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NONLINEAR EQUATIONS OF MOTION FOR THE ELASTIC BENDING AND TORSION OF TWISTED NONUNIFORM ROTOR BLADES

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16. Abstract

    The equations of motion are developed by two complementary methods, Hamilton's principle and the Newtonian method. The former is more mathematically rigorous and systematic, while the latter provides more physical insight. The resulting equations are valid to second order for long, straight, slender, homogeneous, isotropic beams undergoing moderate displacements. The ordering scheme is based on the restriction that squares of the bending slopes, the torsion deformation, and the chord/radius and thickness/radius ratios are negligible with respect to unity. All remaining nonlinear terms are retained. The equations are valid for beams with mass centroid axis and area centroid (tension) axis offsets from the elastic axis, nonuniform mass and stiffness section properties, variable pretwist, and a small precone angle. The strain-displacement relations are developed from an exact transformation between the deformed and undeformed coordinate systems. These nonlinear relations form an important contribution to the final equations. Several nonlinear structural and inertial terms in the final equations are identified that can substantially influence the aeroelastic stability and response of hingeless helicopter rotor blades.

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A   cross-sectional area, equation (35), m^2

$b(T)$,$b(U)$ kinetic and strain energy boundary terms, equations (39) and (59), N-m

$\mathcal{B}$ reference frame which rotates with speed Ω with respect to the stationary inertial frame $\mathcal{R}$

B_1^* blade cross-section integral, equation (35), m^6

B_2^* blade cross-section integral, equation (35), m^5

c   blade chord, m

C_1 blade cross-section integral, equation (35), m^6

C_1^* blade cross-section integral, equation (35), m^5

e   mass centroid offset from elastic axis, positive when in front of the elastic axis, equation (56), m

e_A tensile axis (area centroid) offset from elastic axis, positive when in front of the elastic axis, equation (35), m

E   Young's modulus, N/m^2

G   shear modulus, N/m^2

$\vec{i},\vec{j},\vec{k}$ unit vectors associated with undeformed beam coordinate system, figures 1-3

$\vec{i}',\vec{j}',\vec{k}'$ unit vectors associated with deformed beam coordinate system, figures 2, 3

$\vec{I},\vec{J},\vec{K}$ "fixed" unit vectors with $\vec{K}$ in direction of rotation of $\mathcal{B}$ in $\mathcal{R}$, figure 1

$I_{y'}$ blade cross-section moment of inertia from y' axis, equation (35), m^4

$I_{z'}$ blade cross-section moment of inertia from z' axis, equation (35), m^4

J   torsional rigidity constant, equation (35), m^4

k_A radius of gyration of blade cross section $= \sqrt{\dfrac{I_{y'} + I_{z'}}{A}}$, equation (35), m

k_m	mass radius of gryation of blade cross section $= \sqrt{k_{m_1}^2 + k_{m_2}^2}$, equation (56), m
k_{m_1}	blade cross-section integral, equation (56), m
k_{m_2}	blade cross-section integral, equation (56), m
L_u, L_v, L_w	components of generalized nonconservative forces in x, y, and z directions, equation (60), N/m
m	mass per unit length of the beam, equation (56), kg/m
$\vec{M}$	internal structural moment
$M_x, M_{x'}$	twisting moment about the x and x' axes, figure 4, N-m
$\tilde{M}_{x'}$	$S_{x'} - (P_{x'})'$, equation (93), N-m
$M_y, M_{y'}$	bending moment about y and y' axes, figure 4, N-m
$M_z, M_{z'}$	bending moment about z and z' axes, figure 4, N-m
M_ϕ	generalized nonconservative moment about x' axis, N-m/m
O	fixed point in $\mathcal{B}$ and $\mathcal{R}$ at the root of the blade, figure 1
P_x, P_y, P_z	inertial and aerodynamic forces
$P_{x'}$	warp term in strain energy expression, equation (34), N-m^2
q_x, q_y, q_z	inertial and aerodynamic moments
r	distance along the deformed elastic axis
R	length of blade, m
$\mathcal{R}$	inertial frame
$S_{x'}$	twisting moment arising from shear stress, equation (34), N-m
t	time, sec; also blade thickness, m
T	tension (same as $V_{x'}$), equation (62), N
$\mathcal{T}$	kinetic energy, kg-m^2/sec
$[T]$	transformation matrix
$T_{x'}$	a twisting moment arising from longitudinal stress, equation (34), N-m
u, v, w	elastic displacements in the x,y,z directions, respectively, figures 2, 3

U	strain energy, N-m
$V_x, V_{x'}$	axial force (tension) in the x, x' directions, respectively, N
$V_y, V_{y'}$	shear force in the y, y' directions, respectively, N
$V_z, V_{z'}$	shear force in the z, z' directions, respectively, N
$\vec{V}$	vector velocity of point P in $\mathscr{R}$, m/sec; also internal structural force, N
x, y, z	mutually perpendicular axis system with x along the undeformed blade and y toward the leading edge, figures 1-3, m
x', y', z'	axis system tangent to the deformed body, figures 2, 3, m
x_0, y_0, z_0	coordinates of a point in the undeformed blade, m
x_1, y_1, z_1	coordinates of a point (which was at x_0, y_0, z_0 in the undeformed blade) in the deformed blade, m
$\overline{Y}_u, \overline{Y}_v, \overline{Y}_w, \overline{Y}_\phi$ $Z_u, \overline{Z}_u$ $Z_v, \overline{Z}_v, Z_{v'}, \overline{Z}_{v'}$ $Z_w, \overline{Z}_w, Z_{w'}, \overline{Z}_{w'}$ $Z_\phi, \overline{Z}_\phi$	notation for writing the energy expressions in a concise form, equations (38), (53), and (55)
β_{pc}	precone angle, figure 1, rad
$\delta(\ )$	variation of ( )
δW	virtual work of the nonconservative forces, equation (29), N-m
ε	small parameter, equation (4)
$\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}$ $\varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}$	tensor strain components, equation (5)
$\varepsilon_{xx}, \varepsilon_{x\eta}, \varepsilon_{x\zeta}$	engineering strain components, equation (27)
ζ	sectional coordinate normal to η axis at elastic axis, figure 3, m
$\hat{\zeta}$	$\zeta + \dfrac{\partial \lambda}{\partial \eta}$, equation (32), m
η	sectional coordinate corresponding to major principal axis for a given point on the elastic axis, figure 3, m

$\hat{\eta}$ $\eta - \dfrac{\partial \lambda}{\partial \zeta}$, equation (32), m

$\overline{\theta}, \overline{\zeta}, \overline{\beta}$ Euler angles, see appendix and figure 5, rad

θ pretwist angle, figure 3, rad

$\lambda(\eta, \zeta)$ warp function, m^2

ν Poisson's ratio

ρ density of the homogeneous model material, kg/m^3

$\left. \begin{array}{l} \sigma_{11}, \sigma_{12}, \sigma_{13} \\ \sigma_{22}, \sigma_{23}, \sigma_{33} \end{array} \right\}$ tensor stress components, N/m^2

$\sigma_{xx}, \sigma_{x\eta}, \sigma_{x\zeta}$ engineering stress, N/m^2

ϕ elastic twist about the elastic axis, figure 3, rad

$\overline{\omega}_\phi$ dimensionless (by Ω) uncoupled rotating torsion frequency

Ω blade rotational speed, figure 1, rad

$(\ )'$ $\dfrac{\partial}{\partial x}$, m^{-1}

$(\ )^+$ $\dfrac{\partial}{\partial r}$, m^{-1}

$(\dot{\ })$ $\dfrac{\partial}{\partial t}$, sec^{-1}

$(\vec{\ })$ denotes vector

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SUMMARY

The equations of motion are developed by two complementary methods, Hamilton's principle and the Newtonian method. The former is more mathematically rigorous and systematic, while the latter provides more physical insight. The resulting equations are valid to second order for long, straight, slender, homogeneous, isotropic beams undergoing moderate displacements. The ordering scheme is based on the restriction that squares of the bending slopes, the torsion deformation, and the chord/radius and thickness/radius ratios are negligible with respect to unity. All remaining nonlinear terms are retained. The equations are valid for beams with mass centroid axis and area centroid (tension) axis offsets from the elastic axis, nonuniform mass and stiffness section properties, variable pretwist, and a small precone angle. The strain-displacement relations are developed from an exact transformation between the deformed and undeformed coordinate systems. These nonlinear relations form an important contribution to the final equations. Several nonlinear structural and inertial terms in the final equations are identified that can substantially influence the aeroelastic stability and response of hingeless helicopter rotor blades.

INTRODUCTION

The dynamic response and aeroelastic stability of helicopter rotor blades can be determined in many cases from linear equations of motion. Much work has been done with linear analyses - notably the work of Houbolt and Brooks (ref. 1), which contains a systematic derivation of the partial differential equations for the coupled bending and torsion of twisted nonuniform blades. These equations are not sufficient, however, in cases where nonlinear structural or inertial effects are important. Nonlinearities can be important in determining the dynamic response of both articulated and hingeless (cantilever blades) rotors, and they are especially important in determining the aeroelastic stability of torsionally flexible hingeless rotors.

This report develops a more complete and general nonlinear theory with particular emphasis given to the fundamentals of the nonlinear behavior. The theory is intended for application to long, straight, slender, homogeneous, isotropic beams with moderate displacements and is accurate to second order based on the restriction that squares of bending slopes, twist, t/R, and c/R are small with respect to unity. Radial nonuniformities (mass, stiffness, twist, etc.), chordwise offsets of the mass centroid and tension axes from the elastic axis, precone, and warp of the cross section are included. Other more specialized details are not considered, such as blade root feathering flexibility, torque offset, blade sweep, and droop; nor are configurations considered in which the feathering bearing is replaced with a torsionally flexible strap.

The equations of motion are derived by means of two complementary methods: the variational method based on Hamilton's principle, and the Newtonian method based on the summation of forces and moments acting on a differential blade element. Both methods used together help ensure a more accurate and consistent treatment of the nonlinear terms. The important nonlinear strain-displacement relations, required for both methods, are developed from a classical definition of strain and simplified in accordance with the premise of a long, slender beam subject to moderate displacements. Applications of the equations of motion to rotor blade aeroelastic problems and development of the aerodynamic loads are not included in this report.

Several previous studies have included nonlinearities in the equations of motion and identified their importance in forced response and aeroelastic stability analyses. A brief discussion of this work should help to acquaint the reader with the nonlinearities of interest, the extent to which they are treated in previous work, and the rationale for the present development.

Several approximate analyses have represented the elastic cantilever blade with a rigid blade and a spring restrained hinge. Reference 2 showed that inertial (Coriolis and centrifugal) nonlinearities were important even if only the flap and lead-lag degrees of freedom were retained. Reference 3 (also based on the rigid blade representation) showed that structural bending-torsion nonlinearities were important when blade torsion was included. The flap and lead-lag nonlinear inertial terms were derived for an elastic blade in reference 4; however, the radial displacement and an intermediate equation for radial force equilibrium were required.

The extensive treatment of the elastic bending-torsion equations of motion in reference 1 retained only linear terms in the final equations, although nonlinear terms were included in certain intermediate stages of the derivation. In particular, the present equilibrium equations in terms of force and moment resultants are identical to those of reference 1. The present inertial terms are also identical to those of reference 1 prior to simplification therein. The present work includes a more complete development of the strain-displacement relation, and this is required to obtain the elastic bending-torsion coupling terms. Reference 5 discusses a simplified form of the bending-torsion coupling in the torsion equation - that is, the elastic torque due to the product of flap and lead-lag curvatures. The present derivation includes a more general formulation of these terms as well as the corresponding terms in the bending equations.

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Reference 6 gives a relatively complete derivation (by the Newtonian method) of the nonlinear equations for rotor blade bending, torsion and rotation about flap and lead-lag hinges. Certain second-order nonlinear inertial terms due to elastic deformations of the blade are not included, however, and these can be important for stability of cantilever blade configurations. Reference 7 contains a derivation (also by the Newtonian method) of a system of nonlinear equations for the elastic bending and rigid pitching motion of a cantilever rotor blade. Most of the nonlinear terms are included, but certain inconsistencies in the discarding of higher-order terms resulted in nonself-adjoint inertial and structural operators; that is, the stiffness matrix and mass matrix are not symmetric, and the gyroscopic matrix is not antisymmetric. The nonlinear bending-pitch coupling does not appear explicitly as a structural nonlinearity (as in refs. 5, 6, and the present report) since pitching moments were derived in terms of the aerodynamic forces.

Reference 8 presents an earlier and less rigorous version of the derivations given in the present report. The earlier derivation was based on a simpler development of the strain-displacement relations and several extraneous terms were present in the final equations of motion. Preliminary results for the aeroelastic stability of hingeless rotor blades using the present equations have been reported in reference 9.

After a brief discussion of the coordinate systems, transformations, and the ordering scheme to be used in this report, the strain-displacement relations are developed for an elastic blade. This theory is then simplified to second order for long, slender beams. The derivation of the equations of motion is divided into two parts. Part I gives the derivation of the equations of motion for a rotating beam based on Hamilton's principle. Part II outlines the derivation using the Newtonian method. Both derivations reduce to the results of reference 1 when the nonlinear terms are omitted.

The authors have closely collaborated on the entire report. However, Part I is largely the work of the first author and Part II that of the second author. Both have benefited substantially from D. A. Peters' contributions to the transformation laws and strain-displacement relations. Numerous useful discussions with R. A. Ormiston are also gratefully acknowledged.

COORDINATE SYSTEMS AND TRANSFORMATIONS

Several coordinate systems will be used in the present analysis. The orthogonal axes system X, Y, Z and associated unit vectors $\vec{I}$, $\vec{J}$, $\vec{K}$ (fig. 1) are fixed in an inertial frame $\mathscr{R}$. Orthogonal axes x_p, y, Z are fixed in a reference frame $\mathscr{B}$ which rotates with respect to $\mathscr{R}$ at constant angular velocity $\Omega\vec{K}$. Point 0, a common fixed point of $\mathscr{R}$ and $\mathscr{B}$, is located at the root of the beam. The plane containing X, x_p, Y, and y is called the reference plane, or plane of rotation. The x axis, which lies along the elastic axis of the undeformed beam, is inclined to the plane of rotation (and to the x_p axis) at the precone angle β_{pc}. The orthogonal axes x, y, z, and the corresponding unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, therefore, are also fixed in $\mathscr{B}$. Beam
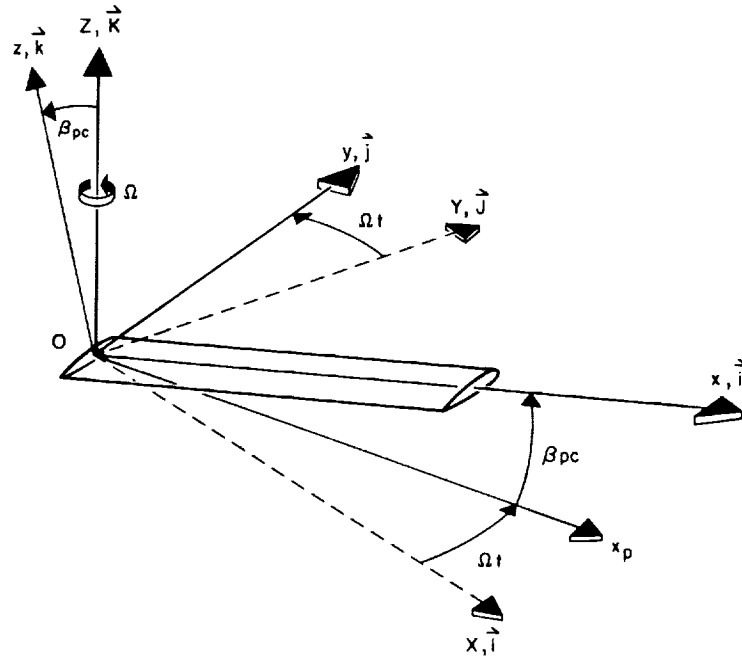
Figure 1.- Undeformed coordinate systems.

bending deformations shown in figure 2 for $\theta = 0$ are described by the displacements of the elastic axis u, v, w, parallel to $\vec{i}$, $\vec{j}$, $\vec{k}$, respectively. A point on the elastic axis that is located at x, 0, 0 in the x, y, z coordinate system before deformation is located at x + u, v, w after deformation.
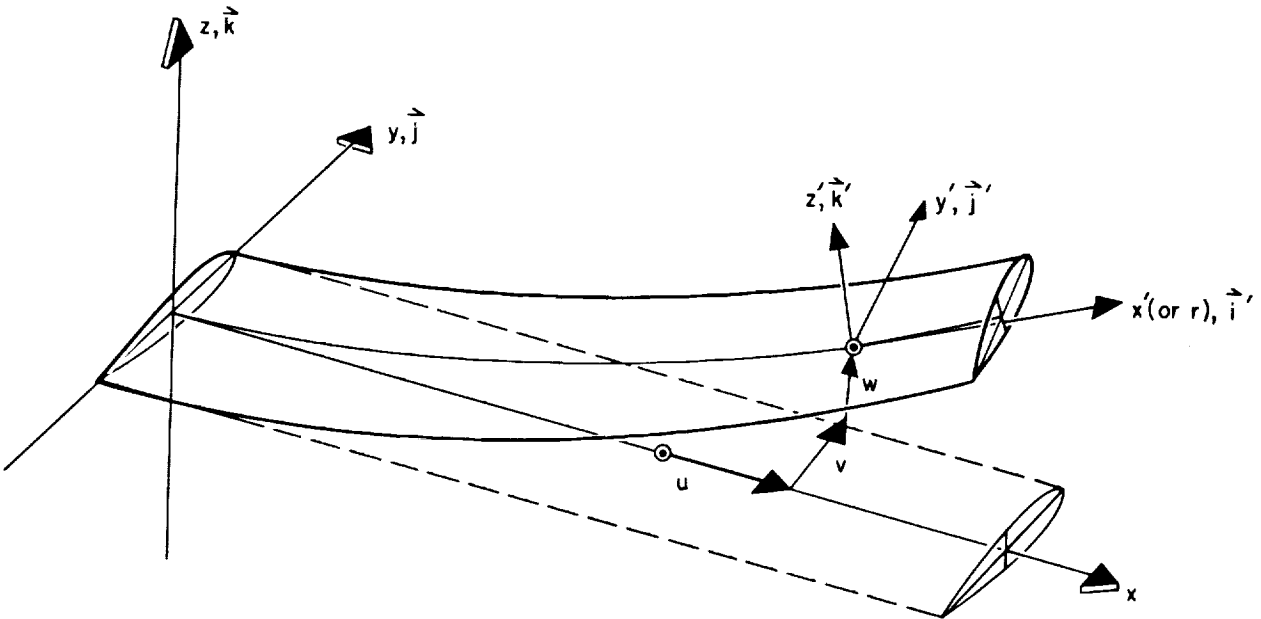

Figure 2.- Elastic displacements.

