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EFFECT OF LIMB DARKENING ON EARTH RADIATION INCIDENT ON A SPHERICAL SATELLITE

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SUMMARY

The thermal radiation from the Earth incident on a spherical satellite depends on the angular distribution of Earth-emitted radiation. (The deviation from a Lambertian distribution is commonly referred to as limb darkening.) An analysis is presented of this dependency, and calculated results are given, based on a published limb-darkening curve for the Earth. The curve was determined from Tiros data, and is a statistical average over the entire globe between $\pm 75^{\circ}$ latitude. The computed effect of limb darkening was 1.8 percent at 900 km altitude, 2.5 percent at 500 km altitude, and 3.0 percent at 300 km altitude. Below 300 km, it increased rapidly with decreasing altitude.

Discussion is included of various other problems inherent in the use of orbiting spheres and stabilized flat plates to measure the heat radiated from the Earth.

INTRODUCTION

This paper discusses the following problem: Given the intensity (power per unit area) of the thermal radiation emitted from the Earth, to what extent is the corresponding irradiation of a spherical satellite a function of the directional distribution of this emitted radiation? The problem arises because it has been proposed to study the energy budget of the Earth through measurements of the temperatures within a fleet of such orbiting spheres; for this purpose, it is obviously necessary that the temperatures be essentially independent of the directional distribution of the Earth-emitted radiation, or else it is necessary to account for such a dependency.

The analysis is made for the cases in which the directional distributions are given as (1) a power series in the zenith angle, and (2) a power series in the cosine of the zenith angle. In order to show the order of magnitude of the effect for a practical case, some calculated results are given in which experimental satellite data for the directional distribution were used.

The following section provides some further background concerning the special purpose of the thermal balance measurements, the desired accuracy, and some possible approaches, with their inherent difficulties. It should also help to put the subject of the present analysis in perspective as one among several troublesome basic questions regarding the use of the sphere as an integrating radiation sensor in attempting to measure the Earth's thermal balance to high accuracy.

PERSPECTIVE OF THE PROBLEM

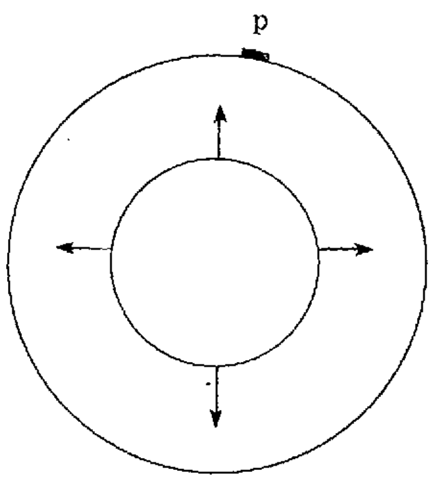
Measuring and understanding the Earth's energy budget have been among the basic problems in meteorology. The recent development of popular concern with the environment, however, has made more general the recognition that gradual modification of the atmosphere and of land and water areas, resulting both from natural causes and, especially, from man's activities, may eventually produce significant, or even profound, changes in the Earth's energy budget and, hence, in the Earth's climate. With the development of space technology, meteorologists recognized the feasibility of using satellites for measuring and even continuously monitoring the Earth's energy budget and also regional energy budgets; and they have accordingly pushed the development of the necessary technology.

High accuracy is necessary if slow modifications and long-term trends are to be recognized – for example, for albedo measurements, the accuracy requirements as estimated by various groups range from 0.1 percent to 1 percent. A fleet of satellites capable of jointly observing all areas of the Earth nearly simultaneously would obviously be needed, but the nature of their measurements remains under discussion. Horizon-to-horizon scanning of the Earth, with appropriate spectral resolution, spatial resolution, and radiometric accuracies, would seem to provide the most thorough and unequivocally interpretable information. Indeed, our present estimates of the Earth's albedo are supported by just such data. However, the dubiousness of achieving long-term reliability of the equipment and instruments, and the practical difficulties in working up huge quantities of data militate against such an approach for a continuous long-term project.

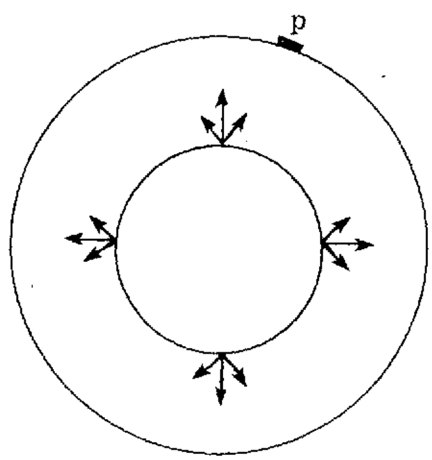
Integrating Radiometer-Type Sensors

Other approaches, presumably simpler and more practical, involve integrating radiometer-type sensors that respond to the total radiation received from the spherical segment of the Earth below them (horizon-to-horizon in all directions). A group of two or three such sensors on each satellite, differently coated so that they have different absorptances for sunlight (direct or Earth-reflected) and also have different absorptances for Earth-emitted thermal radiation, should provide a set of simultaneous equations from which the separate intensities of these two wavelength bands can be extracted. Two such sensor concepts will be briefly discussed in the following paragraphs.

The flat-plate sensor.- Consider an imaginary spherical surface concentric with and surrounding the Earth, as in sketches (a) and (b). All the radiation leaving the Earth, either as reflected sunlight or as Earth emission, must pass outward through this spherical surface; this must be true regardless of the directional distribution of the emitted radiation – for example, whether the radiation is purely radial, as in sketch (a), or is diffuse, as in sketch (b). A small flat-plate sensor p in a circular orbit, stabilized such that it is always normal to the radius, may be considered as an element of



Sketch (a) – Radial emission



Sketch (b) – Diffuse emission

this imaginary surface, intercepting the radiation passing outward at that instant at that location. An adequate number of such sensors, all in circular orbits of the same radius, could thus represent the surrounding imaginary sphere and could, accordingly, adequately sample the outgoing radiation from the Earth, regardless of its directionality. The flat-plate concept thus seems to be the most obvious approach to the integrating radiometer-type sensor mentioned in the preceding paragraph.

