (x, y), represent a parabola:

$$y^2 = constant \, U_\infty^{-1} \nu x = 4 \, a \, x; \quad (7.1.17)$$

$$4a = constant \, U_\infty^{-1} \nu; \quad (7.1.18)$$

the vertex of the parabola is at the origin of the coordinate system (x = 0, y = 0); if a > 0, the parabola is opened to the right; if a < 0 it opens to the left.   The parabola opened to the left can be obtained from the parabola opened to the right by a rotation of the parabola by $180°$ around the vertical y-axis.   Assume that a mass particle is located at the origin of the coordinate Cartesian (x, y) system.   The origin of the system is located at any arbitrary point on the Blasius curve, $f'(\eta) = u U_\infty^{-1}$, function of η.   Through the point (x = 0, y = 0) one can trace two parabolas -- one opened to the right and another opened to the left.   Each parabola represents a possible stream line of the mass particle located at the origin.   If a particle at a certain moment would become a subject of small oscillations around the origin of the (x, y) system, it would oscillate on infinitesimal or small finite

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arcs of one of the two possible parabolas--one opened to the right
or another opened to the left. Thus, in general, through any point
on the Blasius curve $f'(\eta) = uU_\infty^{-1}$, chosen temporarily, as the origin
of the Cartesian coordinate system (x, y), there pass three curves:
(a) the Blasius curve, $f'(\eta) = uU_\infty^{-1}$; (b) a parabola (a streamline),
$f_1(\eta) = \eta = $ constant, or $\eta^2 = $ constant, whose vertex is located at the
origin of the (x, y) coordinate system and which is opened to the right;
and (c) a second parabola (streamline) whose vertex is located also
at the origin of the (x, y) coordinate system but which is opened to
the left. All three curves, treated from the analytical point of view
as functions $(f'(\eta) = uU_\infty^{-1}$, $f_1(\eta) = \eta = $ constant, or before the rationali-
zation process, $\eta^2 = y^2 (U_\infty \nu^{-1}) x^{-1} = $ constant), satisfy the fundamental
nonlinear ordinary equation $f f'' + 2 f''' = 0$, or $f_1 f_1'' + 2 f_1''' = 0$,
proposed and derived in 1908 by Blasius.

The above presentation is a good introduction to the necessity
of application at this point the elements of the Bifurcation Theory.
This will occupy the remaining part of this Section (see "Bifurcation
Theory and Nonlinear Eigenvalue Problems" edited by Joseph B. Keller
and Stuart Antman, New York University, Publisher, W. A. Benjamin,
Inc., New York, 1969, pp. XI to XIV).

By a term "a nonlinear eigenvalue problem" one means the
problem of finding appropriate solutions (more than one) of a nonlinear
equation of the form:

$$F (u, \lambda) = 0. \tag{7.1.19}$$

In Equation (7.1.19) the symbol F denotes a nonlinear operator, depending
upon the parameter λ, which operates on the unknown function or vector u.
Obviously, one of the first questions a reader can ask and which has to be
answered is the following one: suppose that the numerical value of the
parameter λ is given, then the question is, does Equation (7.1.19) have
in this case any solution u ? If it does, the next question arises as to
how many solutions it has, and then how this number of solutions varies

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with λ. Of a particular interest in that respect is the process of bifur-cation whereby a given solution of (7.1.19) splits into two or more solutions as λ passes through a critical value λ_o, called a bifurcation point. The main problem is to determine the properties of the solutions and how they depend upon λ.

To illustrate bifurcation, let it consider the linear eigenvalue problem of the form:

$$Lu = \lambda u. \qquad (7.1.20)$$

Here L is a linear operator operating upon vectors u in some normed linear space and λ is a real number (definitions of norm, measure, space, measure space, metric space, measurable space, and others can be found in Paul R. Halmos' "Measure Theory", University of Chicago, D. Van Nostrand Comp., Inc., Princeton, N. J., 1956). Consider, at first, that the parameter λ in Equation (7.1.20) can take any arbitrary value equal to any real number. Then, obviously, for every real number value of λ, there is a solution of Equation (7.1.20) equal to:

$$u = 0. \qquad (7.1.21)$$

Geometrically, this solution can be represented in the following manner: Assume a two-dimensional space and a Cartesian coordinate system-- on the horizontal axis one will measure the values of numerical values of the parameter λ (λ-axis), on the vertical axis one will measure the value of the solution, Equation (7.1.21). Under the term "norm of a vector" (see John W. Dettman, "Mathematical Methods in Physics and Engineering", McGraw-Hill Book Company, New York, 1969, p. 27), one usually means the length of the vector, i.e., for the vector $\vec{u}$:
The norm is:

$$|| \vec{u} || = (\vec{u}, \vec{u})^{1/2} \qquad (7.1.22)$$

The norm of a vector is usually assumed to represent the value of the vector if and when the vector is a solution of an equation such as Equations (7.1.20) and (7.1.22). The author makes a second assumption

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that on the vertical y-axis of the two-dimensional Cartesian coordinate system considered above, one will measure the norm, $\|\vec{u}\|$, of the vector $\vec{u}$ representing the solution of Equation (7.1.20). Figure 1 represents, in general, a two-dimensional space, $(\lambda, \|u\|)$. In the present case, $\|\vec{u}\| = 0$ is a solution. As the next problem, let us assume that there is a sequence of eigenvalues, $\lambda_1, < \lambda_2 < \lambda_3 < \ldots$, and a corresponding sequence of normalized eigenfunctions $u_1, u_2, u_3, \ldots$, such that both sequences satisfy the linear equation, Equation (7.1.20):

$$L u_j = \lambda_j u_j; \quad \|u_j\| = 1; \quad j = 1, 2, 3, \ldots, \tag{7.1.23}$$

If c is any real number, other possible solutions of Equation (7.1.20) are given by:

$$u = c u_j, \quad j = 1, 2, 3, \ldots \tag{7.1.24}$$

The norm of the solution (7.1.23) is clearly $\|u\| = 0$, while the norm of the solution of Equation (7.1.24) is $\|u\| = c$. A graph of the norms of these solutions is shown in Figure 1. As the figure shows, the solution $u = 0$ splits into two branches at each of the eigenvalues λ_j, located on the horizontal λ-axis. Therefore, the points $u = 0$, $\lambda = \lambda_j$, are designated as the "bifurcation points" of the linear problem given in Equation (7.1.20).

By turn, the author may briefly present the main characteristic features of the nonlinear eigenvalue problem, Equation (7.1.19), which has Equation (7.1.20) as its linearization. An illustrative plot of $\|u\|$ versus λ (called the response diagram) is shown on Figure 2. It demonstrates the following behavior:

(i)   The branches emanating from the eigenvalues of the linear problem are curved.

(ii) There may be no branch emanating from an eigenvalue of the linearized problem. This occurs at λ_2 in Figure 2.

(iii) There may be several branches emanating from an eigenvalue of the linearized problem as from λ_3 in Figure 2.

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(iv)   There may be a secondary bifurcation as on the branch from λ_4.

(v)   The branches from distinct eigenvalues of the linearized problem may be connected. This happens with the branch through λ_5 and λ_6.

(vi)   There may be branches that do not emanate from the eigenvalues of the linearized problem, such as the branch C.

Concrete examples of such response diagrams occur throughout the entire field of the bifurcation theory. In each case some but not necessarily all of these phenomena occur. In the diagrams a quantity which can be either positive or negative is often plotted in place of $\| u \|$.

7.2. Application of the Bifurcation Theory

At the present time, one can notice in world literature a tendency to apply very strong methods to understand better and possibly to solve the problem of turbulence in fluids (liquids and gases). Others use the bifurcation theory and other attacks on the nonlinear eigenvalue problems. These are very difficult and little known fields of the modern applied mathematics. As is clearly seen from Section 7.1, the results of the application of the Bifurcation Theory, from the geometrical-graphical point of view, are usually represented in form of loops, multiple points and multiply-connected particle-paths. The multiply-connected paths often have their origins located at one point on the original curve. As an example, the author can mention the curve $f'(\eta) = u\, U_\infty^{-1}$, discussed thoroughly in this report. It represents the horizontal velocity componenet in the laminar boundary layer flow along an infinitely long flat plate. Through any arbitrarily chosen point on this curve there pass two stream-lines, η^2 = constant. Consequently, at any point on the curve $f'(\eta) = uU_\infty^{-1}$ there meet the three curves discussed above.

The parabolas represent the two possible streamlines, $(\eta^2$ = constant), of a mass-point located initially at the chosen point η = constant on the curve $f'(\eta)$. Consequently, the reader may immediately recognize an important characteristic feature of the bifurcation theory, namely the multiple points located on any curve which represents a possible solution
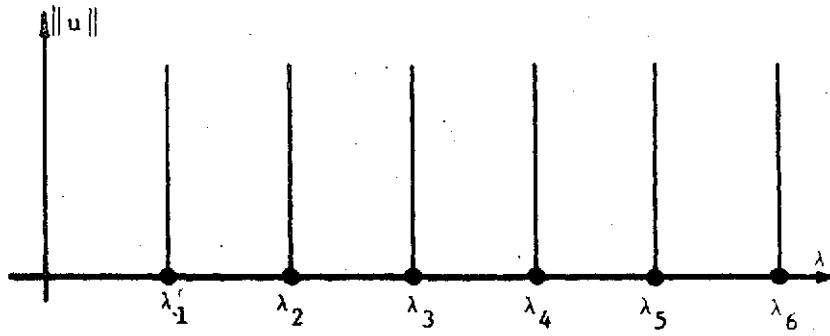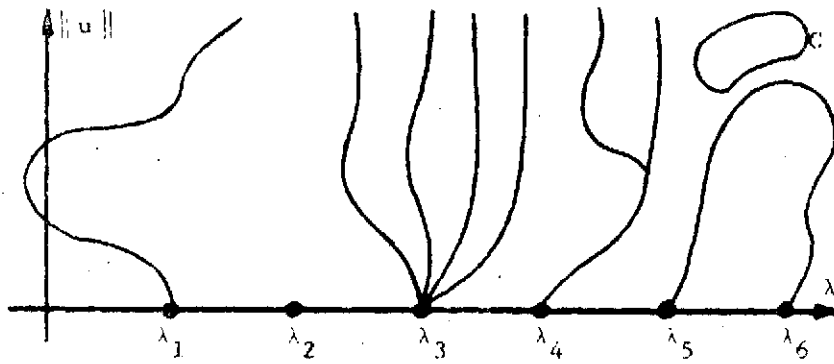
Figure 1.

Figure 2.

of a nonlinear equation. The Blasius equation of the laminar boundary layer along an infinitely long flat plate, $f f'' + 2 f''' = 0$, $f = f(\eta)$, is a highly nonlinear ordinary differential equation in one independent composite variable $\eta = \eta (x, y)$. Consequently. the bifurcation theory may be successfully applied to this equation. The appearance of the multiple points on various curves of the higher order (in the present case of the third order) is obviously a natural phenomenon. The set of streamlines constructed in this work was presented above in detail by means of computer plots. Streamlines traced at various points intersect each other and the pieces of streamlines between the intersection points form zig-zag patterns. The zig-zag patterns represent the final paths of mass particles in the flow under consideration. Due to the intersections of parabolas, there appear the "parabolic loops" in the pattern of the flow. In general, the appearance of various kinds of "loops" like circular, elliptical, parabolic, is one of the most characteristic features of any disturbed flow of gases and liquids. The turbulent flow belongs to the cathegory of disturbed flow systems. The pictures of the surfaces of fluids in the status of turbulence show very clearly the existence of loops of various kinds.