The beam cross section is shown in figure 3 before and after deformation. The point of the cross section through which the elastic axis passes (the shear center) is shown at the intersection of the y and z axes before deformation. The η and ζ axes are the principal axes of the cross section for the
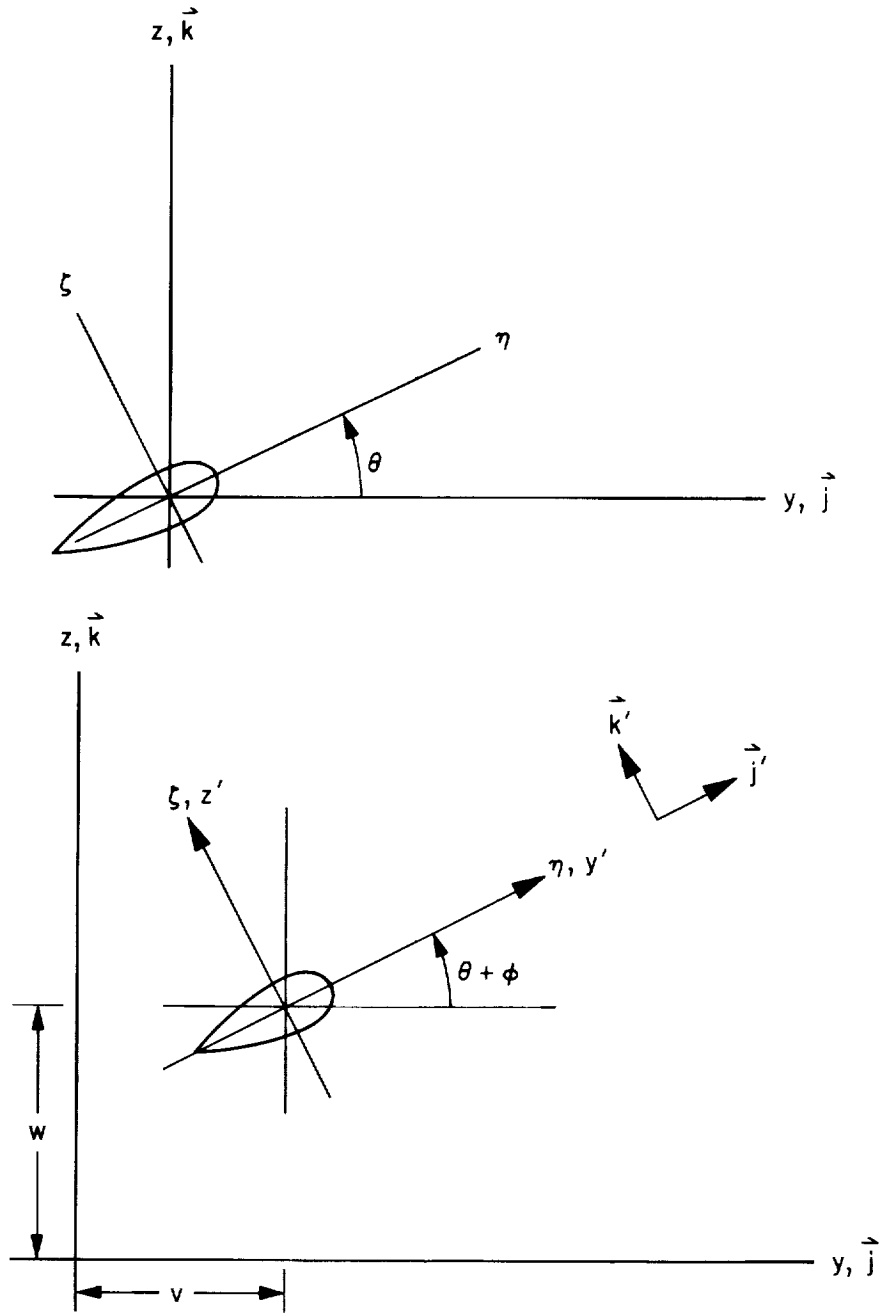

Figure 3.- Cross-section coordinates before and after deformation. (Note: $\vec{j}'$,$\vec{k}'$ are not in the same plane as $\vec{j}$,$\vec{k}$. Their projections are shown in the $\vec{j}$,$\vec{k}$ plane.)

shear center. The cross section is assumed to be symmetric with respect to the η axis. The η and ζ axes are inclined relative to the y and z axes at the "built-in" pitch angle $\theta(x)$. When the beam is deformed, the shear center for the cross section located at x is displaced an amount u in the x direction (not shown), v in the y direction, and w in the z direction. The angle of twist of the cross section changes from θ about the x axis to $\theta + \phi$ about the x' axis.

The deformed beam is shown in figure 4 with force and moment resultants acting on the face of a cross section. At any point along the deformed beam, x' is tangent to the deformed elastic axis. (Note that the distance along the deformed elastic axis is also denoted by r, in addition to x'.) The y' and z' axes are identical to the η and ζ axes, respectively, when the beam is deformed. The unit vectors $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ are parallel to the deformed beam coordinate system axes x', y', z', respectively. Stress resultants and moments are subscripted with x', y', z' to associate them with the deformed beam; for example, $M_{x'}$ is a moment about the x' axis.
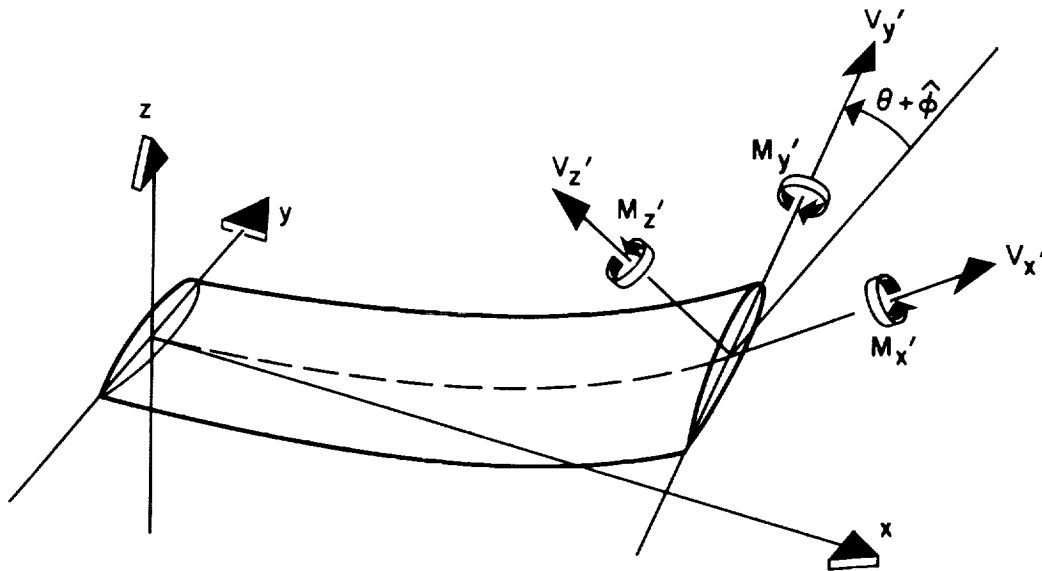

Figure 4.- Beam with resultant forces and moments.

Several coordinate transformations are used, including the simple transformation from $\vec{I}$, $\vec{J}$, $\vec{K}$ to $\vec{i}$, $\vec{j}$, $\vec{k}$. A more difficult transformation (appendix) is one from $\vec{i}$, $\vec{j}$, $\vec{k}$ to $\vec{i}'$, $\vec{j}'$, $\vec{k}'$, which is given to second order by

$$\vec{i}' = \left(1 - \frac{1}{2} v'^2 - \frac{1}{2} w'^2\right)\vec{i} + v'\vec{j} + w'\vec{k} \tag{1a}$$

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$$\vec{j}\,' = -[v'\cos(\theta + \phi) + w'\sin(\theta + \phi)]\vec{i} + \left(1 - \frac{1}{2}v'^2\right)\cos(\theta + \hat{\phi} + v'w')\vec{j}$$

$$+ \left(1 - \frac{1}{2}w'^2\right)\sin(\theta + \hat{\phi})\vec{k} \qquad\qquad (1b)$$

$$\vec{k}\,' = -[-v'\sin(\theta + \phi) + w'\cos(\theta + \phi)]\vec{i} - \left(1 - \frac{1}{2}v'^2\right)\sin(\theta + \hat{\phi} + v'w')\vec{j}$$

$$+ \left(1 - \frac{1}{2}w'^2\right)\cos(\theta + \hat{\phi})\vec{k} \qquad\qquad (1c)$$

where $\hat{\phi}$ (consistent to second order in v', w', ϕ) is given by

$$\hat{\phi} = \phi - \int_0^x \frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx$$

The transformation used in equations (1) may be simplified further in many (but not all) parts of the analysis by replacing $\hat{\phi}$ by ϕ and dropping the squares and products of $\partial v/\partial x$ and $\partial w/\partial x$. The summation of θ and ϕ represents the total pitch angle (built-in pitch plus torsion deformation) with respect to the x' axis; w is the bending deformation (flapping) in z direction; v is the bending deformation (lead-lag) in y direction; and $\partial v/\partial x$, $\partial w/\partial x$ are small rotations about the z and y axes, respectively. It is convenient to carry $\theta + \phi$ through the analysis without making the small-angle approximation until the end of the derivation.

It will be necessary to transform vector components from one coordinate system to another. For example, a resultant moment vector $\vec{M}$ may be written as

$$\vec{M} = M_x\vec{i} + M_y\vec{j} + M_z\vec{k} \qquad\qquad (2a)$$

or

$$\vec{M} = M_{x'}\vec{i}\,' + M_{y'}\vec{j}\,' + M_{z'}\vec{k}\,' \qquad\qquad (2b)$$

in terms of components measured in the undeformed or deformed coordinate systems, respectively. The relations between $M_{x'}$, $M_{y'}$, $M_{z'}$ and M_x, M_y, M_z are determinated by using the unit vector relations of equations (1) and taking appropriate dot products of equations (2). Thus

$$M_X = M_{X'}\vec{i}' \cdot \vec{i} + M_{y'}\vec{j}' \cdot \vec{i} + M_{z'}\vec{k}' \cdot \vec{i}$$

$$= M_{X'}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) - M_{y'}[v' \cos(\theta + \phi) + w' \sin(\theta + \phi)]$$

$$- M_{z'}[-v' \sin(\theta + \phi) + w' \cos(\theta + \phi)] \tag{3a}$$

and, similarly,

$$M_y = M_{X'}v' + \left(1 - \frac{v'^2}{2}\right)[M_{y'} \cos(\theta + \hat{\phi} + v'w') - M_{z'} \sin(\theta + \hat{\phi} + v'w')] \tag{3b}$$

$$M_z = M_{X'}w' + \left(1 - \frac{w'^2}{2}\right)[M_{y'} \sin(\theta + \hat{\phi} + v'w') + M_{z'} \cos(\theta + \hat{\phi} + v'w')] \tag{3c}$$

where the primed notation is introduced for subsequent use.

$$v' = \frac{\partial v}{\partial x} ; \quad w' = \frac{\partial w}{\partial x}$$

Other vectors, such as resultant force $\vec{V}$ and acceleration $\vec{a}$, obey the same transformation laws.

ORDERING SCHEME

In deriving a nonlinear system of equations, it is necessary to neglect higher-order terms to avoid overcomplicating the equations of motion. When neglecting terms within a large system of equations, care must be exercised to ensure that the terms retained constitute self-adjoint structural and inertial operators. These self-adjoint operators lead to symmetric stiffness and mass matrices and an antisymmetric gyroscopic matrix in the modal equations.

A systematic self-consistent set of guidelines has been adopted for determining which terms to retain and which to ignore. The dimensionless axial deflection u/R is generally taken to be of the same order of magnitude as the square of v/R or w/R and thus is small with respect to unity. The elastic twist ϕ is a small angle in the sense that $\sin \phi \simeq \phi$ and $\cos \phi \simeq 1$. The axial coordinate x is of order R and the lateral coordinates are of the same order as the chord c, and thickness t, respectively. Both c and t are assumed to be at the same order of magnitude as v and w. The warp function λ is taken to be of order c times t so that the actual warp displacement will be an order of magnitude less than the axial displacement u. These assumptions can be systematized by introducing ϵ, a parameter of order v/R or w/R. Hence

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$$\frac{u}{R} = O(\varepsilon^2) \qquad \frac{\eta}{R} = O(\varepsilon)$$

$$\frac{v}{R} = O(\varepsilon) \qquad \frac{\zeta}{R} = O(\varepsilon)$$

$$\frac{w}{R} = O(\varepsilon) \qquad \frac{\lambda}{R^2} = O(\varepsilon^2)$$

$$\phi = O(\varepsilon) \qquad \frac{\partial\lambda/\partial\eta}{R} = O(\varepsilon)$$

$$\frac{x}{R} = O(1) \qquad \frac{\partial\lambda/\partial\zeta}{R} = O(\varepsilon)$$

(4)

Within the energy expressions, terms of order ε^2 are ignored with respect to unity. Thus, if the largest terms of the energy expression are $O(\varepsilon^4)$, then all terms of $O(\varepsilon^4)$ are retained (first-order terms), all terms of $O(\varepsilon^5)$ are retained (second-order terms), and generally terms of $O(\varepsilon^6)$ are discarded. There are conditions under which certain $O(\varepsilon^6)$ terms should be retained. These exceptions will be noted below. Since the physical quantities involved in the strain energy and the kinetic energy are fundamentally different, a scaling parameter for their respective orders of magnitude is introduced in the text. This is done so that the ordering scheme may be consistently applied to both strain energy and kinetic energy terms.

In the application of the above guidelines, it is important that the ordering be done within the total energy context, or equivalently within the virtual work expression (e.g., the w equation times δw). Ordering differently in one equation than in another without regard to this consideration can lead to the above-mentioned symmetry problems, and introduce dissipative and circulatory forces into the final equations that are linearized with respect to an equilibrium position. Also, the scheme implies that the same order terms should be retained in the v, w, ϕ equations, but terms of one order less should be retained in the u equation.

STRAIN-DISPLACEMENT RELATIONS

The development of a nonlinear strain-displacement relation is central to both the variational and the Newtonian methods for developing a nonlinear system of equations. The use of this relation, together with the generalized Hooke's law, permits the strain energy, the force resultants, and the moment resultants to be expressed in terms of deformation quantities. Although the primary goal of this paper is to develop a second-order nonlinear theory, we first develop a general nonlinear strain-displacement relation, which is then simplified consistent to second order for deriving the equations of motion. This approach is necessary to ensure that a valid approximation for the

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nonlinear strain-displacement relation is obtained. The general strain-displacement relation would also be useful for future development of a more accurate theory.

Derivation of a General Strain-Displacement Relation

Several different classical definitions of strain may be found in the literature, depending on the mathematical formulation, reference states (based on deformed or undeformed positions), and coordinate systems used. The influence of the definition of strain is considered in the present development. In particular, for nonlinear strain-displacement relations accurate to second order, there is no distinction between two commonly used definitions of strain.

The classic strain tensor ϵ_{ij} in terms of $\vec{r}_1$ and $\vec{r}_0$, the vector positions of the same point on the deformed and undeformed blade, respectively, may be expressed as (ref. 10)

$$d\vec{r}_1 \cdot d\vec{r}_1 - d\vec{r}_0 \cdot d\vec{r}_0 = 2 \lfloor dr \quad d\eta \quad d\zeta \rfloor [\epsilon_{ij}] \begin{Bmatrix} dr \\ d\eta \\ d\zeta \end{Bmatrix} \tag{5}$$

where dr, $d\eta$, $d\zeta$ are increments along the deformed elastic axis and two cross-sectional axes, respectively.

The vector position of a generic point on the undeformed beam is given by (x, y, z) with respect to the unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ where $(x, 0, 0)$ is the elastic axis. The corresponding point on the deformed beam is given by (1) the position of the deformed elastic axis $(x + u, v, w)$ with respect to the undeformed axes $\vec{i}$, $\vec{j}$, $\vec{k}$, and (2) the position of the point relative to the elastic axis $[-\lambda(\theta + \phi)^+, \eta, \zeta]$ with respect to deformed axes $\vec{i}'$, $\vec{j}'$, $\vec{k}'$. Note in particular that $-\lambda(\theta + \phi)^+$, where $(\ )^+ = \partial/\partial r(\ )$, represents an axial position where λ is the warp function; $\lambda(0, 0) = 0$.