Achieving long-term reliability for the stabilization mechanisms of a fleet of such sensors may seem rather formidable, but it is not prohibitive. A rather more basic difficulty is caused by the fact that the Earth-reflected solar radiation has a somewhat variable spectral content that, in any case, differs from that of the direct solar radiation; so that unless the spectral absorptance of the coating is uniform over the entire wavelength range of the solar spectrum (which is unlikely), a certain degree of inaccuracy is introduced by assuming that the coating has a unique solar absorptance, applicable to both direct and reflected sunlight. A similar problem, although probably less serious, exists for the infrared, since the Earth-emitted thermal radiation also has a variable spectral content.

A further basic difficulty would result from nonuniform directional absorptance of the sensor coating: While it is true that the total radiation striking the sensor is the same in sketches (a) and (b), if absorptance decreases with increasing incidence angle, less of the radiation will be absorbed in sketch (b) than in sketch (a), so the readings will be different. Absorptance does, in fact, generally decrease with increasing incidence angle, especially for the longer wavelengths. Hence, accurate interpretation of the readings would require not only that the directional absorptances of the sensor coatings be known over the entire spectral range of interest, but also that the directional-spectral distribution of the radiation striking the sensors also be approximately known or estimable. This latter requirement is not the same as requiring that the results be known beforehand; but it does emphasize the need for special analytical studies to determine the best means of taking this matter into account in interpreting the data, and to estimate the corresponding range of uncertainty in the answer.

The spherical sensor.- Instead of a group of flat plates, one might use a group of thin-walled hollow spheres, well separated from each other. The basic equations for reducing the data from groups of such spheres are given in reference 1. The approach is attractive because a uniformly coated sphere does not have to be stabilized, since it always presents the same aspect to the Earth regardless of its orientation. Furthermore, the directional-absorptance question that was raised for the flat plate seems less important for the sphere, since the sphere presents the same aspect to radiation received from the nadir as to radiation received from the horizon. However, since the angles of incidence of any radiation striking the sphere range from 0° to 90°, one must again have at least an approximate a priori knowledge of the spectral content of the radiation received from the different parts of the Earth's segment beneath it. (Availability of experimental data on the directional-spectral absorptance characteristics of the different coatings used on the spheres is assumed.) In any case, the problems introduced by the nonuniform spectral content of the Earth-emitted radiation and of the Earth-reflected radiation, previously discussed for the flat plate, also exist for the sphere.

Still another complication is introduced if the coating is degraded by solar ultraviolet and particle radiation, with corresponding changes in absorptance characteristics. Reference 1 indicates that absorptance values may be extracted from the flight data, so the degradation might seem to be only an acceptable annoyance; however, unless the surface of the sphere is uniformly exposed to the degrading radiations, the changes in absorptance characteristics will not be uniform over the surface. Uniform exposure might be provided by using magnetic coils in the satellite to interact with the Earth's magnetic field and suitably rotate the satellite, but the basic simplicity of the satellite would thereby be compromised.

A basic source of uncertainty in interpretating the data is that the irradiation by the Earth depends on the directional distribution of the radiation leaving the Earth. It will be remembered that for the flat-plate sensor the amount of radiation received at the flat plate from the Earth is independent of this directional distribution – the flat plate is merely a sample area of the imaginary surrounding sphere (sketches (a) and (b)) and intercepts all the radiation that would be passing outward through that area, regardless of the direction from which the radiation comes. For the spherical sensor, a similar statement cannot be made. If the flat-plate sensor is replaced with a spherical sensor in sketches (a) and (b), it would no longer be true that the radiation reaching the sensor is the same in both sketches. In fact, in a theoretical limiting case, when the sphere is just above the Earth and the directional distribution of radiation in sketch (b) is Lambertian, the sphere would receive twice as much radiation as it would for the purely radial radiation of sketch (a).

Fortunately, this 2:1 uncertainty is not realistic, since (1) the sphere would not be very close to the Earth, because the requirements for a long lifetime and a broad overview require that the satellite be at a fairly high altitude (of the order of, say, 1000 km), and (2) the purely radial distribution in sketch (a) never occurs, and we are, in fact, concerned only with moderate deviations from the Lambertian distribution. Both Earth-emitted thermal radiation and, especially, Earth-reflected solar radiation show deviations from Lambert's law.

The present study concerns specifically the irradiation of a sphere by the Earth-emitted radiation, for which the deviation is often referred to phenomenologically as limb darkening. As in other typical cases of limb darkening, it is characterized not only by reduced infrared brightness of the horizon (compared with the nadir) but also by relatively more of the longer wavelengths in the radiation from the horizon; and it exists because the radiation from the horizon traverses a long path through the cold upper troposphere and lower stratosphere, whereas the radiation from the nadir traverses a shorter path through these cold air layers and, accordingly, remains more nearly like that originally emitted from the warm Earth's surface and lower air layers.

Reference 2 called special attention to the effect of limb darkening on the input to a spherical sensor; and, since then, the Tiros series of satellites has provided data from which the limb darkening of the Earth-atmosphere system has been deduced (refs. 3 and 4). The data from references 3 and 4 have been used in this study to calculate the limb-darkening effect on the irradiation of a spherical satellite. In addition, as mentioned previously, curves and formulas are given herein that may be applied for any specified deviation from the Lambertian radiation law. In the application of these results, it may be desirable to allow for the fact that the wavelengths of the radiation from the limb are larger than those from the nadir; because if the directional absorptance characteristics of the sphere coatings are not constant throughout the entire infrared range of interest, some further correction for limb darkening would be needed.