It seems appropriate now for the author to explain in detail the technique used in this research. Equation derived by Blasius, $f f'' + 2 f''' = 0$ is a highly nonlinear equation. The use of very complicated power expansion of infinite series, asymptotic expansion for large values of η, plus a construction of a common joint point of the series (at which both series furnish the same value of $f'(\eta) = u U_{\infty}^{-1}$) for the horizontal velocity component in the laminar boundary layer along an infinitely long flat plate. Of course this solution satisfies the two required boundary conditions in the flow in question, i.e.,

$$\eta = 0: \quad f = 0, \quad f' = 0; \quad \eta = \infty, \quad f' = 1. \tag{7.2.1}$$

I have proposed two other solutions of this equation in form of two parabolas passing through each point located on the Blasius curve, $f'(\eta) = u U_{\infty}^{-1}$:

one parabola opens to the right and another parabola to the left. The two parabolas do not satisfy the required boundary condtions, Equation (7.2.1), but they satisfy the Blasius equation. The parabolas, traced by computer plotter at successive points of the Blasius curve, $f'(\eta) =$ function of η, intersect each other at many points. This gives the origin to the parabolic loops and to the zig-zag pattern which is perfectly analogous to the oscillograms of turbulence in wind tunnel and in the wake of a cylinder. The results obtained by the author with the use of the computer plotter refer to one single moment (with no time interval taken into account).

One may propose a comparison of these results obtained from two sources. In such a manner one can venture a statement that the geometric-graphical zig-zag plots obtained by means of computer plotters (this is the mathematical side of the technique) represent in the first approximation the oscillograms of turbulence (or of a similar disturbed flow domain) obtained by means of oscillographs (this is the physical side of the technique). In such a manner the author succeeded to obtain in a first approximation some sort of agreement between physics and mathematics. The reason why this technique was chosen is simple: at the present status of applied mathematics the writer was unable to apply directly the bifurcation theory in a purely deterministic sense, to find more than one solution of the nonlinear Blasius equation, and to obtain the sought parabolic (or elliptic or circular or any other) loops which are so characteristic for turbulent domain. Solutions which cannot be obtained by means of purely deter-ministic techniques can be approximated by means of probabilistic geometrical-graphical plots obtained by computer-plotters and wave (quantum) mechanics. In case the first approximative solution of the disturbed (turbulent) flow, as described above, would not be sufficiently accurate, one may try to use higher order approximations. One of these would be to describe the zig-zag pattern obtained above by means of Fourier analysis. A different problem is to learn how to read and to understand the zig-zag patterns obtained above and how to associate

them with the physical, natural problems which are under investigation now (like C.A.T.). This will be done separately.

7.3. References to Section 7

For the sake of convenience to the reader, collected below are some references containing information on the subject of the Bifurcation Theory and its application to the theory of turbulence. In particular, one can refer to the following authors, papers and books: Keller and Antman, Marsdeen, Ruelle and Takens, and Vainberg and Trenogin.

8. LAMINAR FLOW

8.1. Macroscopic Laminar Flow

The investigation begins with macroscopic laminar flow, observable by means of macroscopic instrumentation, with the well-known statement of W. Heisenberg (1948) who said that, "contrary to the common belief, it is the so-called macroscopic laminar flow, and not the turbulent flow, which needs a deep scientific investigation." It has been demonstrated in this investigation that (macroscopic) laminar flow is not at all laminar (in the full meaning of this definition). It possesses vortexes, vortex lines and vortex tubes, and is closer to turbulent flow than to laminar flow in the full meaning of these definitions from the macroscopic point of view. Laminar flow possesses vortexes whose radii are very small. Below a certain length of the radius the vortexes in a fluid medium cannot be seen and visually observed.

This indicates that the present day division of the field of fluid dynamics (laminar and turbulent flows) should be abandoned in favor of a more realistic division: macroscopically visible and observable flow and only microscopically observable flow. Below a certain length of the radius, the vortexes cannot be seen by naked human eye. Special techniques and instrumentation must be used in order to notice, to investigate and to test the phenomenon of turbulence. Another aspect which should be mentioned is approach to fluid dynamics from the side

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of wave mechanics. This field usually refers to very small elements and masses like electrons. To make the approach more practical the writer proposes the approximation in the sense that in place of an electron the writer considers a cluster of mass particles gathered around the electron and guided by it. The classical, deterministic fluid dynamics begins with the concept of mass density of a fluid in question, i.e., mass per cm^3 in standard atmospheric conditions on the earth. This is a large amount of mass. If necessary, they take into consideration the rarefied gas domain, where the amount of mass per cm^3 is appropriately smaller. The approach to fluid dynamics with the use of the wave mechanics (often called quantum fluid dynamics) begins with the concept of electron, next passes to the concept of the cluster of particles gathered around the electron. In such a manner one may approach to the concept of mass density by increasing the amount of the mass of the cluster of mass particles gathered around the electron. The main reasons for using the quantum fluid (or wave) mechanics approach and not the deterministic, classical fluid dynamics approach are purely practical. In some problems (but not all) the wave mechanics gives much better results than the deterministic fluid dynamics. The deterministic, classical fluid dynamics uses as the fundamental system of equations either the spatial form of the Euler's momentum equation (for inviscid, ideal fluids) or the Navier-Stokes equation (for viscous and heat-conducting fluids). Both equations are highly nonlinear and thus cannot be solved exactly (excluding some special simple cases). The wave mechanics (or quantum mechanics) approach uses as the fundamental system the Schroedinger wave equation. This is a linear equation. A system of such equations and their results can be summed and/or multiplied by constants.

In the last few decades the quantum theoretic methods in statistical physics became the most successful techniques used in solving many practical problems.

8.2. Association Between Two Domains: Wave Mechanics (Microscopic) and Classical, Deterministic (Macroscopic) Fluid Dynamics

The reader is asked to give special attention to the way of achieving a passage from the micro-domain to the macro-domain. The passage from the microscopic domain of the validity of the quantum theory, of the Planck constant ($\sim 10^{-27}$) to the macroscopic domain of the validity of the ordinary mechanics (classical or even diabatic) where the variables, like the density of the medium are measured, tested and subjected to the observable, macroscopic experimentation is achieved in a direct way. Namely, each, however small it may be, particle of the medium is subjected directly to the action of the external force fields in the same sense as it is accomplished in the ordinary, macroscopic mechanics but through the use of the Avogadro's Law and Number. It states that if N is the number of molecules in the gas, V the volume of the gas, then the ratio N/V has the same value for different gases at the same p and T (pressure and temperature). Other forms of the Avogadro's law are: The number of molecules in a kilomole, namely, the mass in kilograms of the substance which is equal numerically to its molecular weight is constant; or, the number of molecules present in equal volumes of gases at the same pressure and temperature are equal. The number of particles per unit volume can be found from

$$n = (\rho /A) \ Na$$

where:

ρ = density of the gaseous species (kg/m^3)

A = atomic weight; Na = the Avogadro number 60.2472×10^{25} molecules per kilomole.

Other details of this nature are known from primary courses in physics.

Fundamentally, the method applied in this research follows the ideas expressed in "Feynman diagrams". The "basic advantage of the Feynman diagram technique lies in its intuitive character. Operating with one-particle concepts, one can use the technique to determine the structure of any approximation. Next, one can write down the required

expressions with the aid of correspondence rules. These new methods make it possible not only to solve a large number of problems which did not yield to the old formulation of the theory, but also to obtain many new relations of a general character. At present, these are the most powerful and effective methods available in quantum statistics."
(A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics, translated by Richard A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, N. J., p. V, 1964).

The next practical problem to be explained is the problem of a successful extension of quantum theory to the domain of nuclear dimensions. The writer quotes from D. Bohm, "Quantum Theory," Prentice Hall, 1964, p. 627: "We state that quantum theory has actually evolved in such a way that it implies the need for a new concept or the relation between large scale and small scale properties of a given system. Between others, one may discuss two aspects of this new concept: 1. Quantum theory presupposes a classical level and the correctness of classical concepts in describing this level. 2. The classically definite aspects of large scale systems cannot be deduced from the quantum-mechanical relationships of assumed small-scale elements. Instead, classical definiteness and quantum potentialities complement each other in providing a complete description of the system as a whole. Although these ideas are only implicit in the present form of the quantum theory, we wish to suggest here in a speculative way that the successful extension of quantum theory to the domain of nuclear dimensions may perhaps introduce more explicitly the idea that the nature of what can exist at the nuclear level depends to some extent on the macroscopic environment. In this connection it was shown that the definition of small scale properties of a system is possible only as a result of interaction with large scale systems undergoing irreversible processes. In line with the above suggestion, we propose also that irreversible processes taking place in the large scale environment may also have to appear explicitly in the fundamental equations describing phenomena at the nuclear level."

The part of the suggestion of D. Bohm, which was followed by this investigator literally (almost word by word) is underlined. Instead of upgrading the quantum theory fundamental equations and results from the microscopic, quantum level upward to the level of the macroscopic, observable test-measurable fluid dynamics, the investigator down-graded the equations of Prandtl, Blasius (and others), of the Navier-Stokes class of the macroscopic level, origin and nature, observable, test-measurable-character down to the level of the quantum theory of the microscopic nature and character. This was done directly, without creation of any special idea of the quantum theory philosophy. In the notion of the Planck K-Number, which represents some sort of hypothetical generalization of the Planck constant, the most important variable quantity seems to be the density of the fluid medium in question, but expressed in terms of units of gram-mass. The process of the transformation of the density of a medium from one system of units to another (and vice versa) is very simple and was discussed in Section 2.2.

8.3. Step-By-Step Successive Iterative Method

The fundamental philosophy of the method is proposed by John von Neumann in his "Mathematische Grundlagen der Quanten-Mechanik," Springer Verlag, Berlin, 1932. One operates in the finite phase-space with (p, q) coordinates. The phenomena under consideration refer to a medium without any dissipation; no friction, no viscosity, and no heat conduction take place. Thus one treats only the ideal fluid system. The fundamental equation used is the Schroedinger wave equation, the formal mathematics used is the probability theory, the sought results are presented in the form of the mean values. The domain of the operations is a finite one; in the case of a necessity of extending it up to infinity, one could use a conjecture on the extending the John von Neumann's philiosophy, theory and proofs to function spaces and to fields discussed above. Practically, fluid dynamics at this point will give solutions to problems involving the domains of ideal (perfect) fluids. In this domain the gradient and the curl can be treated separately

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and independently, and the results of all of the operations and manipula-
tions can be added due to the concept of linearity.

In the next step, one superposes upon the domain of an ideal fluid
the viscosity and heat conduction (dissipation, internal transfer of
momentum, and of energy) phenomena. This means that there takes
place a transformation of the ideal fluid into a real one, possessing
the coefficients of viscosity, and of heat conductivity; the fluid may be
subject to some chemical reactions, thermal reactions, and so on.
Any time a new phenomenon of the physical nature appears in the ob-
servation, one can use the results from the previous phase of the research
and superpose upon it the new, additional characteristic properties of
the medium in question. To achieve a great precision and accuracy,
one may apply the successive iteration procedure; as it is well-known,
according to the Seidel hypothesis (or Seidel-Gauss) such a process is
a convergent one (in a given interval) or at least an asymptotic one.
At this point, actually, the adiabatic flow of the medium is transferred
into a diabatic flow.