Let us now express the above mathematically. After deformation, the vector position of a generic point is

$$\vec{r}_1 = \lfloor \vec{i} \quad \vec{j} \quad \vec{k} \rfloor \left[\begin{Bmatrix} x + u \\ v \\ w \end{Bmatrix} + [T]^T \begin{Bmatrix} -\lambda(\theta + \phi)^+ \\ \eta \\ \zeta \end{Bmatrix} \right] \tag{6}$$

where the transformation matrix $[T]$, given in the appendix, relates the deformed axes $(\vec{i}', \vec{j}', \vec{k}')$ and the undeformed axes $(\vec{i}, \vec{j}, \vec{k})$.[2]

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[2]This derivation is simplified if $[T]$ is retained in abstract form throughout. A first-order approximation of $[T]$ introduced at this point could result in an incorrect expression for the shear strain component.

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Before deformation,

$$\vec{r}_0 = \vec{r}_1 \Big|_{u=v=w=\phi=0} \tag{7}$$

Thus, using equation (6)

$$\vec{r}_0 = \lfloor \vec{i} \quad \vec{j} \quad \vec{k} \rfloor \left[\left\{ \begin{matrix} x \\ 0 \\ 0 \end{matrix} \right\} + [T]^T \Big|_{u=v=w=\phi=0} \left\{ \begin{matrix} -\lambda_0 \theta' \\ \eta_0 \\ \zeta_0 \end{matrix} \right\} \right] \tag{8}$$

where

$$\eta_0 = \eta \Big|_{u=v=w=\phi=0} \qquad \zeta_0 = \zeta \Big|_{u=v=w=\phi=0} \qquad \lambda_0 (\eta_0, \zeta_0) = \lambda (\eta, \zeta) \Big|_{u=v=w=\phi=0}$$

The coordinates η and ζ may be regarded as functions of η_0 and ζ_0, respectively.

From equation (A2),

$$[T] \Big|_{u=v=w=\phi=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \tag{9}$$

Thus

$$\vec{r}_0 = \lfloor \vec{i} \quad \vec{j} \quad \vec{k} \rfloor \left\{ \begin{matrix} x - \lambda_0 \theta' \\ \eta_0 \cos \theta - \zeta_0 \sin \theta \\ \eta_0 \sin \theta + \zeta_0 \cos \theta \end{matrix} \right\} \tag{10}$$

The position vector differentials are given by

$$d\vec{r}_0 = \lfloor \vec{i} \quad \vec{j} \quad \vec{k} \rfloor \left\{ \begin{matrix} x^+ (1 - \lambda_0 \theta'') dr - \left(\lambda \eta_0 \dfrac{d\eta_0}{d\eta} d\eta + \lambda \zeta_0 \dfrac{d\zeta_0}{d\zeta} d\zeta \right) \theta' \\ -\theta' x^+ (\eta_0 \sin \theta + \zeta_0 \cos \theta) dr + \dfrac{d\eta_0}{d\eta} \cos \theta \, d\eta - \dfrac{d\zeta_0}{d\zeta} \sin \theta \, d\zeta \\ \theta' x^+ (\eta_0 \cos \theta - \zeta_0 \sin \theta) dr + \dfrac{d\eta_0}{d\eta} \sin \theta \, d\eta + \dfrac{d\zeta_0}{d\zeta} \cos \theta \, d\zeta \end{matrix} \right\} \tag{11}$$

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$$\vec{dr}_1 = [\vec{i}' \quad \vec{j}' \quad \vec{k}'] \left[[T] \begin{Bmatrix} x^+ + u^+ \\ v^+ \\ w^+ \end{Bmatrix} dr + [T][T]^{T^+} \begin{Bmatrix} -\lambda(\theta + \phi)^+ \\ \eta \\ \zeta \end{Bmatrix} dr \right.$$

$$\left. + \begin{Bmatrix} -\lambda(\theta + \phi)^{++} dr - (\lambda_\eta \, d\eta + \lambda_\zeta \, d\zeta)(\theta + \phi)^+ \\ d\eta \\ d\zeta \end{Bmatrix} \right] \qquad (12)$$

where

$$x^+ \equiv \frac{dx}{dr} \; ; \quad \lambda_\eta \equiv \frac{\partial\lambda}{\partial\eta} \; ; \quad \lambda_\zeta \equiv \frac{\partial\lambda}{\partial\zeta} \; ; \quad \lambda_{\eta_0} \equiv \frac{\partial\lambda_0}{\partial\eta_0} \; ; \quad \lambda_{\zeta_0} = \frac{\partial\lambda_0}{\partial\zeta_0}$$

Note that

$$[T][T]^T = \begin{bmatrix} \ddots & & \\ & 1 & \\ & & \ddots \end{bmatrix}$$

Thus

$$[T][T]^{T^+} + [T]^+[T]^T = \begin{bmatrix} \ddots & & \\ & 0 & \\ & & \ddots \end{bmatrix}$$

and

$$[T][T]^{T^+} = -[T]^+[T]^T = \begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix} \qquad (13)$$

where explicit expressions for ω_i, ω_j, ω_k are given in the appendix. Hence,

$$\vec{dr}_1 = \lfloor \vec{i}' \quad \vec{j}' \quad \vec{k}' \rfloor \left[[T] \left\{ \begin{array}{c} x^+ + u^+ \\ v^+ \\ w^+ \end{array} \right\} dr + \begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix} \left\{ \begin{array}{c} -\lambda(\theta+\phi)^+ \\ \eta \\ \zeta \end{array} \right\} dr \right.$$

$$\left. + \left\{ \begin{array}{c} -\lambda(\theta+\phi)^{++} dr - (\lambda_\eta \, d\eta + \lambda_\zeta \, d\zeta)(\theta+\phi)^+ \\ d\eta \\ d\zeta \end{array} \right\} \right] \qquad (14)$$

The expressions for the position vector derivatives can be simplified by eliminating x^+ in equation (11) and $[T]$ in equation (14). On the elastic axis $\eta = d\eta = \zeta = d\zeta = \lambda = \lambda_\eta = \lambda_\zeta = 0$. Thus,

$$\left. \frac{\vec{dr}_1}{dr} \right|_{\eta=\zeta=0} = \lfloor \vec{i}' \quad \vec{j}' \quad \vec{k}' \rfloor [T] \left\{ \begin{array}{c} x^+ + u^+ \\ v^+ \\ w^+ \end{array} \right\} \qquad (15)$$

However, $\vec{dr}_1/dr$ at $\eta = \zeta = 0$ must be $\vec{i}'$ by definition of the $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ system. Therefore,

$$\lfloor \vec{i}' \quad \vec{j}' \quad \vec{k}' \rfloor \left\{ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\} = \lfloor \vec{i}' \quad \vec{j}' \quad \vec{k}' \rfloor [T] \left\{ \begin{array}{c} x^+ + u^+ \\ v^+ \\ w^+ \end{array} \right\} \qquad (16)$$

The first component of this vector equation yields:

$$\sqrt{1 - v^{+2} - w^{+2}} \, (x^+ + u^+) + v^{+2} + w^{+2} = 1$$

Hence,

$$x^+ = \sqrt{1 - v^{+2} - w^{+2}} - u^+ \qquad (17)$$

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Equation (14) for $\vec{dr}_1$ may be now simplified, by using equation (16), to yield

$$\vec{dr}_1 = \lfloor \vec{i}' \quad \vec{j}' \quad \vec{k}' \rfloor \left[\left\{ \begin{matrix} dr \\ 0 \\ 0 \end{matrix} \right\} + \begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix} \left\{ \begin{matrix} -\lambda(\theta + \phi)^+ \\ \eta \\ \zeta \end{matrix} \right\} dr \right.$$

$$\left. + \left\{ \begin{matrix} -\lambda(\theta + \phi)^{++}dr - (\lambda_\eta\, d\eta + \lambda_\zeta\, d\zeta)(\theta + \phi)^+ \\ d\eta \\ d\zeta \end{matrix} \right\} \right] \qquad (18)$$

With equation (17), equation (11) for $\vec{dr}_0$ becomes

$$\vec{dr}_0 = $$

$$\lfloor \vec{i} \quad \vec{j} \quad \vec{k} \rfloor \left\{ \begin{matrix} \left(\sqrt{1 - v^{+2} - w^{+2}} - u^+\right)(1 - \lambda_0\theta'')dr - \theta'\left(\lambda_{\eta_0} \dfrac{d\eta_0}{d\eta} d\eta + \lambda_{\zeta_0} \dfrac{d\zeta_0}{d\zeta} d\zeta\right) \\ -\theta'\left(\sqrt{1 - v^{+2} - w^{+2}} - u^+\right)(\eta_0\, \sin\, \theta + \zeta_0\, \cos\, \theta)dr + \dfrac{d\eta_0}{d\eta} \cos\, \theta\, d\eta - \dfrac{d\zeta_0}{d\zeta} \sin\, \theta\, d\zeta \\ \theta'\left(\sqrt{1 - v^{+2} - w^{+2}} - u^+\right)(\eta_0\, \cos\, \theta - \zeta_0\, \sin\, \theta)dr + \dfrac{d\eta_0}{d\eta} \sin\, \theta\, d\eta + \dfrac{d\zeta_0}{d\zeta} \cos\, \theta\, d\zeta \end{matrix} \right\}$$

$$(19)$$

Equations (18) and (19) can now be substituted into equation (5) to solve for the components of the strain tensor ε_{ij}.

$$2\varepsilon_{11} = (-\eta\omega_k + \zeta\omega_j)^2 + [\lambda\omega_k(\theta + \phi)^+ + \zeta\omega_i]^2 + [\lambda\omega_j(\theta + \phi)^+ + \eta\omega_i]^2$$

$$+ \lambda^2(\theta + \phi)^{++2} + 2(-\eta\omega_k + \zeta\omega_j) - 2\lambda(\theta + \phi)^{++}(1 - \eta\omega_k + \zeta\omega_j)$$

$$+ v^{+2} + w^{+2} - u^{+2} + 2u^+\sqrt{1 - v^{+2} - w^{+2}} - x^{+2}\theta'^2(\eta_0^2 + \zeta_0^2)$$

$$+ x^{+2}(2\lambda_0\theta'' - \lambda_0^2\theta''^2) \qquad (20a)$$

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$$2\varepsilon_{12} = \lambda\lambda_\eta(\theta + \phi)^+ (\theta + \phi)^{++} - \lambda_\eta(\theta + \phi)^+(1 - \eta\omega_k + \zeta\omega_j) - [\lambda\omega_k(\theta + \phi)^+ + \zeta\omega_i]$$

$$+ x^+\theta' \frac{d\eta_0}{d\eta} [\lambda\eta_0(1 - \lambda_0\theta'') + \zeta_0] \tag{20b}$$

$$2\varepsilon_{13} = \lambda\lambda_\zeta(\theta + \phi)^+ (\theta + \phi)^{++} - \lambda_\zeta(\theta + \phi)^+(1 - \eta\omega_k + \zeta\omega_j) + [\lambda\omega_j(\theta + \phi)^+ + \eta\omega_i]$$

$$+ x^+\theta' \frac{d\zeta_0}{d\zeta} [\lambda\zeta_0(1 - \lambda_0\theta'') - \eta_0] \tag{20c}$$

$$2\varepsilon_{22} = 1 - \left(\frac{d\eta_0}{d\eta}\right)^2 + \lambda_\eta^2(\theta + \phi)^{+2} - \lambda_{\eta_0}^2 \theta'^2 \left(\frac{d\eta_0}{d\eta}\right)^2 \tag{20d}$$

$$2\varepsilon_{23} = \lambda_\eta\lambda_\zeta(\theta + \phi)^{+2} - \lambda_{\eta_0}\lambda_{\zeta_0} \left(\frac{d\eta_0}{d\eta}\right)\left(\frac{d\zeta_0}{d\zeta}\right) \theta'^2 \tag{20e}$$

$$2\varepsilon_{33} = 1 - \left(\frac{d\zeta_0}{d\zeta}\right)^2 + \lambda_\zeta^2(\theta + \phi)^{+2} - \lambda_{\zeta_0}^2 \theta'^2 \left(\frac{d\zeta_0}{d\zeta}\right)^2 \tag{20f}$$

where

$$\theta^+ = \theta'x^+ \qquad \theta^{++} = \theta''x^{+2} + \theta'x^{++}$$

Equations (20a)-(20f) are the classical strain components in terms of displacements u, v, w, and ϕ; the warp function λ; the curvatures ω_i, ω_j, and ω_k; the cross-section coordinates η, η_0, ζ, and ζ_0; and the pretwist θ. The warp function is determined through Laplace's equation for the cross section. The curvatures and twist are known (appendix). A relationship between η_0, ζ_0, and η, ζ is needed to express the strain components entirely in terms of displacement variables. This relationship will only be determined for the second-order approximate theory.

Second-Order Approximation for the Strain Components

Equations (20) form the basis for a general nonlinear strain-displacement theory. This theory will now be reduced to second order. First, $d\eta_0/d\eta$ and $d\zeta_0/d\zeta$ must be expressed in terms of the elastic displacements and the known parameters. The assumption of uniaxial stress, valid for long slender beams, ($\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$) and Hooke's law gives

$$\left.\begin{array}{c} \varepsilon_{22} = -\nu\varepsilon_{11} \\ \varepsilon_{33} = -\nu\varepsilon_{11} \end{array}\right\} \tag{21}$$

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Equations (20d) and (20f) are now

$$-2\nu\varepsilon_{11} = 1 - \left(\frac{d\eta_0}{d\eta}\right)^2 + \lambda_\eta{}^2(\theta + \phi)^{+2} - \lambda_{\eta_0}^2\theta'^2\left(\frac{d\eta_0}{d\eta}\right)^2 \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\}$$

$$-2\nu\varepsilon_{11} = 1 - \left(\frac{d\zeta_0}{d\zeta}\right)^2 + \lambda_\zeta{}^2(\theta + \phi)^{+2} - \lambda_{\zeta_0}^2\theta'^2\left(\frac{d\zeta_0}{d\zeta}\right)^2 \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} \qquad (22)$$

and solving for $(d\eta_0/d\eta)$ and $(d\zeta_0/d\zeta)$ gives

$$\left(\frac{d\eta_0}{d\eta}\right)^2 = \frac{1 + 2\nu\varepsilon_{11} + \lambda_\eta{}^2(\theta + \phi)^{+2}}{1 + \lambda_{\eta_0}^2\theta'^2}$$

$$\cong 1 \text{ (to second order)} \qquad (23a)$$

$$\left(\frac{d\zeta_0}{d\zeta}\right)^2 = \frac{1 + 2\nu\varepsilon_{11} + \lambda_\zeta{}^2(\theta + \phi)^{+2}}{1 + \lambda_{\zeta_0}^2\theta'^2}$$

$$\cong 1 \text{ (to second order)} \qquad (23b)$$

Thus to second order η and ζ are equivalent to η_0 and ζ_0, respectively, and $\lambda = \lambda_0$.

The strain components ε_{11}, ε_{12}, and ε_{13} in equations (20) may now be reduced to second order. The expressions for $(d\eta_0/d\eta)$ and $(d\zeta_0/d\zeta)$ are substituted in equations (20), higher-order terms are eliminated and $(\ )'$ is substituted for $(\ )^+$ since $dx/dr \simeq 1$ to second order (eq. (17)).

$$\varepsilon_{11} = u' + \frac{v'^2}{2} + \frac{w'^2}{2} - \lambda\phi'' + (\eta^2 + \zeta^2)\left(\theta'\phi' + \frac{\phi'^2}{2}\right)$$

$$-v''[\eta\cos(\theta + \phi) - \zeta\sin(\theta + \phi)] - w''[\eta\sin(\theta + \phi) + \zeta\cos(\theta + \phi)] \qquad (24)$$

$$\varepsilon_{12} = -\frac{1}{2}(\zeta + \lambda_\eta)\phi' \qquad (25)$$

$$\varepsilon_{13} = \frac{1}{2}(\eta - \lambda_\zeta)\phi' \qquad (26)$$

Although the ϕ'^2 term in equation (24) is formally negligible, since it is of $O(\varepsilon^4)$ compared to other terms of $O(\varepsilon^2)$, it leads to a tension term previously identified in reference 1 that contributes to the elastic torque T_x.

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about the x' axis (see eqs. (33) and (34) in Part I). Hence, it is retained here although for certain applications it could be dropped without loss of accuracy.

The shear strain component ε_{23} is now considered. Equation (20e) shows that ε_{23} is not identically zero, contradicting the previous assumption that $\sigma_{23} = 0$. However, ε_{23} is two orders of magnitude smaller than the shear strains ε_{12} and ε_{13}. Therefore, neglecting ε_{23} and hence σ_{23} is well justified. This completes the reduction to second order of the classical strain components in equations (20).

The relationship between the simplified classical strain tensor and engineering strain is now discussed. Since the uniaxial stress assumption has been invoked ($\sigma_{22} = \sigma_{23} = \sigma_{33} = 0$), the expression for strain energy in terms of the classical strain components is equivalent to the more familiar form of strain energy based on engineering strain (the latter form is used in Part I). Thus, the engineering strain components can be written in terms of the simplified classical strain components.

$$\left.\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{11} \\
\varepsilon_{x\eta} &= 2\varepsilon_{12} \\
\varepsilon_{x\zeta} &= 2\varepsilon_{13}
\end{aligned}\right\} \tag{27}$$

There are other possible definitions of strain that could have been used in this development. Here the strain has been defined on the basis of an increment of deformed length dr. The most frequently used alternative employs the original length dx. By appropriate transformation, the two definitions can be compared. For example, examining the case of longitudinal displacement only, $u \neq 0$, $v = w = \phi = 0$, the present definition of strain gives

$$\varepsilon_{11} = \frac{du}{dr} - \frac{1}{2}\left(\frac{du}{dr}\right)^2 \tag{28a}$$

From the definition of r

$$\frac{dr}{dx} = 1 + \frac{du}{dx}$$

and

$$\frac{du}{dr} = \frac{\dfrac{du}{dx}}{1 + \dfrac{du}{dx}}$$

Thus, the alternative definition of strain gives

$$\varepsilon_{11}^{A} = \varepsilon_{11} \left(\frac{dr}{dx}\right)^2 = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 \tag{28b}$$

which is the more familiar result. To second order, however, the two results are equivalent.