SYMBOLS

A cross-sectional area of spherical satellite, m^2

$A_n(k) = \int_0^{\pi/2} \frac{\phi^n \sin\phi \cos\phi}{\sqrt{1 - k^2 \sin^2\phi}} d\phi$ (eq. (10))

a_n coefficient in power series for $F(\phi)$ (see eq. (8))

D factor measuring effect of limb darkening on Earth's thermal radiation incident on spherical satellite, $\frac{I(k)}{A_0(k)}$

$F(\phi)$ limb-darkening function

G_n coefficient in cosine power series for $F(\phi)$ (see eq. (A2))

h satellite altitude, km

$I(k) = \int_0^{\pi/2} \frac{F(\phi) \sin\phi \cos\phi}{\sqrt{1 - k^2 \sin^2\phi}} d\phi$ (eq. (6))

i intensity of radiation (per steradian) from a surface point (per unit projected area), W/m^2-sr (eq. (1))

$k = \sin\theta$

Q irradiation of spherical satellite by Earth, W (eq. (3))

R radius of Earth, km

S satellite

T temperature of Earth, K

α nadir angle of point on Earth's surface from satellite, radians (see fig. 1)

ϵ total hemispherical emissivity of Earth (considered to be uniform over area seen from satellite)

θ angle between nadir and tangent to Earth's surface from satellite (nadir angle of horizon, see fig. 1), radians

σ Stefan-Boltzmann constant, $5.6696 \times 10^{-8}\ \mathrm{W/m^2\text{-}K^4}$

ϕ angle between zenith and line from Earth's surface to satellite (zenith angle, see fig. 1), radians or deg

ANALYSIS AND DISCUSSION

Basic Equations

The Earth-atmosphere system will be approximated by a sphere of radius R. The radiation leaving any point on the sphere will be assumed to be a function of the zenith angle, and independent of azimuth. The intensity of radiation (per steradian) from a surface point (per unit projected area) may be written as (see ref. 5)

$$i = \frac{\sigma}{\pi} \epsilon T^4 F(\phi) \tag{1}$$

where $F(\phi)$ is the limb-darkening function and indicates the angular distribution of the observed surface brightness. It is a constant, $F(\phi) = 1$, for a Lambertian radiator; and in any case, it satisfies the equation

$$2 \int_0^{\pi/2} F(\phi) \sin \phi \cos \phi \, d\phi = 1 \tag{2}$$

Limb-darkening curves for the Earth are given in references 3, 4, and 6. These curves are proportional to $F(\phi)$, but are normalized such that they are unity at $\phi = 0$.

Now consider a spherical satellite S at height h above the Earth, as shown in figure 1. The radiation from the Earth incident on the satellite is

$$Q = A \int i\,d\Omega = A \int_{\alpha=0}^{\theta} i(2\pi \sin \alpha \, d\alpha)$$

$$= 2A\sigma\epsilon T^4 \int_{\alpha=0}^{\theta} F(\phi) \sin \alpha \, d\alpha \tag{3}$$

where $d\Omega$ is the solid angle subtended at the satellite by an element of the Earth's surface and A is the cross-sectional area of the satellite. It is assumed that the Earth's emittance ϵ and temperature T are constant over the view field. The law of sines applied to triangle OSP gives

$$\frac{\sin \alpha}{R} = \frac{\sin \phi}{R + h}$$

Using this relation to eliminate α from the integrand in equation (3) gives

$$Q = 2A\sigma\epsilon T^4 k^2 \int_0^{\pi/2} \frac{F(\phi) \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi \tag{4}$$

where

$$k = \frac{R}{R + h} = \sin \theta \tag{5}$$

Denote the integral in equation (4) by I(k):

$$I(k) = \int_0^{\pi/2} \frac{F(\phi) \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi \tag{6}$$

It is seen that I(k) is a function of k and a function of the limb-darkening curve $F(\phi)$. The influence of limb darkening on Earth irradiation of a spherical satellite is contained in I(k). Comparison of equations (2) and (6) shows that

$$I(0) = \frac{1}{2}$$

Thus, at large distances from Earth, the incident radiation is independent of the limb-darkening curve, as would be expected, since at large distances the Earth would approximate a point source.

Polynomial Limb-Darkening Function

In reference 3, the limb-darkening curve $F'(\phi)$ is given as

$$F'(\phi) = 1 - 1.116 \times 10^{-4} \phi - 5.873 \times 10^{-5} \phi^2$$

$$+ 1.387 \times 10^{-6} \phi^3 - 1.523 \times 10^{-8} \phi^4$$

where ϕ is expressed in degrees and where the prime indicates that F has been normalized such that it is unity at $\phi = 0$.

Because of the second term in this expansion, a plot of $F'(\phi)$ against ϕ has a finite slope at $\phi = 0$. The existence of such a finite slope seems rather unlikely; for example, if F' is plotted against ϕ as the line SP (fig. 1) swings in a plane through the zenith, there would be a finite discontinuity in the rate of change of F' as the line passes through the zenith ($\phi = 0$). However, the given equation represents a best fit to the data, and altering it in order to eliminate the linear term would hardly be justified. In any case, the finite slope at $\phi = 0$ is quite small.

The expression for $F'(\phi)$, when normalized according to equation (2), becomes

$$F(\phi) = 1.0949 - 0.00700\phi - 0.2113\phi^2 + 0.2856\phi^3 - 0.1797\phi^4 \tag{7}$$

where ϕ is expressed in radians.

Following the form of equation (7), denote

$$F(\phi) = \sum_n a_n \phi^n \tag{8}$$

Then equation (6) reduces to

$$I(k) = \sum_n a_n A_n(k) \tag{9}$$

where

$$A_n(k) = \int_0^{\pi/2} \frac{\phi^n \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi \tag{10}$$

For the cases $k = 0$ and $k = 1$, recursion relations may be readily developed for evaluating this integral. Thus, for $k = 0$, two successive integrations by parts give the relation

$$A_n(0) = \frac{1}{4}\left(\frac{\pi}{2}\right)^n - \frac{n(n - 1)}{4} A_{n-2}(0) \tag{11}$$

which, together with the easily derived

$$\left.\begin{aligned} A_0(0) &= \frac{1}{2} \\ A_1(0) &= \frac{\pi}{8} \end{aligned}\right\} \tag{12}$$

provide an easy recursive determination of all $A_n(0)$.

Similarly, for $k = 1$, two successive integrations by parts give the relation

$$A_n(1) = n\left(\frac{\pi}{2}\right)^{n-1} - n(n - 1)A_{n-2}(1) \tag{13}$$

which, together with the easily derived

$$\left.\begin{aligned} A_0(1) &= 1 \\ A_1(1) &= 1 \end{aligned}\right\} \tag{14}$$

provide an easy recursive determination of all $A_n(1)$. Table I lists $A_n(0)$ and $A_n(1)$ for $n = 0$ to 9.

The case $k = 1$ is a limiting case for zero altitude. In reality, radiation from the atmosphere would render this analysis inaccurate below an altitude of 150 km.