The above method of the successive steps can cover the domains
of an ideal fluid, real fluid, incompressible media, compressible media,
viscous, or heat conductive fluids. Next one can also include subsonic
domain, transonic flow, supersonic flow, or hypersonic domain. The
entire region of the flows in the region of the kinetic theory of gases can
be treated in this manner. To close the list of all of the possibilities in
the field of fluid dynamics, one should mention the flow of electro-
magneto-conducting fluids, plasma fluid dynamics, ion gas dynamics
and others.

The question of validity of the approach to fluid dynamics problems
from the point of view of wave mechanics is discussed very thoroughly.
At certain temperature above T_0 (absolute zero) there appears the
phenomenon of superfluidity, of the mixture of two fluids, and finally
a phase transition at λ -point, below which the superfluid motion is

possible, but above which the superfluid motion is no longer possible and the hydrodynamics of the Bose fluid does not differ from the ordinary hydrodynamics. The definition of small-scale properties of a system is possible only as a result of interaction with large-scale systems which are always undergoing irreversible processes. In line with the suggestion of physicists one may propose that irreversible processes taking place in the large-scale environment may also have to appear explicitly in the fundamental equations describing phenomena at the nuclear level. This fundamental concept is used in this investigation, where the Navier-Stokes equations are expressed by means of the modified form of the wave equation (Schroedinger) with the mass of the electron "m". Next, they are "elevated" from the level of the Planck constant to the level of the Planck K-number which involves the inertial force terms (numerator) in the Reynolds number. The above proposition could be referred to as the quantum-mechanical model of the fluid dynamics with the proper contribution paid to Feynman technique, correspondence rules, and Madelung's proposition.

8.4. Diabatic Flow

The concept of the diabatic flow was developed by the National Aeronautics and Space Administration in the United States during and after the second World War (~ 1946). According to this concept the flow phenomena with heat addition and dissipation can be treated in almost the same manner as the flow phenomena in the classical dissipationless do- mains. By a proper organization of the sequence of events, even the time variation can be taken into account. In each sub-domain one can apply the notion of the friction and of dissipative (negative) forces such as viscosity and heat conduction. Upon treating the dynamic system in question in a manner similar to that in the classical flow system, one passes from one domain to another, and the coefficients (of viscosity and heat conductivity) are changed correspondingly; i.e., they are different from those in the previous domain. This sort of a step-by-step variation of the coefficients and of the analysis by means of the use of the "destruc- tive forces" and of the force potentials allows one to use practically the formalism and the methods of classical mechanics in the domain of

"diabatic" (as contrary to "adiabatic") flow phenomena. The diabatic flow concept will be used primarily in the domains where there appear the friction (viscosity), heat conduction and analogous phenomena. Names associated with the concept of the diabatic flow are those of Hicks, Wasserman, Montgomery and others in N.A.S.A.

More remarks and details on the subject of the association between two domains, quantum mechanics (microscopic) and classical, deterministic (macroscopic) mechanics, including the notion of diabatic flow, are contained below.

8.5. True Nature of the Laminar Flow

Several writers, such as W. Heisenberg, H. Dryden, and others (see list of references, particularly in Dryden's work, 1951), have called attention to the necessity of the investigation of the true nature of the laminar flow in viscous fluids. Quoted below is the opinion of Dryden about Heisenberg's point of view:

The Mechanics of Turbulent Flow

The past few years have seen the introduction of new concepts and considerable progress in studying the mechanics of turbulent flow. The interest taken by such eminent physicists as Heisenberg, Chandrasekhar, Onsager, von Weizsacker, and the great activity of von Kármán, Batchelor, Townsend, and others have been major factors in these developments which can only be sketched in rough outline.

This sketch may be prefaced by a quotation from Heisenberg, as follows:

> A few remarks may be added with regard to the physical picture of turbulence presented in the recent papers. In the earlier years one thought that turbulence was caused by viscosity. This seemed to be true since without viscosity the liquid could theoretically perform all the classical laminar motions, where the liquid glides along the walls; it is only through the viscosity that rotational motions are produced near the walls. At present we know that it is almost the other way round. The liquid without friction is a system with an infinite number of degrees of freedom. It is extremely improbable that only those few degrees of freedom which a laminar motion represents should be excited. As soon as one puts energy into a liquid without

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friction, this energy will be distributed among all degrees of freedom, and what finally results is a certain equilibrium distribution, corresponding to the Maxwellian distribution in gases. It is the viscosity that reduces the number of degrees of freedom, since it damps very quickly all motions in the very small eddies. Therefore only through viscosity is a laminar motion at all possible. Turbulence is an essentially statistical problem of the same type as one meets in statistical mechanics, since it is the problem of distribution of energy among a very large number of degrees of freedom.

This is indeed a refreshing and stimulating reversal of our usual point of view. It is not turbulence but laminar motion that requires explanation.

The writer begins with the conventional approach to the notion of stream function and streamlines in the "laminar" boundary layer along an infinitely long flat plate as formulated by Blasius:

$$\psi = (\nu x\, U_\infty)^{1/2}\, f(\eta) \; ; \qquad \eta = (U_\infty \nu^{-1} x^{-1})^{1/2}\, y \; ; \tag{8.5.1}$$

$$\partial\psi/\partial x = \frac{1}{2}(\nu U_\infty)^{1/2}\, x^{-1/2}\, f(\eta) + (\nu x\, U_\infty)^{1/2}\, f'(\eta)\, \partial\eta/\partial x; \tag{8.5.2}$$

$$\partial\psi/\partial x = \frac{1}{2}(\nu U_\infty x^{-1})^{1/2}\, (f(\eta) - \eta f'(\eta)) = -v, \tag{8.5.3}$$

with:

$$\partial\eta/\partial x = (-\frac{1}{2})\,\eta x^{-1}; \tag{8.5.4}$$

$$\partial\psi/\partial y = (\partial\psi/\partial\eta)(\partial\eta/\partial y) = U_\infty\, f'(\eta) = u \; ; \tag{8.5.5}$$

from the definition of streamlines:

$$\psi = \text{constant}; \quad d\psi = (\partial\psi/\partial x)\, dx + (\partial\psi/\partial y)\, dy = 0 \; ; \tag{8.5.6}$$

one gets:

$$-v\, dx + u\, dy = 0; \quad v/u = dy/dx, \tag{8.5.7}$$

as the equation of streamlines in the "laminar" flow.

As the next step the writer selects various admissible forms of the function $f(\eta)$:

$$\text{(a)} \quad f(\eta) = \eta; \quad \psi = (\nu x\, U_\infty)^{1/2} (U_\infty \nu^{-1} x^{-1})^{1/2}\, y = U_\infty y; \tag{8.5.8}$$

condition for streamlines, ψ = constant:

$$d\psi = (\partial \psi / \partial x) \, dx + (\partial \psi / \partial y) \, dy = 0 + U_\infty \, dy = 0, \qquad (8.5.9)$$

gives, after the integration:

$$U_\infty y = \text{constant}, \qquad (8.5.10)$$

as the equation of the streamlines parallel to the x-axis; the fundamental equation:

$$f f'' + 2 f''' = 0, \quad f = \eta; \; f' = 1; \; f'' = f''' = 0, \qquad (8.5.11)$$

is satisfied; the boundary conditions:

$$\eta = 0: \; f = 0, \; f' = 0; \; \eta = \infty; \; f' = 1, \qquad (8.5.12)$$

may be considered to be satisfied, at least conjecturably;

(b) $f(\eta) = \eta^2$; $\psi = (\nu x U_\infty)^{1/2} (U_\infty \nu^{-1} x^{-1}) y^2 = \text{constant}$; $\qquad (8.5.13)$

$$(\nu x U_\infty)^{1/2} (U_\infty \nu^{-1} x^{-1}) y^2 = U_\infty^{3/2} \nu^{-1/2} x^{-1/2} y^2 = \text{constant}, \quad (8.5.14)$$

represent one possible equation of streamlines. Another possible set of streamlines may be obtained from the condition:

$$\psi = (\nu x U_\infty^{1/2})^{1/2} \eta^2 = C_1 C_2 = C_1 \eta^2 = C_3 = \text{constant};$$

$$(\nu x U_\infty)^{1/2} = C_1 = \text{constant}; \; (U_\infty \nu^{-1} x^{-1}) y^2 = \eta^2 = C_2; \qquad (8.5.15)$$

$$\eta^2 = C_3 / C_1 = (U_\infty \nu^{-1} x^{-1}) y^2 = C_2 = \text{constant};$$

still another possible set of streamlines can be obtained from the squaring of Equation (8.5.14), after rationalization:

$$U_\infty^3 \nu^{-1} x^{-1} y^4 = \text{constant}; \qquad (8.5.16)$$

the writer will concentrate on the last Equation (8.5.15); this is an equation of a parabola of the second degree:

$$y^2 = C_2 U_\infty^{-1} \nu x = 4 \, a \, x; \; C_2 U_\infty^{-1} \nu = 4 a; \qquad (8.5.17)$$

$$a = \frac{1}{4} C_2 U_\infty^{-1} \nu ;$$

this is an equation of a parabola with vertex at the origin of the two-coordinate system (x, y), and focus at (a, 0). The parabola opens to the right if a > 0, and opens to the left if a < 0. Since both C_1 and C_2 may be simultaneously positive or both negative, the parabolas Equation (8.5.17), passing through any point η (or η^2) fixed on the curve $f'(\eta)$ may be opened to the right or to the left. As is already known, the writer traces through each point (η = fixed) two such parabolas, having common vertex at the origin of the two-coordinate (x, y) system (η = temporarily fixed). Let it verify the relation between the function:

$$f(\eta) = \eta^2; \quad f'(\eta) = 2\eta; \quad f''(\eta) = 2; \quad f'''(\eta) = 0, \tag{8.5.18}$$

and the equation:

$$f\,f'' + 2\,f''' = 0; \quad \eta = 0: \quad f = 0; \quad f' = 0; \quad \eta = \infty: f' = 1; \tag{8.5.19}$$

the function η^2 does not satisfy Equation (8.5.19) except at the $\eta = 0$; boundary conditions are satisfied only at $\eta = 0$.

All the groups of streamlines originated at each point (η fixed) located on the curve $f'(\eta)$ form the set of bifurcation curves, and bifurcation loops belonging and attached to the curve $f'(\eta)$. The curve $f'(\eta) = uU_\infty^{-1}$ belongs to the set of bifurcation curves as the base of the points η = fixed. The streamlines of all the groups intersect each other. At the points of intersections the moving particles change the streamlines and pass from one streamline to another. They are subject to a rotation and to change in the direction of motion. An observer may have the impression that they move "aimlessly", but always inside the main stream of flow which follows the direction of the pressure gradient. All these aspects are responsible for the disordered outlook of a turbulent fluid, like the surface of a turbulent fluid spread with aluminium powder (see plate No. 15).

The technique of streamlines with intersection points and with the resulting zig-zag paths is applied by the writer to the well-known laminar flow curve $f'(\eta) = uU_\infty^{-1}$ (Blasius, 1908). The resulting zig-zag

paths are clearly seen on plates No. 21 and No. 31. In plate No. 21, at each point η = fixed (or η^2 = fixed), the parabolas are plotted by the automatic computer-plotter according to the second Equation (8.5.15):

$$(U_\infty \nu^{-1} x^{-1}) y^2 = C_2 \ ; \quad y^2 = C_2 (\nu U_\infty^{-1}) x \ ; \tag{8.5.20}$$

the numerical value of C_2 is assumed to be equal to the numerical value of η^2 at each considered point. This means that C_2 is assumed to be a parametric constant running along the curve $f'(\eta)$ and not an absolute constant.