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PART I

DERIVATION OF EQUATIONS BY HAMILTON'S PRINCIPLE

The equations of motion and boundary conditions for a cantilever beam rotating at constant speed are obtained from Hamilton's principle. These equations are valid for rotor blades that can be represented by a long, slender, homogeneous, isotropic beam. This representation includes spanwise variations in the mass and stiffness properties, variable mass centroid axis and area centroid axis offsets from the elastic axis, variable built-in axial twist, and a small precone angle. The external forces acting on the beam are characterized by a set of generalized distributed loads. The use of an actual helicopter rotor blade as a model would be a more formidable task because of the nonhomogeneous, anisotropic structures found in typical blades. The present structural representation is similar to that of reference 1.

Hamilton's principle may be expressed as

$$\int_{t_1}^{t_2} [\delta(U - T) - \delta W] dt = 0 \tag{29}$$

where U is the strain energy, T is the kinetic energy, and δW is the virtual work of the external forces. Suitable expressions for δU, δT, and δW are now determined and then combined to give the desired equations.

Strain Energy Contributions

The usual expression for strain energy in terms of engineering stresses and strains, defined in equation (27) in terms of the classical strains, is

$$U = \frac{1}{2} \int_0^R \iint_A (\sigma_{xx} \epsilon_{xx} + \sigma_{x\eta} \epsilon_{x\eta} + \sigma_{x\zeta} \epsilon_{x\zeta}) d\eta \, d\zeta \, dx \tag{30}$$

The first variation is

$$\delta U = \int_0^R \iint_A (\sigma_{xx} \delta \epsilon_{xx} + \sigma_{x\eta} \delta \epsilon_{x\eta} + \sigma_{x\zeta} \delta \epsilon_{x\zeta}) d\eta \, d\zeta \, dx \tag{31}$$

where

$$\sigma_{xx} = E \epsilon_{xx}$$

$$\sigma_{x\eta} = G \epsilon_{x\eta}$$

$$\sigma_{x\zeta} = G \epsilon_{x\zeta}$$

and

$$\begin{aligned}
\delta\varepsilon_{xx} = {} & \delta u' + v'\delta v' + w'\delta w' + (\eta^2 + \zeta^2)(\theta + \phi)'\delta\phi' - \lambda\delta\phi'' \\
& -[\eta\cos(\theta+\phi) - \zeta\sin(\theta+\phi)](\delta v'' + w''\delta\phi) \\
& -[\eta\sin(\theta+\phi) + \zeta\cos(\theta+\phi)](\delta w'' - v''\delta\phi) \\
\delta\varepsilon_{x\eta} = {} & -\hat{\zeta}\delta\phi' \\
\delta\varepsilon_{x\zeta} = {} & \hat{\eta}\delta\phi'
\end{aligned} \right\} \tag{32}$$

where we introduce the notation $\hat{\eta} = \eta - \lambda\zeta$; $\hat{\zeta} = \zeta + \lambda\eta$. Since the strains (and hence the stresses) are composed of the sum of terms of order ε^2 and ε^3, any product of stress and strain consists of terms of $O(\varepsilon^4)$, $O(\varepsilon^5)$, and $O(\varepsilon^6)$. Consistent with the ordering assumption ($\varepsilon^2 \ll 1$), $O(\varepsilon^6)$ terms are neglected because of the presence of $O(\varepsilon^4)$ terms. In terms of stress resultants and moments, the strain energy variation becomes

$$\begin{aligned}
\delta U = \int_0^R \{ & V_{x'}(\delta u' + v'\delta v' + w'\delta w') + (S_{x'} + T_{x'})\delta\phi' + P_{x'}\delta\phi'' \\
& + [M_{z'}\cos(\theta+\phi) + M_{y'}\sin(\theta+\phi)](\delta v'' + w''\delta\phi) \\
& + [M_{z'}\sin(\theta+\phi) - M_{y'}\cos(\theta+\phi)](\delta w'' - v''\delta\phi) \}dx
\end{aligned} \tag{33}$$

where the stress resultants and moments are defined by

$$\begin{aligned}
V_{x'} \equiv {} & \iint_A \sigma_{xx}\, d\eta\, d\zeta = EA\left\{ u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A^2\theta'\phi' - e_A[v''\cos(\theta+\phi) + w''\sin(\theta+\phi)] \right\} \\
S_{x'} \equiv {} & \iint_A (\hat{\eta}\sigma_{x\zeta} - \hat{\zeta}\sigma_{x\eta})d\eta\, d\zeta = GJ\phi' \\
T_{x'} \equiv {} & \iint_A (\theta+\phi)'\sigma_{xx}(\eta^2 + \zeta^2)d\eta\, d\zeta = EAk_A^2(\theta + \underline{\phi})'\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \\
& + \underline{\underline{EB_1^*\theta'^2\phi'}} - EB_2^*\theta'(v''\cos\theta + w''\sin\theta) \\
P_{x'} \equiv {} & -\iint_A \lambda\sigma_{xx}\, d\eta\, d\zeta = \underline{\underline{EC_1\phi''}} + EC_1^*(w''\cos\theta - v''\sin\theta) \\
M_{y'} \equiv {} & \iint_A \zeta\sigma_{xx}\, d\eta\, d\zeta = EI_{y'}[v''\sin(\theta+\phi) - w''\cos(\theta+\phi)] - EC_1^*\phi'' \\
M_{z'} \equiv {} & -\iint_A \eta\sigma_{xx}\, d\eta\, d\zeta = EI_{z'}[v''\cos(\theta+\phi) + w''\sin(\theta+\phi)] \\
& - EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) - EB_2^*\theta'\phi'
\end{aligned} \right\} \tag{34}$$

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Three third-order linear terms (doubly underlined) are retained in these equations although, consistent with the present ordering scheme, they could be neglected. These terms (and two subsequent inertial terms) contribute to the final torsion equation. In the special case of rotor blade configurations of very low torsional rigidity, these terms may contribute substantially to the magnitude of the torsion natural frequency; hence, they will be retained. The linear term $EC_1\phi''$ in $P_{x'}$ is usually neglected for beams of closed cross section.

The section integrals in equations (34) are defined as follows:

$$A \equiv \iint_A d\eta\ d\zeta \qquad\qquad Ae_A \equiv \iint_A \eta\ d\eta\ d\zeta$$

$$I_{y'} \equiv \iint_A \zeta^2\ d\eta\ d\zeta \qquad\qquad I_{z'} \equiv \iint_A \eta^2\ d\eta\ d\zeta$$

$$Ak_A^2 \equiv \iint_A (\eta^2 + \zeta^2)d\eta\ d\zeta \qquad J \equiv \iint_A (\hat{\eta}^2 + \hat{\zeta}^2)d\eta\ d\zeta \qquad\qquad (35)$$

$$B_1^* \equiv \iint_A (\eta^2 + \zeta^2)^2\ d\eta\ d\zeta \qquad B_2^* \equiv \iint_A \eta(\eta^2 + \zeta^2)d\eta\ d\zeta$$

$$C_1 \equiv \iint_A \lambda^2\ d\eta\ d\zeta \qquad\qquad C_1^* \equiv \iint_A \zeta\lambda\ d\eta\ d\zeta$$

These integrals are to be evaluated only over the portion of the blade cross section that is structurally effective. The blade cross-section area effective in carrying tension is A; $I_{y'}$ and $I_{z'}$ are the flapwise and chordwise moments of inertia, respectively; Ak_A^2 is the polar moment of inertia; and J is the torsional constant including cross-section warping. Without warping, J is equal to the polar moment of inertia. The tension axis offset from the elastic axis is e_A. C_1 is the warping rigidity, and B_1^* and B_2^* are section constants equivalent to those found in reference 1. C_1^* is not included in previous references.

Because of the antisymmetric character of λ and the assumed symmetry of the cross section about the η axis, the following integrals involving η, ζ are set equal to zero and hence do not appear in equation (34):

$$\iint_A \lambda\ d\eta\ d\zeta = 0 \qquad \iint_A \lambda(\eta^2 + \zeta^2)d\eta\ d\zeta = 0$$

$$\iint_A \zeta\ d\eta\ d\zeta = 0 \qquad\qquad \iint_A \eta\zeta\ d\eta\ d\zeta = 0 \qquad\qquad (36)$$

$$\iint_A \zeta(\eta^2 + \zeta^2)d\eta\ d\zeta = 0 \qquad\qquad \iint_A \eta\lambda\ d\eta\ d\zeta = 0$$

These integrals would be nonzero if the cross section were asymmetric.

Integration by parts of the strain energy yields

$$\delta U = \int_0^R (\overline{Y}_u \delta u + \overline{Y}_v \delta v + \overline{Y}_w \delta w + \overline{Y}_\phi \delta\phi)dx + b(U) \tag{37}$$

where

$$\overline{Y}_u = -(V_{x'})'$$

$$\overline{Y}_v = [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)]'' - (V_{x'}v')'$$

$$\overline{Y}_w = [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)]'' - (V_{x'}w')'$$

$$\overline{Y}_\phi = (P_{x'})'' - (S_{x'} + T_{x'})' - v''[M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)]$$

$$-w''[M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)]$$

$$\left.\right\} \tag{38}$$

and

$$b(U) = V_{x'}\delta u\Big|_0^R + \{V_{x'}v' - [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)]'\}\delta v\Big|_0^R$$

$$+[M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)]\delta v'\Big|_0^R + [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)]\delta w'\Big|_0^R$$

$$+\{V_{x'}w' - [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)]'\}\delta w\Big|_0^R + [S_{x'} + T_{x'} - (P_{x'})']\delta\phi\Big|_0^R$$

$$+P_{x'}\delta\phi'\Big|_0^R \tag{39}$$

It is generally essential to include $b(U)$ in equation (37) when using modal solution methods, as discussed below.

Kinetic Energy Contributions

The position of an arbitrary point after the beam has deformed is given by (x_1, y_1, z_1) where

$$x_1 = x + u - \lambda\phi' - v'[\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)] - w'[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)]$$

$$y_1 = v + \eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)$$

$$z_1 = w + \eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)$$

$$\left.\right\} \tag{40}$$

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or

$$x_1 = x + u - \lambda\phi' - v'(y_1 - v) - w'(z_1 - w) \left.\right\}$$

$$y_1 = v + (y_1 - v) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \tag{41}$$

$$z_1 = w + (z_1 - w) \left.\right\}$$

where $-\lambda\phi'$ is an axial displacement due to warp. The velocity of this point on the blade with respect to $\mathscr{R}$, the inertial frame, is

$$\vec{V} = \frac{\delta\vec{r}}{\delta t} + \Omega\vec{K} \times \vec{r} \tag{42}$$

where $\vec{r} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$, and $\delta/\delta t$ is the derivative in the rotating frame $\mathscr{B}$. The $\Omega\vec{K} \times \vec{r}$ term is the velocity contributed by the rotating coordinate system. The velocity in $\mathscr{B}$ is

$$\frac{\delta\vec{r}}{\delta t} = \dot{x}_1\vec{i} + \dot{y}_1\vec{j} + \dot{z}_1\vec{k} \tag{43}$$

and the rotating coordinate system contribution is

$$\Omega\vec{K} \times \vec{r} = -\Omega y_1 \cos \beta_{pc}\vec{i} + (\Omega x_1 \cos \beta_{pc} - \Omega z_1 \sin \beta_{pc})\vec{j} + \Omega y_1 \sin \beta_{pc}\vec{k} \tag{44}$$

Thus, the total velocity $\vec{V}$ is given by

$$\vec{V} = (\dot{x}_1 - \Omega y_1 \cos \beta_{pc})\vec{i} + (\dot{y}_1 + \Omega x_1 \cos \beta_{pc} - \Omega z_1 \sin \beta_{pc})\vec{j} + (\dot{z}_1 + \Omega y_1 \sin \beta_{pc})\vec{k} \tag{45}$$

The kinetic energy T is the volume integral of $(1/2)\rho\vec{V}\cdot\vec{V}$, or

$$T = \frac{1}{2} \int_0^R \iint_A \rho\vec{V} \cdot \vec{V} \, d\eta \, d\zeta \, dx \tag{46}$$

and the variation is simply

$$\delta T = \int_0^R \iint_A \rho\vec{V} \cdot \delta\vec{V} \, d\eta \, d\zeta \, dx \tag{47}$$

where

$$\delta\vec{V} = (\delta\dot{x}_1 - \Omega \cos \beta_{pc}\delta y_1)\vec{i} + (\delta\dot{y}_1 + \Omega \cos \beta_{pc}\delta x_1$$

$$-\Omega \sin \beta_{pc}\delta z_1)\vec{j} + (\delta\dot{z}_1 + \Omega \sin \beta_{pc}\delta y_1)\vec{k} \tag{48}$$

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The integrand of equation (46) then becomes

$$\vec{V} \cdot \delta\vec{V} = \dot{x}_1\delta\dot{x}_1 - \Omega\dot{x}_1 \cos \beta_{pc}\delta y_1 - \Omega y_1 \cos \beta_{pc}\delta\dot{x}_1 + \Omega^2 y_1 \cos \beta_{pc}\delta y_1$$

$$- +\Omega^2 x_1 \cos^2 \beta_{pc}\delta x_1 - \Omega^2 x_1 \sin \beta_{pc} \cos \beta_{pc}\delta z_1 + \Omega x_1 \cos \beta_{pc}\delta\dot{y}_1$$

$$-\Omega^2 z_1 \sin \beta_{pc} \cos \beta_{pc}\delta x_1 + \Omega^2 z_1 \sin^2 \beta_{pc}\delta z_1 - \Omega z_1 \sin \beta_{pc}\delta\dot{y}_1 + \Omega\dot{y}_1 \cos \beta_{pc}\delta x_1$$

$$-\Omega\dot{y}_1 \sin \beta_{pc}\delta z_1 + \dot{y}_1\delta\dot{y}_1 + \Omega^2 y_1 \sin^2 \beta_{pc}\delta y_1 + \Omega y_1 \sin \beta_{pc}\delta\dot{z}_1$$

$$+\Omega\dot{z}_1 \sin \beta_{pc}\delta y_1 + \dot{z}_1\delta\dot{z}_1 \tag{49}$$

According to the variational method, equation (49) must be integrated in time between two arbitrary points in time, t_1 and t_2. The initial and final values (e.g., $\dot{x}_1\delta x_1\big|_{t_1}^{t_2}$) are taken as zero. Hence, we may anticipate integrating by parts and combine various terms in equations (47) and (49) to obtain

$$\delta T = \int_0^R \iint_A [(-\ddot{x}_1 + 2\Omega\dot{y}_1 \cos \beta_{pc} + \Omega^2 x_1 \cos^2 \beta_{pc} - \Omega^2 z_1 \sin \beta_{pc} \cos \beta_{pc})\delta x_1$$

$$+(\Omega^2 y_1 - \ddot{y}_1 - 2\Omega\dot{x}_1 \cos \beta_{pc} + 2\Omega\dot{z}_1 \sin \beta_{pc})\delta y_1$$

$$+(-\Omega^2 x_1 \sin \beta_{pc} \cos \beta_{pc} + \Omega^2 z_1 \sin^2 \beta_{pc} - 2\Omega\dot{y}_1 \sin \beta_{pc} - \ddot{z}_1)\delta z_1]\rho \, d\eta \, d\zeta \, dx \tag{50}$$

where

$$\dot{x}_1 = \dot{u} - (\dot{v}' + \dot{\phi}w')(y_1 - v) - (\dot{w}' - \dot{\phi}v')(z_1 - w) - \lambda\dot{\phi}'$$

$$\dot{y}_1 = \dot{v} - \dot{\phi}(z_1 - w)$$

$$\dot{z}_1 = \dot{w} + \dot{\phi}(y_1 - v)$$

$$\ddot{x}_1 \cong \ddot{u} - (\ddot{v}' + \ddot{\phi}w' + 2\dot{\phi}\dot{w}')(y_1 - v) - (\ddot{w}' - \ddot{\phi}v' - 2\dot{\phi}\dot{v}')(z_1 - w) - \lambda\ddot{\phi}'$$

$$\ddot{y}_1 \cong \ddot{v} - \ddot{\phi}(z_1 - w)$$

$$\ddot{z}_1 \cong \ddot{w} + \ddot{\phi}(y_1 - v)$$

$$\delta x_1 = \delta u - (y_1 - v)(\delta v' + w'\delta\phi) - (z_1 - w)(\delta w' - v'\delta\phi) - \lambda\delta\phi'$$

$$\delta y_1 = \delta v - (z_1 - w)\delta\phi$$

$$\delta z_1 = \delta w + (y_1 - v)\delta\phi$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \tag{51}$$

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These expressions have been truncated consistent with the ordering scheme $(\epsilon^2 \ll 1)$. Assuming that $\beta_{pc} = 0(\epsilon)$ the variation of the kinetic energy becomes

$$\delta T = \int_0^R \iint_A (Z_u \delta u + Z_v \delta v + Z_w \delta w + Z_\phi \delta\phi$$

$$+ Z_{v'} \delta v' + Z_{w'} \delta w' + Z_{\phi'} \delta\phi')\rho \ d\eta \ d\zeta \ dx \qquad (52)$$

where

$$Z_u = \Omega^2 x + 2\Omega\dot{v}$$

$$Z_v = \Omega^2[v + (y_1 - v)] - \ddot{v} + \ddot{\phi}(z_1 - w) + 2\Omega\beta_{pc}\dot{w}$$

$$\qquad -2\Omega[\dot{u} - \dot{v}'(y_1 - v) - \dot{w}'(z_1 - w)]$$

$$Z_w = -\Omega^2\beta_{pc}x - 2\Omega\beta_{pc}\dot{v} - \ddot{w} - \ddot{\phi}(y_1 - v)$$

$$Z_\phi = -\Omega^2 x(y_1 - v)w' + \Omega^2 x(z_1 - w)v' - \Omega^2[v + (y_1 - v)](z_1 - w)$$

$$\qquad + \ddot{v}(z_1 - w) - \ddot{w}(y_1 - v) - \Omega^2\beta_{pc}x(y_1 - v) - \underline{\underline{\ddot{\phi}[(y_1 - v)^2 + (z_1 - w)^2]}}$$

$$Z_{v'} = -(\Omega^2 x + 2\Omega\dot{v})(y_1 - v)$$

$$Z_{w'} = -(\Omega^2 x + 2\Omega\dot{v})(z_1 - w)$$

$$Z_{\phi'} = -\Omega^2\lambda x$$

$$\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} \qquad (53)$$

The doubly underlined higher-order term could be dropped consistent with the ordering scheme, but this would eliminate torsion inertia from the torsion equation. As discussed previously, this term is important for small values of torsion rigidity and hence is retained.