For $0 < k < 1$, the values of $A_n(k)$ have been evaluated numerically and are shown in figure 2 for values of n from 0 to 9. By use of figure 2, $I(k)$ may be evaluated for any limb-darkening expression of the form of equation (8). It is seen that the values of $A_n(k)$ vary weakly with k for small k, that is, for large altitudes, and vary rapidly for k near 1, that is, for low altitudes.

An alternative method of evaluating $I(k)$ by expanding $F(\phi)$ as a cosine power series is presented in the appendix.

Results Based on Tiros Data

The limb-darkening influence on $I(k)$ corresponding to equation (7) has been computed as a function of k and is shown as the solid curve in figure 3. Comparison with the upper curve of figure 2(a) shows that $I(k)$ and $A_0(k)$ are nearly the same for small k (large satellite altitude) but that the difference becomes appreciable at large k (small satellite altitudes). In the absence of limb darkening, $I(k)$ would be identically $A_0(k)$. A suitable measure of the effect of limb darkening on heat incident on a spherical satellite is the factor

$$D = \frac{I(k)}{A_0(k)}$$

This factor D is plotted as a function of k in figure 4, and as a function of altitude in figure 5 (solid curve in both figures). It is seen that D varies rapidly with altitude below 300 km although, of course, the lower altitudes would not be practical because of air drag. Between 500 and 900 km, the variation is only from 0.976 to 0.984, less than 1 percent. In order to reduce the effect of limb darkening to less than 1 percent, that is, to get a value of D above 0.99, it is necessary to go to an altitude above 2000 km.

The foregoing analysis has shown the main effect of limb darkening on incident radiation from the Earth to a spherical satellite. Some remarks and additional calculations concerning the uncertainty of the results are given in the following section.

Uncertainty of the Limb-Darkening Function

Results which were presented in figures 3 to 5 were computed by using the limb-darkening curve of reference 3, which is a globally averaged curve. Reference 4 presents limb-darkening curves for various regions of the Earth: arctic, antarctic, desert, tropics, and the southern hemisphere. These curves were normalized in accordance with equation (2) and the corresponding incident-heat factors $I(k)$ were computed. Although these curves show appreciable differences, the corresponding incident-heat factors $I(k)$ for the most part show only very slight differences, and accordingly only one $I(k)$ curve is shown in figure 3 for reference 4. For $k \leqq 0.5$, the values of $I(k)$ computed for the various curves are equal, to three decimal places. For $0.5 < k < 0.95$ (300 to 6000 km), the values of $I(k)$ differ by no more than 0.007, but for $k = 1$, the values of $I(k)$ differ by as much as 0.03.

In reference 3, the limb-darkening curves were terminated at $\phi = 78.75^{\circ}$. For the present analysis, the curve fit (eq. (7)) was needed and was used out to $\phi = 90^{\circ}$. Also, some of the curves from reference 4 were extrapolated to $\phi = 90^{\circ}$. The portion of the curve from $\phi = 75^{\circ}$ to 90° is relatively more important at low satellite altitudes than at high altitudes, because the relative contribution from the areas near the horizon is greater. Considerations of the previous paragraph indicate that a moderate inaccuracy in extrapolation is not critical for $k < 0.95$.

In figures 3 to 5, it is seen that the incident heat as computed with the extrapolated curves of reference 3, which is a global average, is less than the incident heat as computed with the limb-darkening curves of reference 4. The limb-darkening curve of reference 3 lies in the midst of the curves of reference 4, for $\phi < 70^{\circ}$, but drops below them for $\phi < 80^{\circ}$. This points out the need for measurements of the limb darkening in the range $70^{\circ} < \phi < 90^{\circ}$.

It was assumed in this study that upwelling radiation is axisymmetric about the vertical. When the satellite is at a fairly high altitude, however, the radiation temperature of the viewed area may be very nonuniform. For example, if its altitude exceeds 500 km, the satellite can receive radiation from both the Torrid Zone and a Frigid Zone simultaneously. This radial and azimuthal nonuniformity would complicate the interpretation of data even if the Earth were a Lambertian radiator; but limb darkening adds a further complication. Finally, the analysis assumed that the satellite was at a clearly definable altitude h above a radiating spherical surface. In reality, radiation is also received from clouds and the atmosphere itself, so that the location of the effective radiating surface is not clearly definable, especially at the horizons.

Thus, the present study involves a number of unanalyzed simplifications, investigation of which could entail considerable effort. Nevertheless, the present analysis shows the magnitude and the main characteristics of the limb-darkening effect on Earth irradiation of a satellite.

CONCLUSIONS

A simple analysis has been made of the effect of limb darkening on Earth irradiation of a spherical satellite. For this study, an estimated average limb darkening based on Tiros data was used. It was concluded that:

1. Limb darkening must be considered for accurate measurements of Earth-emitted heat incident on a spherical satellite.

2. The computed effect of limb darkening was 1.8 percent at 900 km altitude, 2.5 percent at 500 km altitude, and 3.0 percent at 300 km altitude. Below 300 km, it increased rapidly with decreasing altitude.

3. In order to reduce the effect of limb darkening to less than 1 percent, it is necessary to go above 2000 km altitude.

4. Measurements of limb darkening in the range $70^{\circ} < \phi < 90^{\circ}$ are needed.

The basic assumption that the satellite views a well-defined spherical cap, radiating uniformly according to a known limb-darkening curve, is recognized as somewhat unrealistic. Accordingly, although the present study is adequate for estimating the importance of limb darkening, more work is clearly needed to establish a basis for relating the Earth thermal radiation to what is received and absorbed by an orbiting sphere.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., November 8, 1974.