In plate No. 31, different values in $y^2 = 4\,a\,x$ defining the parabolas, representing the streamlines, are used. The use of Equations (8.5.3), (8.5.5), (8.5.7), and the square of Equation (8.5.7) give the following expression for $(^{-1})^2$, Equation (8.5.7):

$$(vu^{-1})^2 = \frac{1}{4} \ [\eta f'(\eta) - f(\eta)]^2 / \{(xU_\infty \nu^{-1}) \ [f'(\eta)]^2 \} \ , \tag{8.5.21}$$

and similarly, in Equation (8.5.17), the expression for 4a:

$$4a = [\eta f'(\eta) - f(\eta)]^2 / \ \{(U_\infty \nu^{-1}) [f'(\eta)]^2 \} \ , \tag{8.5.22}$$

it can be seen that there are little differences between the zig-zag paths in plates No. 21 and No. 31. The curve $f'(\eta) = uU_\infty^{-1}$ was verified experimentally with the use of macroscopic experimental devices by several outstanding scientists during a period of several decades. The writer's results prove without any doubt that the flow in the boundary layer along an infinitely long flat plate, which is laminar from the macroscopic (observable by naked human eye) point of view, is turbulent from the microscopic point of view (the existence of zig-zag paths). This conjecture is in full agreement with the existence of the vortexes and vortex tubes in the laminar boundary layer. The radii of these vortex tubes are very small. They are of the order of 10^{-20} in the cgs-system. In the laminar flow region the vortex tubes are nicely ordered. The thin laminae of the fluid are moving on "rollers" over one another. Prandt-Tietjens describe this phenomenon in the following manner -- there exist two radically different kinds of flow; considering, for instance, the flow through a

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glass tube, using water in which small particles are suspended, it is seen that most often the particles of fluid do not move in paths parallel to the walls of the tube but flow through in a very irregular manner. Besides the principal motion in the direction of the axis of the tube, secondary motions perpendicular to the axis can be observed. This kind of flow is called "turbulent" flow. The majority of cases of fluid flow are of this kind. There is a certain small velocity at which the individual particles of fluid move regularly in paths parallel to the walls of the tube. This is the second kind of flow referred to, commonly called "laminar flow" (Prandtl-Tietjens, p. 14). In the present section the writer has proven without doubt that the so-called macroscopic, laminar flow is turbulent from the microscopic point of view. Here lies the first partial answer to the Heisenberg point of view. "As soon as one puts energy into a liquid, this energy will be distributed among all degrees of freedom, resulting finally in an equilibrium (Maxwellian) distribution in gases. It is the viscosity that reduces the number of degrees of freedom, since it damps very quickly all motions in the very small eddies. There-fore only through viscosity is a laminar motion at all possible."

From plates No. 21 and No. 31, one can notice that the laminar flow curve $f'(\eta)$ is actually a mean value curve of the zig-zag path traced (hypothetically) by a particle or a cluster of particles. And vice versa -- suppose that a particle or a cluster of particles follows the zig-zag paths clearly seen on plates No. 21 and No. 31. If, in each of these cases, one would trace the mean value path between the valleys and the hills of the zig-zag paths, one would obtain the curve $f'(\eta)$. The mean values have to be calculated according to the rules of the probability calculus. Thus, this reasoning serves again as another partial explanation of the Heisenberg approach to the notion of the macroscopic laminar flow. Namely, in a viscous flow along an infinitely long flat plate there is an infinite number of degrees of freedom. The viscosity reduces the number of degrees of freedom and leaves, as undamped, these oscillating clusters of particles which result as the zig-zag paths seen in Plates No. 21 and

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No. 31. Next, if one would trace the mean value path between the valleys and the hills of the zig-zag path in each of these two cases (following the rules of the probability calculus), one would obtain the curve $f'(\eta)$. Therefore, one could define a laminar motion as a mean value path $f'(\eta)$ of the cluster of particles which follow the zig-zag paths of streamlines (broken to pieces) exactly according to the patterns found in plates No. 21 and No. 31.

The next problem and question which can be proposed in connection with the notion of turbulence and turbulent flow, with Heisenberg's definition of the laminar flow, with an infinite number of degrees of freedom, with the above suggested proposition of the laminar motion as a mean value path of a cluster of particles which follow the zig-zag paths of broken-to-pieces-streamlines, and with others, is the following: Is it possible to propose and to discuss some typical and characteristic cases from the infinite number of degrees of freedom proposed in the Heisenberg's approach? This is an important question since in the phenomenon of C.A.T. there appear to be some cases which can be included into the class of typical degrees of freedom. The answer is a positive one, and below the writer will attempt to list a few of the most typical degrees of freedom. The reference system is assumed to be the classical, three-dimensional, Cartesian system of coordinates (x, y, z).

8.6. Elementary Notions of Particle Kinetics [*]

Before continuing the discussion of the laminar flow, it may be of some value to review the particle kinetics of fluids. The motion of waves and groups of waves (mentioned below) refers to the motion of molecules and atoms. The molecule is made up of atoms; the molecules of water vapor, for example, is a cluster of two hydrogen atoms and one oxygen atom. A substance that contains only atoms with identical chemical properties is called an element, while one built of molecules that contain atoms with different chemical properties is called a compound. Elements

[*] This section may be omitted.

smaller than the atom are the proton, the neutron, the electron, the meson, and the positron. The proton has a positive charge of electricity and a mass as large as that of the hydrogen atom; the neutron has no charge and approximately the same mass as that of hydrogen atom; the electron has a negative charge and a mass about 1/1850 of that of the proton; the meson (sometimes called mesotron) has a charge of the same size as that of the electron, may be either positive or negative, and has a mass about 200 times that of an electron; the positron has a positive charge and seems to have a mass about the same size as that of the electron. All atoms have a dense core, or nucleus, surrounded by electrons. The hydrogen atom, which is the simplest of all atoms, has as its nucleus one proton, with an electron outside it. The more common form of the helium atom has a nucleus with a mass about four times that of the hydrogen nucleus, with two electrons outside the nucleus. Because the nuclei of all atoms contain protons, they are positively charged; in a normal atom there are always enough electrons surrounding the nucleus to make the total negative charge on the electrons equal to the positive charge on the nucleus.

The table included in this paper shows the probable structure of the stable forms of a few of the simpler atoms. The two most important quantities used in comparing different atoms are the mass of an atom and the size of the electric charge on the nucleus. Since protons and neutrons have approximately equal masses, and since the mass of an electron is approximately negligible, the masses of the atoms are approximately proportional to whole numbers. The "mass numbers" given in the table are the nearest whole numbers to the exact value of the atomic masses. In each case, as may be seen in the table, the mass number is equal to the total number of protons and neutrons in the nucleus. Since the positive charge on a proton and the negative charge on an electron are numerically equal, and since the total charge of an atom must be zero, the number of electrons surrounding each nucleus must be equal to the number of protons in the nucleus. The "atomic number" of an element

is equal to the number of electrons or to the number of protons in the atom; it is proportional to the magnitude of the electric charge on the nucleus. In a case where two kinds of elements have equal charges on the nucleus and consequently the same atomic number, the elements have identical chemical properties (which depend solely on the magnitude of the electric charge which atoms carry) and cannot be distinguished by any chemical process. Those atoms which have the same kind of chemical properties and the same atomic number, though different atomic masses, are called "isotopes".

Some most characteristic numbers are:

h = Planck' constant = $\approx 6.55 \times 10^{-27}$; cm gram sec;

m = mass of the electron = 0.9107×10^{-27} gram;

mass of a hydrogen atom = 1.66×10^{-24} gram;

mass of the hydrogen molecule is twice that of the atom;

the number of molecules per cubic centimeter in a gas at atmospheric pressure and $0^\circ C$ is 2.69×10^{19} (see Stewart, p. 191).

When considering and discussing the notion of the velocity, one should refer to the so-called "group velocity" (Bohm, p. 64). The "group velocity" denotes the speed of motion of a group of waves collected together in the form of a packet. A wave packet comprises a group of waves of slightly different wave lengths, with phases and amplitudes so chosen that they interfer "constructively" over a small region of space, outside of which they produce an amplitude that rapidly reduces to zero as a result of a "destructive" interference. Further discussion concerning the fundamental nature of matter and energy is omitted; the notion of waves and of groups of waves mentioned above in the description of a wave packet refers to the motion of atoms.

There is a distinction between the groups velocity and the "phase velocity". The phase velocity is precisely the speed with which a point of constant phase moves when ω and k are defined. The symbols used denote the following:

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k denotes the wave vector

$$\omega = k\,c; \quad c = \text{velocity of light} \tag{8.6.1}$$

$$\text{frequency } f = \omega(2\pi)^{-1} = k\,c\,(2\pi)^{-1} = c\,\lambda^{-1} \tag{8.6.2}$$

$$\lambda = \text{wave length} \equiv L; \quad k^2 = k_x^2 + k_y^2 + k_z^2; \quad \omega = c\,k =$$

$$2\pi c L^{-1}(\ell_x^2 + \ell_y^2 + \ell_z^2)^{1/2} \tag{8.6.3}$$

$$k_x = 2\pi L^{-1}\ell_x; \quad k_y = 2\pi L^{-1}\ell_y; \quad k_z = 2\pi L^{-1}\ell_z \tag{8.6.4}$$

$$\ell_x = L\,k_x(2\pi)^{-1}; \quad \ell_y = L\,k_y(2\pi)^{-1}; \quad \ell_z = L\,k_z(2\pi)^{-1};$$

$$\lambda \equiv L = (\ell_x^2 + \ell_y^2 + \ell_z^2)^{1/2}. \tag{8.6.5}$$

This sketchy information may be sufficient for the reader to com-
prehend the general ideas necessary to select the proper size and con-
sistency of the cluster of elements and molecules which may, and should,
be used in place of the mass of electrons in the Schroedinger equation
when passing from the operations in the micro-fluid dynamics (quantum
fluid dynamics) to the macro- or ordinary classical fluid dynamics.
All the above data are important when considering the true nature of the
so-called laminar flow in the macroscopic domain in the Prandtl-Blasius
sense.

8.7. Possible Degrees of Freedom[*]

The possible degrees of freedom in a system of particles repre-
senting a fluid (liquid or gas) can be divided into the groups listed below.
(1) The first group includes curves of one degree of freedom -- simple
curves like sin and cos, having mean value paths always along one of the
axes (x, y, z); vibrating, string, or oscillating, according to triangular
or similar shapes; and multi-triangular formations having multiple-knot
points. The mean value paths are always straight lines located on one
of the axes (x, y, z). Three-dimensional configurations can be decomposed
into three directions (x, y, z), expressed in the form of separate, single
wave functions, $\psi_1(x)$, $\psi_2(y)$, or $\psi_3(z)$, and the results can be summarized.
Any other oscillatory phenomenon can be expressed geometrically in
an arbitrary form such as rectangle, trapezoid, pentagon, hexagon,

[*] This section may be omitted.