Integrating over the blade cross section, equation (52) becomes

$$\delta T = \int_0^R (\bar{Z}_u \delta u + \bar{Z}_v \delta v + \bar{Z}_w \delta w + \bar{Z}_\phi \delta\phi$$

$$+ \bar{Z}_{v'} \delta v' + \bar{Z}_{w'} \delta w')dx \qquad (54)$$

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where

$$\overline{Z}_u = m(\Omega^2 x + 2\Omega\dot{v})$$

$$\overline{Z}_v = m\Omega^2[v + e\cos(\theta + \phi)] + 2m\Omega(\beta_{pc}\dot{w} - \dot{u})$$

$$+ 2me\Omega(\dot{v}'\cos\theta + \dot{w}'\sin\theta) - m\ddot{v} + me\ddot{\phi}\sin\theta$$

$$\overline{Z}_w = -m\beta_{pc}(\Omega^2 x + 2\Omega\dot{v}) - m\ddot{w} - me\ddot{\phi}\cos\theta$$

$$\overline{Z}_\phi = -mk_m^2\ddot{\phi} - m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\cos(\theta + \phi)\sin(\theta + \phi)$$

$$-me\Omega^2 x(w'\cos\theta - v'\sin\theta) - me\Omega^2 v\sin\theta$$

$$-me\Omega^2\beta_{pc}x\cos\theta + me(\ddot{v}\sin\theta - \ddot{w}\cos\theta)$$

$$\overline{Z}_{v'} = -me[\Omega^2 x\cos(\theta + \phi) + 2\Omega\dot{v}\cos\theta]$$

$$\overline{Z}_{w'} = -me[\Omega^2 x\sin(\theta + \phi) + 2\Omega\dot{v}\sin\theta]$$

(55)

and where the sectional integrals are defined by

$$m \equiv \iint_A \rho \, d\eta \, d\zeta \; ; \qquad me \equiv \iint_A \rho\eta \, d\eta \, d\zeta$$

$$mk_{m_1}^2 \equiv \iint_A \rho\zeta^2 \, d\eta \, d\zeta \; ; \qquad mk_{m_2}^2 \equiv \iint_A \rho n^2 \, d\eta \, d\zeta$$

$$k_{m_1}^2 + k_{m_2}^2 = k_m^2 \; ; \qquad \iint_A \rho\zeta \, d\eta \, d\zeta = \iint_A \rho n\zeta \, d\eta \, d\zeta = \iint_A \rho\lambda \, d\eta \, d\zeta = 0$$

(56)

assuming cross-section symmetry about the η axis and an antisymmetric warp function λ. The terms involving $(y_1 - v)$ and $(z_1 - w)$, introduced for convenience in equation (41), are given by

$$\iint_A \rho(y_1 - v)d\eta \, d\zeta = me\cos(\theta + \phi)$$

$$\iint_A \rho(z_1 - w)d\eta \, d\zeta = me\sin(\theta + \phi)$$

$$\iint_A \rho(y_1 - v)(z_1 - w)d\eta \, d\zeta = m(k_{m_2}^2 - k_{m_1}^2)\cos(\theta + \phi)\sin(\theta + \phi)$$

$$\iint_A \rho[(y_1 v)^2 + (z_1 - w)^2]d\eta \, d\zeta = mk_m^2$$

(57)

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The blade mass per unit length is m; the center of mass offset from the elastic axis is e; the polar mass moment of inertia is mk_m^2.

After integration by parts, the variation of the kinetic energy becomes

$$\delta T = \int_0^R [\overline{Z}_u \delta u + (\overline{Z}_v - \overline{Z}_v') \delta v + (\overline{Z}_w - \overline{Z}_w') \delta w + \overline{Z}_\phi \delta \phi] dx + b(T) \qquad (58)$$

where

$$b(T) = \overline{Z}_v' \delta v \Big|_0^R + \overline{Z}_w' \delta w \Big|_0^R \qquad (59)$$

The kinetic energy contributions are equivalent to the inertial terms of Part II. The inertial terms of reference 1, before simplifications were made therein (eqs. B.10-15 of ref. 1), are also equivalent to the present results for $\beta_{pc} = 0$.

Generalized Nonconservative Forces

The virtual work δW of the nonconservative forces may be expressed as

$$\delta W = \int_0^R (L_u \delta u + L_v \delta v + L_w \delta w + M_\phi \delta \phi) dx \qquad (60)$$

where L_u, L_v, L_w, and M_ϕ are the distributed loads that act in the x, y, and z directions and a twisting moment about the elastic axis, respectively. The nonconservative forces include all forces other than the elastic and inertial forces (e.g., aerodynamic or mechanical).

Summary of Partial Differential Equations of Motion
and Boundary Conditions

According to Hamilton's principle (eq. (29)), the strain energy and kinetic energy must be combined into one variational statement. Thus, the relative order of magnitude of the inertial terms with respect to the structural terms must be considered. The ratio of inertial terms to structural terms can be expressed as a dimensionless scaling parameter whose order of magnitude is assumed to be

$$\frac{m\Omega^2 R^2}{EA} = 0(\varepsilon^2)$$

Hence, the order of terms derived from the kinetic energy is, in effect, increased by $0(\varepsilon^2)$ so that the largest terms of the kinetic energy that are $0(\varepsilon^2)$ become $0(\varepsilon^4)$, the same order of magnitude as the largest strain energy terms. Note that the above scaling parameter is essentially the square of the

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ratio of the tip speed to the speed of sound in the blade material. For other applications, the order of magnitude of this scaling parameter may be modified accordingly. Now, for the total energy expression, the first-order terms in either the kinetic energy or strain energy of $O(\epsilon^4)$ are retained along with the second-order terms of $O(\epsilon^5)$. In general, the third-order terms of $O(\epsilon^6)$ are discarded except for certain third-order linear terms as noted above. The variational operations can now be performed on the energy expressions as they are written with no further terms discarded.

By substituting equations (37), (58), and (60) into equation (29) and using equations (34), (38), and (55), one can obtain the total variational equation in terms of u, v, w, and ϕ. For arbitrary, admissible variations δu, δv, δw, $\delta\phi$, the coefficients of the variations must vanish in the integrand for all x from 0 to R and also must vanish in the remaining terms evaluated at 0 and R. The former condition will yield four nonlinear partial differential equations for u, v, w, and ϕ. The latter will specify the boundary conditions at the ends of the beam. The four equations are as follows:

δu equation:

$$-T' - m(\Omega^2 x + \underline{2\Omega\dot{v}}) = \underline{L_u} \tag{61a}$$

δv equation:

$$-(\underline{Tv'})' + \left\{ -EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)\cos(\theta + \underline{\phi}) - EB_2^*\theta'\phi' \cos\theta - EC_1^*\phi'' \sin\theta \right.$$

$$\left. + [EI_{z'} \cos^2(\theta + \underline{\phi}) + EI_{y'} \sin^2(\theta + \underline{\phi})]v'' + (EI_{z'} - EI_{y'})\cos(\theta + \underline{\phi})\sin(\theta + \underline{\phi})w'' \right\}''$$

$$+ \underline{2m\Omega\dot{u}} + m\ddot{v} - me\ddot{\phi} \sin\theta - 2me\Omega(\dot{v}' \cos\theta + \dot{w}' \sin\theta) - m\Omega^2[v + e\cos(\theta + \phi)]$$

$$-2m\Omega\beta_{pc}\dot{w} - \{me[\Omega^2 x \cos(\theta + \phi) + 2\Omega\dot{v} \cos\theta]\}' = L_v \tag{61b}$$

δw equation:

$$-(\underline{Tw'})' + \left\{ -EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \sin(\theta + \underline{\phi}) - EB_2^*\theta'\phi' \sin\theta + EC_1^*\phi'' \cos\theta \right.$$

$$\left. + (EI_{z'} - EI_{y'})\cos(\theta + \underline{\phi})\sin(\theta + \underline{\phi})v'' + [EI_{z'} \sin^2(\theta + \underline{\phi}) + EI_{y'} \cos^2(\theta + \underline{\phi})]w'' \right\}''$$

$$+ m\ddot{w} + me\ddot{\phi} \cos\theta + 2m\Omega\beta_{pc}\dot{v} - \{me[\Omega^2 x \sin(\theta + \phi) + 2\Omega\dot{v} \sin\theta]\}' = L_w - m\Omega^2\beta_{pc}x \tag{61c}$$

$\delta\phi$ equation:

$$-\left[EAk_A{}^2(\theta + \underline{\underline{\phi}})'\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) + \underline{EB_1{}^*\theta'^2\phi'} - EB_2{}^*\theta'(v'' \cos \theta + w'' \sin \theta)\right]'$$

$$-EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)(w'' \cos \theta - v'' \sin \theta) - (GJ\phi')' + [EC_1\phi'' + \underline{\underline{EC_1{}^*(w'' \cos \theta - v'' \sin \theta)}}]''$$

$$+ \underline{(EI_z{}' - EI_y{}')[(w''^2 - v''^2) \cos \theta \sin \theta + v''w'' \cos 2\theta]} + \underline{mk_m{}^2\ddot{\phi}} + \underline{\underline{m\Omega^2\phi(k_{m_2}{}^2 - k_{m_1}{}^2) \cos 2\theta}}$$

$$+ me[\Omega^2 x(w' \cos \theta - v' \sin \theta) - (\ddot{v} - \Omega^2 v) \sin \theta + \ddot{w} \cos \theta]$$

$$= M_\phi - m\Omega^2(k_{m_2}{}^2 - k_{m_1}{}^2) \cos \theta \sin \theta - me\Omega^2\beta_{pc} x \cos \theta \qquad (61d)$$

where

$$T \equiv V_x{}' = EA\left\{u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A{}^2\theta'\phi' - e_A[v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)]\right\} \qquad (62)$$

The boundary conditions, from equations (29), (37), and (58), become

$$b(\mathcal{U}) - b(\mathcal{T}) = 0 \qquad (63)$$

Equations (34), (39), (55), and (59) may be used to express the boundary conditions in terms of the deformation:

$$b(U) - b(T) = T\delta u\Big|_0^R + \Bigg(\underline{Tv'} - \Bigg\{[EI_{z'}\cos^2(\theta + \underline{\phi}) + EI_{y'}\sin^2(\theta + \underline{\phi})]v''$$

$$+ (EI_{z'} - EI_{y'})\cos(\theta + \underline{\phi})\sin(\theta + \underline{\phi})w'' - EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)\cos(\theta + \underline{\phi})$$

$$- EB_2{}^*\theta'\phi'\cos\theta - EC_1{}^*\phi''\sin\theta\Bigg\}' + me[\Omega^2 x\cos(\theta + \phi) + 2\Omega\dot{v}\cos\theta]\Bigg)\delta v\Big|_0^R$$

$$+ \Bigg\{[EI_{z'}\cos^2(\theta + \underline{\phi}) + EI_{y'}\sin^2(\theta + \underline{\phi})]v''$$

$$+ (EI_{z'} - EI_{y'})\cos(\theta + \underline{\phi})\sin(\theta + \underline{\phi})w'' - EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)\cos(\theta + \underline{\phi})$$

$$- EB_2{}^*\theta'\phi'\cos\theta - EC_1{}^*\phi''\sin\theta\Bigg\}\delta v'\Big|_0^R$$

$$+ \Bigg(\underline{Tw'} - \Bigg\{(EI_{z'} - EI_{y'})\cos(\theta + \underline{\phi})\sin(\theta + \underline{\phi})v'' + [EI_{z'}\sin^2(\theta + \underline{\phi})$$

$$+ EI_{y'}\cos^2(\theta + \underline{\phi})]w'' - EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)\sin(\theta + \underline{\phi}) - EB_2{}^*\theta'\phi'\sin\theta$$

$$+ EC_1{}^*\phi''\cos\theta\Bigg\}' + me[\Omega^2 x\sin(\theta + \phi) + 2\Omega\dot{v}\sin\theta]\Bigg)\delta w\Big|_0^R$$

$$+ \Bigg\{(EI_{z'} - EI_{y'})\cos(\theta + \underline{\phi})\sin(\theta + \underline{\phi})v'' + [EI_{z'}\sin^2(\theta + \underline{\phi})$$

$$+ EI_{y'}\cos^2(\theta + \underline{\phi})]w'' - EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)\sin(\theta + \underline{\phi})$$

$$- EB_2{}^*\theta'\phi'\sin\theta + EC_1{}^*\phi''\cos\theta\Bigg\}\delta w'\Big|_0^R$$

$$+ \Bigg\{GJ\phi' + EAk_A{}^2(\underline{\underline{\theta + \phi}})'\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) + \underline{\underline{EB_1{}^*\theta'^2\phi'}}$$

$$- EB_2{}^*\theta'(v''\cos\theta + w''\sin\theta) - [\underline{\underline{EC_1\phi''}} + EC_1{}^*(w''\cos\theta - v''\sin\theta)]'\delta\phi\Big|_0^R$$

$$+ [EC_1\phi'' + EC_1{}^*(w''\cos\theta - v''\sin\theta)]\delta\phi'\Big|_0^R\Bigg\} \tag{64}$$

In the equations as well as the boundary conditions, only the linear expansion of $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$ should be used to avoid complicating the equations with unnecessary small terms. The single underlined terms are nonlinear terms and the double underlined terms are the third-order linear terms discussed earlier. The tension-torsion term

$$- \left[EAk_A{}^2\phi' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \right]'$$

may appear to be a nonlinear term. However, there is a third-order linear approximation based on equations (61a) and (62) such that

$$EAk_A{}^2\phi' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) = k_A{}^2\phi' \int_x^R m\Omega^2 x \; dx$$

$$+ \text{ third- (and higher) order nonlinear terms}$$

The equations of motion may be solved by Galerkin's method provided that natural boundary condition terms in equation (64) that are not identically satisfied by the assumed modal functions are added to the resulting modal equations in the standard manner (ref. 11). This operation may be necessary to insure symmetric structural and inertial operators. Thus, only the geometric boundary conditions (clamped end conditions for a cantilevered beam, e.g.) must be satisfied by the assumed modes to obtain equations equivalent to the Rayleigh-Ritz method (which deals directly with the energy expressions). By incorporating aerodynamic forces into the equations, stability analyses can be performed.

It is convenient to eliminate $u' + (v'^2/2) + (w'^2/2)$ and $\dot{u}$ from the δv, δw, and $\delta\phi$ equations and the boundary conditions before applying a modal solution procedure. From the definition of T in equation (62), $u' + (v'^2/2) + (w'^2/2)$ can be expressed in terms T, v, w, and ϕ, while T may be determined from equation (61a), thus expressing $u' + (v'^2/2) + (w'^2/2)$ in terms of v, w, and ϕ only. Solving for u', integrating over x, and then differentiating with respect to t gives an expression for $\dot{u}$ that may be substituted in equation (61b). The δu contribution to equation (64) must be used in determining $\dot{u}$ and T; $u = 0$ at $x = 0$ and $T = 0$ at $x = R$. Thus, u may be eliminated from the system of equations. T and T' in equations (61b) and (61c) are replaced by equation (61a) for T' and the expression

$$T = \int_x^R [m(\Omega^2 x + \underline{2\Omega\dot{v}}) + \underline{L_u}]dx \tag{65}$$

for T obtained by integrating equation (61a). Thus, there are now three integropartial differential equations in terms of v, w, ϕ, and modal functions are only required for these three deformations. This procedure for eliminating $\dot{u}$ and T is applied to simpler forms of the equations of motion in references 4, 8, and 9.