APPENDIX

ALTERNATIVE EVALUATION OF I(k)

In the evaluation of the effect of limb darkening on the Earth's heat incident on a spherical satellite, the expression

$$I(k) = \int_0^{\pi/2} \frac{F(\phi) \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi \tag{A1}$$

appears. In the main text, $F(\phi)$ was expressed as a power series in ϕ, consistent with Lienesch and Wark (ref. 3). An alternative procedure is to express $F(\phi)$ as a cosine power series, in which case the integrals required to express the heat incident on a satellite can be evaluated in closed form. Thus, let

$$F(\phi) = \sum_{n=0}^{\infty} G_n \cos^n \phi \tag{A2}$$

whence

$$I(k) = \sum_{n=0}^{\infty} G_n L_n(k) \tag{A3}$$

where

$$L_n(k) = \int_0^{\pi/2} \frac{\sin \phi \cos^{n+1} \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi \tag{A4}$$

The law of sines (see p. 8) is used to rewrite the integral in terms of α:

$$L_n(k) = \frac{1}{k^{n+2}} \int_0^{\sin^{-1} k} \left(k^2 - \sin^2 \alpha\right)^{n/2} \sin \alpha \, d\alpha$$

A change of variables is made,

$$x = \cos \alpha$$

so that

$$L_n(k) = \frac{1}{k^{n+2}} \int_b^1 \left(x^2 - b^2\right)^{n/2} dx \tag{A5}$$

where

$$b = \cos\theta = \sqrt{1 - k^2} \tag{A6}$$

Integration of equation (A5) by parts gives

$$(n + 1)L_n = \frac{1}{k^2} - \frac{nb^2}{k^{n+2}}\int_b^1 \left(x^2 - b^2\right)^{\frac{n-2}{2}} dx$$

$$L_n = \frac{1}{k^2(n + 1)} - \frac{nb^2}{k^2(n + 1)} L_{n-2} \tag{A7}$$

Equation (A5) may be integrated for cases $n = 0$ and $n = 1$ to give

$$\left.\begin{aligned} L_0(k) &= \frac{1}{1 + b} \\ L_1(k) &= \frac{1}{2k^3}\left[k + b^2 \ln\left(\frac{b}{1 + k}\right)\right] \end{aligned}\right\} \tag{A8}$$

The L_n may be computed recursively by using equation (A7), with equations (A8) providing the starting values.

Because $F(\phi)$ is defined over a hemisphere, another potentially useful form for $F(\phi)$ is an expansion in terms of Legendre polynomials in $\cos\phi$. The Legendre polynomial may be interchanged with the power series, for example, through use of a table as in reference 7, permitting the results of this appendix to be used.

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TABLE I. VALUES OF $A_n(0)$ AND $A_n(1)$ FOR $n = 0$ to 9

n	$A_n(0)$	$A_n(1)$
0	0.5000	1.0000
1	.3927	1.0000
2	.3668	1.1416
3	.3799	1.4022
4	.4214	1.8040
5	.4913	2.3964
6	.5947	3.2576
7	.7404	4.5020
8	.9410	6.3461
9	1.2277	9.4382