Structure of a Few of the Simpler Atoms

Elements	Atomic Number	Mass Number	Nucleus	Satellites
Hydrogen (two kinds)	1	1	1 protron	1 electron
		2	1 protron, 1 neutron	1 electron
Helium (two kinds)	2	3	2 protrons, 1 neutron	2 electrons
		4	2 protrons, 2 neutrons	2 electrons
Lithium (two kinds)	3	6	3 protrons, 3 neutrons	3 electrons
		7	3 protrons, 4 neutrons	3 electrons
Beryllium	4	9	4 protrons, 5 neutrons	4 electrons
Boron (two kinds)	5	10	5 protrons, 5 neutrons	5 electrons
		11	5 protrons, 6 neutrons	5 electrons
Carbon (two kinds)	6	12	6 protrons, 6 neutrons	6 electrons
		13	6 protrons, 7 neutrons	6 electrons

hexagram, or analogous. These geometrical forms should be located above and below one of the axes, and the mean values should be located on one of the axes. One can propose oscillatory motions constructed in such a manner that the mean values are located on one of the curve-linear axes (segment of a circular arc, say). The general rules of the mean (average) values in the probability calculus are well-known: the average value of the position $\bar{x} = \int_{-\infty}^{\infty} \underline{P}(x) \, x \, dx$, $\underline{P}(x) = \psi^*(x)\psi(x)$; of any function of x: $\bar{f}(x) = \int_{-\infty}^{\infty} \psi^*(x) f(x) \psi(x) \, dx$, $\psi^*(x)$ being complex conjugate; ψ has to be normalized, so that: $\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$, or: $|A|^2 \int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$. A being a suitable constant.

(2) The second group may contain whirls and vortexes of all kinds: circular, elliptical, and spirals, both two- and three-dimensional.

(3) The third group of higher order degrees of freedom may involve resonance phenomena between "n" oscillating clusters, n = 2, 3, ...

(4) The fourth group may involve phenomena due to the intermolecular forces and interference effects between the "n" oscillating clusters.

(5) The fifth group refers to the heat transfer phenomena. Using the analogy between heat transfer, skin friction, and similarity theory, the above cases of the interference and interaction between clusters of particles may be investigated from the point of view of the transverse transfer of momentum (viscosity phenomena) and of the transverse transfer of energy (heat conduction phenomena).

In summary, clusters of particles taking the place of the electron "m" in the Schroedinger equation are subject to various kinds of mechanical-dynamic phenomena. Some are subject to the external force gradient, some are subject to the heat (energy) action, and some are subject to the expansion (density change) phenomena. Analytically, each of the groups discussed above is expressed in terms of the Schroedinger equation (linear), and the results are summed up.

Other writers sometimes divide the shape factors and the intermolecular forces which appear in non-ideal gases into different groups

than those discussed above. Thus chemistry proposes the divisions listed below (as an example see The Properties of Gases and Liquids, Robert C. Reid and Thomas K. Sherwood, McGraw-Hill Book Company, 1966, pp. 46-47):

Shape Factors and Intermolecular Forces. Molecules are not point masses and may have many diverse shapes. Even simple spherical atoms such as argon attract or repel other argon atoms, depending upon the separation distance; in addition, at high pressures, the volume fraction occupied by the atoms themselves may be significant. Molecules behave in a similar way. The fact that real molecules have interacting forces and finite molecular volumes is usually the most important reason why the ideal gas law, $pV = RT$, is not obeyed.

Electrical Forces. Electrical forces of a permanent kind are usually important only in so-called "polar" molecules. Nonpermanent electrical forces, such as electric moments arising from the short-lived perturbation of the electron positions during a collision or near collision, are included above and are present in all real systems. These forces are included in the potential-energy relationships between molecules.

Electrical forces, excluding hydrogen bonding, depend upon the molecular dipole moment, the quadrupole and higher multipole moments, and the polarizability.

Hydrogen Bonding. Hydrogen bonding between molecules usually occurs in systems of a polar nature. Such forces are often considered in the same category as electrical forces.

Quantum Effects. Quantum effects are important only in those molecules where the translational-energy modes must be quantized; such molecules as H_2, He, and Ne at low temperatures are the only ones which have significant quantum effects.

These four effects must be considered when methods are proposed to describe the variation of the P-V-T behavior of a real gas from that of an ideal gas.

In the particle kinetics of plasmas, the polarization force appears

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during a collision of an electron and a neutral particle.

It should be realized that the polarization force is only one of the many forces which act during a collision of an electron and a neutral particle. Other forces which can be significant are listed as follows. The value of η in $F \sim r^{-\eta-1}$ is also given when applicable.

(1) Repulsive polarization force, $\eta = 4$, proportional to r^{-5}. To allow for a screening distance r_p, we write a more correct polarization potential as $V = -A/(r_p^2 + r^2)^2$.

(2) Attractive Coulomb force between electron and atomic nucleus, $\eta = 1$, proportional to r^{-2}.

(3) Repulsive force of the atomic electrons screening the nucleus -- not a simple power dependence. This can be obtained from calculations by Hartree using his expression for the atomic potential.

(4) Short-range repulsive exchange force, due to the exclusion (Pauli) principle for particles of like spin. This force prevents the spatial overlapping of the electron shells of the molecule and a free electron.

(5) Other attraction forces may be important, such as the self-dipole moment of polar molecules, where $\eta = 2$, the charge-induced quadrupole term, and the London dispersion-energy term, both with a potential which varies as $\eta = 6$.

In electron scattering by molecules, additional complications are present which do not exist in electron-atom scattering. For molecules, nearly elastic collisions can occur. In these collisions, the molecule may be excited to higher rotational and vibrational states. In this case, the entire spectrum of the target molecule is involved in the calculation. Furthermore, for molecules, the potential is a function of angular direction as well as of distance r, since the potential in general is not spherically symmetric.

9. OTHER POSSIBLE FORMS OF WAVE EQUATION AND PHYSIO-LOGICAL ASPECTS

9.1. Schroedinger Equation

The wave equation is the linear equation, whereas the equations of the classical, deterministic, macroscopic hydrodynamics are usually nonlinear partial differential equations; this aspect seems to be one of the greatest advantages of using the Madelung's idea and of an association of wave (quantum) mechanics with the macrodynamics following Feynman's technique. One kind of the Schroedinger equation is of the form of the amplitude equation

$$\nabla^2 \psi_o + 8\pi^2 m h^{-2}(W-U)\psi_o = 0 ; \tag{9.1.1}$$

$$\psi = \psi_o \exp(i\,2\pi W h^{-1} t) ; \tag{9.1.2}$$

here the following notation is used:

ψ = wave function;

ψ_o = amplitude (function) ;

W = energy of the system;

U = potential energy of the system.

Often, Schroedinger equation may be used in the one-dimensional form:

$$d^2 u/dx^2 + 8\pi^2 m h^{-2}(E-V)\,u = 0; \quad u = u(x) ; \tag{9.1.3}$$

where

$V \equiv U$ = potential energy function;

$$E \equiv W = \text{total energy} = p^2 (2m)^{-1} + V ; \tag{9.1.4}$$

p = momentum,

$p^2 (2m)^{-1}$ = kinetic energy.

The Schroedinger equation may include the time; in this case it becomes (another form of this equation):

$$\nabla^2 \psi - 8\pi^2 m h^{-2} U \psi - i\,4\pi m h^{-1} \frac{\partial \psi}{\partial t} = 0 . \tag{9.1.5}$$

In general, one usually seeks a finite and continuous solution for any equation in the domain in question. In the case of the Schroedinger equation a solution is possibly only for certain values of the total energy W. These values are called the "eigenvalues" of W. The eigenvalues of W are the values of the energy of the system in question, which the system possesses (or occupies) in its "quantum states." These states may be established and confirmed by means of electroscopic devices. To each eigenvalue of the Schroedinger equation there belongs an "eigen-solution" of the Schroedinger equation; such a solution should be "normalized" and multiplied by the "time factor" $\exp(i\, 2\pi\, W\, h^{-1} t)$; put in such a form, the eigensolution--according to Schroedinger (see Madelung, p. 322)--represents the phenomenon under consideration and gives some physical meaning to the mathematical manipulations performed during the operations. For more details the reader is referred to the references on theoretical physics, wave mechanics, quantum mechanics, and so on.

The Schroedinger equation including directly the time-factor term is given in Equation (9.1.1). According to Madelung (see reference) this equation contains solutions of the "first" equation of Schroedinger, Equation (9.1.1), but above that, it also contains all the linear combinations of the latter. This feature is considered by Madelung to be particularly characteristic. If one puts down:

$$\psi = a \, \exp(i\,\beta) , \qquad\qquad (9.1.6)$$

then a solution of Equation (9.1.1) can be taken only in such a form in which only the function β can depend linearly on the time t, i.e., $\beta = \beta(x, y, z, t)$; whereas a solution of the Schroedinger equation in any form may contain both functions a and β depending both upon the time t.

The reader understands very well that it is impossible for the writer to present here all the details referring to the fundamentals of the theory of the wave mechanics in general and of the Schroedinger equation in particular; consequently, with that in mind, the writer will

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discuss below some more details referring to the form of the Schroedinger equation used in the present research. One can get easily general solutions of the Schroedinger equation, as the linear combinations of particular eigen-solutions (see Madelung, p. 324). For example, let us propose:

$$\psi = a \, \exp(i \, \beta) = \psi_1 + \psi_2 \; ; \tag{9.1.7}$$

$$\psi_1 = c_1 \, a_1 \, \exp(i \, \beta_1) \; ; \tag{9.1.8}$$

$$\psi_2 = c_2 \, a_2 \, \exp(i \, \beta_2) \; ; \tag{9.1.9}$$

ψ_1, ψ_2 = are eigensolutions of Equation (9.1.1) which contain the time factors $\exp(i \, 2\pi \, W \, h^{-1} \, t)$; after some operations one gets:

$$a^2 = c_1^2 a_1^2 + c_2^2 a_2^2 + 2 \, c_1 c_2 a_1 a_2 \cos(\beta_2 - \beta_1) \; ; \tag{9.1.10}$$

after the substitution of the function ϕ in the Schroedinger equation:

$$\phi = - h \, \beta \, (2\pi \, m^{-1}) \; ; \tag{9.1.11}$$

and in the formula:

$$\operatorname{div}(a^2 \, \operatorname{grad} \phi) + \partial a^2 / \partial t = 0 \; , \tag{9.1.12}$$

one gets:

$$a^2 \, \operatorname{grad} \beta = c_1^2 a_1^2 \, \operatorname{grad} \beta_1 + c_2^2 a_2^2 \, \operatorname{grad} \beta_2$$
$$+ c_1 c_2 a_1 a_2 \, \operatorname{grad}(\beta_1 + \beta_2) \cos(\beta_1 - \beta_2) \; , \tag{9.1.13}$$

which demonstrates that the density $(a^2 = \rho)$ contains a periodic term which originates from the difference:

$$\nu = (W_1 - W_2) \, h^{-1} \; , \tag{9.1.14}$$

where the subscripts 1 and 2 refer to the two subsystems, assumed above, and denoted by ψ_1 and ψ_2.

The above discussion shows that electron appearing in the Schroedinger equation may be subject simultaneously to two (or more) kinds of mechanical phenomena (like interactions, interferences); each phenomenon can be represented and expressed physically and next analytically by means of

a separate wave function with the proper adjustment of the external

(action) field of the potential energy function; due to the fact that the

Schroedinger equation is a linear equation, Schroedinger equations and

their results can be summed.