The equations of motion contain the same basic linear terms as those of reference 1 as well as several additional nonlinear terms. A brief discussion of the terms should suffice to familiarize the reader with the equations. Beginning with equation (65), the tension is composed of the centrifugal force

terms $m(\Omega^2 x + 2\Omega\dot{v})$ and the applied load L_u which is often neglected for helicopter applications. The $2m\Omega\dot{v}$ and L_u terms are underlined because they appear as nonlinear terms in the bending equations (δv and δw) through the centrifugal coupling terms $(Tv')'$ and $(Tw')'$. The bracketed terms ($\{\ \}''$) in the bending equations are bending moments. The

$$-EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)\begin{matrix}\cos(\theta+\phi)\\\sin(\theta+\phi)\end{matrix}$$

terms (in the $\begin{matrix}\delta v\\\delta w\end{matrix}$ equation) contain T'' as do

$$-me\left[\Omega^2 x\begin{matrix}\cos(\theta+\phi)\\\sin(\theta+\phi)\end{matrix} + 2\Omega\dot{v}\begin{matrix}\cos\theta\\\sin\theta\end{matrix}\right]'$$

and physically they show that the bending moment due to tension is proportional to $e - e_A$, the distance from the mass centroid axis to the tension axis. The $-EB_2^*\theta'\phi'\begin{matrix}\cos\theta\\\sin\theta\end{matrix}$ and $\mp EC_1^*\phi''\begin{matrix}\sin\theta\\\cos\theta\end{matrix}$ terms represent special twist and warp effect terms. The last terms in the brackets are the conventional bending moment terms modified in the sense that the cross section is inclined at the angle $\theta + \phi$. The coupling of ϕ with v'' and w'' forms the counterpart terms of the product of curvature terms in the torsion equation identified by Mil' $et\ al.$ (ref. 5). To help identify them, these terms are listed separately below. Rearranging the terms with trigonometric substitutions shows that the structural coupling terms are proportional to the difference of the flap and lead-lag bending stiffness:

$$\delta v: \quad \left\{[EI_{z'} - (EI_{z'} - EI_{y'})\sin^2(\theta + \phi)]v'' + \frac{1}{2}(EI_{z'} - EI_{y'})\sin 2(\theta + \phi)w''\right\}''$$

$$\delta w: \quad \left\{\frac{1}{2}(EI_{z'} - EI_{y'})\sin 2(\theta + \phi)v'' + [EI_{y'} + (EI_{z'} - EI_{y'})\sin^2(\theta + \phi)]w''\right\}''$$

$$\delta\phi: \quad (EI_{z'} - EI_{y'})\left[v''w''\cos 2\theta + \frac{1}{2}(w''^2 - v''^2)\sin 2\theta\right]$$

Consider now the remaining terms in the δv equation. The Coriolis term $2m\Omega\dot{u}$ is the counterpart to the $-2m\Omega\dot{v}$ in the δu equation. This term combines with the nonlinear part of $(Tv')'$ to produce Coriolis (gyroscopic) coupling with $(Tw')'$ in the δw equation. With modal functions, these terms form an antisymmetric gyroscopic matrix (ref. 8). The acceleration terms $m\ddot{v}$ and $-me\ddot{\phi}\sin\theta$ are produced by translational and rotational accelerations of the cross section, respectively. The offset Coriolis term, $-2me\Omega(\dot{v}'\cos\theta + \dot{w}'\sin\theta)$, represents the longitudinal velocity of the cross-section mass centroid due to rotation of the cross section during bending. This term also combines with $-\{me[\Omega^2 x\cos(\theta + \phi) + 2\Omega\dot{v}\cos\theta]\}'$ to give an antisymmetric gyroscopic counterpart and a symmetric stiffness counterpart to the $-\{me[\Omega^2 x\sin(\theta + \phi) + 2\Omega\dot{v}\sin\theta]\}'$ term in the δw equation. The

$-m\Omega^2[v + e\cos(\theta + \phi)]$ term is a lateral centrifugal loading due to offset from the elastic axis which passes through the center of rotation. Finally, $-2m\Omega\beta_{pc}\dot{w}$ is a Coriolis term from negative longitudinal velocity due to flap bending in the presence of precone. It is antisymmetric with $2m\Omega\beta_{pc}\dot{v}$ of the δw equation. The only terms not yet discussed in the δw equation are the accelerations $m\ddot{w}$ and $me\ddot{\phi}\cos\theta$ which are identical in origin to their counterpart terms in the δv equation and the negative centrifugal force due to precone, $-m\Omega^2\beta_{pc}x$.

The first term in the torsion equation is the tension-torsion coupling term, which arises from the tendency of a centrifugal force to untwist a pre-twisted blade. The terms involving $B_1{}^*$ and $B_2{}^*$ are special twist-effect terms, which are included in the analysis of reference 1. The terms

$$-EAe_A\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)(w''\cos\theta - v''\sin\theta)$$

and $me\Omega^2x(w'\cos\theta - v'\sin\theta)$ are approximately $-Te_Aw''$ and $T'ew'$ for zero θ and thus create a twisting moment due to tension acting on a deflected beam at a point offset from the elastic axis. The well-known St. Venant torsion rigidity term is $-(GJ\phi')'$. The $[EC_1\phi'' + EC_1{}^*(w''\cos\theta - v''\sin\theta)]''$ terms arise from the longitudinal stress doing work through the warp displacement due to torsion and bending, respectively. The next terms were identified in a more simplified analysis by Mil' (ref. 5). These terms have been shown to be important for rotor blade stability (ref. 8). The torsion inertia $mk_m{}^2\ddot{\phi}$, and the "tennis racquet" terms $m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2)\cos 2\theta$ and $-m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\cos\theta\sin\theta$, are well-known terms that tend to untwist (or, equivalently, stiffen) the rotating blade. The remaining acceleration and centrifugal terms create twisting moments because they are offset from the elastic axis.

These equations in modal form, when linearized with respect to equilibrium for $L_u = L_v = L_w = M_\phi = 0$, have symmetric stiffness and mass matrices and an antisymmetric gyroscopic matrix. This property depends on the retention of only those terms from the substitution of the tension T, that are consistent with the ordering scheme.

PART II

DERIVATION OF EQUATIONS BY NEWTON'S SECOND LAW

In this section, the equations of motion are derived from Newtonian mechanics. The derivation treats the forces and moments (elastic, inertial, or aerodynamic) applied to the rotating beam. We first consider the equilibrium of a deformed blade in terms of its geometry and the resultant forces and moments acting on it. Second, the inertial loadings on the blade are evaluated in terms of the blade deformation. Finally, the resultant structural moments and forces are determined in terms of the blade deformation based on the stress-strain and strain-displacement relations. Combining these three elements yields a set of integropartial differential equations for the longitudinal (radial), flapping, lead-lag, and torsion deformations of the elastic blade.

Equations of Structural Equilibrium

By considering a differential of length dx of the deformed beam, forces and moments may be summed to establish the equations of equilibrium (see fig. 4). For force equilibrium

$$\vec{V} + \frac{\partial \vec{V}}{\partial x} dx - \vec{V} + \vec{p}\, dx = 0$$

or

$$\frac{\partial \vec{V}}{\partial x} + \vec{p} = 0 \tag{66}$$

For moment equilibrium

$$\vec{M} + \frac{\partial \vec{M}}{\partial x} dx - \vec{M} + dx\vec{i}' \times \left(\vec{V} + \frac{\partial \vec{V}}{\partial x} dx\right) + \vec{q}\, dx = 0$$

or, taking the limit as $dx \to 0$,

$$\frac{\partial \vec{M}}{\partial x} + \vec{i}' \times \vec{V} + \vec{q} = 0 \tag{67}$$

where $\vec{p}, \vec{q}$ are "external" forces and moments of inertial or aerodynamic origin (aerodynamic forces and moments are taken as given); $\vec{V}$ and $\vec{M}$ are equipollent internal structural forces and moments acting at the elastic axis. Also note that moments have been summed about a point at x rather than at $x + dx$.[3]

[3]The use of dx rather than distance along the deformed blade might be cause for concern. However, to a consistent approximation of neglecting squares of slopes compared to one they are the same.

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The vector equations of equilibrium may be written in several component forms. Of the various choices possible, the following seems as convenient as any; namely, equations (66) and (67) are expressed in components relative to the undeformed x, y, z axes. The moment equations, in scalar component form, are

$$\frac{\partial M_x}{\partial x} + V_z \frac{\partial v}{\partial x} - V_y \frac{\partial w}{\partial x} + q_x = 0 \tag{68a}$$

$$\frac{\partial M_y}{\partial x} - V_z + V_x \frac{\partial w}{\partial x} + q_y = 0 \tag{68b}$$

$$\frac{\partial M_z}{\partial x} + V_y - V_x \frac{\partial v}{\partial x} + q_z = 0 \tag{68c}$$

This is the same as equation (16) of reference 1, except for a difference in sign convention for M_y and q_y.

Similarly, the force equations are

$$\frac{\partial V_x}{\partial x} + p_x = 0 \tag{69a}$$

$$\frac{\partial V_y}{\partial x} + p_y = 0 \tag{69b}$$

$$\frac{\partial V_z}{\partial x} + p_z = 0 \tag{69c}$$

and these may be integrated, noting that V_y, V_z, V_x are zero at the rotor tip (x = R),

$$V_x = \int_x^R p_x \, dx \tag{70a}$$

$$V_y = \int_x^R p_y \, dx \tag{70b}$$

$$V_z = \int_x^R p_z \, dx \tag{70c}$$

More precisely, the upper integration limit should be the deformed length of the beam rather than the undeformed length R. However, the deformed length is of order $R(1 + \varepsilon_{xx})$ where ε_{xx} (the longitudinal strain) may be neglected for small rotations - for example, $(\partial w/\partial x)^2 \ll 1$.

Anticipating the need for four equilibrium equations for the four deformations u, v, w, and ϕ, the six equilibrium equations (68) and (69)

(or (70)) are reduced to four, eliminating V_y and V_z in the process. Substituting (68b,c) in (68a) and (69b,c) in (68b,c) yields

$$\frac{\partial M_x}{\partial x} + \left(\frac{\partial M_y}{\partial x} + q_y\right)\frac{\partial v}{\partial x} + \left(\frac{\partial M_z}{\partial x} + q_z\right)\frac{\partial w}{\partial x} + q_x = 0 \tag{71a}$$

$$\frac{\partial^2 M_y}{\partial x^2} + p_z + \frac{\partial}{\partial x}\left(V_x \frac{\partial w}{\partial x}\right) + \frac{\partial q_y}{\partial x} = 0 \tag{71b}$$

$$\frac{\partial^2 M_z}{\partial x^2} - p_y - \frac{\partial}{\partial x}\left(V_x \frac{\partial v}{\partial x}\right) + \frac{\partial q_z}{\partial x} = 0 \tag{71c}$$

The first force relation, equation (69a), remains

$$\frac{\partial V_x}{\partial x} + p_x = 0 \tag{71d}$$

Either equation (71d) or its integrated form, equation (70a), may be used to eliminate V_x from the above equations.

There are now four equilibrium equations and seven unknowns, M_y, M_x, M_z, V_x, v, w, and ϕ. The applied loads p_x, p_y, p_z, q_x, q_y, q_z due to inertial and/or aerodynamic loadings are functions of v, w, ϕ, and a longitudinal radial deformation u in the x direction. The inertial relations are developed in the next section, and the aerodynamic loadings are assumed to be known. In addition, four force and moment deformation relations required to express M_x, M_y, M_z, V_x in terms of u, v, w, ϕ are developed in a later section. These latter relations are most easily developed in terms of the moment and force components in the deformed body axis system, x', y', z'. Hence, the equations of equilibrium will be developed for this axis system.

The necessary moment transformation between the deformed and undeformed axis systems was given by equations (2) and (3); an identical transformation holds for the force components $V_{x'}$, V_x, etc. Applying the transformation to equations (71) yields

$$\left\{ M_{x'}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) - M_{y'}[v'\cos(\theta + \phi) + w'\sin(\theta + \phi)] \right.$$

$$\left. + M_{z'}[v'\sin(\theta + \phi) - w'\cos(\theta + \phi)] \right\}' + v'q_y$$

$$+ v'\left\{ M_{x'}v' + \left(1 - \frac{v'^2}{2}\right)[M_{y'}\cos(\theta + \hat{\phi} + v'w') - M_{z'}\sin(\theta + \hat{\phi} + v'w')] \right\}'$$

$$+ w'q_z + w'\left\{ M_{x'}w' + \left(1 - \frac{w'^2}{2}\right)[M_{y'}\sin(\theta + \hat{\phi}) + M_{z'}\cos(\theta + \hat{\phi})] \right\}' + q_x = 0 \tag{72a}$$

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$$\left\{ M_{x'}v' + \left(1 - \frac{v'^2}{2}\right) \left[M_{y'}\, \cos(\theta + \hat{\phi} + v'w') - M_{z'}\, \sin(\theta + \hat{\phi} + v'w')\right] \right\}''$$

$$+ (V_x w')' + p_z + q_{y'} = 0 \qquad (72b)$$

$$\left\{ M_{x'}w' + \left(1 - \frac{w'^2}{2}\right)\left[M_{y'}\, \sin(\theta + \hat{\phi}) + M_{z'}\, \cos(\theta + \hat{\phi})\right] \right\}'' - (V_x v')' - p_y + q_{z'} = 0 \qquad (72c)$$

where

$$V_x = \int_x^R p_x\, dx$$

Note that ( )' is substituted for $\partial/\partial x$; this should cause no confusion with the primes that denote the deformed body axis system. Equation (72a) may be simplified further by first rewriting it as

$$M_{x'}' + \left[v'(M_{x'}v')' - \left(M_{x'}\, \frac{v'^2}{2}\right)'\right] + \left[w'(M_{x'}w')' - \left(M_{x'}\, \frac{w'^2}{2}\right)'\right]$$

$$+ v'\left[\left(1 - \frac{v'^2}{2}\right)M_{y'}\, \cos(\theta + \hat{\phi} + v'w')\right]' + w'\left[\left(1 - \frac{w'^2}{2}\right)M_{y'}\, \sin(\theta + \hat{\phi})\right]'$$

$$- \left\{M_{y'}\left[v'\, \cos(\theta + \phi) + w'\, \sin(\theta + \phi)\right] - M_{z'}\left[v'\, \sin(\theta + \phi) - w'\, \cos(\theta + \phi)\right]\right\}'$$

$$- v'\left[\left(1 - \frac{v'^2}{2}\right)M_{z'}\, \sin(\theta + \hat{\phi} + v'w')\right]' + w'\left[\left(1 - \frac{w'^2}{2}\right)M_{z'}\, \cos(\theta + \hat{\phi})\right]$$

$$+ q_x + v'q_y + w'q_z = 0 \qquad (73)$$

Consistent with the ordering scheme, the second and third terms may be neglected compared to the first. Also, $\hat{\phi}$ may be replaced by ϕ, and the squares and products of v' and w' may be ignored with respect to unity in equations (72) and (73). Expanding derivatives and cancelling terms in equation (73) then gives

$$M_{x'}' - M_{y'}\left[v''\, \cos(\theta + \phi) + w''\, \sin(\theta + \phi)\right] + M_{z'}\left[v''\, \sin(\theta + \phi) - w''\, \cos(\theta + \phi)\right]$$

$$+ q_x + v'q_y + w'q_z = 0 \qquad (74)$$

Equations (72b) and (72c) may be simplified further since the product of torque $M_{x'}$ and a bending slope v' or w' may be neglected compared to a bending moment $M_{x'}$ or $M_{z'}$. This is physically evident since a point load on

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the rotor blade will have a smaller associated moment arm ($\sim$ chord or thickness) for producing torques than for producing bending moments ($\sim$ radius). This may be shown formally by integrating equations (72a-c), multiplying equation (72a) by v' or w', and subtracting the result from the equation (72b) or (72c). This calculation eliminates the terms $M_{x'}v'$ and $M_{x'}w'$ in equations (72b) and (72c), respectively. The additional terms introduced by the subtraction will be negligible consistent to second order provided that the applied torque loading is small compared to the applied bending moments:

$$q_x \ll q_y, \; q_z$$

Therefore, equations (72b) and (72c) may be written

$$[M_{y'} \cos(\theta + \phi) - M_{z'} \sin(\theta + \phi)]'' + (V_x w')' + p_z + q_y' = 0 \qquad (75)$$

$$[M_{y'} \sin(\theta + \phi) + M_{z'} \cos(\theta + \phi)]'' - (V_x v')' - p_y + q_z' = 0 \qquad (76)$$

The linear approximations for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$ should also be used in equations (72):

$$\left. \begin{aligned} \cos(\theta + \phi) &\cong \cos\theta - \phi \sin\theta \\ \sin(\theta + \phi) &\cong \sin\theta + \phi \cos\theta \end{aligned} \right\} \qquad (77)$$

In addition to equations (74)-(76) for moment equilibrium, an equation for $V_{x'}$ is needed if the tension changes significantly, as a function of the rotor blade deformation, from the nominal centrifugal force value due to blade rotation. The equations for $V_{x'}$, $V_{y'}$, and $V_{z'}$ are obtained by transforming equations (69a-c) to the deformed axis system, giving

$$V_{x'}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) - V_{y'}[v' \cos(\theta + \phi) + w' \sin(\theta + \phi)]$$

$$+ V_{z'}[v' \sin(\theta + \phi) - w' \cos(\theta + \phi)] = \int_x^R p_x \, dx \qquad (78a)$$

$$V_{x'}v' + \left(1 - \frac{v'^2}{2}\right)[V_{y'} \cos(\theta + \hat{\phi} + v'w') - V_{z'} \sin(\theta + \hat{\phi} + v'w')] = \int_x^R p_y \, dx \qquad (78b)$$

$$V_{x'}w' + \left(1 - \frac{w'^2}{2}\right)[V_{y'} \sin(\theta + \hat{\phi}) + V_{z'} \cos(\theta + \hat{\phi})] = \int_x^R p_z \, dx \qquad (78c)$$

The required equation for $V_{x'}$ is obtained by substituting $V_{y'}$ and $V_{z'}$ from equations (78b) and (78c) into equation (78a). After neglecting higher-order terms consistent to second order, $V_{x'}$ becomes

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$$V_{x'} = \int_x^R p_x \, dx + v' \int_x^R p_y \, dx + w' \int_x^R p_z \, dx \qquad (79)$$

Equations (74)-(76) and (79) are the principal expressions for the equilibrium of internal and external forces and moments acting on the blade. For rotor blade applications, the second and third terms are negligible to second order; thus, $V_{x'} = V_x$.