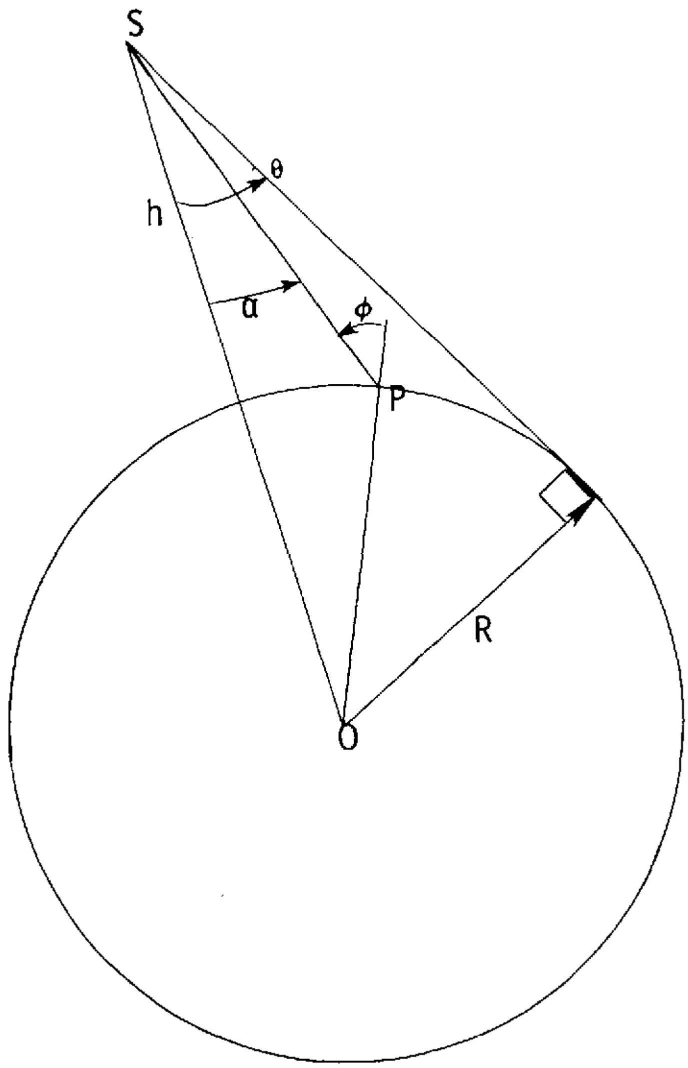
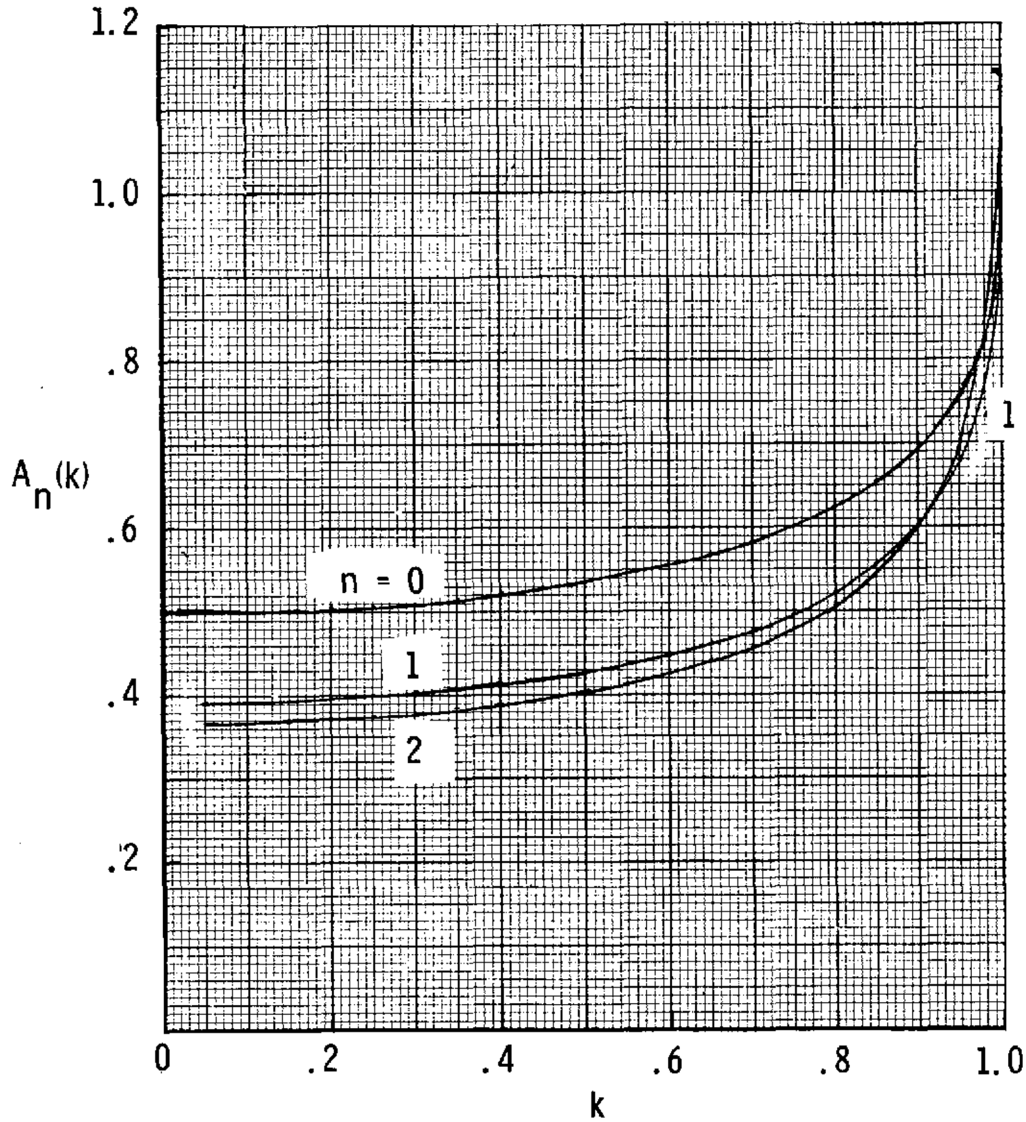
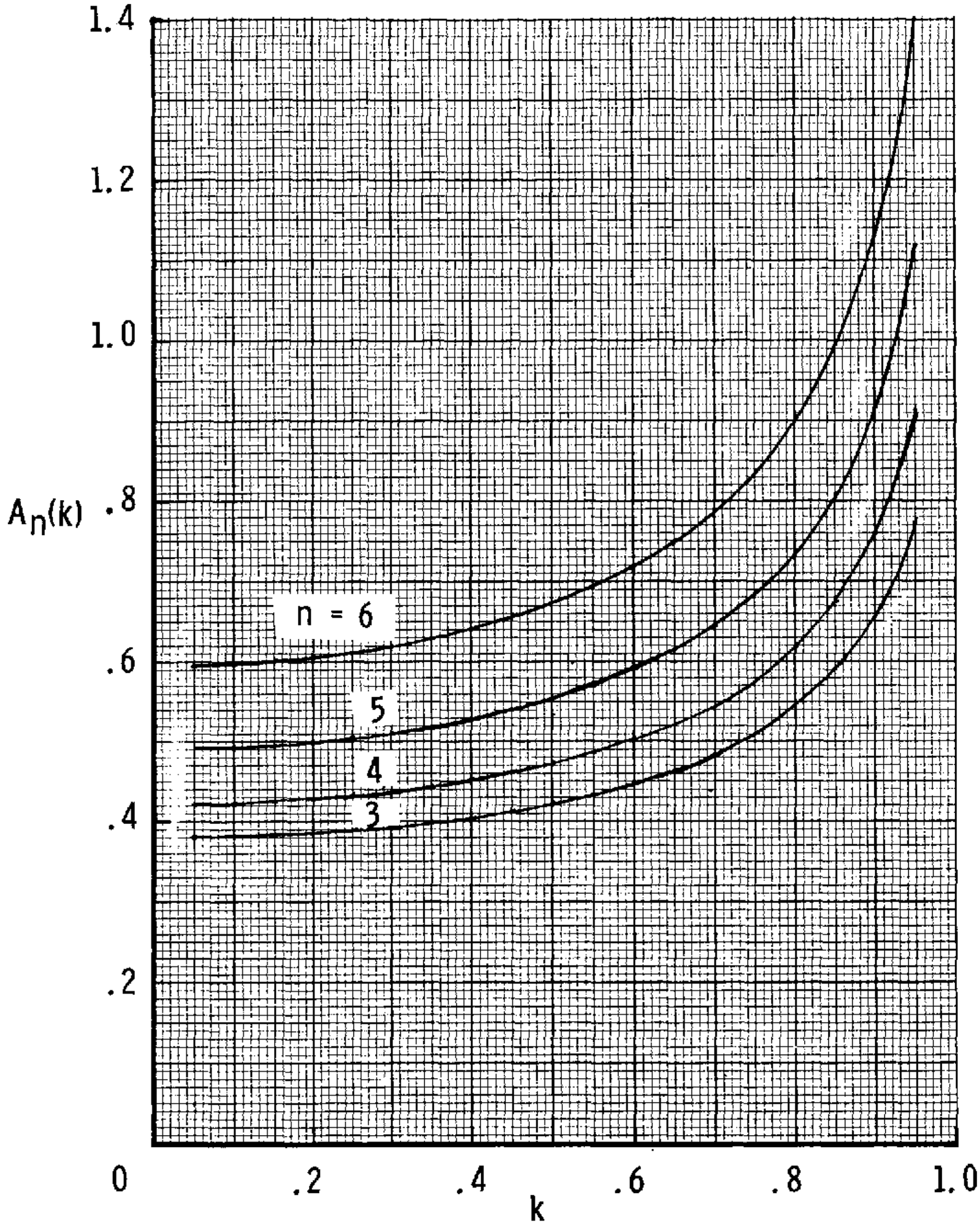


Figure 1.- Geometry for radiation from Earth to spherical satellite.