The purpose of Madelung's approach was to associate the wave

mechanics of Schroedinger with the classical fluid dynamics in the

macroscopic sense. This can be achieved theoretically. The discussion,

presented below, clearly demonstrates that this can be also achieved

practically. For the simplicity sake, the writer will reduce the con-

siderations below to the case of an incompressible, steady flow

phenomenon. In this case, one obtains from the Schroedinger equation:

$$\nabla^2 \psi - 8\pi^2 m h^{-2} U\psi - i(4\pi m)h^{-1} \partial\psi/\partial t = 0 ; \qquad (9.1.15)$$

with:

$$\psi = a \exp(i\beta), \quad a = a(x,y,z,t) ; \quad \beta = \beta(x,y,z,t) , \qquad (9.1.16)$$

and with

$$\phi = -\beta h (2\pi m)^{-1} ; \qquad (9.1.17)$$

one gets two equations:

$$\text{div}(a^2 \text{ grad } \phi) + \partial a^2/\partial t = 0 ; \quad a^2 = \rho ; \qquad (9.1.18)$$

$$\partial\phi/\partial t + \frac{1}{2}(\text{grad } \phi)^2 + U m^{-1} - a^{-1}\nabla^2 a h^2 (8\pi^2 m^2)^{-1} = 0; \qquad (9.1.19)$$

application of the ∇ operation to Equation (9.1.19) furnishes with:

$$\nabla\phi = \vec{V}(x,y,z) = \text{velocity vector} ; \qquad (9.1.20)$$

$$\nabla(\vec{V}^2) = 2\vec{V} \cdot \nabla\vec{V} + 2\vec{V} \times (\nabla\times\vec{V}) ; \qquad (9.1.21)$$

the form:

$$\partial\vec{V}/\partial t + \frac{1}{2}\text{grad}(\vec{V}^2) + m^{-1}\text{grad } U$$

$$- \text{grad}(a^{-1}\nabla^2 a) h^2 (8\pi^2 m^2)^{-1} = 0 ; \qquad (9.1.22)$$

Madelung assumed that $\nabla \times \vec{V} = 0$ and consequently the remaining terms

in Equation (9.1.21) give the form which appears in the classical

equation of an ideal fluid in the macroscopic scale. A very peculiar

configuration of factors can be seen in Equations (9.1.17) and (9.1.22).

Namely, there are two factors, both very characteristic, which appear in the Schroedinger equation:

m = mass of the electron = ~ 0.9107 x 10^{-27} gr, (gram-mass);

h = ~ 6.625 x 10^{-27}, (gr-cm-sec) (gram-mass) (or ~6.54 x 10^{-27});

due to their smallness, they emphasize the microscopic aspects of the wave (quantum) mechanics; by the proper configuration of the factor coefficients in the expressions for the function ϕ, (Equation (9.1.17), and in the last term of Equation (9.1.22) one can notice that the influence of the smallness of these two factors vanishes; namely, the ratio of $(h\,m^{-1})$ in Equation (9.1.17) is equal to a constant $(6.625 \times 0.9107^{-1})$ and $h^2 (m^2)^{-1}$ in Equation (9.1.22) is equal to the above constant squared. Put in such a form, the Schroedinger wave equation can be applied to the description of the flow phenomena in the fully macroscopic domain. The only difference is that as the classical fluid dynamics operates in and deals with the classical, deterministic mathematics, the wave mechanics operates in and deals with the probability calculus (and statistics, particularly when the problem of the collection of data is important). The last term in Equation (9.1.22), $(a^{-1}\nabla^2 a)\,h^2\,(8\pi^2 m^2)^{-1}$ is related by Madelung to the well-known term in the classical fluid dynamics of gases, expressing the action of the static pressure, i.e.:

$$a^{-1}\nabla^2 a h^2 (8\pi^2 m^2)^{-1} \quad \rightarrow \quad \int \rho^{-1}\,dp \,. \qquad (9.1.23)$$

In the case of the stationary conditions the first term in Equation (9.1.22) can be neglected; in the case of an incompressible fluid medium, the last term in Equation (9.1.22) can be neglected.

The entire procedure discussed and explained in the present section refers actually to an ideal fluid, i.e., a fluid in which there do not appear phenomena due to the presence of the viscosity (transverse transport of momentum) and of the heat conductivity (transverse transport of energy); both transport phenomena take place on a macroscopic scale

by the assumption, accepted by the writer (diabatic flow).

All fluids (both gases and liquids) have some coefficients of viscosity (and heat conductivity) different from zero; there do not exist real fluids at the temperatures above the λ -point (at which there occurs a phase transition in helium, say) which do not possess the coefficient of viscosity and/or the coefficient of conductivity different from zero; this causes the appearance of the dissipation phenomena; in turn this causes the necessity of application of the idea of the "diabatic" flow or a flow with heat addition or subtraction; this notion was discussed already in some previous sections of this project.

The writer does not intend to go very thoroughly into the discussion of "hidden variables" introduced by the late John von Neumann (see references). It is enough to mention that the literature on "hidden variables" increases every year and the reader is referred to the open literature on the subject. This item was already mentioned in previous sections of this project.

9.2. Physiological Aspects in Fluid Dynamics (Turbulence in Particular) and True Role of Reynolds Number [*]

Typical values of the coefficients of kinematic viscosity are:

water at T = 20°C: $\nu = 1.01 \times 10^{-6}$ me^2/sec; (9.2.1)

air : $\nu = 14.9 \times 10^{-6}$ me^2/sec; (9.2.2)

according to the tests in physiology, the visual acuity of human eye is associated with the angle of one minute. A typical Re Number associated with these values can be calculated in the following manner (for the illustrative purposes). Let

$U_\infty \approx 1$ cm/sec;

$\nu \approx 10^{-7}$ cm^2/sec; (air); $\approx 10^{-6}$ cm^2/sec (water);

a radian is the angle at the center of a circle subtended by an arc equal to the radius, hence

$ds = R\,d\theta$, or $s = R\theta$. (9.2.3)

[*] Thanks are due to Prof. L. Wolterink, Physiol. Dept., MSU, for the data.

One minute corresponds to the value of 0.000291 radians, thus in (cgs)-system, let assume:

$$L = 0.000291 \text{ cm};$$ (9.2.4)

and

$$\text{Re} = U_\infty \times L/\nu = 1 \times 0.000291/10^{-7} = 10^7 \times 0.000291$$

$$\approx 2.91 \times 10^3 \approx 2910 \text{ (for air)}$$

$$\text{or} \approx 291 \text{ (for water)} ;$$ (9.2.5)

the typical values of $\overset{J}{\text{Re}}$, at which the so-called laminar flow in the domain of the deterministic, macroscopic fluid dynamics, breaks down and the turbulent flow appears, is $\text{Re}_{cr} \approx 1000$ to $10,000$. Thus, the values in Equation (9.2.5) lie in the limits given by Reynolds and many other investigators. One can clearly see that the lower limits of the visual observations of flow domains are dictated by the physiological aspects of the human eye and the possibility of visual observations (unless some magnifying instruments are used). It is also clear that the waves in the zig-zag patterns on the plots should be made very dense, much more dense than was done on the present plots. The limitations are due to the physical thickness of the plotter pen and necessary clearance between various parts of the instruments (i.e., the plotter itself and the plotter pen).

9.3. Description of Plots

Included in this section are plots referring to the flow in the boundary layer along an infinitely long flat plate. This will allow one to better see and understand the nature of turbulence, at least in some particular cases.

Plot No. 21: The curve $f'(\eta) = u \, U_\infty^{-1} \, (=UU_\infty^{-1})$ (Blasius 1908), can be distinctly seen on the left hand side of the plot. The coordinate system is as follows: on the horizontal axis there is measured the composite coordinate η (ETA) in the interval from $\eta = 0.0$ to $\eta = 8.8$; on the vertical axis there are measured the values of the function $f'(\eta)$ as the function

of η in the interval from $f'(\eta) = 0.00$ to $f'(\eta) = 1.00$; the resulting curve, $f'(\eta)$ = function of η, is the well-known Blasius' curve. To complete the description, the function uU_∞^{-1} denotes the horizontal velocity component in the "laminar" boundary layer along an infinitely long flat plate in the macroscopic, deterministic fluid dynamics. Next one chooses a number of points, η, on the curve $f'(\eta)$; at each of these points one assumes a fixed coordinate system (x,y). Through each point a parabola, given in Equation (8.5.20), is plotted. These parabolas are very flat, as explained above. The points of intersections of the parabolas (streamlines), connected together, form the zig-zag path of a cluster of particles. The zig-zag paths (unseen by naked human eye in normal conditions), clearly seen on plot No. 21, are done completely automatically by the plotter.

Plot No. 31: This plot is the same as plot No. 21, but the parabolas are plotted according to the formula, $y^2 = 4ax$, Equation (8.5.22). The above two plots prove without doubt that the boundary layer, laminar from the deterministic, macroscopic point of view, is a turbulent one from the microscopic, wave mechanics point of view.

Plate No. 19. Four oscillograms of turbulence in windtunnel and in the wake of a cylinder; these picutres are taken from actual physical tests. They are cited here for the purpose of comparison.

    1a. First from the top: Turbulence in windtunnel, Time 0.4 sec, approximately; Relative Amplification = 64; Wake behind cylinder;

    1b. Second from the top: Center of Wake; $2\frac{1}{2}$" behind 13/16" cylinder; Relative Amplification = 1; Time = 0.3 sec, approximately;

    1c. Third from the top: 13/32" Laterally from center of Wake; $2\frac{1}{2}$" behind 13/16" cylinder; Relative Amplification = 1; Time = 0.3 sec, approximately;

ld. Fourth from the top: $1\frac{1}{2}$" Laterally from center of Wake; $2\frac{1}{2}$" behind 13/16" cylinder; Relative Amplification = 8; Time = 0.3 sec, approximately.

Picture No. 15. Surface of a liquid covered by aluminum powder in a turbulent state.

9.4. Concluding Remarks

It is increasingly obvious that outside purely mechanical aspects (like gradient of pressure) in any fluid system should be included with other characteristic properties when considering a fluid domain-- turbulence in particular: (a) physiological aspects--they may limit the possibility of visual observations by naked human eye; (b) special instrumentation to increase the visual observations, particularly at low and very low Reynolds numbers; this point is difficult since such instrumentation may not always be available; (c) an imperative necessity of including into the analysis the detailed interference and inter- correlation action of inter-particle forces is obvious; this point may be very difficult since all the data, particularly at various altitudes in the atmosphere (troposphere, stratosphere, ozonosphere, ionosphere) might not always be available; and (d) turbulence is a space-time (four dimensional) problem, hence the necessity of a three-dimensional analysis is more than imperative.

10. MODERN TASK OF COMPUTER

10.1. New Tool for Aerodynamists

Recent developments of high-speed digital computers have opened the door to simulating fluid flow by numerical computation. The pro- gress toward the ultimate objective has been limited by the speed-cost characteristics of the computers. The computational process employs a finite difference scheme to solve the partial differential equations of the model of the flow field. In the absence of exact numerical solutions, the wind tunnel has been the aerodynamist's pre-eminent tool for simulating fluid flow. Its importance has increased with the complexity

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of the vehicles being designed and the expanding domain of their flight regimes. Thus, around 100 wind tunnel hours was sufficient for this task when designing the Douglas DC-3 and the Boeing B-17 prototypes; but some 60,000 hours will be needed for the NASA space shuttle program (see References, paper by B. M. Elson). However, tunnels are relatively slow and expensive to operate. Average cost is estimated at around $1,000 per hour. On this basis, the cost of the 40,000 hours of tunnel time planned for the USAF/Rockwell International B-1 program will be in the area of $40 million. Recent expectations are that Illiac 4 and other new super-computers will reduce the time and cost of calculating fluid flow. More importantly, the computers may be able to simulate flows that cannot possibly or practically be simulated in tunnels, including those:

(a)   at flight Reynolds numbers,

(b)   not affected by tunnel walls or model supports, or distorted aeroelastic deflections,

(c)   that account for flight air chemistry, and

(d)   at flight entry velocities for any planetary atmosphere.