Inertial Loading

The present derivation of the inertia loads closely follows appendix B of reference 1 except that nonlinear terms are retained in accordance with the ordering scheme. The deformed position of a generic point of the rotor blade will be expressed with respect to the rotating undeformed, nonpreconed x_p, y, Z coordinate system. Note that *here* (but not elsewhere in this report), u, v, w, ϕ, p_x, p_y, p_z, q_x, q_y, q_z are measured with respect to a nonpreconed rather than a preconed coordinate system; the nonpreconed coordinate system corresponds to that of reference 1. The deformations and loadings with respect to the two coordinate systems are related by the following transformation, if the precone angle β_{pc} is small:

$$\left.\begin{array}{ll}
u_{pc} = u + \beta_{pc}w & px_{pc} = p_x + \beta_{pc}p_z \\[2mm]
v_{pc} = v & qx_{pc} = q_x + \beta_{pc}q_z \\[2mm]
w_{pc} = w & pz_{pc} = p_z - \beta_{pc}p_x \\[2mm]
\phi_{pc} = \phi & qz_{pc} = q_z - \beta_{pc}q_x \\[2mm]
x_{pc} = x & py_{pc} = p_y; \quad qy_{pc} = q_y
\end{array}\right\} \qquad (80)$$

The inertia terms are derived using the nonpreconed variables. In the final equations for the inertial loadings, however, equation (80) will be used to transform the forces, moments, and deformations into preconed coordinate system consistent with the remainder of this report.

As a result of equation (80), the position of a generic point with respect to the x_p, y, Z axes (see eqs. (40)) becomes

$$x_1 = x + u - v'(y_1 - v) - (w' + \beta_{pc})(z_1 - w - \beta_{pc}x) - \lambda\phi' \qquad (81a)$$

$$y_1 = v + y \cos(\theta + \phi) - \zeta \sin(\theta + \phi) \qquad (81b)$$

$$z_1 = w + \beta_{pc}x + y_1 \sin(\theta + \phi) + \zeta \cos(\theta + \phi) \qquad (81c)$$

Substituting equations (81b) and (81c) into equation (81a) yields

$$x_1 = x + u - \lambda\phi' - v'[\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)] - (w' + \beta_{pc})[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)]$$

The time derivatives are

$$\dot{x}_1 = \dot{u} - [\dot{v}' + (w' + \beta_{pc})\dot{\phi}][\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)]$$

$$+ (v'\dot{\phi} - \dot{w}')[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)] - \lambda\dot{\phi}'$$

$$\ddot{x}_1 = \ddot{u} - [\ddot{v}' + \dot{w}'\dot{\phi} + (w' + \beta_{pc})\ddot{\phi} + (\dot{w}' - v'\dot{\phi})\dot{\phi}][\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)]$$

$$- \lambda\ddot{\phi}' + \{[\dot{v}' + (w' + \beta_{pc})\dot{\phi}]\dot{\phi} + \dot{v}'\dot{\phi} + v'\ddot{\phi} - \ddot{w}'\}[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)]$$

(83a)

$$\dot{y}_1 = \dot{v} - \dot{\phi}[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)]$$

$$\ddot{y}_1 = \ddot{v} - \dot{\phi}^2[\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)] - \ddot{\phi}[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)]$$

(83b)

$$\dot{z}_1 = \dot{w} + \dot{\phi}[\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)]$$

$$\ddot{z}_1 = \ddot{w} - \dot{\phi}^2[\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)] + \ddot{\phi}[\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)]$$

(83c)

The components of acceleration in the $\vec{i}, \vec{j}, \vec{k}$ coordinate system, including the usual contributions due to rotation Ω, are (ref. 1)

$$a_x = \ddot{x}_1 - \Omega^2 x_1 - 2\Omega\dot{y}_1$$

$$a_y = \ddot{y}_1 - \Omega^2 y_1 + 2\Omega\dot{x}_1$$

$$a_z = \ddot{z}_1$$

(84)

The inertial forces $\vec{p}^I$ and moments $\vec{q}^I$ may now be evaluated in a D'Alembert sense using equations (81)-(84) and the following definitions

$$p_x{}^I \equiv -\iint_A \rho a_x \, d\eta \, d\zeta$$

$$p_y{}^I \equiv -\iint_A \rho a_y \, d\eta \, d\zeta$$

$$p_z{}^I \equiv -\iint_A \rho a_z \, d\eta \, d\zeta$$

(85)

$$q_x{}^I \equiv \iint_A \rho [a_y(z_1 - w) - a_z(y_1 - v)] d\eta \, d\zeta$$

$$q_y{}^I \equiv -\iint_A \rho a_x(z_1 - w) d\eta \, d\zeta$$

$$q_z{}^I \equiv \iint_A \rho a_x(y_1 - v) d\eta \, d\zeta$$

Note that the present sign convention for $q_y{}^I$ is different from reference 1. Equations (81)-(84) are substituted into equations (85) and (86), and higher order terms are discarded. In reference 1, it is assumed that

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$$\cos(\theta + \phi) \cong \cos \theta$$

$$\sin(\theta + \phi) \cong \sin \theta$$

and all products in u, v, w, ϕ and their derivatives are ignored - that is, equations (85) and (86) are linearized. This is not the same as simply assuming ϕ, v', etc., are small and using the small-angle approximation. Here we apply a consistent set of simplifying approximations to equations (85) and (86). In ordering the various terms it is noted that $(e/R)^2$, $(c/R)^2$, $(t/R)^2$ are small compared to one; hence, products of these quantities with themselves or with v/R, w/R or ϕ can be neglected with respect to unity. The order of magnitude estimates $(\ )' \sim 1/R$, $(\cdot) \sim \Omega$ are also used.

The evaluation of $\vec{p}^I$ and $\vec{q}^I$ from equations (85) and (86) and the transformation from nonpreconed to preconed axes leads to the same inertial terms that were developed in Part I. To second order, the results for the nonpreconed variables are

$$\left. \begin{aligned}
p_x^{\ I} &= m(\Omega^2 x + 2\Omega \dot{v}) \\[2mm]
p_y^{\ I} &= -m\{\ddot{v} - e\ddot{\phi} \sin \theta - \Omega^2[v + e \cos(\theta + \phi)] \\[2mm]
&\quad + 2\Omega[\dot{u} - e(\dot{v}' \cos \theta + \dot{w}' \sin \theta)]\} \\[2mm]
p_z^{\ I} &= -m(\ddot{w} + e\ddot{\phi} \cos \theta)
\end{aligned} \right\} \qquad (87)$$

$$\left. \begin{aligned}
q_x^{\ I} &= m\{e[(\ddot{v} - \Omega^2 v)\sin(\theta + \phi) - \ddot{w} \cos(\theta + \phi) + 2\Omega\dot{u} \sin \theta] \\[2mm]
&\quad \underline{\underline{-k_m^2\ddot{\phi}}} - \Omega^2 (k_{m_2}^2 - k_{m_1}^2)\cos(\underline{\underline{\theta + \phi}})\sin(\underline{\underline{\theta + \phi}}) \\[2mm]
&\quad -2\Omega[(k_{m_2}^2 - k_{m_1}^2)\dot{v}' \sin \theta \cos \theta + \dot{w}'(k_{m_2}^2 \sin^2 \theta + k_{m_1}^2 \cos^2 \theta)]\} \\[2mm]
q_y^{\ I} &= me[\Omega^2 x \sin(\theta + \phi) + 2\Omega\dot{v} \sin \theta] \\[2mm]
q_z^{\ I} &= -me[\Omega^2 x \cos(\theta + \phi) + 2\Omega\dot{v} \cos \theta]
\end{aligned} \right\} \qquad (88)$$

Everywhere in equations (87) and (88), the usual linearized expressions for $\sin(\theta + \phi)$, $\cos(\theta + \phi)$ should be used. Additional simplifications will be made when these terms are compared to the elastic forces. For example, the third-order terms of $q_x^{\ I} + v'q_y^{\ I} + w'q_z^{\ I}$ are neglected except the double underlined ones.

The correspondence between the present inertial forces and those of Part I is shown by

$$\left. \begin{aligned}
q_x^{\ I} + v'q_y^{\ I} + w'q_z^{\ I} &\leftrightarrow \overline{Z}_\phi \\[2mm]
p_z^{\ I} + (q_y^{\ I})' &\leftrightarrow \overline{Z}_w - \overline{Z}_{w'}' \\[2mm]
p_y^{\ I} - (q_z^{\ I})' &\leftrightarrow \overline{Z}_v - \overline{Z}_{v'}'
\end{aligned} \right\} \qquad (89)$$

Force-Deformation and Moment-Deformation Relations

The force and moment resultants are expressed in terms of the deformations by resolving the distributed stresses into a resultant force and moment system acting at the elastic axis. These resultants are essentially the same as those obtained in Part I. Here, they are derived from a physical point of view, whereas in the variational method they arise as mathematical groups of terms. The resultant axial force is

$$
V_{x'} \equiv \iint_A \sigma_{xx} \, d\eta \, d\zeta = \iint_A E \varepsilon_{xx} \, d\eta \, d\zeta
$$

$$
= EA \left\{ u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A^2 \theta' \phi' - e_A [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \right\} \tag{90}
$$

The bending moments are

$$
M_{y'} \equiv \iint_A \zeta \sigma_{xx} \, d\eta \, d\zeta = \iint_A E \zeta \varepsilon_{xx} \, d\eta \, d\zeta
$$

$$
= EI_{y'} [v'' \sin(\theta + \phi) - w'' \cos(\theta + \phi)] - EC_1^* \phi'' \tag{91}
$$

$$
M_{z'} \equiv -\iint_A \eta \sigma_{xx} \, d\eta \, d\zeta = \iint_A E \eta \varepsilon_{xx} \, d\eta \, d\zeta
$$

$$
= -EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) - EB_2^* \theta' \phi' + EI_{z'} [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \tag{92}
$$

The twisting moment with respect to the x' axis is more complicated because of the interaction of longitudinal and shear strains with the warp. Brunelle (ref. 12) has identified the twisting moment by use of the variational method. For the assumptions of the present work, his equation takes on a slightly different form

$$
\tilde{M}_{x'} = \iint_A \left[\eta \sigma_{x\zeta} - \zeta \sigma_{x\eta} + \lambda \left(\frac{\partial \sigma_{x\eta}}{\partial \eta} + \frac{\partial \sigma_{x\zeta}}{\partial \zeta} \right) \right] d\eta \, d\zeta + \frac{\partial}{\partial x} \iint_A \lambda \sigma_{xx} \, d\eta \, d\zeta
$$

$$
= S_{x'} - (P_{x'})' \tag{93}
$$

where

$$
\left.
\begin{aligned}
S_{x'} &\equiv \iint_A [(\eta - \lambda_\zeta) \sigma_{x\zeta} - (\zeta + \lambda_\eta) \sigma_{x\eta}] \, d\eta \, d\zeta = GJ \phi' \\
P_{x'} &\equiv -\iint_A \lambda \sigma_{xx} \, d\eta \, d\zeta = \underline{\underline{EC_1 \phi''}} + EC_1^* (w'' \cos \theta - v'' \sin \theta)
\end{aligned}
\right\} \tag{94}
$$

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The contribution of the P_X, term is clearly related to the work done by the longitudinal stress acting through the virtual displacement in the longitudinal direction due to warp. Moreover, as shown in reference 1 (based on a more detailed discussion in ref. 13), there is an additional twisting moment due to longitudinal stress:

$$T_X' \equiv (\theta + \phi)' \iint_A (\eta^2 + \zeta^2)\sigma_{xx} \, d\eta \, d\zeta$$

$$= EAk_A^2(\theta + \underline{\underline{\phi}})' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) + \underline{\underline{EB_1^*\theta'^2\phi'}} - EB_2^*\theta'(v'' \cos\theta + w'' \sin\theta) \qquad (95)$$

This expression represents the component of force due to longitudinal stress normal to the elastic axis with moment arm $\sqrt{\eta^2 + \zeta^2}$. The total resultant twisting moment is now expressed by combining equations (93) and (95):

$$M_X' = T_X' + S_X' - (P_X')' = GJ\phi' + EAk_A^2(\theta + \underline{\underline{\phi}})' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right)$$

$$+\underline{\underline{EB_1^*\theta'^2\phi'}} - EB_2^*\theta'(v'' \cos\theta + w'' \sin\theta)$$

$$-[EC_1\phi'' + EC_1^*(w'' \cos\theta - v'' \sin\theta)]' \qquad (96)$$

The double underlined terms may be neglected according to the ordering scheme. However, as discussed above, these terms are important for configurations with low torsion stiffness. The various section constants in the above resultants are defined in Part I, equation (35).

Equations (90)-(96) give the desired force- and moment-deformation relations. They can be simplified by making the usual small-angle assumptions:

$$\cos(\theta + \phi) = \cos\theta - \phi\sin\theta$$

$$\sin(\theta + \phi) = \sin\theta + \phi\cos\theta$$

In general, no additional approximations can be made, although for special blade cross sections, several of the blade section constants may reduce to zero - for example, for a doubly symmetric cross section $e_A = B_2^* = 0$.

Final Equations and Boundary Conditions

The four equations of equilibrium, the inertial loadings, and the force- and moment-deformation relations may be combined to give the final equations of motion in terms of the deformations u, v, w, ϕ. That is, equations (87), (88), transformed to the preconed axes, (90), (91), (92), and (96) are substituted into equations (74)-(76) and (79). The resulting equations are simplified consistent to second order by discarding higher-order terms in

accordance with the ordering scheme. The final equations are identical to those obtained by the variational method in Part I, and are not repeated here.

As shown in Part I, the boundary conditions for the variational method were obtained as a natural by-product of the derivation of the equations of motion, without requiring any independent consideration. For the Newtonian method, special attention must be given to the boundary conditions, which are of two types. The geometric conditions, normally applied at $x = 0$, are simply

$$u = v = w = \phi = v' = w' = 0 \quad \text{for } x = 0 \tag{97}$$

The natural conditions for the free end $x = R$ may be expressed as $\vec{M} = \vec{V} = 0$ or, since $V_x = V_{x'}$ to second order,

$$M_{x'} = M_{y'} = M_{z'} = V_{x'} = V_y = V_z = 0 \quad \text{for } x = R \tag{98}$$

In addition, the work done by the longitudinal stresses acting through the warp displacements must be zero at $x = R$. Thus

$$P_{x'} = 0 \quad \text{for } x = R \tag{99}$$

Restrained warping at the hub implies that

$$\phi' = 0 \quad \text{for } x = 0 \tag{100}$$

That is, there can be no warp displacement at $x = 0$. These conditions reduce to the standard cantilever boundary conditions when warp effects (C_1 and C_1^*) are neglected. Recall from equations (68) and (69) that the shears V_y, V_z may be expressed in terms of $V_{x'}$, $M_{y'}$, $M_{z'}$, q_y, and q_z. When $x = R$, $V_{x'} = 0$ and the shears may be written (for $q_x \ll q_y$, q_z)

$$\left. \begin{array}{l} V_y = -M_{y'}' \sin(\theta + \phi) - M_{z'}' \cos(\theta + \phi) - q_z = 0 \\[2mm] V_z = M_{y'}' \cos(\theta + \phi) - M_{z'}' \sin(\theta + \phi) + q_y = 0 \end{array} \right\} \quad \text{for } x = R \tag{101}$$

The boundary conditions (97)-(101), when expressed in terms of the deformations u, v, w, and ϕ, are equivalent to equation (64) - the boundary conditions as derived from the variational method.

CONCLUDING REMARKS

Nonlinear equations of motion for the elastic bending and torsion of twisted nonuniform rotor blades have been derived by two complementary methods. When deriving a system of nonlinear equations it is advantageous to check the resulting equations in this manner. The use of the Newtonian method alone, for example, may give rise to nonsymmetric structural or inertial operators. However, with the variational method, a consistent set of equations will

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automatically result if the energy expressions are accurate to the desired order of magnitude. With the Newtonian method, the boundary conditions must be established on the basis of physical reasoning. The variational method leads to the appropriate boundary conditions as a by-product of the derivation of the equations of motion. The two methods are complementary in that the Newtonian method provides clearer physical understanding of the force components and reactions, while the variational method is more precise mathematically - that is, there is less chance of inadvertently leaving out important terms or including unnecessary ones.

In the resulting system of equations, several important nonlinear terms are identified. First, the centrifugal term proportional to lead-lag velocity in the tension equation combines with the centrifugal coupling terms in the bending equations to produce nonlinear flap-lag inertial terms. The longitudinal velocity in the lead-lag equation, a Coriolis term, is expressed in terms of bending quantities as another nonlinear flap-lag inertial term. These terms, when linearized with respect to equilibrium, are antisymmetric gyroscopic terms and significantly influence hingeless rotor stability and forced response phenomena. Second, the nonlinear bending-torsion coupling term in the torsion equation is written in a form similar to the one identified by Mil' in reference 5. The twisting moment arises from bending in two directions and is proportional to the difference in bending stiffness and the product of curvatures. The counterpart nonlinear bending-torsion coupling terms in the bending equations appear in the form of a change in elastic coupling due to elastic twist. These bending-torsion coupling terms are also important in determining the aeroelastic stability of hingeless rotors. References 8 and 9 give a preliminary discussion of the application of the present equations to a stability analysis of uniform blades.

The effect of warp interacting with the longitudinal stress is included for completeness, although it can be neglected without essential loss of accuracy for most applications involving closed cross sections.

An important step in the derivation of the equation is the development of a nonlinear strain-displacement relation based on an exact coordinate transformation from the undeformed system to the deformed system. This general relation between strain and displacement may be useful for extending the equations to include higher-order terms than these considered herein.