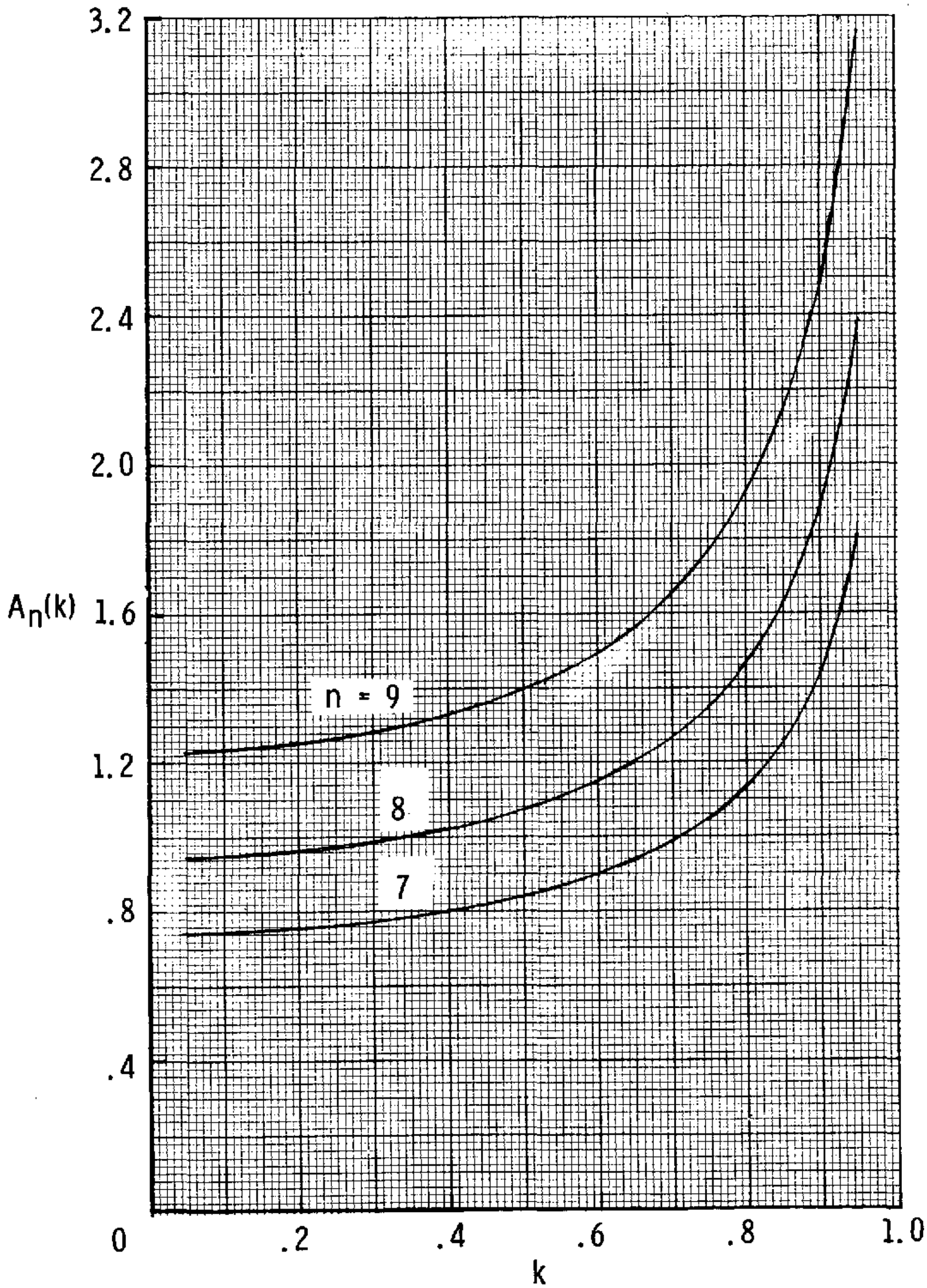
(a) n = 0, 1, and 2.

Figure 2.- A_n as a function of k. $A_n(k) = \int_0^{\pi/2} \frac{\phi^n \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi$.



(b) n = 3, 4, 5, and 6.

Figure 2.- Continued.



(c) n = 7, 8, and 9.

Figure 2.- Concluded.

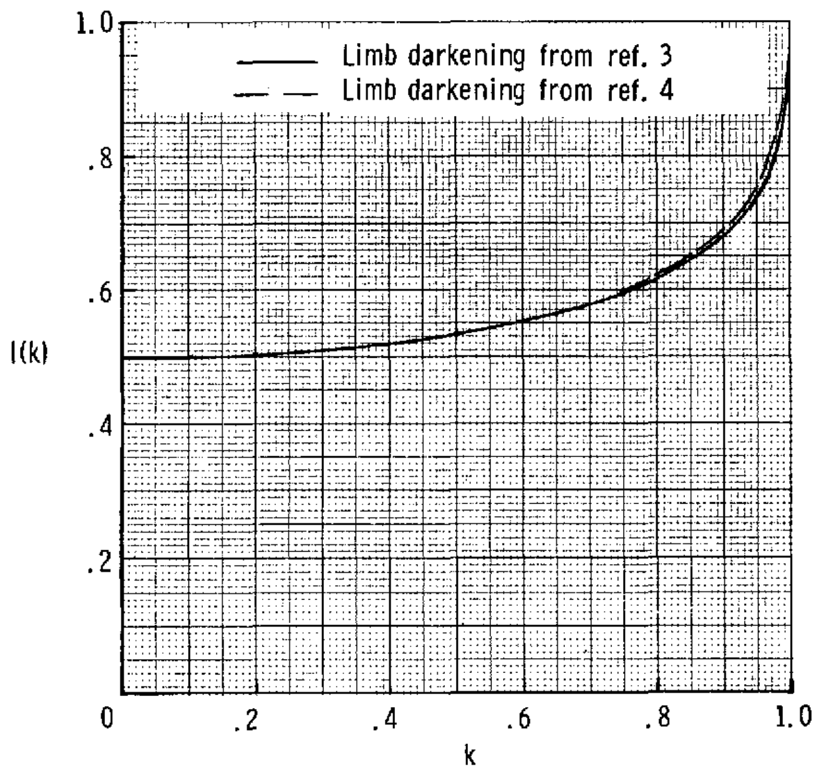


Figure 3.- Incident-heat factor I for spherical satellite as a function of k.