There are predictions that Illiac's high speed should enable the computation of flow about two- and three-dimensional shapes with hitherto unattainable levels of approximation. During the 1960s, flow could be computed to only a very crude approximation. Only slender configurations and small angles of attack could be analyzed. Gases had to be treated as perfect gases, behaving according to physical laws that real gases only approximately conform to. The transonic, hypersonic and separated flows could not be computed. Some authors refer to this lowest order of approximation as the inviscid linearized model.

Illiac's speed will enable scientists to predict inviscid nonlinear flow for wing and aircraft of current interest, and to progress to the third stage of approximation, the simulation of time-averaged viscous flow. An approximation of this order, involving viscous flow about a

simple two-dimensional airfoil, was the objective in the recent demonstration by a group in NASA (Ames Research Center). Illiac's results were better than a 13.1 edge over the known 7600 computer. It would have taken several hours on the 7600 computer to get reasonable convergence in the computation, but the same number of problem iterations required only minutes on the Illiac 4. Obviously, the most required simulations in that respect are those about three-dimensional bodies. The pacing item in this work, and the limiting factor in the accuracy achieved, is the mathematical modeling of turbulence, a result of viscosity in gases (Elson, see References). The writer underlines the statements in Elson's description which he feels are very important.

According to Elson, when the time dependent expressions are inserted into the Navier-Stokes equations, the simulation of the realistic, physical, flight phenomena would be complete. But even Illiac 4 operations would require a prohibitive length of time. Possibly, some new computers should now be proposed. Consequently, one could speculate that computers may never eliminate wind tunnel completely. One may propose a sort of cooperation of computers with a wind tunnel. For more details of this nature, the reader is referred to the paper by Elson. Illiac 4 may become a particularly useful source of preliminary design information, and the next section of this report is devoted to this problem. One more item should be mentioned--the reaction of the aerospace companies. Elson states that to date the Illiac program reflects a lack of appreciation of the contribution of computational aerodynamics. This points to a new missionary role of NASA in the United States. Elson quotes F. J. deMeritte (NASA headquarters) as stating a view that the big computers (like Illiac 4 and next generations) may become publicly owned national facilities.

10.2. Present Research as a Part of Modern Tool of Aerodynamics

Some aerodynamists may try to oppose the ideas developed in this research report. Although the nonlinear aspects of the equations of

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fluid dynamics can be taken care of in the present research, nevertheless the approach is not 100 % exact. The tool of probability, of mean values, have to be applied. The writer would like to demonstrate and to emphasize the extremely useful, practical, and profitable way in which the present approach could and, perhaps, should be used. Suppose that a new design is suggested and all the attacks on it (analylical, experimental, wind-tunnel, computer) are comtemplated. Then from the analytical point of view, one can attack the problem according to the ideas developed in this report, including the notion of stream function, streamlines, zig-zag paths, plots of turbulence and the derivation of all the possible mechanical data. This approach will demonstrate, as clearly as the probabilistic approach (with mean values) may permit, all the possible strong, weak, and "neutral" aspects of the design in question. As a matter of fact, the entire spectrum of all the characteristic details and features can be shown to the chief designer and his group. The reader must keep in mind that the wave equation of Schroedinger is used by the writer in a form which is a linear partial differential equation. Consequently, a set of such equations can be summarized and/or multiplied by a constant or a set of constants. One Schroedinger equation can represent only one particular characteristic property of the design in question. Consequently, the entire scheme of design can be represented by a set of seperate Schroedinger equations. Each part of the design has to be designed according to two separate, at first independent, schemes: (1) one scheme will consist of the geometry of the object, drafting boards, draftsmen, small calculators or small computers, a certain number of computer programmers to operate these small computers, and the chief designer and his staff; (2) the second scheme will consist of the numerical analysis of the design, of big computers like Illiac 4, of computer programmers, of physicists and mathematicians, of applied physicists, applied mathematicians and technicians, and the chief analyst and his staff.

The experimental part of the design and the instrumentation devices may not be included in the above two schemes. They may be included in a separate division. Depending upon the decision, the preliminary design may be achieved in group (scheme) 1 or group (scheme) 2, or both. After this moment the above two schemes may stop being completely independent and may become inter-dependent and interfering schemes. One will effect another; one will interfere or interact with (or upon) the other. A geometry of some design detail may effect the numerical and computer analysis. The results may require some changes in the design which, in turn, may require some changes in the numerical-computational analysis. During this analysis, the chief analyst may oscillate between the calculations based upon the wave mechanics (quantum), probabilistic (with mean values) approach on one side (discussed in the present report), and the rigorous, macroscopic, deterministic approach on the other side. The first approach involves the linear systems which can be summarized, thus furnishing directly a spectrum of the influence of all, and of each, particular design element upon the whole, total design of the entire problem in question. Any partial change in a design of a sub-element can be achieved without introducing any changes whatsoever in the design of other particular design elements. This is so, provided that the interference effects (with other design sub-elements) either do not exist or do not appear, or may be neglected or easily solved. The fact is that quite often the interference effects, if taken into consideration, introduce some sort of non-linearity into the system of partial differential system. Consequently, any change in any design of sub-elements can be easily introduced. The new numerical, final result can be directly calculated and added to the entire, total sum of all the other numerical results of all other design sub-elements. The total, entire cost and time of the total design, consisting of n-elements (or n-sub-elements), will be appropriately adjusted without any influence whatsoever upon the designs,

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costs and time of all the other sub-elements. Of course, the inclusion of the interference and interaction effects may often change the entire design, cost and time.

The second approach, presently being used, follows the standard technique involving the nonlinear, macroscopic systems of the classical, deterministic fluid dynamics. The well known Navier-Stokes system of equations is very often used. This is a highly nonlinear system about which modern mathematics can say little. It is not even possible to predict the number of solutions (only more than one) and their possible forms. There are only a few exceptions. In case of some changes in the design of larger elements or of a single sub-element, the entire expensive and time consuming process of the solution of nonlinear system must be repeated from beginning to end. This must be done whenever any other change is done.

In conclusion, it seems that the approach discussed in the present research report should be used, not only in simple cases like the flow in the boundary layer along an infinitely long flat plate or similar and analogous cases, but whenever and wherever there is a possibility of the association of the nonlinear equations of the macroscopic, deterministic fluid dynamics with the linear Schroedinger equation of the wave mechanics. Calculation of the friction drag coefficients may serve as one example. But attention is called to the fact that the present research refers only to the two-dimensional phenomena in steady state. The three-dimensional phenomena and the nonsteady cases are not treated and should be treated in the future. Again, NASA may become the pioneering institution in the sense that it must persuade industry to use the proposition of Elson and Chapman. After the entire spectrum of design is presented in the manner discussed above and adjusted to all the plans and requirements, the best design becomes the model for the wind-tunnel tests.

From the standpoint of physics the phenomenon of turbulence is a four-dimensional, i.e., three-dimensional space plus time.

Consequently, the generalization of the two-dimensional approach, presented above, to three-dimensions is an imperative one. Next, the generalization to the fourth dimension time will be an absolute necessity; even in a short time interval, the turbulence may vary with time, in some instances very strongly and violently. A subdivision of time interval into very small sub-intervals, much smaller than one second, becomes a standard procedure.

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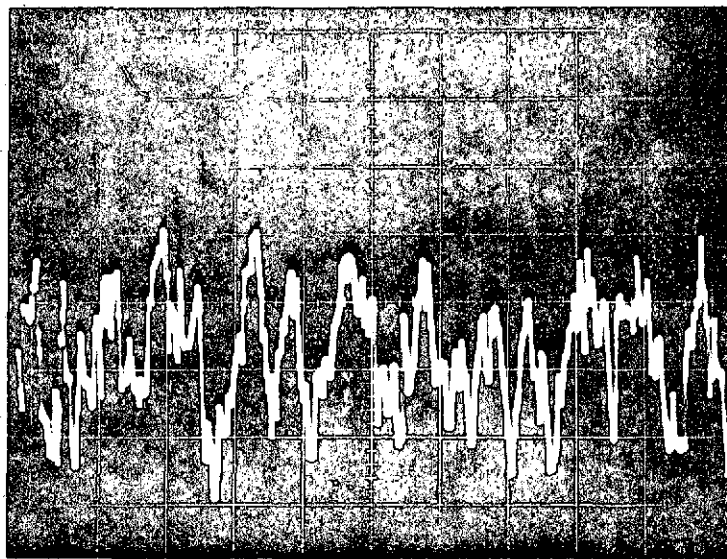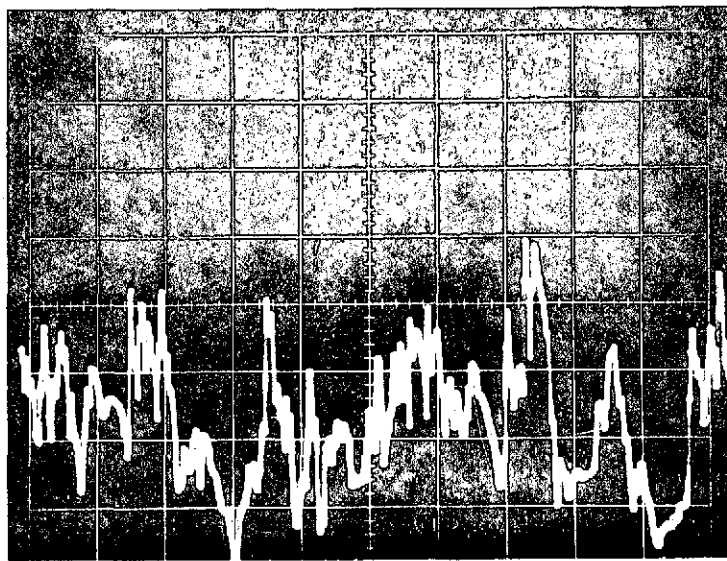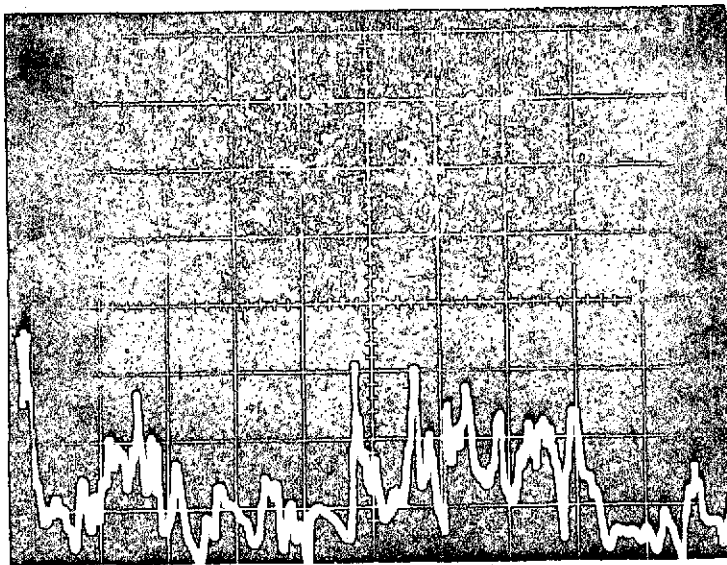Yih, Chia-Shun, Fluid Mechanics, McGraw-Hill Book Company, 1969.