Future improvements in the equations should include the incorporation of torque offset, root-pitching motion with elastic restraint, droop, pre-lag, and other parameters (discussed in ref. 9) to describe more completely the behavior of hingeless rotor blades. With the addition of aerodynamic forces, the equations of motion may be used for stability analyses and frequency response studies of hingeless helicopter rotor blades.

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APPENDIX

DEFORMED BLADE COORDINATE SYSTEM TRANSFORMATION

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In the section devoted to the development of the strain-displacement relations, the transformation [T], given to second order in equations (1), is of central importance. Here [T] is determined in terms of u, v, w, and ϕ based upon the development of reference 14. [T] is the transformation between the undeformed blade coordinate system $\vec{i}$, $\vec{j}$, $\vec{k}$, and the deformed blade system $\vec{i}'$, $\vec{j}'$, $\vec{k}'$:

$$\begin{Bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{Bmatrix} = [T] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \tag{A1}$$

In terms of Euler angles (fig. 5), [T] may be expressed as

$$[T] = \begin{bmatrix} \cos\bar\beta\cos\bar\zeta & \cos\bar\beta\sin\bar\zeta & \sin\bar\beta \\ -\sin\bar\theta\sin\bar\beta\cos\bar\zeta -\cos\bar\theta\sin\bar\zeta & \cos\bar\theta\cos\bar\zeta -\sin\bar\zeta\sin\bar\beta\sin\bar\theta & \cos\bar\beta\sin\bar\theta \\ -\cos\bar\theta\sin\bar\beta\cos\bar\zeta +\sin\bar\theta\sin\bar\zeta & -\sin\bar\theta\cos\bar\zeta -\sin\bar\zeta\sin\bar\beta\cos\bar\theta & \cos\bar\beta\cos\bar\theta \end{bmatrix} \tag{A2}$$

The Euler angles are taken in the order $\bar\zeta$, $\bar\beta$, $\bar\theta$. They uniquely define the orientation of the blade principal axes with respect to the undeformed coordinate system. Rotor blade equations are normally written in terms of v, w, and ϕ, rather than $\bar\zeta$, $\bar\beta$, $\bar\theta$. As shown in figure 5, the Euler angles $\bar\zeta$ and $\bar\beta$ can be easily expressed in terms of v and w, yielding

$$[T] = \begin{bmatrix} \sqrt{1-v^{+2}-w^{+2}} & v^+ & w^+ \\ -[\sin\bar\theta w^+\sqrt{1-v^{+2}-w^{+2}}+\cos\bar\theta v^+]/\sqrt{1-w^{+2}} & [\cos\bar\theta\sqrt{1-v^{+2}-w^{+2}}-\sin\bar\theta v^+w^+]/\sqrt{1-w^{+2}} & \sin\bar\theta\sqrt{1-w^{+2}} \\ -[\cos\bar\theta w^+\sqrt{1-v^{+2}-w^{+2}}+\sin\bar\theta v^+]/\sqrt{1-w^{+2}} & -[\sin\bar\theta\sqrt{1-v^{+2}-w^{+2}}+\cos\bar\theta v^+w^+]/\sqrt{1-w^{+2}} & \cos\bar\theta\sqrt{1-w^{+2}} \end{bmatrix} \tag{A3}$$
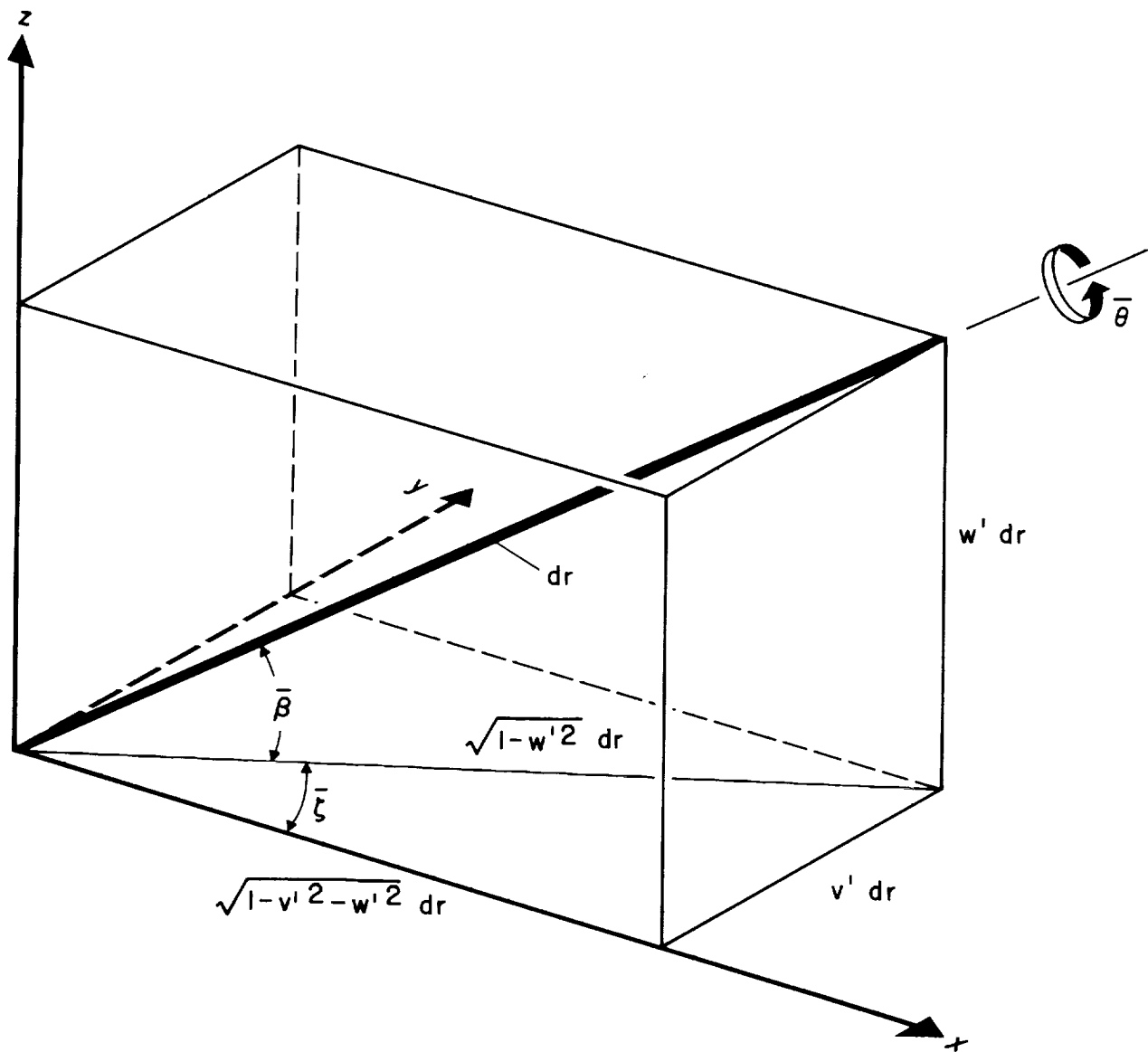
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Figure 5.- Deformation and Euler angles.

The determination of the third Euler angle $\bar{\theta}$, however, requires the formulation and solution of a differential equation for $[T]$. Consider the small rotation $\vec{\omega}dr$ of the blade-fixed coordinate system, which occurs as r goes through the increment dr. (To second order there is no distinction between distance along the deformed dr and undeformed dx elastic axes; hence, $dr \approx dx$.) The vector components of the rate of rotation $\vec{\omega}$ can be identified as the torsional rotation rate, $\omega_i = (\theta + \phi)^+$,[1] and the bending curvature ω_j and ω_k as shown in figure 6. This infinitesimal rotation can be written in terms of the transformation matrix $[T]$, giving the differential equation

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[1]Note that in this purely geometric exercise there is no distinction between pretwist θ and elastic twist ϕ.
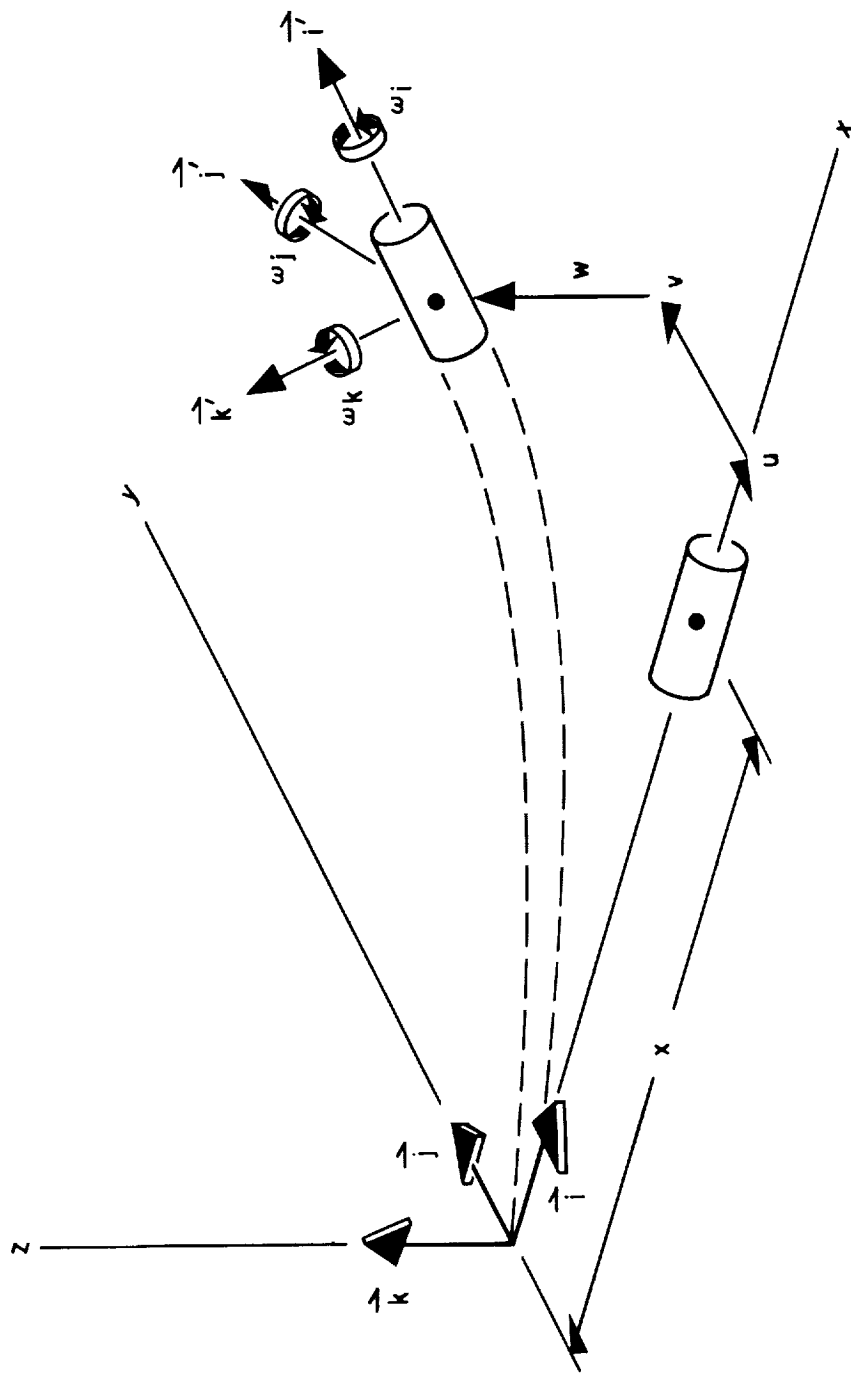
Figure 6.- Transformation geometry.

$$[T]^+ = \begin{bmatrix} 0 & \omega_k & -\omega_j \\ -\omega_k & 0 & \omega_i \\ \omega_j & -\omega_i & 0 \end{bmatrix} [T]$$

which can be solved by applying the identity $[T]^{-1} = [T]^T$:

$$\begin{bmatrix} 0 & \omega_k & -\omega_j \\ -\omega_k & 0 & \omega_i \\ \omega_j & -\omega_i & 0 \end{bmatrix} = [T]^+ [T]^T \qquad (A4)$$

Formally, there are nine equations; however, there are only three independent ones due to the usual orthonormality conditions on the transformation matrix. Substitution of equation (A3) into (A4), eliminating ω_j and ω_k and identifying $\omega_i = (\theta + \phi)^+$, yields the exact solution

$$\bar{\theta} = \theta + \phi - \int_0^r \frac{v^{++}w^+ \, dr}{\sqrt{1 - v^{+2} - w^{+2}}} - \int_0^r \frac{w^{+2}w^{++}v^+ \, dr}{(1 - w^{+2})\sqrt{1 - v^{+2} - w^{+2}}} \qquad (A5)$$

where the pretwist and twist are both taken to be zero at the root. To second order, equation (A5) may be written

$$\bar{\theta} = \theta + \phi - \int_0^x v''w' \, dx \qquad (A6)$$

Thus, from (A6) and (A3), [T] may be written to second order as

$$[T] = \begin{bmatrix} 1 - \dfrac{v'^2}{2} - \dfrac{w'^2}{2} & v' & w' \\ -[v'\cos(\theta+\phi) + w'\sin(\theta+\phi)] & \cos(\theta+\hat{\phi}+v'w')\left(1 - \dfrac{v'^2}{2}\right) & \sin(\theta+\hat{\phi})\left(1 - \dfrac{w'^2}{2}\right) \\ [v'\sin(\theta+\phi) - w'\cos(\theta+\phi)] & -\sin(\theta+\hat{\phi}+v'w')\left(1 - \dfrac{v'^2}{2}\right) & \cos(\theta+\hat{\phi})\left(1 - \dfrac{w'^2}{2}\right) \end{bmatrix}$$

$$(A7)$$

where

$$\hat{\phi} \equiv \phi - \int_0^x v''w' \, dx$$

Equation (A7) is the desired result, compare equations (1).

It is also useful to record here expressions for ω_j and ω_k, which can be obtained directly from (A3) and (A4):

$$\left.\begin{aligned}
\omega_j &= \frac{-\cos\overline{\theta}w^{++}\sqrt{1 - v^{+2} - w^{+2}} + \sin\overline{\theta}(v^+w^+w^{++} + v^{++} - v^{++}w^{+2})}{\sqrt{1 - v^{+2} - w^{+2}}\,\sqrt{1 - w^{+2}}} \\[2em]
\omega_k &= \frac{\cos\overline{\theta}(v^+w^+w^{++} + v^{++} - v^{++}w^{+2}) + \sin\overline{\theta}w^{++}\sqrt{1 - v^{+2} - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}\,\sqrt{1 - w^{+2}}}
\end{aligned}\right\} \tag{A8}$$

To second order, these may be simplified as

$$\left.\begin{aligned}
\omega_j &= v''\sin(\theta + \phi) - w''\cos(\theta + \phi) \\
\omega_k &= v''\cos(\theta + \phi) + w''\sin(\theta + \phi)
\end{aligned}\right\} \tag{A9}$$

REFERENCES

1. Houbolt, J. C.; and Brooks, G. W.: Differential Equations of Motion for Combined Flapwise Bending, Chordwise Bending, and Torsion of Twisted Nonuniform Rotor Blades. NACA Rep. 1346, 1958.

2. Ormiston, R. A.; and Hodges, D. H.: Linear Flap-Lag Dynamics of Hingeless Helicopter Rotor Blades in Hover. *J. Am. Helicopter Soc.*, vol. 17, no. 2, April 1972, pp. 2-14.

3. Huber, H. B.: Effect of Torsion-Flap-Lag Coupling on Hingeless Rotor Stability. Am. Helicopter Soc. Preprint 731, May 1973.

4. Hodges, D. H.; Ormiston, R. A.: Nonlinear Equations for Bending of Rotating Beams With Application to Linear Flap-Lag Stability of Hingeless Rotors. NASA TM X-2770, May 1973.

5. Mil', M. L., et al.: *Helicopters: Calculation and Design,* vol. I, Aerodynamics. NASA TT F-494, Sept. 1967, pp. 430-432.

6. Arcidiacono, P. J.: Steady Flight Differential Equations of Motion for a Flexible Helicopter Blade With Chordwise Mass Unbalance. USAAVLABS TR68-18A, vol. I, Feb. 1969.

7. Friedmann, P.; and Tong, P.: Dynamic Nonlinear Elastic Stability of Helicopter Rotor Blades in Hover and in Forward Flight. NASA CR-114485, May 1972.

8. Hodges, D. H.: Nonlinear Bending and Torsion Equations for Rotating Beams With Application to Linear Stability of Hingeless Rotors. PhD Thesis, Stanford University, Department of Aeronautics and Astronautics, Dec. 1972.

9. Hodges, D. H.: and Ormiston, R. A.: Stability of Elastic Bending and Torsion of Uniform Cantilevered Rotor Blades in Hover. AIAA Paper no. 73-405, March 1973.

10. Novozhilov, V. V.: *Foundations of the Nonlinear Theory of Elasticity.* Translated edition by Graylock Press, Rochester, New York, 1953.

11. Duncan, W. J.: Galerkin's Method in Mechanics and Differential Equations. ARC R and M 1798, Aug. 1937.

12. Brunelle, E. J.: Dynamical Torsion Theory of Rods Deduced From the Equations of Linear Elasticity, *AIAA J.,* vol. 10, no. 4, April 1972, pp. 524-526.

13. Timoshenko, S.; and Goodier, J. N.: *Theory of Elasticity.* Second edition, McGraw-Hill Book Co., Inc., 1951, pp. 333-336.

14. Peters, D. A.; and Ormiston, R. A.: The Effects of Second-Order Blade Bending on the Angle of Attack of Hingeless Rotor Blades. *J. Am. Helicopter Soc.*, vol. 18, no. 4, Oct. 1973, pp. 45-48.