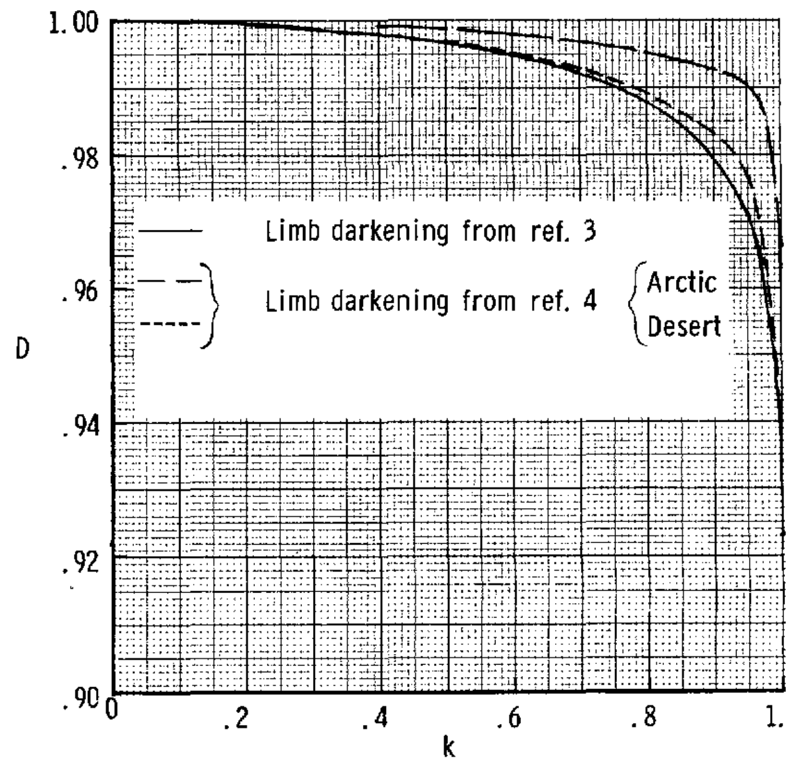


Figure 4.- Factor D measuring effect of limb darkening on Earth radiation incident on spherical satellite as a function of k.

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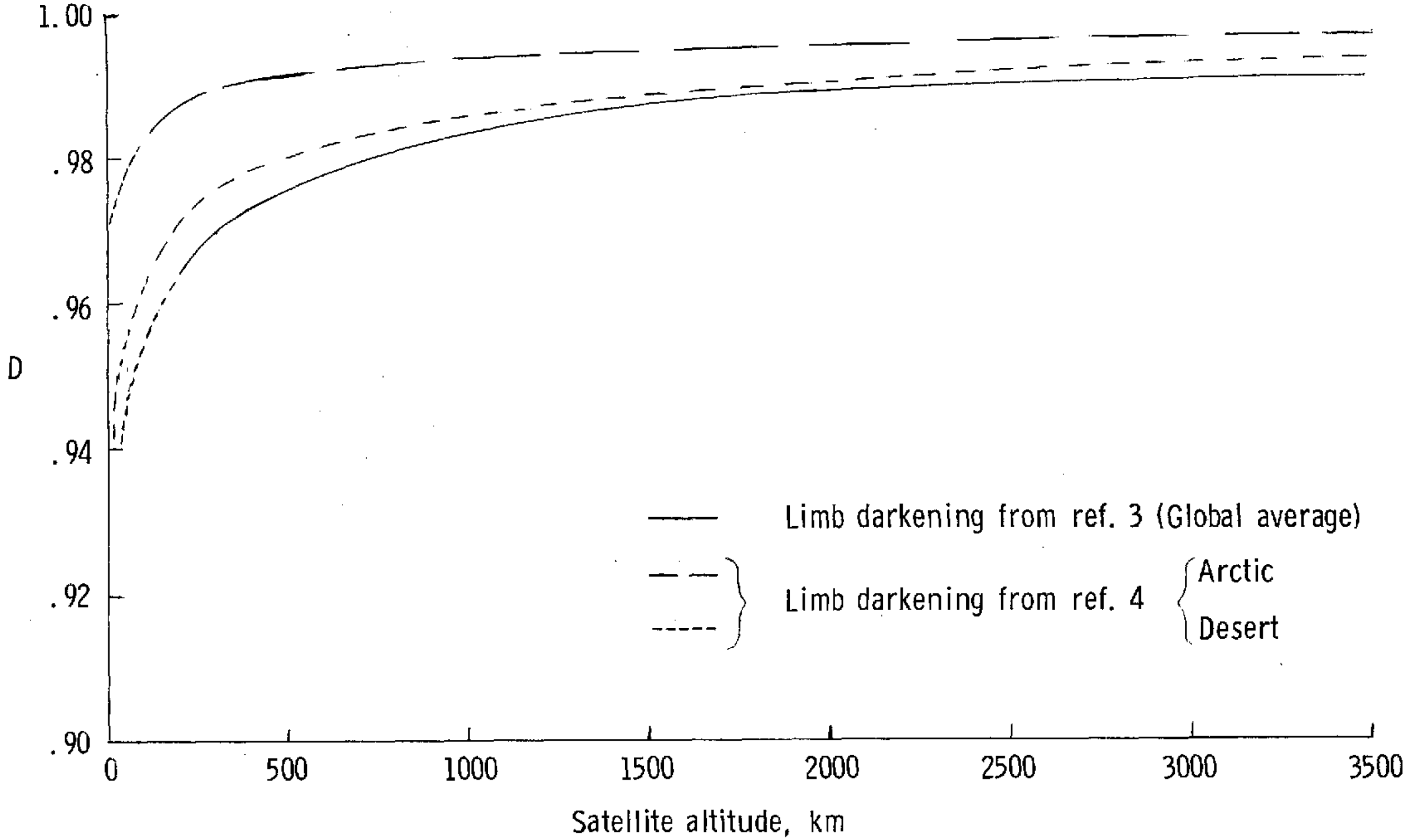

Figure 5.- Factor D, $D = \frac{I}{A_0}$, the effect of limb darkening on Earth radiation incident on a spherical satellite as a function of altitude.