Picture No. 15

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Picture No. 19

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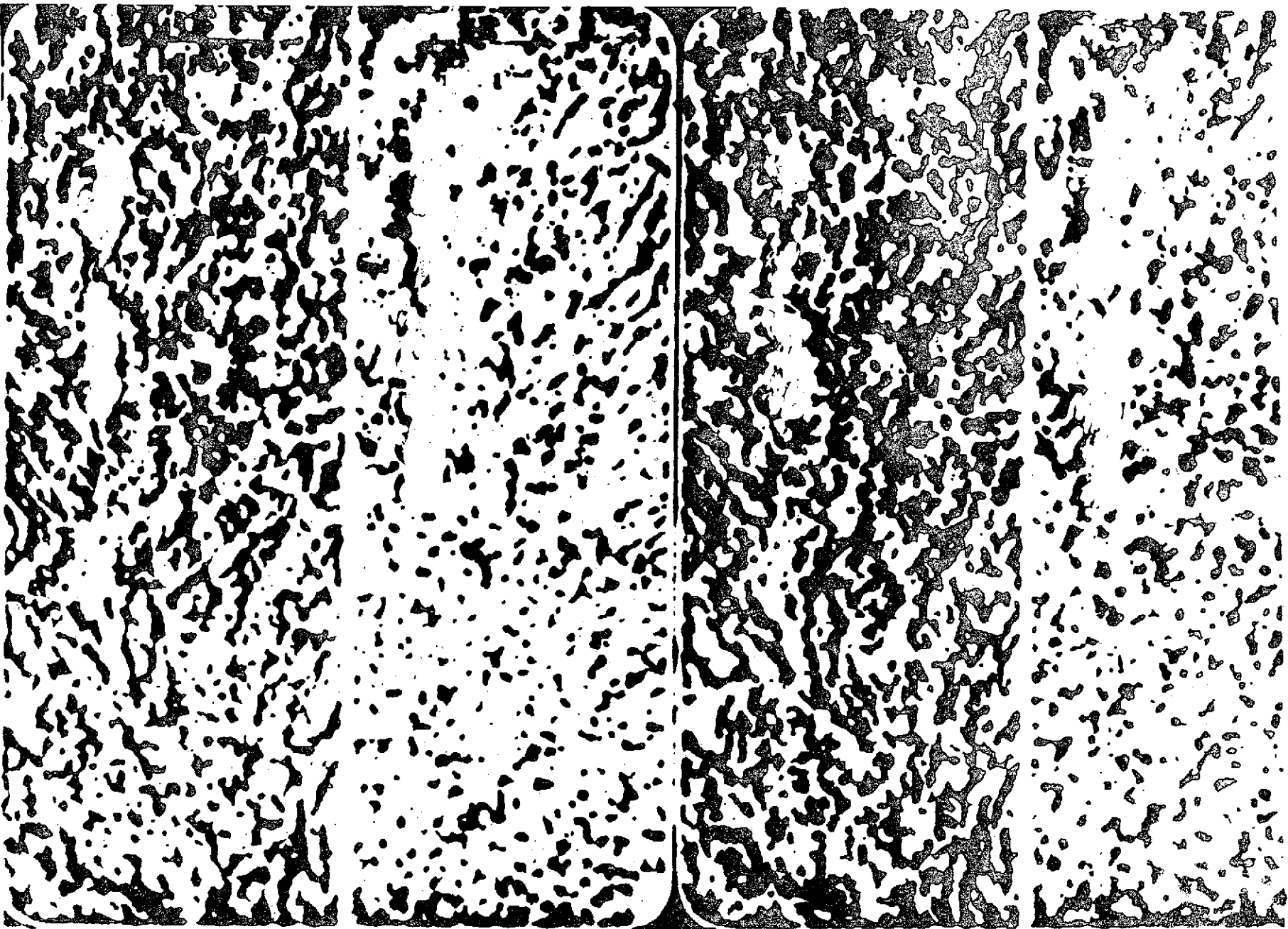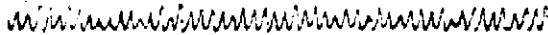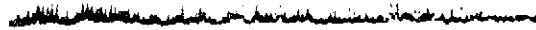
Picture No. 20.   Turbulent free shear layer (composed by Prof. J. Foss).
Umax = 40 f. p. s.

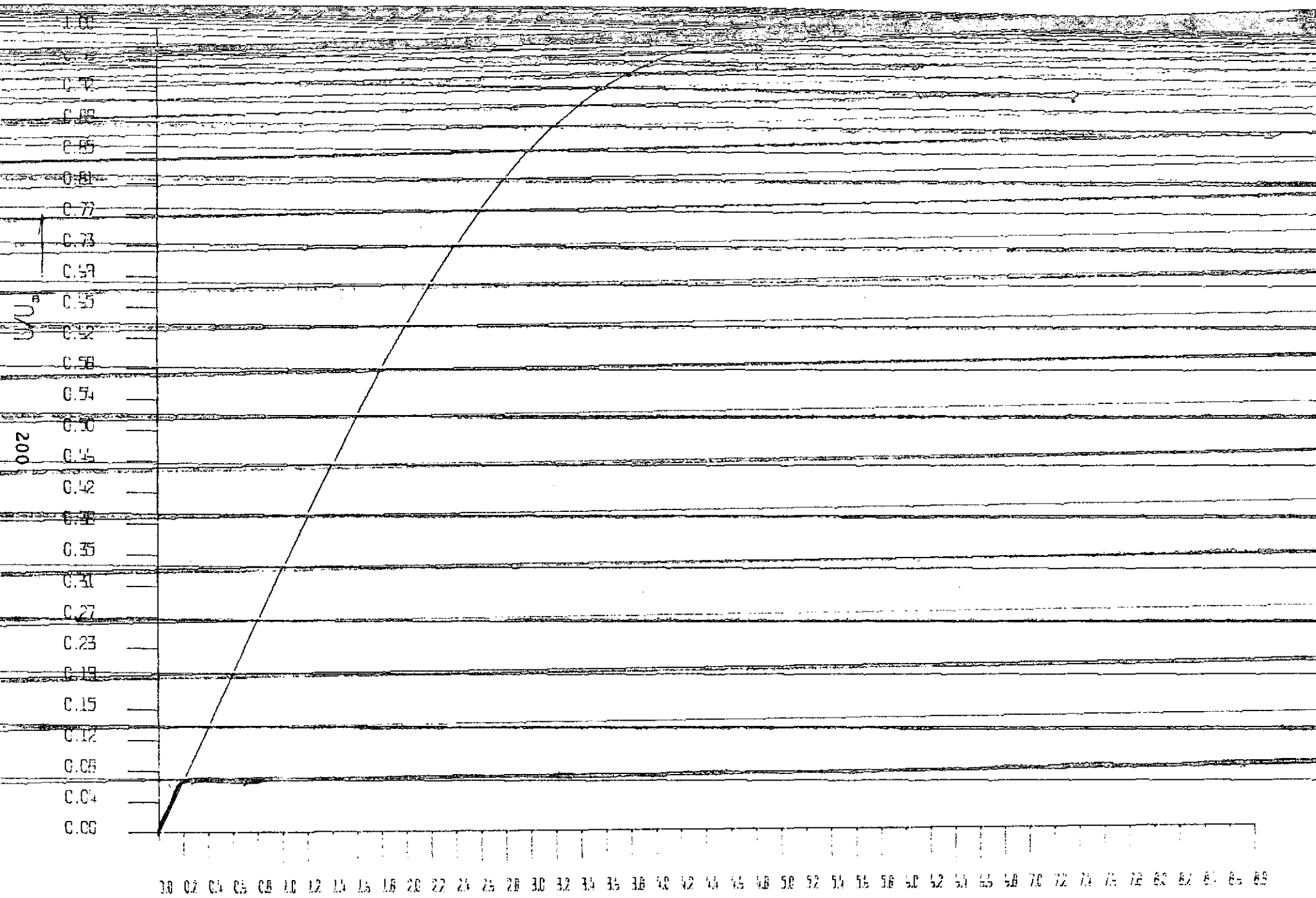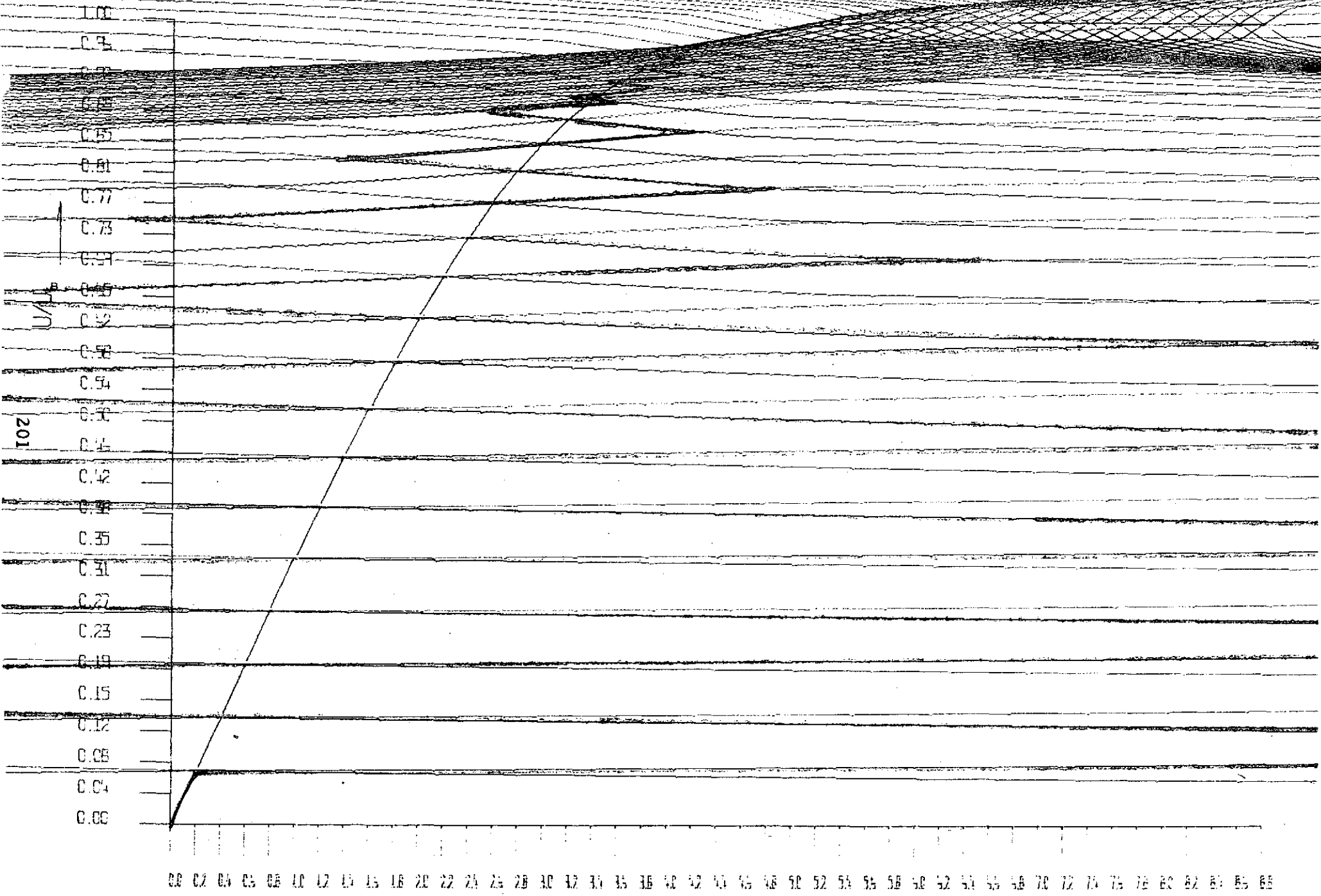# VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ALONG A PLATE

Plate No. 21

PREPARED BY D WIERENGA

VELOCITY DISTRIBUTION IN THE BOUNDARY LAYER ALONG A PLATE

Plate No. 31

ETA